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On the negativity of higher order derivatives of Dirichlet's energy in plateau's problem

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Abstract. We calculate higher order derivatives of Dirichlet's Energy at a branched minimal surface in the direction of Forced Jacobi Fields discovered by the author and R. Böhme. We show that, under certain conditions these derivatives can be made negative, while all lower order derivatives vanish. This is the first time that derivatives of order greater than three have been calculated.

1. Introduction and intrinsic derivatives

In 1932 Jesse Douglas [7] and in 1942 Richard Courant [6] both thought they had produced minimizers for energy (area) which possessed interior branch points. The example of Douglas was shown by Radó to be incorrect [13]. Only in 1970 did Courant's example come into question, when Osserman claimed to have proved that all absolute minimizers of E had to be immersed on the interior of the unit disc [12]. However, Osserman had overlooked the need in his proof (a local cutting and pasting argument) to distinguish between true and false branch points (the latter are those whose image locally is still an embedded surface).

In 1973 Alt [1,2] and Gulliver [8] independently proved that absolute minima of energy (area) had no interior branch points. In Gulliver and Lesley [9] extend this result to show the absence of boundary branch points for minima in the case when the boundary curve is real analytic. In [17] Wienholtz pointed out that the discontinuous reparametrization used by Gulliver in [8] did not exist. However, Gulliver and Lesley correct this in [9].

In Gulliver et al. [10], proved that *all* minimal surfaces bounded by rectifiable Jordan curves in $\mathbb{R}^3$ do not have any false interior branch points, even if they are not minima of energy (or area). This filled the hole in Osserman's 1969 argument. In 1980 Beeson [4] gave another argument for the absence of true interior branch points The original Osserman result on the absence of true interior branch points was extended to area minimizing surfaces in 3-manifolds by Micallef and White [11]. In Alt and Tomi [3], gave conditions where one could prove the absence of interior and boundary branch points to minimal surfaces with free boundaries.

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In 1998 Wienholtz [17] presented conditions which ruled out boundary branch points for minimal surfaces spanning sufficiently smooth contours in $\mathbb{R}^3$. In 2001, Stefan Hildebrandt pointed out to the author that all of the current proofs were too long and too technical for presentation in a text on the subject and challenged him to come up with a direct, clear analytic proof that involved no cutting or pasting. This paper is an answer to that challenge. In a future paper, we extend Wienholtz's results on the non-existence of boundary branch points.

In Böhme and the author [5] showed that for the generic contour Γ, all minimal surfaces spanning Γ were either immersed up to the boundary, or had only simple (order 1) interior branch points. In [14] the author proved a normal form theorem for Dirichlet's energy in the neighborhood of a generic branched minimal surface in $\mathbb{R}^3$ and further proved that the winding number about such surfaces was $\pm 2^p$, where p is the number of branch points.

As a trivial consequence of this normal form result, it is evident that the generic branch minimal surface in $\mathbb{R}^3$ cannot be a relative minimum. The strategy of the author's proof was to calculate the third (intrinsic) derivative of Dirichlet's energy. All higher order derivatives are non-intrinsic, and thus can be quite complicated.

In this paper we compute a formula for all higher order derivatives of Dirichlet's energy in the direction of Forced Jacobi Fields discovered by the author and R. Böhme. We show that, under certain conditions, one can be made negative while all lower order derivatives vanish.

The author wishes to thank Stefan Hildebrandt for carefully checking this entire approach to branch points.

1.1. Notations and conventions

Let Γ be a C^∞ contour in $\mathbb{R}^3$ which is the image of a differentiable immersion of the unit circle S^1 into $\mathbb{R}^3$. Let D be the closed unit disc in $\mathbb{R}^2$. A disc minimal surface X spanning Γ is a map $X : D \to \mathbb{R}^3$ such that

(i) $\Delta X = 0$ (each component of X is harmonic)
(ii) $X_u \cdot X_v = 0$ (the coordinates in $\mathbb{R}^2$ being labelled by $u \cdot v$)
(iii) $\|X_u\|^2 = \|X_v\|^2$
(iv) $X : S^1 \to \Gamma$

X is said to be classical disc minimal surface if Γ is the image of a differentiable embedding and $X : S^1 \to \Gamma$ is a homeomorphism. Conditions (ii) and (iii) imply that the surface is conformally parameterized and (i)–(iii) are the Euler equations of Dirichlet's energy,

$$E(X) := \frac{1}{2} \int_D \nabla X \cdot \nabla X du\, dv.$$

A point $z_0 \in D^o$ is called an interior branch point if both $X_u(z_0) = X_v(z_0) = 0$. In this case z_0 is a zero of the homomorphic function $F(z) = \frac{1}{2}(X_u - iX_v)$. Then $F(z) = (z - z_0)^\lambda G(z)$, $G(z_0) \neq 0$. The integer λ is called the order of the branch point z_0.

For the purposes of future calculations, we shall consider X as a map from S^1 into $\mathbb{R}^3$, and by $\hat{X}$ or HX the harmonic extension of X to the unit disc. Thus, in this notation, the minimal surface is actually $\hat{X}$.

Denote by $\hat{X}_z := \frac{\partial}{\partial z}\hat{X}$ the complex derivative of the harmonic map $\hat{X}$, $\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)$.

The Forced Jacobi Fields associated to a minimal surface X are the harmonic extensions $\hat{h}$ of maps $h : S^1 \to \mathbb{R}^3$, $h(p) \in T_{x(p)}\Gamma$, the tangent space to Γ at $X(p)$ with

$$h = ReK, \quad K(z) := izF(z)\mu(z) \text{ on } S^1$$
$$F(z) = \frac{1}{2}(\hat{X}_u - iX_v), \quad Re := \text{real part},$$

where $\mu(z)$ is a meromorphic function on D, real on $S^1 = \partial D$ with poles at the zeros of F of orders not exceeding the orders of the associated zeros. Thus $F(z)\mu(z)$ can be considered a global holomorphic map of D^o into $\mathbb{C}^3$. Thus

$$\hat{h} = Re\{izF(z)\mu(z)\}.$$

The space $J(X)$ of all such h is called the space of *Forced Jacobi Fields*. Their importance arises from the fact that

Theorem 1.1. *The space $J(X) \subset$ Ker $D^2E(\hat{X})$, the kernel of the second variation of Dirichlet's energy at a minimal surface $\hat{X}$.*

We prove this in formula (2.11). These fields were discovered independently by Böhme and Tromba. Their existence arises from the action of the conformal group and the presence of branch points.

We shall be taking higher order derivatives in the direction of such $h \in J(X)$.

2. The first five intrinsic derivatives of Dirichlet's energy in the direction of forced Jacobi fields

Recall that

$$E(\hat{X}) = \frac{1}{2}\int_D \nabla\hat{X}\cdot\nabla\hat{X}.$$

Clearly the first derivative [5,14] of E in the direction $\hat{h}$ is given by

$$DE(\hat{X})\hat{h} = \int_D \nabla\hat{X}\cdot\nabla\hat{h}$$

which after integration by parts yields

$$DE(\hat{X})\hat{h} = \int_{S^1} \hat{X}_r \cdot h\, d\theta, \tag{2.1}$$

r, θ denoting partial derivatives w.r.t. the polar coordinates r and θ. Here $h(1,\theta) \in T_{X(1,\theta)}\Gamma$ is a tangent vector to the manifold [5] of all harmonic surfaces spanning Γ at X. Thus on S^1, $h = \phi X_\theta$, ϕ defined uniquely away from the zeros of X_θ, ϕ a real valued function C^∞ away from the zeros. In what follows we shall often identify h with ϕ.

With this in mind (2.1) can be written as a complex line integral

$$DE(\hat{X})\hat{h} = 2Re\int z\hat{X}_z \cdot \hat{X}_z \phi\, dz \tag{2.2}$$

where $X_z \cdot X_z := \sum_{j=1}^{3} \left(X_z^j\right)^2$.

To prove formula (2.2) note that on S^1 $z\hat{X}_z = \frac{1}{2}\left(\frac{\partial}{\partial r} - i\,\frac{\partial}{\partial \theta}\right)$, $dz = izd\theta$.
From (2.1)

$$DE(\hat{X})\hat{h} = \int_{S^1} \hat{X}_r \cdot X_\theta \phi\, d\theta.$$

Now,

$$\begin{aligned}
2\,Re\int z\hat{X}_z \cdot \hat{X}_z\phi\, dz &= 2\,Re\int (z\hat{X}_z)\cdot(z\hat{X}_z)\phi\,\frac{dz}{z} \\
&= 2\,Re\int i(z\hat{X}_z)\cdot(z\hat{X}_z)\phi\,\frac{dz}{iz} = -2\,Im\int (z\hat{X}_z)\cdot(z\hat{X}_z)\phi\, d\theta \\
&= -\frac{1}{2}\,Im\int\left\{\left(\frac{\partial}{\partial r} - i\,\frac{\partial}{\partial\theta}\right)\hat{X}\right\}\cdot\left\{\left(\frac{\partial}{\partial r} - i\,\frac{\partial}{\partial\theta}\right)\hat{X}\right\}\,\phi\, d\theta \\
&= \int \hat{X}_r \cdot X_\theta\phi\, d\theta
\end{aligned}$$

which proves (2.2).

Formula (2.2) will be our starting point for calculating all higher order derivatives of Dirichlet's energy E.

In order to carry out these computations in the most efficient manner, we will need to develop a few techniques. We will consider variations of a minimal surface $X : S^1 \to \mathbb{R}^3$ as functions of a real variable $t \in (-\delta, \delta)$. Thus we have $X(t) : S^1 \to \Gamma$ a smooth 1-parameter family of maps. Thus, through harmonic extension $\hat{X}(t) : D \to \mathbb{R}^3$ is a 1-parameter family of harmonic surfaces spanning Γ. We define $X(t)$ by

$$X(t)\left(e^{i\theta}\right) := X\left(e^{i\gamma(t,\theta)}\right) \tag{2.3}$$

where $\gamma : (-\delta, \delta) \times [0, 2\pi) \to R$ a C^∞ function in t and θ, which is 2π "shift" periodic in θ for all t, $(\gamma(t, \theta + 2\pi) = \gamma(t, \theta) + 2\pi)$ and also with

$$\gamma(0,\theta) = \theta. \tag{2.4}$$

Given a minimal surface X, let

$$DX(p) : T_pS^1 \to R^3$$

be the derivative of X at a point $p \in S^1$. Then from (2.3) it follows that

$$\begin{aligned}\frac{\partial}{\partial t} X(t)\left(e^{i\theta}\right) &= DX\left(e^{i\gamma(t,\theta)}\right)\left[ie^{i\gamma(t,\theta)}\right]\frac{\partial\gamma}{\partial t}\\ &= \left\{DX\left(e^{i\gamma(t,\theta)}\right)\left[ie^{i\gamma(t,\theta)}\frac{\partial\gamma}{\partial\theta}\right]\left(\frac{\partial\gamma}{\partial\theta}\right)^{-1}\right\}\frac{\partial\gamma}{\partial t}\\ &= X_\theta\phi \end{aligned} \tag{2.5}$$

where

$$\phi(t,\theta) := \left(\frac{\partial\gamma}{\partial\theta}\right)^{-1}\frac{\partial\gamma}{\partial t}. \tag{2.6}$$

Assume there is an interval $(-\delta, \delta)$ so that for $t \in (-\delta, \delta)$

$$\frac{\partial\gamma}{\partial\theta} > 0.$$

Thus $\hat{h} := H\{X_\theta\phi\}$ is, through a variation of the boundary values of X, and admissible variation of the harmonic surface $X(t)$. Thus to re-iterate and summarize: We have defined a 1-parameter variation of a minimal surface $X(t)$ with

$$\begin{aligned}\frac{\partial}{\partial t} E(\left(\hat{X}(t)\right) &= 2\,Re\int_z z\hat{X}(t)_z \cdot \hat{X}(t)_z\phi\, dz\\ &= 2\,Re\int z\,(HX(t))_z \cdot (HX(t))_z\,\phi\, dz \end{aligned} \tag{2.7}$$

where, from (2.4) $X(0) = X$.

Lemma 2.1. *By choosing the derivatives of* $\frac{\partial\gamma}{\partial t}$ *at* $t = 0$, *we may arbitrarily select the derivatives of* $\frac{\partial\phi}{\partial t}$ *at* $t = 0$.

Proof. The proof follows easily from induction by differentiating (2.7) and using the fact that $\frac{\partial\gamma}{\partial\theta} = 1$ at $t = 0$. □

Let us return, for the moment, to formula (2.5). Now

$$X(t)_\theta = 2\,Re\left\{iz\hat{X}(t)_z\right\}.$$

From this and the fact that ϕ is real valued we see that

$$\frac{\partial}{\partial t}X(t) = 2\,Re\left(iz\hat{X}(t)_z\phi\right) \tag{2.8}$$

which is the fundamental formula permitting us to calculate all higher order derivatives of Dirichlet's energy. We begin with the second and third derivatives.

We first need a variation of formula (2.8). Since $\frac{\partial}{\partial t}$ and harmonic extension H (or $\wedge$) commute

$$\frac{\partial}{\partial t}\hat{X}(t) = H\left[2\,Re\left(iz\hat{X}(t)_z\phi\right)\right] \tag{2.9}$$

and

$$\left\{\frac{\partial}{\partial t}\,\hat{X}(t)\right\}_z = \frac{\partial}{\partial t}\hat{X}(t)_z = \left\{H\left[2\,Re(iz\hat{X}(t)_z\phi)\right]\right\}_z. \tag{2.10}$$

Now a straight forward differentiation of (2.7) yields

$$\begin{aligned}\frac{\partial^2 E}{\partial t^2} &= 4\,Re\int z\left\{\frac{\partial\hat{X}}{\partial t}\,(t)\right\}_z\cdot\hat{X}(t)_z\phi\,dz\\ &\quad+2\,Re\int z\hat{X}(t)_z\cdot\hat{X}(t)_z\,\frac{\partial\phi}{\partial t}\,dz. \end{aligned}\tag{2.11}$$

At $t=0$, since $X=X(0)$ is a minimal surface the second integral vanishes. If ϕ represents a Forced Jacobi Field (FJF) direction $\hat{X}_z\phi$ is holomorphic and the first integral is the complex line integral of a holomorphic function and vanishes by Cauchy's integral theorem. Thus, as it must, the second variation of Dirichlet's energy at $t=0$ vanishes in all FJF directions.

We now compute the third derivative of Dirichlet's energy in FJF directions. Using (2.7) and (2.11)

$$\begin{aligned}\frac{\partial^3 E}{\partial t^3} &= 4Re\int z\left\{\frac{\partial\hat{X}}{\partial t}\right\}_z\cdot\left\{\frac{\partial\hat{X}}{\partial t}\right\}_z\phi\,dz\\ &\quad+4Re\int z\left\{\frac{\partial^2\hat{X}}{\partial t^2}\right\}_z\cdot\hat{X}(t)_z\phi\,dz+8\,Re\int z\left\{\frac{\partial\hat{X}}{\partial t}\right\}_z\cdot\hat{X}(t)_z\,\frac{\partial\phi}{\partial t}\,dz\\ &\quad+2Re\int z\hat{X}(t)_z\cdot\hat{X}(t)_z\,\frac{\partial^2\phi}{\partial t^2}\,dz. \end{aligned}\tag{2.12}$$

At $t=0$ and if $\hat{X}_z\phi$ is holomorphic the second integral vanishes by Cauchy's integral theorem. By (2.10) at $t=0$

$$\left\{\frac{\partial\hat{X}}{\partial t}\right\}_z = \left\{H\,\frac{\partial X}{\partial t}\right\}_z = \left\{H[2\,Re\,iz\hat{X}_z\phi]\right\}_z.$$

But again since $\hat{X}_z\phi$ is holomorphic at $t=0$ this equals

$$\frac{1}{2}\left\{2iz\hat{X}_z\phi\right\}_z = \left\{iz\hat{X}_z\phi\right\}_z = \left\{\frac{\partial\hat{X}}{\partial t}\right\}_z. \tag{2.13}$$

Since X is minimal

$$\left\{\hat{X}_z\right\}\cdot\left\{\frac{\partial\hat{X}}{\partial t}\right\}_z = \hat{X}_z\cdot\left\{iz\hat{X}_z\phi\right\}_z \equiv 0$$

and thus the third term in (2.12) vanishes. The last term vanishes since $\hat{X}_z\cdot\hat{X}_z\equiv 0$.

Thus, the third derivative, at $t = 0$ reduces to:

$$\frac{\partial^3 E}{\partial t}\bigg|_{t=0} = 4\, Re \int z \left\{\frac{\partial \hat{X}}{\partial t}\right\}_z \cdot \left\{\frac{\partial \hat{X}}{\partial t}\right\}_z \phi\, dz. \tag{2.14}$$

Substituting (2.13) we obtain

$$\begin{aligned}\frac{\partial^3 E}{\partial t^3}\bigg|_{t=0} &= 4\, Re \int z \left\{iz\hat{X}_z\phi\right\}_z \cdot \left\{iz\hat{X}_z\phi\right\}_z \phi\, dz \\ &= -4\, Re \int z^3 \hat{X}_{zz} \cdot \hat{X}_{zz}\phi^3\, dz \end{aligned} \tag{2.15}$$

which is the formula obtained for the third "intrinsic" derivative in [14] and [16]. This derivation is, however, much simpler and more direct, and will serve as the foundation of future calculations.

2.1. The fourth and fifth derivatives of Dirichlet's energy

We now seek a way of choosing the derivatives of ϕ in such a way as to make all higher order derivatives as simple as possible.

From formula (2.12) we calculate

$$\begin{aligned}\frac{\partial^4 E}{\partial t^4} &= 12\, Re \int z \left\{\frac{\partial^2 \hat{X}}{\partial t^2}\right\}_z \cdot \left\{\frac{\partial \hat{X}}{\partial t}\right\}_z \phi\, dz \\ &\quad + 4 Re \int z \left\{\frac{\partial^3 \hat{X}}{\partial t^3}\right\}_z \cdot \hat{X}(t)_z \phi\, dz + 12\, Re \int z \left\{\frac{\partial \hat{X}}{\partial t}\right\}_z \cdot \left\{\frac{\partial \hat{X}}{\partial t}\right\}_z \frac{\partial \phi}{\partial t}\, dz \\ &\quad + 12\, Re \int z \left\{\frac{\partial^2 \hat{X}}{\partial t^2}\right\}_z \cdot \hat{X}(t)_z \frac{\partial \phi}{\partial t}\, dz + 12\, Re \int z \left\{\frac{\partial \hat{X}}{\partial t}\right\}_z \cdot \hat{X}(t)_z \frac{\partial^2 \phi}{\partial t^2}\, dz \\ &\quad + 2\, Re \int z\hat{X}_z \cdot \hat{X}_z \frac{\partial^3 \phi}{\partial t^3}\, dz. \end{aligned} \tag{2.16}$$

Applying the same reasoning as with the third derivative, we see that, at $t = 0$ the fourth derivative is given by

$$\begin{aligned}\frac{\partial^4 E}{\partial t^4}\bigg|_{t=0} &= 12\, Re \int \left\{\frac{\partial^2 \hat{X}}{\partial t^2}\right\}_z \cdot \left\{z\left(\frac{\partial \hat{X}}{\partial t}\right)_z \phi + z\hat{X}_z \frac{\partial \phi}{\partial t}\right\} dz \\ &\quad + 12\, Re \int z \left\{\frac{\partial \hat{X}}{\partial t}\right\}_z \cdot \left\{\frac{\partial \hat{X}}{\partial t}\right\}_z \frac{\partial \phi}{\partial t}\, dz. \end{aligned} \tag{2.17}$$

A straight forward calculation, using the same reasoning shows that at $t = 0$

$$\begin{aligned}\frac{\partial^5 E}{\partial t^5} &= 16\, Re \int \left(\frac{\partial^3 \hat{X}}{\partial t^3}\right)_z \cdot \left(z\left(\frac{\partial \hat{X}}{\partial t}\right)_z \phi + z\hat{X}_z \frac{\partial \phi}{\partial t}\right) dz \\ &+12\, Re \int z \left(\frac{\partial^2 \hat{X}}{\partial t^2}\right)_z \cdot \left(z\left(\frac{\partial^2 \hat{X}}{\partial t^2}\right)_z \phi + 4z\left(\frac{\partial \hat{X}}{\partial t}\right)_z \frac{\partial \phi}{\partial t} + 2z\hat{X}_z \frac{\partial^2 \phi}{\partial t^2}\right) dz \\ &+24\, Re \int z\left(\frac{\partial \hat{X}}{\partial t}\right)_z \cdot \left(\frac{\partial \hat{X}}{\partial t}\right)_z \frac{\partial^2 \phi}{\partial t^2}\, dz. \end{aligned} \tag{2.18}$$

Now consider (2.17) and (2.18). We shall show that, under certain conditions, the derivatives of ϕ, $\frac{\partial \phi}{\partial t}$ can be chosen at $t = 0$ so that the formulas for the fourth and fifth derivatives greatly simplify, namely

$$\left.\frac{\partial^4 E}{\partial t^4}\right|_{t=0} = 12\, Re \int z \left\{\frac{\partial \hat{X}}{\partial t}\right\}_z \cdot \left\{\frac{\partial \hat{X}}{\partial t}\right\}_z \frac{\partial \phi}{\partial t}\, dz \tag{2.19}$$

which by (2.13) and as in (2.15)

$$= -12\, Re \int z^3 \hat{X}_{zz} \cdot \hat{X}_{zz} \phi^2 \frac{\partial \phi}{\partial t}\, dz$$

and

$$\frac{\partial^5 E}{\partial t^5} = 12\, Re \int \left(\frac{\partial^2 \hat{X}}{\partial t^2}\right) \cdot \left(z\left(\frac{\partial^2 \hat{X}}{\partial t^2}\right)_z \phi + 4z\left(\frac{\partial \hat{X}}{\partial t}\right)_z \frac{\partial \phi}{\partial t}\right) dz. \tag{2.20}$$

How do we choose $\frac{\partial \phi}{\partial t}$ at $t = 0$? Choose $\frac{\partial \phi}{\partial t}$ so that

$$z\left(\frac{\partial \hat{X}}{\partial t}\right)_z \phi + z\hat{X}_z \frac{\partial \phi}{\partial t} \tag{2.21}$$

is holomorphic. Then formulas (2.19) and (2.20) for the fourth and fifth derivatives hold. In this case

$$\begin{aligned}\left(\frac{\partial^2 \hat{X}}{\partial t^2}\right) &= \left\{4H\left[Re\, iz\left\{Re\, H\left(iz\hat{X}_z\phi\right)\right\}_z \phi + 2\left\{Re\, iz\hat{X}_z \frac{\partial \phi}{\partial t}\right\}\right]\right\}_z \\ &= \left\{iz\left[iz\hat{X}_z\phi\right]_z \phi + iz\hat{X}_z \frac{\partial \phi}{\partial t}\right\}_z . \end{aligned} \tag{2.22}$$

By the way choosing $\frac{\partial \phi}{\partial t}$ so that (2.21) is holomorphic, implies, as an easy exercise shows, that

$$\begin{aligned}\left(\frac{\partial^2 X}{\partial t^2}\right)_z \cdot X_z &= -z^2 \hat{X}_{zzz} \cdot \hat{X}_z \phi^2 = z^2 \hat{X}_{zz} \cdot \hat{X}_{zz} \phi^2 \\ &= -\left\{iz\hat{X}_z\phi\right\} \cdot \left\{iz\hat{X}_z\phi\right\}_z \\ &= -\left(\frac{\partial \hat{X}}{\partial t}\right)_z \cdot \left(\frac{\partial \hat{X}}{\partial t}\right)_z . \end{aligned}$$

which is needed to deduce formula (2.20) for the fifth derivative. How do we proceed in general?

2.2. The strategy

The strategy to find the first non-vanishing derivative *which can be made* negative is to

I. Decide the candidate L for which, at $t = 0$,

$$\frac{\partial^\mu E}{\partial t^\mu} = 0, \quad \mu < L$$

and

$$\frac{\partial^L E}{\partial t^L} < 0.$$

II. Select $\frac{\partial^\beta \phi}{\partial t^\beta}$ at $t = 0$, so that, in forming derivatives of lower order than L, poles are removed and these derivatives vanish!

III. With respect to a special linear coordinate system in $\mathbb{R}^3$ we may write

$$\hat{X}_z = (A_1 z^n + A_2 z^{n+1} + \ldots, R_m z^m + \ldots)$$

where $A_i \in \mathbb{C}^2$, $R_m \in \mathbb{C}$, $m \neq 0$, $A_1 \neq 0$. n is called the "order" of the branch point and m the "index". We show that, at $t = 0$

$$\frac{\partial^L E}{\partial t^L} = Re \int c^L K (R_m^2/z) dz,$$

c an arbitrary non zero complex number, $K \in \mathbb{C}$.

IV. Prove $K \neq 0$.

The need to remove poles can be made clear from (2.22), and formula (2.17) for the fourth derivative. In (2.17) we have the term

$$12\, Re \int z \left\{ \frac{\partial^2 \hat{X}}{\partial t^2} \right\}_z \cdot \left\{ z \left(\frac{\partial \hat{X}}{\partial t} \right)_z \phi + z \hat{X}_z \frac{\partial \phi}{\partial t} \right\} dz$$

Considering (2.22), this integral is of the form

$$Re \int \{2H\,[Reif(z)]\}_z\, f(z) dz \tag{2.23}$$

where above

$$\text{if}(z) := iz \left[iz \hat{X}_z \phi \right]_z \phi + iz \hat{X}_z \frac{\partial \phi}{\partial t}.$$

Assume $f(z)$ *had* poles, say

$$f(z) = \Sigma\, a_k/z^k = a_k/z^k$$

using the Einstein summation convention. Then an easy calculation shows

$$\left[H\left\{2Re[ia_k/z^k]\right\}\right]_z = -ik\overline{a}_k z^{k-1}.$$

Therefore in this case, (2.23) becomes

$$Re\int -ik\overline{a}_k z^{k-1} a_l z^l dz$$

and by Cauchy's theorem this equals

$$Re\int ik\overline{a}_k a_l \delta_{kl} z^{k-1} z^l\, dz = 2\pi k|a_k|^2 > 0.$$

Thus, not removing the poles (which gives a zero result) would give us something *strictly* positive, defeating the "ultimate" goal of showing that the presence of branch points implies that the minimal surface cannot be a relative minima for either energy or area.

For the sake of completeness, we work out the Wienholtz result for the third derivative as it applies to minimal surfaces in $\mathbb{R}^3$. We therefore assume that

$$2m - 2 < 3n.$$

From (2.15), at $t = 0$

$$\frac{\partial^3 E}{\partial t^3} = -4Re\int z^3 \hat{X}_{zz}\cdot\hat{X}_{zz}\phi^3 dz$$
$$z^3\hat{X}_{zz}\cdot\hat{X}_{zz} = (m-n)^2 R_m^2 z^{2m+1}.$$

Note that, since $m > n$, $(m-n)^2 > 0$. Choose

$$\phi_0 := c/z^{n+1} + \overline{c}z^{n+1}$$
$$\phi_1 := c/z^k + \overline{c}z^k$$

$c \in \mathbb{C}$, where

$$k = (2m+2) - 2(n+1) = (2m-2) - 2(n-1) < n+2.$$

Thus $k \le n+1$.

If $k = n+1$, set $\phi := \phi_0$. If $k < n+1$, set $\phi = \epsilon\phi_0 + \phi_1$, $\epsilon > 0$. If $k = n+1$, $3(n+1) = 2m+2$, and at $t = 0$

$$\frac{\partial^3 E}{\partial t^3} = -4Re\int c^3(m-n)^2(R_m^2/z)dz$$

which is negative, after an appropriate choice of c.

If $k < n+1$, $3(n+1) > (2m+2)$

$$\phi^3 = \epsilon^3 c^3/z^{3n+3} + 3\epsilon^2 c^3/z^{2n+2+k} + \dots$$

where $\dots$ denotes pole terms of lower order. Then, again, by Cauchy's integral theorem, at $t = 0$

$$\frac{\partial^3 E}{\partial t^3} = -12Re\,\epsilon^2\int c^3(m-n)^2(R_m^2/z)dz + o(\epsilon^2) < 0$$

after an appropriate choice of $(a - ib)$ and for $\epsilon > 0$ sufficiently small.

3. The main theorem

Let $X : D \to \mathbb{R}^3$ be a minimal surface with an interior branch point, which without loss of generality we may assume is at the origin. The invariance of Dirichlet's energy under the action of the conformal group permits us to do this. Write

$$\hat{X}_z = (A_1 z^n + A_2 z^{n+1} + \dots, R_m z^m + R_{m+1} z^{m+1} + \dots) \tag{3.1}$$

where $A_i \in \mathbb{C}^2$, $R_j \in \mathbb{C}$, $R_m \neq 0$, $A_1 \neq 0$ and where we assume $2m - 2 \geq 3n$. In the case $2m - 2 < 3n$, as we have seen the third derivative can be made negative and if $2m - 2 \geq 3n$ the third derivative vanishes in all Forced Jacobi Field directions [17]. Again, n is called the *order* of the branch point and m its *index*. The question is then how many derivatives can be made zero in these directions, what is the first non-vanishing derivative, and can we make it "negative"?

We note here that the methodology of calculating the higher order derivatives does not actually involve the boundary contour Γ, but only a change of parametrization of our minimal surface on S^1.

Definition 3.1. We say that the origin is an exceptional branch point if $(m + 1) = k(n + 1)$ where k is an integer.

We now state the central result of our paper:

Theorem 3.1. *Let X be a non-planar minimal surface with an interior branch point of order n and index m. If n is odd and m is even, then the $(m + 1)$-st derivative of Dirichlet's energy can be made negative while all lower order derivatives vanish.*

We shall prove this theorem assuming $n \geq 3$. The case $n = 1$ being proved in [15]

Remark 1. Such a branch point cannot be an exceptional branch point.

We proceed with some preliminaries:

The minimal surface equation

$$\hat{X}_z \cdot \hat{X}_z \equiv 0$$

implies certain relations on the coefficients $\{A_j\}$. In particular

$$A_1 \cdot A_1 = 0 \quad \text{and} \quad A_1 \cdot A_j = 0\,, \quad 1 \leq j \leq 2m - 2n \tag{3.2}$$

where $A \cdot B = \alpha_1\beta_1 + \alpha_2\beta_2$ if $A = (\alpha_1, \alpha_2)$, $B = (\beta_1, \beta_2)$. It is easily to see that $A_1 \cdot A_j = 0$ implies that

$$\begin{cases} A_j = v_j A,\ v_j \in \mathbb{C},\ 1 \leq j \leq 2m - 2n \\ A_1 \cdot A_{sm-2n+1} = -R_m^2/2 \end{cases} \tag{3.3}$$

Using (3.2) we see that

$$\hat{X}_{zz} \cdot \hat{X}_{zz} = (m - n)^2 R_m^2 z^{2m-2} + \dots$$

We now discuss a method for calculating the Lth derivative of Dirichlet's energy in the presence of an interior branch point of order n and index m, where L is odd and begin by reminding our readers of Leibniz's formula on differentiation for the product of two differentiable functions f and g of a real variable t, namely

$$\frac{\partial^n(fg)}{\partial t^n} = \sum_{r=0}^{n} \frac{n!}{(n-r)!} \frac{\partial^{n-r} f}{\partial t^{n-r}} \frac{\partial^r g}{\partial t^r}. \tag{3.4}$$

In view of (3.4)

$$\begin{aligned}\frac{\partial}{\partial t^n}\left\{\hat{X}_z \cdot \hat{X}_z\right\}\phi &= \sum_{\beta=0}^{n} \frac{n!}{(n-\beta)!\beta!} \frac{\partial^{n-\beta}}{\partial t^{n-\beta}}\left(\hat{X}_z \cdot \hat{X}_z\right)\frac{\partial^\beta \phi}{\partial t^\beta}\\ &= \sum_{\alpha=0}^{n-\beta}\sum_{\beta=0}^{n} \frac{n!(n-\beta)!}{(n-\beta)!\beta!(n-\beta-\alpha)!}\\ &\quad \times \left(\frac{\partial^{n-\beta-\alpha}\hat{X}}{\partial t^{n-\beta-\alpha}}\right)_z \cdot \left(\frac{\partial^\alpha \hat{X}}{\partial t^\alpha}\right)_z \frac{\partial^\beta \phi}{\partial t^\beta}.\end{aligned}$$

Set $n := L-1$, $M = L-(\alpha+\beta+1) = n-(\alpha+\beta)$ and order the sum of decreasing M we obtain for L odd:

$$\begin{cases} \dfrac{\partial^L E}{\partial t^L} = \\ 4Re \displaystyle\int z\left(\frac{\partial^{L-1}\hat{X}}{\partial t^{L-1}}\right)_z \cdot \hat{X}_z \phi\, dz + \\ 4\,\dfrac{(L-1)!}{(L-2)!}\, Re \displaystyle\int \left(\frac{\partial^{L-2}\hat{X}}{\partial t^{L-2}}\right)_z \cdot \left(z\left(\frac{\partial \hat{X}}{\partial t}\right)_z \phi + z\hat{X}_z \frac{\partial \phi}{\partial t}\right)_z dz \\ +\, 4 \displaystyle\sum_{M>\frac{L-1}{2}}^{L-3} \frac{(L-1)!}{M!(L-M-1)!}\, Re \int \left(\frac{\partial^M \hat{X}}{\partial t^M}\right)_z \\ \cdot \left(\displaystyle\sum_{\alpha=0}^{L-M-1} \tfrac{(L-M-1)!}{\alpha!(L-M-1-\alpha)!}\, z\left(\tfrac{\partial^\alpha \hat{X}}{\partial t^\alpha}\right)_z \tfrac{\partial^\beta \phi}{\partial t^\beta}\right) dz + \end{cases} \tag{3.5}$$

$$\begin{cases} \displaystyle\sum_{M=2}^{\frac{L-1}{2}} \frac{2(L-1)!}{M!M!}\, Re \int \left(\frac{\partial^M \hat{X}}{\partial t^M}\right)_z \\ \cdot \left(\displaystyle\sum_{\alpha=0}^{M} \tfrac{M!}{\alpha!\beta!}\, \psi(M,\alpha)\left(\tfrac{\partial^\alpha \hat{X}}{\partial t^\alpha}\right)_z \tfrac{\partial^\beta \phi}{\partial t^\beta}\right) dz \\ +\, \dfrac{2(L-1)!}{(L-3)!}\, Re \displaystyle\int z\left(\frac{\partial \hat{X}}{\partial t}\right)_z \cdot \left(\frac{\partial \hat{X}}{\partial t}\right)_z \frac{\partial^{L-3}\phi}{\partial t^{L-3}}\, dz + \end{cases} \tag{3.6}$$

$$\begin{cases} \tfrac{4(L-1)!}{(L-2)!}\, Re \int z\left(\tfrac{\partial \hat{X}}{\partial t}\right)_z \cdot \hat{X}_z \tfrac{\partial^{L-2}\phi}{\partial t^{L-2}}\, dz \\ +\, 2\, Re \displaystyle\int z\hat{X}_z \cdot \hat{X}_z \frac{\partial^{L-1}\phi}{\partial t^{L-1}}\, dz \end{cases} \tag{3.7}$$

where $\alpha + \beta + M = L - 1$ and $\psi(M, \alpha) = \begin{cases} 1 \text{ if } \alpha = M \\ 2 \text{ if } \alpha \neq M \end{cases}$. Note that at $t = 0$ the two terms of (3.7) vanish. We may assume that $L \geq 5$ and that $n \geq 3$, m even, n odd.

Since $L > 4$, $2m - 2 \geq 3n > 9$ implying that $m > 6$. Again, let $z\hat{X}_z = \left(A_1 z^{n+1} + \ldots A_{2m-2n+1} z^{2m-n+1} + \ldots, R_m z^{m+1} + \ldots\right)$. Choose

$$\phi := (a - ib)/z^2 + (a + ib)z^2$$

as our initial condition. Then, at $t = 0$,

$$\begin{aligned}\left(\frac{\partial \hat{X}}{\partial t}\right)_z &= (iz\hat{X}_z\phi)_z \\ &= (a - ib)\Big(i(n-1)A_1 z^{n-2} + inA_2 z^{n-1} + \ldots \\ &\quad i(2m - n - 1)A_{2m-2n+1} z^{2m-n-2} + \ldots,\ i(m-1)R_m z^{m-2}\Big) \qquad (3.8)\end{aligned}$$

and

$$\begin{aligned}z\left(\frac{\partial \hat{X}}{\partial t}\right)_z \phi = (a - ib)^2 &\Big(i(n-1)A_1 z^{n-3} + inA_2 z^{n-2} + \ldots \\ &i(2m - n - 1)A_{2m-2n+1} z^{2m-n-3} + \ldots, i(m-1)R_m z^{m-3} + \ldots\Big).\end{aligned}$$

If $n \geq 3$, there are "no" poles in this expression. This gives us the freedom to choose $\frac{\partial \phi}{\partial t}$ at $t = 0$ by setting

$$\frac{\partial \phi}{\partial t} := (a - ib)^2(c + id)/z^4 + (a + ib)^2(c - id)z^4. \qquad (3.9)$$

Since $n \geq 3$, at $t = 0$

$$z\hat{X}_z \frac{\partial \phi}{\partial t} \quad \text{is holomorphic,}$$

and at $t = 0$ (cf. (2.22))

$$\begin{aligned}\left(\frac{\partial^2 \hat{X}}{\partial t^2}\right)_z &= i\left\{z\left(\frac{\partial \hat{X}}{\partial t}\right)_z \phi + zX_z \frac{\partial \phi}{\partial t}\right\}_z \\ &= -(a - ib)^2\Big((n-1)(n-3)A_1 z^{n-4} + \ldots \\ &\quad (2m - n - 1)(2m - n - 3)A_{2m-2n+1} z^{2m-n-4} + \ldots, \\ &\quad (m-1)(m-3)R_m z^{m-4} + \ldots\Big) \\ &\quad + i(c + id)(a - ib)^2\Big((n-3)A_1 z^{n-4} + \ldots (2m - n - 3) \\ &\quad A_{2m-2n+1} z^{2m-n-4} + \ldots,\ (m-3)R_m z^{m-4} + \ldots\Big)\end{aligned}$$

$$= (a-ib)^2 \Big([-(n-1)(n-3) + i(n-3)(c+id)]\, A_1 z^{n-4} + \ldots$$
$$[-(2m-n-1)(2m-n-3) + i(2m-n-3)(c+id)]$$
$$A_{2m-2n+1} z^{2m-n-4} + \ldots,$$
$$[-(m-1)(m-3) + i(m-3)(c+id)]\, R_m z^{m-3} + \ldots \Big). \tag{3.10}$$

Now, if $\frac{\partial^2\phi}{\partial t^2}$ is chosen according to the pole removal methodology

$$\left(\frac{\partial^3 \hat{X}}{\partial t^3}\right)_z = i\left\{\left(\frac{\partial^2 \hat{X}}{\partial t^2}\right)_z \phi + 2\left(\frac{\partial \hat{X}}{\partial t}\right)_z \frac{\partial \phi}{\partial t} + z\hat{X}_z \frac{\partial^2\phi}{\partial t^2}\right\}_z. \tag{3.11}$$

Let us examine this a little more closely. At $t = 0$,

$$z\left(\frac{\partial \hat{X}}{\partial t}\right)_z \frac{\partial \phi}{\partial t} = (a-ib)\left(2i(n-1)A_1 z^n + \ldots, 2i(m-1)R_m z^m + \ldots\right)\frac{\partial \phi}{\partial t}$$

where $\frac{\partial^2\phi}{\partial t^2}$ necessarily has a pole of order 6 if $n = 3$; there is no pole in the third complex component ($m \geq 6$) but there might be one in the first complex components.

If $n - 5 < 0$, we then remove the pole by an appropriate choice of $\frac{\partial^2\phi}{\partial t^2}$ at $t = 0$, namely,

$$\frac{\partial^2\phi}{\partial t^2} := -2i(n-1)(a-ib)^3(c+id)/z^6 + 2i(n-1)(a+ib)^3(c-id)z^6. \tag{3.12}$$

If $n - 5 \geq 0$, $n + 1 \geq 6$ and $z\hat{X}_z \frac{\partial^2\phi}{\partial t^2}$ with $\frac{\partial^2\phi}{\partial t^2}$ defined by (3.12) is holomorphic and (3.11) remains valid. The next lemmas show that we can continue in this manner.

Assume that $L = m + 1$ is odd.

Lemma 3.1. *If* $1 \leq \beta \leq \frac{L-3}{2}$ *or equivalently* $\alpha \geq \frac{L-1}{2}$, $\left(\frac{\partial^\alpha X}{\partial t^\alpha}\right)_z$ *can be defined inductively by the pole removal technique*

$$\begin{cases} \left(\frac{\partial \hat{X}}{\partial t}\right)_z = \left\{iz\hat{X}_z\phi\right\}_z \\ \left(\frac{\partial^\gamma \hat{X}}{\partial t^\gamma}\right)_z = \left\{i\sum_{\alpha=0}^{\gamma-1} \frac{(\alpha-1)!}{\alpha!(\gamma-\alpha-1)!}\, z\left(\frac{\partial^\alpha \hat{X}}{\partial t^\alpha}\right)_z \frac{\partial^\beta \phi}{\partial t^\beta}\right\}_z \\ \gamma = 1, \ldots, (L-1)/2. \end{cases} \tag{3.13}$$

Proof. During the process, there will be sufficiently many A_j's, so long as the power associated to $A_{2m-2n+1}$ does not become negative (no pole associated to $A_{2m-2n+1}$). Since at each α^{th} stage in defining $\left(\frac{\partial^\alpha \hat{X}}{\partial t^\alpha}\right)_z$ the powers are reduced by 2α, we must check that the terms $z\left(\frac{\partial^\alpha \hat{X}}{\partial t^\alpha}\right)_z \frac{\partial^\beta \phi}{\partial t^\beta}$ have no poles associated to $A_{2m-2n+1}$. Looking only at $z\left(\frac{\partial^\alpha \hat{X}}{\partial t^\alpha}\right)\phi$, $\alpha \leq \left(\frac{L-3}{2}\right)$ we must have

$$2m - n - 2\alpha - 1 = 2m - n + 1 - 2(\alpha + 1) \geq 0$$

or

$$2m - n + 1 - 2\left(\frac{L-1}{2}\right) \geq 0.$$

But

$$2m - n + 1 - 2\left(\frac{L-1}{2}\right) = (m - n) + 1 > 0.$$

We must also check that during this process no pole is introduced into the third complex component. Again, we look at

$$z\left(\frac{\partial^\alpha \hat{X}}{\partial t^\alpha}\right)_z \phi,$$

$\alpha \leq \left(\frac{L-3}{2}\right)$. Then the order of the zero of the R_m term is

$$m - 2\alpha - 1 = (m+1) - 2(\alpha + 1) = (m+1) - (L-1) = 1,$$

so there is no pole.

Lemma 3.2. *In removing all poles we may choose*

$$\frac{\partial^\beta \phi}{\partial t^\beta} := k(a - ib)^{\beta+1}(c + id)/z^{2(\beta+1)} + \ldots \tag{3.14}$$

i.e. In the highest order pole $\frac{\partial^\beta \phi}{\partial t^\beta}$ *can be chosen to be linear in* $c + id$.

Proof. It is here that we assume n odd. We use induction on β. Assume true for $\beta \leq \alpha$.

Case a. $n \geq 2\alpha$. Then the first complex components of $\left(\frac{\partial^\alpha \hat{X}}{\partial t^\alpha}\right)_z$ have a zero of order $n - 2\alpha > 0$ ($n \neq 2\alpha$). Then $\frac{\partial^\alpha \phi}{\partial t^\alpha}$ must be chosen so that

$$\sum_{\gamma=0}^{\alpha} \frac{\alpha!}{\gamma!(\alpha-\gamma)!} z\left(\frac{\partial^\gamma \hat{X}}{\partial t^\gamma}\right)_z \frac{\partial^\beta \phi}{\partial t^\beta}, \quad \gamma + \beta = \alpha$$

is holomorphic. Also for $\gamma \leq \alpha$, the first complex component of $\left(\frac{\partial^\gamma \hat{X}}{\partial t^\gamma}\right)_z$ has a zero of order $n - 2\gamma > 0$.

Thus if $\beta \leq \alpha$

$$z\left(\frac{\partial^\gamma \hat{X}}{\partial t^\gamma}\right)_z \frac{\partial^\beta \phi}{\partial t^\beta}$$

has a "zero" of order $n - 2\gamma - 2\beta - 1 = n - 2\alpha - 1 \geq 0$; i.e. there is no pole. Then $n + 1 \geq 2\alpha + 2$ and we may choose $\frac{\partial^\alpha \phi}{\partial t^\alpha}$ according to (3.14), since $z\hat{X}_z \frac{\partial^\alpha \phi}{\partial t^\alpha}$ will also be holomorphic and $\left(\frac{\partial^{\alpha+1}\hat{X}}{\partial t^{\alpha+1}}\right)_z$ may again be defined by (3.13).

Case b. Now suppose that α is the first integer such that $n - 2\alpha < 0$. Then

$$\sum_{\gamma=0}^{\alpha} \frac{\alpha!}{\gamma!(\alpha-\gamma)!} z \left(\frac{\partial^\gamma \hat{X}}{\partial t^\gamma}\right)_z \frac{\partial^\beta \phi}{\partial t^\beta}$$
$$= z\left(\frac{\partial^\alpha \hat{X}}{\partial t^\alpha}\right)_z \phi + \sum_{\gamma=1}^{\alpha-1} \frac{\alpha!}{\gamma!(\alpha-\gamma)!} z \left(\frac{\partial^\gamma \hat{X}}{\partial t^\gamma}\right)_z \frac{\partial^\beta \phi}{\partial t^\beta} + z\hat{X}_z \frac{\partial^\alpha \phi}{\partial t^\alpha}. \tag{3.15}$$

Each $z\left(\frac{\partial^\gamma \hat{X}}{\partial t^\gamma}\right)_z \frac{\partial^\beta \phi}{\partial t^\beta}$ has a pole of order $-(1+n-2\gamma-2\beta-2) = 2\alpha+1-n \geq 2$ but $z\left(\frac{\partial^\alpha \hat{X}}{\partial t^\alpha}\right)_z \phi$ has a pole of order at most 1. Therefore, in order to kill this pole, $\frac{\partial^\alpha \phi}{\partial t^\alpha}$ must have a pole of order at most $n+2$. In order to kill all other poles in (3.15), $\frac{\partial^\alpha \phi}{\partial t^\alpha}$ must have a pole of order k so that $k - (n+1) \geq 2\alpha + 1 - n$, or $k \geq 2(\alpha+1)$, as expected.

But $n - 2\alpha \leq -1$ so $n + 2 \leq 2\alpha + 1 < 2(\alpha+1)$ and the order of the pole of $\frac{\partial^\alpha \phi}{\partial t^\alpha}$ needed to kill the pole of order 1 term (which need not be linear in $c + id$) is strictly less than the order of the pole need to kill the poles in all other terms.

We now continue in this manner. For $\alpha + 2$,

$$\sum_{\gamma=c}^{\alpha+1} \frac{(\alpha+1)!}{\gamma!\beta!} \left(\frac{\partial^\gamma \hat{X}}{\partial t^\gamma}\right)_z \frac{\partial^\beta \phi}{\partial t^\beta} = z\left(\frac{\partial^{\alpha+1} \hat{X}}{\partial t^{\alpha+1}}\right)_z \phi + (\alpha+1) z \left(\frac{\partial^\alpha \hat{X}}{\partial t^\alpha}\right)_z \frac{\partial \phi}{\partial t}$$
$$+ \sum_{\gamma=1}^{\alpha-1} \frac{(\alpha+1)!}{\gamma!\beta!} z \left(\frac{\partial^\gamma \hat{X}}{\partial t^\gamma}\right)_z \frac{\partial^\beta \phi}{\partial t^\beta} + zX_z \frac{\partial^{\alpha+1} \phi}{\partial t^{\alpha+1}}.$$

The first term on the right has a pole of order at most 1 and the second term a pole of order at most 3. To kill the pole of order three $\frac{\partial^{\alpha+1}\phi}{\partial t^{\alpha+1}}$ must have a pole of order k so that $k - (n+1) = 3$ or $k = n + 4$. But $n + 4 \leq 2\alpha + 3 < 2(\alpha+2)$ which is needed to kill poles other than in the first two terms. This choice of $\frac{\partial^{\alpha+1}\phi}{\partial t^{\alpha+1}}$ can then be made according to (3.14).

Now consider the Lth derivative ($L = m + 1$) given by formulas (3.5) through (3.7). Then, by Cauchy's integral theorem, the first terms (3.5) being boundary integrals of holomorphic functions vanish. We are left with the terms

$$\frac{2m!}{(\frac{m}{2})!\,(\frac{m}{2})!} Re \int z \left(\frac{\partial^{m/2} \hat{X}}{\partial t^{m/2}}\right)_z \cdot \left(\frac{\partial^{m/2} \hat{X}}{\partial t^{m/2}}\right)_z \phi \, dz \tag{3.16}$$

$$\frac{4m!}{(\frac{m}{2})!\,(\frac{m}{2})!} \sum_{\alpha=0}^{M-1} Re \int z \left(\frac{\partial^{m/2} \hat{X}}{\partial t^{m/2}}\right)_z \cdot \left(\frac{\partial^\alpha \hat{X}}{\partial t^\alpha}\right)_z \frac{\partial^\beta \phi}{\partial t^\beta} \, dz \tag{3.17}$$

where $\beta \geq 1$, plus the remaining terms

$$\sum_{M=2}^{\frac{S-3}{2}} \frac{2m!}{M!M!} Re \int z \left(\frac{\partial^M \hat{X}}{\partial t^M}\right)_z \cdot \left(\sum_{\alpha=0}^{M} \frac{M!}{\alpha!\beta!} \psi(M,\alpha) z \left(\frac{\partial^\alpha \hat{X}}{\partial t^\alpha}\right) \frac{\partial^\beta \phi}{\partial t^\beta}\right) dz.$$

With the exception of (3.16) we claim that each other integrand is a polynomial in the variable $(c+id)$ of degree $m-2$ (each $\left(\frac{\partial^\alpha \hat{X}}{\partial t^\alpha}\right)_z$ is a polynomial of degree $\alpha - 1$ in $c+id$) with no constant term.

Lemma 3.3. *Term* (3.16) *takes the form*

$$\frac{2m!}{\left(\frac{m}{2}\right)!\left(\frac{m}{2}\right)!} Re \int (\kappa R_m^2/z)(a-ib)^{m+1} dz + Re \int (\nu R_m^2)/z(a-ib)^{m+1} dz$$

$\kappa = (i)^m (m-1)^2 (m-3)^2 \ldots 3^2 \cdot 1$ *and* ν *is a polynomial in* $(c+id)$ *with no constant term.*

Proof. We will use induction but first we need another

Lemma 3.4. *The contributions of the first complex components to the integral of the product*

$$\frac{2m!}{\left(\frac{m}{2}\right)!\left(\frac{m}{2}\right)!} Re \int z \left(\frac{\partial^{m/2} \hat{X}}{\partial t^{m/2}}\right)_z \cdot \left(\frac{\partial^{m/2} \hat{X}}{\partial t^{m/2}}\right)_z \phi \, dz \tag{3.18}$$

is zero.

Proof. The terms with the lowest powers of z in the product are of the form

$$\text{const} \cdot A_j \cdot A_{2m-2n+1} z^{2m-n-2(m/2)-1+1} = \text{const}\, z^{m-n}, \quad m-n > 0.$$

Now to continue with Lemma 3.3.

We know from Lemma 3.4 that the first two complex components play no role in (3.18) and from earlier discussions (3.18) takes the form

$$Re \int (\mu R_m^2/z)(a-ib)^{m+1} dz.$$

Now the third complex component of $\left(\frac{\partial \hat{X}}{\partial t}\right)_z$ is

$$(a-ib) i (m-1) R_m z^{m-2} + \ldots.$$

Assume that the third complex component of $\left(\frac{\partial^\alpha \hat{X}}{\partial t^\alpha}\right)_z$ is of the form $(a-ib)^\alpha$ $\left\{i^\alpha (m-1)(m-3)\ldots(m-2\alpha+1) R_m z^{m-2\alpha} + \ldots\right\}$ plus a polynomial in $(c+id)$ with no constant term denoted simply by P. Then if $\alpha + 1 \leq m/2$, by (3.13)

$$\left(\frac{\partial^{\alpha+1}\hat{X}}{\partial t^{\alpha}}\right)_z = i\left\{z\left(\frac{\partial^{\alpha}\hat{X}}{\partial t^{\alpha}}\right)_z \phi + \dot{P}(a-ib)^{\alpha+1}\right\}_z$$
$$= (a-ib)^{\alpha+1} i \left\{i^{\alpha}(m-1)(m-3)\ldots(m-2\alpha+1)R_m z^{m-2\alpha-1}\right.$$
$$\left. + P\cdot(a-ib)^{\alpha+1}\right\}_z$$
$$= (a-ib)^{\alpha+1} i^{\alpha+1}(m-1)(m-3)\ldots(m-2(\alpha+1)+1)\, R_m z^{m-2(\alpha+1)}$$
$$+ P_z(a-ib)^{\alpha+1}.$$

Setting $\alpha+1 = m/2$ completes the proof of the lemma 3.3.

We can now complete the proof of theorem 3.1.

All other terms in the $(m+1)$th derivative are of the form $\int(\nu R_m^2/z)(a-ib)^{m+1}dz$, ν a polynomial in $(c+id)$ with no constant term.

Summarizing, the $(m+1)$th derivative of Dirichlet's integral takes the form

$$\frac{2m!}{\left(\frac{m}{2}\right)!\left(\frac{m}{2}\right)!}\, Re\int(\kappa R_m^2/z)(a-ib)^{m+1}dz + \int(\nu R_m^2/z)(a-ib)^{m+1}dz$$

where ν is a polynomial in $(c+id)$ with no constant term. Now set $(c+id) = 0$. Then the Lth derivative is given by the explicit formula

$$\frac{\partial^L E}{\partial t^L} = \frac{2m!}{\left(\frac{m}{2}\right)!\left(\frac{m}{2}\right)!}\, Re\int(\kappa R_m^2/z)(a-ib)^{m+1}dz$$
$$\kappa = (i)^m(m-1)^2(m-3)^2\ldots 3^2\cdot 1$$

which is non-zero if $a-ib \neq 0$ or in fact negative for appropriate choice of $a-ib$.

For $L < m+1$, $2L < 2m+2$ implying that all lower order derivatives with order L initial conditions $\phi := (a-ib)/z^2 + (a+ib)z^2$ vanish.

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