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Pointwise estimates via parabolic potentials for a class of doubly nonlinear parabolic equations with measure data

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Abstract. On a cylindrical domain E_T, we consider doubly nonlinear parabolic equations, whose prototype is $\partial_t u - \operatorname{div}(|u|^{m-1}|Du|^{p-2}Du) = \mu$, where μ is a non-negative Radon measure having finite total mass $\mu(E_T)$. The central objective is to establish pointwise estimates for weak solutions in terms of nonlinear parabolic potentials in the doubly degenerate case $(p \geq 2, m > 1)$. Moreover, we will prove the sharpness of the estimates by giving an optimal Lorentz space criterion regarding the local uniform boundedness of weak solutions and by comparing them to the decay of the Barenblatt solution.

1. Introduction and main result

In this paper, we study potential estimates for doubly nonlinear parabolic equations with measure data. Such equations arise in the field of plasma physics, ground water surveys, or the motion of viscous fluids, but also in the modeling of an ideal gas flowing isoentropically in a homogeneous porous medium. In this introductory section, we describe the treated problem and specify some notations. Further, we explain the notion of weak solutions, mention the main results, and comment on the proof strategies as well as the history of potential estimates.

1.1. Setting

We consider a class of nonhomogeneous doubly nonlinear parabolic equations

$$\partial_t u - \operatorname{div}\big(\mathbf{A}(x, t, u, Du)\big) = \mu \tag{1.1}$$

in a space-time cylinder $E_T := E \times (0, T)$, where $E \subset \mathbb{R}^n$ is an open bounded set, $n \geq 2$, $T > 0$, and $\mu \in \mathcal{M}^+(E_T)$ is a non-negative Radon measure on E_T with finite total mass $\mu(E_T) < \infty$.

Our aim is to establish pointwise estimates in terms of nonlinear parabolic potentials, where the main tasks are to identify the decent potential for the doubly nonlinear parabolic context and to construct intrinsic cylinders that suitably reflect

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the geometry of the equations under consideration. The optimality of our estimates will be explained by deducing a sharp Lorentz space criterion for μ providing the local boundedness of u, and by comparing the behavior of the potential to the decay of the Barenblatt fundamental solution.

Throughout this paper, the vector field $\mathbf{A}\colon E_T \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be a Carathéodory function, i. e. it is measurable with respect to $(x,t) \in E_T$ for all $(u,\xi) \in \mathbb{R} \times \mathbb{R}^n$ and continuous with respect to $(u,\xi) \in \mathbb{R} \times \mathbb{R}^n$ for almost every $(x,t) \in E_T$. Moreover, we want $\mathbf{A}$ to satisfy the ellipticity condition

$$\mathbf{A}(x,t,u,\xi) \cdot \xi \geq C_0 |u|^{m-1} |\xi|^p, \tag{1.2}$$

together with the growth condition

$$|\mathbf{A}(x,t,u,\xi)| \leq C_1 |u|^{m-1} |\xi|^{p-1} \tag{1.3}$$

for any $u \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, and almost every $(x,t) \in E_T$, where $C_0 > 0$ and $C_1 > 0$ are fixed constants, $p \geq 2$ and $m > 1$.

1.2. Some remarks on doubly nonlinear parabolic equations

The model example for equations treated in the sequel is given by the doubly nonlinear parabolic equation

$$\partial_t u - \operatorname{div}\big(|u|^{m-1} |Du|^{p-2} Du\big) = \mu \quad \text{in } E_T, \tag{1.4}$$

whose modulus of ellipticity is $|u|^{m-1}|Du|^{p-2}$. For $p > 2$, $m > 1$, this quantity vanishes if u or $|Du|$ become 0, which is why we call the equation doubly degenerate, whereas in the singular-degenerate situation $p < 2$, $m > 1$, the coefficient $|Du|^{p-2}$ tends to ∞ and $|u|^{m-1} \to 0$ as $|u| \to 0$, $|Du| \to 0$. According to that approach, the cases $p > 2, m < 1$ and $p < 2, m < 1$ are named degenerate-singular and doubly singular, respectively. Apart from that, one can categorize the solutions with regard to their support after finite time and speed of propagation, where the equation is referred to as of the type of slow, normal, or fast diffusion, depending on whether $p + m$ is larger than, equal to, or less than 3. Both classifications can also be found in [17, p. 23].

Lately, several authors examined doubly nonlinear parabolic equations because of their physical and mathematical interest, though, substantial parts of the recent research were not on equations of the above universal form, but rather on specific examples like (1.4) with either $\mu \equiv 0$, $p+m = 3$, or other simplifications of (1.1). For instance, Hölder regularity and Harnack's inequality for bounded weak solutions were established in [16,17,34,44] and [24,43]. What is more, [33,35,40] are concerned with the asymptotic behavior of solutions to doubly nonlinear parabolic equations for certain values of the quantity $p+m$, and the local boundedness of the gradient of a solution to the homogeneous equation was shown in [36] under the additional assumption that u is strictly positive. Existence and uniqueness results for the Cauchy–Dirichlet problem with an inhomogeneity $\mu \in L^\infty(E_T, \mathbb{R}_{\geq 0})$ were developed in [18–20] and generalized in [38,39] to Lebesgue integrable functions and Radon measures as right-hand sides.

However, since the proof strategies are quite sophisticated due to the inherent difficulty of a double nonlinearity, the achievements, especially for the measure-valued equation, are relatively sparse, and, to the author's knowledge, there is no theory for potential estimates regarding doubly nonlinear parabolic equations in the literature up to now. Nevertheless, single special cases of (1.1) like the porous medium equation ($p = 2$) or the p-Laplacian equation ($m = 1$) have been intensively studied and we refer the interested reader to [9,21,41,42] and the lists of references therein.

1.3. Notations

As to the notation, we always write $z = (x, t)$ for a point $z \in \mathbb{R}^{n+1} \cong \mathbb{R}^n \times \mathbb{R}$. As is customary, we call $q' := \frac{q}{q-1} \in [1, \infty]$ the Hölder conjugate of $q \in [1, \infty]$. By $\{u > \ell\}$, we express the superlevel set $\{(x, t) \in E_T : u(x, t) > \ell\}$ where the function u exceeds the level $\ell > 0$, and we address the positive part of u as $u_+ := \max\{u, 0\}$. We denote the weak spatial derivative of the function u by $Du = D_x u = (D_{x_1} u, D_{x_2} u, \dots, D_{x_n} u)$, and $\partial_t = \frac{\partial}{\partial t}$ is the operator for the time derivative. Besides, by $\mathcal{M}^+(E_T)$, we mean the set of all non-negative Radon measures, and $c \equiv c(\cdot)$ stands for a constant, which may vary from line to line and depend only on the parameters in brackets. Finally, for $(x_0, t_0) \in E_T$, r, $\theta > 0$, and $\nu > 0$, we define the parabolic cylinder

$$Q_{r,\theta} \equiv Q_{r,\theta}(x_0, t_0) := B_r(x_0) \times (t_0 - \theta, t_0) \tag{1.5}$$

and write $\nu Q_{r,\theta} := Q_{\nu r, \nu^p \theta}$ for its rescaled associate.

1.4. Weak solutions

In this section, we specify the notion of weak solutions to the Cauchy–Dirichlet problem associated to the doubly nonlinear parabolic equation (1.1), which is given by

$$\begin{cases} \partial_t u - \operatorname{div}\big(\mathbf{A}(x, t, u, Du)\big) = \mu & \text{in } E_T, \\ u = 0 & \text{on } \Gamma_T, \end{cases} \tag{1.6}$$

where $\Gamma_T := [\overline{E} \times \{0\}] \cup [\partial E \times (0, T)]$ denotes the parabolic boundary of E_T.

Definition 1.1. Let $\beta := \frac{m-1}{p-1}$. A non-negative function $u : E_T \to \mathbb{R}$ satisfying

$$u \in C^0\big([0, T]; L^{\beta+2}(E)\big) \text{ and } u^{\beta+1} \in L^p\big((0, T); W_0^{1,p}(E)\big) \tag{1.7}$$

is termed a weak solution to the Cauchy–Dirichlet problem (1.6) if and only if the identity

$$\iint_{E_T} \big[-u\partial_t \varphi + \mathbf{A}(x, t, u, Du) \cdot D\varphi\big]\, dz = \iint_{E_T} \varphi \, d\mu \tag{1.8}$$

holds true for any testing function $\varphi \in C^1(\overline{E_T})$ vanishing on $[\overline{E} \times \{T\}] \cup [\partial E \times (0,T)]$. In (1.8), the symbol Du has to be understood in the sense of

$$Du := \tfrac{1}{\beta+1}\chi_{\{u>0\}}u^{-\beta}Du^{\beta+1}.$$

Remark 1.2. For the interested reader, we remark that, apart from our definition of weak solutions, which is also employed in [15,19,33,35,39,42,44], for instance, there is another concept of weak solutions in the context of doubly nonlinear parabolic equations (see [4,10,18,38]), where our regularity assumptions (1.7) are replaced by

$$u \in C^0\big([0,T];L^2(E)\big) \quad \text{and} \quad u^{\alpha+1} \in L^p\big((0,T);W_0^{1,p}(E)\big) \tag{1.9}$$

with $\alpha := \frac{m-1}{p}$.

1.5. Main results

We now state the central results of this paper. The parabolic potential $\mathbf{P}_p^\mu$ appearing in the following theorem was originally introduced in [31], where potential estimates of the form (1.10) were proven for the evolutionary p-Laplacian; see also Sect. 1.7 for a discussion of the history of potential estimates. The proof of Theorem 1.3 will be performed in Sect. 4.

Theorem 1.3. *Let u be a weak solution to the Cauchy–Dirichlet problem (1.6) for the doubly nonlinear parabolic equation (1.1) in the sense of Definition 1.1 and suppose that the ellipticity and growth properties (1.2) and (1.3) for the Carathéodory-regular vector field $\mathbf{A}$ are in force. Then, for any $\lambda \in (0, \frac{1}{n}]$, almost every $z_0 \in E_T$, and every parabolic cylinder $Q_{r,\theta}(z_0) \Subset E_T$ as introduced in (1.5), where $r, \theta > 0$ additionally fulfill $r^2 \le \theta$ in the case $p = 2$, the potential estimate*

$$\begin{aligned} u(z_0) \le c\Bigg[&\bigg(\frac{1}{r^{n+p}}\iint_{Q_{r,\theta}(z_0)} u^{m-1+(1+\lambda)(p-1)}\,dz\bigg)^{\frac{1}{1+\lambda(p-1)}} \\ &+1+\Big[\frac{r^p}{\theta}\Big]^{\frac{1}{p+m-3}} + \mathbf{P}_p^\mu(z_0;r)\Bigg] \end{aligned} \tag{1.10}$$

holds with a constant $c \equiv c(n,m,p,C_0,C_1,\lambda)$. The parabolic potential $\mathbf{P}_p^\mu$ will be defined in Sect. 2.1.

A few remarks on the above theorem are necessary. First, the additional assumption $r^2 \le \theta$ in the case $p = 2$ guarantees that the condition $Q_{r,\omega r^p}(z_0) \subset E_T$ in (2.2) is satisfied for $\omega = 1$; see Sect. 2.1 for the details.

Next, the sharpness of the potential estimate (1.10) can be seen, for example, by looking at the fundamental solution $\mathcal{B}_{m,p}$, which is the explicit very weak solution to the equation

$$\partial_t u - \operatorname{div}(|u|^{m-1}|Du|^{p-2}Du) = \delta \quad \text{in } \mathbb{R}^n \times [0,\infty)$$

with δ being the Dirac measure charging the origin in $\mathbb{R}^{n+1}$. According to [14, Sect. 2] or [41, Sect. 12.2.1], the so-called Barenblatt solution is given by

$$\mathcal{B}_{m,p}(x,t) = \chi_{\{t>0\}}(t)\, t^{-n\varsigma}\Big[C - k\Big|\frac{x}{t^{\varsigma}}\Big|^{\frac{p}{p-1}}\Big]_+^{\frac{p-1}{m(p-1)-1}} \tag{1.11}$$

for any $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, where $\varsigma^{-1} := p+n[m(p-1)-1]$, $k := \frac{m(p-1)-1}{mp}\varsigma^{1/(p-1)}$, and $C > 0$. Our estimate shows the correct decay at the origin in the sense that it reflects the same structure as $\mathcal{B}_{m,p}$. More precisely, we can directly read off from (1.11) that the Barenblatt solution satisfies

$$\mathcal{B}_{m,p}(0,t_0) \lesssim t_0^{-\frac{n}{p+n[m(p-1)-1]}} \tag{1.12}$$

for $t_0 > 0$, and we will prove in Sect. 5 that the potential $\mathbf{P}_p^{\delta}$ from Theorem 1.3 provides in (1.10) the same behavior at the origin as exhibited by $\mathcal{B}_{m,p}$. Under this point of view, it means that our pointwise estimate is the best possible.

Moreover, we can infer from (1.10) that $u \in L^{\infty}_{\mathrm{loc}}(E_T)$ for weak solutions u to the Cauchy–Dirichlet problem (1.6), given that there exists a radius $r > 0$ such that $z \mapsto \mathbf{P}_p^{\mu}(z;r)$ is locally bounded in E_T. In particular, Theorem 1.3 allows us to formulate a sharp Lorentz space criterion, which ensures the local boundedness of the potential $\mathbf{P}_p^{\mu}(\cdot;r)$ for small radii r under the assumption $\mu \in L^{q_1,q_2}_{\mathrm{loc}}\big(E; L^{q_1,\infty}_{\mathrm{loc}}((0,T))\big)$ with $q_1 = \frac{n+p}{p}$ and $q_2 = \frac{n+p}{n(p-1)+p}$; see Sect. 2.4 for the definition and basic properties of Lorentz spaces and Sect. 6 for the rigorous proof of the following statement.

Theorem 1.4. *Suppose that the Carathéodory-regular vector field* $\mathbf{A}$ *fulfills the conditions* (1.2) *and* (1.3). *If*

$$\mu \in L^{q_1,q_2}_{\mathrm{loc}}\big(E; L^{q_1,\infty}_{\mathrm{loc}}((0,T))\big) \tag{1.13}$$

for $q_1 = \frac{n+p}{p}$ *and* $q_2 = \frac{n+p}{n(p-1)+p}$, *then, any weak solution* u *to the Cauchy–Dirichlet problem* (1.6) *is locally uniformly bounded.*

The condition (1.13) is satisfied for any Lebesgue function $\mu \in L^{\frac{n+p}{p}+\varepsilon}_{\mathrm{loc}}(E_T)$, where $\varepsilon > 0$ is arbitrary. For the p-Laplacian, optimal Lorentz space criteria guaranteeing the local boundedness of solutions and their gradients were established in [8, Thm. 4.7, Thm. 4.9] in the elliptic setting, and [28, cond. (1.26)] treats the local gradient boundedness in the parabolic context. Yet, up to now, no results of that kind for solutions u to the evolutionary p-Laplacian are present in the literature, at least to the author's knowledge. The only computations related to that subject can be found in [31, Rem. 1.3.2], however, the problem is that a Lorentz space criterion with possibly negative exponents is deduced. It seems that major modifications in their argumentation are necessary to ensure the well-definedness of the condition appearing there. Our Theorem 1.4 gives the desired result, not only for the evolutionary p-Laplacian, but even for doubly nonlinear parabolic equations. The exponent q_1 in (1.13) is optimal, which indicates the optimality of the potential $\mathbf{P}_p^{\mu}$ in (1.10). The minimality of q_1 can be retrieved, for instance, from [9, Rem. 3.1, p. 122].

Coming back to Theorem 1.3, we note that the existence of a weak solution cannot be guaranteed as long as a general Radon measure $\mu \in \mathcal{M}^+(E_T)$ without any further qualities is considered. Using $u^{\alpha+1}$ in the regularity assumptions (see Remark 1.2), the existence of less regular very weak solutions to the Cauchy–Dirichlet problem (1.6) was established in [39] under the additional monotonicity condition

$$\big[\mathbf{A}(x,t,u,\xi_1) - \mathbf{A}(x,t,u,\xi_2)\big] \cdot (\xi_1 - \xi_2) \geq C_2|u|^{m-1}|\xi_1 - \xi_2|^p$$

for any $u \in \mathbb{R}$, ξ_1, $\xi_2 \in \mathbb{R}^n$, and almost every $(x,t) \in E_T$ with a fixed constant $C_2 > 0$. If actually $\mu \in L^s(E_T, \mathbb{R}_{\geq 0})$ for

$$s = 1 + \frac{n}{n(p+m-2)+2p},$$

one can prove the existence of weak solutions in the sense of (1.9) (see [39, Rem. 4.3]), whereas [38] supplies the existence of weak solutions in the sense of Definition 1.1, provided that $\mu \in L^{\tilde{s}}(E_T, \mathbb{R}_{\geq 0})$ for

$$\tilde{s} = 1 + \frac{n}{p\big(n + \frac{2p+m-3}{p+m-2}\big) - n} > s.$$

Hence, the pointwise bound (1.10) has to be interpreted as an a priori estimate. By an approximation argument (see [4, Chap. 7]), the regularity result (1.10) can be transferred to very weak solutions.

1.6. Proof strategies

Our proof techniques are an adaption of the methods launched in [23] for elliptic p-Laplacian equations and [4–6,29–32] for the parabolic setting of equations of p-Laplacian and porous medium type. Our result, Theorem 1.3, is in perfect accordance with the ones from those papers and it is based on the notion of the parabolic potential $\mathbf{P}_p^\mu$ defined in Sect. 2.1 (see also [31]) and a sophisticated construction of intrinsic cylinders (see (3.2)). The intrinsic scaling approach was introduced by DiBenedetto (see [9]) and reflects the lack of homogeneity of the problem by rescaling the dimensions of the cylinders to compensate the degeneracy of the considered equation. In our context, i.e. in the case of a doubly nonlinear parabolic equation like (1.4), the appropriate intrinsic correction is $a^{1-m}d^{2-p}r^p$, where a neutralizes the degeneracy of u, and the factor d makes up for the absence of homogeneity with regard to $|Du|$.

The proof of the pointwise estimate (1.10) consists of establishing a Caccioppoli type inequality on such intrinsic cylinders in Sect. 3 and choosing adequate sequences of numbers a_j and d_j (see Sect. 4). Applying the energy estimate, we will receive a uniform estimate from above for a_j by iteration of recursive bounds. One of the key ingredients when proving the latter is the growth bound (4.10), which allows to replace d_j by d_{j-1} and is, to the author's knowledge, new in the literature (a similar argument for a_j was used in [5, Sect. 4.2]). In the end, we will have shown that $u(z_0)$ can be bounded from above by the limit a_∞, which will prove Theorem 1.3.

1.7. Potential estimates

The research on potential estimates was initiated by [22,23] with the investigation of solutions to stationary p-Laplacian equations. Since then, the outcome was extended in various respects, which we will briefly comment on in the following. In the case $m = 1$, $p = 2$, we almost arrive at the linear parabolic zero order Riesz potential estimate from [13]. The only difference springs from the integral $c[\frac{1}{r^{n+2}} \iint_{Q_{r,\theta}} u^{1+\lambda}\, dz]^{\frac{1}{1+\lambda}}$, where the parameter $\lambda > 0$ can be chosen arbitrarily small, but we are not permitted to let $\lambda \searrow 0$ because the constant c blows up in the limit. However, this curiosity is not new and conforms with the prominent estimates for p-Laplacian and porous medium type equations discussed below where the classical bound cannot be completely recovered by letting $\lambda \searrow 0$ as well. Next, for $m = 1$, our conclusion reduces to the known estimate from [31] for the degenerate situation of the parabolic p-Laplacian equation. Earlier, an analogue for time-independent Radon measures was derived in [32], and pointwise estimates in the singular p-Laplacian context involving a Radon measure defined on E can be found in [29]. Setting $p = 2$ in (1.10), the pointwise estimates for degenerate porous medium type equations from [4,30] can be reattained. Note that our estimate and the one from [4] do not comprise the sup-term from [30] on the right-hand side. Seen from this perspective, they are more natural since the famous bound from [2] can be retrieved in the case $\mu \equiv 0$. In [37], the results for degenerate porous medium type equations were generalized in the sense that vector fields $\mathbf{A}$ satisfying even more universal structure conditions were treated, and, recently, also the singular range for porous medium type equations could be coped with (see [5,6]). Nevertheless, potential estimates for doubly nonlinear parabolic equations are not covered in the literature up to now. Finally, we shall mention that all results presented here are estimates for the solution u itself, and we refer the reader to [25–28] for gradient estimates for the p-Laplacian, which we will not dwell on in this paper.

2. Preliminaries

In this section, we will provide various tools, which will be needed later in the proof. We will display the parabolic potential initially introduced in [31], cite an evolutionary version of the Gagliardo–Nirenberg inequality, analyze some auxiliary functions, define a time mollification procedure for functions in $L^1(E_T)$, and list the basic knowledge as regards Lorentz function spaces.

2.1. Nonlinear parabolic potentials

For the construction of the parabolic potential $\mathbf{P}^{\mu}_{p}$, we first define the mapping $\mathbf{i}_p \colon (0,\infty) \to [0,\infty]$ by

$$\mathbf{i}_p(\omega) := \begin{cases} (p-2)\,\omega^{-\frac{1}{p-2}} & \text{for } p > 2, \\ \begin{cases} \infty & \text{if } \omega \in (0,1), \\ 0 & \text{if } \omega \in [1,\infty) \end{cases} & \text{for } p = 2 \end{cases} \tag{2.1}$$

and observe that $p \mapsto \mathbf{i}_p(\omega)$ is continuous for any fixed $\omega > 0$. Next, we remember the definition of the parabolic cylinders from (1.5) and set

$$\mathbf{D}_p^{\mu}(z_0; r) := \inf_{\omega>0} \left\{ \mathbf{i}_p(\omega) + \tfrac{1}{(p-1)^{p-1}} r^{-n} \mu(Q_{r,\omega r^p}(z_0)) \colon Q_{r,\omega r^p}(z_0) \subset E_T \right\} \tag{2.2}$$

for a point $z_0 \in E_T$ and a radius $r > 0$. Obviously, for small radii $r > 0$ such that $Q_{r,r^2}(z_0) \subset E_T$, we have $\mathbf{D}_2^{\mu}(z_0; r) = r^{-n}\mu(Q_{r,r^2}(z_0))$. Moreover, for $p > 2$, we note that the infimum in (2.2) is attained at some $\omega > 0$ because the function under the infimum is continuous in ω, and ω is bounded since E_T is bounded. We remark that our definition of $\mathbf{D}_p^{\mu}$ differs in a factor $\frac{1}{2}$ from the definition in [31] as forward-backward cylinders of the form $B_r(x_0) \times (t_0 - \theta,\ t_0 + \theta)$ are considered there, and the condition $Q_{r,\omega r^p}(z_0) \subset E_T$ in (2.2) is implicitly assumed also in [31]. Furthermore, we note that the scaling factor r^p in time is typical when dealing with estimates for u as opposed to gradient estimates where the canonic scaling is r^2. Finally, we define the nonlinear parabolic potential $\mathbf{P}_p^{\mu}$ with respect to the Radon measure μ by

$$\mathbf{P}_p^{\mu}(z_0; r) := \sum_{j=0}^{\infty} \mathbf{D}_p^{\mu}(z_0; r_j), \tag{2.3}$$

where the sequence of radii $(r_j)_{j\in\mathbb{N}_0}$ is given by $r_j := \frac{r}{2^j}$ for any $j \in \mathbb{N}_0$. We realize that

$$\mathbf{P}_2^{\mu}(z_0; r) = \sum_{j=0}^{\infty} r_j^{-n} \mu(Q_{r_j, r_j^2})$$

for any $r > 0$ small enough, which is why there exists some constant $c > 1$ such that

$$c^{-1}\mathbf{P}_2^{\mu}(z_0; r) \le \int_0^r \frac{\mu(Q_{\varrho,\varrho^2})}{\varrho^n} \frac{d\varrho}{\varrho} \le c\mathbf{P}_2^{\mu}(z_0; r),$$

i.e. for $p = 2$, the parabolic potential $\mathbf{P}_2^{\mu}$ is equivalent to the truncated Riesz potential from [4–6,13,28,30]. If μ is independent of time, the infimum in (2.2) is attained at

$$\omega = \left[\tfrac{1}{(p-1)^{p-1}} r^{p-n} \mu(B_r)\right]^{-\frac{p-2}{p-1}}$$

provided that $r > 0$ is such that $B_r(x_0) \times (t_0 - \omega r^p, t_0) \subset E_T$. Therefore, we have

$$\mathbf{P}_p^{\mu}(x_0; r) = \sum_{j=0}^{\infty} \left[\frac{\mu(B_{r_j})}{r_j^{n-p}}\right]^{\frac{1}{p-1}},$$

which means that $\mathbf{P}_p^{\mu}$ equals the elliptic Wolff potential as defined in [31], which is in turn equivalent to the elliptic Wolff potential in integral notation from [12,13,22,23,27,29,32]. In the light of the foregoing comments, our definition of the parabolic potential $\mathbf{P}_p^\mu$ is natural because it reduces to the parabolic Riesz potential as $p \searrow 2$ and, for any $p \geq 2$, to the known elliptic Wolff potential when the Radon measure μ is time-independent. Beyond that, the usage of the potential $\mathbf{P}_p^\mu$ is justified in view of the facts that it allows to retrieve from Theorem 1.3 the behavior of the Barenblatt solution (see Sect. 5) and grants sharp Lorentz space estimates (see Theorem 1.4). Note that our potential does not depend on the value of m, which harmonizes with the prior estimates for porous medium type equations (see [4–6,30]).

2.2. *Auxiliary lemmata*

In this section, we will study some auxiliary functions. Before that, we cite a parabolic Sobolev embedding (cf. [9, Prop. 3.1, p. 7]), which we will employ in Sect. 4.5.

Lemma 2.1. *Let* $1 < p < \infty$, $0 < \ell < \infty$, *and* $Q_{r,\theta}(z_0) \subset E_T$ *be a parabolic cylinder as in* (1.5) *with* $z_0 \in E_T$ *and* $r, \theta > 0$. *Then, there exists a constant* $c \equiv c(n, p, \ell)$ *such that for every* $u \in L^\infty\big((t_0-\theta, t_0); L^\ell(B_r(x_0))\big) \cap L^p\big((t_0-\theta, t_0); W^{1,p}(B_r(x_0))\big)$ *there holds the Gagliardo–Nirenberg inequality*

$$\iint_{Q_{r,\theta}} |u|^q \, dz \leq c\Big[\sup_{t\in(t_0-\theta,\, t_0)} \int_{B_r\times\{t\}} |u|^\ell \, dx\Big]^{\frac{p}{n}} \cdot \iint_{Q_{r,\theta}} \Big[\big|\tfrac{1}{r}u\big|^p + |Du|^p\Big] dz, \tag{2.4}$$

where q is given by $q = \frac{p(n+\ell)}{n}$.

Next, we define the auxiliary functions G_λ, V_λ and W_λ, which will turn up later in the proof.

Definition 2.2. For $\lambda \in (0, 1)$ and $s \geq 0$, we define the functions G_λ, V_λ and W_λ by

$$\begin{aligned}
G_\lambda(s) &:= \int_0^s \big[1-(1+\sigma)^{-\lambda}\big]\, d\sigma = s - \tfrac{1}{1-\lambda}\big[(1+s)^{1-\lambda}-1\big],\\
V_\lambda(s) &:= \int_0^s \sigma^{\frac{m-1}{p}}(1+\sigma)^{-\frac{1+\lambda}{p}}\, d\sigma,\\
W_\lambda(s) &:= \int_0^s (1+\sigma)^{-\frac{1+\lambda}{p}}\, d\sigma = \tfrac{p}{p-1-\lambda}\big[(1+s)^{\frac{p-1-\lambda}{p}}-1\big].
\end{aligned}$$

We now mention one lemma for each of those auxiliary functions containing some characteristics, which are required afterwards. The proofs can be adapted from [4, Sect. 2.3].

Lemma 2.3. *For any* $\varepsilon \in (0, 1]$ *and* $s \geq 0$, *there holds*

$$s \leq \varepsilon + c_\varepsilon G_\lambda(s)$$

for a constant $c_\varepsilon \equiv \frac{c(\lambda)}{\varepsilon}$.

Lemma 2.4. *For any $\varepsilon \in (0, 1]$ and $s \geq 0$, there hold*

$$V_\lambda(s) \leq \frac{p}{p-2+m-\lambda} s^{\frac{p-2+m-\lambda}{p}} \tag{2.5}$$

and

$$s^{m-1+(1+\lambda)(p-1)} \leq \varepsilon^{(1+\lambda)(p-1)} s^{m-1} + c_\varepsilon V_\lambda(s)^{\frac{p[m-1+(1+\lambda)(p-1)]}{p-2+m-\lambda}}, \tag{2.6}$$

where the constant $c_\varepsilon \equiv c_\varepsilon(m, p, \lambda, \varepsilon)$ blows up as $\varepsilon^{-(1+\lambda)\frac{m-1+(1+\lambda)(p-1)}{p-2+m-\lambda}}$ in the limit $\varepsilon \searrow 0$.

Lemma 2.5. *For any $\varepsilon \in (0, 1]$ and $s \geq 0$, there hold*

$$W_\lambda(s) \leq \frac{p}{p-1-\lambda} s^{\frac{p-1-\lambda}{p}} \tag{2.7}$$

and

$$s^{(1+\lambda)(p-1)} \leq \varepsilon^{(1+\lambda)(p-1)} + c_\varepsilon W_\lambda(s)^{\frac{p(1+\lambda)(p-1)}{p-1-\lambda}}, \tag{2.8}$$

where the constant $c_\varepsilon \equiv c_\varepsilon(p, \lambda, \varepsilon)$ blows up as $\varepsilon^{-\frac{(1+\lambda)^2(p-1)}{p-1-\lambda}}$ in the limit $\varepsilon \searrow 0$.

2.3. Mollification in time

We will now introduce an averaging process in time and on its basis develop the regularized version (2.10) of the weak formulation (1.8).

Definition 2.6. For $v \in L^1(E_T)$, we define the mollification in time by

$$[\![v]\!]_h(\cdot, t) := \frac{1}{h}\int_0^t e^{\frac{s-t}{h}} v(\cdot, s)\, ds$$

and its time reversed analogue by

$$[\![v]\!]_{\bar{h}}(\cdot, t) := \frac{1}{h}\int_t^T e^{\frac{t-s}{h}} v(\cdot, s)\, ds$$

for any $h > 0$ and $t \in [0, T]$. Likewise, one can define the time regularization of a vector-valued function $v' \in L^1(E_T, \mathbb{R}^n)$.

For the main properties of this mollification, we refer to [7, Appendix B] and remark that $[\![\cdot]\!]_{\bar{h}}$ has similar characteristics as $[\![\cdot]\!]_h$. In particular, we remember that, for $u \in L^p(E_T)$, we have $\partial_t [\![u]\!]_h \in L^p(E_T)$, and the identity

$$\partial_t [\![u]\!]_h = \tfrac{1}{h}(u - [\![u]\!]_h) \tag{2.9}$$

holds. One can now derive the regularized variant (2.10) of the weak formulation (1.8) (see [4, p. 3293] or [37, Thm. 2.10]). The time mollification procedure from Definition 2.6 allows us to insert in (2.10) testing functions whose time derivative does not need to exist. In other words, Lemma 2.7 admits testing functions containing the solution u itself, avoiding an appearance of the quantity $\partial_t u$.

Lemma 2.7. *If $u\colon E_T \to \mathbb{R}$ is a weak solution to the Cauchy–Dirichlet problem (1.6) in the sense of Definition 1.1, then, its time mollification $[\![u]\!]_h$ fulfills the averaged equation*

$$\iint_{E_T} \big[\partial_t [\![u]\!]_h \varphi + [\![\mathbf{A}(x,t,u,Du)]\!]_h \cdot D\varphi\big]\, dz = \iint_{E_T} [\![\varphi]\!]_{\bar{h}}\, d\mu \tag{2.10}$$

for any testing function $\varphi \in C^\infty(\overline{E_T})$ with compact support in E_T.

2.4. Lorentz spaces

In this section, we assume that $\Omega \subset \mathbb{R}^d$ is a measurable set, and $\mathcal{L}^d$ denotes the Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. For a measurable function $u\colon \Omega \to \mathbb{R}$, we define the nonincreasing rearrangement $u^*\colon [0,\infty] \to [0,\infty]$ by

$$u^*(s) := \inf\big\{\sigma \geq 0\colon \mathcal{L}^d\big(\{x \in \Omega\colon |u(x)| > \sigma\}\big) \leq s\big\}$$

for any $s \in [0,\infty]$, and its average $u^{**}\colon [0,\infty] \to [0,\infty]$ by

$$u^{**}(s) := \frac{1}{s}\int_0^s u^*(\tilde{s})\, d\tilde{s}$$

for any $s \in [0,\infty]$. Obviously, u^* is nonincreasing, which also implies

$$u^{**}(s) \geq u^*(s) \tag{2.11}$$

for any $s \in [0,\infty]$. For $0 < p,\ q \leq \infty$, we say that u belongs to the Lorentz space $L^{p,q}(\Omega)$ if and only if the Lorentz quasi-norm

$$\|u\|_{L^{p,q}(\Omega)} := \begin{cases} \left[\displaystyle\int_0^\infty \big[s^{\frac{1}{p}} u^{**}(s)\big]^q \frac{ds}{s}\right]^{\frac{1}{q}} & \text{for } 0 < q < \infty, \\ \displaystyle\sup_{s>0} s^{\frac{1}{p}} u^{**}(s) & \text{for } q = \infty \end{cases}$$

is finite. As usual, $L^{p,q}_{\mathrm{loc}}(\Omega)$ indicates the space of functions with $\|u\|_{L^{p,q}(\Omega')} < \infty$ for any $\Omega' \Subset \Omega$. In the case $\Omega = E_T$, we can take $U \subset E$ and $0 \leq t_1 < t_2 \leq T$, and define the Lorentz-Bochner space $L^{p,q}\big(U; L^{p,q}((t_1,t_2))\big)$ as the space of all functions $(\underline{u}(x))(t) := u(x,t)$ such that

$$\Big\| \|\underline{u}\|_{L^{p,q}((t_1,t_2))} \Big\|_{L^{p,q}(U)} < \infty.$$

As is customary, we identify $\underline{u}$ with u and do not distinguish between them in the notation. Lorentz spaces refine the classical Lebesgue function spaces since $L^{p,p}(\Omega) = L^p(\Omega)$ for $p > 1$. For the interested reader, there is a wide-ranging literature on the properties of Lorentz spaces; see for instance [1, Chap. 7]. In particular, we will need the inclusions

$$\begin{aligned} L^{p,q}(\Omega) &\subset L^{p,r}(\Omega) \quad \text{for } 0 < p \leq \infty,\ 0 < q \leq r \leq \infty, \\ L^{r,s}(\Omega) &\subset L^{p,q}(\Omega) \quad \text{for } 0 < p < r \leq \infty,\ 0 < q,\ s \leq \infty, \end{aligned} \tag{2.12}$$

which can be found in [3, Sect. IV.4], and from [11, ineq. (33)], we cite the inequality

$$\int_A |u(x)|\,dx \le \int_0^{\mathcal{L}^d(A)} u^*(s)\,ds \tag{2.13}$$

for any measurable set $A \subset \Omega$.

3. Energy estimates

In this chapter, we will establish a Caccioppoli type inequality. For that purpose, let $z_0 = (x_0, t_0) \in E_T$ be a fixed point, $a,\ d > 0$, and define

$$\pi_p := \begin{cases} (p-2)^{p-2} & \text{if } p > 2, \\ 1 & \text{if } p = 2. \end{cases} \tag{3.1}$$

This parameter will compensate the constant in (4.6) arising from the definition of $\mathbf{i}_p$ in (2.1). We will work on intrinsic parabolic cylinders with the structure

$$Q_\varrho^{(a,d)}(z_0) := B_\varrho(x_0) \times \Lambda_\varrho^{(a,d)}(t_0), \tag{3.2}$$

where

$$\Lambda_\varrho^{(a,d)}(t_0) := (t_0 - t_\varrho^{(a,d)}, t_0) := (t_0 - \pi_p a^{1-m} d^{2-p} \varrho^p, t_0).$$

These cylinders are natural as they take into account the scaling behavior of the considered doubly nonlinear parabolic equations. Henceforth, we will use the abbreviations $B_\varrho := B_\varrho(x_0)$, $\Lambda_\varrho^{(a,d)} := \Lambda_\varrho^{(a,d)}(t_0)$, $Q_\varrho^{(a,d)} := Q_\varrho^{(a,d)}(z_0)$, $Q_{\varrho,+}^{(a,d)} := Q_\varrho^{(a,d)} \cap \{u > a\}$, and $B_\varrho^+(t) := B_\varrho \cap \{u(\cdot, t) > a\}$. Moreover, we define the number

$$\nu := 4^{-\frac{1}{p}}, \tag{3.3}$$

where the necessity of choosing the value of ν that way will become comprehensible when proving (4.7). In the remainder of this chapter, we will show the following inequality.

Lemma 3.1. *Let π_p as in (3.1), ν as in (3.3), and $\lambda \in (0, 1)$. Suppose further that $z_0 \in E_T$ and $\varrho,\ a,\ d > 0$ are such that $Q_\varrho^{(a,d)} \subset E_T$. Then, for any weak solution u to the Cauchy–Dirichlet problem (1.6), the energy estimate*

$$\begin{aligned} &\sup_{t \in \Lambda_{\nu\varrho}^{(a,d)}} \int_{B_{\nu\varrho}^+(t)} G_\lambda\Big(\frac{u-a}{d}\Big)\,dx \\ &\quad + \iint_{Q_{\nu\varrho,+}^{(a,d)}} \Big[d^{p+m-3} \Big| DV_\lambda\Big(\frac{u-a}{d}\Big)\Big|^p + a^{m-1} d^{p-2} \Big| DW_\lambda\Big(\frac{u-a}{d}\Big)\Big|^p \Big]\,dz \\ &\quad \le \frac{c d^{p-2}}{\varrho^p} \iint_{Q_{\varrho,+}^{(a,d)}} u^{m-1} \Big(1 + \frac{u-a}{d}\Big)^{(1+\lambda)(p-1)}\,dz + \frac{c\mu(Q_\varrho^{(a,d)})}{d} \end{aligned} \tag{3.4}$$

holds with a constant $c \equiv c(p, C_0, C_1, \lambda)$, where G_λ, V_λ and W_λ are given in Definition 2.2.

Proof. In the regularized weak formulation (2.10), we choose the testing function $\varphi := \eta^p \zeta_\varepsilon v$, where

$$v := g(u) := 1 - \left[1 + \frac{(u-a)_+}{d}\right]^{-\lambda}$$

and $\eta \in C_0^1(B_\varrho, [0,1])$ is such that $\eta \equiv 1$ on $B_{\nu\varrho}$ and $|D\eta| \le \frac{2}{(1-\nu)\varrho}$ on B_ϱ. The cut-off function in time $\zeta_\varepsilon \in W_0^{1,\infty}(\mathbb{R}, [0,1])$ satisfies

$$\zeta_\varepsilon(t) := \begin{cases} 0 & \text{for } t \in (-\infty, t_0 - t_\varrho^{(a,d)}] \cup [\tau, \infty), \\ \frac{4}{3}\big(t_\varrho^{(a,d)}\big)^{-1}\big[t - (t_0 - t_\varrho^{(a,d)})\big] & \text{for } t \in (t_0 - t_\varrho^{(a,d)}, t_0 - t_{\nu\varrho}^{(a,d)}), \\ 1 & \text{for } t \in [t_0 - t_{\nu\varrho}^{(a,d)}, \tau - \varepsilon], \\ -\frac{1}{\varepsilon}(t-\tau) & \text{for } t \in (\tau - \varepsilon, \tau), \end{cases}$$

where $\tau \in \Lambda_{\nu\varrho}^{(a,d)}(t_0)$ and $0 < \varepsilon < \tau - [t_0 - t_{\nu\varrho}^{(a,d)}]$. Furthermore, we denote by ζ the pointwise limit of ζ_ε as $\varepsilon \searrow 0$. In the sequel, we will analyze all terms appearing in Lemma 2.7. As g is increasing, the identity (2.9) implies

$$\partial_t [\![u]\!]_h \big(g(u) - g([\![u]\!]_h)\big) = \tfrac{1}{h}\big(u - [\![u]\!]_h\big)\big(g(u) - g([\![u]\!]_h)\big) \ge 0,$$

which yields

$$\begin{aligned}
\iint_{Q_\varrho^{(a,d)}} \partial_t [\![u]\!]_h \varphi \, dz &\ge \iint_{Q_\varrho^{(a,d)}} \eta^p \zeta_\varepsilon \partial_t [\![u]\!]_h g([\![u]\!]_h)\, dz \\
&= \iint_{Q_\varrho^{(a,d)}} \eta^p \zeta_\varepsilon \partial_t \left[\int_a^{[\![u]\!]_h} g(\sigma)\, d\sigma\right] dz \\
&= -\iint_{Q_\varrho^{(a,d)}} \eta^p \partial_t \zeta_\varepsilon \int_a^{[\![u]\!]_h} g(\sigma)\, d\sigma \, dz \\
&= -\frac{4a^{m-1} d^{p-2}}{3\pi_p \varrho^p} \int_{t_0 - t_\varrho^{(a,d)}}^{t_0 - t_{\nu\varrho}^{(a,d)}} \int_{B_\varrho} \eta^p \int_a^{[\![u]\!]_h} g(\sigma)\, d\sigma \, dx\, dt \\
&\quad + \frac{1}{\varepsilon} \int_{\tau-\varepsilon}^{\tau} \int_{B_\varrho} \eta^p \int_a^{[\![u]\!]_h} g(\sigma)\, d\sigma\, dx\, dt \\
&=: \mathbf{I}(h) + \mathbf{II}(h, \varepsilon).
\end{aligned}$$

First, we will turn towards the integral $\mathbf{II}(h,\varepsilon)$. Passing to the limits $\varepsilon \searrow 0$ and $h \searrow 0$, we receive

$$\begin{aligned}
&\lim_{h\searrow 0} \lim_{\varepsilon \searrow 0} \mathbf{II}(h,\varepsilon) \\
&\quad = \lim_{h\searrow 0} \lim_{\varepsilon \searrow 0} ⨍_{\tau-\varepsilon}^{\tau} \int_{B_\varrho} \eta^p \int_a^{[\![u]\!]_h} g(\sigma)\, d\sigma\, dx\, dt \\
&\quad = \lim_{h\searrow 0} \int_{B_\varrho} \eta^p \int_a^{[\![u]\!]_h(\cdot,\tau)} \left[1 - \left(1 + \frac{(\sigma-a)_+}{d}\right)^{-\lambda}\right] d\sigma\, dx
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\searrow 0} d \int_{B_\varrho} \eta^p \left[\frac{\big([\![u]\!]_h(\cdot,\tau)-a\big)_+}{d} - \tfrac{1}{1-\lambda}\left(\Big(1+\frac{\big([\![u]\!]_h(\cdot,\tau)-a\big)_+}{d}\Big)^{1-\lambda} - 1\right)\right] dx \\
&= d \int_{B_\varrho} \eta^p \left[\frac{\big(u(\cdot,\tau)-a\big)_+}{d} - \tfrac{1}{1-\lambda}\left(\Big(1+\frac{\big(u(\cdot,\tau)-a\big)_+}{d}\Big)^{1-\lambda} - 1\right)\right] dx \\
&= d \int_{B_\varrho^+(\tau)} \eta^p G_\lambda\Big(\frac{u-a}{d}\Big)\, dx
\end{aligned}
$$

for a. e. $\tau \in \Lambda_{\nu\varrho}^{(a,d)}$ due to the Lebesgue differentiation theorem. Next, in order to find a bound for the term $\mathbf{I}(h)$, we note that $\eta \le 1$ and

$$
\left| \int_a^{[\![u]\!]_h} g(\sigma)\, d\sigma \right| \le \big([\![u]\!]_h - a\big)_+ .
$$

Indeed, if $z \in Q_\varrho^{(a,d)}$ is such that $[\![u]\!]_h(z) < a$, the function g vanishes on the whole interval $\big([\![u]\!]_h(z), a\big)$, and otherwise, we can estimate $|g| \le 1$. Now, we can treat $\mathbf{I}(h)$ by using the above inequality and subsequently letting $h \searrow 0$. This results in

$$
\begin{aligned}
\lim_{h\searrow 0} |\mathbf{I}(h)| &\le \lim_{h\searrow 0} \frac{c a^{m-1} d^{p-2}}{\varrho^p} \int_{t_0 - t_\varrho^{(a,d)}}^{t_0} \int_{B_\varrho} \big([\![u]\!]_h - a\big)_+ \, dx\, dt \\
&= \frac{c d^{p-1}}{\varrho^p} \iint_{Q_{\varrho,+}^{(a,d)}} a^{m-1} \frac{u-a}{d}\, dz \\
&\le \frac{c d^{p-1}}{\varrho^p} \iint_{Q_{\varrho,+}^{(a,d)}} u^{m-1} \Big(1+\frac{u-a}{d}\Big)^{(1+\lambda)(p-1)} dz
\end{aligned}
$$

with a constant $c \equiv c(p)$, where we observe that $u > a$ on $Q_{\varrho,+}^{(a,d)}$ and $(1+\lambda)(p-1) \ge 1$ for the last step. In the following, we will deal with the diffusion part from (2.10). Again building the limits $\varepsilon \searrow 0$ and $h \searrow 0$, we get

$$
\begin{aligned}
&\lim_{h\searrow 0} \lim_{\varepsilon\searrow 0} \iint_{Q_{\varrho,+}^{(a,d)}} [\![\mathbf{A}(x,t,u,Du)]\!]_h \cdot D\varphi\, dz \\
&= \iint_{Q_{\varrho,+}^{(a,d)}} \eta^p \zeta \mathbf{A}(x,t,u,Du) \cdot Dv\, dz \\
&\quad + p \iint_{Q_{\varrho,+}^{(a,d)}} \eta^{p-1} \zeta v \mathbf{A}(x,t,u,Du) \cdot D\eta\, dz \\
&=: \mathbf{III} + \mathbf{IV}.
\end{aligned}
$$

Before considering the term $\mathbf{IV}$, we will treat the integral $\mathbf{III}$. Having in mind the ellipticity assumption (1.2), we compute for the latter

$$
\begin{aligned}
\mathbf{III} &= \frac{\lambda}{d} \iint_{Q_{\varrho,+}^{(a,d)}} \eta^p \zeta \Big(1+\frac{u-a}{d}\Big)^{-(1+\lambda)} \mathbf{A}(x,t,u,Du) \cdot Du\, dz \\
&\ge \frac{\lambda C_0}{d} \iint_{Q_{\varrho,+}^{(a,d)}} \eta^p \zeta \frac{u^{m-1} |Du|^p}{(1+\frac{u-a}{d})^{1+\lambda}}\, dz.
\end{aligned}
$$

For the other summand, we exploit in turn the fact that $|v| \leq 1$, the growth condition (1.3), the bound $|D\eta| \leq \frac{2}{(1-\nu)\varrho}$, Young's inequality, and $|\zeta| \leq 1$ to conclude that

$$
\begin{aligned}
|\mathbf{IV}| \leq & p \iint_{Q_{\varrho,+}^{(a,d)}} \eta^{p-1} \zeta v |\mathbf{A}(x,t,u,Du)||D\eta|\,dz \\
\leq & \frac{2pC_1}{(1-\nu)\varrho} \iint_{Q_{\varrho,+}^{(a,d)}} \eta^{p-1} \zeta u^{m-1} |Du|^{p-1}\,dz \\
\leq & \frac{1}{p'} \frac{\lambda C_0}{d} \iint_{Q_{\varrho,+}^{(a,d)}} \eta^p \zeta \frac{u^{m-1}|Du|^p}{(1+\frac{u-a}{d})^{1+\lambda}}\,dz \\
& + \frac{cd^{p-1}}{\varrho^p} \iint_{Q_{\varrho,+}^{(a,d)}} u^{m-1} \Big(1+\frac{u-a}{d}\Big)^{(1+\lambda)(p-1)}\,dz
\end{aligned}
$$

with a constant $c \equiv c(p, C_0, C_1, \lambda)$. It remains to estimate the integral involving the Radon measure μ, where we use $|\varphi| \leq 1$ to derive

$$
\lim_{h \searrow 0} \lim_{\varepsilon \searrow 0} \iint_{Q_{\varrho,+}^{(a,d)}} [\![\varphi]\!]_{\bar{h}}\,d\mu \leq \mu\big(Q_{\varrho,+}^{(a,d)}\big).
$$

Combining the results obtained so far and modifying the domains of integration of the left-hand side integrals in a way that we can discard the cut-off functions η and ζ, we receive

$$
\begin{aligned}
& \int_{B_{\nu\varrho}^+(\tau)} G_\lambda\Big(\frac{u-a}{d}\Big)\,dx + \frac{\lambda C_0}{pd^2} \int_{t_0-t_{\nu\varrho}^{(a,d)}}^{\tau} \int_{B_{\nu\varrho}^+(t)} \frac{u^{m-1}|Du|^p}{(1+\frac{u-a}{d})^{1+\lambda}}\,dx\,dt \\
& \leq \frac{cd^{p-2}}{\varrho^p} \iint_{Q_{\varrho,+}^{(a,d)}} u^{m-1}\Big(1+\frac{u-a}{d}\Big)^{(1+\lambda)(p-1)}\,dz + \frac{\mu(Q_\varrho^{(a,d)})}{d}
\end{aligned}
\tag{3.5}
$$

for a. e. $\tau \in \Lambda_{\nu\varrho}^{(a,d)}$ with a constant $c \equiv c(p, C_0, C_1, \lambda)$. Building the supremum over all τ in the first and letting $\tau \nearrow t_0$ in the second term, we infer that

$$
\sup_{t \in \Lambda_{\nu\varrho}^{(a,d)}} \int_{B_{\nu\varrho}^+(t)} G_\lambda\Big(\frac{u-a}{d}\Big)\,dx + \frac{1}{d^2} \iint_{Q_{\nu\varrho,+}^{(a,d)}} \frac{u^{m-1}|Du|^p}{(1+\frac{u-a}{d})^{1+\lambda}}\,dz
$$

can be bounded from above up to a constant by the right-hand side of (3.5). Then, taking into account that on the set $Q_{\nu\varrho,+}^{(a,d)}$ there holds

$$
\begin{aligned}
& d^{p+m-3}\Big|DV_\lambda\Big(\frac{u-a}{d}\Big)\Big|^p + a^{m-1}d^{p-2}\Big|DW_\lambda\Big(\frac{u-a}{d}\Big)\Big|^p \\
& = \Big[d^{p+m-3}\Big(\frac{u-a}{d}\Big)^{m-1} + a^{m-1}d^{p-2}\Big]\Big(1+\frac{u-a}{d}\Big)^{-(1+\lambda)} \frac{|Du|^p}{d^p} \\
& = \big[(u-a)^{m-1} + a^{m-1}\big]\Big(1+\frac{u-a}{d}\Big)^{-(1+\lambda)} \frac{|Du|^p}{d^2} \\
& \leq 2u^{m-1}\Big(1+\frac{u-a}{d}\Big)^{-(1+\lambda)} \frac{|Du|^p}{d^2},
\end{aligned}
$$

we can rewrite the diffusion term, and (3.4) is proven. □

4. Potential estimates: the proof of Theorem 1.3

Proof of Theorem 1.3. In this section, we will perform the proof of Theorem 1.3. We will proceed as described in Sect. 1.6.

4.1. Choice of parameters

Let $z_0 = (x_0, t_0) \in E_T$ be an arbitrary point and $r, \theta > 0$ such that $Q_{r,\theta} \equiv Q_{r,\theta}(z_0) \Subset E_T$ with the additional assumption $r^2 \le \theta$ when $p = 2$. Moreover, let $\lambda \in (0, \frac{1}{n}]$, and $\kappa \in (0, 1)$ be a fixed parameter, which will be specified later. For $j \in \mathbb{N}_0$, we define radii $r_j := \frac{r}{2^j}$ and determine positive numbers a_j and d_{j-1} inductively as follows. To get in a position where we can prove certain cylinder inclusions (see Sect. 4.2), we set

$$d_{-1} := 2\Big[\pi_p \frac{r^p}{\theta}\Big]^{\frac{1}{p+m-3}} \quad \text{and} \quad a_0 := \max\left\{1, \Big[\pi_p \frac{r^p}{\theta}\Big]^{\frac{1}{p+m-3}}\right\} \tag{4.1}$$

for π_p as in (3.1), and suppose for some $j \in \mathbb{N}_0$ that a_k and d_{k-1} have already been selected for any $0 \le k \le j$. In order to choose a_{j+1} and d_j, we recall the definitions of $\mathbf{i}_p$ and $\mathbf{D}_p^\mu$ from (2.1) and (2.2), and let

$$\begin{aligned}\omega_j := \sup \Big\{&\omega > 0 \colon \mathbf{i}_p(\omega) + \tfrac{1}{(p-1)^{p-1}} (\tfrac{1}{\nu} r_j)^{-n} \mu\big(\tfrac{1}{\nu} Q_{r_j, \omega r_j^p}\big) = \mathbf{D}_p^\mu(z_0; \tfrac{1}{\nu} r_j),\\ &\tfrac{1}{\nu} Q_{r_j, \omega r_j^p} \subset E_T\Big\}\end{aligned} \tag{4.2}$$

with ν as in (3.3). We remark that, for a fixed $j \in \mathbb{N}_0$, such a number ω_j exists by the very definition of $\mathbf{D}_p^\mu$, and ω_j is uniformly bounded with respect to j by the assumption $\frac{1}{\nu} Q_{r_j, \omega r_j^p} \subset E_T$ and the fact that $r_j \le r$ for any $j \in \mathbb{N}_0$. Moreover, we set

$$\hat{d}_j := \max\left\{\tfrac{1}{2} d_{j-1}, \mathbf{i}_p(\omega_j)\right\}$$

and define

$$Q_j^{(d)} := B_j \times \Lambda_j^{(d)} := B_{r_j}(x_0) \times (t_0 - \pi_p a_j^{1-m} d^{2-p} r_j^p, t_0)$$

and

$$\mathbf{K}_j(d) := \frac{d^{p-2}}{r_j^{n+p}} \iint_{Q_j^{(d)} \cap \{u > a_j\}} u^{m-1} \Big(\frac{u - a_j}{d}\Big)^{(1+\lambda)(p-1)} dz \tag{4.3}$$

for $d \ge \hat{d}_j$. Note that $\mathbf{K}_j(d) \to 0$ as $d \to \infty$. Now, if $\mathbf{K}_j(\hat{d}_j) \le \kappa$, we define $d_j := \hat{d}_j$, whereas in the situation that $\mathbf{K}_j(\hat{d}_j) > \kappa$ holds true, we first observe that $d \mapsto \mathbf{K}_j(d)$ is a continuous and decreasing function. Thus, there exists some $\hat{d} > \hat{d}_j$ such that $\mathbf{K}_j(\hat{d}) = \kappa$, and we choose $d_j := \hat{d}$. In the latter case, we obviously get $d_j > \hat{d}_j$ and $\mathbf{K}_j(d_j) = \kappa$. Having fixed d_j, we introduce the abbreviations $Q_j := Q_j^{(d_j)}$, $\Lambda_j := \Lambda_j^{(d_j)}$, $L_j := Q_j \cap \{u > a_j\}$, $\frac{1}{\nu} L_j := (\frac{1}{\nu} Q_j) \cap \{u > a_j\}$, and $L_j(t) := B_j \cap \{u(\cdot, t) > a_j\}$ for $t \in \Lambda_j$. Eventually, we set $a_{j+1} := a_j + d_j$ and become aware of the fact that

$$\mathbf{K}_j(d_j) = \frac{d_j^{p-2}}{r_j^{n+p}} \iint_{L_j} u^{m-1}\Big(\frac{u-a_j}{d_j}\Big)^{(1+\lambda)(p-1)} dz \le \kappa. \tag{4.4}$$

4.2. Cylinder inclusions

To start with, we claim that

$$\tfrac{1}{\nu} Q_j \subset \tfrac{1}{\nu} Q_{r_j, \omega_j r_j^p} \subset E_T \tag{4.5}$$

for any $j \in \mathbb{N}_0$, where ν is as in (3.3). The second inclusion is obvious from the definition of ω_j from (4.2), and, since $a_j \ge a_0 \ge 1$ and $d_j \ge \mathbf{i}_p(\omega_j)$, the first inclusion is a consequence of

$$\pi_p a_j^{1-m} d_j^{2-p} r_j^p \le \pi_p \mathbf{i}_p(\omega_j)^{2-p} r_j^p \le \omega_j r_j^p. \tag{4.6}$$

Next, we will show that

$$Q_{j+1} \subset \nu Q_j \tag{4.7}$$

for any $j \in \mathbb{N}_0$ and ν as in (3.3). Clearly, there holds $B_{r_{j+1}} \subset B_{\nu r_j}$, and, as $a_{j+1} \ge a_j$, $d_{j+1} \ge \hat{d}_{j+1} \ge \frac{1}{2} d_j$, and $r_{j+1} = \frac{1}{2} r_j$, we also know that

$$a_{j+1}^{1-m} d_{j+1}^{2-p} r_{j+1}^p \le a_j^{1-m} \big(\tfrac{1}{2} d_j\big)^{2-p} \big(\tfrac{1}{2} r_j\big)^p = a_j^{1-m} d_j^{2-p} (\nu r_j)^p$$

such that (4.7) is proven. Finally, we will argue that

$$Q_j \subset Q_{r,\theta} \tag{4.8}$$

for any $j \in \mathbb{N}_0$. By an inductive application of (4.7), we find that $Q_j \subset Q_0$. Hence, (4.8) follows once we have asserted $Q_0 \subset Q_{r,\theta}$. However, this relation results from the inequality

$$\pi_p a_0^{1-m} d_0^{2-p} \le \pi_p a_0^{1-m} \big(\tfrac{1}{2} d_{-1}\big)^{2-p} \le \pi_p \Big[\pi_p \frac{r^p}{\theta}\Big]^{\frac{1-m}{p+m-3}} \Big[\pi_p \frac{r^p}{\theta}\Big]^{\frac{2-p}{p+m-3}} = \frac{\theta}{r^p},$$

where we have inserted (4.1).

4.3. Growth bounds for d_j

In this section, we will assume that

$$\mathbf{K}_j(d_j) = \kappa \tag{4.9}$$

and establish the growth bound

$$d_j \le c^* d_{j-1} \tag{4.10}$$

for any $j \in \mathbb{N}$, where $c^* := 2^{\frac{n+p}{(1+\lambda)(p-1)-(p-2)}} > 1$ is a constant. For that purpose, we use the definition of $\mathbf{K}_j$ from (4.3) and the facts that $r_j = \frac{1}{2} r_{j-1}$ and $a_j \ge a_{j-1}$ to compute

$$\begin{aligned}\mathbf{K}_j(c^* d_{j-1}) &= \frac{(c^* d_{j-1})^{p-2}}{r_j^{n+p}} \iint_{Q_j^{(c^* d_{j-1})} \cap \{u > a_j\}} u^{m-1} \Big(\frac{u - a_j}{c^* d_{j-1}} \Big)^{(1+\lambda)(p-1)} dz \\ &\le \frac{d_{j-1}^{p-2}}{r_{j-1}^{n+p}} \iint_{Q_j^{(c^* d_{j-1})} \cap \{u > a_j\}} u^{m-1} \Big(\frac{u - a_{j-1}}{d_{j-1}} \Big)^{(1+\lambda)(p-1)} dz.\end{aligned}$$

Now, if we keep in mind that $B_j \subset B_{j-1}$, $a_j^{1-m}(c^* d_{j-1})^{2-p} r_j^p \le a_{j-1}^{1-m} d_{j-1}^{2-p} r_{j-1}^p$, and $\{u > a_j\} \subset \{u > a_{j-1}\}$, the last integral can be bounded from above by $\mathbf{K}_{j-1}(d_{j-1})$, which is in turn smaller than or equal to κ by virtue of (4.4). Thus, in view of (4.9), we have proven that $\mathbf{K}_j(c^* d_{j-1}) \le \mathbf{K}_j(d_j)$, which implies (4.10) since $\mathbf{K}_j$ is a decreasing function.

4.4. Preliminary estimates

For any $j \in \mathbb{N}$, the estimates

$$\frac{u - a_{j-1}}{d_{j-1}} = \frac{u - a_j}{d_{j-1}} + 1 \ge 1 \tag{4.11}$$

and

$$\frac{u - a_j}{d_j} \le \frac{u - a_{j-1}}{d_j} \le 2 \frac{u - a_{j-1}}{d_{j-1}} \tag{4.12}$$

hold true on the set $\{u > a_j\}$ by the definitions of a_j and d_j. Until the end of this section, we will again assume that (4.9) is valid for any $j \in \mathbb{N}$. Hence, due to (4.11), (4.10), the induction hypotheses $r_j = \frac{1}{2} r_{j-1}$ and $L_j \subset L_{j-1}$, and the inequality (4.4), we have

$$\begin{aligned}\frac{d_j^{p-2}}{r_j^{n+p}} \iint_{L_j} u^{m-1}\, dz &\le \frac{d_j^{p-2}}{r_j^{n+p}} \iint_{L_j} u^{m-1} \Big(\frac{u - a_{j-1}}{d_{j-1}} \Big)^{(1+\lambda)(p-1)} dz \\ &\le \frac{c d_{j-1}^{p-2}}{r_{j-1}^{n+p}} \iint_{L_{j-1}} u^{m-1} \Big(\frac{u - a_{j-1}}{d_{j-1}} \Big)^{(1+\lambda)(p-1)} dz \\ &\le c\kappa \end{aligned} \tag{4.13}$$

with a constant $c \equiv c(n, p, \lambda)$. Next, we note that, by (4.5), the cylinder $\frac{1}{\nu}Q_j$ is contained in E_T and apply the energy estimate (3.4) on $\frac{1}{\nu}Q_j$ with the parameters (a, d) replaced by (a_j, d_j), where $j \in \mathbb{N}$. Then, (3.4) reads

$$\begin{aligned}
&\sup_{t\in\Lambda_j} \int_{L_j(t)} G_\lambda\Big(\frac{u-a_j}{d_j}\Big)\,dx \\
&\quad + \iint_{L_j} \Big[d_j^{p+m-3}\Big|DV_\lambda\Big(\frac{u-a_j}{d_j}\Big)\Big|^p + a_j^{m-1}d_j^{p-2}\Big|DW_\lambda\Big(\frac{u-a_j}{d_j}\Big)\Big|^p\Big]\,dz \\
&\le \frac{cd_j^{p-2}}{r_j^p}\iint_{\frac{1}{\nu}L_j} u^{m-1}\Big(1+\frac{u-a_j}{d_j}\Big)^{(1+\lambda)(p-1)}\,dz + \frac{c\mu(\frac{1}{\nu}Q_j)}{d_j} \\
&\le \frac{cd_{j-1}^{p-2}}{r_{j-1}^p}\iint_{L_{j-1}} u^{m-1}\Big(3\frac{u-a_{j-1}}{d_{j-1}}\Big)^{(1+\lambda)(p-1)}\,dz + \frac{c\mu(\frac{1}{\nu}Q_j)}{d_j} \\
&\le cr_j^n\kappa + \frac{c\mu(\frac{1}{\nu}Q_j)}{d_j}
\end{aligned} \tag{4.14}$$

with a constant $c \equiv c(n, p, C_0, C_1, \lambda)$, where, in the second last step, we have used (4.10), (4.11) and (4.12), and enlarged the domain of integration. After that, the final bound is an easy consequence of (4.4).

4.5. Recursive bounds for d_j

In this section, we will show that

$$d_j \le \tfrac{1}{2}d_{j-1} + c\mathbf{D}_p^\mu\left(z_0; \tfrac{1}{\nu}r_j\right) \tag{4.15}$$

for any $j \in \mathbb{N}$ and

$$\begin{aligned}
d_0 \le \tfrac{1}{2}d_{-1} &+ c\Big[\frac{1}{r^{n+p}}\iint_{Q_{r,\theta}} u^{m-1+(1+\lambda)(p-1)}\,dz\Big]^{\frac{1}{1+\lambda(p-1)}} \\
&+ c\mathbf{D}_p^\mu(z_0; \tfrac{1}{\nu}r)
\end{aligned} \tag{4.16}$$

with a constant $c \equiv c(n, m, p, C_0, C_1, \lambda)$, where ν is defined in (3.3). First, we fix $j \in \mathbb{N}$ and prove (4.15). To that end, we can assume without loss of generality that $d_j > \hat{d}_j$ since otherwise we had $d_j = \hat{d}_j$ such that either $d_j = \frac{1}{2}d_{j-1}$ or $d_j = \mathbf{i}_p(\omega_j)$ holds, which both instantly yield (4.15). However, as a result of starting from the premise that $d_j > \hat{d}_j$, we can expect $d_j > \frac{1}{2}d_{j-1}$, $d_j > \mathbf{i}_p(\omega_j)$, and (4.9) to be valid. Therefore, we can proceed as follows:

$$\begin{aligned}
\kappa = \mathbf{K}_j(d_j) &= \frac{d_j^{p-2}}{r_j^{n+p}}\iint_{L_j}\big[(u-a_j)+a_j\big]^{m-1}\Big(\frac{u-a_j}{d_j}\Big)^{(1+\lambda)(p-1)}\,dz \\
&\le \frac{cd_j^{p+m-3}}{r_j^{n+p}}\iint_{L_j}\Big(\frac{u-a_j}{d_j}\Big)^{m-1+(1+\lambda)(p-1)}\,dz
\end{aligned}$$

$$
\begin{aligned}
&+\frac{c a_j^{m-1} d_j^{p-2}}{r_j^{n+p}} \iint_{L_j} \Big(\frac{u-a_j}{d_j}\Big)^{(1+\lambda)(p-1)} dz \\
&=: \mathcal{J} + \tilde{\mathcal{J}},
\end{aligned} \tag{4.17}
$$

where $c \equiv c(m)$ is a constant. To begin with, we estimate the integral $\mathcal{J}$ and use (2.6) for some fixed $\varepsilon \in (0, 1)$ and (4.13) to get

$$
\begin{aligned}
\mathcal{J} \leq &\frac{c d_j^{p+m-3}}{r_j^{n+p}} \bigg[\varepsilon^{(1+\lambda)(p-1)} \iint_{L_j} \Big(\frac{u-a_j}{d_j}\Big)^{m-1} dz \\
&+ c_\varepsilon \iint_{L_j} V_\lambda\Big(\frac{u-a_j}{d_j}\Big)^{\frac{p[m-1+(1+\lambda)(p-1)]}{p-2+m-\lambda}} dz \bigg] \\
\leq &c\varepsilon^{(1+\lambda)(p-1)}\kappa + \frac{c_\varepsilon d_j^{p+m-3}}{r_j^{n+p}} \iint_{Q_j} V_\lambda\Big(\tfrac{(u-a_j)_+}{d_j}\Big)^{\frac{p[m-1+(1+\lambda)(p-1)]}{p-2+m-\lambda}} dz
\end{aligned}
$$

with constants $c \equiv c(n, m, p, \lambda)$ and $c_\varepsilon \equiv c_\varepsilon(m, p, \lambda, \varepsilon)$. Applying the Gagliardo–Nirenberg inequality (2.4) with $\ell = \frac{pn\lambda}{p-2+m-\lambda}$, we find

$$
\begin{aligned}
\mathcal{J} \leq &c\varepsilon^{(1+\lambda)(p-1)}\kappa + c_\varepsilon \bigg[\sup_{t\in\Lambda_j} \tfrac{1}{r_j^n} \int_{B_j\times\{t\}} \Big| V_\lambda\Big(\tfrac{(u-a_j)_+}{d_j}\Big)\Big|^{\frac{pn\lambda}{p-2+m-\lambda}} dx \bigg]^{\frac{p}{n}} \\
&\cdot \frac{d_j^{p+m-3}}{r_j^n} \iint_{Q_j} \bigg[\tfrac{1}{r_j^p} \Big| V_\lambda\Big(\tfrac{(u-a_j)_+}{d_j}\Big)\Big|^p + \Big| DV_\lambda\Big(\tfrac{(u-a_j)_+}{d_j}\Big)\Big|^p \bigg] dz \\
=: &c\varepsilon^{(1+\lambda)(p-1)}\kappa + c_\varepsilon \mathcal{J}_1(\mathcal{J}_2 + \mathcal{J}_3)
\end{aligned} \tag{4.18}
$$

with constants $c \equiv c(n, m, p, \lambda)$ and $c_\varepsilon \equiv c_\varepsilon(n, m, p, \lambda, \varepsilon)$ with the obvious labeling of $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{J}_3$. In the sequel, we will separately estimate the appearing terms, starting with $\mathcal{J}_1$. Employing (2.5), the Hölder inequality (note that $\lambda \leq \frac{1}{n}$), Lemma 2.3 for some $\varepsilon_1 \in (0, 1)$ to be chosen later, and (4.14), we arrive at

$$
\begin{aligned}
\mathcal{J}_1 &\leq c\bigg[\sup_{t\in\Lambda_j} \frac{1}{r_j^n} \int_{L_j(t)} \Big(\frac{u-a_j}{d_j}\Big)^{\lambda n} dx \bigg]^{\frac{p}{n}} \\
&\leq c\bigg[\sup_{t\in\Lambda_j} \frac{1}{r_j^n} \int_{L_j(t)} \frac{u-a_j}{d_j} dx \bigg]^{p\lambda} \\
&\leq c\varepsilon_1^{p\lambda} + c\varepsilon_1^{-p\lambda}\bigg[\sup_{t\in\Lambda_j} \frac{1}{r_j^n} \int_{L_j(t)} G_\lambda\Big(\frac{u-a_j}{d_j}\Big) dx \bigg]^{p\lambda} \\
&\leq c\varepsilon_1^{p\lambda} + c\varepsilon_1^{-p\lambda}\bigg[\kappa + \frac{\mu(\frac{1}{\nu}Q_j)}{d_j r_j^n} \bigg]^{p\lambda}
\end{aligned} \tag{4.19}
$$

with a constant $c \equiv c(n, m, p, C_0, C_1, \lambda)$. The next step is to estimate $\mathcal{J}_2$ via (2.5), (4.12), and (4.10). Additionally enlarging the domain of integration and the exponent from $p-1-\lambda$ to $(1+\lambda)(p-1)$ (recall that (4.11) holds) and exploiting (4.4), we deduce

$$\begin{aligned}\mathcal{J}_2 &\le \frac{cd_j^{p+m-3}}{r_j^{n+p}}\iint_{L_j}\Big(\frac{u-a_j}{d_j}\Big)^{p-2+m-\lambda}dz\\ &\le \frac{cd_{j-1}^{p-2}}{r_{j-1}^{n+p}}\iint_{L_j}u^{m-1}\Big(\frac{u-a_{j-1}}{d_{j-1}}\Big)^{p-1-\lambda}dz\\ &\le \frac{cd_{j-1}^{p-2}}{r_{j-1}^{n+p}}\iint_{L_{j-1}}u^{m-1}\Big(\frac{u-a_{j-1}}{d_{j-1}}\Big)^{(1+\lambda)(p-1)}dz\\ &\le c\kappa \end{aligned}\tag{4.20}$$

with a constant $c\equiv c(n,\ m,\ p,\ \lambda)$. Finally, we use (4.14) to obtain

$$\mathcal{J}_3=\frac{d_j^{p+m-3}}{r_j^n}\iint_{L_j}\Big|DV_\lambda\Big(\frac{u-a_j}{d_j}\Big)\Big|^p dz\le c\Big[\kappa+\frac{\mu(\frac{1}{\nu}Q_j)}{d_jr_j^n}\Big]$$

with a constant $c\equiv c(n,\ p,\ C_0,\ C_1,\ \lambda)$. Inserting the estimates for $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{J}_3$ in (4.18), we conclude that

$$\mathcal{J}\le c\varepsilon^{(1+\lambda)(p-1)}\kappa+c_\varepsilon\Big[\varepsilon_1^{p\lambda}+\varepsilon_1^{-p\lambda}\Big(\kappa+\frac{\mu(\frac{1}{\nu}Q_j)}{d_jr_j^n}\Big)^{p\lambda}\Big]\Big[\kappa+\frac{\mu(\frac{1}{\nu}Q_j)}{d_jr_j^n}\Big]$$

with constants $c\equiv c(n,\ m,\ p,\ \lambda)$ and $c_\varepsilon\equiv c_\varepsilon(n,\ m,\ p,\ C_0,\ C_1,\ \lambda,\ \varepsilon)$. Our next aim is to analogously estimate the term $\tilde{\mathcal{J}}$ from (4.17). An application of (2.8), (4.13), and the Gagliardo–Nirenberg inequality (2.4) with the choice $\ell=\frac{pn\lambda}{p-1-\lambda}$ yields

$$\begin{aligned}\tilde{\mathcal{J}}\le& c\varepsilon^{(1+\lambda)(p-1)}\kappa+\frac{c_\varepsilon a_j^{m-1}d_j^{p-2}}{r_j^{n+p}}\iint_{Q_j}W_\lambda\Big(\tfrac{(u-a_j)_+}{d_j}\Big)^{\frac{p(1+\lambda)(p-1)}{p-1-\lambda}}dz\\ \le& c\varepsilon^{(1+\lambda)(p-1)}\kappa+c_\varepsilon\Big[\sup_{t\in\Lambda_j}\tfrac{1}{r_j^n}\int_{B_j\times\{t\}}\Big|W_\lambda\Big(\tfrac{(u-a_j)_+}{d_j}\Big)\Big|^{\frac{pn\lambda}{p-1-\lambda}}dx\Big]^{\frac{p}{n}}\\ &\cdot\frac{a_j^{m-1}d_j^{p-2}}{r_j^n}\iint_{Q_j}\Big[\tfrac{1}{r_j^p}\Big|W_\lambda\Big(\tfrac{(u-a_j)_+}{d_j}\Big)\Big|^p+\Big|DW_\lambda\Big(\tfrac{(u-a_j)_+}{d_j}\Big)\Big|^p\Big]dz\\ =:& c\varepsilon^{(1+\lambda)(p-1)}\kappa+c_\varepsilon\tilde{\mathcal{J}}_1(\tilde{\mathcal{J}}_2+\tilde{\mathcal{J}}_3)\end{aligned}$$

with constants $c\equiv c(n,\ m,\ p,\ \lambda)$ and $c_\varepsilon\equiv c_\varepsilon(n,\ m,\ p,\ \lambda,\ \varepsilon)$. Estimating $\tilde{\mathcal{J}}_1$ via (2.7), we see that this integral can be bounded as in (4.19). Moreover, using (2.7) and the fact that we can replace a_j by u, we can copy the arguments from (4.20) to find that the bound for $\mathcal{J}_2$ is valid also for $\tilde{\mathcal{J}}_2$, and $\tilde{\mathcal{J}}_3$ can be estimated in the same manner as $\mathcal{J}_3$. Therefore, if we plug the previous results in (4.17), we have shown that

$$\begin{aligned}\kappa \leq & c\varepsilon^{(1+\lambda)(p-1)}\kappa + c_\varepsilon\Big[\varepsilon_1^{p\lambda} + \varepsilon_1^{-p\lambda}\Big(\kappa + \frac{\mu(\frac{1}{\nu}Q_j)}{d_j r_j^n}\Big)^{p\lambda}\Big]\Big[\kappa + \frac{\mu(\frac{1}{\nu}Q_j)}{d_j r_j^n}\Big] \\ \leq & \big[c\varepsilon^{(1+\lambda)(p-1)} + c_\varepsilon\varepsilon_1^{p\lambda} + c_\varepsilon\varepsilon_1^{-p\lambda}\kappa^{p\lambda}\big]\kappa \\ & + c_\varepsilon\varepsilon_1^{-p\lambda}\Big[\frac{\mu(\frac{1}{\nu}Q_j)}{d_j r_j^n} + \Big(\frac{\mu(\frac{1}{\nu}Q_j)}{d_j r_j^n}\Big)^{1+p\lambda}\Big]\end{aligned}$$

with constants $c \equiv c(n, m, p, \lambda)$ and $c_\varepsilon \equiv c_\varepsilon(n, m, p, C_0, C_1, \lambda, \varepsilon)$. Choosing ε such that $c\varepsilon^{(1+\lambda)(p-1)} = \frac{1}{6}$, then ε_1 such that $c_\varepsilon\varepsilon_1^{p\lambda} = \frac{1}{6}$, and lastly κ such that $c_\varepsilon\varepsilon_1^{-p\lambda}\kappa^{p\lambda} = \frac{1}{6}$ ensures that $\varepsilon, \varepsilon_1, \kappa \in (0, 1)$ only depend on n, m, p, C_0, C_1 and λ, and the preceding inequality simplifies to

$$\kappa \leq c\Big[\frac{\mu(\frac{1}{\nu}Q_j)}{d_j r_j^n} + \Big(\frac{\mu(\frac{1}{\nu}Q_j)}{d_j r_j^n}\Big)^{1+p\lambda}\Big] \tag{4.21}$$

with a constant $c \equiv c(n, m, p, C_0, C_1, \lambda)$. Distinguishing the cases $\mu(\frac{1}{\nu}Q_j) \leq d_j r_j^n$ and $\mu(\frac{1}{\nu}Q_j) > d_j r_j^n$, we observe that (4.21) and (4.5) imply

$$d_j \leq c r_j^{-n}\mu\big(\tfrac{1}{\nu}Q_{r_j,\omega_j r_j^p}\big)$$

such that (4.15) follows. In the remainder of this section, we will explain the bound (4.16) for d_0. Exactly as in the argument for $j \geq 1$, we can assume that $d_0 > \hat{d}_0$. In particular, this means that we can take for granted that we have $\mathbf{K}_0(d_0) = \kappa$, which is equivalent to

$$d_0^{(1+\lambda)(p-1)-(p-2)} = \frac{1}{\kappa r^{n+p}}\iint_{L_0} u^{m-1}(u - a_0)^{(1+\lambda)(p-1)}\,dz.$$

We note that $(1 + \lambda)(p - 1) - (p - 2) = 1 + \lambda(p - 1)$ is positive and use the estimate $u - a_0 < u$, valid on the domain of integration. Furthermore, we employ (4.8) and recall that κ only depends on n, m, p, C_0, C_1 and λ. Hence, we receive

$$d_0 \leq c\Big[\frac{1}{r^{n+p}}\iint_{Q_{r,\theta}} u^{m-1+(1+\lambda)(p-1)}\,dz\Big]^{\frac{1}{1+\lambda(p-1)}},$$

which proves (4.16).

4.6. Potential estimates

For any $\ell \geq 2$, we derive

$$\begin{aligned}a_\ell - a_0 = \sum_{j=0}^{\ell-1} d_j &\leq d_0 + \tfrac{1}{2}\sum_{j=0}^{\ell-2} d_j + c\sum_{j=1}^{\ell-1}\mathbf{D}_p^\mu\big(z_0; \tfrac{1}{\nu}r_j\big) \\ &\leq d_0 + \tfrac{1}{2}a_{\ell-1} + c\mathbf{P}_p^\mu\big(z_0; \tfrac{1}{\nu}r\big)\end{aligned}$$

with a constant $c \equiv c(n, m, p, C_0, C_1, \lambda)$ by the definition of $a_{j+1} = a_j + d_j$ and (4.15). Besides, appealing to the fact that $a_{\ell-1} \le a_\ell$, the recursive bound (4.16), and the definition (4.1) of a_0, we infer

$$
\begin{aligned}
a_\ell \le 2 + 4\Big[\pi_p \frac{r^p}{\theta}\Big]^{\frac{1}{p+m-3}} + c\Big[\frac{1}{r^{n+p}} \iint_{Q_{r,\theta}} u^{m-1+(1+\lambda)(p-1)}\, dz\Big]^{\frac{1}{1+\lambda(p-1)}} \\
+ c\mathbf{P}_p^\mu\big(z_0; \tfrac{1}{\nu} r\big).
\end{aligned}
$$

Estimating r by $\frac{1}{\nu} r$ and subsequently substituting $\frac{1}{\nu} r$ by r, we have shown that

$$
\begin{aligned}
a_\infty &:= \lim_{\ell\to\infty} a_\ell \\
&\le c\Bigg[1 + \Big[\frac{r^p}{\theta}\Big]^{\frac{1}{p+m-3}} + \Big[\frac{1}{r^{n+p}} \iint_{Q_{r,\theta}} u^{m-1+(1+\lambda)(p-1)}\, dz\Big]^{\frac{1}{1+\lambda(p-1)}} + \mathbf{P}_p^\mu(z_0; r)\Bigg]
\end{aligned}
$$

is finite such that $d_j = a_{j+1} - a_j \to 0$ as $j \to \infty$. Now, pick an arbitrary Lebesgue point z_0 of u. Then, due to (4.4), we find

$$
\begin{aligned}
&\Big(\frac{u(z_0)}{a_\infty}\Big)^{m-1} \big(u(z_0) - a_\infty\big)_+^{(1+\lambda)(p-1)} \\
&= \lim_{j\to\infty} ⨌_{Q_j} \Big(\frac{u}{a_j}\Big)^{m-1} \big(u - a_j\big)_+^{(1+\lambda)(p-1)}\, dz \\
&= c \lim_{j\to\infty} d_j^{(1+\lambda)(p-1)} \frac{d_j^{p-2}}{r_j^{n+p}} \iint_{L_j} u^{m-1} \Big(\frac{u - a_j}{d_j}\Big)^{(1+\lambda)(p-1)}\, dz \\
&\le c\kappa \lim_{j\to\infty} d_j^{(1+\lambda)(p-1)},
\end{aligned}
$$

where $c \equiv c(n, p)$ is a constant. Since $d_j \to 0$ as $j \to \infty$, we have $u(z_0) \le a_\infty$, and Theorem 1.3 is proven. □

5. Comparison with the Barenblatt solution

In this section, we will consider the model equation

$$
\partial_t u - \operatorname{div}(|u|^{m-1}|Du|^{p-2}Du) = \delta, \tag{5.1}
$$

where δ is the Dirac measure on $\mathbb{R}^{n+1}$ charging the origin. We want to test our potential estimate (1.10) against the explicit very weak solution to (5.1), the so-called Barenblatt solution $\mathcal{B}_{m,p}$ given in (1.11), by analyzing the behavior at the origin. Since, for $p = 2$, the potential $\mathbf{P}_2^\mu$ is equivalent to the truncated Riesz potential, and the problem reduces to the porous medium situation, which was already studied in [4, p. 3289], we will concentrate on the case $p > 2$ here. For $z_0 = (0, t_0)$ with some $t_0 > 0$, we introduce the abbreviations

$$
u := \mathcal{B}_{m,p}(0, t_0), \quad \sigma := m(p-1) - 1 \quad \text{and} \quad b(r) := \frac{\delta(Q_{r, u^{-\sigma} r^p}(z_0))}{u^{2-p} r^n},
$$

and choose $\omega := b(r)^{-\frac{p-2}{p-1}}$ in (2.2), where $r > 0$ is such that $Q_{r,\omega r^p}(z_0) \subset E_T$. By the definition of the nonlinear parabolic potential from (2.3), we have

$$\mathbf{P}_p^\mu(z_0; r) \le c \sum_{j=0}^{\infty} \Big[b(r_j)^{\frac{1}{p-1}} + r_j^{-n} \delta\Big(Q_{r_j, b(r_j)^{-\frac{p-2}{p-1}} r_j^p}(z_0) \Big) \Big]$$

with a constant $c \equiv c(p)$, where $r_j = \frac{r}{2^j}$ for any $j \in \mathbb{N}_0$. Estimating the sum by an integral, we obtain

$$\mathbf{P}_p^\mu(z_0; r) \le c \int_0^\infty \Big[u^{\frac{p-2}{p-1}} \varrho^{-\frac{n}{p-1}} \delta\big(Q_{\varrho, u^{-\sigma}\varrho^p}(z_0) \big) + \varrho^{-n} \delta\Big(Q_{\varrho, b(\varrho)^{-\frac{p-2}{p-1}} \varrho^p}(z_0) \Big) \Big] \frac{d\varrho}{\varrho}.$$

Now, the origin is contained in the cylinder $Q_{\varrho, u^{-\sigma}\varrho^p}(z_0)$ if and only if $\varrho > \sqrt[p]{u^\sigma t_0}$. Therefore, the above integral simplifies to

$$\begin{aligned} &\int_{\sqrt[p]{u^\sigma t_0}}^\infty \Big[u^{\frac{p-2}{p-1}} \varrho^{-\frac{n}{p-1}} + \varrho^{-n} \delta\Big(Q_{\varrho, b(\varrho)^{-\frac{p-2}{p-1}} \varrho^p}(z_0) \Big) \Big] \frac{d\varrho}{\varrho} \\ &\quad \le u^{\frac{p-2}{p-1}} \int_{\sqrt[p]{u^\sigma t_0}}^\infty \varrho^{-\frac{n}{p-1}-1}\, d\varrho + \int_{\sqrt[p]{u^\sigma t_0}}^\infty \varrho^{-n-1}\, d\varrho, \end{aligned}$$

where we have trivially estimated the Dirac measure by 1 in the second step. Hence, we deduce the bound

$$\mathbf{P}_p^\mu(z_0; r) \le c \max \Big\{ t_0^{-\frac{n}{p(p-1)}} u^{\frac{p(p-2)-n[m(p-1)-1]}{p(p-1)}},\ t_0^{-\frac{n}{p}} u^{-\frac{n[m(p-1)-1]}{p}} \Big\}$$

with a constant $c \equiv c(n, p)$. Considering in (1.10) only the bound from above coming from the potential $\mathbf{P}_p^\mu$, we infer that

$$u \le c t_0^{-\frac{n}{p+n[m(p-1)-1]}}$$

such that our potential estimate yields the same decay as displayed in (1.12) by the Barenblatt solution.

6. Lorentz space criteria: the proof of Theorem 1.4

Proof of Theorem 1.4. The local boundedness of u follows from Theorem 1.3 once we have established the local uniform boundedness of $\mathbf{P}_p^\mu$. As, for $p = 2$, the potential $\mathbf{P}_p^\mu$ is equivalent to the well-understood Riesz potential, we will not dwell on this case and only deal with $p > 2$ in this section. Here, we assume that the measure μ has some density $\mu(x,t)\,dx\,dt$, which we do not rename. For a fixed $0 < \varepsilon \ll 1$ and a point $(x_0, t_0) \in E_T$, we will consider $r > 0$ and $\omega > 0$ both small enough such that $B_r(x_0) \Subset E$ and $(t_0 - \omega r^p, t_0) \subset (\varepsilon, T - \varepsilon) =: J_\varepsilon$. More precisely, we will choose ω in dependence on r in the proof and write w_r to emphasize the dependence on r. In view of (2.2), we will have to ensure the existence of a number $R > 0$ such that $Q_{r,\omega_r r^p}(z_0) \Subset E_T$ holds for any $r \in (0, R)$. First, by (2.13), we deduce

$$\int_{t_0-\omega_r r^p}^{t_0} \mu(x,t)\,dt \le \omega_r r^p \mu^{**}(x, \omega_r r^p) \le (\omega_r r^p)^{\frac{n}{n+p}} \|\mu\|_{L^{q_1,\infty}(J_\varepsilon)}(x)$$

for any $0 < \omega_r < \frac{t_0-\varepsilon}{r^p}$ and $x \in B_r(x_0)$, where $q_1 = \frac{n+p}{p}$. From the above inequality, we infer by another application of (2.13) that

$$\begin{aligned} r^{-n}\mu(Q_{r,\omega_r r^p}) &= r^{-n}\int_{B_r(x_0)}\int_{t_0-\omega_r r^p}^{t_0} \mu(x,t)\,dt\,dx \\ &\le \alpha_n (\omega_r r^p)^{\frac{n}{n+p}} \|\mu\|^{**}_{L^{q_1,\infty}(J_\varepsilon)}(\alpha_n r^n) \end{aligned}$$

with $\alpha_n = \mathcal{L}^n(B_1(0))$ being the volume of the unit ball. Inserted in (2.2), this gives

$$\mathbf{D}_p^\mu(z_0; r) \le c \inf_{\omega>0} \left\{ \omega^{-\frac{1}{p-2}} + (\omega r^p)^{\frac{n}{n+p}} \|\mu\|^{**}_{L^{q_1,\infty}(J_\varepsilon)}(\alpha_n r^n) \colon Q_{r,\omega r^p}(z_0) \subset E_T \right\} \tag{6.1}$$

with a constant $c \equiv c(n, p)$. Now, let $-p < \psi_1 < 0$ and $\psi_2 < 0$ be constants, which will be specified later in dependence on n and p. Obviously, $\varrho^{p+\psi_1}$ vanishes in the limit $\varrho \searrow 0$. Further, we may assume without loss of generality that there exists some number $\bar\varrho > 0$ such that $\|\mu\|^{*}_{L^{q_1,\infty}(J_\varepsilon)}(\alpha_n \varrho^n)$ is strictly positive for any $\varrho \in [0, \bar\varrho]$ since otherwise, we had $\|\mu\|^{*}_{L^{q_1,\infty}(J_\varepsilon)} \equiv 0$ by monotonicity. Then, due to (2.11), we also know that $\|\mu\|^{**}_{L^{q_1,\infty}(J_\varepsilon)}(\alpha_n \varrho^n) > 0$ for any $\varrho \in [0, \bar\varrho]$. As a consequence, we see that

$$\varrho^{p+\psi_1}\big[\|\mu\|^{**}_{L^{q_1,\infty}(J_\varepsilon)}(\alpha_n \varrho^n)\big]^{\psi_2} \searrow 0 \ \text{ as } \ \varrho \searrow 0.$$

Therefore, we can find a radius $R > 0$ such that the choice

$$\omega_r = r^{\psi_1}\big[\|\mu\|^{**}_{L^{q_1,\infty}(J_\varepsilon)}(\alpha_n r^n)\big]^{\psi_2} \tag{6.2}$$

is admissible in (6.1) for any $r \in (0, R)$. To obtain an upper bound for the potential $\mathbf{P}_p^\mu$, we insert ω_r as in (6.2) in (6.1) and integrate the inequality with respect to r. Then, the first term on the right-hand side of (6.1) reads

$$\begin{aligned} &\int_0^r \left[s^{-\frac{\psi_1}{p-2}} \big[\|\mu\|^{**}_{L^{q_1,\infty}(J_\varepsilon)}(\alpha_n s^n)\big]^{-\frac{\psi_2}{p-2}} \right] \frac{ds}{s} \\ &\le c \int_0^\infty \left[s^{\frac{\psi_1}{n\psi_2}} \|\mu\|^{**}_{L^{q_1,\infty}(J_\varepsilon)}(s) \right]^{-\frac{\psi_2}{p-2}} \frac{ds}{s} \\ &= c \left\| \|\mu\|_{L^{q_1,\infty}(J_\varepsilon)} \right\|^{-\frac{\psi_2}{p-2}}_{L^{\frac{n\psi_2}{\psi_1}, -\frac{\psi_2}{p-2}}(B_r(x_0))} =: \mathcal{J}_1 \end{aligned} \tag{6.3}$$

with a constant $c \equiv c(n, p, \psi_1)$, where the inequality follows by an easy substitution. Analogously, the other summand from (6.1) can be estimated from above by

$$c\int_0^\infty \left[s^{(p+\psi_1)/\left[(n+p)(1+\frac{n}{n+p}\psi_2)\right]}\|\mu\|^{**}_{L^{q_1,\infty}(J_\varepsilon)}(s)\right]^{1+\frac{n}{n+p}\psi_2}\frac{ds}{s}$$
$$= c\left\|\|\mu\|_{L^{q_1,\infty}(J_\varepsilon)}\right\|^{1+\frac{n}{n+p}\psi_2}_{L^{(n+p)(1+\frac{n}{n+p}\psi_2)/(p+\psi_1),\,1+\frac{n}{n+p}\psi_2}(B_r(x_0))} =: \mathcal{J}_2 \tag{6.4}$$

with a constant $c \equiv c(n,\ p,\ \psi_1)$. From (6.3) and (6.4), we can conclude the estimate

$$\mathbf{P}^\mu_p(z_0; r) = \sum_{j=0}^\infty \mathbf{D}^\mu_p(z_0; r_j) \le \mathcal{J}_1 + \mathcal{J}_2 \tag{6.5}$$

for the potential $\mathbf{P}^\mu_p$. Note that the parameters ψ_1 and ψ_2 are free up to now. Our aim is to establish conditions for those parameters which admit a uniform bound for $\mathbf{P}^\mu_p$ in terms of the Lorentz quasi-norm of μ with optimal exponents. More precisely, we want to estimate $\mathcal{J}_1$ and $\mathcal{J}_2$ by some positive power of

$$\left\|\|\mu\|_{L^{q_1,\infty}(J_\varepsilon)}\right\|_{L^{q_1,q_2}(B_r(x_0))} \tag{6.6}$$

with $q_1 = \frac{n+p}{p}$ and $q_2 = \frac{n+p}{n(p-1)+p}$. The quantity in (6.6) is finite by our assumption (1.13). For the following argumentation, we recall the inclusions (2.12). In order to estimate $\mathcal{J}_1$ by (6.6), we require $-\frac{\psi_2}{p-2} \ge \frac{n+p}{n(p-1)+p}$ and $0 < \frac{n\psi_2}{\psi_1} \le \frac{n+p}{p}$, or, equivalently,

$$\psi_2 \le -\frac{(n+p)(p-2)}{n(p-1)+p} \quad\text{and}\quad \psi_1 \le \frac{np}{n+p}\psi_2. \tag{6.7}$$

Further, we establish restrictions which allow us to estimate $\mathcal{J}_2$ by (6.6). Here, we need to assume

$$1 + \tfrac{n}{n+p}\psi_2 \ge \frac{n+p}{n(p-1)+p} \quad\text{and}\quad 0 < \frac{(n+p)(1+\frac{n}{n+p}\psi_2)}{p+\psi_1} \le \frac{n+p}{p}. \tag{6.8}$$

Joining $(6.7)_1$ and $(6.8)_1$, we obtain

$$\psi_2 = -\frac{(n+p)(p-2)}{n(p-1)+p}. \tag{6.9}$$

What is more, since $1 + \frac{n}{n+p}\psi_2 = \frac{n+p}{n(p-1)+p} > 0$ and $p + \psi_1 > 0$, we can rewrite $(6.8)_2$ and combine it with $(6.7)_2$ to get

$$\psi_1 = -\frac{np(p-2)}{n(p-1)+p}. \tag{6.10}$$

Therefore, choosing ψ_1 and ψ_2 as in (6.9) and (6.10), we have shown that the right-hand side of (6.5) is bounded by some power of (6.6), which ensures the uniform boundedness of $\mathbf{P}^\mu_p(z_0; r)$ for small radii $r > 0$. In view of (1.10), this finishes the proof. □

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