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Quantitative properties of the non-properness set of a polynomial map

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Abstract. Let f be a generically finite polynomial map $f : \mathbb{C}^n \to \mathbb{C}^m$ of algebraic degree d. Motivated by the study of the Jacobian Conjecture, we prove that the set S_f of non-properness of f is covered by parametric curves of degree at most $d-1$. This bound is best possible. Moreover, we prove that if $X \subset \mathbb{R}^n$ is a closed algebraic set covered by parametric curves, and $f : X \to \mathbb{R}^m$ is a generically finite polynomial map, then the set S_f of non-properness of f is also covered by parametric curves. Moreover, if X is covered by parametric curves of degree at most d_1, and the map f has degree d_2, then the set S_f is covered by parametric curves of degree at most $2d_1d_2$. As an application of this result we show a real version of the Białynicki-Birula theorem: Let G be a real, non-trivial, connected, unipotent group which acts effectively and polynomially on a connected smooth algebraic variety $X \subset \mathbb{R}^n$. Then the set $Fix(G)$ of fixed points has no isolated points.

1. Introduction

Let $f : X \to Y$ be a generically finite polynomial map between affine varieties.

Definition 1.1. We say that f is *proper at a point* $y \in Y$ if there exists an open neighborhood U of y such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is a proper map. The set of points at which f is not proper is denoted by S_f.

The set S_f was first introduced by the first author in [5] (see also [6,7]). It is a good measure of non-properness of the map f, and it has interesting applications in pure and applied mathematics [4,9,13]. The first author proved the following property of the set S_f when the base field is $\mathbb{C}$.

Theorem 1.2. (Theorem 4.1 [8]) *Let X be an affine variety over $\mathbb{C}$, and let $f : X \to \mathbb{C}^m$ be a generically finite polynomial map. If X is $\mathbb{C}$-uniruled (covered by polynomially parametric curves), then the set S_f is also $\mathbb{C}$-uniruled.*

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The first aim of this paper is to give a numerical form of Theorem 1.2. We introduce the notion of degree of uniruledness and we estimate this degree in some cases. In particular if f is a generically finite polynomial map $f : \mathbb{C}^n \to \mathbb{C}^m$ of algebraic degree d, we prove that the set S_f of non-properness of the map f is covered by parametric curves of degree at most $d - 1$. This bound is best possible.

The second aim of our paper is to generalize Theorem 1.2 to the field of real numbers (see Theorem 4.11):

Let X be a closed algebraic set over $\mathbb{R}$, and let $f : X \to \mathbb{R}^m$ be a generically finite polynomial map. If X is $\mathbb{R}$-uniruled, then the set S_f is also $\mathbb{R}$-uniruled.

Our third aim is to prove a real counterpart of the following theorem of Białynicki-Birula.

Theorem 1.3. ([2]) *If a connected, unipotent, algebraic group acts on an irreducible affine algebraic variety $X \subset \mathbb{C}^n$, then the set $Fix(G)$ of fixed points of this action has no isolated points.*

The proof from [2] is cohomological, and it cannot be extended to the real case. In the last section, as an application of our methods, we modify our approach from [10] and we give a real counterpart of the result of Białynicki-Birula (Corollary 5.3):

Let G be a real, non-trivial, connected, unipotent group which acts effectively and polynomially on a connected smooth closed algebraic variety $X \subset \mathbb{R}^n$. Then the set $Fix(G)$ is $\mathbb{R}$-uniruled. In particular, it has no isolated points.

2. Preliminaries

Unless stated otherwise, $\mathbb{K}$ is an arbitrary algebraically closed field (the real field case is explained in Sect. 4). All affine varieties are considered to be embedded in an affine space.

The study of uniruled varieties in projective geometry, that is, varieties possessing a covering by rational curves, has a long history. In affine geometry it is more natural to consider polynomially parametric curves (see the definition below) than rational ones. Therefore in [6] (see also [13]) the first author defined $\mathbb{K}$-uniruled varieties as those which are covered by polynomially parametric curves. In [10] we refined this definition for countable fields. In this paper we introduce and study the corresponding quantitative parameter, the degree of $\mathbb{K}$-uniruledness.

Definition 2.1. An irreducible affine curve $\Gamma \subset \mathbb{K}^m$ is called a *polynomially parametric curve of degree at most d*, if there exists a non-constant polynomial map $f : \mathbb{K} \to \Gamma$ of degree at most d (by the degree of $f = (f_1, \ldots, f_m)$ we mean $\max_i \deg f_i$). A curve is *polynomially parametric* if it is polynomially parametric of some degree.

We have the following equivalences (see also [10, Proposition 2.4]).

Proposition 2.2. *Let $X \subset \mathbb{K}^m$ be an irreducible affine variety of dimension n, and let d be a constant. The following conditions are equivalent:*

(1) *for every* $x \in X$ *there exists a polynomially parametric curve* $l_x \subset X$ *of degree at most* d *passing through* x,
(2) *there exists an open, non-empty subset* $U \subset X$ *such that for every* $x \in U$ *there exists a polynomially parametric curve* $l_x \subset X$ *of degree at most* d *passing through* x,
(3) *there exists an affine variety* W *of dimension* $\dim X - 1$ *and a dominant polynomial map* $\phi : \mathbb{K} \times W \ni (t, w) \mapsto \phi(t, w) \in X$ *such that* $\deg_t \phi \leq d$.

Proof. The implication (1) $\Rightarrow$ (2) is obvious. To prove (2) $\Rightarrow$ (1) suppose that $X = \{x \in \mathbb{K}^m : f_1(x) = 0, \ldots, f_r(x) = 0\}$. For a point $a = (a_1, \ldots, a_m) \in \mathbb{K}^m$ and $B = (b_{1,1} : \cdots : b_{d,m}) \in \mathbb{P}^M$, where $M = dm - 1$, let

$$\varphi_{a,b} : \mathbb{K} \ni t \mapsto (a_1 + b_{1,1}t + \cdots + b_{1,d}^d t^d, \ldots, a_m + b_{m,1}t + \cdots + b_{m,d}^d t^d) \in \mathbb{K}^m$$

be a polynomially parametric curve. Note that for every dm-tuple $b = (b_{1,1}, \ldots, b_{m,d})$ we have $\varphi_{a,\lambda b}(t) = \varphi_{a,b}(\lambda t))$ for every $\lambda \in \mathbb{K}^*$, hence the image of $\varphi_{a,b}$ depends only on the class $[b] = B \in \mathbb{P}^M$ but not on b. We will identify $\varphi_{a,b}$ with the curve $\varphi_{a,b}(\mathbb{K})$.

Consider the following variety and projection:

$$\mathbb{K}^m \times \mathbb{P}^M \supset V = \{(a, b) \in \mathbb{K}^m \times \mathbb{P}^M : \forall_{t,i}\ f_i(\varphi_{a,b}(t)) = 0\} \ni (a, b) \to a \in \mathbb{K}^m.$$

Note that $f_i(\varphi_{a,b}(t)) = \sum_k \alpha_{i,k}(a, b)t^k$, hence the equations $\{f_i(\varphi_{a,b}(t)) \equiv 0\}$ split into finite number of equations $\alpha_{i,k}(a, b) = 0$, which are homogeneous with respect to b.

From the definition, $(a, b) \in V$ if and only if the polynomially parametric curve $\varphi_{a,b}$ is contained in X. Hence the image of the projection is contained in X and contains U, since through every point of U passes a polynomially parametric curve of degree at most d. But since the projective space $\mathbb{P}^M$ is complete and V is closed, we find that the image is closed, and hence it is the whole X.

Let us prove (2) $\Rightarrow$ (3). For some affine chart $V_j = V \cap \{b_j = 1\}$ the above map is dominant. We consider the dominant map

$$\Phi : \mathbb{K} \times V_j \ni (t, \phi) \mapsto \phi(t) \in X.$$

After replacing V_j by some irreducible component $Y \subset \mathbb{K}^m$ ($\dim(Y) = s$) the map remains dominant. On an open subset of X fibers of the map $\Phi' = \Phi|_{\mathbb{K}\times Y}$ are of pure dimension $s + 1 - n$; let x be one of such points. From the construction of the set V we know that the fiber $F = \Phi'^{-1}(x)$ does not contain any line of type $\mathbb{K} \times \{y\}$, so in particular the image F' of F under the projection $\mathbb{K} \times Y \to Y$ (which is a constructible subset of Y) has the same dimension. For a general linear subspace $L \subset \mathbb{K}^m$ of dimension $m + n - s - 1$ the set $L \cap F'$ is 0-dimensional (indeed, F' contains an open and dense subset of $\overline{F'}$). Let us fix such an L, and let R be any irreducible component of $L \cap Y$ intersecting F'. Now the map $\Phi'|_{\mathbb{K}\times R} : \mathbb{K} \times R \to X$ satisfies the assertion, since it has one fiber of dimension 0 (over x) and the dimension of R is $n - 1$. Indeed, in this case we have $\dim \mathbb{K} \times R = \dim X$ and since the fibers of Φ' have generically dimension 0, the map Φ' has to be dominant.

To prove the implication (3) $\Rightarrow$ (2) it is enough to notice that for every $w \in W$ the map $\phi_w : \mathbb{K} \ni t \mapsto \phi(t, w) \in X$ is a polynomially parametric curve of degree at most d or it is constant. The image of ϕ contains an open dense subset, so after excluding the points with infinite preimages (a closed set of codimension at most one) we get an open set U with required properties. $\square$

Definition 2.3. We say that an affine variety X has *degree of* $\mathbb{K}$*-uniruledness at most* d if all its irreducible components satisfy the conditions of Proposition 2.2. An affine variety is called $\mathbb{K}$*-uniruled* if it has some degree of $\mathbb{K}$-uniruledness.

To simplify our statements we say that the empty set has degree of $\mathbb{K}$-uniruledness zero, in particular it is $\mathbb{K}$-uniruled.

Example 2.4. Let $X \subset \mathbb{K}^n$ be a general hypersurface of degree $d < n$. It is well-known (see [11, Exercise V.4.4.3, p. 269] that X is covered by affine lines, therefore its degree of $\mathbb{K}$-uniruledness is one.

For uncountable (algebraically closed) fields there is also another characterization of $\mathbb{K}$-uniruled varieties (see [13, Theorem 3.1]).

Proposition 2.5. *Let* $\mathbb{K}$ *be an uncountable algebraically closed field, and let* $X \subset \mathbb{K}^m$ *be an affine variety. The following conditions are equivalent:*

(1) X *is* $\mathbb{K}$*-uniruled,*
(2) *for every* $x \in X$ *there exists a polynomially parametric curve* $l_x \subset X$ *passing through* x*,*
(3) *there exists an open, non-empty subset* $U \subset X$ *such that for every* $x \in U$ *there exists a polynomially parametric curve* $l_x \subset X$ *passing through* x.

3. The complex field case

In the whole section we assume that the base field is $\mathbb{C}$. The condition that a map is not finite at a point y is equivalent to it being locally non-proper in the topological sense (there is no neighborhood U of y such that $f^{-1}(\overline{U})$ is compact). This characterization gives the following:

Proposition 3.1. ([5]) *Let* $f : X \to Y$ *be a generically finite map between affine varieties. Then* $y \in S_f$ *if and only if there exists a sequence* (x_n) *in* X*, such that* $x_n \to \infty$ *and* $f(x_n) \to y$.

In particular, for a polynomial map $f : \mathbb{C}^n \to \mathbb{C}^n$, $y \in S_f$ if and only if either $\dim f^{-1}(y) > 0$, or $f^{-1}(y) = \{x_1, \ldots, x_r\}$ is a finite set, but $\sum_{i=1}^{r} \mu_{x_i}(f) < \mu(f)$, where μ denotes multiplicity. In other words, f is not proper at y if f is not a local analytic covering over y.

Theorem 3.2. *Suppose* $f : \mathbb{C}^n \to \mathbb{C}^m$ *is a generically finite polynomial map of degree* d*. Then the set* S_f *is covered by parametric polynomial curves of degree at most* $d - 1$.

Proof. Let $y \in S_f$; by an affine transformation we can assume that $y = O = (0, 0, ..., 0) \in \mathbb{C}^m$. For the same reason we can assume that $O \notin f^{-1}(S_f)$. By Proposition 3.1 there exists a sequence of points $x_k \to \infty$ such that $f(x_k) \to O$. Let us consider the line $L_k(t) = tO + (1-t)x_k = (1-t)x_k,\ t \in \mathbb{C}$. Set $l_k(t) = f(L_k(t))$. Of course we have $\deg l_k \leq d$ for every k. Moreover, we can assume that $\deg l_k > 0$, because infinite fibers cover only a nowhere dense subset of $\mathbb{C}^n$. Each curve l_k is given by m polynomials of one variable:

$$l_k(t) = \left(\sum_{i=0}^{d} a_i^1(k)t^i, \ldots, \sum_{i=0}^{d} a_i^m(k)t^i \right).$$

Hence l_k corresponds to the uniquely determined point

$$(a_0^1(k), \ldots, a_d^1(k); a_0^2(k), \ldots, a_d^2(k); \ldots; a_0^m(k), \ldots, a_d^m(k)) \in \mathbb{C}^N.$$

Since for each i, $a_0^i(k) \to 0$ as $k \to \infty$, we can change the parametrization of l_k by setting $t \to \lambda_k t$ in such a way that $\|l_k\| = 1$ for $k \gg 0$ (we consider here l_k as an element of $\mathbb{C}^N$ with Euclidean norm). Now, since the unit sphere is compact, it is easy to see that there exists a subsequence (l_{k_r}) of (l_k) which converges to a polynomial map $l : \mathbb{C} \to \mathbb{C}^m$ with $l(0) = O$ and $\deg l \leq d$. Moreover, l is non-constant, because $\|l\| = 1$ and $l(0) = O$. We can also assume that the limit $\lim_{k\to\infty} \lambda_k = \lambda$ exists in the compactification of the field $\mathbb{C}$. We consider two cases:

(1) λ is finite: then $L_k(\lambda_k t) = (1 - \lambda_k t)x_k \to \infty$ for $t \neq \lambda^{-1}$.

(2) $\lambda = \infty$; then $\|L_k(\lambda_k t)\| \geq (|\lambda_k t| - 1)\|x_k\|$, and hence $\|L_k(\lambda_k t)\| \to \infty$ for every $t \neq 0$.

On the other hand, $f(L_k(\lambda_k t)) = l_k(\lambda_k t) \to l(t)$; using once more Proposition 3.1 this means that the curve l is contained S_f, and so we see that S_f is covered by parametric polynomial curves of degree at most d.

Now we show that $\deg l < d$. The idea of the proof is as follows: Note that every curve l_k passes through the point $f(O)$, but the curve $l = \lim\ l_k$ does not. In fact $f(O)$ does not belong to S_f so the line l (which is included in S_f) cannot pass through $f(O)$. Thus if $l_k(t_k) = f(O)$, then $\lim\ t_k = \infty$.

On the other hand we show that if $\deg l = d$, then we can bound all t_k, and consequently we get a contradiction.

Assume that $\deg\ l = d$. Hence we can assume $\deg\ l_k = d$ for all k. Let $l(t) = (l_1(t), \ldots, l_m(t))$ and $l_k(t) = (l_1^k(t), \ldots, l_m^k(t))$. We can assume that the component $l_1(t)$ has maximal degree. Denote $f(O) = a = (a_1, \ldots, a_m)$. All roots of the polynomial $l_1(t) - a_1$ are contained in the interior of some disc $D = \{t \in \mathbb{C} : |t| < R\}$. Let $\epsilon = \inf\{|l_1(t) - a_1| : t \in \partial D\}$. For $k \gg 0$ we have $|(l_1 - a_1) - (l_1^k - a_1)|_D < \epsilon$. Consequently, by the Rouché Theorem these polynomials have the same number of zeros (counted with multiplicities) in D. In particular, the zeros of $l_1^k - a_1$ are uniformly bounded. All curves L_k pass through O, so all l_k pass through $a = f(O)$. This means that there is a sequence t_k such that $l_k(t_k) = a$. We have just shown that $|t_k| < R$, since t_k is a root of the polynomial $l_1^k - a_1$. So we can assume that the sequence t_k converges to some t_0. When we pass to the limit we get $l(t_0) = a$, which is a contradiction, since $a = f(O) \notin S_f$. Hence $\deg l < d$.

□

Now let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map with non-vanishing jacobian. The famous Jacobian Conjecture asserts that in this case f is a diffeomorphism (see e.g. [1,15]). Despite many efforts the conjecture is still wide open. The main obstruction for its solution is related to the set S_f of non-properness of the map f. Van den Dries and McKenna proved in 1990 that there is no counterexample to the Jacobian Conjecture for which the set S_f is a union of hyperplanes (see [14]). This suggests that we could solve the Jacobian Conjecture if we had some information about the geometry of the set S_f. On the other hand, it is well-known that we can reduce the algebraic degree of the map f to degree 3 (see [1,3]). The price we have to pay for this reduction is that in practice we have to consider all possible dimensions, even if we try to solve the problem for a fixed dimension. Theorem 3.2 gives the following characterization of the set S_f for generically finite cubic maps $f : \mathbb{C}^n \to \mathbb{C}^n$.

Corollary 3.3. *Suppose $f : \mathbb{C}^n \to \mathbb{C}^n$ is a generically finite cubic map. Then the set S_f is covered by lines and parabolas. Moreover, if f is quadratic, then S_f is covered only by lines.*

Theorem 3.4. *Let $X = \mathbb{C} \times W \subset \mathbb{C} \times \mathbb{C}^n$ be an affine cylinder and let $f : \mathbb{C} \times W \ni (t, w) \to (f_1(t, w), \dots, f_m(t, w)) \in \mathbb{C}^m$ be a generically finite polynomial map. Assume that $\deg_t f_i \le d$ for every $1 \le i \le n$. Then the set S_f has degree of $\mathbb{C}$-uniruledness at most d.*

Proof. Let $y \in S_f$; by an affine transformation we can assume that $y = O = (0, 0, ..., 0) \in \mathbb{C}^m$. By Proposition 3.1 there exists a sequence $(a_k, w_k) \in \mathbb{C} \times W$ such that $(a_k, w_k) \to \infty$ and $f(a_k, w_k) \to y$. Let us consider the line $L_k(t) = ((1 - t)a_k, w_k),\ t \in \mathbb{C}$. Set $l_k(t) = f(L_k(t))$. We can assume that $\deg l_k > 0$, because infinite fibers cover only nowhere dense subset of X. Each curve l_k is given by m polynomials of one variable:

$$l_k(t) = \left(\sum_{i=0}^{d} a_i^1(k)t^i, \dots, \sum_{i=0}^{d} a_i^m(k)t^i \right).$$

As before, l_k corresponds to the single point

$$(a_0^1(k), \dots, a_d^1(k); a_0^2(k), \dots, a_d^2(k); \dots; a_0^m(k), \dots, a_d^m(k)) \in \mathbb{C}^N.$$

Since for each i, $a_0^i(k) \to 0$ as $k \to \infty$ we can change the parametrization of l_k by setting $t \to \lambda_k t$ in such a way that $\|l_k\| = 1$ for $k \gg 0$ (we consider here l_k as an element of $\mathbb{C}^N$ with Euclidean norm). Now, since the unit sphere is compact, there exists a subsequence (l_{k_r}) of (l_k) which is convergent to a polynomial map $l : \mathbb{C} \to \mathbb{C}^m$ with $l(0) = O$. Moreover, l is non-constant, because $\|l\| = 1$ and $l(0) = O$. We can also assume that the limit $\lim_{k\to\infty} \lambda_k = \lambda$ exists in the compactification of the field $\mathbb{C}$. We consider two cases:

(1) λ is finite; then $L_k(\lambda_k t) = ((1-\lambda_k t)a_k, w_k) \to \infty$ for $t \neq \lambda^{-1}$.
(2) $\lambda = \infty$; then $\|L_k(\lambda_k t)\| \geq \max((|\lambda_k t|-1)|a_k|, \|w_k\|)$, and $\|L_k(\lambda_k t)\| \to \infty$ for every $t \neq 0$.

On the other hand, $f(L_k(\lambda_k t)) = l_k(\lambda_k t) \to l(t)$; using once more Proposition 3.1 we find that the curve l is contained in S_f, and so S_f has degree of $\mathbb{C}$-uniruledness at most d. □

Corollary 3.5. *Let $f = (f_1, \ldots, f_m) : \mathbb{C}^n \to \mathbb{C}^m$ be a generically finite map with $d = \min_j \max_i \deg_{x_j} f_i$. Then the set S_f has degree of $\mathbb{C}$-uniruledness at most d.*

Proof. Assume that $d = \max_i \deg_{x_1} f_i$. Then $f : \mathbb{C} \times \mathbb{C}^{n-1} \to \mathbb{C}^m$ and we can apply Theorem 3.4 for $W = \mathbb{C}^{n-1}$. □

Let us recall (see [8]) that for a generically finite polynomial map $f : X \to Y$ with X being $\mathbb{C}$-uniruled the set S_f is also $\mathbb{C}$-uniruled. We have the following "quantitative" counterpart of this result:

Theorem 3.6. *Let X be an affine variety with degree of $\mathbb{C}$-uniruledness at most d_1, and let $f : X \to \mathbb{C}^m$ be a generically finite map of degree d_2. Then the set S_f has degree of $\mathbb{C}$-uniruledness at most $d_1 d_2$.*

Proof. By Definition 2.3 there exists an affine variety W with $\dim W = \dim X - 1$ and a dominant polynomial map $\phi : \mathbb{C} \times W \to X$ of degree at most d_1 in the first coordinate. The equality $\dim \mathbb{C} \times W = \dim X$ implies that ϕ is generically finite, hence so is $f \circ \phi : \mathbb{C} \times W \to \mathbb{C}^m$, which is of degree at most $d_1 d_2$ in the first coordinate. By Theorem 3.4, $S_{f\circ\phi}$ has degree of $\mathbb{C}$-uniruledness at most $d_1 d_2$. We have the inclusion $S_f \subset S_{f\circ\phi}$, and from Theorem 1.2 we know that if non-empty, both sets are of pure dimension $\dim X - 1$, so each component of S_f is a component of $S_{f\circ\phi}$. This implies the assertion. □

Example 3.7. Let $f : \mathbb{C}^n \ni (x_1, \ldots, x_n) \mapsto (x_1, x_1 x_2, \ldots, x_1 x_n) \in \mathbb{C}^n$. We have $\deg f = 2$ and $S_f = \{x \in \mathbb{C}^n : x_1 = 0\}$. The set S_f has degree of $\mathbb{C}$-uniruledness 1. This shows that in general Theorems 3.2, 3.4 and Corollary 3.5 cannot be improved.

Example 3.8. For $n > 2$ let $X = \{x \in \mathbb{C}^n : x_1 x_2 = 1\}$, and $f : X \ni (x_1, \ldots, x_n) \mapsto (x_2, \ldots, x_n) \in \mathbb{C}^{n-1}$. The variety X has degree of $\mathbb{C}$-uniruledness 1. Moreover, $\deg f = 1$ and $S_f = \{x \in \mathbb{C}^{n-1} : x_1 = 0\}$. So the set S_f has degree of $\mathbb{C}$-uniruledness 1. This shows that in general Theorems 3.4 and 3.6 cannot be improved.

Remark 3.9. By the Lefschetz Principle all the results of this section remain true for an arbitrary algebraically closed field of characteristic zero.

4. The real field case

In the whole section we assume that the base field is $\mathbb{R}$. Let us recall that by a *real polynomially parametric curve of degree at most d* in a semialgebraic set $X \subset \mathbb{R}^n$ we mean the image of a non-constant real polynomial map $f : \mathbb{R} \to X$ of degree at most d. In general a real polynomially parametric curve need not be algebraic, but only semialgebraic. The real counterpart of Proposition 2.2 is the following.

Proposition 4.1. *Let $X \subset \mathbb{R}^n$ be a closed semialgebraic set, and let d be a constant. The following conditions are equivalent:*

(1) *for every $x \in X$ there exists a polynomially parametric curve $l_x \subset X$ of degree at most d passing through x,*
(2) *there exists a dense subset $U \subset X$ such that for every $x \in U$ there is a polynomially parametric curve $l_x \subset X$ of degree at most d passing through x,*
(3) *for every polynomial map $f : X \to \mathbb{R}^m$, and every sequence $x_k \in X$ such that $f(x_k) \to a \in \mathbb{R}^m$ there exists a semialgebraic curve W and a generically finite polynomial map $\phi : \mathbb{R} \times W \ni (t, w) \mapsto \phi(t, w) \in X$ such that $\deg_t \phi \leq d$, and there exists a sequence $y_k \in \mathbb{R} \times W$ such that $f(\phi(y_k)) \to a$. Moreover, if $x_k \to \infty$, then also $\phi(y_k) \to \infty$.*

Proof. First we prove the implication (2) $\Rightarrow$ (1). Suppose that $X = \{x \in \mathbb{R}^n : f_1(x) = 0, \ldots, f_r(x) = 0\, g_1(x) \geq 0, ..., g_s(x) \geq 0\}$. For $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and $b = (b_{1,1}, \ldots, b_{d,n}) \in \mathbb{R}^M$, where $M = dn$, let

$$\varphi_{a,b}(t) = (a_1 + b_{1,1}t + \cdots + b_{1,d}^d t^d, \ldots, a_n + b_{n,1}t + \cdots + b_{n,d}^d t^d)$$

be a polynomially parametric curve. If there exists a polynomially parametric curve of degree at most d passing through a, then after reparametrization we can assume that it is $\varphi_{a,b}$ for $\sum_{i,j} b_{i,j}^2 = 1$. This means that $b \in S_M(0, 1)$, where S_M denotes the unit sphere in $\mathbb{R}^M$. Consider the semialgebraic set

$$V = \{(a, b) \in \mathbb{R}^n \times S_M(0, 1) : \forall_{t,i}\ f_i(\varphi_{a,b}(t)) = 0, \forall_{t,j}\ g_j(\varphi_{a,b}(t)) \geq 0\}.$$

The definition of the set V says that for $(a, b) \in V$ the polynomially parametric curve $\varphi_{a,b}(t)$ is contained in X. It is easy to see that V is closed. For any $a \in X$, by the assumption there is a sequence $a_k \to a$ such that for every k there is a polynomially parametric curve $\varphi_{a_k,b_k} \in V$. We can assume that $\|a_k\| < \|a\| + 1$ for all k. Since V is closed and the sequence $((a_k, b_k)) \subset V$ is bounded, there is a subsequence (a_{k_r}, b_{k_r}) which converges to $(a, b) \in V$. Now the polynomially parametric curve $\varphi_{a,b} \subset X$ of degree at most d passes through a.

We prove (1) $\Rightarrow$ (3). Consider the semialgebraic set V as above. We have the surjective map

$$\Phi : \mathbb{R} \times V \ni (t, \varphi_{a,b}) \to \varphi_{a,b}(t) \in X.$$

Let $f : X \to \mathbb{R}^m$ be a polynomial map, and suppose $f(x_k) \to a \in \mathbb{R}^m$ for a sequence $x_k \in X$. Set $g = f \circ \Phi$. Hence there exists a sequence $z_k \in \mathbb{R} \times V$ such that $g(z_k) \to a$. By the curve selection lemma there is a semialgebraic curve $W_1 \subset \mathbb{R} \times V$ such that $a \in \overline{g(W_1)}$. Set $W_2 = p_2(W_1)$, where $p_2 : \mathbb{R} \times V \to V$ is the projection. If W_2 is a curve then let $W := W_2$, if it is a point we take as W any semialgebraic curve in V which contains the point $\pi(W_1)$. Now W and $\Phi|_{\mathbb{R}\times W}$ are as required.

Finally, to prove (3) $\Rightarrow$ (2) it is enough to take as f the identity in the third condition. $\square$

Definition 4.2. We say that a closed semialgebraic set X has *degree of* $\mathbb{R}$-*uniruledness at most* d if it satisfies the conditions of Proposition 4.1. A closed semialgebraic set is called $\mathbb{R}$-*uniruled* if it has some degree of $\mathbb{R}$-uniruledness.

Example 4.3. Let $X = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$. It is easy to check that the degree of $\mathbb{R}$-uniruledness of X is 2. It has a ruling $\{(a, t^2) : a \geq 0\}$.

Let $X \subset \mathbb{R}^n$ be a closed semialgebraic set, and let $f : X \to \mathbb{R}^m$ be a polynomial map. As in the complex case, we say that it is not proper at a point $y \in \mathbb{R}^m$ if there is no neighborhood U of y such that $f^{-1}(\overline{U})$ is compact. As before, we denote by S_f the set of all points $y \in \overline{f(X)}$ at which f is not proper. This set is also closed and semialgebraic [7]. We have:

Theorem 4.4. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a generically finite polynomial map of degree d. Then the set S_f has degree of $\mathbb{R}$-uniruledness at most $d - 1$.*

Theorem 4.5. *Let $X = \mathbb{R} \times W \subset \mathbb{R} \times \mathbb{R}^n$ be a closed semialgebraic cylinder and let $f : R \times W \ni (t, w) \mapsto (f_1(t, w), \ldots, f_m(t, w)) \in \mathbb{R}^m$ be a generically finite polynomial map. Assume that* $\deg_t f_i \leq d$ *for every i. Then the set S_f has degree of $\mathbb{R}$-uniruledness at most d.*

Corollary 4.6. *Let $L = \phi(\mathbb{R})$ be a polynomially parametric curve of degree D. Let $X = L \times W \subset \mathbb{R} \times \mathbb{R}^n$ be a closed semialgebraic cylinder and let $f : L \times W \ni (x, w) \mapsto (f_1(x, w), \ldots, f_m(x, w)) \in \mathbb{R}^m$ be a generically finite polynomial map. Assume that* $\deg_t f_i \leq d$ *for every i. Then the set S_f has degree of $\mathbb{R}$-uniruledness at most dD.*

Proof. For the proof it is enough to note that the mapping $\mathbb{R} \times W \ni (t, w) \mapsto (\phi(t), w) \in L \times W$ is proper and generically-finite. $\square$

Corollary 4.7. *Let $f = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a generically finite polynomial map with $d = \min_j \max_i \deg_{x_j} f_i$. Then the set S_f has degree of $\mathbb{R}$-uniruledness at most d.*

The proofs of these facts are exactly the same as in the complex case. To prove a real analog of Theorem 3.6 we need some ideas from [10]. Let X be a smooth complex projective surface, and let $D = \sum_{i=1}^n D_i$ be a simple normal crossing divisor on X (we consider only reduced divisors). Let graph(D) be the graph of D, with vertices D_i, and one edge between D_i and D_j for each point of intersection of D_i and D_j.

Definition 4.8. We say that D a simple normal crossing divisor on a smooth surface X is *a tree* if graph(D) is a tree (it is connected and acyclic).

The following fact is obvious from graph theory.

Proposition 4.9. *Let X be a smooth projective surface and $D \subset X$ be a divisor which is a tree. If $D', D'' \subset D$ are connected divisors without common components, then D' and D'' have at most one point in common.*

Definition 4.10. Let $X \subset \mathbb{R}^n$ ($X \subset \mathbb{P}^n$)be an algebraic variety. Hence we have a natural embedding $X \subset \mathbb{C}^n$ ($X \subset \mathbb{P}^n(C)$). By *the complexification* X^c of the variety X we mean the Zariski closure of X in $\mathbb{C}^n$ ($\mathbb{P}^n(\mathbb{C})$).

Now we are ready to prove a real counterpart of Theorem 3.6. In particular we show that for a generically finite map $f : X \to Y$ of real algebraic sets, the set S_f is also $\mathbb{R}$-uniruled, provided X is.

Theorem 4.11. *Let $X \subset \mathbb{R}^n$ be a closed algebraic set with degree of $\mathbb{R}$-uniruledness at most d_1, and let $f : X \to \mathbb{R}^m$ be a generically finite polynomial map of degree d_2. Then the set S_f is also $\mathbb{R}$-uniruled. Moreover, its degree of $\mathbb{R}$-uniruledness is at most $2d_1d_2$.*

Proof. Let $a \in S_f$ and let $x_k \in X$ be a sequence of points such that $f(x_k) \to a$ and $x_k \to \infty$. By Proposition 4.1 there exists a semialgebraic curve $W \subset \mathbb{R}^Q$ and a generically finite polynomial map $\phi : \mathbb{R} \times W \ni (t, w) \to \phi(t, w) \in X$ such that $\deg_t \phi \le d_1$, and there exists a sequence $(y_k) \subset \mathbb{R} \times W$ such that $f(\phi(y_k)) \to a$ and $y_k \to \infty$. In particular $a \in S_{f \circ \phi}$. If we knew that the mapping ϕ is proper, then $S_{f \circ \phi} \subset S_f$ and we are done by Theorem 4.5. However, in general it is not true. Our idea is to obtain a suitable compactification ϕ' of the map ϕ, and then to derive all information from the fact that $S_{f \circ \phi'} \subset S_{f \circ \phi}$ and $S_{f \circ \phi'} \subset S_f$.

Let $\Gamma \subset \mathbb{R}^Q$ be the Zariski closure of W. We can assume that Γ is smooth and irreducible. Denote $Z := \mathbb{R} \times \Gamma$. We have the induced map $\phi : Z \to X$. Hence we also have the induced complex map $\phi^c : Z^c := \mathbb{C} \times \Gamma^c \to X^c$, where Z^c, X^c denote the complexification of Z and X respectively. Note that we can resolve the complex singularities of Γ^c and this process does not affect the real structure of the curve Γ. Hence we can assume that Γ^c is smooth.

Let $\overline{\Gamma^c}$ be a smooth completion of Γ^c and let us write $\overline{\Gamma^c} \backslash \Gamma = \{a_1, ..., a_l\}$. Let $\mathbb{P}^1 \times \overline{\Gamma^c}$ be a projective completion of Z^c. The divisor $D = \overline{Z^c} \backslash Z^c = \infty \times \overline{\Gamma^c} + \sum_{i=1}^{l} \mathbb{P}^1 \times \{a_i\}$ is a tree. The map ϕ induces a rational map $\phi : \overline{Z^c} \dashrightarrow \overline{X^c}$, where $\overline{X^c}$ denotes the projective closure of X^c. We can resolve the points of indeterminacy of this map (see e.g., [12, Theorem 3, p. 254]):

$$
\begin{tikzcd}
(\overline{Z^c})_m \arrow[d, "\pi_m"] \arrow[dddr, "\phi'"] & \\
(\overline{Z^c})_{m-1} \arrow[d, "\pi_{m-1}"] & \\
\vdots \arrow[d, "\pi_1"] & \\
\overline{Z^c} \arrow[r, dashed, "\phi"] & \overline{X^c}
\end{tikzcd}
\qquad \pi = \pi_1 \circ \cdots \circ \pi_m
$$

Note that we can first resolve the real points of indeterminacy. After this process the variety $H := \pi^{-1}(\overline{Z})$ still has a structure of a real variety. Further we will call all points which are over $\overline{Z}$ real points. Note that there is a Zariski open neighborhood $U \subset \overline{Z^c}$ of $\overline{Z}$ such that the on $\pi^{-1}(U)$ we have the operation of complex conjugation of points. Moreover, $\mathbb{R} \times \Gamma \subset H$.

Let $Q := (\overline{Z^c})_m \cap \phi'^{-1}(X^c)$. Then the map $\phi' : Q \to X^c$ is proper. Moreover, $Q = (\overline{Z^c})_m \backslash \phi'^{-1}(\overline{X^c} \backslash X^c)$. The divisor $D_1 = \phi'^{-1}(\overline{X^c} \backslash X^c)$ is connected as the complement of a semi-affine variety $\phi'^{-1}(X^c)$ (for details see [6, Lemma 4.5]). Note that the divisor $D' = \pi^*(D)$ is a tree. Hence the divisor $D_1 \subset D'$ is also a tree.

Note that the map $f' = f \circ \phi'$ is determined on the set $Q^r := H \cap Q$ and now the mapping $\phi' : Q^r \to X$ is proper. The mapping f' has a natural extension to the set Q and we will consider the regular complex map $f' : Q \to \mathbb{C}^m$. This map induces a rational map from $P := (\overline{Z^c})_m$ to $\mathbb{P}^m(\mathbb{C})$. As before we can resolve its points of indeterminacy:

$$\begin{tikzcd}
P_k \arrow[d, "\rho_k"] \arrow[dddr, "F"] & \\
P_{k-1} \arrow[d, "\rho_{k-1}"] & \\
\vdots \arrow[d, "\rho_1"] & \\
P \arrow[r, dashed, "f'"] & \mathbb{P}^m(\mathbb{C})
\end{tikzcd}$$

Again we can first resolve the real points of indeterminancy. After this process the variety $\psi^{-1}(H)$ still has the structure of a real variety. In particular there is a Zariski open neighborhood $V \subset P_k$ of $\psi^{-1}(H)$ such that on V we have the operation of complex conjugation of points.

Note that the divisor $D_1' = \psi^*(D_1)$ is a tree. Let $\infty' \times \overline{\Gamma}$ denote the proper transform of $\infty \times \overline{\Gamma}$. It is an easy observation that $F(\infty' \times \overline{\Gamma}) \subset \pi_\infty$, where π_∞ denotes the hyperplane at infinity of $\mathbb{P}^m(\mathbb{C})$. Now $S_{f'} = F(D_1' \backslash F^{-1}(\pi_\infty))$. The curve $L = F^{-1}(\pi_\infty)$ is connected (by the same argument as above). Now by Proposition 4.9 every irreducible curve $l \subset D_1'$ (note that necessarily $l \cong \mathbb{P}^1(\mathbb{C})$) which does is not contained in L has at most one point in common with L. Let $R \subset S_{f'}$ be an irreducible component. Hence R is a curve. There is a curve $l \subset D_1'$, which has exactly one point in common with L, such that $R = F(l \backslash L)$. If l is given by blowing up a real point, then L also has a real point in common with l (because otherwise there are two conjugate common points of l and L). When we restrict to the real model l^r of l we have $l^r \backslash L \cong \mathbb{R}$. Hence if we restrict our considerations only to the real points and to the set Q^r, we see that the set S of non-proper points of the map $f'|_{Q^r}$ is a union of polynomially parametric curves $F(l^r \backslash L),\ l \subset D_1', \psi(l) \subset H$. Of course $a \in S \subset S_f$. Similarly the set $S_{f \circ \phi}$ is a union of polynomially parametric curves $F(l^r \backslash L),\ l \subset \psi^*(D'), \pi(\psi(l)) \subset \overline{Z}$. Hence we can say that every "irreducible component" of the set of non-proper points of $f'|_{Q^r}$ is also an 'irreducible' component of $S_{f \circ \phi}$. Moreover $a \in S_{f'|_{Q^r}} \subset S_f$. In particular there is a real parametric curve $F(l^r \backslash L) \subset S_f$ which contains the point a and which is covered by curves lying in $S_{f \circ \phi}$. Now we can finish the proof by invoking Theorem 4.5 and Lemma 4.12 below. □

Lemma 4.12. *Let $\psi : \mathbb{R} \to \mathbb{R}^m$ be a polynomially parametric curve. If there exist polynomially parametric curves $\phi_i : \mathbb{R} \to \mathbb{R}^m,\ i = 1, ..., n$, of degree at most d with $\psi(\mathbb{R}) \subset \bigcup_{i=1}^n \phi_i(\mathbb{R})$, then $\psi(\mathbb{R})$ has degree of $\mathbb{R}$-uniruledness at most $2d$.*

Proof. Indeed, let $\psi(t) = (\psi_1(t), ..., \psi_m(t))$ and let X denote the Zariski closure of $\psi(\mathbb{R})$. Consider the field $L = \mathbb{R}(\psi_1, ..., \psi_m)$. By the Lüroth Theorem there exists a rational function $g(t) \in \mathbb{R}(t)$ such that $L = \mathbb{R}(g(t))$. In particular there exist $f_1, ..., f_m \in \mathbb{R}(t)$ such that $\psi_i(t) = f_i(g(t))$ for $i = 1, ..., m$. In fact, we have two induced maps $\overline{f} : \mathbb{P}^1(\mathbb{R}) \to \overline{X} \subset \mathbb{P}^m(\mathbb{R})$ and $\overline{g} : \mathbb{P}^1(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R})$. Here $\overline{X}$ denotes the projective closure of X. Moreover, $\overline{f} \circ \overline{g} = \overline{\psi}$. Let A_∞ denote the unique point at infinity of $\overline{X}$ and let $\infty = \overline{f}^{-1}(A_\infty)$. Then $\overline{g}^{-1}(\infty) = \infty$, i.e., $g \in \mathbb{R}[t]$. Similarly $f_i \in \mathbb{R}[t]$. Hence $\psi = f \circ g$, where $f : \mathbb{R} \to X$ is a birational and polynomial mapping and $g : \mathbb{R} \to \mathbb{R}$ is a polynomial mapping.

Now if $\deg g = 1$ then $f : \mathbb{R} \to \mathbb{R}^n$ covers the whole $\psi(\mathbb{R})$. Otherwise we can compose f with a suitable polynomial of degree two to obtain the whole $\psi(\mathbb{R})$ as image.

Let $\phi_i := \phi$ be a curve which has infinitely many points in common with $\psi(\mathbb{R})$. In the same way as above we have $\phi = f' \circ g'$, where $f' : \mathbb{R} \to X$ is a birational and polynomial mapping and $g' : \mathbb{R} \to \mathbb{R}$ is a polynomial mapping. In particular $f^{-1} \circ f : \mathbb{R} \to \mathbb{R}$ is a polynomial automorphism, i.e., $f(t) = f'(at + b),\ a \in \mathbb{R}^*, b \in \mathbb{R}$. Hence we can compose f' with a suitable polynomial of degree one or two to obtain the whole $\psi(\mathbb{R})$ as image. In any case $\psi(\mathbb{R})$ has a parametrization of degree bounded by $2 \deg f \leq 2d$. □

Corollary 4.13. *Let X be a closed algebraic set which is $\mathbb{R}$-uniruled and let $f : X \to \mathbb{R}^m$ be a generically finite polynomial map. Then every connected component of the set S_f is unbounded.*

5. An application of the real field case

As an application we give a real counterpart of a theorem of Białynicki-Birula [2].

Theorem 5.1. *Let G be a real, non-trivial, connected, unipotent group, which acts effectively and polynomially on a closed algebraic $\mathbb{R}$-uniruled set $X \subset \mathbb{R}^n$. Then the set $Fix(G)$ of fixed points of this action, is also $\mathbb{R}$-uniruled. In particular, it has no isolated points.*

Proof. First of all let us recall that a connected unipotent group has a normal series

$$0 = G_0 \subset G_1 \subset \cdots \subset G_r = G,$$

where $G_i/G_{i-1} \cong G_a = (\mathbb{R}, +, 0)$. By induction on $\dim G$ we can easily reduce the problem to $G = G_a$. Indeed, assume the conclusion holds for $G = G_a$. Take a unipotent group G with $\dim G = n$ and assume that the conclusion holds in dimension $n - 1$. There is a normal subgroup G_{n-1} of dimension $n - 1$ such that $G/G_{n-1} = G_a$. Moreover, the set $R := Fix(G_{n-1})$ is $\mathbb{R}$-uniruled by our hypothesis. Consider the induced action of the group $G_a = G/G_{n-1}$ on R. The set of fixed points of this action is $\mathbb{R}$-uniruled and it coincides with $Fix(G)$.

Hence assume that $G = G_a$. Let D be the degree of $\mathbb{R}$-uniruledness of X. Choose $a \in Fix(G)$. Let $\phi : G \times X \ni (g, x) \mapsto \phi(g, x) \in X$ be a polynomial action of G on X. This action also induces a polynomial action of the complexification $G^c = (\mathbb{C}, +)$ of G on X^c. We will denote this action by $\overline{\phi}$. Assume that $\deg_g \phi \le d$. By Definition 4.2 it is enough to prove that there exists a polynomially parametric curve $S \subset Fix(G)$ passing through a of degree bounded by dD. Let L be a polynomially parametric curve in X passing through a. If it is contained in $Fix(G)$, then the assertion is true. Otherwise consider a closed semialgebraic surface $Y = L \times G$. There is a natural $G-$action on Y: for $h \in G$ and $y = (l, g) \in Y$ we set $h(y) = (l, hg) \in Y$. Consider the map

$$\Phi : L \times G \ni (x, g) \to \phi(g, x) \in X.$$

It is a generically finite polynomial map. Observe that it is G-invariant, which means $\Phi(gy) = g\Phi(y)$. This implies that the set S_Φ of points at which Φ is not finite is G-invariant. Indeed, it is enough to show that the complement of this set is G-invariant. Let Φ be finite at $x \in X$. Then there is an open neighborhood U of x such that $\Phi : \Phi^{-1}(U) \to U$ is finite. Now we have the following diagram:

$$\begin{tikzcd}
\Phi^{-1}(U) \arrow[r, "g"] \arrow[d, "\Phi"] & \Phi^{-1}(gU) = g\Phi^{-1}(U) \arrow[d, "\Phi"] \\
U \arrow[r, "g"] & gU
\end{tikzcd}$$

This shows that if Φ is finite over U, then it is finite over gU. In particular this implies that the set S_Φ is G-invariant. Let $S_\Phi = S_1 \cup \ldots \cup S_k$ be a decomposition of S_Φ into polynomially parametric curves (see Corollary 4.6). Since S_Φ is G-invariant, each curve S_i is also G-invariant. Note that the point a belongs to S_Φ, because the fiber over a has infinitely many points. We can assume that $a \in S_1$. Let us note that a is also a fixed point for G^c. Let $x \in S_1$; we want to show that $x \in Fix(G)$. Indeed, the set S_1^c is also G^c-invariant and if $x \notin Fix(G)$ then $G^c.x = S_1^c$ and a would be in the orbit of x, which is a contradiction. Hence $S_1 \subset Fix(G)$ and we conclude by Proposition 4.1 and Corollary 4.6. □

Corollary 5.2. *Let G be a real, non-trivial, connected, unipotent group which acts effectively and polynomially on a closed algebraic set $X \subset \mathbb{R}^n$. If the set $Fix(G)$ of fixed points of this action, is nowhere dense in X, then it is $\mathbb{R}$-uniruled.*

Corollary 5.3. *Let G be a real, non-trivial, connected, unipotent group which acts effectively and polynomially on a connected smooth closed algebraic variety $X \subset \mathbb{R}^n$. Then the set $Fix(G)$ is $\mathbb{R}$-uniruled.*

References

[1] Bass, H., Connell, E., Wright, D.: The Jacobian conjecture: reduction of degree and formal expansion of the inverse. Bull. Am. Math. Soc. (N.S.) **7**(2), 287–330 (1980)
[2] Białynicki-Birula, A.: On fixed point schemes of actions of multiplicative and additive groups. Topology **12**, 99–103 (1973)
[3] Drużkowski, L.: An effective approach to Keller's Jacobian conjecture. Math. Ann. **264**(3), 303–313 (1983)
[4] Hà, H.V., Pham, T.S.: Representations of positive polynomials and optimization on noncompact semialgebraic sets. SIAM J. Optim. **20**, 3082–3103 (2010)
[5] Jelonek, Z.: The set of points at which the polynomial mapping is not proper. Ann. Polon. Math. **58**, 259–266 (1993)
[6] Jelonek, Z.: Testing sets for properness of polynomial mappings. Math. Ann. **315**, 1–35 (1999)
[7] Jelonek, Z.: Geometry of real polynomial mappings. Math. Z. **239**, 321–333 (2002)
[8] Jelonek, Z.: On the Russell problem. J. Algebra **324**(12), 3666–3676 (2010)
[9] Jelonek, Z., Kurdyka, K.: On asymptotic critical values of a complex polynomial. J. Reine Angew. Math. **565**, 1–11 (2003)
[10] Jelonek, Z., Lasoń, M.: The set of fixed points of a unipotent group. J. Algebra **322**, 2180–2185 (2009)
[11] Kollar, J.: Rational Curves on Algebraic Varieties. Springer, Berlin (1999)
[12] Shafarevich, I.: Basic Algebraic Geometry 1. Springer, Berlin (1994)
[13] Stasica, A.: Geometry of the Jelonek set. J. Pure Appl. Algebra **198**, 317–327 (2005)
[14] van den Dries, L., McKenna, K.: Surjective polynomial maps, and a remark on the Jacobian problem. Manuscr. Math. **67**(1), 1–15 (1990)
[15] van den Essen, A.: Polynomial Automorphisms and the Jacobian Conjecture. Birkhauser Verlag, Basel (2000)