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Multistage Vertex Cover

Till Fluschnik[1] · Rolf Niedermeier[1] · Valentin Rohm[1] · Philipp Zschoche[1]

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Abstract

The NP-complete VERTEX COVER problem asks to cover all edges of a graph by a small (given) number of vertices. It is among the most prominent graph-algorithmic problems. Following a recent trend in studying temporal graphs (a sequence of graphs, so-called layers, over the same vertex set but, over time, changing edge sets), we initiate the study of MULTISTAGE VERTEX COVER. Herein, given a temporal graph, the goal is to find for each layer of the temporal graph a small vertex cover *and* to guarantee that two vertex cover sets of every two consecutive layers differ not too much (specified by a given parameter). We show that, different from classic VERTEX COVER and some other dynamic or temporal variants of it, MULTISTAGE VERTEX COVER is computationally hard even in fairly restricted settings. On the positive side, however, we also spot several fixed-parameter tractability results based on some of the most natural parameterizations.

Keywords Parameterized algorithmics · NP-completeness · Temporal graphs · Data reduction

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✉ Till Fluschnik
till.fluschnik@tu-berlin.de

✉ Philipp Zschoche
zschoche@tu-berlin.de

Rolf Niedermeier
rolf.niedermeier@tu-berlin.de

Valentin Rohm
valentinl.rohm@campus.tu-berlin.de

[1] Algorithmics and Computational Complexity, Technische Universität Berlin, Berlin, Germany

1 Introduction

Vertex Cover asks, given an undirected graph G and an integer $k \geq 0$, whether at most k vertices can be deleted from G such that the remaining graph contains no edge. Vertex Cover is NP-complete and it is a formative problem of algorithmics and combinatorial optimization. We study a *time-dependent*, "*multistage*" version, namely a variant of Vertex Cover on temporal graphs. A *temporal graph* $\mathcal{G}$ is a tuple $(V, \mathcal{E}, \tau)$ consisting of a set V of vertices, a discrete time-horizon τ, and a set of temporal edges $\mathcal{E} \subseteq \binom{V}{2} \times \{1, \ldots, \tau\}$. Equivalently, a temporal graph $\mathcal{G}$ can be seen as a vector $(G_1, \ldots, G_\tau)$ of static graphs (*layers*), where each graph is defined over the same vertex set V. Then, our specific goal is to find a small vertex cover S_i for each layer G_i such that the size of the symmetric difference $S_i \triangle S_{i+1} = (S_i \setminus S_{i+1}) \cup (S_{i+1} \setminus S_i)$ of the vertex covers S_i and S_{i+1} of every two consecutive layers G_i and G_{i+1} is small. Formally, we thus introduce and study the following problem (see Fig. 1 for an illustrative example).

Multistage Vertex Cover
Input: A temporal graph $\mathcal{G} = (V, \mathcal{E}, \tau)$ and two integers $k \in \mathbb{N}$, $\ell \in \mathbb{N}_0$.
Question: Is there a sequence $\mathcal{S} = (S_1, \ldots, S_\tau)$ such that

(i) for all $i \in \{1, \ldots, \tau\}$, it holds true that $S_i \subseteq V$ is a size-at-most-k vertex cover for layer G_i, and
(ii) for all $i \in \{1, \ldots, \tau - 1\}$, it holds true that $S_i \triangle S_{i+1} \leq \ell$?

Throughout this paper we assume that $0 < k < |V|$ because otherwise we have a trivial instance. In our model, we follow the recently proposed *multistage* [2–14] view on classical optimization problems on temporal graphs.

In general, the motivation behind a multistage variant of a classical problem such as Vertex cover is that the environment changes over time (here reflected by the changing edge sets in the temporal graph) and a corresponding adaptation of the current solution comes with a cost. In this spirit, the parameter ℓ in the definition of Multistage Vertex Cover allows to model that only moderate changes concerning the solution vertex set may be wanted when moving from one layer to the subsequent one. Indeed, in this sense ℓ can be interpreted as a parameter measuring the degree of (non-)conservation [15, 16].

It is immediate that Multistage Vertex Cover is NP-hard as it generalizes Vertex Cover ($\tau = 1$). We will study its parameterized complexity regarding the problem-specific parameters k, τ, ℓ, and some of their combinations, as well as restrictions to temporal graph classes [17, 18].

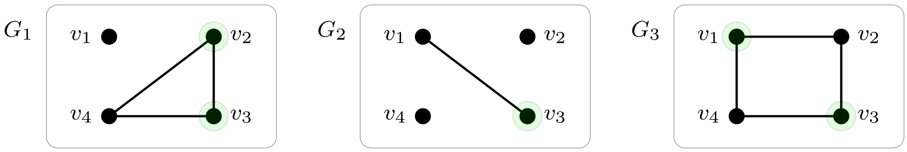

Fig. 1 An illustrative example with temporal graph $\mathcal{G} = (G_1, G_2, G_3)$ over the vertex set $V = \{v_1, \ldots, v_4\}$. A solution $\mathcal{S} = (\{v_2, v_3\}, \{v_3\}, \{v_1, v_3\})$ for $k = 2$ and $\ell = 1$ is highlighted

Related Work The literature on vertex covering is extremely rich, even when focusing on parameterized complexity studies. Indeed, VERTEX COVER can be seen as "drosophila" of parameterized algorithmics. Thus, we only consider VERTEX COVER studies closely related to our setting. First, we mention in passing that VERTEX COVER is studied in dynamic graphs [19, 20] and graph stream models [21]. More importantly for our work, Akrida et al. [22] studied a variant of VERTEX COVER on temporal graphs. Their model significantly differs from ours: they want an edge to be covered at least once over every time window of some given size Δ. That is, they define a temporal vertex cover as a set $S \subseteq V \times \{1, \dots, \tau\}$ such that, for every time window of size Δ and for each edge $e = \{v, w\}$ appearing in a layer contained in the time window, it holds that $(v, t) \in S$ or $(w, t) \in S$ for some t in the time window with $(e, t) \in \mathcal{E}$. For their model, Akrida et al. ask whether such an S of small cardinality exists. Note that if $\Delta > 1$, then for some $t \in \{1, \dots, \tau\}$ the set $S_t := \{v \mid (v, t) \in S\}$ is not necessarily a vertex cover of layer G_t. For $\Delta = 1$, each S_t must be a vertex cover of G_t. However, in Akrida et al.'s model the size of each S_t as well as the size of the symmetric difference between each S_t and S_{t+1} may strongly vary. They provide several hardness results and algorithms (mostly referring to approximation or exact algorithms, but not to parameterized complexity studies).

A second related line of research, not directly referring to temporal graphs though, studies reconfiguration problems which arise when we wish to find a step-by-step transformation between two feasible solutions of a problem such that all intermediate results are feasible solutions as well [23, 24]. Among other reconfiguration problems, Mouawad et al. [25, 26] studied VERTEX COVER RECONFIGURATION: given a graph G, two vertex covers S and T each of size at most k, and an integer τ, the question is whether there is a sequence $(S = S_1, \dots, S_\tau = T)$ such that each S_t, $1 \leq t \leq \tau$, is a vertex cover of size at most k. The essential difference to our model is that from one "sequence element" to the next only one vertex may be changed and that the input graph does not change over time. Indeed, there is an easy reduction of this model to ours while the opposite direction is unlikely to hold. This is substantiated by the fact that Mouawad et al. [25] showed that VERTEX COVER RECONFIGURATION is fixed-parameter tractable when parameterized by vertex cover size k while we show W[1]-hardness for the corresponding case of MULTISTAGE VERTEX COVER.

Finally, there is also a close relation to the research on dynamic parameterized problems [16, 27]. Krithika et al. [27] studied DYNAMIC VERTEX COVER where one is given two graphs on the same vertex set and a vertex cover for one of them together with the guarantee that the cardinality of the symmetric difference between the two edge sets is upper-bounded by a parameter d. The task then is to find a vertex cover for the second graph that is "close enough" (measured by a second parameter) to the vertex cover of the first graph. They show fixed-parameter tractability and a linear kernel with respect to d.

Our Contributions Table 1 summarized our results, focusing on the three perhaps most natural parameters. We highlight a few specific results. MULTISTAGE VERTEX COVER remains NP-hard even if every layer consists of only one edge; not surprisingly, the corresponding hardness reduction exploits an unbounded number τ

Table 1 Overview of our results. The column headings describe the restrictions on the input and each row corresponds to a parameter. p-NP-hard, PK, and NoPK abbreviate para-NP-hard, polynomial-size problem kernel, and no problem kernel of polynomial size unless coNP $\subseteq$ NP/poly

	general layers $0 \le \ell < 2k$	$\ell \ge 2k$	tree layers $0 \le \ell < 2k$	one-edge layers $\ell = 1$
	NP-hard	NP-hard	NP-hard (Theorem 4.1(i))	NP-hard (Theorem 4.1(ii))
τ	p-NP-hard (Theorem 4.1)	p-NP-hard (Theorem 4.1)	p-NP-hard (Theorem 4.1)	FPT, PK (Observation 6.1)
k	XP, W[1]-h., (Theorem 5.1)	FPT†, NoPK (Observation 3.5, Theorem 6.1)	XP, W[1]-h. (Theorem 5.1, Corollary 5.3)	*open*, NoPK (Theorem 6.1)
$k + \tau$	FPT, PK (Theorem 6.2)	FPT, PK (Theorem 6.2)	FPT, PK (Theorem 6.2)	FPT, PK (Theorem 6.2)

of time layers. If there are one two layers, however, one of them being a tree and the other being a path, then again MULTISTAGE VERTEX COVER already becomes NP-hard. MULTISTAGE VERTEX COVER parameterized by solution size k is fixed-parameter tractable if $\ell \ge 2k$, but becomes W[1]-hard if $\ell < 2k$. Considering the tractability results for DYNAMIC VERTEX COVER [27] and VERTEX COVER RECONFIGURATION [25], this hardness is surprising; it is our most technical result. Furthermore, MULTISTAGE VERTEX COVER parameterized by k with $\ell \ge 2k$ does not admit a problem kernel of polynomial size unless coNP $\subseteq$ NP/poly. Finally, for the combined parameter $k + \tau$ we obtain polynomial-sized problem kernels (and thus fixed-parameter tractability) in *all* cases without any further constraints.

Outline In Section 2, we provide some preliminaries. For MULTISTAGE VERTEX COVER, we provide some first and general observations in Section 3, study the parameterized complexity regarding k in Section 5, and discuss the possibilities for efficient data reduction in Section 6. We conclude in Section 7.

2 Preliminaries

We denote by $\mathbb{N}$ and $\mathbb{N}_0$ the natural numbers excluding and including zero, respectively. For two sets A and B, we denote by $A \triangle B := (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$ the symmetric difference of A and B, and by $A \uplus B$ the disjoint union of A and B. For static graphs, we use standard notations [28].

Temporal Graphs A temporal graph $\mathcal{G}$ is a tuple $(V, \mathcal{E}, \tau)$ consisting of the set V of vertices, the set $\mathcal{E}$ of temporal edges, and a discrete time-horizon τ. A temporal edge e

is an element in $\binom{V}{2} \times \{1, \ldots, \tau\}$. Equivalently, a temporal graph $\mathcal{G}$ can be defined as a vector of static graphs $(G_1, \ldots, G_\tau)$, where each graph is defined over the same vertex set V. We also denote by $V(\mathcal{G})$, $\mathcal{E}(\mathcal{G})$, and $\tau(\mathcal{G})$ the set of vertices, the set of temporal edges, and the discrete (and finite) time-horizon of $\mathcal{G}$, respectively. The *underlying graph* $\mathrm{G}_\downarrow = \mathrm{G}_\downarrow(\mathcal{G})$ of a temporal graph $\mathcal{G}$ is the static graph with vertex set $V(\mathcal{G})$ and edge set $\{e \mid \exists t \in \{1, \ldots, \tau(\mathcal{G})\} : (e, t) \in \mathcal{E}(\mathcal{G})\}$.

Parameterized Complexity Theory Let Σ be a finite alphabet. A parameterized problem L is a subset $L \subseteq \{(x, k) \in \Sigma^* \times \mathbb{N}_0\}$. An instance $(x, k) \in \Sigma^* \times \mathbb{N}_0$ is a yes-instance of L if and only if $(x, k) \in L$ (otherwise, it is a no-instance). Two instances (x, k) and (x', k') of parameterized problems L, L' are *equivalent* if $(x, k) \in L \iff (x', k') \in L'$. A parameterized problem L is fixed-parameter tractable (FPT) if for every input (x, k) one can decide whether $(x, k) \in L$ in $f(k) \cdot |x|^{O(1)}$ time, where f is some computable function only depending on k. A parameterized problem L is in XP if for every instance (x, k) one can decide whether $(x, k) \in L$ in time $|x|^{f(k)}$ for some computable function f only depending on k. A W[1]-hard parameterized problem is fixed-parameter intractable unless FPT=W[1].

Given a parameterized problem L, a *kernelization* is an algorithm that maps any instance (x, k) of L in time polynomial in $|x| + k$ to an instance (x', k') of L (the problem *kernel*) such that

(i) $(x, k) \in L \iff (x', k') \in L$, and
(ii) $|x'| + k' \leq f(k)$ for some computable function f (the *size* of the problem kernel) only depending on k.

3 Basic Observations

In this section, we state some preliminary simple-but-useful observations on Multistage Vertex Cover and its relation to Vertex Cover.

Observation 3.1 Every instance $(\mathcal{G}, k, \ell)$ of Multistage Vertex Cover with $k \geq \sum_{i=1}^{\tau(\mathcal{G})} |E(G_i)|$ is a yes-instance.

Proof Clearly, a graph with m edges always admits a vertex cover of size m. Hence, there is a vertex cover $S \subseteq V$ of size k of $\mathrm{G}_\downarrow(\mathcal{G})$, and hence, S is a vertex cover for each layer. The vector $(S_1, \ldots, S_\tau)$ with $S_i = S$ for all $i \in \{1, \ldots, \tau\}$ is a solution for every $\ell \geq 0$. □

Next, we state that if we are facing a yes-instance, then we can assume that there exists a solution where each layer's vertex cover is either of size k or $k - 1$.

Observation 3.2 Let $(\mathcal{G}, k, \ell)$ be an instance of Multistage Vertex Cover. If $(\mathcal{G}, k, \ell)$ is a yes-instance, then there is a solution $\mathcal{S} = (S_1, \ldots, S_\tau)$ such that $|S_1| = k$ and $k - 1 \leq |S_i| \leq k$ for all $i \in \{1, \ldots, \tau\}$.

Proof We first show that there is a solution $\mathcal{S} = (S_1, \ldots, S_\tau)$ for $I := (\mathcal{G}, k, \ell)$ such that $|S_1| = k$. (Recall that we assume $k < |V(\mathcal{G})|$.) Towards a contradiction assume that such a solution does not exist. Let $\mathcal{S} = (S_1, \ldots, S_\tau)$ be a solution such that $|S_1|$ is maximal over all solutions for I. Let $i \in \{1, \ldots, \tau\}$ be the maximum index such that $S_j \subseteq S_{j-1}$, for all $j \in \{2, \ldots, i\}$. If $i = \tau$, then we have that $|S_j| \leq |S_1| < k$ for all $j \in \{1, \ldots, \tau\}$. Hence, we can find a subset $X \subseteq V \setminus S_1$ such that $(S_1 \cup X, \ldots, S_\tau \cup X)$ is a solution. This contradicts $|S_1|$ being maximal. Now let $i < \tau$. Hence, there is a vertex $v \in S_{i+1} \setminus S_i$. Now we can adjust the solution by adding v to S_j for all $j \in \{1, \ldots, i\}$. This contradicts $|S_1|$ being maximal. Hence, there is a solution $\mathcal{S} = (S_1, \ldots, S_\tau)$ such that $|S_1| = k$.

Let Ψ be the set of solutions such that the first vertex cover is of size k. Assume towards a contradiction that all solutions in Ψ contain a vertex cover smaller than $k-1$. Let $\Psi_i \subseteq \Psi$ be the set of solutions such that for each $(S_1, \ldots, S_\tau) \in \Psi_i$ we have that $|S_i| < k - 1$ and $|S_j| \geq k - 1$ for all $j \in \{1, \ldots, i - 1\}$. Let $i \in \{1, \ldots, \tau\}$ be maximal such that $\Psi_i \neq \emptyset$. Furthermore, let $\mathcal{S} = (S_1, \ldots, S_\tau) \in \Psi_i$ such that $|S_i|$ is maximal over all solutions in Ψ_i. Hence, there is a vertex $v \in S_{i-1} \setminus S_i$. We distinguish two cases.

(a): Assume that there is a $p \in \{i + 1, \ldots, \tau\}$ such that there is a $w \in S_p \setminus S_{p-1}$ and $S_j \subseteq S_{j-1}$ for all $j \in \{i + 1, \ldots, p - 1\}$. The idea now is to keep v and add w in the i-th layer and then remove v in the p-th layer. We can achieve this by simply setting $S_q := S_q \cup \{v, w\}$ for all $q \in \{i, \ldots, p - 1\}$.

(b): Assume that $S_j \subseteq S_{j-1}$ for all $j \in \{i + 1, \ldots, \tau\}$. In this case we take an arbitrary vertex $w \in V \setminus S_i$ and set $S_q := S_q \cup \{v, w\}$ for all $q \in \{i, \ldots, \tau\}$.

In either of the cases (a) or (b), the obtained solution either contradicts that $|S_i|$ is maximal, or that i is maximal, or that every solution in Ψ contains a vertex cover of size smaller than $k - 1$. □

With the next two observations, we show that the special case of MULTISTAGE VERTEX COVER with $\ell = 0$ is equivalent to VERTEX COVER under polynomial-time many-one reductions.

Observation 3.3 There is a polynomial-time algorithm that maps any instance $(G = (V, E), k)$ of VERTEX COVER to an equivalent instance $(\mathcal{G}, k, \ell)$ of MULTISTAGE VERTEX COVER where $\ell = 0$ and every layer G_i contains only one edge.

Proof Let the edges $E = \{e_1, \ldots, e_m\}$ of G be ordered in an arbitrary way. Set $\tau = m$ and $\ell = 0$. Set $G_i = (V, \{e_i\})$ for each $i \in \{1, \ldots, \tau\}$. We claim that $(G = (V, E), k)$ is a `yes`-instance of VERTEX COVER if and only if $(\mathcal{G}, k, \ell)$ is a `yes`-instance of MULTISTAGE VERTEX COVER.

($\Rightarrow$) Let S be a vertex cover of G of size at most k. Set $S_i := S$ for all $i \in \{1, \ldots, \tau\}$. Clearly, S_i is a vertex cover of G_i for all $i \in \{1, \ldots, \tau\}$ of size at most k. Moreover, by construction, $S_i \triangle S_{i+1} = 0$ for all $i \in \{1, \ldots, \tau - 1\}$. Hence, $(S_1, \ldots, S_\tau)$ forms a solution to $(\mathcal{G}, k, \ell)$.

($\Leftarrow$) Let $\mathcal{S} = (S_1, \ldots, S_\tau)$ be a solution to $(\mathcal{G}, k, \ell)$. Observe that $|\bigcup_i S_i| \leq k$. It follows that there are at most k vertices covering all edges of the layers G_i, that is, $E = \bigcup_{i=1}^{\tau} E(G_i)$, and hence they cover all edges of G. □

Observation 3.4 There is a polynomial-time algorithm that maps any instance $(\mathcal{G}, k, \ell)$ of MULTISTAGE VERTEX COVER with $\ell = 0$ to an equivalent instance (G, k) of VERTEX COVER.

Proof Now let $(\mathcal{G} = (V, \mathcal{E}, \tau), k, 0)$ be an arbitrary instance of MULTISTAGE VERTEX COVER. Construct the instance $(\mathrm{G}_\downarrow, k)$ of VERTEX COVER. We claim that $(\mathcal{G}, k, 0)$ is a yes-instance if and only if $(\mathrm{G}_\downarrow, k)$ is a yes-instance.

($\Leftarrow$) Let $S \subseteq V$ be a vertex cover of size at most k. Since S is a vertex cover for $\mathrm{G}_\downarrow$, S covers each layer of $\mathcal{G}$. Hence, $S_i := S$ for all $i \in \{1, \ldots, \tau\}$ forms a solution to $(\mathcal{G}, k, 0)$.

($\Rightarrow$) Let $(S_1, \ldots, S_\tau)$ be a solution to $(\mathcal{G}, k, 0)$. Clearly, since $\ell = 0$, we have that $S_i = S_j$ for all $i, j \in \{1, \ldots, \tau\}$. Thus, $S := S_1$ is a vertex cover for $\mathrm{G}_\downarrow$, and hence the claim follows. □

Finally, the special case of MULTISTAGE VERTEX COVER with $\ell \geq 2k$ (that is, where vertex covers of any two consecutive layers can be even disjoint) is Turing-reducible to VERTEX COVER.

Observation 3.5 Any instance $(\mathcal{G}, k, \ell)$ of MULTISTAGE VERTEX COVER with $\ell \geq 2k$ and $\mathcal{G} = (G_1, \ldots, G_\tau)$ can be solved by deciding each instance of the set $\{(G_i, k) \mid 1 \leq i \leq \tau\}$ of VERTEX COVER-instances.

Proof For each of the layers G_i, $i \in \{1, \ldots, \tau\}$, we can construct an instance of VERTEX COVER of the form (G_i, k). We can solve each instance independently, since the symmetric difference of any two size-at-most-k solutions is at most $2k \leq \ell$. □

4 Hardness for Restricted Input Instances

MULTISTAGE VERTEX COVER is NP-hard as it generalizes VERTEX COVER ($\tau = 1$). In this section, we prove that MULTISTAGE VERTEX COVER remains NP-hard on inputs with only two layers (one consisting of a path and the other consisting of a tree), and on inputs where every layer contains only one edge.

Theorem 4.1 MULTISTAGE VERTEX COVER *is NP-hard even if*

(i) $\tau = 2$, $\ell = 0$, *and the first layer is a path and the second layer is a tree, or*
(ii) *every layer contains only one edge and* $\ell \leq 1$.

Remark 4.1 Theorem 4.1(i) is tight regarding τ since VERTEX COVER (i.e., MULTISTAGE VERTEX COVER with $\tau = 1$) on trees is solvable in linear time. Theorem 4.1(ii) is tight regarding ℓ because if $\ell > 1$, then Observation 3.5 is applicable.

It is known that Vertex Cover remains NP-complete on cubic Hamiltonian graphs when a Hamiltonian cycle is additionally given as part of the input [29]: [1]

Hamiltonian Cubic Vertex Cover (HCVC)

Input: An undirected, cubic, Hamiltonian graph $G = (V, E)$, an integer $k \in \mathbb{N}$, and a Hamiltonian cycle $C = (V, E')$ of G.

Question: Is there a set $S \subseteq V$ such that S is a size-at-most-k vertex cover for G?

To prove Theorem 4.1(i), we give a polynomial-time many-one reduction from HCVC to Multistage Vertex Cover with two layers, one being a path, the other being a tree.

Proposition 4.1 *There is a polynomial-time algorithm that maps any instance* $(G = (V, E), k, C)$ *of* HCVC *to an equivalent instance* $(\mathcal{G}, k', \ell')$ *of* Multistage Vertex Cover *with* $\tau = 2$ *and the first layer* G_1 *being a path and the second layer* G_2 *being a tree.*

Proof Let $e \in E(C)$ be some edge of C, and let $P = C - e$ be the Hamiltonian path obtained from C when removing e. Let $E_1 := E(P)$, and $E_2 := E \setminus E(P)$. Set initially $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$. Note that G_1 is a path. Moreover, observe that G_2 is the disjoint union of $|V|/2 - 2$ paths of length one and one path of length three: the graph $G - E(C)$ is a matching of size $|V|/2$. This is because each vertex is of degree three in $G and$ each vertex is adjacent to two vertices in C. Thus, all vertices in $G - E(C)$ have degree one. Since $G - E(C) = G_2 - e$, edge e connects two paths of length one to one path of length three in G_2. Add two special vertices z, z' to V. In G_1, connect z with z' and with one endpoint of P. In G_2, connect z with z' and with exactly one vertex of each connected component. Set $k' = k + 1$ and $\ell' = 0$. We claim that $(G = (V, E), k, C)$ is a yes-instance if and only if $(\mathcal{G}, k', \ell')$ is a yes-instance.

($\Rightarrow$) Let S' be a vertex cover of G of size at most k. We claim that $S' := S \cup \{z\}$ is a vertex cover for both G_1 and G_2. Observe that $G_1[E_1]$ and $G_2[E_2]$ are subgraphs of G, and hence all edges are covered by S'. Moreover, all edges in $G_i - E_i$, $i \in \{1, 2\}$, are incident with z and hence covered by S'.

($\Leftarrow$) Let (S_1, S_2) be a minimal solution to $(\mathcal{G}, k', \ell')$ with $S' := S_1 = S_2$ and $|S'| \leq k'$. We can assume that $z \in S'$ since the edge $\{z, z'\}$ is present in both G_1 and G_2, and exchanging z in z' does not cover less edges. Moreover, we can assume that not both z and z' are in S' due to the minimality of S'. Let $S := S' \setminus \{z\}$. Observe that S covers all edges in $E_1 \cup E_2$ and, hence, S forms a vertex cover of G of size at most $k = k' - 1$. □

Note that with Observation 3.3, we already proved that Multistage Vertex Cover is NP-hard even if $\ell = 0$ and each layer contains only one edge. In order to prove Theorem 4.1(ii) (with $\ell = 1$), we adjust the polynomial-time many-one reduction behind Observation 3.3.

[1] A graph is cubic if each vertex is of degree exactly three; a graph is Hamiltonian if it contains a subgraph being a Hamiltonian cycle, that is, a cycle that visits each vertex in the graph exactly once.

Proposition 4.2 *There is a polynomial-time algorithm that maps any instance* $(G = (V, E), k)$ *of* Vertex Cover *to an equivalent instance* $(\mathcal{G}, k', \ell')$ *of* Multistage Vertex Cover *where* $\ell' = 1$ *and every layer* G_i *contains only one edge.*

Proof Let the edges $E = \{e_1, \ldots, e_m\}$ of G be arbitrarily ordered. Set $\tau = 2m$. Set $V' = V \cup W$, where $W = \{w_1, \ldots, w_{2m}\}$. Set $G_{2i-1} = (V', \{e_i\})$ and $G_{2i} = (V', \{w_i, w_{i+m}\})$ for each $i \in \{1, \ldots, m\}$. Set $k' = k + 1$ and $\ell' = 1$. We claim that $(G = (V, E), k)$ is a yes-instance of Vertex Cover if and only if $(\mathcal{G}, k', \ell')$ is a yes-instance of Multistage Vertex Cover.

($\Rightarrow$) Let S be a vertex cover of G of size at most k. Set $S_{2i-1} := S$, and $S_{2i} := S \cup \{w_i\}$ for all $i \in \{1, \ldots, m\}$. Clearly, S_i is a vertex cover of G_i for all $i \in \{1, \ldots, \tau\}$ of size at most $k' = k + 1$. Moreover, by construction, $S_i \triangle S_{i+1} \leq 1$ for all $i \in \{1, \ldots, 2\tau - 1\}$. Hence, $(S_1, \ldots, S_\tau)$ forms a solution to $(\mathcal{G}, k', \ell')$.

($\Leftarrow$) Let $\mathcal{S} = (S_1, \ldots, S_\tau)$ be a solution to $(\mathcal{G}, k', \ell')$. Observe that $|\bigcup_i S_i| \leq k + \tau$. We know that $|W \cap \bigcup_i S_i| \geq \tau$. It follows that there are at most k vertices covering all edges of the layers G_{2i-1}, that is, covering $E = \bigcup_{i=1}^m E(G_{2i-1})$, and, hence, covering all edges of G. $\square$

Theorem 4.1 directly follows from Proposition 4.1 and 4.2.

5 Parameter Vertex Cover Size

In this section, we study the parameter size k of the vertex cover of each layer for Multistage Vertex Cover. Vertex Cover and Vertex Cover Reconfiguration [25], when parameterized by the vertex cover size, are fixed-parameter tractable. We prove that this is no longer true for Multistage Vertex Cover (unless FPT = W[1]).

Theorem 5.1 Multistage Vertex Cover *parameterized by k is in XP and W[1]-hard.*

We first present the XP-algorithm (Section 5.1), and then prove the W[1]-hardness (Section 5.2) and discuss its implications.

5.1 XP-Algorithm

Here, we prove the following.

Proposition 5.1 *Every instance* $(\mathcal{G}, k, \ell)$ *of* Multistage Vertex Cover *can be decided in* $O(\tau(\mathcal{G}) \cdot |V(\mathcal{G})|^{2k+1})$ *time.*

In a nutshell, to prove Proposition 5.1 we first consider for each layer all vertex subsets of size at most k that form a vertex cover. Second, we find a sequence of vertex covers for all layers such that the sizes of the symmetric differences for every two consecutive solutions is at most ℓ. We show that the second step can be solved via computing a source-sink path in an auxiliary directed graph that we call *configuration graph* (see Fig. 2 for an illustrative example).

Definition 5.1 Given an instance $I = (\mathcal{G}, k, \ell)$ of MULTISTAGE VERTEX COVER, the *configuration graph* of I is the directed graph $D = (V, A, \gamma)$ with $V = V_1 \uplus \cdots \uplus V_\tau \uplus \{s, t\}$, being equipped with a function $\gamma : V \to \{V' \subseteq V(\mathcal{G}) \mid |V'| \leq k\}$ such that

(i) for every $i \in \{1, \ldots, \tau(\mathcal{G})\}$, it holds true that S is a vertex cover of G_i of size exactly $k-1$ or k if and only if there is a vertex $v \in V_i$ with $\gamma(v) = S$,
(ii) there is an arc from $v \in V$ to $w \in V$ if and only if $v \in V_i$, $w \in V_{i+1}$, and $\gamma(v) \triangle \gamma(w) \leq \ell$, and
(iii) there is an arc (s, v) for all $v \in V_1$ and an arc (v, t) for all $v \in V_\tau$.

Note that Mouawad et al. [25] used a similar configuration graph to show fixed-parameter tractability of VERTEX COVER RECONFIGURATION parameterized by the vertex cover size k. In the multistage setting, the configuration graph is too large for showing fixed-parameter tractability regarding k. However, we show an XP-algorithm regarding k to construct the configuration graph.

Lemma 5.1 *The configuration graph of an instance* $(\mathcal{G}, k, \ell)$ *of* MULTISTAGE VERTEX COVER, *where* $\mathcal{G}$ *has n vertices and time horizon* τ,

(i) *can be constructed in* $O(\tau \cdot n^{2k+1})$ *time, and*

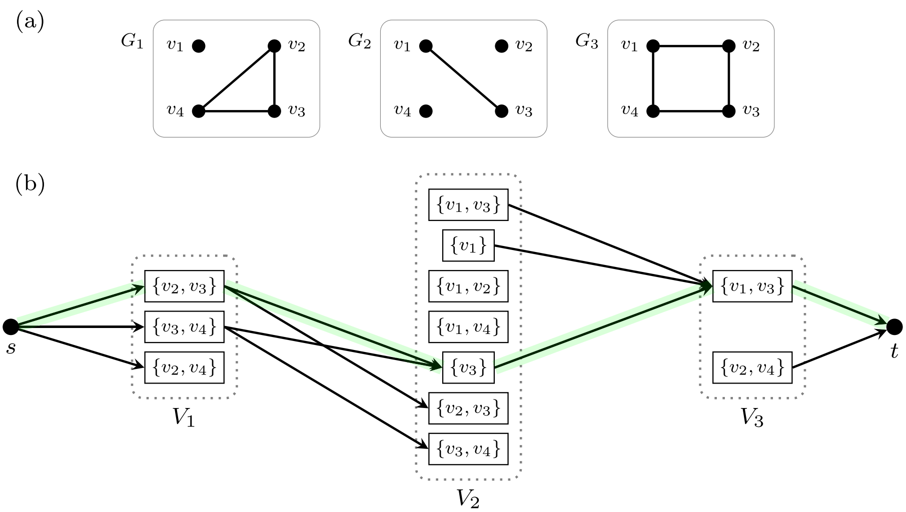

Fig. 2 Illustrative example of a configuration graph. (**a**) Temporal graph instance $I = (\mathcal{G}, k, \ell)$ from Fig. 1 with $\mathcal{G} = (G_1, G_2, G_3)$, $k = 2$, and $\ell = 1$. (**b**) Configuration graph of I from (a); a directed s-t path is highlighted corresponding to the solution depicted in Fig. 1

(ii) *contains at most* $\tau \cdot 2n^k + 2$ *vertices and* $(\tau - 1)n^{2k} + 4n^k$ *arcs.*

Proof Compute the set $\mathcal{S} = \{V' \subseteq V(\mathcal{G}) \mid k-1 \leq |V'| \leq k\}$ in $O(n^k)$ time. For each layer G_i and each set $S \in \mathcal{S}$, check in $O(|E(G_i)|)$ time whether S is a vertex cover for G_i. Let $\mathcal{S}_i \subseteq \mathcal{S}$ denote the set of vertex covers of size $k-1$ or k of layer G_i. For each $S \in \mathcal{S}_i$, add a vertex v to V_i and set $\gamma(v) = S$. Lastly, add the vertices s and t. Hence, we can construct the vertex set V of the configuration graph D of size $\tau \cdot 2n^k + 2$ in $O(n^{k+2} \cdot \tau)$ time. For every $i \in \{1, \ldots, \tau - 1\}$, and every $v \in V_i$ and $w \in V_{i+1}$, check whether $\gamma(v) \triangle \gamma(w) \leq \ell$ in $O(k)$ time. If this is the case, then add the arc (v, w). The latter steps can be done in $O(n^{2k+1} \cdot (\tau - 1))$ time, because there are at most n^{2k} arcs from V_i to V_{i+1}, for all $i \in \{1, \ldots \tau - 1\}$. Finally, add the arc (s, v) for each $v \in V_1$ and the arc (v, t) for each $v \in V_\tau$ in $O(n^k)$ time, because $|V_1|, |V_\tau| \leq n^k$. This finishes the construction of $D = (V = V_1 \uplus \cdots \uplus V_\tau \uplus \{s, t\}, A, \gamma)$. Note that we added at most $(\tau - 1)n^{2k} + 4n^k$ arcs to D. □

The crucial observation is that we can decide any instance by checking for an s-t path in its configuration graph.

Lemma 5.2 *A* Multistage Vertex Cover*-instance* $I = (\mathcal{G}, k, \ell)$ *is a* `yes`*-instance if and only if there is an* s-t *path in the configuration graph* D *of* I.

Proof Let $D = (V = V_1 \uplus \cdots \uplus V_\tau \uplus \{s, t\}, A, \gamma)$.

($\Rightarrow$) Let $(S_1, \ldots, S_\tau)$ be a solution to $(\mathcal{G}, k, \ell)$. By Observation 3.2, we can assume without loss of generality that $k - 1 \leq |S_i| \leq k$, for all $i \in \{1, \ldots, \tau\}$. Hence for each S_i, there is a $v_i \in V_i$ such that $\gamma(v_i) = S_i$, for all $i \in \{1, \ldots, \tau\}$. Note that the arc (v_i, v_{i+1}) is contained in A for each $i \in \{1, \ldots, \tau - 1\}$ since $\gamma(v_i) \triangle \gamma(v_{i+1}) = S_i \triangle S_{i+1} \leq \ell$. Hence, $P = (\{v_1, \ldots, v_\tau\} \cup \{s, t\}, \{(s, v_1), (v_\tau, t)\} \cup \bigcup_{i=1}^{\tau-1} \{(v_i, v_{i+1})\})$ is an s-t path in D.

($\Leftarrow$) Let $P = (\{v_1, \ldots, v_\tau\} \cup \{s, t\}, \{(s, v_1), (v_\tau, t)\} \cup \bigcup_{i=1}^{\tau-1} \{(v_i, v_{i+1})\})$ be an s-t path in D. We claim that $(\gamma(v_i))_{i \in \{1,\ldots,\tau\}}$ forms a solution to $(\mathcal{G}, k, \ell)$. First, note that for all $i \in \{1, \ldots, \tau\}$, $\gamma(v_i)$ is a vertex cover for G_i of size at most k. Moreover, for all $i \in \{1, \ldots, \tau - 1\}$, $\gamma(v_i) \triangle \gamma(v_{i+1}) \leq \ell$ since the arc (v_i, v_{i+1}) is present in D. This finishes the proof. □

Eventually, we are ready to prove Proposition 5.1.

Proof of Proposition 5.1 First, compute the configuration graph D of the instance $(\mathcal{G} = (V, \mathcal{E}, \tau), k, \ell)$ of Multistage Vertex Cover in $O(\tau \cdot |V|^{2k+1})$ time (Lemma 5.1(i)). Then, find an s-t path in D with a breadth-first search in $O(\tau \cdot |V|^{2k})$ time (Lemma 5.1 (ii)). If an s-t path is found, then return `yes`, otherwise return `no` (Lemma 5.2). □

Remark 5.1 The reason why the algorithm behind Proposition 5.1 is only an XP-algorithm and not an FPT-algorithm regarding k stems from the fact that we do not have a better upper bound on the number of vertices in the configuration graph for

$(\mathcal{G}, k, \ell)$ than $O(\tau(\mathcal{G}) \cdot |V(\mathcal{G})|^k)$. This is because we check for each subset of $V(\mathcal{G})$ of size k or $k-1$ whether it is a vertex cover in some layer.

This changes if we consider MINIMAL MULTISTAGE VERTEX COVER where we additionally demand the i-th set in the solution to be a *minimal* vertex cover for the layer G_i. Here, we can enumerate for each layer G_i all minimal vertex covers of size at most k (and hence all candidates for the i-th set of the solution) with the folklore search-tree algorithm for vertex cover. This leads to $O(2^k \tau(\mathcal{G}))$ many vertices in the configuration graph (for MINIMAL MULTISTAGE VERTEX COVER) and thus to fixed-parameter tractability of MINIMAL MULTISTAGE VERTEX COVER parameterized by the vertex cover size k.

It is unlikely (unless FPT=W[1]), however, that one can substantially improve the algorithm behind Proposition 5.1, as we show next.

5.2 Presumable Fixed-Parameter Intractability

Here, we show that MULTISTAGE VERTEX COVER is W[1]-hard when parameterized by k. This hardness result is established by the following parameterized reduction from the W[1]-complete CLIQUE problem [30], where, given an undirected graph G and a positive integer k, the question is whether G contains a clique of size k (that is, k vertices that are pairwise adjacent and k is the parameter).

Proposition 5.2 *There is an algorithm that maps any instance (G, k) of* CLIQUE *in polynomial time to an equivalent instance $(\mathcal{G}, k', \ell)$ of* MULTISTAGE VERTEX COVER *with $k' = 2\binom{k}{2} + k + 1$, $\ell = 2$, and each layer of $\mathcal{G}$ being a forest with $O(k^4)$ edges.*

In the remainder of this section, we prove Proposition 5.2. We next give the construction of the MULTISTAGE VERTEX COVER instance, then prove the forward (Section 5.2.1) and backward (Section 5.2.2) direction of the equivalence, and finally (in Section 5.2.3) put the pieces together and derive two corollaries.

We construct an instance of MULTISTAGE VERTEX COVER from an instance of CLIQUE as follows (see Fig. 3 for an illustrative example).

Construction 5.1 Let $(G = (V, E), k)$ be an instance of CLIQUE with $m := |E|$ and $E = \{e_1, \dots, e_m\}$. Let

$$K := \binom{k}{2}, \ k' := 2K + k + 1, \ \text{and} \quad \kappa := K + k + 3.$$

We construct a MULTISTAGE VERTEX COVER instance $(\mathcal{G}, k', \ell)$ with $\ell := 2$, where we construct the temporal graph $\mathcal{G} = (V', \mathcal{E}, \tau)$ as follows. Let V' be initialized to $V \cup E$ (note that E simultaneously describes the edge set of G and a vertex subset of $\mathcal{G}$). We add the following vertex sets

$$U^t := \{u_j^t \mid j \in \{1, \dots, K\}\} \text{ for every } t \in \{1, \dots, \kappa + 1\}, \text{ and}$$
$$C := \{c_1, \dots, c_{2m\kappa+1}\} \text{ (we refer to } C \text{ as the set of } \textit{center} \text{ vertices).}$$

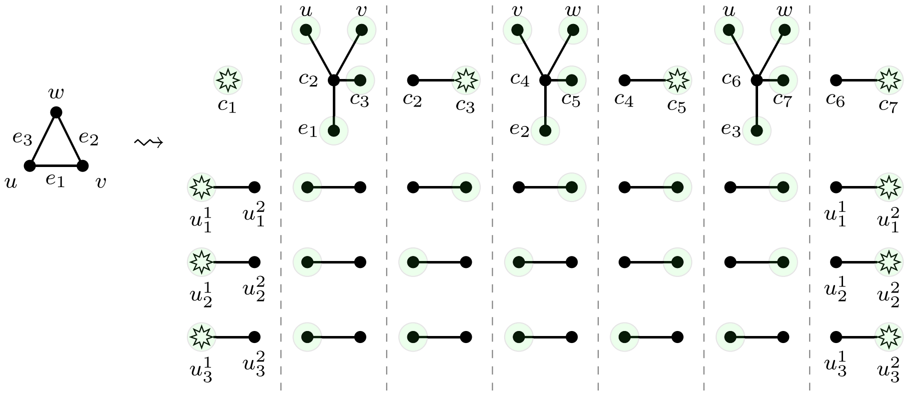

Fig. 3 Illustration of Construction 5.1 on an example graph (left-hand side) and the first seven layers of the obtained graph (right-hand side). Dashed vertical lines separate layers, and for each layer all present edges (but only their incident vertices) are depicted. Star-shapes illustrate star graphs with $k' + 1$ leaves. Vertices in a solution (layers' vertex covers) are highlighted

Let $\mathcal{E}$ be initially empty. We extend the set V' and define $\mathcal{E}$ through the $\tau := 2m\kappa + 1$ layers we construct in the following.

1. In each layer G_i with i being odd, make c_i the center of a star with $k'+1$ leaves.[2]
2. In each layer G_{2mj+1}, $j \in \{0, \dots, \kappa\}$, make each vertex in U^{j+1} the center of a star with $k' + 1$ leaves.
3. For each $j \in \{0, \dots, \kappa - 1\}$, in each layer G_{2mj+i} with $i \in \{1, \dots, 2m + 1\}$, make u_x^{j+1} adjacent to u_x^{j+2} for each $x \in \{1, \dots, K\}$.
4. For each even i, add the edge $\{c_i, c_{i+1}\}$ to G_i and to G_{i+1}.
5. For each $j \in \{0, \dots, \kappa - 1\}$, for each $i \in \{1, \dots, m\}$, in G_{2mj+2i}, make c_{j2m+2i} adjacent to $e_i = \{v, w\}$, v, and w.

This finishes the construction of $\mathcal{G}$. ♦

The construction essentially repeats the same gadget (which we call *phase*) κ times, where the layer $2m \cdot i + 1$ is simultaneously the last layer of phase i and the first layer of phase $i + 1$. In the beginning of phase i, a solution has to contain the vertices of U^i. The idea now is that during phase i one has to exchange the vertices of U^i with the vertices of U^{i+1}.

It is not difficult to see that the instance in Construction 5.1 can be computed in polynomial time. Hence, it remains to prove the equivalence stated in Proposition 5.2. We prove the forward and the backward direction in Sections 5.2.1 and 5.2.2, respectively, and finally prove Proposition 5.2 in Section 5.2.3.

[2] A star (graph) is a tree where at most one vertex (the so-called center) has degree larger than one.

5.2.1 Forward Direction

The forward direction of Proposition 5.2 is—in a nutshell—as follows: If $V' \cup E'$ with $V' \subseteq V$ and $E' \subseteq E$ corresponds to the vertex set and edge set of a clique of size k, then there are K layers in each phase covered by $V' \cup E'$. Hence, having K layers where no vertices from C have to be exchanged, in each phase t we can exchange all vertices from U^t to U^{t+1}. Starting with set $S_1 := U^1 \cup V' \cup E' \cup \{c_1\}$ then yields a solution.

Lemma 5.3 *Let (G, k) be an instance of* CLIQUE *and $(\mathcal{G}, k', \ell)$ be the instance of* MULTISTAGE VERTEX COVER *resulting from Construction 5.1. If (G, k) is a* `yes`*-instance, then $(\mathcal{G}, k', \ell)$ is a* `yes`*-instance.*

Proof Let $G' = (V', E')$ be the clique of size k in G. We construct a solution $\mathcal{S} = (S_1^1, \ldots, S_{2m}^1, S_{2m+1}^1 = S_1^2, \ldots, S_{2m+1}^\kappa = S_1^{\kappa+1})$ for $(\mathcal{G}, k', \ell)$ in the following way. For each $t \in \{1, \ldots, \kappa + 1\}$ we set $S_1^t := V' \cup E' \cup U^t \cup \{c_{(t-1)2m+1}\}$, which is a vertex cover of size k' for $G_{(t-1)2m+1}$.

Now, for each $t \in \{1, \ldots, \kappa\}$, we iteratively construct vertex covers for the layers $(t-1)2m + 2$ until $t2m$ in the following way. Let $T := (t-1) \cdot 2m$. Let $i \in \{1, \ldots, 2m-1\}$, and assume that the set S_i^t is already constructed and is a vertex cover for G_{T+i} (this is possible due to the definition of S_1^t). We distinguish two cases.

Case 1: i is odd. We know that $c_{T+i} \in S_i^t$. If $(S_i^t \setminus \{c_{T+i}\}) \cup \{c_{T+i+2}\}$ is a vertex cover for G_{T+i+1}, then we set $S_{i+1}^t := (S_i^t \setminus \{c_{T+i}\}) \cup \{c_{T+i+2}\}$. Otherwise we set $S_{i+1}^t := (S_i^t \setminus \{c_{T+i}\}) \cup \{c_{T+i+1}\}$. In both cases S_{i+1}^t is a vertex cover for G_{T+i+1} and either $S_{i+1}^t \cap C = \{c_{T+i+1}\}$ or $S_{i+1}^t \cap C = \{c_{T+i+2}\}$.

Case 2: i is even. We know that c_{T+i} or c_{T+i+1} is in S_i^t. If $c_{T+i} \in S_i^t$, then we set $S_{i+1}^t = (S_i^t \setminus \{c_{T+i}\}) \cup \{c_{T+i+1}\}$, which is a vertex cover for G_{T+i+1}. If $c_{T+i+1} \in S_i^t$, then S_i^t is already a vertex cover for G_{T+i+1} and the vertices in $V' \cup E'$ cover all edges incident with c_{T+i} in the graph G_{T+i}. In this case we say that G' *covers* the layer $T + i$ and set $S_{i+1}^t = (S_i^t \setminus \{u_j^t\}) \cup \{u_j^{t+1}\}$, where u_j^t is an arbitrary vertex in $S_i^t \cap U^t$.

Observe that the clique G' covers K even-numbered layers in each phase. Hence, we replace, during phase $t \in \{1, \ldots, \kappa\}$ (that is, from layer $(t-1)2m+1$ to $t2m+1$), the vertices U^t with the vertices U^{t+1}. This also implies that the symmetric difference of two consecutive sets in $\mathcal{S}$ is exactly $2 = \ell$. It follows that $\mathcal{S}$ is a solution for $(\mathcal{G}, k', \ell)$. □

5.2.2 Backward Direction

In this section, we prove the backward direction for the proof of Proposition 5.2. We first show that if an instance of MULTISTAGE VERTEX COVER computed by Construction 5.1 is a `yes`-instance, then it is safe to assume that two vertices are neither deleted from nor added to a vertex cover in a consecutive step (we refer to these solutions as *smooth*, see Definition 5.2). Moreover, a vertex from the vertex set C is only exchanged with another vertex from C and, at any time, there is exactly one vertex from C contained in the solution (similarly to the constructed solution in Lemma 5.3. We call these (smooth) solutions *one-centered* (Definition 5.3). We then prove that there must be a phase t for any one-centered solution where at least $\binom{k}{2}$ times a vertex from "past" sets $U_{t'}$, $t' \leq t$, is deleted. This at hand, we prove that such a phase witnesses a clique of size k.

The fact that a solution needs to contain at least one vertex from C at any time immediately follows from the fact that there is either an edge between two vertices in C or there is a vertex in C which is the center of a star with $k'+1$ leaves.

Observation 5.2 Let $(\mathcal{G}, k', \ell)$ from Construction 5.8 be a yes-instance. Then for each solution $(S_1, \ldots, S_\tau)$ it holds true that $|S_i \cap C| \geq 1$ for all $i \in \{1, \ldots, \tau(\mathcal{G})\}$.

In the remainder of this section, we denote the vertices which are removed from the set S_{i-1} and added to the next set S_i in a solution $\mathcal{S} = (\ldots, S_{i-1}, S_i, \ldots)$ by

$$S_{i-1} \diamond S_i := (S_{i-1} \setminus S_i, S_i \setminus S_{i-1}).$$

If $S_{i-1} \setminus S_i$ or $S_i \setminus S_{i-1}$ have size one, then we will omit the brackets of the singleton.

Definition 5.2 A solution $\mathcal{S} = (S_1, \ldots, S_\tau)$ for $(\mathcal{G}, k', \ell)$ from Construction 5.8 is *smooth* if for all $i \in \{2, \ldots, \tau\}$ we have $|S_{i-1} \setminus S_i| \leq 1$ and $|S_i \setminus S_{i-1}| \leq 1$.

In fact, if there is a solution, then there is also a smooth solution.

Observation 5.3 Let $(\mathcal{G}, k', \ell = 2)$ from Construction 5.1 be a yes-instance. Then there is a smooth solution $(S_1, \ldots, S_\tau)$.

Proof By Observation 3.1, we know that there is a solution $\mathcal{S} = (S_1, \ldots, S_\tau)$ such that $|S_1| = k'$ and $k' - 1 \leq |S_i| \leq k'$ for all $i \in \{1, \ldots, \tau\}$. Hence, for all $i \in \{2, \ldots, \tau\}$ it holds true that $\big||S_i| - |S_{i-1}|\big| \leq 1$. By Construction 5.8, we have that $|S_i \triangle S_{i-1}| \leq \ell = 2$, for all $i \in \{2, \ldots, \tau\}$. It follows that $|S_{i-1} \setminus S_i| \leq 1$ and $|S_i \setminus S_{i-1}| \leq 1$, and thus, $\mathcal{S}$ is a smooth solution. □

Our next goal is to prove the existence of the following type of solutions.

Definition 5.3 A smooth solution $\mathcal{S} = (S_1, \ldots, S_\tau)$ for $(\mathcal{G}, k', \ell)$ from Construction 5.1 is *one-centered* if

1. for all $i \in \{1, \ldots, \tau\}$ it holds true that $|S_i \cap C| = 1$, and
2. for all $i \in \{2, \ldots, \tau\}$ and $S_{i-1} \diamond S_i = (\alpha, \beta)$ it holds true that $\alpha \in C \iff \beta \in C$.

With the next two lemmata, we prove that in case of a yes-instance, a one-centered solution exists. We first show that if the output instance of Construction 5.1

is a yes-instance, then there is a solution where $c_1 \in C$ is the only vertex from C in the first set of the solution.

Lemma 5.4 *Let $(\mathcal{G}, k', \ell)$ from Construction 5.1 be a yes-instance. Then there is a smooth solution $(S_1, \ldots, S_\tau)$ for $(\mathcal{G}, k', \ell)$ such that $S_1 \cap C = \{c_1\}$.*

Proof Suppose towards a contradiction that such a smooth solution does not exist. Since we know from Observation 5.2 that there is at least one smooth solution, our assumption means that, in every smooth solution the first vertex cover S_1 contains at least two vertices from C (due to Observation 5.1, S_1 must contain at least one). Let Ψ be the set of smooth solutions with $|S_1 \cap C|$ being minimal, where S_1 is the first vertex cover. Let $\mathcal{S} = (S_1, \ldots, S_\tau) \in \Psi$ be a smooth solution such that the value $i := \min\{j \in \{1, \ldots, \tau\} \mid c_j \in S_1 \setminus \{c_1\}\}$ is maximal. Let $\mathcal{S}' = (S'_1, \ldots, S'_\tau)$ be initially $\mathcal{S}$.

Suppose that c_i was moved out of the solution before the i-th layer. That is, there is a $j \in \{1, \ldots, i-1\}$ such that $S_j \diamond S_{j+1} = (c_i, \alpha)$. Let $j' := \min\{j \in \{1, \ldots, i-1\} \mid S_j \diamond S_{j+1} = (c_i, \alpha)\}$ be the smallest among them. Then, set $S'_q := S_q \setminus \{c_i\}$ for all $q \in \{1, \ldots, j'-1\}$ to get a feasible solution (note that $S'_{j'-1} \diamond S'_{j'} = (\emptyset, \alpha)$ is feasible since $|S'_{j'-1}| \leq k-1$). This contradicts the minimality of $\mathcal{S}$ regarding $|S_1 \cap C|$.

Hence, suppose that there is no such j, that is, there is no $j \in \{1, \ldots, i-1\}$ such that $S_j \diamond S_{j+1} = (c_i, \alpha)$. If $S_{i^*} \setminus \{c_i\}$ is a vertex cover of layer G_{i^*} for all $i^* \leq i$, then setting $S'_q := S_q \setminus \{c_i\}$, for all $q \in \{1, \ldots, p\}$ with $p := \max\{p' \in \{1, \ldots, \tau\} \mid \forall q \in \{1, \ldots, p'\} : c_i \in S_q\}$, yields a feasible solution. This contradicts the minimality of $\mathcal{S}$ regarding $|S_1 \cap C|$.

Finally, suppose that there is no $j \in \{1, \ldots, i-1\}$ such that $S_j \diamond S_{j+1} = (c_i, \alpha)$ (and hence $c_i \in S_i$) and $S_{i^*} \setminus \{c_i\}$ is no vertex cover of layer G_{i^*}, with $i^* \leq i$ smallest possible. Note that $i^* \in \{i-1, i\}$, and we distinguish the two cases.

Case 1: $i^* = i$. Let $S_{i-1} \diamond S_i = (\alpha, \beta)$ for some α, β (each being possibly the empty set). Note that $\beta \neq c_i$. Then for all $q \in \{1, \ldots, i-1\}$ do the following (we distinguish two cases):

Case 1.1: $\beta = c_r$ with $r < i$. We remove c_i from the first $i-1$ vertex covers and β from all vertex covers after i containing β. Formally, set $S'_q := S_q \setminus \{c_i\}$ and $S'_{q'} := S_{q'} \setminus \{\beta\}$ (i.e. $S'_{i-1} \diamond S'_i = (\alpha, c_i)$) for all $q' \in \{i, \ldots, p\}$ with $p := \max\{p' \in \{1, \ldots, \tau\} \mid \forall p'' \in \{i, \ldots, p'\} : \beta \in S_{p''}\}$. Recall that c_i is dispensable from the first $i-1$ vertex covers, $S_i \setminus \{\beta\}$ is a vertex cover of G_i (since, if i is odd, then c_i is the center of a star, or if i even, then β is isolated), and β is isolated in all layers after the i-th one. Moreover, from the $(i-1)$-st to i-th vertex cover, we exchange α by c_i instead of α by β. Hence, the obtained sequence is a solution. This contradicts the minimality of $\mathcal{S}$ regarding $|S_1 \cap C|$.

Case 1.2: $\beta = c_r$ with $r > i$, or $\beta \notin C$. We replace c_i by β in the first $i-1$ vertex covers. Formally, set $S'_q := (S_q \setminus \{c_i\}) \cup \{\beta\}$ (note that $S'_i = S_i$ and hence $S'_{i-1} \diamond S'_i = (\alpha, c_i)$). Note that if there is a $p \in \{2, \ldots, i-1\}$ with $S_{p-1} \diamond S_p = (\beta, x)$ or $S_{p-1} \diamond S_p = (x, \beta)$, then we get $S'_{p-1} \diamond S'_p = (\emptyset, x)$ and $S'_{p-1} \diamond S'_p = (x, \emptyset)$, respectively. Recall that c_i is dispensable from the first $i-1$ vertex covers, and β is still contained in each vertex cover it contained before the modification. Moreover, from the $(i-1)$-st to the i-th vertex cover, we exchange α by c_i instead of α by β. Hence, the obtained sequence is a solution. In the case of $\beta = c_r$ with $r > i$, this contradicts the fact that c_i is maximal regarding i. In the case of $\beta \notin C$, this contradicts the minimality of $\mathcal{S}$ regarding $|S_1 \cap C|$.

Case 2: $i^* = i-1$. It follows that i is odd. Let $S_{i-2} \diamond S_{i-1} = (\alpha, \beta)$ for some α, β (each being possibly the empty set). Note that $\beta \neq c_i$. Since $S_{i-1} \setminus \{c_i\}$ is not a vertex cover of G_{i-1}, we have that $c_{i-1} \notin S_{i-1}$. It follows that $\beta \neq c_{i-1}$. Then for all $q \in \{1, \dots, i-2\}$ do the following (we distinguish two cases):

Case 2.1: $\beta = c_r$ with $r < i-1$. We remove c_i from the first $i-2$ vertex covers and β from all vertex covers after $(i-2)$-nd containing β. Formally, set $S'_q := S_q \setminus \{c_i\}$ and $S'_{q'} := S_{q'} \setminus \{\beta\}$ (i.e. $S'_{i-2} \diamond S'_{i-1} = (\alpha, c_i)$) for all $q' \in \{i-1, \dots, p\}$ with $p := \max\{p' \in \{1, \dots, \tau\} \mid \forall p'' \in \{i-1, \dots, p'\} : \beta \in S_{p''}\}$. Recall that c_i is not part of any of the first $i-2$ vertex covers and β is isolated in all layers after the $(i-2)$-nd one. Moreover, from the $(i-2)$-nd to i-th vertex cover, we exchange α by c_i instead of α by β. Hence, the obtained sequence is a solution. This contradicts the minimality of $\mathcal{S}$ regarding $|S_1 \cap C|$.

Case 2.2: $\beta = c_r$ with $r > i$, or $\beta \notin C$. We replace c_i by β in the first $i-2$ vertex covers. Formally, set $S'_q := (S_q \setminus \{c_i\}) \cup \{\beta\}$ (note that $S'_{i-1} = S_{i-1}$ and hence $S'_{i-2} \diamond S'_{i-1} = (\alpha, c_i)$). Note that if there is a $p \in \{2, \dots, i-1\}$ with $S_{p-1} \diamond S_p = (\beta, x)$ or $S_{p-1} \diamond S_p = (x, \beta)$, then we get $S'_{p-1} \diamond S'_p = (\emptyset, x)$ and $S'_{p-1} \diamond S'_p = (x, \emptyset)$, respectively. Note that c_i is isolated in the first $i-2$ layers, and β is still contained in each vertex cover it was contained in before the modification. Moreover, from the $(i-2)$-nd to the $(i-1)$-st vertex cover, we exchange α by c_i instead of α by β. Hence, the obtained sequence is a solution. If $\beta = c_r$ with $r > i$, then this contradicts the fact that c_i is maximal regarding i. If $\beta \notin C$, then this contradicts the minimality of $\mathcal{S}$ regarding $|S_1 \cap C|$.

In every case, we obtain a contradiction, concluding the proof. □

Next, we show that there are solutions such that whenever we remove a vertex in the set of center vertices C from the vertex cover, then we simultaneously add another vertex from C to the vertex cover. Formally, we prove the following.

Lemma 5.5 *Let $(\mathcal{G}, k', \ell)$ from Construction 5.1 be a* yes*-instance. Then there is a smooth solution $(S_1, \dots, S_\tau)$ with $S_1 \cap C = \{c_1\}$ such that for all $i \in \{1, \dots, \tau\}$ with $S_{i-1} \diamond S_i = (\alpha, c)$ and $c \in C$ we also have $\alpha \in C$.*

Proof Suppose towards a contradiction the contrary. That is, let for every smooth solution $(S_1, \dots, S_\tau)$ exist an $i \in \{1, \dots, \tau\}$ with $S_{i-1} \diamond S_i = (\alpha, c)$ and $c \in C$ and $\alpha \notin C$. Let Ψ be the non-empty (due to Lemma 5.4) set of smooth solutions $(S_1, \dots, S_\tau)$ with $|S_1 \cap C| = 1$. Let $\Psi' \subseteq \Psi$ be the set of smooth solutions that maximizes the first index i with $S_{i-1} \diamond S_i = (\alpha, c_q)$ with $c_q \in C$ and $\alpha \notin C$. Among those solutions, consider $\mathcal{S} = (S_1, \dots, S_\tau) \in \Psi'$ to be the one with q being maximal.

Note that due to Observation 5.10, we have that $|S_{i-1} \cap C| \geq 1$. Let $S'_j := S_j$ for all $j \in \{1, \dots, \tau\}$.

Case 1: $i > 1$ is odd. Since c_i is the center of a star in layer i, c_i has to be in S_i. We distinguish three subcases regarding the relation of q and i, that is, the cases of q being smaller than, equal to, or larger than i.

Case 1.1: $q < i$. We remove c_q from the longest sequence of vertex covers containing c_q starting from the i-th vertex cover. Formally, set $S'_j := (S_j \setminus \{c_q\})$ (i.e., $S'_{i-1} \diamond S'_i = (\alpha, \emptyset)$) for all $j \in \{i, \dots, q'\}$ with $q' := \max\{q'' \in \{i, \dots, \tau\} \mid \forall j \in \{i, \dots, q''\} : c_q \in S_j\}$. Note that c_q with $q < i$ is isolated in all layers after the i-th one and $S_i \setminus \{c_q\}$ is a vertex cover of layer G_i since $c_i \in S_i$. It follows that $(S'_1, \dots, S'_\tau)$ is again a feasible smooth solution contradicting i being maximal.

Case 1.2: $q = i$. We have $c_i \notin S_{i-1}$. Since the edge $\{c_{i-1}, c_i\}$ must be covered in layer G_{i-1}, it follows that $c_{i-1} \in S_{i-1}$. Moreover, $c_{i-1} \in S_i$ (and possibly more subsequent vertex covers) since $\alpha \notin C$. As c_{i-1} is isolated in all layers after the i-th and $c_i \in S_i$, we remove c_{i-1} and add α in the i-th and subsequent layers (i.e., instead of exchanging α with c_q from the $(i-1)$-st to the i-th layer, we exchange c_{i-1} with c_q). Formally, set $S'_p := (S_p \setminus \{c_{i-1}\}) \cup \{\alpha\}$ (i.e., $S'_{i-1} \diamond S'_i = (c_{i-1}, c_q)$) for all $p \in \{i, \dots, j\}$, where $j > i$ is minimal such that $S_{j-1} \diamond S_j = (c_{i-1}, x)$, or τ if such a j does not exist. If there is a minimal $j > i$ such that $S_{j-1} \diamond S_j = (c_{i-1}, x)$, then set $S'_p := (S_p \setminus \{\alpha\})$ (i.e., $S'_{j-1} \diamond S'_j = (\alpha, x)$) for all $p \in \{j, \dots, q'\}$ with $q' := \max\{q'' \in \{i, \dots, \tau\} \mid \forall p \in \{i, \dots, q''\} : \alpha \in S_p\}$. Suppose that between i and j, there are j_1 and j_2 such that $S_{j_1-1} \diamond S_{j_1} = (y, \alpha)$ and $S_{j_2-1} \diamond S_{j_2} = (\alpha, y')$. Note that $S'_{j_1-1} \diamond S'_{j_1} = (y, \emptyset)$ and $S'_{j_1-1} \diamond S'_{j_1} = (\emptyset, y')$. It follows that $(S'_1, \dots, S'_\tau)$ is again a feasible smooth solution, contradicting i being maximal.

Case 1.3: $q > i$. Then $c_i \in S_{i-1}$. Let $q^* \leq q$ be the smallest index with $S_{q^*} \setminus \{c_q\}$ being no vertex cover of G_{q^*}. Note that $q^* \in \{q-1, q\}$. Observe that if no such q^* exists, then we can exclude c_q from all vertex cover starting from the i-th one. Formally, set $S'_j := S_j \setminus \{c_q\}$ (i.e., $S'_{i-1} \diamond S'_i = (\alpha, \emptyset)$) for all $j \in \{i, \dots, q'\}$ where $q' := \max\{q'' \in \{i, \dots, \tau\} \mid \forall j \in \{i, \dots, q''\} : c_q \in S_j\}$, contradicting the fact that i is maximal. We distinguish two cases:

Case 1.3.1: $q^* = q$. Let $S_{q-1} \diamond S_q = (\beta, d)$. Note that if $d = c_q$, then we remove c_q from all vertex covers before the q-th one, yielding a contradiction to i being maximal. We distinguish two cases regarding d.

Case 1.3.1.1: $d = c_p$ with $p < q$. Instead of adding c_q to the i-th vertex cover, we add c_q instead of d to the q-th vertex cover (recall that c_q is dispensable from all vertex covers before the q-th). Note that if q is even, then d is isolated in all layers after the $(q-1)$-st one, and if q is odd, then $S_q \setminus \{d\}$ is a vertex cover of G_q (since c_q is added to S_q) and d is isolated in all layers after the q-th one. Formally, set $S'_j := S_j \setminus \{c_q\}$ (i.e., $S'_{i-1} \diamond S'_i = (\alpha, \emptyset)$) for all $j \in \{i, \dots, q-1\}$. Moreover, set $S'_j :=$

$(S_j \setminus \{d\}) \cup \{c_q\}$ (i.e., $S'_{q-1} \diamond S'_q = (\beta, c_q)$) for all $j \in \{q, \dots, q'\}$ with $q' := \max\{q'' \in \{q, \dots, \tau\} \mid \forall j \in \{q, \dots, q''\} : d \in S_j\}$. The obtained sequence is a solution.

Case 1.3.1.2: $d \notin C$ or if $d = c_p$, then $p > q$. Instead of introducing c_q in the i-th and d in the q-th vertex cover, we swap their timings and introduce d in the i-th vertex cover and c_q in the q-th vertex cover. Set $S'_j = (S_j \setminus \{c_q\}) \cup \{d\}$ (i.e., $S'_{i-1} \diamond S'_i = (\alpha, d)$) for all $j \in \{i, \dots, q-1\}$. Moreover, set $S'_j = S_j \cup \{c_q\}$ (i.e., $S'_{q-1} \diamond S'_q = (\beta, c_q)$ or $S'_{q-1} \diamond S'_q = (\beta, \emptyset)$) for all $j \in \{q, \dots, q'\}$ with $q' := \max\{q'' \in \{q, \dots, \tau\} \mid \forall j \in \{q, \dots, q'\} : c_q \in S_j\}$. Recall that c_q is dispensable from all vertex covers before the q-th one. Hence, the obtained sequence is a solution.

In either case, we have that $(S'_1, \dots, S'_\tau)$ is a feasible solution contradicting either i being maximal ($d \notin C$, or $d = c_p$ with $p < q$) or q being maximal ($d = c_p$ with $p > q$).

Case 1.3.2: $q^* = q - 1$. It follows that q must be odd and that $c_{q-1} \notin S_{q-1}$. Let $S_{q-2} \diamond S_{q-1} = (\beta, d)$. Note that $d \neq c_{q-1}$ and that if $d = c_q$, then we remove c_q from all vertex covers before the $(q-1)$-st one, yielding a contradiction to i being maximal. We distinguish two cases regarding d.

Case 1.3.2.1: $d = c_p$ with $p < q - 1$. Instead of adding c_q to the i-th vertex cover, we add c_q instead of d to the $(q-1)$-st vertex cover. Formally, set $S'_j = S_j \setminus \{c_q\}$ (i.e., $S'_{i-1} \diamond S'_i = (\alpha, \emptyset)$) for all $j \in \{i, \dots, q-2\}$. Moreover, set $S'_j = (S_j \setminus \{d\}) \cup \{c_q\}$ (i.e., $S'_{q-2} \diamond S'_{q-1} = (\beta, c_q)$) for all $j \in \{q-1, \dots, q'\}$ with $q' := \max\{q'' \in \{q-1, \dots, \tau\} \mid \forall j \in \{q-1, \dots, q''\} : d \in S_j\}$. Since $d = c_p$ for $p < q-1$ and q is odd, d is isolated in all layers after the $(q-2)$-nd. Hence, the obtained sequence is a solution.

Case 1.3.2.2: $d \notin C$ or if $d = c_p$, then $p > q$. Instead of introducing c_q in the i-th and d in the $(q-1)$-st vertex cover, we swap their timings and introduce d in the i-th vertex cover and c_q in the $(q-1)$-st vertex cover. Set $S'_j := (S_j \setminus \{c_q\}) \cup \{d\}$ (i.e., $S'_{i-1} \diamond S'_i = (\alpha, d)$) for all $j \in \{i, \dots, q-2\}$. Moreover, set $S'_j := S_j \cup \{c_q\}$ (i.e., $S'_{q-2} \diamond S'_{q-1} = (\beta, c_q)$ or $S'_{q-2} \diamond S'_{q-1} = (\beta, \emptyset)$) for all $j \in \{q-1, \dots, q'\}$ with $q' := \max\{q'' \in \{q-1, \dots, \tau\} \mid \forall j \in \{q-1, \dots, q'\} : c_q \in S_j\}$. Note that c_q is isolated before the $(q-1)$-st layer, and d appears in a superset of vertex covers after the modification. Hence, the obtained sequence is a solution.

In either case, we have that $(S'_1, \dots, S'_\tau)$ is a feasible solution contradicting either i being maximal ($d \notin C$, or $d = c_p$ with $p < q$) or q being maximal ($d = c_p$ with $p > q$).

Case 2: $i > 1$ is even. Then $c_{i-1} \in S_{i-1}$ and $c_q \in \{c_i, c_{i+1}\}$. Since c_{i-1} is isolated in all layers after the $(i-1)$-st, we can exchange c_{i-1} instead of α by c_q. Formally, set $S'_j := (S_j \setminus \{c_{i-1}\}) \cup \{\alpha\}$ (i.e., $S'_{i-1} \diamond S'_i = (c_{i-1}, c_q)$) for all $j \in \{i, \dots, q'\}$

with $q' := \max\{q'' \in \{i, \ldots, \tau\} \mid \forall j \in \{i, \ldots, q''\} : c_{i-1} \in S_j\}$. Then $(S'_1, \ldots, S'_\tau)$ is a feasible solution contradicting i being maximal.

□

Combining Observation 5.1 and Lemma 5.5, we can assume that for every given `yes`-instance, there is a solution which is one-centered.

Corollary 5.1 *Let $(\mathcal{G}, k', \ell)$ from Construction 5.1 be a `yes`-instance. Then there is a solution $\mathcal{S}$ which is one-centered.*

In the remainder of this section, for each $t \in \{1, \ldots, \kappa + 1\}$ let the union of all U^i be denoted by

$$\widehat{U}_t := \bigcup_{i=1}^{t} U^i.$$

We introduce further notation regarding a one-centered solution $\mathcal{S} := (S_1^1, \ldots, S_{2m+1}^1 = S_1^2, \ldots, \ldots, \ldots, S_1^\kappa, \ldots, S_{2m+1}^\kappa)$ for $(\mathcal{G}, k', \ell)$. Here, S_i^t is the i-th set of phase t and thus the $(2m(t-1) + i)$-th set of $\mathcal{S}$. The set

$$Y_i^t := \{e_j \in S_i^t \cap E \mid 2j \geq i\} \tag{1}$$

is the set of vertices e_j from E in S_i^t such that the corresponding layer for e_j in phase t is not before the layer i in phase t. The set

$$F_i^t := \{j > i \mid S_{j-1}^t \diamond S_j^t = (u, \beta) \text{ with } u \in \widehat{U}_t\} \tag{2}$$

is the set of layers from $\mathcal{G}$ in phase t where a vertex from $\widehat{U}_t$ is not carried over to the next layer's vertex cover. We now show that there is a phase t where $|F_1^t| \geq K$.

Lemma 5.6 *Let $\mathcal{S} = (S_1^1, \ldots, S_{2m+1}^1 = S_1^2, \ldots, \ldots, \ldots, S_1^\kappa, \ldots, S_{2m+1}^\kappa)$ be a one-centered solution to $(\mathcal{G}, k', \ell)$ from Construction 5.1. Then, there is a $t \in \{1, \ldots, \kappa\}$ such that $|F_1^t| \geq K$.*

Proof Suppose towards a contradiction that for all $t \in \{1, \ldots, \kappa\}$ it holds true that $|F_1^t| < K$. Then, for each $i \in \{2, \ldots, \kappa + 1\}$, we have that $|S_1^i \cap \widehat{U}_{i-1}| \geq i - 1$. Since $\mathcal{S}$ is a solution, we know that $U^{\kappa+1} \subseteq S_1^{\kappa+1}$ and hence $|S_1^{\kappa+1} \cap U^{\kappa+1}| = K$. Thus, we have that

$$|S_1^{\kappa+1}| \geq |S_1^{\kappa+1} \cap U^{\kappa+1}| + |S_1^{\kappa+1} \cap \widehat{U}_\kappa| \geq K + \kappa - 1 = 2K + k + 2 > k',$$

contradicting $\mathcal{S}$ being a solution. □

In the remainder of this section, the value

$$f_i^t := |S_i^t \cap \widehat{U}_{\kappa+1}| - K \tag{3}$$

describes the number of vertices in $\widehat{U}_{\kappa+1}$ which we could remove from S_i^t such that S_i^t is still a vertex cover for $G_{2m(t-1)+i}$ (the i-th layer of phase t). Observe that

$f_i^t \geq 0$ for all $t \in \{1, \ldots, \kappa\}$ and all $i \in \{1, \ldots, 2m+1\}$, because we need in each layer exactly K vertices from $\widehat{U}_{\kappa+1}$ in the vertex cover.

We now derive an invariant which must be true in each phase.

Lemma 5.7 *Let* $\mathcal{S} = (S_1^1, \ldots, S_{2m+1}^1 = S_1^2, \ldots, \ldots, S_1^\kappa, \ldots, S_{2m+1}^\kappa)$ *be a one-centered solution to* $(\mathcal{G}, k', \ell)$ *from Construction 5.1. Then, for all* $t \in \{1, \ldots, \kappa\}$ *and all* $i \in \{1, \ldots, 2m+1\}$*, it holds true that* $|F_i^t| - |Y_i^t| \leq f_i^t$*.*

Proof Let $t \in \{1, \ldots, \kappa\}$ be arbitrary but fixed. For all $i \in \{1, \ldots, 2m+1\}$ let

$$\varepsilon_i := |F_i^t| - |Y_i^t| - f_i^t.$$

We claim that $\varepsilon_i - \varepsilon_{i-1} \geq 0$ for all $i \in \{1, \ldots, 2m+1\}$. Since $\mathcal{S}$ is one-centered, in Table 2 all relevant tuples for $S_{i-1}^t \diamond S_i^t$ are shown. As each relevant tuple results in $\varepsilon_i - \varepsilon_{i-1} \in \{0, 1, 2\}$, the claim follows.

We want to prove that $\varepsilon_i \leq 0$ for all $i \in \{1, \ldots, 2m+1\}$. So, assume towards a contradiction that there is a $j \in \{1, \ldots, 2m+1\}$ such that $\varepsilon_j > 0$. Since $\varepsilon_i - \varepsilon_{i-1} \geq 0$ for all $i \in \{1, \ldots, 2m+1\}$, we have that $\varepsilon_{2m+1} > 0$, which is equivalent to $|F_{2m+1}^t| - |Y_{2m+1}^t| > f_{2m+1}^t$. By definition, we have that $|Y_{2m+1}^t| = 0$ (see (1)) and $|F_{2m+1}^t| = 0$ (see (2)). Moreover, since $\mathcal{S}$ is a solution and each vertex cover needs at least K vertices from $\widehat{U}_\tau$, we have that $f_{2m+1}^t \geq 0$. It follows that $0 = |F_{2m+1}^t| - |Y_{2m+1}^t| > f_{2m+1}^t \geq 0$, yielding a contradiction. □

Next, we prove that in a phase t with $|F_1^t| \geq K$, there are at most k vertices from V contained in the union of the vertex covers of phase t.

Lemma 5.8 *Let* $\mathcal{S} = (S_1^1, \ldots, S_{2m+1}^1 = S_1^2, \ldots, \ldots, S_1^\kappa, \ldots, S_{2m+1}^\kappa)$ *be a one-centered solution to* $(\mathcal{G}, k', \ell)$ *from Construction 5.1, and let* $t \in \{1, \ldots, \kappa\}$ *be such that* $|F_1^t| \geq K$*. Then,* $|\bigcup_{i=1}^{2m+1} S_i^t \cap V| \leq k$*.*

Table 2 Overview of all tuples of $S_{i-1}^t \diamond S_i^t$ relevant in the proof of Lemma 5.17 and their possible values of $\varepsilon_i - \varepsilon_{i-1} = |F_i^t| - |F_{i-1}^t| - (|Y_i^t| - |Y_{i-1}^t|) - (f_i^t - f_{i-1}^t)$

$S_{i-1}^t \diamond S_i^t$		$\lvert F_i^t\rvert - \lvert F_{i-1}^t\rvert$	$-(\lvert Y_i^t\rvert - \lvert Y_{i-1}^t\rvert)$	$-(f_i^t - f_{i-1}^t)$	$\varepsilon_i - \varepsilon_{i-1}$
(u, β)	$\beta \in E$	$\in \{-1, 0\}$	$\in \{0, 1\}$	1	$\in \{0, 1, 2\}$
	$\beta \in \widehat{U}_{\kappa+1}$	$\in \{-1, 0\}$	1	0	$\in \{0, 1\}$
	$\beta \in V, \beta = \emptyset$	$\in \{-1, 0\}$	1	1	$\in \{1, 2\}$
(α, u)	$\alpha \in E$	0	$\in \{1, 2\}$	-1	$\in \{0, 1\}$
	$\alpha \in V, \alpha = \emptyset$	0	1	-1	0
(α, v)	$\alpha \in E$	0	$\in \{1, 2\}$	0	$\in \{1, 2\}$
	$\alpha \in V, \alpha = \emptyset$	0	1	0	1
(α, e)	$\alpha \in V$	0	1	0	1
	$\alpha \in E, \alpha = \emptyset$	0	$\in \{0, 1\}$	0	$\in \{0, 1\}$

In the tuples, u, v, and e represent some vertices from $\widehat{U}_{\kappa+1}$, V, and E, respectively

Proof From Lemma 5.7, we know that $|Y_1^t| \geq K - f_1^t$. Let

$$|Y_1^t| = K - f_1^t + \lambda$$

for some $\lambda \in \mathbb{N}_0$, and let $\varepsilon_i = |F_i^t| - |Y_i^t| - f_i^t$, for all $i \in \{1, \ldots, 2m+1\}$.

We now show that there are at most λ layers where we exchange a vertex currently in the vertex cover with a vertex in V. Let $i \in \{2, \ldots, 2m+1\}$ such that $S_{i-1}^t \diamond S_i^t = (\alpha, v)$ with $v \in V$. From Table 2 (recall that one-centered solutions are smooth), we know that $\varepsilon_i \geq \varepsilon_{i-1} + 1$.

Assume towards a contradiction that there are $\lambda + 1$ many of these exchanges. Then, there is a $j \in \{1, \ldots, 2m+1\}$ such that

$$\begin{aligned}\varepsilon_j &\geq \varepsilon_1 + \lambda + 1 = |F_1^t| - |Y_1^t| - f_1^t + \lambda + 1 \\ &\geq K - (K - f_1^t + \lambda) - f_1^t + \lambda + 1 \geq 1 \iff |F_j^t| - |Y_j^t| > f_j^t.\end{aligned}$$

This contradicts the invariant of Lemma 5.7.

In the beginning of phase t, we have at most $k - \lambda$ vertices from V in the vertex cover because

$$|S_1^t \cap V| \leq K + k - |Y_1^t| - f_1^t = K + k - (K - f_1^t + \lambda) - f_1^t = k - \lambda.$$

Since there are at most λ many exchanges $S_{i-1}^t \diamond S_i^t = (\alpha, v)$ where $v \in V$ and $i \in \{2, \ldots, 2m+1\}$, we know that the vertex set $\bigcup_{i=1}^{2m+1} S_i^t \cap V$ is of size at most k. $\square$

We are set to prove the backward direction of Proposition 5.2.

Lemma 5.9 *Let (G, k) be an instance of* Clique *and $(\mathcal{G}, k', \ell)$ be the instance of* Multistage Vertex Cover *resulting from Construction 5.1. If $(\mathcal{G}, k', \ell)$ is a* `yes`*-instance, then (G, k) is a* `yes`*-instance.*

Proof Let $(\mathcal{G}, k', \ell)$ be a `yes`-instance. From Corollary 5.16 it follows that there is a one-centered solution $\mathcal{S} = (S_1^1, \ldots, S_{2m+1}^1 = S_1^2, \ldots\ldots\ldots, S_1^\kappa, \ldots, S_{2m+1}^\kappa)$ for $(\mathcal{G}, k', \ell)$. By Lemma 5.6, there is a $t \in \{1, \ldots, \kappa\}$ such that $|F_1^t| \geq K = \binom{k}{2}$. By Lemma 5.8, we know that $|\bigcup_{i=1}^{2m+1} S_i^t \cap V| \leq k$. Now we identify the clique of size k in G. Since $|F_1^t| \geq K$, we know that, by Construction 5.1, at least $K = \binom{k}{2}$ layers are covered by vertices in $V \cup E \cup \widehat{U}_{\kappa+1} \cup \{c_{2j+1}^t \mid j \in \{1, \ldots, m\}\}$ in phase t. Note that each of these layers corresponds to an edge $e = \{v, w\}$ in G and that we need in particular the vertices v and w in the vertex cover. Since we have at most k vertices in $\bigcup_{i=1}^{2m+1} S_i^t \cap V$, these vertices induce a clique of size k in G. $\square$

5.2.3 Proof of Proposition 5.2 and two corollaries

We proved the forward and backward direction of Proposition 5.2 in Sections 5.2.1 and 5.2.2, respectively. It remains to put everything together.

Proof of Proposition 5.2 Let (G, k) be an instance of Clique and $(\mathcal{G}, k', \ell)$ be the instance of Multistage Vertex Cover resulting from Construction 5.1. Observe that Construction 5.1 runs in polynomial time, and that each layer of $\mathcal{G}$ is a forest with $O(k'^2)$ edges. We know that if (G, k) is a yes-instance of CLIQUE, then $(\mathcal{G}, k', \ell)$ is a yes-instance of MULTISTAGE VERTEX COVER (Lemma 5.3), and vice versa (Lemma 5.9). Finally, the W[1]-hardness of CLIQUE [30] regarding k and the fact that $k' \in O(k^2)$ then finish the proof. □

From a motivation point of view, it is natural to assume that the change over time modeled by the temporal graph is rather of *evolutionary* character, meaning that the difference of a layer to its predecessor is limited. However, Proposition 5.2 gives a bound (in terms of the desired vertex cover size in input instance) on the number of edges of each layer. In particular, we also obtain the following W[1]-hardness.

Corollary 5.2 MULTISTAGE VERTEX COVER *parameterized by the maximum number* $\max_{i\in\{1,\ldots,\tau\}} |E(G_i)|$ *of edges in a layer is W[1]-hard, even if each layer is a forest.*

Thus, we cannot hope for fixed-parameter tractability of MULTISTAGE VERTEX COVER when parameterized for example by the combination of k and the maximum size of symmetric difference between two consecutive layers.

Furthermore, we can turn the instance $(\mathcal{G}, k', \ell)$ resulting from Construction 5.1 into an equivalent instance $(\mathcal{G}', k'', \ell)$ where each layer is a tree as follows. Set $k'' := k' + 1$. Add a vertex x to $\mathcal{G}$. In each layer of $\mathcal{G}$, make x the center of a star with k'' (new) leaf vertices, and connect x with exactly one vertex of each connected component. Note that, in every solution, x is contained in a vertex cover for each layer in $\mathcal{G}'$.

Corollary 5.3 MULTISTAGE VERTEX COVER *parameterized by k is W[1]-hard, even if each layer is a tree.*

Note, however, that in Corollary 5.3, $\max_{i\in\tau} |E(G_i)|$ is unbounded and we cannot hope to strengthen the reduction in this sense because if each layer is a tree, then we have exactly $|V| - 1$ edges in each layer. This would contradict Proposition 5.1.

6 On Efficient Data Reduction

In this section, we study the possibility of efficient and effective data reduction for MULTISTAGE VERTEX COVER when parameterized by k, τ, and $k + \tau$, that is, the possible existence of problem kernels of polynomial size. We prove that unless coNP ⊆ NP/poly, MULTISTAGE VERTEX COVER admits no problem kernel of size polynomial in k even on quite restricted inputs (Section 6.1). Yet, when combining k and τ, we prove a problem kernel of size $O(k^2\tau)$ (Section 6.2). Moreover, we prove a problem kernel of size 5τ when each layer consists of only one edge (Section 6.3). Recall (from Theorem 4.1) that MULTISTAGE VERTEX COVER is para-NP-hard regarding τ even if each layer is a tree.

6.1 No Problem Kernel of Size Polynomial in k for Restricted Input Instances

In this section, we prove the following.

Theorem 6.1 *Unless* coNP $\subseteq$ NP/*poly,* MULTISTAGE VERTEX COVER *admits no polynomial kernel when parameterized by k, even*

(i) *if each layer consists of one edge and* $\ell = 1$*, or*
(ii) *if each layer is planar*[3] *and* $\ell \geq 2k$*.*

Recall that MULTISTAGE VERTEX COVER parameterized by k is fixed-parameter tractable in case of (ii) (see Observation 3.5), while we left open whether it also holds true in case (i).

We prove Theorem 6.1 using AND-compositions [31].

Definition 6.1 An *AND-composition* for a parameterized problem L is an algorithm that, given p instances $(x_1, k), \dots, (x_p, k)$ of L, computes in time polynomial in $\sum_{i=1}^{p} |x_i|$ an instance (y, k') of L such that

1. $(y, k') \in L$ if and only if $(x_i, k) \in L$ for all $i \in \{1, \dots, p\}$, and
2. k' is polynomially upper-bounded in k.

The following is the crucial connection to polynomial kernelization: If a parameterized problem whose unparameterized version is NP-hard admits an AND-composition, then coNP $\subseteq$ NP/poly [32]. Note that coNP $\subseteq$ NP/poly implies a collapse of the polynomial-time hierarchy to its third level [33].

In the proof of Theorem 6.1(i), we use an AND-composition. The idea is to take p instances of MULTISTAGE VERTEX COVER on the same vertex set with $\ell = 1$ and identical k, and stack all these instances one after the another in the time dimension. Here, we connect the i-th instance with $(i + 1)$-st instance by just repeating the first layer of the $(i + 1)$-st instance so often such that there is enough time to *transfer* from a solution of the i-th instance to a solution of the $(i + 1)$-st instance without violating the upper bound on the symmetric difference between two consecutive vertex covers. Formally, we use the following construction.

Construction 6.1 Let $(\mathcal{G}_1, k, \ell), \dots, (\mathcal{G}_p, k, \ell)$ be p instances of MULTISTAGE VERTEX COVER where $\ell = 1$ and each layer of each $\mathcal{G}_q = (V, \mathcal{E}_q, \tau_q)$, $q \in \{1, \dots, p\}$, consists of one edge. We construct an instance $(\mathcal{G} = (V, \mathcal{E}, \tau), k, \ell)$ of MULTISTAGE VERTEX COVER as follows. Denote by $(G^i_1, \dots, G^i_{\tau_i})$ the sequence of layers of $\mathcal{G}_i$. Initially, let $\mathcal{G}$ be the temporal graph with layer sequence $((G^i_j)_{1 \leq j \leq \tau_i})_{1 \leq i \leq p}$. Next, for each $i \in \{1, \dots, p - 1\}$, insert between $G^i_{\tau_i}$ and G^{i+1}_1 the sequence $(H^i_1, H^i_2, \dots, H^i_{2k}) := (G^i_{\tau_i}, G^{i+1}_1, \dots, G^{i+1}_1)$. This finishes the construction. Note that $\tau := 2k(p - 1) + \sum_{i=1}^{p} \tau_i$.

[3] A graph is planar if it can be drawn on the plane such that no two edges cross each other.

In the next two propositions, we prove that Construction 6.1 forms AND-compositions, then used in the proof of Theorem 6.1(i).

Proposition 6.1 MULTISTAGE VERTEX COVER *where each layer consists of one edge and* $\ell = 1$ *admits an AND-composition when parameterized by* k.

Proof We AND-compose MULTISTAGE VERTEX COVER where each layer consists of one edge. Let $I_1 = (\mathcal{G}_1 = (V, \mathcal{E}_1, \tau_1), k, \ell), \ldots, I_p = (\mathcal{G}_p = (V, \mathcal{E}_p, \tau_p), k, \ell)$ be p instances of MULTISTAGE VERTEX COVER with $\ell = 1$ where each layer consists of one edge. Apply Construction 6.1 to obtain instance $I = (\mathcal{G} = (V, \mathcal{G}, \tau), k, \ell)$ of MULTISTAGE VERTEX COVER. We claim that I is a $\texttt{yes}$-instance if and only if I_i is a $\texttt{yes}$-instance for all $i \in \{1, \ldots, p\}$.

($\Rightarrow$) If I is a $\texttt{yes}$-instance, then for each $i \in \{1, \ldots, p\}$, the subsequence of the solution restricted to the layers $(G^i_j)_{1 \le j \le \tau_i}$ forms a solution to I_i.

($\Leftarrow$) Let $(S^i_1, \ldots, S^i_{\tau_i})$ be a solution to I_i for each $i \in \{1, \ldots, p\}$. Clearly, $(S^i_1, \ldots, S^i_{\tau_i})$ forms a solution to the layers $(G^i_j)_{1 \le j \le \tau_i}$. For H^i_1, let $T^i_1 = S^i_{\tau_i} \setminus \{v\}$ for some v such that the unique edge of H^i_1 is still covered. Next, set $T^i_2 = T^i_1 \cup \{w\}$, where $w \in S^{i+1}_1$ with w being incident with the unique edge of H^i_2. Now, over the next $2k - 2$ layers, transform T^i_2 into S^{i+1}_1 by first removing layer by layer the vertices in $T^i_2 \setminus S^{i+1}_1$ (at most $k - 1$ many vertices), and then layer by layer add the vertices in $S^{i+1}_1 \setminus T^i_2$ (again, at most $k - 1$ vertices). This forms a solution to I. $\square$

Turning a set of input instances of MULTISTAGE VERTEX COVER with only one layer ($\tau = 1$) additionally being a planar graph into a sequence gives an AND-composition to be used in the proof of Theorem 6.1(ii).

Proposition 6.2 MULTISTAGE VERTEX COVER *where each layer is planar and* $\ell \ge 2k$ *admits an AND-composition when parameterized by* k.

Proof We AND-compose MULTISTAGE VERTEX COVER where the temporal graph has only one layer which is planar (and $\ell \ge 2k$) into MULTISTAGE VERTEX COVER with $\ell \ge 2k$. Let $(G_1, k, \ell'), \ldots, (G_p, k, \ell')$ be p instances of MULTISTAGE VERTEX COVER with one layer being a planar graph. Construct a temporal graph $\mathcal{G}$ with layers $(G_1, \ldots, G_p)$. Set $\ell := 2k$. This finishes the construction. It is not difficult to see that $(\mathcal{G}, k, \ell)$ is a $\texttt{yes}$-instance of MULTISTAGE VERTEX COVER if and only if (G_i, k) is a $\texttt{yes}$-instance of VERTEX COVER for all $i \in \{1, \ldots, p\}$. $\square$

Proposition 6.1 and 6.2 at hand, we are set to prove this section's main result.

Proof of Theorem 6.1 Using Drucker's result [32] for AND-compositions, Proposition 6.1 and 6.2 prove Theorem 6.1(i) and (ii), respectively. Recall that MULTISTAGE VERTEX COVER where each layer consists of one edge (Theorem 4.1) and MULTISTAGE VERTEX COVER on one layer being a planar graph (basically, VERTEX COVER on planar graphs) [34] are NP-hard. $\square$

6.2 A Problem Kernel of Size $O(k^2\tau)$

Multistage Vertex Cover remains NP-hard for $\tau = 2$, even if each layer is a tree (Theorem 4.1). Moreover, Multistage Vertex Cover does not admit a problem kernel of size polynomial in k, even if each layer consists of only one edge (Theorem 6.1). Yet, when combining both parameters, we obtain a problem kernel of cubic size.

Theorem 6.2 *There is an algorithm that maps any instance $(\mathcal{G}, k, \ell)$ of* Multistage Vertex Cover *in $O(|V(\mathcal{G})|^2\tau)$ time to an instance $(\mathcal{G}', k, \ell)$ of* Multistage Vertex Cover *with at most $2k^2\tau(\mathcal{G})$ vertices and at most $k^2\tau(\mathcal{G})$ temporal edges.*

To prove Theorem 6.2, we apply three polynomial-time data reduction rules. These reduction rules can be understood as temporal variants of the folklore reduction rules for Vertex Cover. Our first reduction rule is immediate.

Reduction Rule 6.1 (Isolated vertices) *If there is some vertex $v \in V$ such that $e \cap \{v\} = \emptyset$ for all $e \in E(G_\downarrow)$, then delete v.*

For Vertex Cover, when asking for a vertex cover of size q, there is the well-known reduction rule dealing with high-degree vertices: If there is a vertex v of degree larger than q, then delete v and its incident edges and decrease q by one. For Multistage Vertex Cover a high-degree vertex can only appear in some layers, and hence deleting this vertex is in general not correct. However, the following is a temporal variant of the high-degree rule (see Fig. 4 for an illustration).

Reduction Rule 6.2 (High degree) *If there exists a vertex v such that there is an inclusion-maximal subset $\emptyset \neq J \subseteq \{1, \ldots, \tau\}$ such that $\deg_{G_i}(v) > k$ for all $i \in J$, then add a vertex w_v to V and for each $i \in J$, remove all edges incident to v in G_i, and add the edge $\{v, w_v\}$.*

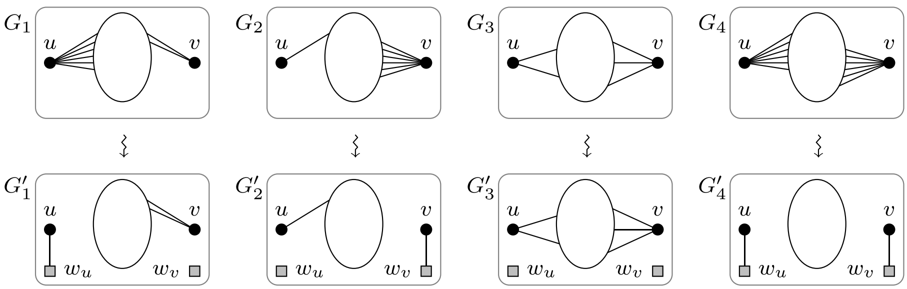

Fig. 4 Illustration of Reduction Rule 6.2, exemplified for *two* vertices u, v and $k = 5$. Each ellipse for a graph G_i and G'_i, respectively, represents $G_i - \{u, v\}$ and $G'_i - \{u, v, w_u, w_v\}$. The vertices w_v, w_u (gray squares) are introduced by the application of Reduction Rule 6.8. Note that u (v) has a high degree in G_1 (G_2) and G_4

We now show how Reduction Rule 6.2 can be applied and that it does not turn a `yes`-instance into a `no`-instance or vice versa.

Lemma 6.1 *Reduction Rule 6.2 is correct and exhaustively applicable in $O(|V|^2 \tau)$ time.*

Proof (*Correctness*) Let $I = (\mathcal{G}, k, \ell)$ be an instance with $\mathcal{G} = (G_1, \ldots, G_\tau)$, and let $I' = (\mathcal{G}', k, \ell)$ be the instance with $\mathcal{G}' = (G'_1, \ldots, G'_\tau)$ obtained from I applying Reduction Rule 6.2 with vertex v and index set J. We prove that I is a `yes`-instance if and only if I' is a `yes`-instance.

($\Rightarrow$) Let $(S_1, \ldots, S_\tau)$ be a solution to I. Observe that for all $i \in J$, $\deg_{G_i}(v) > k$ and hence $v \in S_i$. It follows that $(S_1, \ldots, S_\tau)$ is a solution to I'.

($\Leftarrow$) Let $(S'_1, \ldots, S'_\tau)$ be a solution to I'. Observe that for each $i \in J$, $S'_i \cap \{v, w_v\} \neq \emptyset$. Set $S_i := (S'_i \setminus \{w_v\}) \cup \{v\}$ for all $i \in J$. Note that S_i is a vertex cover for G_i since v covers all its incident edges and $S_i \setminus \{v\}$ is a vertex cover for $G_i - \{v\} = G'_i - \{v, w_v\}$. For each $i \in \{1, \ldots, \tau\} \setminus J$, set $S_i := S'_i$ if $w_v \notin S'_i$, and $S_i := (S'_i \setminus \{w_v\}) \cup \{v\}$ otherwise. Note that S_i is a vertex cover of $G_i = G'_i - \{w_v\}$. Finally, observe that $|S_i| \leq |S'_i|$ for all $i \in \{1, \ldots, \tau\}$, and that $|S_i \triangle S_{i+1}| \leq \ell$ for all $i \in \{1, \ldots, \tau - 1\}$. It follows that $(S_1, \ldots, S_\tau)$ is a solution to I.

(*Running time*) For each vertex, we count the number of edges in each layer. If there are more than k edges in one layer, then we remember the index of the layer. For each layer, we compute for each vertex the degree and make the modification. Once for some v vertex w_v is introduced, we add a pointer from v to w_v, and add the edge $\{v, w_v\}$ in subsequent layers when needed. Hence, in each layer we touch each edge at most twice, yielding $O(|V(\mathcal{G})|^2)$ time per layer. □

Similarly as in the reduction rules for VERTEX COVER, we now count the number of edges in each layer: if more than k^2 edges are contained in one layer, then no set of k vertices, each of degree at most k, can cover more than k^2 edges.

Reduction Rule 6.3 (`no`-instance) *If neither Reduction Rule 6.1 nor Reduction Rule 6.2 is applicable and there is a layer with more than k^2 edges, then output a trivial* `no`*-instance.*

We are ready to prove that when none of the Reduction Rules 6.1, 6.2 and 6.3 can be applied, then the instance contains "few" vertices and temporal edges.

Lemma 6.2 *Let $(\mathcal{G}, k, \ell)$ be an instance of* MULTISTAGE VERTEX COVER *such that none of Reduction Rules 6.1, 6.2 and 6.3 is applicable. Then $\mathcal{G}$ consists of at most $2k^2\tau(\mathcal{G})$ vertices and $k^2\tau(\mathcal{G})$ temporal edges.*

Proof Since none of Reduction Rules 6.1 and 6.2 is applicable, for each layer it holds true that there is no isolated vertex and no vertex of degree larger than k. Since Reduction Rule 6.3 is not applicable, each layer consists of at most k^2 edges. Hence, there are at most $k^2\tau$ temporal edges in $\mathcal{G}$. Consequently, due to Reduction Rule 6.1, there are at most $2k^2\tau$ vertices in $\mathcal{G}$. □

We are ready to prove the main result of this section.

Proof of Theorem 6.2 Given an instance $I = (\mathcal{G}, k, \ell)$ of Multistage Vertex Cover, apply Reduction Rules 6.1, 6.2 and 6.3 exhaustively in $O(|V(\mathcal{G})|^2\tau(\mathcal{G}))$ time either to decide that I is a trivial `no`-instance or to obtain an instance $(\mathcal{G}', k, \ell)$ equivalent to I. Due to Lemma 6.2, $\mathcal{G}'$ consists of at most $2k^2\tau(\mathcal{G})$ vertices and at most $k^2\tau(\mathcal{G})$ temporal edges. □

6.3 A Problem Kernel of Size 5τ

Multistage Vertex Cover, even when each layer is a tree, does not admit a problem kernel of any size in τ unless P = NP. Yet, when each layer consists of only one edge, then each instance of Multistage Vertex Cover contains at most τ edges and, hence, at most 2τ non-isolated vertices. Thus, Multistage Vertex Cover admits a straight-forward problem kernel of size linear in τ.

Observation 6.3 Let $(\mathcal{G}, k, \ell)$ be an instance of Multistage Vertex Cover where each layer consists of one edge. Then we can compute in $O(|V(\mathcal{G})| \cdot \tau)$ time an instance $(\mathcal{G}', k, \ell)$ of size at most $5\tau(\mathcal{G})$.

Proof Let $(\mathcal{G}, k, \ell)$ be an instance of Multistage Vertex Cover where each layer of $\mathcal{G} = (V, \mathcal{E}, \tau)$ consists of one edge. Observe that we can immediately output a trivial `yes`-instance if $k \geq \tau$ (Observation 3.1) or $\ell \geq 2$ (Observation 3.5). Hence, assume that $k \leq \tau - 1$ and $\ell \leq 1$. Apply Reduction Rule 6.1 exhaustively on $(\mathcal{G}, k, \ell)$ to obtain $(\mathcal{G}', k, \ell)$. Since there are τ edges in $\mathcal{G}$, there are at most 2τ vertices in $\mathcal{G}'$. It follows that the size of $(\mathcal{G}', k, \ell)$ is at most 5τ. □

7 Conclusion

We introduced Multistage Vertex Cover, proved its NP-hardness even on very restricted input instances, and studied its parameterized complexity regarding the natural parameters k, ℓ, and τ (all given as part of the input). The technical highlight is the W[1]-hardness described in Section 5.2 which, because it holds on very restricted instances of Multistage Vertex Cover, may turn out to be useful to provide W[1]-hardness results for other problems in the multistage setting. We leave open whether Multistage Vertex Cover parameterized by the vertex cover size bound k is fixed-parameter tractable when each layer consists of only one edge (see Table 1). Moreover, it is open whether Multistage Vertex Cover remains NP-hard on two layers each being a path (which would strengthen Theorem 4.1(i)).

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References

1. Fluschnik, T., Niedermeier, R., Rohm, V., Zschoche, P.: Multistage vertex cover. In: Proceeding of 14th IPEC, LIPIcs, vol. 148, pp. 14:1–14:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik (2019)
2. Bampis, E., Escoffier, B., Lampis, M., Paschos, V.T.: Multistage matchings. In: Proc. of 16th SWAT, of LIPIcs Schloss Dagstuhl - Leibniz-Zentrum für Informatik, vol. 101, pp. 7:1–7:13 (2018). https://doi.org/10.4230/LIPIcs.SWAT.2018.7
3. Gupta, A., Talwar, K., Wieder, U.: Changing bases: Multistage optimization for matroids and matchings. In: Proc. of 41st ICALP, of LNCS, Springer, vol. 8572, pp. 563–575 (2014). https://doi.org/10.1007/978-3-662-43948-7_47
4. Bampis, E., Escoffier, B., Teiller, A.: Multistage knapsack. In Proc. of 44th MFCS of LIPIcs. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, vol. 138, pp. 22:1–22:14 (2022). https://doi.org/10.4230/LIPIcs.MFCS.2019.22
5. Eisenstat, D., Mathieu, C., Schabanel, N.: Facility location in evolving metrics. In: Proc. of 41st ICALP, LNCS, Springer, pp. 459–470 (2014). https://doi.org/10.1007/978-3-662-43951-7_39
6. Bampis, E., Escoffier, B., Kononov, A.V.: LP-based algorithms for multistage minimization problems. In: Proc. of 18th WAOA, LNCS, Springer, vol. 12806, pp. 1–15 (2020). https://doi.org/10.1007/978-3-030-80879-2_1
7. Heeger, K., Himmel, A.-S., Kammer, F., Niedermeier, R., Renken, M., Sajenko, A.: Multistage graph problems on a global budget. Theor. Comput. Sci. **868**, 46–64 (2021). https://doi.org/10.1016/j.tcs.2021.04.002
8. Fluschnik, T., Niedermeier, R., Schubert, C., Zschoche, P.: Multistage s-t path: Confronting similarity with dissimilarity in temporal graphs. In: Proc. of 31st ISAAC, LIPIcs, Schloss Dagstuhl - Leibniz-Zentrum für Informatik, vol. 181, pp. 43:1–43:16. dagstuhl (2020). https://doi.org/10.4230/LIPIcs.ISAAC.2020.43
9. Chimani, M., Troost, N., Wiedera, T.: Approximating multistage matching problems. In: Proc. of 32nd IWOCA, LNCS, Springer, pp. 558–570 (2021). https://doi.org/10.1007/978-3-030-79987-8_39
10. Fluschnik, T.: A multistage view on 2-satisfiability of LNCS, Springer, vol. 12701, pp. 231–244 (2021). https://doi.org/10.1007/978-3-030-75242-2_16
11. Bampis, E., Escoffier, B., Schewior, K., Teiller, A.: Online multistage subset maximization problems. Algorithmica **83**(8), 2374–2399 (2021). https://doi.org/10.1007/s00453-021-00834-7
12. Bredereck, R., Fluschnik, T., Kaczmarczyk, A.: Multistage committee election. CoRR abs/2005.02300, arXiv:2005.02300 (2020)
13. Kellerhals, L., Renken, M., Zschoche, P.: Parameterized algorithms for diverse multistage problems. In: Proc. of 29th ESA of LIPIcs. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, vol. 204, pp. 55:1–55:17 (2021). https://doi.org/10.4230/LIPIcs.ESA.2021.55
14. Fluschnik, T., Kunz, P.: Bipartite temporal graphs and the parameterized complexity of multistage 2-coloring. CoRR abs/2111.09049, arXiv:2111.09049 (2021)
15. Hartung, S., Niedermeier, R.: Incremental list coloring of graphs, parameterized by conservation. Theor. Comput. Sci. **494**, 86–98 (2013)
16. Abu-Khzam, F.N., Egan, J., Fellows, M.R., Rosamond, F.A., Shaw, P.: On the parameterized complexity of dynamic problems. Theor. Comput. Sci. **607**, 426–434 (2015). https://doi.org/10.1016/j.tcs.2015.06.053
17. Casteigts, A., Flocchini, P., Quattrociocchi, W., Santoro, N.: Time-varying graphs and dynamic networks. International Journal of Parallel, Emergent and Distributed Systems **27**(5), 387–408 (2012). https://doi.org/10.1080/17445760.2012.668546
18. Fluschnik, T., Molter, H., Niedermeier, R., Renken, M., Zschoche, P.: Temporal graph classes: A view through temporal separators. Theor. Comput. Sci. **806**, 197–218 (2020)
19. Iwata, Y., Oka, K.: Fast dynamic graph algorithms for parameterized problems. In: Proc. of 12th SWAT of LNCS, Springer, vol. 8503, pp. 241–252 (2014)

20. Alman, J., Mnich, M., Williams, V.V.: Dynamic parameterized problems and algorithms. ACM T. Algorithms **16**(4), 45:1–45:46, (2020). https://doi.org/10.1145/3395037
21. Chitnis, R., Cormode, G., Esfandiari, H., Hajiaghayi, M., McGregor, A., Monemizadeh, M., Vorotnikova, S.: Kernelization via sampling with applications to finding matchings and related problems in dynamic graph streams. In: Proc. of 27th SODA, SIAM, pp. 1326–1344 (2016). https://doi.org/10.1137/1.9781611974331.ch92
22. Akrida, E.C., Mertzios, G.B., Spirakis, P.G., Zamaraev, V.: Temporal vertex cover with a sliding time window. J. Comput. Syst. Sci. **107**, 108–123 (2020). https://doi.org/10.1016/j.jcss.2019.08.002
23. Ito, T., Demaine, E.D., Harvey, N.JA., Papadimitriou, C.H., Sideri, M., Uehara, R., Uno, Y.: On the complexity of reconfiguration problems. Theor. Comput. Sci. **412**(12-14), 1054–1065 (2011)
24. Gopalan, P., Kolaitis, P.G., Maneva, E., Papadimitriou, C.H.: The connectivity of boolean satisfiability: computational and structural dichotomies. SIAM J. Comput. **38**(6), 2330–2355 (2009)
25. Mouawad, A.E., Nishimura, N., Raman, V., Simjour, N., Suzuki, A.: On the parameterized complexity of reconfiguration problems. Algorithmica **78**(1), 274–297 (2017)
26. Mouawad, A., Nishimura, N., Raman, V., Siebertz, S.: Vertex cover reconfiguration and beyond. Algorithms **11**(2), 20 (2018)
27. Krithika, R., Sahu, A., Tale, P.: Dynamic parameterized problems. Algorithmica **80**(9), 2637–2655 (2018)
28. Diestel, R. Graph theory. GTM, 5th edn., vol. 173. Springer, Berlin (2016)
29. Fleischner, H., Sabidussi, G., Sarvanov, V.I.: Maximum independent sets in 3- and 4-regular Hamiltonian graphs. Discrete Math. **310**(20), 2742–2749 (2010)
30. Downey, R.G., Fellows, M.R.: Parameterized complexity. Monographs in Computer Science. Springer, Berlin (1999)
31. Bodlaender, H.L., Downey, R.G., Fellows, M.R., Hermelin, D.: On problems without polynomial kernels. J. Comput. Syst. Sci. **75**(8), 423–434 (2009). https://doi.org/10.1016/j.jcss.2009.04.001
32. Drucker, A.: New limits to classical and quantum instance compression. SIAM J. Comput. **44**(5), 1443–1479 (2015). https://doi.org/10.1137/130927115
33. Yap, C.-K.: Some consequences of non-uniform conditions on uniform classes. Theor. Comput. Sci. **26**, 287–300 (1983)
34. Garey, M.R., Johnson, D.S., Stockmeyer, L.J.: Some simplified NP-complete graph problems. Theor. Comput. Sci. **1**(3), 237–267 (1976). https://doi.org/10.1016/0304-3975(76)90059-1