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The Number of Clones Determined by Disjunctions of Unary Relations

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Abstract
We consider finitary relations (also known as crosses) that are definable via finite disjunctions of unary relations, i.e. subsets, taken from a fixed finite parameter set Γ. We prove that whenever Γ contains at least one non-empty relation distinct from the full carrier set, there is a countably infinite number of polymorphism clones determined by relations that are disjunctively definable from Γ. Finally, we extend our result to finitely related polymorphism clones and countably infinite sets Γ. These results address an open problem raised in Creignou, N., et al. Theory Comput. Syst. **42**(2), 239–255 (2008), which is connected to the complexity analysis of the satisfiability problem of certain multiple-valued logics studied in Hähnle, R. Proc. 31st ISMVL 2001, 137–146 (2001).

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1 Introduction

Constraint Satisfaction Problems (CSPs) offer a uniform framework to study algorithmic problems. In one of the simplest forms one is given a conjunctive formula

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over some chosen parameter set of finitary relations Q and is asked to decide whether the formula is satisfiable. Even in this basic manifestation many important problems can be encoded as CSPs, for instance, graph k-colourability, unrestricted Boolean satisfiability (SAT), Boolean 3-satisfiability (3-SAT) and further variants of this problem, solvability of sudokus, the n-queens problem, more generally the exact cover problem, and many others.

There is an active stream of research producing results regarding the decision complexity of CSPs and its variants, using various approaches [2–9, 12, 13, 15–17, 19, 25, 28, 30, 34]. Moreover, recently there have been credible claims regarding the solution [14, 40, 41] of Feder and Vardi's famous CSP dichotomy conjecture [22] stating that any CSP on a finite set can either be decided in polynomial time (is *tractable*) or is NP-complete, otherwise.

A more algebraic formulation of a CSP is given by fixing a relational structure $\mathbb{A}$ of finite signature on a finite set A with set of basic relations Q. The question to decide is, for any finite relational structure $\mathbb{B}$ of the same signature as $\mathbb{A}$, whether there is a homomorphism from $\mathbb{B}$ to $\mathbb{A}$. A basic reduction result attributed to Jeavons [26] implies that any two CSPs on the same carrier set A parametrized by finite sets of relations Q_1 and Q_2 sharing the same polymorphism clone (see Section 2) have the same complexity behaviour (up to polynomial time many-one reductions). This means, as far as their complexity is concerned, it is not necessary to examine more CSPs than there are finitely related clones on a given set.

Creignou et al. [18] studied CSPs involving so-called *clausal constraints* over totally ordered finite domains. Their problem can be understood within the previously introduced CSP paradigm as a CSP given by a finite set of relations of the form

$$\left\{(x_1,\dots,x_p,y_1,\dots,y_q) \mid x_1 \geq a_1 \vee \dots \vee x_p \geq a_p \vee y_1 \leq b_1 \vee \dots \vee y_q \leq b_q\right\}.$$

These are called *clausal relations* (over chains) and have been studied more extensively in [10, 11, 37–39].

One of the main motivations for the CSPs studied in [18] comes from many-valued logic, more precisely, from regular signed logic over totally ordered finite sets of truth values, as described in [23, 24]. In fact, the satisfiability problem associated with regular signed conjunctive normal form formulae over chains can be expressed as a CSP over clausal relations (or with respect to clausal constraints). In [18] a complete classification of complexity was achieved in terms of the involved clausal patterns, establishing a dichotomy between tractability and NP-completeness. The authors left open the problem to algebraically describe all CSPs on the same domain whose complexity is equivalent to one of their problems via Jeavons's reduction, with particular focus on the tractable cases. This problem is in fact asking for a description of all clones (with tractable CSP) associated with clausal relations, called *clausal clones* in [37]. Since there is a continuum of clones on finite, at least three-element sets, a first necessary step to ensure the feasibility of answering such a question is to determine the cardinality of the lattice of all clausal clones, which is another open problem from [38], see also [11, Section 1].

We address the open problem from [18] by answering this feasibility question in the affirmative way. It turns out that solving this problem is less convoluted when generalizing from clausal relations to relations defined as disjunctions over arbitrary subsets, not just upsets or downsets of finite total orders. In this way, for finite carrier sets, we include in particular *all* signed logics discussed in [24]: general regular signed logic with respect to any order, monosigned logic and full signed logic. Moreover, relations defined via disjunctions of unary relations have seen applications in general algebra for more than three decades, see, e.g., [36]. If n is the arity of such a relation, it is also known under the term *n-dimensional cross* [27, 32]. When the unary relations of a cross are subuniverses of a given algebra, crosses have recently become prominent as a means to characterize the non-existence of *n-dimensional cube terms* in idempotent varieties [27]. Symmetric crosses, i.e., disjunctions of only one non-trivial subuniverse, have appeared earlier as so-called *cube term blockers* [29].

Our main result is that for any finite non-trivial set of unary relations Γ on any carrier set, there are exactly $\aleph_0$-many polymorphism clones determined by relations that are disjunctively definable over Γ (i.e., by crosses over Γ).

We achieve this theorem by relating the number of such clones to the number of downsets of a certain order on finite powers of the natural numbers. The latter can be bounded above by $\aleph_0$ using Dickson's Lemma. Finally, we prove that our bound is tight by exhibiting an infinite chain of clones whenever the parameter set Γ contains a non-empty relation γ distinct from the full domain. As a by-product we even obtain a better understanding of the ordered set of all polymorphism clones of crosses over Γ.

On a concluding note, we observe that our result extends to non-trivial countably infinite sets of unary relations Γ, when one is only interested in polymorphism clones of finite subsets of disjunctively definable relations over Γ.

The sections of this paper should be read in consecutive order. The following one introduces some notation and prepares basic definitions and facts concerning clone and order theory. Section 3 links polymorphism clones of relations being disjunctively definable over unary relations with downsets of a poset on $\mathbb{N}^I$, and the final section concludes the task by counting these and deriving our results.

2 Preliminaries

2.1 Sets, Functions and Relations

We write $\mathbb{N} = \{0, 1, 2, \dots\}$ for the set of *natural numbers* and use $\mathbb{N}_+$ for $\mathbb{N} \setminus \{0\}$. *Inclusion* between sets A and B is denoted by $A \subseteq B$, as opposed to *proper inclusion* $A \subset B$ or $A \subsetneq B$. The *powerset* $\mathfrak{P}(A)$ is the set of all subsets of A; $\mathfrak{P}_{\text{fin}}(A)$ the set of all finite subsets of A. The *cardinality* of A is written as $|A|$ and we say that A is *countable* if $|A| \leq \aleph_0$. It is convenient for us to use John von Neumann's model of $\mathbb{N}$ where each $n \in \mathbb{N}$ is the set $n = \{0, \dots, n-1\}$. If $f: A \longrightarrow B$ and $g: B \longrightarrow C$ are functions, their *composition* is the map $g \circ f: A \longrightarrow C$ given by $g \circ f(a) =$

$g(f(a))$ for all $a \in A$. Moreover, if $U \subseteq A$ and $V \subseteq B$, we write $f[U]$ for the *image* $\{f(u) \mid u \in U\}$ of U under f and $f^{-1}[V]$ for the *preimage* $\{a \in A \mid f(a) \in V\}$ of V with respect to f. The *image of* f is denoted by $\operatorname{im} f := f[A]$; the *kernel of* f, denoted by $\ker f$, is the equivalence relation $\{(a_1, a_2) \in A^2 \mid f(a_1) = f(a_2)\}$. Clearly, every member $f(a)$ of the image of f is in one-to-one correspondence with the kernel class $[a]_{\ker f}$, i.e., we have a bijection between $\operatorname{im} f$ and the *factor set* $A/\ker f$ of all equivalence classes by the kernel of f. We shall denote this fact also by $\operatorname{im} f \cong A/\ker f$.

If I and A are sets, the direct (Cartesian) power A^I is the set of all maps $f : I \longrightarrow A$. This is also true for finite powers, i.e. A^n where $n \in \mathbb{N}$: we understand n-tuples over A as maps from n to A; in particular all notions defined for maps, such as composition, image, preimage, kernel also make sense for tuples. Formally, an n-tuple $x \in A^n$ is given as $x = (x(0), \dots, x(n-1))$, however, if no confusion is to be expected, we shall also refer to the entries of a tuple by some other indexing, e.g. $x = (x_1, \dots, x_n)$ or $x = (a, b, c)$ etc. Moreover, an *n-ary relation on A* is just any subset $\rho \subseteq A^n$ of n-tuples; an n-ary operation on A is any function $f : A^n \longrightarrow A$. We collect all *finitary (excluding nullary) operations on* A in the set $\mathcal{O}_A = \bigcup_{n \in \mathbb{N}_+} A^{A^n}$. Multi-ary functions $f : B^n \longrightarrow C$ and $g_1, \dots, g_n : A^m \longrightarrow B$ can be composed in the following way: putting $f \circ (g_1, \dots, g_n)(a) := f(g_1(a), \dots, g_n(a))$ for every $a \in A^m$ defines a function $f \circ (g_1, \dots, g_n) : A^m \longrightarrow C$. This works for functions and tuples, too. Namely, we say that an n-ary function f *preserves* an m-ary relation $\rho \subseteq A^m$ and write $f \rhd \rho$ if for all $(r_1, \dots, r_n) \in \rho^n$ we have $f \circ (r_1, \dots, r_n) \in \rho$. Then for a set $Q \subseteq \bigcup_{m \in \mathbb{N}_+} \mathfrak{P}(A^m)$ of finitary relations on A, we define the set $F = \operatorname{Pol}_A Q$ of all *polymorphisms of* Q to be $\{f \in \mathcal{O}_A \mid \forall \rho \in Q : f \rhd \rho\}$. This set of operations forms a *clone on* A, i.e., it is closed under composition (viz. $f \circ (g_1, \dots, g_n) \in F$ whenever $f \in F$ is n-ary and $g_1, \dots, g_n \in F$ are m-ary) and contains all projections $e_i^{(n)} : A^n \longrightarrow A, 0 \leq i < n, n \in \mathbb{N}_+$, given by $e_i^{(n)}(x_0, \dots, x_{n-1}) = x_i$ for each tuple $(x_0, \dots, x_{n-1}) \in A^n$. A clone on A is *finitely related* if it can be obtained as $\operatorname{Pol}_A Q$ for a finite set Q of finitary relations. More information on the importance of finitely related clones in the context of CSP can be found in [8].

Note that the preservation relation $\rhd$ between finitary operations and relations, and the Galois correspondence induced by it, is fundamental for the study of clones on finite sets [33, 35]. In point of fact, this paper is concerned with counting the number of Galois closed sets for a variant of this Galois correspondence where the relational side is restricted to a certain subset $\mathrm{DD}(\Gamma)$ to be defined in Section 3. For more background information and basic facts concerning Galois connections in general, see [20, p. 155 et seqq.].

2.2 Ordered Sets

If $\mathbb{P} = (P, \leq)$ is a *partially ordered set* (*poset*), i.e. $\leq$ is a binary reflexive, antisymmetric and transitive relation on P, then a subset $X \subseteq P$ is said to be a *downset of* $\mathbb{P}$ (occasionally called *order ideal*), if it is closed w.r.t. taking lower bounds. This means that with every member $x \in X$ the *principal downset* $\downarrow_{\mathbb{P}} \{x\} := \{y \in P \mid$

$y \le x\}$ *generated by* x is a subset of X. We denote the *set of all downsets of* $\mathbb{P}$ *by* $\mathrm{DS}(\mathbb{P})$.

It is easy to see that $\mathrm{DS}(\mathbb{P})$ forms a closure system on P, the associated closure operator maps any set $Y \subseteq P$ to its closure under lower bounds $\downarrow_{\mathbb{P}} Y := \bigcup_{y\in Y} \downarrow_{\mathbb{P}} \{y\}$, that is, $\{z \in P \mid \exists y \in Y : z \le y\}$, the least downset of $\mathbb{P}$ containing Y. This set is also referred to as *downset generated by* Y. Clearly, a set $Y \subseteq P$ is a downset if and only if $\downarrow_{\mathbb{P}} Y = Y$.

The dual notion of a downset is that of an *upset* (*order filter*), which is a subset $X \subseteq P$ of a poset $\mathbb{P} = (P, \le)$ that is closed under upper bounds. Again the collection $\mathrm{US}(\mathbb{P})$ of all upsets of $\mathbb{P}$ forms a closure system and the corresponding closure operator $\uparrow_{\mathbb{P}}$ is given by adding all upper bounds. Obviously, complementation establishes a one-to-one correspondence between $\mathrm{DS}(\mathbb{P})$ and $\mathrm{US}(\mathbb{P})$.

A (*homo*)*morphism* from a poset $\mathbb{P}$ to $\mathbb{Q}$ is any monotone map $h : P \longrightarrow Q$, i.e. one being compatible with the respective order relations. A morphism $h : \mathbb{P} \longrightarrow \mathbb{Q}$ is a *retraction* if there exists a homomorphism $\tilde{h} : \mathbb{Q} \longrightarrow \mathbb{P}$ that is a right-inverse to h, i.e., satisfies $h \circ \tilde{h} = \mathrm{id}_{\mathbb{Q}}$. An *isomorphism* between posets $\mathbb{P}$ and $\mathbb{Q}$ is a retraction $h : \mathbb{P} \longrightarrow \mathbb{Q}$ that also has a left-inverse, i.e. any order preserving and order reflecting bijection.

Without proof we present the following basic lemmas.

Lemma 1 *If* $h : \mathbb{P} \longrightarrow \mathbb{Q}$ *is a homomorphism between posets* $\mathbb{P}$ *and* $\mathbb{Q}$ *and a subset* $Y \in \mathrm{DS}(\mathbb{Q})$ *is a downset, then so is the preimage* $h^{-1}[Y] \in \mathrm{DS}(\mathbb{P})$.

Lemma 2 *If* $h : \mathbb{P} \longrightarrow \mathbb{Q}$ *is an isomorphism between posets* $\mathbb{P}$ *and* $\mathbb{Q}$, *then the map* $H : \mathrm{DS}(\mathbb{P}) \longrightarrow \mathrm{DS}(\mathbb{Q})$ *given by* $H(X) := h[X]$ *is a bijection. In particular we have* $|\mathrm{DS}(\mathbb{P})| = |\mathrm{DS}(\mathbb{Q})|$.

If $\mathbb{P} = (P, \le)$ is a poset and $Y \subseteq P$ is a subset, then $\le\restriction_Y := \le \cap (Y \times Y)$ denotes the *restricted order relation*, i.e. the order of the *subposet* $\mathbb{Y} = (Y, \le\restriction_Y)$ *of* $\mathbb{P}$ *induced by* Y.

Lemma 3 *If* $\mathbb{P}$ *is a poset and* $X, Y \in \mathrm{DS}(\mathbb{P})$, *then* $X \cap Y \in \mathrm{DS}(\mathbb{Y})$ *for* $\mathbb{Y} = (Y, \le\restriction_Y)$.

Next, we consider a special order on powers of $\mathbb{N}$. For any set I we denote the order of $(\mathbb{N}, \le)^I$, which is given by ordering I-tuples by $\le$ pointwise, also by $\le$. For any element $x \in \mathbb{N}^I$, we refer by its *support* to the set $\mathrm{supp}(x) := \{i \in I \mid x(i) \ne 0\}$, i.e. the preimage $x^{-1}[\mathbb{N} \setminus \{0\}]$. Relating two elements $x, y \in \mathbb{N}^I$ if $\mathrm{supp}(x)$ contains $\mathrm{supp}(y)$ defines a quasiorder on $\mathbb{N}^I$, i.e. a reflexive and transitive binary relation. Intersecting this quasiorder with the pointwise order $\le$, we obtain the poset $\mathbb{P} = (\mathbb{N}^I, \sqsubseteq)$, where $x \sqsubseteq y$ holds for $x, y \in \mathbb{N}^I$ exactly if $x \le y$ and $\mathrm{supp}(y) \subseteq \mathrm{supp}(x)$. If $x \sqsubseteq y$ then for every $i \in \mathrm{supp}(x)$ we have $0 < x(i) \le y(i)$, i.e. $i \in \mathrm{supp}(y)$. Hence, the condition $x \sqsubseteq y$ is equivalent to $x \le y$ and $\mathrm{supp}(x) = \mathrm{supp}(y)$.

In the following sections, we shall be interested in the downsets of the poset $(\mathbb{N}^I, \sqsubseteq)$, mostly for finite I. In order to count these we need information about the number of downsets of finite powers of $(\mathbb{N}, \le)$, which is a consequence of Dickson's

Lemma [21]. The original formulation [21, Lemma A, p. 414] is a statement about polynomials in a finite number of indeterminates; the variant we need here states that $(\mathbb{N}, \leq)^I$ is a so-called well-partial order (see e.g. [1, Definition 5.4.3, p. 113]) and can, for instance, be found in [31, p. 50].

Lemma 4 (**Dickson's Lemma**) *The poset* $(\mathbb{N}, \leq)^I$ *is a well-partial order, that is, for every sequence of tuples* $(x_i)_{i\in\mathbb{N}} \in (\mathbb{N}^I)^{\mathbb{N}}$ *there exist indices* $i < j$ *such that* $x_i \leq x_j$.

This statement implies that $(\mathbb{N}, \leq)^I$ has a countably infinite number of downsets whenever I is a finite non-empty set.

Lemma 5 *We have* $\left|\mathrm{DS}\left((\mathbb{N}, \leq)^I\right)\right| = \aleph_0$ *for all finite* $I \neq \emptyset$.

Proof Let $\mathbb{P} := (\mathbb{N}, \leq)^I$. Complementation bijectively maps downsets of $\mathbb{P}$ to upsets and vice versa. Hence, consider any $F \in \mathrm{US}(\mathbb{P})$; its set of minimal elements $M(F)$ certainly forms an anti-chain, so by the contrapositive of Lemma 4, the subset $M(F) \subseteq \mathbb{N}^I$ must be finite. Moreover, since $(\mathbb{N}, \leq)^I$ does not have any infinite (strictly) descending chains, every element $x \in F$ satisfies $m \leq x$ for some $m \in M(F)$. In other words, we have $F = \uparrow_{\mathbb{P}} M(F)$, showing that the map $M\colon \mathrm{US}(\mathbb{P}) \longrightarrow \mathfrak{P}_{\mathrm{fin}}(\mathbb{N}^I)$ is injective. Finally, the set $\mathfrak{P}_{\mathrm{fin}}(\mathbb{N}^I)$ of all finite subsets of $\mathbb{N}^I$ is countably infinite for $\mathbb{N}$ is. So we have $|\mathrm{DS}\,(\mathbb{P})| = |\mathrm{US}(\mathbb{P})| \leq \aleph_0$.

For the converse it suffices to exhibit an infinite subset of $\mathrm{DS}\,(\mathbb{P})$, for instance, infinitely many principal downsets $\{\downarrow_{\mathbb{P}} \{(n, \ldots, n)\} | n \in \mathbb{N}\}$. ☐

3 Clones Determined by Disjunctions of Unary Predicates

We are interested in clones that are determined by relations that are disjunctively definable by unary predicates. Throughout our whole study we shall fix a (mostly finite) parameter set Γ of unary relations on a given carrier set A that we call *(unary) relational language*. If $n \in \mathbb{N}_+$ and $(\gamma_1, \ldots, \gamma_n) \in \Gamma^n$ is any n-tuple of basic relations from Γ, the n-ary relation $\rho = R(\gamma_1, \ldots, \gamma_n) = \left\{(x_1, \ldots, x_n) \in A^n \,\middle|\, \bigvee_{1\leq i\leq n} x_i \in \gamma_i \right\}$ is said to be *disjunctively definable from* Γ. Relations constructed in this way have been denoted as $\mathrm{Cross}(\gamma_1, \ldots, \gamma_n)$ in [27, 32]. The set $\mathrm{DD}(\Gamma)$ consists exactly of all relations definable in this manner. Note that $\mathrm{DD}(\emptyset) = \emptyset$ since we consider only non-empty disjunctions, i.e., disjunctively definable relations of positive arity.

We quickly observe that $R(\gamma_1, \ldots, \gamma_n) = A^n$ holds if and only if at least one of the relations $\gamma_1, \ldots, \gamma_n$ equals the full set A. Namely, if for every $i \in \{1, \ldots, n\}$ the basic unary relation $\gamma_i \subsetneq A$ is proper, and $x_i \in A \setminus \gamma_i$, then $(x_1, \ldots, x_n) \notin R(\gamma_1, \ldots, \gamma_n)$, so $R(\gamma_1, \ldots, \gamma_n) \subsetneq A^n$. Also, note that $R(\gamma_1, \ldots, \gamma_n) = \emptyset$ if and only if $\gamma_1 = \cdots = \gamma_n = \emptyset$.

It is useful to see that the basic unary relations defining some n-ary relation $\rho \in \mathrm{DD}(\Gamma)$ can be uniquely reconstructed from ρ in every interesting case, i.e. whenever $\rho \neq A^n$.

Lemma 6 *The parameter reconstruction map*

$$\begin{array}{rccc} p\colon & \mathrm{DD}(\Gamma) & \longrightarrow & \bigcup_{n\in\mathbb{N}_+} \Gamma^n \\ & \rho = R(\gamma_1,\dots,\gamma_n) & \longmapsto & \begin{cases} (\gamma_1,\dots,\gamma_n) & \text{if } \rho \subsetneq A^n \\ (A,\dots,A) & \text{else,} \end{cases} \end{array}$$

is well-defined.

Proof By definition of $\mathrm{DD}(\Gamma)$ every disjunctively definable relation ρ has a parameter representation $\rho = R(\gamma_1,\dots,\gamma_n)$. We need to prove that the latter is unique, whenever $\rho \neq A^n$. For this consider parameters $(\gamma_1,\dots,\gamma_n), (\gamma_1',\dots,\gamma_n') \in \Gamma^n$ such that $R(\gamma_1,\dots,\gamma_n) = R(\gamma_1',\dots,\gamma_n') \subsetneq A^n$. We now show that $\gamma_i = \gamma_i'$ holds for every $i \in \{1,\dots,n\}$. Since our assumption is symmetric, it suffices to fix $i \in \{1,\dots,n\}$ and to prove that $\gamma_i \subseteq \gamma_i'$. Because $\rho = R(\gamma_1',\dots,\gamma_n') \neq A^n$, we have $\gamma_j' \subsetneq A$ for all $j \in \{1,\dots,n\}$ and so we can pick $x_j \in A \setminus \gamma_j'$ for $j \neq i$. If now $x_i \in \gamma_i$, our assumption entails $(x_1,\dots,x_n) \in R(\gamma_1,\dots,\gamma_n) = R(\gamma_1',\dots,\gamma_n')$. By the choice of the x_j for all $j \neq i$, we must have $x_i \in \gamma_i'$, proving $\gamma_i \subseteq \gamma_i'$. □

It is an obvious consequence of the preceding lemma that the parameter reconstruction map provides a one-sided inverse to the construction of relations from unary predicates. That is to say, we have $R(p(\rho)) = \rho$ for all $\rho \in \mathrm{DD}(\Gamma)$. This inverse is uniquely determined for the non-trivial relations ρ, and it chooses a canonical representative of all possible parametrizations when ρ is a full power of A.

Based on the parameter reconstruction p from Lemma 6, we can define the *pattern* of a disjunctively definable relation. Intuitively it counts how often each $\gamma \in \Gamma$ occurs in the parameter tuple $p(\rho)$. More formally, for $\rho \in \mathrm{DD}(\Gamma)$ with reconstructed parameters $p(\rho) = (\gamma_1,\dots,\gamma_n)$, i.e. $\rho = R(\gamma_1,\dots,\gamma_n)$, the tuple $\mathrm{pt}(\rho) \in \mathbb{N}^\Gamma$ maps every $\gamma \in \Gamma$ to $\mathrm{pt}(\rho)(\gamma) := \left|p(\rho)^{-1}[\{\gamma\}]\right|$ where for both the parameter tuple $p(\rho) \in \Gamma^n$ and the pattern $\mathrm{pt}(\rho) \in \mathbb{N}^\Gamma$ we make use of the ambiguous interpretation as a tuple and as a map. Note that the length of the parameter tuple $p(\rho)$ depends on the arity of ρ, while the length of the pattern only depends on $|\Gamma|$, which is usually finite (at least $\mathrm{pt}(\rho)$ has finite support in Γ) and normally not varying for our considerations. The pattern of a disjunctively definable relation roughly carries the same information as *clausal patterns* [18, Section 2] do for clausal relations; however, since we are dealing with a more generic situation, the notion of clausal pattern had to be adapted and generalized.

Next we study how the polymorphism clone of a disjunctively definable relation ρ over Γ changes when duplicating one of its unary parameter relations. Since polymorphism clones are not affected by variable permutations of their defining relations, we can restrict our attention to duplicating the first parameter of $\rho \in \mathrm{DD}(\Gamma)$. Letting $p(\rho) = (\gamma_1, \dots, \gamma_n)$, such a duplication apparently increases the pattern of $\rho = R(\gamma_1, \dots, \gamma_n)$ in precisely one place, while preserving all the other values:

$$\mathrm{pt}(R(\gamma_1, \gamma_1, \dots, \gamma_n))(\gamma) = \begin{cases} \mathrm{pt}(R(\gamma_1, \dots, \gamma_n))(\gamma) + 1 & \text{if } \gamma = \gamma_1, \\ \mathrm{pt}(R(\gamma_1, \dots, \gamma_n))(\gamma) & \text{otherwise.} \end{cases} \qquad (\dagger)$$

Lemma 7 *For arbitrary unary relations $\gamma_1, \dots, \gamma_n \subseteq A$ we have*

$$\mathrm{Pol}_A\{R(\gamma_1, \gamma_1, \dots, \gamma_n)\} \subseteq \mathrm{Pol}_A\{R(\gamma_1, \dots, \gamma_n)\}.$$

Proof Let $k \in \mathbb{N}$ and $f \in \mathrm{Pol}_A\{R(\gamma_1, \gamma_1, \dots, \gamma_n)\}$ be a k-ary polymorphism, and consider $r_1, \dots, r_k \in R(\gamma_1, \dots, \gamma_n)$. Let $s_j \in A^{n+1}$ arise from r_j by duplicating the first entry of the tuple (for each $j \in \{1, \dots, k\}$). Then, clearly, $s_1, \dots, s_k \in R(\gamma_1, \gamma_1, \dots, \gamma_n)$, thus $x := f \circ (s_1, \dots, s_k) \in R(\gamma_1, \gamma_1, \dots, \gamma_n)$ as f preserves $R(\gamma_1, \gamma_1, \dots, \gamma_n)$. Since the first and second entry of x are identical and the last n entries of x coincide with $y := f \circ (r_1, \dots, r_k)$, we obtain $y \in R(\gamma_1, \dots, \gamma_n)$. Therefore, we have demonstrated that $f \in \mathrm{Pol}_A\{R(\gamma_1, \dots, \gamma_n)\}$. □

Since the preservation property remains unaffected by variable permutations of relations, we have the following immediate corollary.

Corollary 8 *If $\rho, \rho' \in \mathrm{DD}(\Gamma)$ are such that ρ' arises from ρ by a finite number of applications of the operations of duplicating some parameters and of rearranging the order of parameters, then* $\mathrm{Pol}_A\{\rho'\} \subseteq \mathrm{Pol}_A\{\rho\}$.

Observe that if ρ' arises from $\rho \in \mathrm{DD}(\Gamma)$ as described in the previous corollary, then $\mathrm{pt}(\rho) \leq \mathrm{pt}(\rho')$ holds with respect to the pointwise order of tuples. The increases occur exactly in the places where parameter relations have been duplicated. The only exception to this is the case when ρ is a full power of A, where we may duplicate some (non-canonical) parameter relation $\gamma_1 \neq A$, but observe an increase of $\mathrm{pt}(\rho)(A)$. This issue does not occur if we only duplicate canonical parameters as computed by $p(\rho)$.

Moreover, in this process, no new basic unary relations can be introduced for ρ' that have not already been present as parameters of ρ. This means, if $\mathrm{pt}(\rho)(\gamma) = 0$ for some $\gamma \in \Gamma$, the same must be true for $\mathrm{pt}(\rho')(\gamma)$. In other words, we have that $\mathrm{supp}(\mathrm{pt}(\rho')) \subseteq \mathrm{supp}(\mathrm{pt}(\rho))$. Combining the previous observations we conclude that $\mathrm{pt}(\rho) \sqsubseteq \mathrm{pt}(\rho')$ must be satisfied when transmuting $\rho \rightsquigarrow \rho'$.

In fact the converse is also true, which is the reason for the following crucial lemma, relating the order $\sqsubseteq$ on patterns of disjunctively definable relations and the inclusion of their corresponding polymorphism clones. Note that, as Corollary 8, this lemma does not really depend on the finiteness of Γ, it only depends on the fact that the supports $\mathrm{supp}(\mathrm{pt}(\rho)), \mathrm{supp}(\mathrm{pt}(\rho')) \subseteq \Gamma$ are finite.

Lemma 9 *For $\rho, \rho' \in \mathrm{DD}(\Gamma)$ satisfying $\mathrm{pt}(\rho) \sqsubseteq \mathrm{pt}(\rho')$ we have the dual inclusion* $\mathrm{Pol}_A\{\rho'\} \subseteq \mathrm{Pol}_A\{\rho\}$.

Proof If $\rho, \rho' \in \mathrm{DD}(\Gamma)$ are such that $\mathrm{pt}(\rho) \sqsubseteq \mathrm{pt}(\rho')$, then by verifying that the assumptions of Corollary 8 are fulfilled, we see that $\mathrm{Pol}_A\{\rho'\} \subseteq \mathrm{Pol}_A\{\rho\}$. In more detail, from $\mathrm{pt}(\rho) \sqsubseteq \mathrm{pt}(\rho')$ we get that $\mathrm{supp}(\mathrm{pt}(\rho)) = \mathrm{supp}(\mathrm{pt}(\rho'))$, so in the parameter tuples $p(\rho)$ and $p(\rho')$ the same relations from Γ occur. Since $\mathrm{pt}(\rho) \leq \mathrm{pt}(\rho')$, those that are actually present, occur possibly a few more times in the pattern of ρ' than in that of ρ and are perhaps associated with different coordinates in ρ and ρ'. However this means exactly that ρ' can be obtained from ρ by duplicating and permuting parameters in a finite number of steps (since $\mathrm{supp}(\mathrm{pt}(\rho)) = \mathrm{supp}(\mathrm{pt}(\rho'))$ is finite). □

Now, finally, in order to bound the number of polymorphism clones given by sets Q of relations that are disjunctively definable from Γ, we associate with each such set a downset of $(\mathbb{N}^\Gamma, \sqsubseteq)$, namely the downset generated by the associated patterns. More formally, we define the encoding

$$\begin{aligned} I\colon \mathfrak{P}(\mathrm{DD}(\Gamma)) &\longrightarrow \mathrm{DS}\left(\mathbb{N}^\Gamma, \sqsubseteq\right) \\ Q &\longmapsto {\downarrow_{(\mathbb{N}^\Gamma,\sqsubseteq)}} \mathrm{pt}[Q] = \bigcup_{\rho\in Q} {\downarrow_{(\mathbb{N}^\Gamma,\sqsubseteq)}} \mathrm{pt}(\rho). \end{aligned}$$

With the help of Lemma 9 we can now prove the following relationship:

Proposition 10 *For* $Q_1, Q_2 \subseteq \mathrm{DD}(\Gamma)$ *satisfying* $I(Q_1) \subseteq I(Q_2)$ *we have the inclusion* $\mathrm{Pol}_A Q_2 \subseteq \mathrm{Pol}_A Q_1$.

Proof For $Q_1, Q_2 \subseteq \mathrm{DD}(\Gamma)$ assume $I(Q_1) \subseteq I(Q_2)$. Let $f \in \mathrm{Pol}_A Q_2$ and $\rho \in Q_1$ be chosen arbitrarily. Abbreviating $\mathbb{P} := (\mathbb{N}^\Gamma, \sqsubseteq)$, we have

$$\mathrm{pt}(\rho) \in {\downarrow_{\mathbb{P}}} \mathrm{pt}[Q_1] = I(Q_1) \subseteq I(Q_2) = {\downarrow_{\mathbb{P}}} \mathrm{pt}[Q_2] = \bigcup_{\rho'\in Q_2} {\downarrow_{\mathbb{P}}} \mathrm{pt}(\rho').$$

Hence, there is some $\rho' \in Q_2$ such that $\mathrm{pt}(\rho) \in {\downarrow_{\mathbb{P}}} \mathrm{pt}(\rho')$, i.e., $\mathrm{pt}(\rho) \sqsubseteq \mathrm{pt}(\rho')$. Now Lemma 9 implies $f \in \mathrm{Pol}_A Q_2 \subseteq \mathrm{Pol}_A\{\rho'\} \subseteq \mathrm{Pol}_A\{\rho\}$. □

More important for our target is actually the image $\{\mathrm{Pol}_A Q | Q \subseteq \mathrm{DD}(\Gamma)\}$ of the map $\mathrm{Pol}\colon \mathfrak{P}(\mathrm{DD}(\Gamma)) \longrightarrow \{\mathrm{Pol}_A Q | Q \subseteq \mathrm{DD}(\Gamma)\}$, whose cardinality is equal to that of the factor set $\mathfrak{P}(\mathrm{DD}(\Gamma))/\ker \mathrm{Pol}$ by the kernel. As a straightforward corollary of Proposition 10 the latter is closely related to the kernel of I.

Corollary 11 *On* $\mathfrak{P}(\mathrm{DD}(\Gamma))$ *we have the following inclusion between equivalence relations:* $\ker I = \{(Q_1, Q_2) \in \mathfrak{P}(\mathrm{DD}(\Gamma))^2 \,|\, I(Q_1) = I(Q_2)\} \subseteq \ker \mathrm{Pol}$.

This result allows us to establish an upper bound on the number of clones determined by relations that are disjunctively definable over Γ.

Corollary 12 *We have* $|\mathrm{imPol}| \leq |\mathrm{im} I| \leq \left|\mathrm{DS}\left(\mathbb{N}^\Gamma, \sqsubseteq\right)\right|$.

Proof Since $\ker I \subseteq \ker \mathrm{Pol}$, there is a canonical well-defined surjection from the factor set $\mathfrak{P}(\mathrm{DD}(\Gamma))/\ker I$ onto $\mathfrak{P}(\mathrm{DD}(\Gamma))/\ker \mathrm{Pol}$, so

$$\mathrm{imPol} \cong \mathfrak{P}(\mathrm{DD}(\Gamma))/\ker \mathrm{Pol} \twoheadleftarrow \mathfrak{P}(\mathrm{DD}(\Gamma))/\ker I \cong \mathrm{im} I \subseteq \mathrm{DS}\left(\mathbb{N}^\Gamma, \sqsubseteq\right),$$

telling us that the cardinality of $|\mathrm{im}\,\mathrm{Pol}| = |\mathfrak{P}(\mathrm{DD}(\Gamma))/\ker \mathrm{Pol}|$ is bounded above by that of $|\mathfrak{P}(\mathrm{DD}(\Gamma))/\ker I| = |\mathrm{im} I| \leq \left|\mathrm{DS}\left(\mathbb{N}^\Gamma, \sqsubseteq\right)\right|$. □

An alternative proof of the previous fact can be obtained by noting that the following map ψ, representing disjunctively definable clones as downsets of patterns, is injective. Moreover, it even embeds the whole ordered structure of such clones.

Proposition 13 *The map*

$$\begin{array}{rccc} \psi\colon & (\{\mathrm{Pol}_A Q \mid Q \subseteq \mathrm{DD}(\Gamma)\}, \supseteq) & \longrightarrow & \left(\mathrm{DS}\left(\mathbb{N}^\Gamma, \sqsubseteq\right), \subseteq\right) \\ & F = \mathrm{Pol}_A Q & \longmapsto & \mathrm{pt}\left[F'\right], \end{array}$$

where $F' = \{\rho \in \mathrm{DD}(\Gamma) \mid \forall f \in F\colon f \rhd \rho\}$, is a well-defined order embedding, and it makes the following diagram commute:

$$\begin{array}{ccc} \mathfrak{P}(\mathrm{DD}(\Gamma)) & \xrightarrow{\quad I \quad} & \mathrm{DS}\left(\mathbb{N}^\Gamma, \sqsubseteq\right) \\ \downarrow \mathrm{Pol} & \swarrow \phi & \uparrow \psi \\ \mathrm{im}\,\mathrm{Pol} & \xleftarrow[\mathrm{id}_{\mathrm{imPol}}]{} & \mathrm{im}\,\mathrm{Pol} \end{array}$$

where ϕ is any factor map (cf. Corollary 11) satisfying $\phi(I(Q)) = \mathrm{Pol}_A Q$ on the image of I and being defined as, e.g., $\phi(U) = \mathrm{Pol}_A \emptyset$ anywhere else.

Proof In this proof, let us abbreviate $Q^{\backprime} := \mathrm{Pol}_A Q$ for any set $Q \subseteq \mathrm{DD}(\Gamma)$. This is to emphasize that the pair $({}^{\backprime}, {}')$ forms an (antitone) Galois connection between finitary operations on A and disjunctively definable relations with respect to Γ. To demonstrate that ψ is well-defined, we need to show that $\mathrm{pt}\left[F'\right] \subseteq \mathbb{N}^\Gamma$ is a downset with respect to $\sqsubseteq$ given $F = Q^{\backprime}$ for some $Q \subseteq \mathrm{DD}(\Gamma)$. So let $\rho_1 \in F'$ be m-ary and $x \in \mathbb{N}^\Gamma$ with $x \sqsubseteq \mathrm{pt}(\rho_1)$. Thus $\mathrm{pt}(\rho_1)$ has height m above the zero tuple, so it is sufficient to consider the case where x is a lower cover of $\mathrm{pt}(\rho_1)$, i.e., there is exactly one $\gamma_1 \in \Gamma$ such that $x(\gamma_1) = \mathrm{pt}(\rho_1)(\gamma_1) - 1 > 0$, and x coincides with $\mathrm{pt}(\rho_1)$ everywhere else. Since F' is closed under forming relations with permuted coordinates and this operation does not change the pattern of a disjunctively definable relation, we can assume that $\rho_1 = R(\gamma_1, \gamma_1, \gamma_2, \ldots, \gamma_{m-1})$ for (not necessarily distinct) $\gamma_2, \ldots, \gamma_{m-1} \in \Gamma$. Now the relation $\rho_2 = R(\gamma_1, \gamma_2, \ldots, \gamma_{m-1}) \in \mathrm{DD}(\Gamma)$ satisfies $\mathrm{pt}(\rho_2) = x$ (cf. the observation made in (†)) and, by Lemma 7, $\{\rho_1\}^{\backprime} \subseteq \{\rho_2\}^{\backprime}$, so $\rho_2 \in \{\rho_2\}^{\backprime\prime} \subseteq \{\rho_1\}^{\backprime\prime} \subseteq F'^{\backprime\prime} = F'$ and $x \in \mathrm{pt}\left[F'\right]$.

It is obvious that ψ is order preserving. To prove that it is order reflecting, consider $F_1 = Q_1^{\backprime}$ and $F_2 = Q_2^{\backprime}$ for $Q_1, Q_2 \subseteq \mathrm{DD}(\Gamma)$ such that $\psi(F_1) \subseteq \psi(F_2)$ and any $\rho_1 \in F_1'$. Then we have $\mathrm{pt}(\rho_1) \in \psi(F_1) \subseteq \psi(F_2) = \mathrm{pt}\left[F_2'\right]$, so there is some $\rho_2 \in F_2'$ such that $\mathrm{pt}(\rho_1) = \mathrm{pt}(\rho_2)$. Applying Lemma 9 twice, we obtain $\{\rho_1\}^{\backprime} = \{\rho_2\}^{\backprime}$ and hence $\rho_1 \in \{\rho_1\}^{\backprime\prime} = \{\rho_2\}^{\backprime\prime} \subseteq F_2'^{\backprime\prime} = F_2'$. Thus $F_1' \subseteq F_2'$ and $F_1 = F_1'^{\backprime} \supseteq F_2'^{\backprime} = F_2$.

Order reflection clearly implies that ψ is injective. It remains to be shown that $\phi(\psi(F)) = F$ for every $F = Q^{\backprime}$ where $Q \subseteq \mathrm{DD}(\Gamma)$. As $\psi(F) \in \mathrm{DS}\left(\mathbb{N}^\Gamma, \sqsubseteq\right)$, we have $\psi(F) = {\downarrow_{(\mathbb{N}^\Gamma, \sqsubseteq)}}\, \psi(F) = {\downarrow_{(\mathbb{N}^\Gamma, \sqsubseteq)}}\, \mathrm{pt}\left[F'\right] = I(F')$, so $\phi(\psi(F)) = \phi(I(F')) = F'^{\backprime} = F$. □

4 Results

By Corollary 12 from the previous section we have transferred the task of counting the number of clones given by relations that are disjunctively definable over a fixed (commonly finite) set of unary predicates Γ to the investigation of the downsets of the poset $\left(\mathbb{N}^{\Gamma}, \sqsubseteq\right)$. This is a special case of the poset $\left(\mathbb{N}^{I}, \sqsubseteq\right)$ from Section 2, where we now have the additional assumption that I is finite. Concerning downsets of this poset, we first verify the following general facts.

Lemma 14 *Let $\mathbb{P} = \left(\mathbb{N}^{I}, \sqsubseteq\right)$ be the poset defined in* Section 2.

(a) *The set $Y_{\mathrm{F}} := \{x \in \mathbb{N}^{I} | \mathrm{supp}(x) = I\}$ is a downset of $\mathbb{P}$, and the induced subposet $\mathbb{Y}_{\mathrm{F}} = \left(Y_{\mathrm{F}}, \sqsubseteq\restriction_{Y_{\mathrm{F}}}\right)$ satisfies $\mathbb{Y}_{\mathrm{F}} \cong \left((\mathbb{N}_{+})^{I}, \leq\right) \cong \left(\mathbb{N}^{I}, \leq\right)$.*
(b) *For every $J \subseteq I$ the set $Y_{\subseteq J} := \{x \in \mathbb{N}^{I} | \mathrm{supp}(x) \subseteq J\}$ is a downset of $\mathbb{P}$; the induced subposet $\mathbb{Y}_{\subseteq J} = \left(Y_{\subseteq J}, \sqsubseteq\restriction_{Y_{\subseteq J}}\right)$ is isomorphic to $\left(\mathbb{N}^{J}, \sqsubseteq\right)$.*
(c) *We have $\mathbb{N}^{I} = Y_{\mathrm{F}} \cup \bigcup_{i \in I} Y_{\subseteq I\setminus\{i\}}$.*

Proof (a) If $x \in Y_{\mathrm{F}}$ and $y \in \mathbb{N}^{I}$ satisfies $y \sqsubseteq x$, then $I = \mathrm{supp}(x) = \mathrm{supp}(y)$. Hence, $y \in Y_{\mathrm{F}}$, and so $Y_{\mathrm{F}} \in \mathrm{DS}\,(\mathbb{P})$. Moreover, for $x \in \mathbb{N}^{I}$, we have $x \in Y_{\mathrm{F}}$ if and only if $x(i) \neq 0$ for all $i \in I$, i.e. if $x \in (\mathbb{N}_{+})^{I}$. Thus the identical map induces an isomorphism $\mathrm{id}\colon \mathbb{Y}_{\mathrm{F}} \longrightarrow \left((\mathbb{N}_{+})^{I}, \leq\right)$ since all elements $x, y \in (\mathbb{N}_{+})^{I}$ automatically satisfy $\mathrm{supp}(x) = I = \mathrm{supp}(y)$. Besides, the map $h\colon \left((\mathbb{N}_{+})^{I}, \leq\right) \longrightarrow \left(\mathbb{N}^{I}, \leq\right)$ given by $h(x) = (x(i) - 1)_{i \in I}$ obviously is an isomorphism, too.

(b) Consider $J \subseteq I$ and $x \in Y_{\subseteq J}$. Any $y \sqsubseteq x$ fulfils $\mathrm{supp}(y) = \mathrm{supp}(x) \subseteq J$, so $y \in Y_{\subseteq J}$. Thus, $Y_{\subseteq J} \in \mathrm{DS}\,(\mathbb{P})$. Define $\mathrm{pr}_J\colon Y_{\subseteq J} \longrightarrow \mathbb{N}^{J}$ by letting $\mathrm{pr}_J(x) := (x(j))_{j \in J}$; conversely, for every element $y \in \mathbb{N}^{J}$ define $\mathrm{emb}_J(y)$ by $\mathrm{emb}_J(y)(i) := y(i)$ if $i \in J$ and $\mathrm{emb}_J(y)(i) := 0$, otherwise. The map $\mathrm{emb}_J\colon \mathbb{N}^{J} \longrightarrow Y_{\subseteq J}$ is inverse to pr_J since every $x \in Y_{\subseteq J}$ satisfies $\mathrm{supp}(x) \subseteq J$, i.e. every $i \in I \setminus J$ does not belong to $\mathrm{supp}(x)$ and thus $x(i) = 0$. For the same reason, $\mathrm{supp}(x) = \mathrm{supp}(\mathrm{pr}_J(x))$ holds for all $x \in Y_{\subseteq J}$, whence pr_J is a homomorphism. Furthermore, we have the equality $\mathrm{supp}(y) = \mathrm{supp}(\mathrm{emb}_J(y))$ for all $y \in \mathbb{N}^{J}$, so emb_J is a homomorphism, too.

(c) The inclusion $\mathbb{N}^{I} \supseteq Y_{\mathrm{F}} \cup \bigcup_{i \in I} Y_{\subseteq I\setminus\{i\}}$ holds by definition. Moreover, if $x \in \mathbb{N}^{J} \setminus Y_{\mathrm{F}}$, then $\mathrm{supp}(x) \subsetneq I$, so there exists some $i \in I \setminus \mathrm{supp}(x)$, i.e. $\mathrm{supp}(x) \subseteq I \setminus \{i\}$. This proves that $x \in \bigcup_{i \in I} Y_{\subseteq I\setminus\{i\}}$.

□

Theorem 15 *We have $\left|\mathrm{DS}\left(\mathbb{N}^{I}, \sqsubseteq\right)\right| = \aleph_0$ for all finite $I \neq \emptyset$.*

Proof Since $\sqsubseteq \subseteq \leq$, we clearly have $\mathrm{DS}\left(\mathbb{N}^{I}, \leq\right) \subseteq \mathrm{DS}\left(\mathbb{N}^{I}, \sqsubseteq\right)$, which together with Lemma 5 proves that $\left|\mathrm{DS}\left(\mathbb{N}^{I}, \sqsubseteq\right)\right| \geq \left|\mathrm{DS}\left((\mathbb{N}, \leq)^{I}\right)\right| = \aleph_0$ for all finite non-empty I. We shall prove by induction on $|I|$ that $\left|\mathrm{DS}\left(\mathbb{N}^{I}, \sqsubseteq\right)\right| \leq \aleph_0$ holds for all finite sets I. The basis is the case $|I| = 0$, i.e. $I = \emptyset$. Then $|\mathbb{N}^{I}| = 1$, so we are dealing with a finite poset having only finitely many downsets. Now assume $|I| > 0$, i.e. $I \neq \emptyset$, and suppose we know the truth of the claim already for all finite J, $|J| < |I|$. In particular, we have the induction hypothesis for all $I \setminus \{i\}$ where $i \in I$. We define

$$\begin{aligned}\delta:\ \mathrm{DS}\left(\mathbb{N}^I, \sqsubseteq\right) &\longrightarrow \mathrm{DS}\left(\mathbb{Y}_{\mathrm{F}}\right) \times \prod\nolimits_{i\in I} \mathrm{DS}\left(\mathbb{Y}_{\subseteq I\setminus\{i\}}\right)\\ X &\longmapsto \left(X \cap Y_{\mathrm{F}}, \left(X \cap Y_{\subseteq I\setminus\{i\}}\right)_{i\in I}\right).\end{aligned}$$

By Lemma 14(a) and (b) in combination with Lemma 3, this map is well-defined. Moreover, due to Lemma 14(c), we have

$$(X \cap Y_{\mathrm{F}}) \cup \bigcup_{i\in I} \left(X \cap Y_{\subseteq I\setminus\{i\}}\right) = X \cap \left(Y_{\mathrm{F}} \cup \bigcup_{i\in I} Y_{\subseteq I\setminus\{i\}}\right) = X \cap \mathbb{N}^I = X,$$

so δ is injective. Hence, $\left|\mathrm{DS}\left(\mathbb{N}^I, \sqsubseteq\right)\right| \leq \left|\mathrm{DS}\left(\mathbb{Y}_{\mathrm{F}}\right) \times \prod_{i\in I} \mathrm{DS}\left(\mathbb{Y}_{\subseteq I\setminus\{i\}}\right)\right|$. Since for $i \in I$, we have $\mathbb{Y}_{\subseteq I\setminus\{i\}} \cong \left(\mathbb{N}^{I\setminus\{i\}}, \sqsubseteq\right)$ by Lemma 14(b), Lemma 2 together with the induction hypothesis yields $\left|\mathrm{DS}\left(\mathbb{Y}_{\subseteq I\setminus\{i\}}\right)\right| = \left|\mathrm{DS}\left(\mathbb{N}^{I\setminus\{i\}}, \sqsubseteq\right)\right| \leq \aleph_0$. Similarly, we have $\mathbb{Y}_{\mathrm{F}} \cong (\mathbb{N}, \leq)^I$ by Lemma 14(a), wherefore Lemmas 2 and 5 jointly imply that $|\mathrm{DS}\left(\mathbb{Y}_{\mathrm{F}}\right)| = \left|\mathrm{DS}\left((\mathbb{N}, \leq)^I\right)\right| = \aleph_0$. Consequently, as a finite product of countable sets, one of which is infinite, the co-domain of δ has cardinality $\aleph_0$, whence $\left|\mathrm{DS}\left(\mathbb{N}^I, \sqsubseteq\right)\right|$ is countable. □

As a result the number of clones on a fixed set determined by disjunctions of finitely many unary predicates is countable.

Theorem 16 *For every finite unary relational language* Γ *we have*

$$|\{\mathrm{Pol}_A Q | Q \subseteq \mathrm{DD}(\Gamma)\}| \leq \aleph_0.$$

Proof From Corollary 12 we know $|\{\mathrm{Pol}_A Q | Q \subseteq \mathrm{DD}(\Gamma)\}| \leq \left|\mathrm{DS}\left(\mathbb{N}^\Gamma, \sqsubseteq\right)\right|$. By Theorem 15 the latter is countably infinite if $\Gamma \neq \emptyset$, and it has two elements if $\Gamma = \emptyset$ (in this case we are dealing only with the clone of all operations). □

This already demonstrates that a classification of such clones (as requested in [18, Section 6]) is not a hopeless task.

As a final step we wish to show that in all relevant cases, our cardinality bound from Theorem 16 is tight. This generalizes an argument given in [37, Proposition 3.1] regarding the number of clones determined by (mixed) clausal relations (an important subcase of the relations considered in [18]).

Proposition 17 *If* Γ *is any unary relational language with carrier set* A *containing a non-trivial basic unary relation* $\gamma \in \Gamma$ *such that* $\emptyset \neq \gamma \subsetneq A$, *then the lattice* $(\{\mathrm{Pol}_A Q | Q \subseteq \mathrm{DD}(\Gamma)\}, \subseteq)$ *contains a strictly descending* ω*-chain of finitely related clones over* $\mathrm{DD}(\Gamma)$.

Proof The proof of this fact is constructive. Fix $\gamma \in \Gamma$ with the properties claimed in the proposition. For every $m \in \mathbb{N}_+$ we let $\rho_m := R(\gamma, \ldots, \gamma)$, where the parameter

γ occurs exactly m times. By Lemma 7, we have $\mathrm{Pol}_A\{\rho_m\} \subseteq \mathrm{Pol}_A\{\rho_{m-1}\}$ for all $m \in \mathbb{N}$, $m \geq 2$. We only have to prove that these inclusions are strict. This will be done by exhibiting an m-ary function $f \in \mathrm{Pol}_A\{\rho_{m-1}\}$ that does not preserve ρ_m.

Since $\gamma \subsetneq A$, there is an element $0 \in A$ such that $0 \notin \gamma$. Moreover, as γ is not empty, there is some element $1 \in A\setminus\{0\}$ such that $1 \in \gamma$. We define $f(x_1, \dots, x_m) = 0$ if at least $m-1$ entries in $(x_1, \dots, x_m)$ are equal to 0 and we put $f(x_1, \dots, x_m) = 1$ everywhere else.

Let $e_j \in A^m$ be the tuple whose j-th entry is 1 and which is 0 otherwise. Since $1 \in \gamma$, we have $e_1, \dots, e_m \in \rho_m$. However, applying f yields $f \circ (e_1, \dots, e_m) = \mathbf{0}$, the tuple containing only zeros. As $0 \notin \gamma$, we have $\mathbf{0} \notin \rho_m$ and hence $f \notin \mathrm{Pol}_A\{\rho_m\}$.

On the other hand, if $r_1, \dots, r_m \in \rho_{m-1}$ and we consider these tuples as columns of an $((m-1) \times m)$-matrix, then each column contains a non-zero entry (because $\mathbf{0} \notin \rho_{m-1}$). As $m > m-1$, by the pigeonhole principle there must be one row x of the matrix that contains at least two entries distinct from zero. In other words, there cannot be more than $m-2$ zero entries in x. Now applying f to x yields $f(x) = 1 \in \gamma$. Therefore, $f \circ (r_1, \dots, r_m) \in \rho_{m-1}$. □

Combining Theorem 16 and Proposition 17 we can pinpoint the exact number of clones determined by disjunctions of non-trivial unary relations from a finite parameter set Γ. This answers, in particular, the question regarding the number of clausal clones on finite sets that was stated to be open in [11, 38].

Corollary 18 *If Γ is a finite unary relational language with carrier set A containing a non-trivial basic unary relation $\gamma \in \Gamma$ such that $\emptyset \neq \gamma \subsetneq A$, then we have $|\{\mathrm{Pol}_A Q | Q \subseteq \mathrm{DD}(\Gamma)$ finite$\}| = |\{\mathrm{Pol}_A Q | Q \subseteq \mathrm{DD}(\Gamma)\}| = \aleph_0$.*

Remark 19 When focussing only on finitely related polymorphism clones, we can extend the scope of our arguments a little bit. That is to say, if we restrict the encoding map I (cf. Proposition 10) to $\mathfrak{P}_{\mathrm{fin}}(\mathrm{DD}(\Gamma))$ and only consider clones $\mathrm{Pol}_A Q$ given by a finite subset $Q \subseteq \mathrm{DD}(\Gamma)$, we can still obtain the upper bound $\aleph_0$ on their cardinality when Γ is countably infinite. Namely, if $Q = \{\rho_1, \dots, \rho_N\}$ for some $N \in \mathbb{N}$, then $I(Q) = \bigcup_{i=1}^{N} \downarrow_{(\mathbb{N}^\Gamma, \sqsubseteq)} \mathrm{pt}(\rho_i)$ is a finitely generated downset, where for each $i \in \{1, \dots, N\}$ the tuples in the principal downset $\downarrow_{(\mathbb{N}^\Gamma, \sqsubseteq)} \mathrm{pt}(\rho_i)$ have a fixed finite support $J_i \subseteq \Gamma$. Hence, all tuples in $I(Q)$ have their support within the finite set $J(Q) = \bigcup_{i=1}^{N} J_i \subseteq \Gamma$. By projecting the patterns to these indices (as in the proof of Lemma 14 this does not change the support of the tuples), we obtain a different encoding $K(Q) = \bigcup_{i=1}^{N} \downarrow_{(\mathbb{N}^{J(Q)}, \sqsubseteq)} \mathrm{pr}_{J(Q)}\mathrm{pt}(\rho_i)$, which is a downset of $\left(\mathbb{N}^{J(Q)}, \sqsubseteq\right)$. Therefore, $K(Q) \in \bigcup_{J \in \mathfrak{P}_{\mathrm{fin}}(\Gamma)} \mathrm{DS}\left(\mathbb{N}^J, \sqsubseteq\right)$. Since Γ is countable, $\mathfrak{P}_{\mathrm{fin}}(\Gamma)$ is countable and the codomain of K is a countable union of countably infinite sets (see Theorem 15), hence countably infinite. With analogous arguments as in Proposition 10 through Corollary 12 we can thus show that $|\{\mathrm{Pol}_A Q | Q \subseteq \mathrm{DD}(\Gamma) \text{ finite}\}| \leq |\mathrm{im} K| \leq \aleph_0$. Concerning the analogue of Proposition 10 we note that $\emptyset \neq K(Q_1) \subseteq K(Q_2)$ implies $J(Q_1) = J(Q_2)$ and so $\mathrm{pr}_{J(Q_1)}\mathrm{pt}(\rho) \sqsubseteq \mathrm{pr}_{J(Q_2)}\mathrm{pt}(\rho')$ yields $\mathrm{pt}(\rho) \sqsubseteq \mathrm{pt}(\rho')$ for $\rho \in Q_1$ and any $\rho' \in Q_2$. As $\mathrm{supp}(\mathrm{pt}(\rho)) = \mathrm{supp}(\mathrm{pt}(\rho')) \subseteq J(Q_2)$ is finite, Corollary 8 and Lemma 9 are applicable to derive the conclusion of Proposition 10.

Tightness of the cardinality bound is again provided by Proposition 17, which does not require finiteness of Γ.

Combining these explanations with Corollary 18 we can strengthen our result as follows:

Corollary 20 *If* Γ *is a countable unary relational language on the carrier set* A *containing a non-trivial basic unary relation* $\gamma \in \Gamma$ *such that* $\emptyset \neq \gamma \subsetneq A$*, then we have* $|\{\mathrm{Pol}_A Q \mid Q \subseteq \mathrm{DD}(\Gamma)$ *finite*$\}| = \aleph_0$.

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