Product structure and regularity theorem for totally nonnegative flag varieties

Huanchen Bao[1] · Xuhua He[2]

Abstract

The totally nonnegative flag variety was introduced by Lusztig. It has enriched combinatorial, geometric, and Lie-theoretic structures. In this paper, we introduce a (new) J-total positivity on the full flag variety of an arbitrary Kac-Moody group, generalizing the (ordinary) total positivity.

We show that the J-totally nonnegative flag variety has a cellular decomposition into totally positive J-Richardson varieties. Moreover, each totally positive J-Richardson variety admits a favorable decomposition, called a product structure. Combined with the generalized Poincare conjecture, we prove that the closure of each totally positive J-Richardson variety is a regular CW complex homeomorphic to a closed ball. Moreover, the J-total positivity on the full flag provides a model for the (ordinary) totally nonnegative partial flag variety. As a consequence, we prove that the closure of each (ordinary) totally positive Richardson variety is a regular CW complex homeomorphic to a closed ball, confirming conjectures of Galashin, Karp and Lam in (Adv. Math. 351:614–620, 2019). We also show that the link of the totally nonnegative part of U^- for any Kac-Moody group forms a regular CW complex. This generalizes the result of Hersh (Invent. Math. 197(1):57–114, 2014) for reductive groups.

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Contents

✉ X. He
xuhuahe@hku.hk

H. Bao
huanchen@nus.edu.sg

1 Department of Mathematics, National University of Singapore, Singapore, Singapore

2 Department of Mathematics and New Cornerstone Science Laboratory, The University of Hong Kong, Pokfulam, Hong Kong, Hong Kong SAR, China

1 Introduction

1.1 Totally nonnegative flag varieties of reductive groups

The theory of total positivity on the reductive groups G and their partial flag varieties $\mathcal{P}_K$ was introduced by Lusztig in the seminal work [25]. The totally nonnegative partial flag variety $\mathcal{P}_{K,\geqslant 0}$ is a "remarkable polyhedral subspace" (cf. [25]). It has many nice combinatorial, geometric, and Lie-theoretic properties. And it has been used in many other areas, such as cluster algebras [9], the Grassmann polytopes [22], the physics of scattering amplitudes [1].

We give a quick review of $\mathcal{P}_{K,\geqslant 0}$. Let G be a connected reductive group, split over $\mathbb{R}$ and $B^{\pm} = TU^{\pm}$ be the Borel and opposite Borel subgroups of G. The full flag variety $\mathcal{B} = G/B^+$ admits the decompositions into Schubert cells and opposite Schubert cells, both indexed by the Weyl group W of G. The intersection of a Schubert cell $B^+ w B^+/B^+$ with an opposite Schubert cell $B^- v B^+/B^+$ is called an (open) Richardson variety, and is denoted by $\mathcal{B}_{v,w}$. The variety $\mathcal{B}_{v,w}$ is nonempty if and only if $v \leqslant w$, where $\leqslant$ is the Bruhat order on W.

Let I be the set of simple roots in G. Let $P_K^+ \supset B^+$ be the standard parabolic subgroup associated to a subset K of I. For the partial flag $\mathcal{P}_K = G/P_K^+$, we have the decomposition into the projected Richardson varieties $\mathcal{P}_K = \bigsqcup_{\alpha \in Q_K} \mathcal{P}_{K,\alpha}$. The definition and the closure relation of the projected Richardson varieties are more complicated and we skip the details in the introduction.

Let $U_{\geqslant 0}^-$ be the totally nonnegative part of U^-. The totally nonnegative part $\mathcal{P}_{K,\geqslant 0}$ of the partial flag variety $\mathcal{P}_K$ is by definition, the closure of $U_{\geqslant 0}^- P_K^+/P_K^+$ in $\mathcal{P}_K$. In the case where $\mathcal{P}_K$ is the Grassmannian, $\mathcal{P}_{K,\geqslant 0}$ is the totally nonnegative Grassmannian [35].

The totally positive projected Richardson variety $\mathcal{P}_{K,\alpha,>0}$ is, by definition, the intersection of the totally nonnegative partial flag $\mathcal{P}_{K,\geqslant 0}$ with the projected Richardson variety $\mathcal{P}_{K,\alpha}$. We then have the stratification

$$\mathcal{P}_{K,\geqslant 0} = \bigsqcup_{\alpha \in Q_K} \mathcal{P}_{K,\alpha,>0}.$$

We have many remarkable properties on the totally positive projected Richardson varieties.

(1) $\mathcal{P}_{K,\geqslant 0}$ admits a natural monoid action of $G_{\geqslant 0}$ and a natural duality (see [23]);
(2) $\mathcal{P}_{\geqslant 0}$ admits a representation-theoretic interpretation via canonical basis (see [23] and [24]);
(3) $\mathcal{P}_{K,\alpha,>0}$ is a cell and is a connected component of $\mathcal{P}_{K,\alpha}(\mathbb{R})$ (see [38]);
(4) The closure of $\mathcal{P}_{K,\alpha,>0}$ is a union of $\mathcal{P}_{K,\alpha',>0}$ for some α' (see [39]);
(5) The cell decomposition $\overline{\mathcal{P}_{K,\alpha,>0}} = \bigsqcup_{\alpha'} \mathcal{P}_{K,\alpha',>0}$ is a regular CW complex.

The last property is called the regularity theorem of $\mathcal{P}_{K,\geqslant 0}$. In particular, the closure $\overline{\mathcal{P}_{K,\alpha,>0}}$ is homeomorphic to a closed ball. It was conjectured by Postnikov for totally nonnegative Grassmannian and by Williams [44] for totally nonnegative partial flag varieties of split real reductive groups. Important progress has been made in [11, 13, 36, 40, 41]. It was finally established by Galashin, Karp and Lam [12].

1.2 Totally nonnegative Kac-Moody flag varieties

The theory of total positivity on the reductive groups and their flag varieties have been generalized to arbitrary Kac-Moody groups by Lusztig in a series of papers [27–30] and [31], and by us in [2]. For the full flag variety of an arbitrary Kac-Moody group, we proved in [2] that the totally nonnegative flag variety $\mathcal{B}_{\geqslant 0}$ has a representation-theoretic interpretation, is a union of totally positive Richardson varieties, and each totally positive Richardson variety is a cell.

However, the closure relations among the cells and the geometric structure of these closures were not established. For reductive groups, there is a natural duality coming from $B^+ \leftrightarrow B^-$, which plays a significant role in establishing geometric properties of the flag varieties. Such duality does not exist for Kac-Moody groups, which leads to extra difficulty in the study of totally nonnegative flag varieties for the general Kac-Moody groups than the reductive groups. We shall overcome the obstacles and establish results in the general setting of J-total positivity using the "product structure".

1.3 J-Total positivity

Unless otherwise stated, in the rest of this paper we assume that G is a Kac-Moody group, split over $\mathbb{R}$. We fix a subset J of I. Let ${}^J B^+ \subset P_J^+$ be the Borel subgroup opposite to B^+ and ${}^J B^- \subset P_J^-$ be the Borel subgroup opposite to B^-. The ${}^J B^+$-orbits on $\mathcal{B} = G/B^+$ are called the J-Schubert cells and ${}^J B^-$-orbits on $\mathcal{B} = G/B^+$ are called the opposite J-Schubert cells, respectively. For $v, w \in W$, the open J-Richardson variety is defined to be

$$^J\mathcal{B}_{v,w} = {}^J B^+ w B^+ / B^+ \bigcap {}^J B^- v B^+ / B^+.$$

It is known that $^J\mathcal{B}_{v,w} \neq \emptyset$ if and only if $v \,{}^J\!\leqslant w$, where $^J\!\leqslant$ is the J-twisted Bruhat order.

Our motivation to study the J-Richardson varieties comes from the partial flag varieties. The projected Richardson varieties in a partial flag variety $\mathcal{P}_K$ and their geometric structures come from the projection map $\mathcal{B} \to \mathcal{P}_K$. Roughly speaking, the projection map $\mathcal{B} \to \mathcal{P}_K$ folds the Richardson varieties in a rather complicated way,

which makes the projected Richardson varieties rather complicated to study. In [3] we introduced an "atlas model" $\mathcal{P}_K \dashrightarrow \tilde{\mathcal{B}}$ of the partial flag variety and regarded the projected Richardson varieties in $\mathcal{P}_K$ as certain J-Richardson varieties in the full flag variety $\tilde{\mathcal{B}}$ of another (larger) Kac-Moody group.

However, the "atlas model" $\mathcal{P}_K \dashrightarrow \tilde{\mathcal{B}}$ is not compatible with the total positivity on $\mathcal{P}_K$ and the ordinary total positivity $\tilde{\mathcal{B}}$. This should not be a surprise, as the ordinary total positivity on $\tilde{\mathcal{B}}$ is suitable for the decomposition into the (ordinary) Richardson varieties, not the J-Richardson varieties. To provide a "model" for the total positivity on $\mathcal{P}_K$, we introduce the J-total positivity. The J-total positivity on the flag variety is "compatible" with the stratification by J-Richardson varieties.

It is worth mentioning that when $J = \emptyset$, the J-Schubert (resp. opposite J-Schubert) varieties are just the Schubert (resp. opposite Schubert) varieties; the J-total positivity coincides with the ordinary total positivity. Therefore, our main results apply to the setting of the ordinary total positivity. If the Weyl group W_J is finite, then the J-total positivity can be obtained from the ordinary total positivity by multiplying $\dot{w}_J$ on the left, where w_J is the longest element of W_J. In general, the J-total positivity is quite different from the ordinary total positivity.

1.4 The main results on the J-total positivity

We set

$$^J U_{\geqslant 0}^- = \{h_1 \pi_J(h_2)^{-1} h_2; h_1 \in U_{J,\geqslant 0}^-, h_2 \in U_{\leqslant 0}^-\}.$$

Here U_J^- is the unipotent radical of the opposite Borel subgroup in the Levi subgroup L_J of G and π_J is the projection map from the opposite parabolic subgroup P_J^- to its Levi subgroup L_J. We define the J-totally nonnegative flag variety

$$^J\mathcal{B}_{\geqslant 0} = \overline{^J U_{\geqslant 0}^- \cdot B^+} \subset \mathcal{B}.$$

For any $w_1 {}^J\!\leqslant w_2$, we set $^J\mathcal{B}_{w_1,w_2,>0} = {}^J\mathcal{B}_{\geqslant 0} \bigcap {}^J\mathcal{B}_{w_1,w_2}$. We call $^J\mathcal{B}_{w_1,w_2,>0}$ the totally positive J-Richardson variety.[1] Note that the definition of $^J U_{\geqslant 0}^-$ is a mixture of the totally positive and totally negative parts $U_{\leqslant 0}^-$ of U^-. The J-total positivity is more difficult to study than the ordinary total positivity on $\mathcal{B}$. Some major differences between the J-total positivity and the ordinary total positivity are

- the totally nonnegative flag $\mathcal{B}_{\geqslant 0}$ admits a natural action of the totally nonnegative monoid $G_{\geqslant 0}$, while the J-totally nonnegative flag $^J\mathcal{B}_{\geqslant 0}$ only admits a natural action of totally nonnegative submonoid $L_{J,\geqslant 0}$;
- the totally nonnegative flag $\mathcal{B}_{\geqslant 0}$ has a nice representation-theoretic interpretation via Lusztig's canonical basis. In contrast, the positivity property of the canonical basis is not preserved for the J-total positivity.

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[1]This should be called J-totally positive J-Richardson variety to be precise. But since we never consider the interaction between the ordinary total positivity and J-Richardson varieties, this should not cause any confusion.

It is worth mentioning that the symmetry (of $G_{\geqslant 0}$), the representation-theoretic interpretation and the duality $B^+ \leftrightarrow B^-$ play a crucial role in the previous study of the totally nonnegative flag $\mathcal{B}_{\geqslant 0}$ of reductive groups. However, none of these features are available for the J-totally nonnegative flag variety of a general Kac-Moody group. Thus we need to develop a new strategy to study the J-total positivity.

Our starting point is the open covering $\mathcal{B} = \bigcup_{w \in W} wU^-B^+/B^+$ and the isomorphisms

$$^J c_w : wU^-B^+/B^+ \cong {}^JB^+wB^+/B^+ \times {}^JB^-wB^+/B^+.$$

The idea of such isomorphism dates back to Kazhdan and Lusztig [17], see also [18].

Our first main result on J-total positivity is the following. Part (2) is new even for the ordinary total positivity for the full flag variety of reductive groups.

Theorem A (Proposition 5.9, Theorem 5.10) *Let $w_1 \, {}^J\!\leqslant w_3 \, {}^J\!\leqslant w_2$. Then*
(1) ${}^J\mathcal{B}_{w_1,w_2,>0} \subset w_3 U^-B^+/B^+.$
(2) *The map ι_{w_3} induces an isomorphism*

$$^J\mathcal{B}_{w_1,w_2,>0} \cong {}^J\mathcal{B}_{w_1,w_3,>0} \times {}^J\mathcal{B}_{w_3,w_2,>0}.$$

We call the isomorphism in part (2) of Theorem A *the product structure* of ${}^J\mathcal{B}_{w_1,w_2,>0}$. If we fix w_3, but let w_1 and w_2 vary, then we obtain an isomorphism

$$\bigsqcup_{w_1 \, {}^J\!\leqslant w_3 \, {}^J\!\leqslant w_2} {}^J\mathcal{B}_{w_1,w_2,>0} \cong \bigsqcup_{w_1 \, {}^J\!\leqslant w_3} {}^J\mathcal{B}_{w_1,w_3,>0} \times \bigsqcup_{w_3 \, {}^J\!\leqslant w_2} {}^J\mathcal{B}_{w_3,w_2,>0}. \qquad (\star)$$

We call it the *product structure* of $\bigsqcup_{w_1 \, {}^J\!\leqslant w_3 \, {}^J\!\leqslant w_2} {}^J\mathcal{B}_{w_1,w_2,>0}$. It allows us to understand ${}^J\mathcal{B}_{w_1,w_2,>0}$ and its closure inductively. We can translate a geometric/topological question of calculating the closure to an algebraic question of calculating the image under the map ${}^J c_w$. As consequences of the product structure $(\star)$ we obtain

Theorem B (Theorem 5.10) *Let $w_1 \, {}^J\!\leqslant w_2$. Then*
(1) ${}^J\mathcal{B}_{w_1,w_2,>0}$ *is a cell and is a connected component of* ${}^J\mathcal{B}_{w_1,w_2}(\mathbb{R})$.
(2) *The closure of* ${}^J\mathcal{B}_{w_1,w_2,>0}$ *is* $\bigsqcup_{w_1 \leqslant w_1' \, {}^J\!\leqslant w_2' \leqslant w_2} {}^J\mathcal{B}_{w_1',w_2',>0}.$

Taking $J = \emptyset$, Theorem B generalizes the works of Rietsch in [38, 39] from reductive groups to Kac-Moody groups.

We also prove that

Theorem C (Proposition 8.1) *The Birkhoff-Bruhat atlas of* [3] *sends a totally positive cell $\mathcal{P}_{K,\alpha,>0}$ isomorphically to a totally positive cell J-Richardson variety ${}^J\tilde{\mathcal{B}}_{w_1,w_2,>0}$ for certain J and $w_1, w_2 \in \tilde{W}$.*

Note that the partial flag variety $\mathcal{P}_{K,\geqslant 0}$ does not have an obvious product structure as in Theorem A. This is reflected combinatorially on the lack of symmetry on the

face poset of $\mathcal{P}_{K,\geqslant 0}$. Theorem C allows us to study (inductively) the complicated totally positive projected Richardson varieties on $\mathcal{P}_K$ using the product structure coming from totally positive J-Richardson varieties on $\tilde{\mathcal{B}}$. This is a key ingredient in our proof of the regularity theorem for the totally positive projected Richardson varieties on $\mathcal{P}_K$, which we will discuss in the next subsection.

The J-total positivity for the full flag variety of any Kac-Moody group will also be applied to the study of the total positivity in many other spaces, such as the double flag varieties, the Bott-Samelson varieties, the double Bruhat cells and the wonderful compactifications. This will be done in future works.

1.5 Regularity theorem

We establish the regularity theorem for the links of the identity in the totally positive cells in U^-, the (ordinary) totally positive cells in the partial flag varieties, and the totally positive J-Richardson varieties in the full flag varieties.

Theorem D (Theorem 5.3, Theorem 5.7, Theorem 5.10, Theorem 5.11) *All the following three spaces are regular CW complexes homeomorphic to closed balls*:

(1) *the link of the identity in* $\overline{U^-_{w,>0}}$, *for any* $w \in W$;
(2) *the totally nonnegative projected Richardson variety* $\overline{\mathcal{P}_{K,\alpha,>0}}$;
(3) *the totally nonnegative J-Richardson variety* $\overline{{}^J\mathcal{B}_{w_1,w_2,>0}}$.

For reductive groups, the regularity of the link was first established by Hersh in [15]; the regularity of $\overline{\mathcal{P}_{K,\alpha,>0}}$ was established by Galashin, Karp and Lam in [12]. The generalization of regularity theorems in [12, 15] for Kac-Moody groups was conjectured by Galashin, Karp and Lam in [12, conjecture 10.2]. Theorem D (1) & (2) proves the conjectures, and part (3) is a new regularity result.

To prove regularity theorems, we follow [12] for the use of the generalized Poincaré conjecture [10, 42] and [34] as well as some general results on the poset topology. One then needs to show that each space $\overline{Y}$ above is a topological manifold with boundary $\overline{Y} - Y$. In the case where $Y = \mathcal{P}_{K,\alpha,>0}$ for a reductive group, [12] proved this result by constructing the Fomin-Shapiro atlas. Such construction relies on the affine model [12, §7] and a detailed study of the admissible functions [12, §5 & §6]. Another crucial fact is that the maps involved in the construction are the restrictions of smooth maps. Such construction of the Fomin-Shapiro atlas does not work for the ordinary total positivity for the Kac-Moody groups of infinite types nor the J-total positivity.

Instead, we use the product structure to study (inductively) the open neighborhood of a point in each space $\overline{Y}$ in Theorem D. For the J-Richardson varieties, the product structure $(\star)$ is established in Theorem A. The product structure of the links in the totally positive cells in U^- is inherited from the product structure of the (ordinary) total positivity on the full flag variety, which is also established in Theorem A.

As to the partial flag varieties, we do not have a obvious product structure. However, by Theorem C, the "atlas model" $\mathcal{P}_K \dashrightarrow \tilde{\mathcal{B}}$ in [2] translates the local structure of the totally nonnegative projected Richardson varieties in $\mathcal{P}_K$ homeomorphically

to the local structure of a J-Richardson variety in $\tilde{\mathcal{B}}$. And thus we may use the product structure of the J-total positivity to understand the local structures of the totally positive projected Richardson varieties and establish the desired regularity theorem.

2 Preliminaries

Throughout this paper, unless stated otherwise, for any ind-scheme X over $\mathbb{R}$, we shall simply denote by X its set of $\mathbb{C}$-valued points, and denote by $X(\mathbb{R})$ its set of $\mathbb{R}$-valued points. For any topological subspace Y of $X(\mathbb{R})$, we denote by $\overline{Y}$ the closure with respect to the Hausdorff topology.

2.1 Minimal Kac-Moody groups

Let I be a finite set and $A = (a_{ij})_{i,j \in I}$ be a symmetrizable generalized Cartan matrix in the sense of [16, §1.1]. A *Kac-Moody root datum* associated to A is a sextuple $\mathcal{D} = (I, A, X, Y, (\alpha_i)_{i \in I}, (\alpha_i^\vee)_{i \in I})$, where X is a free $\mathbb{Z}$-module of finite rank with $\mathbb{Z}$-dual Y, such that the elements α_i of X and $\alpha_i^\vee$ of Y satisfy $\langle \alpha_j^\vee, \alpha_i \rangle = a_{ij}$ for $i, j \in I$. The *split minimal Kac-Moody group* G over $\mathbb{R}$ associated to the Kac-Moody root datum $\mathcal{D}$ is the split group over $\mathbb{R}$ generated by the split torus T associated to Y and the root subgroup $U_{\pm \alpha_i}$ for $i \in I$, subject to the Tits relations [43]. Let $U^+ \subset G$ (resp. $U^- \subset G$) be the subgroup generated by U_{α_i} (resp. $U_{-\alpha_i}$) for $i \in I$. Let $B^\pm \subset G$ be the Borel subgroup generated by T and $U^\pm$, respectively.

For any $K \subset I$, let L_K be the subgroup of G generated by T and $U_{\pm \alpha_i}$ for $i \in K$. Let P_K^+ be the standard parabolic subgroup of G generated by B^+ and $U_{-\alpha_i}$ for $i \in K$ and P_K^- be the opposite parabolic subgroup of G generated by B^- and U_{α_i} for $i \in K$. Let $U_{P_K^\pm}$ be the unipotent radical of $P_K^\pm$. We have the Levi decomposition $P_K^\pm = L_K \ltimes U_{P_K^\pm}$.

For each $i \in I$, we fix isomorphisms $x_i : \mathbb{R} \to U_{\alpha_i}$ and $y_i : \mathbb{R} \to U_{-\alpha_i}$ such that the map

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \longmapsto x_i(a), \quad \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \longmapsto \alpha_i^\vee(a), \quad \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \longmapsto y_i(c)$$

defines a homomorphism $SL_2 \to G$.

2.2 Weyl groups

Let W be the Weyl group of G. It is a Coxeter group with the set of generators $\{s_i\}_{i \in I}$. Let ℓ be the length function on W and $\leqslant$ be the Bruhat order on W. We have natural actions of W on both X and Y.

For any $J \subset I$, let W_J be the subgroup of W generated by s_j for $j \in J$. This is the Weyl group of the parabolic subgroup P_J^+ of G. Let W^J be the set of minimal-length coset representatives of W/W_J and $^J W$ be the set of minimal-length coset representatives of $W_J \backslash W$. The multiplication gives a natural bijection $W_J \times {}^J W \cong W$. For any $w \in W$, let w_J be the unique element in W_J and $^J w$ be the unique element in $^J W$ with $w = w_J {}^J w$.

For any $i \in I$, we set $\dot{s}_i = x_i(1)y_i(-1)x_i(1) \in G(\mathbb{R})$. Let $w \in W$. By [32, Proposition 7.57], for any reduced expression $w = s_{i_1}s_{i_2} \cdots s_{i_n}$ of w, the element $\dot{s}_{i_1}\dot{s}_{i_2} \cdots \dot{s}_{i_n}$ of G is independent of the choice of the reduced expression. We denote this element by $\dot{w}$.

2.3 The flag varieties

Let $\mathcal{B} = G/B^+$ be the thin (full) flag variety of G (see [20]). For any $w \in W$, we set $\mathring{\mathcal{B}}_w = B^+\dot{w}B^+/B^+$ and $\mathring{\mathcal{B}}^w = B^-\dot{w}B^+/B^+$. We denote by $\mathcal{B}_w$ and $\mathcal{B}^w$ the Zariski closure of $\mathring{\mathcal{B}}_w$ and $\mathring{\mathcal{B}}^w$ in $\mathcal{B}$, respectively. For $v, w \in W$, let $\mathring{\mathcal{B}}_{v,w} = \mathring{\mathcal{B}}_w \bigcap \mathring{\mathcal{B}}^v$. This is an (open) Richardson variety of $\mathcal{B}$. It is known that $\mathring{\mathcal{B}}_{v,w} \neq \emptyset$ if and only if $v \leqslant w$. In this case, $\dim \mathring{\mathcal{B}}_{v,w} = \ell(w) - \ell(v)$. We have the decomposition $\mathcal{B} = \bigsqcup_{v \leqslant w} \mathring{\mathcal{B}}_{v,w}$. Moreover, for any $v \leqslant w$, the Zariski closure of $\mathring{\mathcal{B}}_{v,w}$ is $\mathcal{B}_w \bigcap \mathcal{B}^v = \bigsqcup_{v \leqslant v' \leqslant w' \leqslant w} \mathring{\mathcal{B}}_{v',w'}$.

2.4 Regular CW complexes

Let X be a Hausdorff space. We call a finite disjoint union $X = \bigsqcup_{\alpha \in Q} X_\alpha$ a *regular CW complex* if it satisfies the following two properties.

(1) For each $\alpha \in Q$, there exists a homeomorphism from a closed ball to $\overline{X}_\alpha$ mapping the interior of the ball to X_α.
(2) For each α, there exists $Q' \subset Q$, such that $\overline{X}_\alpha = \bigsqcup_{\beta \in Q'} X_\beta$.

The face poset of X is the poset $(Q, \leqslant)$, where $\beta \leqslant \alpha$ if and only if $X_\beta \subset \overline{X}_\alpha$.

We refer to [3, §4] and the references therein for the definitions of graded, thin, and shellable posets. We have the following result (see [5]).

Theorem 2.1 *Suppose that X is a regular CW complex with face poset Q. If $Q \bigsqcup \{\hat{0}, \hat{1}\}$ (adjoining a minimum $\hat{0}$ and a maximum $\hat{1}$) is graded, thin, and shellable, then X is homeomorphic to a sphere of dimension rank$(Q) - 1$.*

2.5 The Poincare conjecture

Recall that an n-dimension topological manifold with boundary is a Hausdorff space X such that every point $x \in X$ has an open neighborhood homeomorphic to either $\mathbb{R}^n$, or $\mathbb{R}_{\geqslant 0} \times \mathbb{R}^{n-1}$ mapping x to a point in $\{0\} \times \mathbb{R}^{n-1}$. In the latter case, we say that x is on ∂X, the boundary of X.

The following theorem can be derived from the generalized poincare conjecture and Brown's collar theorem. We refer to [12, §3.2] for details and history.

Theorem 2.2 *Let X be a compact n-dimensional topological manifold with boundary, such that ∂X is homeomorphic to an $(n-1)$-dimensional sphere and $X - \partial X$ is homeomorphic to an n-dimensional open ball. Then X is homeomorphic to an n-dimensional closed ball D^n.*

3 J-Richardson varieties

3.1 The partial order $^J\!\leqslant$ on W

Following [4] and [3, §2.3 & Proposition 4.6], we define the J-twisted length $^J\ell$ and the J-twisted Bruhat order $^J\!\leqslant$ on W as follows. For $w \in W$,

$$^J\ell(w) = \ell(^J w) - \ell(w_J).$$

For $w, w' \in W$, $w' \,^J\!\leqslant w$ if there exists $u \in W_J$ with $w_J \leqslant w'_J u^{-1}$, $u \,^J w' \leqslant \,^J w$. This is a special case of the twisted Bruhat order considered in [7]. We say $w' \,^J\!< w$ if $w' \,^J\!\leqslant w$ and $w' \neq w$. It follows from [7, Proposition 1.7] that the poset $(W, \,^J\!\leqslant)$ is graded. By [7, Proposition 1.1], if $v \,^J\!\leqslant w$, then any maximal chain is of length $^J\ell(w) - \,^J\ell(v)$.

We define the poset

$$^JQ = \{(v, w) \in W \times W \mid v \,^J\!\leqslant w\}, \text{ where } (v', w') \,^J\!\preceq (v, w), \text{ if } v \,^J\!\leqslant v' \,^J\!\leqslant w' \,^J\!\leqslant w.$$

We also define a poset $^J\hat{Q} = \,^JQ \bigsqcup \{\hat{0}\}$, where $\hat{0}$ is the new minimal element.

Proposition 3.1 (1) *The poset $(W, \,^J\!\leqslant)$ graded, thin and shellable. In particular, any convex interval of W is graded, thin and shellable.*
(2) *The poset $(^J\hat{Q}, \,^J\!\preceq)$ is graded, thin and shellable. In particular, any convex interval of $^J\hat{Q}$ is graded, thin and shellable.*

Proof It follows from [7, Proposition 1.7 & 2.5 & 3.9] that the poset $(W, \,^J\!\leqslant)$ is graded, thin and EL-shellable.

We next equip $W \times W$ with the partial order $^J\!\preceq$ such that $(v_1, w_1) \preceq (v_2, w_2)$ if $v_1 \,^J\!\leqslant v_2$ and $w_2 \,^J\!\leqslant w_1$. It follows as a special case of (1) that $(W \times W, \,^J\!\preceq)$ is graded, thin and EL-shellable. Note that poset ^JQ is a closed interval in $(W \times W, \preceq)$. Thus the poset ^JQ is graded, thin, and shellable.

It is easy to see that $(^J\hat{Q}, \,^J\!\preceq)$ is graded and thin. The EL-shellability of $^J\hat{Q}$ can be proved similar to [44] (see also [3, §4]). $\qquad\square$

3.2 The J-Richardson varieties

We follow [3, §2.3] to introduce the J-Richardson varieties. Let $J \subset I$. Let $B_J^{\pm} = L_J \bigcap B^{\pm}$ and $U_J^{\pm} = L_J \bigcap U^{\pm}$. Set

$$^JB^+ = B_J^- \ltimes U_{P_J^+}, \quad ^JB^- = B_J^+ \ltimes U_{P_J^-}.$$

We set $^JU^+ = U_J^- U_{P_J^+}$ and $^JU^- = U_J^+ U_{P_J^-}$. Then $^JU^{\pm}$ is the unipotent radical of $^JB^{\pm}$. For $v, w \in W$, we define, respectively, the J-Schubert cell, the opposite J-Schubert cell and the open J-Richardson variety by

$$^J\mathring{\mathcal{B}}_w = \,^JB^+ \dot{w} B^+ / B^+, \quad ^J\mathring{\mathcal{B}}^v = \,^JB^- \dot{v} B^+ / B^+, \quad ^J\mathring{\mathcal{B}}_{v,w} = \,^J\mathring{\mathcal{B}}_w \bigcap \,^J\mathring{\mathcal{B}}^v.$$

By [3, Proposition 2.4], ${}^{J}\mathring{\mathcal{B}}_{v,w} \neq \emptyset$ if and only if $v \stackrel{J}{\leqslant} w$. We have the decomposition $\mathcal{B} = \bigsqcup_{v \stackrel{J}{\leqslant} w} {}^{J}\mathring{\mathcal{B}}_{v,w}$. By [4, Theorem 4], the Zariski closure of ${}^{J}\mathring{\mathcal{B}}_{v,w}$, denoted by ${}^{J}\mathcal{B}_{v,w}$, is contained in ${}^{J}\mathcal{B}_{w} \bigcap {}^{J}\mathcal{B}^{v} = \bigsqcup_{v \stackrel{J}{\leqslant} v' \stackrel{J}{\leqslant} w' \stackrel{J}{\leqslant} w} {}^{J}\mathring{\mathcal{B}}_{v',w'}$. We will show later in Proposition 7.4 that ${}^{J}\mathcal{B}_{v,w}$ equals ${}^{J}\mathcal{B}_{w} \bigcap {}^{J}\mathcal{B}^{v}$.

If $J = \emptyset$, then $v \stackrel{J}{\leqslant} w$ if and only if $v \leqslant w$. In this case, ${}^{J}\mathring{\mathcal{B}}_{v,w} = \mathcal{B}_{v,w}$. If $J = I$, then $v \stackrel{J}{\leqslant} w$ if and only if $w \leqslant v$. In this case, ${}^{J}\mathring{\mathcal{B}}_{v,w} = \mathcal{B}_{w,v}$.

3.3 Some isomorphisms on $\mathcal{B}$

For any $r \in W$, we have isomorphisms

$$(\dot{r}U^{-}\dot{r}^{-1} \bigcap {}^{J}U^{+}) \times (\dot{r}U^{-}\dot{r}^{-1} \bigcap {}^{J}U^{-}) \longrightarrow \dot{r}U^{-}\dot{r}^{-1}, \quad (g_1, g_2) \longmapsto g_1 g_2;$$

$$(\dot{r}U^{-}\dot{r}^{-1} \bigcap {}^{J}U^{-}) \times (\dot{r}U^{-}\dot{r}^{-1} \bigcap {}^{J}U^{+}) \longrightarrow \dot{r}U^{-}\dot{r}^{-1}, \quad (h_1, h_2) \longmapsto h_1 h_2.$$

We define morphisms of ind-varieties

$$
{}^{J}\sigma_{r,-} : \dot{r}U^{-}\dot{r}^{-1} \to \dot{r}U^{-}\dot{r}^{-1} \bigcap {}^{J}U^{-}, \qquad g_1 g_2 \longmapsto g_2,
$$

$$
{}^{J}\sigma_{r,+} : \dot{r}U^{-}\dot{r}^{-1} \to \dot{r}U^{-}\dot{r}^{-1} \bigcap {}^{J}U^{+}, \qquad h_1 h_2 \longmapsto h_2.
$$

We have the following isomorphism as a special case of [18, Lemma 2.2]:

$$
{}^{J}\sigma_r = ({}^{J}\sigma_{r,+}, {}^{J}\sigma_{r,-}) : \dot{r}U^{-}\dot{r}^{-1} \xrightarrow{\sim} (\dot{r}U^{-}\dot{r}^{-1} \bigcap {}^{J}U^{+}) \times (\dot{r}U^{-}\dot{r}^{-1} \bigcap {}^{J}U^{-}). \quad (3.1)
$$

By (3.1), for any $r \in W$, the map $g\dot{r}B^{+}/B^{+} \mapsto \left({}^{J}\sigma_{r,+}(g)\dot{r} \cdot B^{+}/B^{+}, {}^{J}\sigma_{r,-}(g)\dot{r} \cdot B^{+}/B^{+}\right)$ for $g \in \dot{r}U^{-}\dot{r}^{-1}$ defines an isomorphism

$$
{}^{J}c_r = ({}^{J}c_{r,+}, {}^{J}c_{r,-}) : \dot{r}U^{-}B^{+}/B^{+} \xrightarrow{\sim} {}^{J}\mathring{\mathcal{B}}_{r} \times {}^{J}\mathring{\mathcal{B}}^{r}. \quad (3.2)
$$

The map ${}^{J}c_r$ sends ${}^{J}\mathring{\mathcal{B}}_{v,w} \bigcap (\dot{r}B^{-}B^{+}/B^{+})$ to ${}^{J}\mathring{\mathcal{B}}_{v,r} \times {}^{J}\mathring{\mathcal{B}}_{r,w}$ for any $v \stackrel{J}{\leqslant} w$. The isomorphism in (3.2) restricts to an isomorphism

$$
{}^{J}\mathring{\mathcal{B}}_{v,w} \bigcap (\dot{r}B^{-}B^{+}/B^{+}) \xrightarrow{\sim} {}^{J}\mathring{\mathcal{B}}_{v,r} \times {}^{J}\mathring{\mathcal{B}}_{r,w}. \quad (3.3)
$$

This also shows that

(a) ${}^{J}\mathring{\mathcal{B}}_{v,w} \bigcap (\dot{r}B^{-}B^{+}/B^{+}) \neq \emptyset$ if and only if $v \stackrel{J}{\leqslant} r \stackrel{J}{\leqslant} w$.

3.4 Some general results

Note that the map ${}^{J}c_r$ in (3.2) is defined over $\mathbb{R}$.

Lemma 3.2 *Let* $v \stackrel{J}{\leqslant} u \stackrel{J}{\leqslant} w$. *Let* $Y \subset {}^{J}\mathring{\mathcal{B}}_{v,w}(\mathbb{R})$ *with* $\dot{u}B^{+}/B^{+} \in \overline{Y}$. *Then*

(1) *for any* $v \leqslant v' \leqslant u$, *we have* $\overline{Y} \bigcap {}^{J}\mathring{\mathcal{B}}_{v',u} = \overline{{}^{J}c_{u,+}(Y \bigcap (\dot{u}B^{-}B^{+}/B^{+}))} \bigcap {}^{J}\mathring{\mathcal{B}}_{v',u};$

(2) *for any $u \leqslant w' \leqslant w$, we have $\overline{Y} \bigcap {}^{J}\mathring{\mathcal{B}}_{u,w'} = \overline{{}^{J}c_{u,-}(Y \bigcap (\dot{u}B^{-}B^{+}/B^{+}))} \bigcap {}^{J}\mathring{\mathcal{B}}_{u,w'}.$*

Remark 3.3 Similar results hold if the Hausdorff closure is replaced by the Zariski closure.

Proof Set $Y' = Y \bigcap (\dot{u}B^{-}B^{+}/B^{+})$. Then $\overline{Y} \bigcap (\dot{u}B^{-}B^{+}/B^{+}) = \overline{Y'} \bigcap (\dot{u}B^{-}B^{+}/B^{+})$. We have the following isomorphism via restriction

$$\overline{Y'} \bigcap \dot{u}B^{-}B^{+}/B^{+} \cong \overline{{}^{J}c_{u,+}(Y')} \times \overline{{}^{J}c_{u,-}(Y')}.$$

Since $\dot{u}B^{+}/B^{+} \in \overline{Y} \bigcap (\dot{u}B^{-}B^{+}/B^{+}) \subset \overline{Y'}$, we must have $\dot{u}B^{+}/B^{+} \in \overline{{}^{J}c_{u,+}(Y')}$ and $\dot{u}B^{+}/B^{+} \in {}^{J}c_{u,-}(Y')$. Since the isomorphism is stratified, we have

$$\overline{Y'} \bigcap {}^{J}\mathring{\mathcal{B}}_{v',u} \cong (\overline{{}^{J}c_{u,+}(Y')} \bigcap {}^{J}\mathring{\mathcal{B}}_{v',u}) \times \dot{u}B^{+}/B^{+} \cong \overline{{}^{J}c_{u,+}(Y')} \bigcap {}^{J}\mathring{\mathcal{B}}_{v',u}.$$

The composition is actually the identity map. Now part (1) follows. Part (2) is proved in the same way. $\qquad\square$

Lemma 3.4 *Let $v \stackrel{J}{\leqslant} u \stackrel{J}{\leqslant} w$ and Y be a connected component of ${}^{J}\mathring{\mathcal{B}}_{v,w}(\mathbb{R})$. If $Y \subset \dot{u}B^{-}B^{+}/B^{+}$, then*

(1) *$\dot{u}B^{+}/B^{+} \in \overline{Y}$;*
(2) *$\overline{Y} \bigcap {}^{J}\mathring{\mathcal{B}}_{v,u} = {}^{J}c_{u,+}(Y)$ is a connected component of ${}^{J}\mathring{\mathcal{B}}_{v,u}(\mathbb{R})$;*
(3) *$\overline{Y} \bigcap {}^{J}\mathring{\mathcal{B}}_{u,w} = {}^{J}c_{u,-}(Y)$ is a connected component of ${}^{J}\mathring{\mathcal{B}}_{u,w}(\mathbb{R})$.*

Proof Let μ be a dominant regular coweight. Then for any $g \in U^{-}$, we have $\lim_{t \to 0} \mu(t)g\mu(t)^{-1} = 1$. Set $\mu' = u(\mu)$. Thus $\lim_{t \to 0} \mu'(t)\dot{u}gB^{+}/B^{+} = \dot{u}B^{+}/B^{+}$ for any $g \in U^{-}$. Since Y is a connected component of ${}^{J}\mathring{\mathcal{B}}_{v,w}(\mathbb{R})$, it is stable under the action of $\mu'(\mathbb{R}_{>0})$. This shows that $\dot{u}B^{+}/B^{+} \in \overline{Y}$. Part (1) is proved.

Thanks to (3.3), we see that ${}^{J}c_{u,+}(Y)$ is a connected component in ${}^{J}\mathring{\mathcal{B}}_{v,u}(\mathbb{R})$. Therefore ${}^{J}c_{u,+}(Y)$ is closed in ${}^{J}\mathring{\mathcal{B}}_{v,u}$ and thus equals to $\overline{{}^{J}c_{u,+}(Y)} \bigcap {}^{J}\mathring{\mathcal{B}}_{v,u}$. Hence by Lemma 3.2, we have ${}^{J}c_{u,+}(Y) = \overline{Y} \bigcap {}^{J}\mathring{\mathcal{B}}_{v,u} = \overline{{}^{J}c_{u,+}(Y)} \bigcap {}^{J}\mathring{\mathcal{B}}_{v,u}$. We proved part (2). Part (3) can be proved similarly. $\qquad\square$

Now we prove the main result of this section.

Theorem 3.5 *Let $Y_{v,w}$ be a connected component of ${}^{J}\mathring{\mathcal{B}}_{v,w}(\mathbb{R})$. We define*

$$Y_{v',w'} = \overline{Y_{v,w}} \bigcap {}^{J}\mathring{\mathcal{B}}_{v',w'}, \text{ for any } v \stackrel{J}{\leqslant} v' \stackrel{J}{\leqslant} w' \stackrel{J}{\leqslant} w.$$

Assume that for any $v \stackrel{J}{\leqslant} v' \stackrel{J}{\leqslant} u \stackrel{J}{\leqslant} w' \stackrel{J}{\leqslant} w$, we have $Y_{v',w'} \subset \dot{u}B^{-}B^{+}/B^{+}$. Then the map ${}^{J}c_{u}$ restricts to an isomorphism

$${}^{J}c_{u} : Y_{v',w'} \cong Y_{v',u} \times Y_{u,w'}.$$

Remark 3.6 We refer to the isomorphism above as a *product structure*. It is also worth pointing out that the assumption $Y_{v',w'} \subset \dot{u} B^- B^+ / B^+$ is a very strong condition and most of the connected components of ${}^J \mathring{\mathcal{B}}_{v,w}(\mathbb{R})$ do not satisfy this assumption. For example, in type G_2, $\mathring{\mathcal{B}}_{1,w_0}(\mathbb{R})$ has 11 connected components (see [37]) and only 4 of them satisfy this assumption.

Proof The proof consists of the following steps

(i) for any $v \overset{J}{\leqslant} u \overset{J}{\leqslant} w$, we have ${}^J c_u : Y_{v,w} \cong Y_{v,u} \times Y_{u,w}$;

(ii) for any $v \overset{J}{\leqslant} v' \overset{J}{\leqslant} u \overset{J}{\leqslant} w$, we have ${}^J c_u : Y_{v',w} \cong Y_{v',u} \times Y_{u,w}$;

(iii) for any $v \overset{J}{\leqslant} u \overset{J}{\leqslant} w' \overset{J}{\leqslant} w$, we have ${}^J c_u : Y_{v,w'} \cong Y_{v,u} \times Y_{u,w'}$;

(iv) for any $v \overset{J}{\leqslant} v' \overset{J}{\leqslant} u \overset{J}{\leqslant} w' \overset{J}{\leqslant} w$, we have ${}^J c_u : Y_{v',w'} \cong Y_{v',u} \times Y_{u,w'}$.

Thanks to Lemma 3.4 (at u), we see

$$Y_{u,w} = {}^J c_{u,-}(Y_{v,w}) \quad \text{and} \quad Y_{v,u} = {}^J c_{u,+}(Y_{v,u}), \text{ for any } v \overset{J}{\leqslant} u \overset{J}{\leqslant} w.$$

Part (i) follows. We also see that $Y_{u,w}$ and $Y_{v,u}$ are connected components of ${}^J \mathring{\mathcal{B}}_{u,w}(\mathbb{R})$ and ${}^J \mathring{\mathcal{B}}_{v,u}(\mathbb{R})$, respectively.

Let $u = v'$ in part (i). Applying Lemma 3.4 (at w') to $Y_{v',w}$, we obtain that

$$\overline{Y_{v',w}} \bigcap {}^J \mathring{\mathcal{B}}_{v',w'} = {}^J c_{w',+}(Y_{v',w}).$$

We finally apply Lemma 3.2 (at v') to obtain that

$$Y_{v',w'} = \overline{Y_{v,w}} \bigcap {}^J \mathring{\mathcal{B}}_{v',w'} = \overline{{}^J c_{v',-}(Y_{v,w})} \bigcap {}^J \mathring{\mathcal{B}}_{v',w'} = \overline{Y_{v',w}} \bigcap {}^J \mathring{\mathcal{B}}_{v',w'}$$

$$= {}^J c_{w',+}(Y_{v',w}) = {}^J c_{w',+}({}^J c_{v',-}(Y_{v,w})).$$

One may prove similarly that $Y_{v',w'} = {}^J c_{v',-}({}^J c_{w',+}(Y_{v,w}))$, $Y_{w',w} = {}^J c_{w',-}({}^J c_{v',-}(Y_{v,w}))$ and $Y_{v,v'} = {}^J c_{v',+}({}^J c_{w',+}(Y_{v,w}))$.

Now part (ii) and (iii) follow. We also see that $Y_{v',w'}$ is a connected component of ${}^J \mathring{\mathcal{B}}_{v',w'}(\mathbb{R})$.

Let $v' \overset{J}{\leqslant} v'' \overset{J}{\leqslant} w'' \overset{J}{\leqslant} w$. Thanks to Lemma 3.2, we have

$$\overline{Y_{v',w}} \bigcap {}^J \mathring{\mathcal{B}}_{v'',w''} = \overline{{}^J c_{v'',-}(Y_{v',w})} \bigcap {}^J \mathring{\mathcal{B}}_{v'',w''} = \overline{Y_{v'',w}} \bigcap {}^J \mathring{\mathcal{B}}_{v'',w''}$$

$$= {}^J c_{w'',+}(Y_{v'',w}) = {}^J c_{w'',+}({}^J c_{v'',-}(Y_{v,w}))$$

$$= Y_{v'',w''}.$$

Now we see that $Y_{v',w}$ satisfies the same assumptions as $Y_{v,w}$, hence (i), (ii), (iii) apply to the new space $\overline{Y_{v',w}}$. Now part (iii) for the space $\overline{Y_{v',w}}$ is the same as part (iv) for the original space $\overline{Y_{v,w}}$. In particular, we have

$$\overline{Y_{v',w'}} \bigcap {}^J \mathring{\mathcal{B}}_{v'',w''} = Y_{v'',w''}. \tag{3.4}$$

We hence finish the proof. $\qquad\square$

Let us draw some consequences from Theorem 3.5 and its proof.

Corollary 3.7 *Retain the assumptions in Theorem 3.5.*

(1) *For any $v \overset{J}{\leqslant} v' \overset{J}{\leqslant} w' \overset{J}{\leqslant} w$, the subspace $Y_{v',w'}$ is a connected component in $^J\mathring{\mathcal{B}}_{v',w'}(\mathbb{R})$.*

(2) *For any $v \overset{J}{\leqslant} v' \overset{J}{\leqslant} w' \overset{J}{\leqslant} w$, we have*

$$\overline{Y_{v',w'}} = \bigsqcup_{v' \overset{J}{\leqslant} v'' \overset{J}{\leqslant} w'' \overset{J}{\leqslant} w'} Y_{v'',w''}.$$

(3) *We have $Y_{v',w'} \cong \mathbb{R}_{>0}^{\,^J\ell(w') - ^J\ell(v')}$.*

Proof (1) has been establish in the proof of Theorem 3.5. By §3.2 and (3.4), we have

$$\overline{Y_{v',w'}} = \bigsqcup_{v' \overset{J}{\leqslant} v'' \overset{J}{\leqslant} w'' \overset{J}{\leqslant} w'} (\overline{Y_{v',w'}} \bigcap {}^J\mathring{\mathcal{B}}_{v'',w''}) = \bigsqcup_{v' \overset{J}{\leqslant} v'' \overset{J}{\leqslant} w'' \overset{J}{\leqslant} w'} Y_{v'',w''}.$$

Part (2) is proved.

We show (3). It follows from [4, Theorem 5 & Corollary 6] that

$${}^J\mathring{\mathcal{B}}_{v',w'}(\mathbb{R}) \cong \mathbb{R}^{\times}, \text{ if } v' \overset{J}{\leqslant} w' \text{ and } {}^J\ell(w') - {}^J\ell(v') = 1.$$

In this case, by part (1) we have $Y_{v',w'} \cong \mathbb{R}_{>0}$. Recall §3.1 that the poset $(W, \overset{J}{\leqslant})$ is graded. Now the general case follows from Theorem 3.5 by induction on $^J\ell(w') - {}^J\ell(v')$. $\qquad\square$

4 A regular CW complex

The main result of this section is the following theorem.

Theorem 4.1 *We fix $v \overset{J}{\leqslant} w$. Let $Y_{v,w}$ be a connected component of $^J\mathring{\mathcal{B}}_{v,w}(\mathbb{R})$. We define*

$$Y_{v',w'} = \overline{Y_{v,w}} \bigcap {}^J\mathring{\mathcal{B}}_{v',w'}(\mathbb{R}) \text{ for any } v \overset{J}{\leqslant} v' \overset{J}{\leqslant} w' \overset{J}{\leqslant} w.$$

Assume that for any $v \overset{J}{\leqslant} v' \overset{J}{\leqslant} u \overset{J}{\leqslant} w' \overset{J}{\leqslant} w$, we have $Y_{v',w'} \subset \dot{u}B^-B^+/B^+$. Then $\overline{Y_{v,w}} = \bigsqcup_{(v',w')} Y_{v',w'}$ is a regular CW complex homeomorphic to a closed ball of dimension $^J\ell(w) - {}^J\ell(v)$.

4.1 Links

Links can be defined for arbitrary Whitney stratified spaces; see [15, Definition 4.25]. We shall not consider this abstract definition here, but follow [8, Theorem 1.2] and [12, §3.1] instead.

We denote by X^{++} the set of dominant regular weights of G. For $\lambda \in X^{++}$, we denote by V^λ the highest weight simple G-module over $\mathbb{C}$ with highest weight λ. Let η_λ be a highest weight vector. We denote by $V^\lambda(\mathbb{R})$ the $\mathbb{R}$-subspace of V^λ spanned by the canonical basis.

For any $v' \in W$ with $(v', w) \in {}^J Q$, we consider the embedding

$$
{}^J\mathring{\mathcal{B}}^{v'}(\mathbb{R}) \xrightarrow{(\dot{v}')^{-1} \cdot -} U^- B^+/B^+ \cong U^- \xrightarrow{u \mapsto u \cdot \eta_\lambda} V^\lambda. \tag{4.1}
$$

The image of ${}^J\mathring{\mathcal{B}}^{v'}(\mathbb{R}) \bigcap {}^J\mathcal{B}_{v,w}$ lies in a finite-dimensional subspace $L^{v'} \subset V^\lambda(\mathbb{R})$; cf. [4, Theorem 5]. We identify ${}^J\mathring{\mathcal{B}}^{v'}(\mathbb{R})$ with the image. We equip $L^{v'}$ with the standard Euclidean norm with respect to the canonical basis. We define the links[2] $Lk_?(\overline{Y_{v,w}})$ (via the embedding above) by

$$
Lk_{v',w'}(\overline{Y_{v,w}}) = Y_{v',w'} \bigcap \{||x|| = 1 | x \in L^{v'}\}, \text{ for any } w' \text{ such that } v'{}^J < w'{}^J \leqslant w,
$$

$$
Lk_{v'}(\overline{Y_{v,w}}) = \bigsqcup_{v'{}^J < w'{}^J \leqslant w} Lk_{v',w'}(\overline{Y_{v,w}}) = \overline{Y_{v,w}} \bigcap \{||x|| = 1 | x \in L^{v'}\}.
$$

We simply write $Lk_? = Lk_?(\overline{Y_{v,w}})$ if there is no confusing.

For any dominant regular coweight μ, we consider the natural $\mathbb{R}^\times$-action on $\mathcal{B}(\mathbb{R})$ via the coweight $v'(\mu)$. Note that the action is compatible with the $\mathbb{R}^\times$-action on $V^\lambda(\mathbb{R})$ via the dominant regular coweight μ through the embedding (4.1). We shall abuse notations and denote both actions by ϑ_μ.

It is clear ${}^J\mathring{\mathcal{B}}_{v',w'}(\mathbb{R})$ is stable under the action of ϑ_μ and any connected component of ${}^J\mathring{\mathcal{B}}_{v',w'}(\mathbb{R})$ is stable under the action of $\mathbb{R}_{>0}$. This defines a contractive flow on the space $L^{v'}$ in the sense of [12, Definition 2.2], since μ is dominant regular.

The following results are proved in [12, Lemma 3.4 & Proposition 3.5].

(a) For any $x \in \overline{Y_{v,w}} \bigcap {}^J\mathring{\mathcal{B}}^{v'}(\mathbb{R})$ such that $x \notin Y_{v',v'} = \dot{v}' B^+/B^+$, there is a unique $t_1(x) \in \mathbb{R}_{>0}$ such that $\vartheta_\mu(t_1(x))x \in Lk_{v'}$. Moreover, the map $x \to t_1(x)$ is continuous.

(b) We have a stratified isomorphism $\overline{Y_{v,w}} \bigcap {}^J\mathring{\mathcal{B}}^{v'}(\mathbb{R}) \cong \mathrm{Cone}(Lk_{v'})$ such that $Y_{v',v'}$ maps to the cone point and $Y_{v',w'} \cong Lk_{v',w'} \times \mathbb{R}_{>0}$, $x \mapsto (\vartheta_\mu(t_1(x))x, 1/t_1(x))$ for $w' \neq v'$. Here for any topological space A, the cone over A is defined by $\mathrm{Cone}(A) = (A \times \mathbb{R}_{\geqslant 0})/(A \times \{0\})$.

We denote by D^n the closed ball of dimension n. Note that $\mathrm{Cone}(D^n) \cong \mathbb{R}^n \times \mathbb{R}_{\geqslant 0}$.

Proposition 4.2 *For any* $v'{}^J < u{}^J \leqslant w$, *we have stratified isomorphisms*

$$
Lk_{v'} \bigcap \dot{u} U^- B^+/B^+ = \bigsqcup_{u{}^J \leqslant w'{}^J \leqslant w} Lk_{v',w'}
$$

$$
\cong Lk_{v',u} \times (Y \bigcap {}^J\mathring{\mathcal{B}}^u(\mathbb{R})) \cong Lk_{v',u} \times \mathrm{Cone}(Lk_u).
$$

---

[2] In fact, the definition of links is independent of the choice of $\lambda \in X^{++}$ up to a stratified homeomorphism. We do not use this fact in this paper.

Proof We write ϑ for ϑ_μ. Recall the assumption that $Y_{v',w'} \subset \dot u U^- B^+/B^+$ for $v'\,{}^J\!\leqslant u\,{}^J\!\leqslant w'$. Hence $Lk_{v'} \bigcap \dot u U^- B^+/B^+ = \bigsqcup_{u\,{}^J\!\leqslant w'\,{}^J\!\leqslant w} Lk_{v',w'}$. The last isomorphism follows from §4.1 (b). We construct the second isomorphism.

We first define a morphism α as the following composition

$$Lk_{v',u} \times (\overline{Y_{v,w}} \bigcap {}^J\!\mathring{\mathcal{B}}^u(\mathbb{R})) \;\hookrightarrow\; Y_{v',u} \times (\overline{Y_{v,w}} \bigcap {}^J\!\mathring{\mathcal{B}}^u(\mathbb{R}))$$

$$\xrightarrow{{}^J\!c_u^{-1}} \bigsqcup_{u\,{}^J\!\leqslant w'\,{}^J\!\leqslant w} Y_{v',w'} \xrightarrow{\pi} \bigsqcup_{u\,{}^J\!\leqslant w'\,{}^J\!\leqslant w} Lk_{v',w'}.$$

Here the second map comes from Theorem 3.5. We next construct the inverse. We then define a morphism β as follows

$$\bigsqcup_{u\,{}^J\!\leqslant w'\,{}^J\!\leqslant w} Lk_{v',w'} \;\hookrightarrow\; \bigsqcup_{u\,{}^J\!\leqslant w'\,{}^J\!\leqslant w} Y_{v',w'}$$

$$\xrightarrow{{}^J\!c_u} Y_{v',u} \times (\overline{Y_{v,w}} \bigcap {}^J\!\mathring{\mathcal{B}}^u(\mathbb{R})) \xrightarrow{\phi} Lk_{v',u} \times (\overline{Y_{v,w}} \bigcap {}^J\!\mathring{\mathcal{B}}^u(\mathbb{R})),$$

where $\phi(x,y) = (\vartheta(t_1(x))x, \vartheta(t_1(x))y)$. We claim α and β give the desired isomorphism. The compatibility with the stratification is clear. We show that they are inverse to each other.

We first show $\beta \circ \alpha = \mathrm{id}$. Let $(x,y) \in Lk_{v',u} \times (\overline{Y_{v,w}} \bigcap {}^J\!\mathring{\mathcal{B}}^u(\mathbb{R}))$. Then let $z = {}^J\!c_u^{-1}((x,y))$. Then $\alpha(x,y) = \pi(z) = \vartheta(t_1(z))z$. Since ${}^J\!c_u$ is $\mathbb{R}_{>0}$-equivariant, we have ${}^J\!c_u(\vartheta(t_1(z))z) = (\vartheta(t_1(z))x, \vartheta(t_1(z))y)$. By the uniqueness in §4.1 (a), we have $\vartheta(t_1(\vartheta(t_1(z))x))x = x \in Lk_{v',u}$, hence $t_1(\vartheta(t_1(z))x) = t_1(z)^{-1}$. Therefore we have $\phi(\vartheta(t_1(z))x, \vartheta(t_1(z))y) = \beta(z) = (x,y)$.

We next show $\alpha \circ \beta = \mathrm{id}$. Let $z \in \bigsqcup_{u\,{}^J\!\leqslant w'\,{}^J\!\leqslant w} Lk_{v',w'}$ and $(x,y) = {}^J\!c_u(z)$. Then $\beta(z) = (\vartheta(t_1(x))x, \vartheta(t_1(x))y)$. Then since ${}^J\!c_u$ is T-equivariant, we obtain ${}^J\!c_u^{-1}(\beta(z)) = \vartheta(t_1(x))z$. Then by the uniqueness in §4.1 (a), we must have $\pi(\vartheta(t_1(x))z) = z$, that is $\alpha \circ \beta(z) = z$.

We finish the proof. $\qquad\square$

Corollary 4.3 *Let* $v'\,{}^J\!< w'$. *We have* $Lk_{v',w'} \cong \mathbb{R}_{>0}^{{}^J\!\ell(w') - {}^J\!\ell(v') - 1}$.

Proof Thanks to §4.1 (b) and Proposition 4.2, it suffices to show $Lk_{v',w'}$ is a point when ${}^J\!\ell(w') - {}^J\!\ell(v') = 1$. The latter statement follows from Theorem 3.5 and direct computation. $\qquad\square$

Proposition 4.4 *For* $v\,{}^J\!\leqslant v'\,{}^J\!< w$, $Lk_{v'} = \bigsqcup_{v'\,{}^J\!< w'\,{}^J\!\leqslant w} Lk_{v',w'}$ *is a regular CW complex homeomorphic to a closed ball of dimension* ${}^J\!\ell(w) - {}^J\!\ell(v') - 1$.

Proof We prove by induction on ${}^J\!\ell(w) - {}^J\!\ell(v')$. When ${}^J\!\ell(w) - {}^J\!\ell(v') = 1$, $Lk_{v'}$ is a point by Corollary 4.3. In the induction process, we shall consider $Lk_?(\overline{Y_{v,w'}})$ for $w'\,{}^J\!\leqslant w$ as well. Note that $Lk_{v'}(\overline{Y_{v,w'}})$ is a subspace of $Lk_{v'}(\overline{Y_{v,w}})$ and $Lk_{v',w''}(\overline{Y_{v,w'}}) = Lk_{v',w''}(\overline{Y_{v,w}})$ for $v'\,{}^J\!< w''\,{}^J\!\leqslant w'$.

We first show

(a) $Lk_{v'}$ is a topological manifold with boundary $\partial Lk_{v'} = \bigsqcup_{v'^J < w'^J < w} Lk_{v',w'}$.

We have $Lk_{v',w} = Lk_{v'} \cap \dot{w}U^- B^+ / B^+ \cong \mathbb{R}_{>0}^{{}^J\ell(w) - {}^J\ell(v') - 1}$. Now for any u with $v'^J < u^J < w$, we apply the stratified isomorphism in Proposition 4.2. We have

$$Lk_{v'} \cap \dot{u}U^- B^+ / B^+ = \bigsqcup_{u^J \leqslant w'^J \leqslant w} Lk_{v',w'} \cong Lk_{v',u} \times \text{Cone}(Lk_u)$$

$$\cong \mathbb{R}_{>0}^{{}^J\ell(u) - {}^J\ell(v') - 1} \times \text{Cone}(D^{{}^J\ell(w) - {}^J\ell(u) - 1})$$

$$\cong \mathbb{R}_{>0}^{{}^J\ell(w) - {}^J\ell(v') - 2} \times \mathbb{R}_{\geqslant 0}.$$

Here $Lk_u \cong D^{{}^J\ell(w) - {}^J\ell(u) - 1}$ is obtained via the induction hypothesis since ${}^J\ell(w) - {}^J\ell(u) < {}^J\ell(w) - {}^J\ell(v')$. This shows that $Lk_{v'}$ is a topological manifold with boundary and $Lk_{v',u}$ lies on the boundary for $u \neq w'$. This proves (a).

We next prove

(b) $\partial Lk_{v'} = \bigsqcup_{v'^J < w'^J < w} Lk_{v',w'}$ is a regular CW complex homeomorphic to a sphere of dimension ${}^J\ell(w) - {}^J\ell(v') - 2$.

By induction hypothesis $Lk_{v'}(\overline{Y_{v,w'}})$ is a regular CW complex homeomorphic to a closed ball of dimension ${}^J\ell(w) - {}^J\ell(v') - 1$, for any $v'^J < w'^J < w$. Therefore $\partial Lk_{v'}$ is a regular CW complex with the face poset $(\{w' | v'^J < w'^J < w\}, {}^J\leqslant)$. It is clear after adding a new minimal $\hat{0}$ and $\hat{1}$, the poset is $\{w' | v'^J \leqslant w'^J \leqslant w\}$, which is graded, thin, and shellable by Proposition 3.1. Hence by Theorem 2.1, $\partial Lk_{v'}$ is homeomorphic to a sphere of dimension ${}^J\ell(w) - {}^J\ell(v') - 2$. This proves (b).

The statement now follows from (a), (b) and Theorem 2.2. $\qquad\square$

4.2 Proof of Theorem 4.1

Set $Y = \overline{Y_{v,w}}$. The outline of the proof is similar to the proof of Proposition 4.4. We prove by induction on ${}^J\ell(w) - {}^J\ell(v)$. The base case ${}^J\ell(w) - {}^J\ell(v) = 0$ is trivial, since $Y_{v,v} = \dot{v}B^+ / B^+$ is a single point.

We first show that

(a) Y is a topological manifold with boundary

$$\partial Y = \bigsqcup_{v^J \leqslant v'^J \leqslant w'^J \leqslant w, (v,w) \neq (v',w')} Y_{v',w'}.$$

The proof of the claim is divided into several cases depending on (v', w').

(i) It follows from Corollary 3.7 that $Y_{v,w} \cong \mathbb{R}_{>0}^{{}^J\ell(w) - {}^J\ell(v)}$ is the open cell.

(ii) We consider the case $(v', w') = (v, v)$. By Proposition 4.4 and §4.1 (b), we have

$$Y \cap \dot{v}B^- B^+ / B^+ = Y \cap {}^J\mathring{\mathcal{B}}^v \cong \text{Cone}(D^{{}^J\ell(w) - {}^J\ell(v) - 1}),$$

$$Y_{v,v} \longmapsto \text{cone point}.$$

This shows that $Y_{v,v}$ lies on the boundary.

(iii) We consider the case $(v', w') = (w, w)$. Similar to (ii), we can establish (via a variation of Proposition 4.4 and §4.1 (b))

$$Y \bigcap \dot{w} B^- B^+ / B^+ = Y \bigcap {}^J \mathring{\mathcal{B}}_w \cong \mathrm{Cone}(D^{{}^J \ell(w) - {}^J \ell(v) - 1}),$$

$$Y_{w,w} \longmapsto \text{ cone point.}$$

This shows that $Y_{w,w}$ lies on the boundary.

(iv) We next consider the case when $v' \neq v$. We further assume $v' \neq w$, otherwise we are done by (iii). We apply the stratified isomorphism in Theorem 3.5 to obtain

$$Y \bigcap \dot{v}' B^- B^+ / B^+ = \bigsqcup_{v \overset{J}{\leqslant} v'' \overset{J}{\leqslant} v' \overset{J}{\leqslant} w'' \overset{J}{\leqslant} w} Y_{v'', w''}$$

$$\cong \bigsqcup_{v \overset{J}{\leqslant} v'' \overset{J}{\leqslant} v'} Y_{v'', v'} \times \bigsqcup_{v' \overset{J}{\leqslant} w'' \overset{J}{\leqslant} w} Y_{v', w''}$$

Now induction applies to the spaces $\overline{Y_{v'', v'}}$ and $\overline{Y_{v', w''}}$. In particular, they are both topological manifolds with the expected boundaries. Therefore

$$Y \bigcap \dot{v}' B^- B^+ / B^+ \cong \mathrm{Cone}(D^{{}^J \ell(v') - {}^J \ell(v) - 1}) \times \mathrm{Cone}(D^{{}^J \ell(w) - {}^J \ell(v') - 1})$$

$$\cong (\mathbb{R}^{{}^J \ell(v') - {}^J \ell(v) - 1}_{>0} \times \mathbb{R}_{\geqslant 0}) \times (\mathbb{R}^{{}^J \ell(w) - {}^J \ell(v') - 1}_{>0} \times \mathbb{R}_{\geqslant 0})$$

$$\cong \mathbb{R}^{{}^J \ell(w) - {}^J \ell(v) - 1}_{>0} \times \mathbb{R}_{\geqslant 0}$$

with $Y_{v', w'}$ lying on the boundary.

(v) The final case $w' \neq w$ is similar to (iv).

Now we finish the proof of (a).

We next show that

(b) ∂Y is a regular CW complex homeomorphic to a sphere of dimension ${}^J \ell(w) - {}^J \ell(v) - 1$.

It follows by the induction hypothesis that $\overline{Y_{v', w'}}$ is a regular CW complex homeomorphic to a closed ball of dimension ${}^J \ell(w') - {}^J \ell(v')$ if $v \overset{J}{\leqslant} v' \overset{J}{\leqslant} w' \overset{J}{\leqslant} w$ and $(v, w) \neq (v', w')$. Hence ∂Y is a regular CW complex with the face poset $\{(v', w') | (v', w') \overset{J}{\preceq} (v, w), (v', w') \neq (v, w)\} \subset {}^J Q$. By adding a new maximal element and a new minimal element we obtain the new poset $\{(v', w') \in {}^J Q | (v', w') \overset{J}{\preceq} (v, w)\} \bigsqcup \{\hat{0}\} \subset {}^J \hat{Q}$. Thanks to Proposition 3.1, this is graded, thin and shellable. By Theorem 2.1, ∂Y is a regular CW complex homeomorphic to a sphere of dimension ${}^J \ell(w) - {}^J \ell(v) - 1$. This finishes the proof of (b).

The statement now follows from (a), (b) and Theorem 2.2.

5 Main results

We collect the main results of this paper. We shall first prove the main results on the ordinary totally nonnegative flag variety $\mathcal{B}_{\geqslant 0}$ in §5.1. In §5.2 and §5.3, we will

prove the main results on the totally nonnegative partial flag variety and the J-totally nonnegative flag variety, respectively. The proofs rely on Proposition 5.6 and Proposition 5.9 (both marked with ♣) respectively, which will be established in §6 to §8.

5.1 Ordinary total positivity

5.1.1 Totally nonnegative part of G

We follow [23] and [27]. The generalization to Kac-Moody groups is straightforward. Let $U_{\geqslant 0}^+$ be the submonoid of G generated by $x_i(a)$ for $i \in I$ and $a \in \mathbb{R}_{>0}$ and $U_{\geqslant 0}^-$ be the submonoid of G generated by $y_i(a)$ for $i \in I$ and $a \in \mathbb{R}_{>0}$. Let $T_{>0}$ be the identity component of $T(\mathbb{R})$. Let $G_{\geqslant 0}$ be the submonoid of G generated by $U_{\geqslant 0}^{\pm}$ and $T_{>0}$. By [27, §2.5], $G_{\geqslant 0} = U_{\geqslant 0}^+ T_{>0} U_{\geqslant 0}^- = U_{\geqslant 0}^- T_{>0} U_{\geqslant 0}^+$.

Let $w \in W$ and $w = s_{i_1} s_{i_2} \cdots s_{i_n}$ be a reduced expression of w. Set

$$U_{w,>0}^+ = \{x_{i_1}(a_1) x_{i_2}(a_2) \cdots x_{i_n}(a_n) | a_1, a_2, \ldots, a_n \in \mathbb{R}_{>0}\};$$

$$U_{w,>0}^- = \{y_{i_1}(a_1) y_{i_2}(a_2) \cdots y_{i_n}(a_n) | a_1, a_2, \ldots, a_n \in \mathbb{R}_{>0}\}.$$

By [27, Lemma 2.3 (b)], $U_{w,>0}^{\pm}$ is independent of the choice of reduced expression of w. Moreover, by [27, §2.5 (d) & (e)], we have $U_{\geqslant 0}^{\pm} = \bigsqcup_{w \in W} U_{w,>0}^{\pm}$. Let $U_{w,\geqslant 0}^{\pm} = \overline{U_{w,>0}^{\pm}}$ be the closure of $U_{w,>0}^{\pm}$. Then $U_{w,\geqslant 0}^{\pm} = \bigsqcup_{w' \leqslant w} U_{w',>0}^{\pm}$ by [23, Proposition 4.2].

5.1.2 Totally nonnegative flag varieties

Let $\mathcal{B}_{\geqslant 0} = \overline{U_{\geqslant 0}^- \cdot B^+}$ be the closure of $U_{\geqslant 0}^- \cdot B^+$ in $\mathcal{B}$ with respect to the Hausdorff topology. For any $v \leqslant w$, let

$$\mathcal{B}_{v,w,>0} = \mathring{\mathcal{B}}_{v,w} \bigcap \mathcal{B}_{\geqslant 0}.$$

Hence $\mathcal{B}_{\geqslant 0} = \bigsqcup_{v \leqslant w} \mathcal{B}_{v,w,>0}$.

Let $\mathbf{w} = s_{i_1} s_{i_2} \cdots s_{i_n}$ be a reduced expression of $w \in W$. A subexpression of $\mathbf{w}$ is $t_{i_1} t_{i_2} \cdots t_{i_n}$, where $t_{i_j} = 1$ or s_{i_j} for any j. For any $v \leqslant w$, there exists a unique positive subexpression $\mathbf{v}_+ = t_{i_1} t_{i_2} \cdots t_{i_n}$ for v in $\mathbf{w}$ in the sense of [33, Lemma 3.5]. Following [33, Definition 5.1], we set

$$G_{\mathbf{v}_+,\mathbf{w},>\mathbf{0}} = \{g_1 g_2 \cdots g_n | g_j = \dot{s}_{i_j}, \text{ if } t_{i_j} = 1; \text{ and } g_j \in y_{i_j}(\mathbb{R}_{>0}), \text{ if } t_{i_j} = s_{i_j}\}.$$

Note that the obvious map $\mathbb{R}_{>0}^{\ell(w)-\ell(v)} \to G_{\mathbf{v}_+,\mathbf{w},>\mathbf{0}}$ is a homeomorphism.

By [33, Theorem 11.3] for reductive groups and [2, Theorem 4.10] for Kac-Moody groups, we have the following parametrization result.

(a) Let $v \leqslant w$. For any reduced expression $\mathbf{w}$ of w, the map $g \mapsto g \cdot B^+$ gives a homeomorphism

$$G_{\mathbf{v}_+,\mathbf{w},>\mathbf{0}} \cong \mathcal{B}_{v,w,>0}. \tag{5.1}$$

In particular, $\mathcal{B}_{v,w,>0} \cong \mathbb{R}_{>0}^{\ell(w)-\ell(v)}$ is a topological cell.

We recall the monoid actions $*$, $\circ_l$ and $\circ_r$ of W in [2, §5]. We have $s_i * w = \max\{w, s_i w\}$, $s_i \circ_l w = \min\{w, s_i w\}$, and $w \circ_r s_i = \min\{w, w s_i\}$ for any simple reflection s_i.

Lemma 5.1 *Let $v \leqslant r \leqslant w$ in W. We have*

$$\mathcal{B}_{v,w,>0} \subset \dot{r} B^- B^+ / B^+.$$

Proof We argue by induction on $\ell(w)$. Let $w' = s_{i_1} w$ with the reduced expression $\mathbf{w}' = s_{i_2} \cdots s_{i_n}$. Set $v' = s_{i_1} \circ_l v$ and $r' = s_{i_1} \circ_l r$. It follows from [14, Lemma 2] that $v' \leqslant r' \leqslant w'$. We divide the computation into several cases.

- If $r \leqslant s_{i_1} r$ or $r' = r$, then $v \leqslant r \leqslant s_{i_1} w$. In this case $g_1 \in y_{i_1}(\mathbb{R}_{>0})$ and $\dot{r}^{-1} g_1 \dot{r} \in U^-(\mathbb{R})$. Hence $\dot{r}^{-1} G_{\mathbf{v}_+,\mathbf{w},>\mathbf{0}} \subset U^-(\mathbb{R}) \dot{r}^{-1} G_{\mathbf{v}_+,\mathbf{w}',>0}$.
- If $r \geqslant s_{i_1} r$ and $g_1 = \dot{s}_{i_1}$, then $v' = s_{i_1} v$. Therefore $\dot{r}^{-1} G_{\mathbf{v}_+,\mathbf{w},>\mathbf{0}} = (\dot{r}')^{-1} G_{\mathbf{v}'_+,\mathbf{w}',>0}$.
- Assume $r \geqslant s_{i_1} r$ and $g_1 \in y_{i_1}(\mathbb{R}_{>0})$. Then $G_{\mathbf{v}_+,\mathbf{w},>\mathbf{0}} = y_{i_1}(\mathbb{R}_{>0}) G_{\mathbf{v}_+,\mathbf{w}',>0}$. For any $a \in \mathbb{R}_{>0}$, we have $\dot{s}_{i_1}^{-1} y_{i_1}(a) = \alpha_{i_1}^\vee(a^{-1}) y_{i_1}(-a) x_{i_1}(a^{-1})$. Then we have $x_{i_1}(a^{-1}) G_{\mathbf{v}_+,\mathbf{w}',>0} \subset G_{\mathbf{v}'_+,\mathbf{w}',>0} B^+$ by [2, Proposition 5.2]. Therefore $\dot{r}^{-1} G_{\mathbf{v}_+,\mathbf{w},>\mathbf{0}} \subset B^- (\dot{r}')^{-1} G_{\mathbf{v}'_+,\mathbf{w}',>0} B^+$.

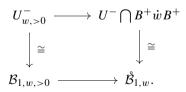The statement then follows from inductive hypothesis on w'. $\qquad\square$

5.1.3 Main results on $\mathcal{B}_{\geqslant 0}$

We apply results in Theorem 3.5 and Theorem 4.1 for the case $J = \emptyset$ to prove the main result for $\mathcal{B}_{\geqslant 0}$.

Proposition 5.2 *Let $v, w \in W$ with $v \leqslant w$. Then*

(1) *$\mathcal{B}_{v,w,>0}$ is a connected component of $\mathring{\mathcal{B}}_{v,w}(\mathbb{R})$.*
(2) *We have $\mathcal{B}_{v,w,\geqslant 0} = \overline{\mathcal{B}_{v,w,>0}} = \bigsqcup_{v \leqslant v' \leqslant w' \leqslant w} \mathcal{B}_{v',w',>0}$.*

Proof We first consider the $v = 1$ case. We have a commutative diagram

$$
\begin{array}{ccc}
U^-_{w,>0} & \longrightarrow & U^- \bigcap B^+ \dot{w} B^+ \\
\Big\downarrow{\cong} & & \Big\downarrow{\cong} \\
\mathcal{B}_{1,w,>0} & \longrightarrow & \mathring{\mathcal{B}}_{1,w}.
\end{array}
$$

By [26, §6.3], $U^-_{w,>0}$ is a connected component of $U^-(\mathbb{R}) \bigcap B^+ \dot{w} B^+$. Thus $\mathcal{B}_{1,w,>0}$ is a connected component of $\mathring{\mathcal{B}}_{1,w}(\mathbb{R})$.

Let $v_1 \leqslant w_1 \leqslant w$. By Lemma 5.1, $\overline{\mathcal{B}_{1,w,>0}} \bigcap \mathring{\mathcal{B}}_{v_1,w_1} \subset \mathcal{B}_{v_1,w_1,>0} \subset \dot{u} U^- B^+ / B^+$ for any $u \in W$ with $v_1 \leqslant u \leqslant w_1$. Hence the assumption in Theorem 3.5 is satisfied for $Y_{1,w} = \mathcal{B}_{1,w,>0}$. By Corollary 3.7, $\overline{\mathcal{B}_{1,w,>0}} \bigcap \mathring{\mathcal{B}}_{v_1,w_1} \subset \mathcal{B}_{v_1,w_1,>0}$ is a connected component of $\mathring{\mathcal{B}}_{v_1,w_1}(\mathbb{R})$. By (5.1), $\mathcal{B}_{v_1,w_1,>0}$ is connected. Hence $\mathcal{B}_{v_1,w_1,>0} =$

$\overline{\mathcal{B}_{1,w,>0}} \bigcap \mathring{\mathcal{B}}_{v_1,w_1}$ and it is a connected component of $\mathring{\mathcal{B}}_{v_1,w_1}(\mathbb{R})$. Moreover, part (2) for $v = 1$ now follows from Theorem 3.5, Theorem 4.1 and Corollary 3.7 for $Y_{1,w} = \mathcal{B}_{1,w,>0}$.

Now we consider the general case. We have already proved that $Y_{v,w} = \mathcal{B}_{v,w,>0}$ is a connected component of $\mathring{\mathcal{B}}_{v,w}(\mathbb{R})$. By Lemma 5.1 and Corollary 3.7, for $v_1, w_1 \in W$ with $v \leqslant v_1 \leqslant w_1 \leqslant w$, $\overline{\mathcal{B}_{v,w,>0}} \bigcap \mathring{\mathcal{B}}_{v_1,w_1} \subset \mathcal{B}_{v_1,w_1,>0}$ is a connected component of $\mathring{\mathcal{B}}_{v_1,w_1}(\mathbb{R})$. Hence $\overline{\mathcal{B}_{v,w,>0}} \bigcap \mathring{\mathcal{B}}_{v_1,w_1} = \mathcal{B}_{v_1,w_1,>0}$. Now part (2) for arbitrary $v \in W$ follows from Theorem 3.5, Corollary 3.7 for $Y_{v,w} = \mathcal{B}_{v,w,>0}$. $\qquad \square$

We have verified the assumption in Theorem 3.5 for $Y_{v,w} = \mathcal{B}_{v,w,>0}$. The following theorem follows from Theorem 3.5, Theorem 4.1, Corollary 3.7, and Proposition 4.4 for $Y_{v,w} = \mathcal{B}_{v,w,>0}$.

Theorem 5.3 *Let $v, w \in W$ with $v \leqslant w$.*

(1) *For any $u \in W$ with $v \leqslant u \leqslant w$, the map c_u restricts to an isomorphism*

$$c_u : \mathcal{B}_{v,w,>0} \cong \mathcal{B}_{v,u,>0} \times \mathcal{B}_{u,w,>0}.$$

(2) *$\mathcal{B}_{v,w,\geqslant 0}$ is a regular CW complex homeomorphic to a closed ball of dimension $\ell(w) - \ell(v)$.*

Remark 5.4 Part (2) of Theorem 5.3 proves [12, Conjecture 10.2 (2)].

5.2 Totally nonnegative partial flag varieties

5.2.1 Partial flag varieties

Let $K \subset I$ and $\mathcal{P}_K = G/P_K^+$ be the partial flag variety. Let $Q_K = \{(v, w) \in W \times W^K | v \leqslant w\}$. Define the partial order $\preceq$ on Q_K by $(v', w') \preceq (v, w)$ if there exists $u \in W_K$ with $v \leqslant v'u \leqslant w'u \leqslant w$. For any $(v, w) \in Q_K$, set

$$\mathring{\mathcal{P}}_{K,(v,w)} = pr_K(\mathring{\mathcal{B}}_{v,w}) \text{ and } \mathcal{P}_{K,(v,w)} = pr_K(\mathcal{B}_{v,w}),$$

where $pr_K : \mathcal{B} \to \mathcal{P}_K$ is the projection map. Then $\mathcal{P}_{K,(v,w)}$ is the (Zariski) closure of $\mathring{\mathcal{P}}_{K,(v,w)}$ in $\mathcal{P}_K$. We call $\mathring{\mathcal{P}}_{K,(v,w)}$ an *open projected Richardson variety* and $\mathcal{P}_{K,(v,w)}$ a *closed projected Richardson variety*. Note that $pr_K : \mathring{\mathcal{B}}_{v,w} \to \mathring{\mathcal{P}}_{K,(v,w)}$ is an isomorphism for $(v, w) \in Q_K$.

By [19, Proposition 3.6], we have

$$\mathcal{P}_K = \bigsqcup_{(v,w) \in Q_K} \mathring{\mathcal{P}}_{K,(v,w)} \quad \text{and} \quad \mathcal{P}_{K,(v,w)} = \bigsqcup_{(v',w') \in Q_K,(v',w') \preceq (v,w)} \mathring{\mathcal{P}}_{K,(v',w')}.$$

$$(5.2)$$

5.2.2 Total positivity on $\mathcal{P}_K$

Let $\mathcal{P}_{K,\geqslant 0} = \overline{U_{\geqslant 0}^- P_K^+ / P_K^+}$ be the closure of $U_{\geqslant 0}^- P_K^+ / P_K^+$ in $\mathcal{P}_K$ with respect to the Hausdorff topology. It is easy to see that $\mathcal{P}_{K,\geqslant 0} = pr_K(\mathcal{B}_{\geqslant 0})$. For $(v, w) \in Q_K$, we further define $\mathcal{P}_{K,(v,w),>0} = \mathcal{P}_{K,\geqslant 0} \bigcap \mathring{\mathcal{P}}_{K,(v,w)}$.

Proposition 5.5 *Let $(v, w) \in Q_K$. Then we have*

(1) *The map pr_K restricts to an isomorphism $\mathcal{B}_{v,w,>0} \cong \mathcal{P}_{K,(v,w),>0}$;*
(2) $\overline{\mathcal{P}_{K,(v,w),>0}} = \bigsqcup_{(v',w') \preceq (v,w) \text{ in } Q_K} \mathcal{P}_{K,(v',w'),>0}$;
(3) $\mathcal{P}_{K,(v,w),>0}$ *is a connected component of $\mathring{\mathcal{P}}_{K,(v,w)}(\mathbb{R})$;*
(4)

$$\begin{cases} \mathcal{P}_{K,(v,w),>0} \subset \dot{r} U^- P_K^+ / P_K^+, & \text{if } (r, r) \preceq (v, w) \in Q_K; \\ \mathcal{P}_{K,(v,w),>0} \bigcap \dot{r} U^- P_K^+ / P_K^+ = \emptyset, & \text{otherwise.} \end{cases}$$

Proof Let $v', w' \in W$ with $v' \leqslant w'$. We write $w' = (w')^K w'_K$ with $(w')^K \in W^K$ and $w'_K \in W_K$. Set $v'_1 = (w'_K)^{-1} \circ_r v'$. Then $v'_1 \leqslant (w')^K$. Set $v'_2 = (v'_1)^{-1} v'$. We fix a reduced expression $(\mathbf{w'})^{\mathbf{K}}$ of $(w')^K$ and a reduced expression $\mathbf{w'_K}$ of w'_K. Then $(\mathbf{w'})^{\mathbf{K}} \mathbf{w'_K}$ is a reduced expression of w'. Let $(\mathbf{v'_1})_+$ be the positive subexpression of v'_1 in $(\mathbf{w'})^{\mathbf{K}}$ and $(\mathbf{v'_2})_+$ be the positive subexpression of v'_2 in $\mathbf{w'_K}$. It is easy to see that $(\mathbf{v'_1})_+(\mathbf{v'_2})_+$ is the positive subexpression of v' in $(\mathbf{w'})^{\mathbf{K}} \mathbf{w'_K}$. By definition,

$$pr_K(\mathcal{B}_{v',w',>0}) = pr_K(G_{(\mathbf{v'_1})_+(\mathbf{v'_2})_+,(\mathbf{w'})^{\mathbf{K}}\mathbf{w'_K},>0} \cdot B^+) = pr_K(G_{(\mathbf{v'_1})_+,(\mathbf{w'})^{\mathbf{K}},>0} \cdot B^+)$$

$$= pr_K(\mathcal{B}_{v'_1,(w')^K,>0}) \subset \mathcal{P}_{K,(v'_1,(w')^K)}.$$

In particular,

$$pr_K(\mathcal{B}_{\geqslant 0}) = \bigcup_{v' \leqslant w' \text{ in } W} pr_K(\mathcal{B}_{v',w',>0}) = \bigcup_{v' \leqslant w' \text{ in } W} pr_K(\mathcal{B}_{v'_1,(w')^K,>0}).$$

Since $(v'_1, (w')^K) \in Q_K$, we have $pr_K(\mathcal{B}_{\geqslant 0}) = \bigcup_{(v',w') \in Q_K} pr_K(\mathcal{B}_{(v',w'),>0})$. For any $(v', w') \in Q_K$, $pr_K(\mathcal{B}_{(v',w'),>0}) \subset \mathcal{P}_{K,(v',w')}$. Thus the union

$$\bigcup_{(v',w') \in Q_K} pr_K(\mathcal{B}_{(v',w'),>0})$$

is a disjoint union and $pr_K(\mathcal{B}_{(v',w'),>0}) = \mathcal{P}_{K,(v',w'),>0}$ for any $(v', w') \in Q_K$.

Thus we have $\mathcal{P}_{K,(v,w),>0} = pr_K(\mathcal{B}_{v,w,>0})$. Note that pr_K restricts to $\mathring{\mathcal{B}}_w$ is an isomorphism to its image. Thus $pr_K : \mathcal{B}_{v,w,>0} \cong \mathcal{P}_{K,(v,w),>0}$. Part (1) is proved.

We have

$$\overline{\mathcal{P}_{K,(v,w),>0}} = pr_K(\overline{\mathcal{B}_{v,w,>0}}) = \bigsqcup_{v \leqslant v'' \leqslant w'' \leqslant w} pr_K(\mathcal{B}_{v'',w'',>0}).$$

Let $(v', w') \in Q_K$ with $(v', w') \preceq (v, w)$. Then there exists $u \in W_K$ such that $v \leqslant v'u \leqslant w'u \leqslant w$. Let $u' \leqslant u$ with $v' * u = v'u'$. Then $v \leqslant v'u \leqslant v' * u = v'u' \leqslant$

$w'u' \leqslant w'u \leqslant w$. We have $pr_K(\mathcal{B}_{v'u',w'u',>0}) = pr_K(\mathcal{B}_{v',w',>0}) = \mathcal{P}_{K,(v',w'),>0}$. Thus $\mathcal{P}_{K,(v',w'),>0} \subset \overline{\mathcal{P}_{K,(v,w),>0}}$.

On the other hand, for any $v' \leqslant w'$ in W with $v \leqslant v' \leqslant w' \leqslant w$, we have $v \leqslant v' = v'_1 v'_2 \leqslant (w')^K v'_2 \leqslant (w')^K w'_K = w' \leqslant w$, where v'_1 and v'_2 are defined above. Thus $(v'_1, (w')^K) \preceq (v, w)$ and $pr_K(\mathcal{B}_{v',w',>0}) = \mathcal{P}_{v'_1,(w')^K,>0}$. Part (2) is proved.

Finally part (3) and part (4) follow from §3.3(a), Proposition 5.2 (1) and Lemma 5.1. □

5.2.3 Main results on $\mathcal{P}_{K,\geqslant 0}$

We collect the main results on $\mathcal{P}_{K,\geqslant 0}$ in this subsection. The proof relies on the following result, which will be proved in §8.6.

Proposition 5.6 (♣) *Let $(v, w) \in Q_K$. Then $\overline{\mathcal{P}_{K,(v,w),>0}}$ is a topological manifold with boundary $\partial \overline{\mathcal{P}_{K,(v,w),>0}} = \bigsqcup_{(v',w')<(v,w)\in Q_K} \mathcal{P}_{K,(v',w'),>0}$.*

Now we prove the main result.

Theorem 5.7 *Let $(v, w) \in Q_K$. Then $\overline{\mathcal{P}_{K,(v,w),>0}} = \bigsqcup_{(v',w')\preceq(v,w) \text{ in } Q_K} \mathcal{P}_{K,(v',w'),>0}$ is a regular CW complex homeomorphic to a closed ball of dimension $\ell(w) - \ell(v)$.*

Remark 5.8 This proves [12, Conjecture 10.2 (3)].

Proof The proof is similar to the proof of Theorem 4.1. We prove by induction on $\ell(w) - \ell(v)$. The base case when $\ell(w) - \ell(v) = 0$ is trivial.

It follows by induction that $\overline{\mathcal{P}_{K,(v',w'),>0}}$ is a regular CW complex homeomorphic to a closed ball of dimension $\ell(w') - \ell(v')$ for any $(v', w') \in Q_K$ with $\ell(w') - \ell(v') < \ell(w) - \ell(v)$. Therefore by Proposition 5.6, $\partial \overline{\mathcal{P}_{K,(v,w),>0}} = \bigsqcup_{(s,t)<(v,w)\in Q_K} \mathcal{P}_{K,(s,t),>0}$ is a regular CW complex. Its face poset is $\{(v', w')|(v', w') < (v, w)\} \subset Q_K$. By adding the maximal element (v, w) and the minimal element $\{\hat{0}\}$, we obtain the new poset $\{(v', w')|(v', w') \leqslant (v, w)\} \bigsqcup \{\hat{0}\} \subset Q_K \bigsqcup \{\hat{0}\}$.

By [3, Theorem 4.1], this poset is graded, thin, and shellable. By Theorem 2.1, $\partial \overline{\mathcal{P}_{K,(v,w),>0}}$ is a regular CW complex homeomorphic to a sphere of dimension $\ell(w) - \ell(v) - 1$. By Proposition 5.6 and (5.1), $\mathcal{P}_{K,(v,w),>0}$ is a topological cell. Now the theorem follows from Theorem 2.2. □

5.3 J-Total positivity

5.3.1 The totally nonpositive part $\mathcal{B}_{\leqslant 0}$

Let $\iota : G \to G$ be the unique group automorphism that is identity on T and maps $x_i(a)$ to $x_i(-a)$ and $y_i(a)$ to $y_i(-a)$ for any $i \in I$, $a \in \mathbb{R}$. Let $U^-_{\leqslant 0} = (U^-_{\geqslant 0})^{-1} = \iota(U^-_{\geqslant 0})$ be the submonoid of U^- generated by $y_i(a)$ for $i \in I$ and $a \in \mathbb{R}_{<0}$. Since B^+ is stable under ι, we denote the induced automorphism on $\mathcal{B}$ by ι as well.

Similar to the definition of $\mathcal{B}_{\geqslant 0}$, let $\mathcal{B}_{\leqslant 0}$ be the closure of $U^-_{\leqslant 0}B^+/B^+$ with respect to the Hausdorff topology. For any $v \leqslant w$, we set $\mathcal{B}_{v,w,<0} = \mathcal{B}_{\leqslant 0} \bigcap \mathring{\mathcal{B}}_{v,w}$. It is clear that we have isomorphisms $\iota : \mathcal{B}_{\geqslant 0} \cong \mathcal{B}_{\leqslant 0}$, and $\iota : \mathcal{B}_{v,w,>0} \cong \mathcal{B}_{v,w,>0}$.

We fix a reduced expression $\mathbf{w}$. Let $\mathbf{v}_+$ be the unique positive subexpression for v in $\mathbf{w}$. We define $G_{\mathbf{v}_+,\mathbf{w},<0}$ in the similar way as $G_{\mathbf{v}_+,\mathbf{w},>0}$ in §5.1.1, but using $y_i(\mathbb{R}_{<0})$ instead of $y_i(\mathbb{R}_{>0})$ and $\dot{s}_i^{-1}$ instead of $\dot{s}_i$. It is clear that $\iota(G_{\mathbf{v}_+,\mathbf{w},<0}) = G_{\mathbf{v}_+,\mathbf{w},>0}$ and $G_{\mathbf{v}_+,\mathbf{w},<0}B^+/B^+ = \mathcal{B}_{v,w,<0}$.

5.3.2 J-Totally nonnegative flag varieties

Let $J \subset I$. Let $U_{J,\geqslant 0}^-$ be the submonoid of U^- generated by $y_i(a)$ for $i \in J$ and $a \in \mathbb{R}_{>0}$. Since $U_{\geqslant 0}^- = \bigsqcup_{w \in W} U_{w,>0}^-$ and $U_{J,\geqslant 0}^- = \bigsqcup_{w \in W_J} U_{w,>0}^-$, we have $U_{J,\geqslant 0}^- = U_{\geqslant 0}^- \cap L_J$. Moreover, let $\pi_J : P_J^- \to L_J$ be the projection map. Then we have $\pi_J(U_{\geqslant 0}^-) = U_{J,\geqslant 0}^-$ and $\pi_J(U_{\leqslant 0}^-)^{-1} = U_{J,\geqslant 0}^-$. Set

$$^J U_{\geq 0}^- = \{h_1 \pi_J(h_2)^{-1} h_2 | h_1 \in U_{J,\geqslant 0}^-, h_2 \in U_{\leqslant 0}^-\}.$$

If $J = I$, then $^J U_{\geq 0}^- = U_{\geqslant 0}^-$. If $J = \emptyset$, then $^J U_{\geq 0}^- = U_{\leqslant 0}^-$.

For $v \in W_J$ and $w \in {}^J W$, we set

$$^J U_{v,w,>0}^- = \{h_1 \pi_J(h_2)^{-1} h_2 | h_1 \in U_{v,>0}^-, h_2 \in U_{w,<0}^-\}.$$

Then $^J U_{v,w,>0}^- \cong U_{v,>0}^- \times U_{w,<0}^- \cong \mathbb{R}_{>0}^{\ell(v)+\ell(w)}$ is a cell and

$$^J U_{\geq 0}^- = \bigsqcup_{v \in W_J, w \in {}^J W} {}^J U_{v,w,>0}^-.$$

One may also see that each cell $^J U_{v,w,>0}^-$ is locally closed in U^- and thus $^J U_{\geq 0}^-$ is a constructible subset of U^-. It is worth pointing out that $^J U_{\geq 0}^-$, in general, is not closed in U^-.

We define the J-totally nonnegative flag variety $^J \mathcal{B}_{\geqslant 0}$ to be the closure of $^J U_{\geq 0}^- B^+/B^+$ in $\mathcal{B}$ with respect to the Hausdorff topology. Note that if $v_1 \leqslant v_2$, $w_1 \leqslant w_2$, then $^J U_{v_1,w_1,>0}^-$ is contained in the Hausdorff closure of $^J U_{v_2,w_2,>0}^-$. Thus

$$^J \mathcal{B}_{\geqslant 0} = \bigcup_{v \in W_J, w \in {}^J W} \overline{{}^J U_{v,w,>0}^- B^+/B^+}.$$

For $w_1 {}^J\!\leqslant w_2$, let $^J \mathcal{B}_{w_1,w_2,>0} = {}^J \mathring{\mathcal{B}}_{w_1,w_2} \cap {}^J \mathcal{B}_{\geqslant 0}$. We call $^J \mathcal{B}_{w_1,w_2,>0}$ the *totally positive J-Richardson variety*. Then $^J \mathcal{B}_{\geqslant 0} = \bigsqcup_{w_1 {}^J\!\leqslant w_2} {}^J \mathcal{B}_{w_1,w_2,>0}$.

If $J = \emptyset$, then $^J \mathcal{B}_{\geqslant 0} = \mathcal{B}_{\leqslant 0}$ and $^J \mathcal{B}_{w_1,w_2,>0} = \mathcal{B}_{w_1,w_2,<0}$. If $J = I$, then $^J \mathcal{B}_{\geqslant 0} = \mathcal{B}_{\geqslant 0}$ and $^J \mathcal{B}_{w_1,w_2,>0} = \mathcal{B}_{w_2,w_1,>0}$.

5.3.3 Main results on $^J \mathcal{B}_{\geqslant 0}$

We collect the main results on $^J \mathcal{B}_{\geqslant 0}$ in this subsection. The proof relies on the following result, which will be proved in §8.5.

Proposition 5.9 (♣) *Let $v {}^J\!\leqslant w$ in W. We have*

(1) ${}^J\mathcal{B}_{v,w,\geqslant 0} = \overline{{}^J\mathcal{B}_{v,w,>0}} = \bigsqcup_{v \leqslant v' \leqslant w' \leqslant w} {}^J\mathcal{B}_{v',w',>0}$.

(2) ${}^J\mathcal{B}_{v,w,>0}$ is a connected component of ${}^J\mathring{\mathcal{B}}_{v,w}(\mathbb{R})$.

(3) For any $u \in W$ with $v \, {}^J\!\leqslant u \, {}^J\!\leqslant w$, we have ${}^J\mathcal{B}_{v,w,>0} \subset \dot{u}U^- \cdot B^+$.

Combining Proposition 5.9 with Theorem 3.5 and Theorem 4.1, we have the main result for the J-total positivity.

Theorem 5.10 Let $v \, {}^J\!\leqslant w$. Then

(1) For any $u \in W$ with $v \, {}^J\!\leqslant u \, {}^J\!\leqslant w$, the map ${}^J c_u$ restricts to an isomorphism

$$ {}^J c_u : {}^J\mathcal{B}_{v,w,>0} \cong {}^J\mathcal{B}_{v,u,>0} \times {}^J\mathcal{B}_{u,w,>0}. $$

(2) ${}^J\mathcal{B}_{v,w,\geqslant 0}$ is a regular CW complex homeomorphic to a closed ball of dimension ${}^J\ell(w) - {}^J\ell(v)$.

5.4 Links

In this subsection, we consider the link of the identity in subspaces of U^-. The cases we consider here can be regarded as the special cases of the links of some totally positive (ordinary, J-, projected) Richardson varieties in the previous subsections.

5.4.1 The unipotent radical of B^-

Let $\lambda \in X^{++}$ and $w \in W$. We consider the embedding $U^- \xrightarrow{u \mapsto u \cdot \eta_\lambda} V^\lambda$. We identify $U^-(\mathbb{R})$ with its image. The image of $U^-(\mathbb{R}) \bigcap B^+ \dot{w} B^+$ is contained in a finite dimensional subspace $L \subset V^\lambda(\mathbb{R})$. We equip L with the standard Euclidean norm. For any $1 < w' \leqslant w$, we define

$$ Lk(U^-_{w',>0}) = U^-_{w',>0} \bigcap \{||x|| = 1 | x \in L\}. $$

Let $Lk(\overline{U^-_{w,>0}}) = \overline{U^-_{w,>0}} \bigcap \{||x|| = 1 | x \in L\}$. Since the $\overline{U^-_{w,>0}} = \bigsqcup_{w' \leqslant w} U^-_{w',>0}$, we have $Lk(\overline{U^-_{w,>0}}) = \bigsqcup_{1 < w' \leqslant w} Lk(U^-_{w',>0})$.

Theorem 5.11 $Lk(\overline{U^-_{w,>0}}) = \bigsqcup_{1 < w' \leqslant w} Lk(U^-_{w',>0})$ is a regular CW complex homeomorphic to a closed ball of dimension $\ell(w) - 1$.

Remark 5.12 This proves [12, Conjecture 10.2 (1)], and generalizes the main result in [15].

Proof We have a stratified isomorphism $Lk(\overline{U^-_{w',>0}}) \cong Lk_1(\overline{\mathcal{B}_{1,w',>0}})$ as defined in §4.1. We have verified the assumptions in Theorem 4.1 for $Y_{1,w} = \mathcal{B}_{1,w,>0}$ in Lemma 5.1 and Proposition 5.2. Hence the theorem follows by Proposition 4.4. $\qquad\square$

5.4.2 The unipotent radical of P_K^-

Let $K \subset I$. We consider the map

$$pr_K : U^- \longrightarrow U^-/U_K^- \cong U_{P_K^-}, \quad g \longmapsto g\pi_K(g)^{-1}.$$

Then $pr_K(U_{w,>0}^-) = pr_K(U_{w^K,>0}^-)$, for any $w \in W$. We define

$$Lk(\overline{pr_K(U_{w,>0}^-)}) = \overline{pr_K(U_{w,>0}^-)} \bigcap \{||x|| = 1 | x \in L\}.$$

We have $Lk(\overline{pr_K(U_{w,>0}^-)}) \cong Lk_1(\overline{\mathcal{P}_{K,(1,w^K),>0}})$, where $Lk_1(\overline{\mathcal{P}_{K,(1,w^K),>0}})$ can be defined entirely similarly to §4.1 using a singular dominant weight λ. Therefore, via the compatibility in Proposition 8.1, we actually have a stratified isomorphism $Lk(\overline{pr_K(U_{w,>0}^-)}) \cong Lk_1(\overline{\mathcal{P}_{K,(1,w^K),>0}})$. Note that

$$\overline{pr_K(U_{w,>0}^-)} \supsetneq \bigsqcup_{w' \in W^K, w' \leqslant w} pr_K(U_{w',>0}^-).$$

The remaining strata of $\overline{pr_K(U_{w,>0}^-)}$ corresponds to $\mathcal{P}_{K,(v,w'),>0}$ for $v \in W_J$, $w' \in W^K$ with $w'v \leqslant w^K$.

Thanks to the atlas map in §7.3, Proposition 4.4 and Theorem 5.10, we conclude that

(a) $Lk(\overline{pr_K(U_{w,>0}^-)})$ (with the stratification arising arising from Q_K) is a regular CW complex homeomorphic to a closed ball of dimension $\ell(w) - 1$.

5.4.3 Links in J-total positivity

Let $J \subset I$ and $\lambda \in X^{++}$. Let $v \in W_J$ and $w \in {}^J W$. By [4, Theorem 5], the image of $U^-(\mathbb{R}) \bigcap {}^J B^- \dot{v} B^+ \bigcap {}^J B^+ \dot{w} B^+$ under the embedding $U^- \xrightarrow{u \mapsto u \cdot \eta_\lambda} V^\lambda$ lies in a finite-dimensional subspace $L \subset V^\lambda(\mathbb{R})$.

We define

$$Lk(\overline{{}^J U_{v,w,>0}^-}) = \overline{{}^J U_{v,w,>0}^-} \bigcap \{||x|| = 1 | x \in L\}.$$

We have $Lk(\overline{{}^J U_{v,w,>0}^-}) \cong Lk_1(\overline{{}^J \mathcal{B}_{v,w,>0}})$. Note that

$$\overline{{}^J U_{v,w,>0}^-} \supsetneq \bigsqcup_{v' \leqslant v, w' \leqslant w} {}^J U_{v',w',>0}^-.$$

The remaining pieces of $\overline{{}^J U_{v',w',>0}^-}$ correspond to certain J-Richardson varieties arising from the twisted Bruhat order ${}^J \leqslant$.

Thanks to Proposition 4.4 and Theorem 5.10, we conclude that

(a) $Lk(\overline{{}^J U_{v,w,>0}^-})$ (with the stratification arising from ${}^J \leqslant$) is a regular CW complex homeomorphic to a closed ball of dimension $\ell(vw) - 1$.

6 J-Total positivity on $\mathcal{B}$

6.1 The main result

The main purpose of this section is to give an explicit description of $^J\mathcal{B}_{u,w,>0}$ in the special case where $w \in {}^J W$.

Let $w \in {}^J W$. Note that $u \, {}^J\!\leqslant w$ if and only if $^J u \leqslant w$. We fix a reduced expression $\mathbf{w}$. Let $\mathbf{^J u_+}$ be the unique positive subexpression for $^J u$ in $\mathbf{w}$. Set

$$
{}^J G_{u,\mathbf{w},>0} = \{ h_1 \pi_J (h_2 \, {}^J\dot{u}^{-1})^{-1} h_2 \mid h_1 \in U^-_{u_J,>0}, h_2 \in G_{\mathbf{J u_+},\mathbf{w},<0} \}
$$

$$
\cong U^-_{u_J,>0} \times G_{\mathbf{J u_+},\mathbf{w},<0}.
$$

Note that $^J G_{u,\mathbf{w},>0} \cdot B^+/B^+$ is connected and $^J G_{u,\mathbf{w},>0} \cdot B^+/B^+ \subset {}^J\mathring{\mathcal{B}}_{u,w}$. Now we state the main result of this section.

Proposition 6.1 *Let $w \in {}^J W$ and $u \in W$ with $u \, {}^J\!\leqslant w$. Then*
(1) $^J G_{u,\mathbf{w},>0} \cdot B^+/B^+$ *is a connected component of $^J\mathring{\mathcal{B}}_{u,w}(\mathbb{R})$.*
(2) *For any $w' \in W$ with $u \, {}^J\!\leqslant w' \, {}^J\!\leqslant w$, we have $^J\mathcal{B}_{u,w',>0} = \overline{{}^J\mathcal{B}_{u,w,>0}} \bigcap {}^J\mathring{\mathcal{B}}_{u,w'}$.*
(3) *Let $\mathbf{w}$ be a reduced expression of w. Then the following map is an isomorphism:*

$$
{}^J G_{u,\mathbf{w},>0} \longrightarrow {}^J\mathcal{B}_{u,w,>0}, \qquad g \longmapsto g \cdot B^+/B^+.
$$

This proposition proves a special case of Proposition 5.9. We outline our strategy of the proof. Part (1) follows from §6.2.1. Part (2) follows from Corollary 6.6. We remark that §3.4 plays a crucial role in the proof. Finally, Part (3) is proved in §6.5.

6.2 Connected components

Recall that $^J B^+ = U^-_J \ltimes T U_{P^+_J}$ and $B^- = U^-_J \ltimes T U_{P^-_J}$. Let $p_{+,J} : {}^J B^+ \to U^-_J$ be the projection map. For any $w \in {}^J W$, we define

$$
\phi_{w,J} : {}^J B^+ \dot{w} \cdot B^+/B^+ \longrightarrow B^+ \dot{w} \cdot B^+/B^+,
$$

$$
b\dot{w} \cdot B^+/B^+ \longmapsto p_{+,J}(b)^{-1} b\dot{w} \cdot B^+/B^+.
$$

We have the isomorphism $^J B^+ \dot{w} \cdot B^+/B^+ \xrightarrow{\sim} U^-_J \times (B^+ \dot{w} \cdot B^+/B^+)$. For any $u \in {}^J W$, we define

$$
\psi^{u,J} : B^- \dot{u} \cdot B^+/B^+ \longrightarrow {}^J B^- \dot{u} \cdot B^+/B^+,
$$

$$
b\dot{u} \cdot B^+/B^+ \longmapsto \pi_J(b)^{-1} b\dot{u} \cdot B^+/B^+.
$$

It is easy to see that the maps $\phi_{w,J}$ and $\psi^{u,J}$ are well-defined. In the special case where $u, w \in {}^J W$ with $u \leqslant w$, $\phi_{w,J} : {}^J\mathring{\mathcal{B}}_{u,w} \to \mathring{\mathcal{B}}_{u,w}$ is inverse to $\psi^{u,J} : \mathring{\mathcal{B}}_{u,w} \to {}^J\mathring{\mathcal{B}}_{u,w}$ and hence we have an isomorphism $^J\mathring{\mathcal{B}}_{u,w} \cong \mathring{\mathcal{B}}_{u,w}$.

Lemma 6.2 *Let $w \in W$ and $h_{1,n}, h_{2,n} \in U^-_{w,\geqslant 0}$ be two sequences such that $\lim_{n\to\infty} h_{1,n}h_{2,n}$ exists. Then we have convergent subsequences $\{h_{1,n'}\}$ and $\{h_{2,n'}\}$.*

Remark 6.3 Note that all limits are in $U^-_{\geqslant 0}$, provided they exist.

Proof We first illustrate the case where $h_{1,n} = y_i(a_n)$ for some $i \in I$ and $a_n \in \mathbb{R}_{\geqslant 0}$. It suffices to prove $\{a_n\}$ is bounded above. Consider the group morphism $r_i : U^- \to \mathbb{R}$ mapping $y_i(a)$ to a, and $y_j(b)$ to 0 for $j \neq i$. Then it is clear if $\{a_n\}$ is unbounded, then $r_i(h_{1,n}h_{2,n}) \geqslant a_n$ would diverge.

Now we consider the general case. Let $w = s_{i_1} \cdots s_{i_k}$ be reduced. Then we may write $h_{1,n}$ as $y_{i_1}(a_{1,n}) \cdots y_{i_k}(a_{k,n})$, where $i_1, \ldots, i_k \in I$ and $a_{1,n}, \ldots, a_{k,n} \in \mathbb{R}_{\geqslant 0}$. Set $h'_{1,n} = y_{i_1}(a_{1,n})$ and $h'_{2,n} = y_{i_2}(a_{2,n}) \cdots y_{i_k}(a_{k,n})h_{2,n}$. By assumption, $\lim_{n\to\infty} h'_{1,n}h'_{2,n} = \lim_{n\to\infty} h_{1,n}h_{2,n}$ exists. Then there exists convergent subsequnces $\{a_{1,n'}\}$ and $\{h'_{2,n'_1}\}$. Applying the same argument to h'_{2,n'_1}, we conclude that there exists convergent subsequences $\{a_{2,n'}\}, \ldots, \{a_{k,n'}\}$ and $\{h_{2,n'}\}$. By definition, the sequence $h_{1,n'} = y_{i_1}(a_{1,n'}) \cdots y_{i_k}(a_{k,n'})$ also converges. $\qquad\square$

6.2.1 Proof of Proposition 6.1 (1)

Let

$$\pi : U^+_J \dot{u}_J U^-_{P^-_J} {}^J\dot{u} \cdot B^+/B^+ = {}^J\mathring{\mathcal{B}}^u \to \mathring{\mathcal{B}}_{u_J} = U^+ \dot{u}_J \cdot B^+/B^+$$

be the projection map. We show that

(a) ${}^J G_{u,\mathbf{w},>\mathbf{0}} \cdot B^+/B^+ = \{z \in {}^J\mathring{\mathcal{B}}_{u,w} | \phi_{w,J}(z) \in \mathcal{B}_{J_{u,w},<0}, \pi(z) \in \mathcal{B}_{1,u_J,>0}\}$.

For any $z \in {}^J G_{u,\mathbf{w},>\mathbf{0}} \cdot B^+/B^+$, $\phi_{w,J}(z) \in \mathcal{B}_{J_{u,w},<0}$ and $\pi(z) \in \mathcal{B}_{1,u_J,>0}$. Now let $z \in {}^J\mathring{\mathcal{B}}_{u,w}$ with $\phi_{w,J}(z) \in \mathcal{B}_{J_{u,w},<0}$ and $\pi(z) \in \mathcal{B}_{1,u_J,>0}$. Since $\phi_{w,J}(z) \in \mathcal{B}_{J_{u,w},<0}$, we have $z = hgB^+/B^+$ for some $h \in U^-_J$ and $g \in G_{J_{u,w},<0}$. Then $\pi(z) = h\pi_J(g^J\dot{u}^{-1})B^+/B^+ \in \mathcal{B}_{1,u_J,>0}$. Since $h\pi_J(g^J\dot{u}^{-1}) \in U^-_J$, we have $h\pi_J(g^J\dot{u}^{-1}) \in U^-_{u_J,>0}$. This shows that $z \in {}^J G_{u,\mathbf{w},>\mathbf{0}}$. (a) is proved.

We then show that

(b) ${}^J G_{u,\mathbf{w},>\mathbf{0}} \cdot B^+/B^+$ is open in ${}^J\mathring{\mathcal{B}}_{u,w}(\mathbb{R})$.

The image of ${}^J\mathring{\mathcal{B}}_{u,w}$ under the isomorphism ${}^J B^+ \dot{w} \cdot B^+/B^+ \xrightarrow{\sim} U^-_J \times (B^+ \dot{w} \cdot B^+/B^+)$ is in $U^-_J \times (B^+ \dot{w} \cdot B^+/B^+ \cap U^-_J \cdot {}^J\mathring{\mathcal{B}}^u)$, hence in $U^-_J \times (B^+ \dot{w} \cdot B^+/B^+ \cap P^-_J {}^J\dot{u} \cdot \mathring{\mathcal{B}})$. Note that $U^-_J \times \mathring{\mathcal{B}}_{J_{u,w}}$ is open in $U^-_J \times (B^+ \dot{w} \cdot B^+/B^+ \cap P^-_J {}^J\dot{u} \cdot \mathring{\mathcal{B}})$ and $U^-_J(\mathbb{R}) \times \mathcal{B}_{J_{u,w},<0}$ is open in $U^-_J(\mathbb{R}) \times \mathring{\mathcal{B}}_{J_{u,w}}(\mathbb{R})$. Similarly we see that $\mathcal{B}_{1,u_J,>0}$ is open in $U^+(\mathbb{R})\dot{u}_J \cdot B^+/B^+$. (b) is proved.

Finally we show that

(c) ${}^J G_{u,\mathbf{w},>\mathbf{0}} \cdot B^+/B^+$ is closed in ${}^J\mathring{\mathcal{B}}_{u,w}(\mathbb{R})$.

Let $z \in \overline{{}^J G_{u,\mathbf{w},>\mathbf{0}} \cdot B^+/B^+} \cap {}^J\mathring{\mathcal{B}}_{u,w}$. We assume $z = \lim_{n\to\infty} h_{1,n}h_{2,n}h_{3,n}B^+/B^+$, where $h_{1,n} \in U^-_{u_J,>0}$, $h_{3,n} \in G_{J_{u,w},<0}$ and $h_{2,n} = \pi_J(h_{3,n}{}^J\dot{u}^{-1})^{-1} \in U^-_{J,\geqslant 0}$.

Thanks to the isomorphism ${}^J B^+ \dot{w} \cdot B^+/B^+ \xrightarrow{\sim} U^-_J \times (B^+ \dot{w} \cdot B^+/B^+)$, $\lim_{n\to\infty} h_{1,n}h_{2,n}$ and $\lim_{n\to\infty} h_{3,n}B^+/B^+$ exist. Moreover, $\lim_{n\to\infty} h_{3,n}B^+/B^+ \in \mathcal{B}_{\geqslant 0} \cap \mathring{\mathcal{B}}_w \cap P^-_J {}^J\dot{u}B^+/B^+$. Thus $\lim_{n\to\infty} h_{3,n}B^+/B^+ = hB^+/B^+$ for some $h \in$

$G_{v\,^J u, \mathbf{w}, <0}$ with $v \in W_J$. Thanks to Lemma 6.2, we can find convergent subsequences of $h_{1,n}$ and $h_{2,n}$. Without loss of generality, we can assume both $\lim_{n\to\infty} h_{1,n}$ and $\lim_{n\to\infty} h_{2,n}$ exist and hence converge in $U^-_{J,\geqslant 0}$.

Hence $\lim_{n\to\infty} h_{2,n} h_{3,n} B^+/B^+$ exists and is in $\overline{U^-_{P_J^-}\,^J \dot{u} B^+/B^+} \cap P_J^- \dot{u} B^+/B^+$. Since $^J u \in\,^J W$, $U^-_{P_J^-}\,^J \dot{u} B^+/B^+$ is closed in $P_J^- \dot{u} B^+/B^+$. So $\lim_{n\to\infty} h_{2,n} h_{3,n} B^+/B^+ \in U^-_{P_J^-}\,^J \dot{u} B^+/B^+$. This shows that $v = 1$ and $\lim_{n\to\infty} h_{2,n} = \pi_J(h\,^J \dot{u}^{-1})^{-1}$. Finally note that

$$\pi(z) = \lim_{n\longrightarrow\infty} \pi(h_{1,n} h_{2,n} h_{3,n} B^+/B^+)$$

$$= \lim_{n\longrightarrow\infty} h_{1,n} B^+/B^+ \in U^-_{J,\geqslant 0} B^+/B^+ \cap U^+ \dot{u}_J \cdot B^+/B^+.$$

We have $\lim_{n\to\infty} h_{1,n} \in U^-_{u_J, >0}$. We conclude that $z \in\,^J G_{u,\mathbf{w}, >\mathbf{0}} \cdot B^+/B^+$.

6.3 Inside open subspaces

We have the following simple equalities on the product of x_i with y_i for $i \in I$. Let $a, b, c > 0$ and $i \neq j$ in I. Then

$$x_i(a) y_j(\pm b) = y_j(\pm b) x_i(a),$$

$$x_i(a) y_i(b+c) = y_i\big(\frac{b+c}{a(b+c)+1}\big)\alpha_i^\vee(a(b+c)+1)$$

$$\times x_i\big(\frac{a}{a(b+c)+1}\big),\tag{6.1}$$

$$x_i\big(\frac{a}{a(b+c)+1}\big) y_i(-c) = y_i\big(\frac{-c(a(b+c)+1)}{ab+1}\big)\alpha_i^\vee\big(\frac{a(b)+1}{a(b+c)+1}\big)$$

$$\times x_i\big(\frac{a}{ab+1}\big).$$

By direct calculation using the equalities (6.1), we have for any $i \in I$, $a > 0$ and $g_1 \in U^-_{w, <0}$, $x_i(a)\pi_J(g)^{-1} g \in \pi_J(g_2)^{-1} g_2 B^+$ for some $g_2 \in U^-_{w, <0}$. Thus

(a) Let $w \in W$. Then for any $g \in U^-_{w, <0}$ and $b \in B^+_{\geqslant 0}$, $b\pi_J(g)^{-1} g \in \pi_J(g')^{-1} g' t U^+$ for some $g' \in U^-_{w, <0}$ and $t \in T_{>0}$.

We also have some results on the product of totally nonnegative part of $U^\pm$ with certain Weyl group elements. Let $w, w_1, w_2 \in W$ be such that $w_1 w_2 = w$ and $\ell(w_1) + \ell(w_2) = \ell(w)$ and $h \in U^-_{w, >0}$, $b \in U^+_{w^{-1}, >0}$. By [12, Lemma 5.6] and its proof, we have

$$\dot{w}^{-1} h \in (U^- \cap \dot{w}^{-1} U^+ \dot{w}) U^+_{w^{-1}, >0} T_{>0};$$

$$\dot{w}_1^{-1} h \in (U^- \cap \dot{w}_1^{-1} U^+ \dot{w}_1) U^-_{w_2, >0} U^+_{w_1^{-1}, >0} T_{>0};\tag{6.2}$$

$$\dot{w} b \in (U^+ \cap \dot{w} U^- \dot{w}^{-1}) U^-_{w, >0} T_{>0};$$

$$\dot{w}_2 b \in (U^+ \bigcap \dot{w}_2 U^- \dot{w}_2) U^+_{w_1^{-1}, >0} U^-_{w_2, >0} T_{>0}.$$

More generally, we have
(b) if $w_1 \leqslant w$, then $\dot{w}_1^{-1} h \in U^- U^+_{w_1^{-1}, >0} T_{>0}$.

Lemma 6.4 *Let $v \in W_J$, $w \in {}^J W$ and $u \in W$ with $v {}^J\!\leqslant u {}^J\!\leqslant w$. Then we have*

$$^J U^-_{v,w,>0} \subset \dot{u} U^- B^+.$$

Proof Since $v {}^J\!\leqslant u {}^J\!\leqslant w$, we have $u_J \leqslant v$ and ${}^J u \leqslant w$. Let $h \in U^-_{v,>0}$ and $g \in U^-_{w,<0}$. We have $\dot{u}_J^{-1} h = h_1 b_1 t_1$ for some $h_1 \in U_J^-$, $b_1 \in U^+_{J,\geqslant 0}$ and $t_1 \in T_{>0}$. By §6.3 (a), $b_1 t_1 \pi_J(g)^{-1} g \in \pi_J(g_1)^{-1} g_1 B^+$ for some $g_1 \in U^-_{w,<0}$.

Via a variation of §6.3 (b) (for $g_1 \in U^-_{\leqslant 0}$), we have ${}^J \dot{u}^{-1} g_1 \in U^- B^+$. Thus

$$\dot{u}^{-1} h \pi_J(g)^{-1} g = {}^J \dot{u}^{-1} h_1 b_1 t_1 \pi_J(g)^{-1} g \in {}^J \dot{u}^{-1} \pi_J(g_1)^{-1} g_1 B^+ \subset U^- B^+.$$

This finishes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 6.5 *Let $u \in W$ and $w \in {}^J W$ with $u {}^J\!\leqslant w$. Let $v \in W_J$ with $u_J \leqslant v$. Then*

$$^J c_{u,-} ({}^J U^-_{v,w,>0} \cdot B^+/B^+) = {}^J \mathcal{B}_{u,w,>0}.$$

In particular, ${}^J U^-_{v,w,>0} \cdot B^+/B^+ = {}^J \mathcal{B}_{v,w,>0}$.

Proof For $x \in W_J$ and $y \in {}^J W$ with $x {}^J\!\leqslant y$, we simply write ${}^J \mathring{\mathcal{B}}'_{x,y,>0}$ for ${}^J U^-_{x,y,>0} \cdot B^+/B^+$. Recall that ${}^J \mathcal{B}_{u,w,>0} = \bigcup_{v' \in W_J, w' \in {}^J W} (\overline{{}^J \mathring{\mathcal{B}}'_{v',w',>0}} \bigcap {}^J \mathring{\mathcal{B}}_{u,w})$. By Lemma 3.4 and Lemma 6.4 we have

$$^J c_{u,-} ({}^J \mathring{\mathcal{B}}'_{v,w,>0}) = \overline{{}^J \mathring{\mathcal{B}}'_{v,w,>0}} \bigcap {}^J \mathring{\mathcal{B}}_{u,w} \subset {}^J \mathcal{B}_{u,w,>0}.$$

In particular, ${}^J c_{u,-} ({}^J \mathring{\mathcal{B}}'_{v,w,>0})$ is a connected component of ${}^J \mathring{\mathcal{B}}_{u,w}(\mathbb{R})$. Note that for any $v' \in W_J$, $w' \in {}^J W$, we can always find $v'' \in W_J$, $w'' \in {}^J W$ such that $v' \leqslant v''$, $w' \leqslant w''$, $u_J \leqslant v''$, $u {}^J\!\leqslant w''$. In particular, ${}^J \mathring{\mathcal{B}}'_{v',w',>0} \subset \overline{{}^J \mathring{\mathcal{B}}'_{v'',w'',>0}}$. Without loss of generality, we can further assume $v \leqslant v''$ and $w \leqslant w''$. Let us fix such v'' and w''. Then it remains to show:

(a) $\overline{{}^J \mathring{\mathcal{B}}'_{v'',w'',>0}} \bigcap {}^J \mathring{\mathcal{B}}_{u,w} = \overline{{}^J \mathring{\mathcal{B}}'_{v'',w,>0}} \bigcap {}^J \mathring{\mathcal{B}}_{u,w}$;
(b) $\overline{{}^J \mathring{\mathcal{B}}'_{v'',w,>0}} \bigcap {}^J \mathring{\mathcal{B}}_{u,w} = \overline{{}^J \mathring{\mathcal{B}}'_{v,w,>0}} \bigcap {}^J \mathring{\mathcal{B}}_{u,w}$.

Note that ${}^J \mathring{\mathcal{B}}'_{v'',w'',>0} \subset \dot{w} U^- B^+/B^+$ and ${}^J \mathring{\mathcal{B}}'_{v'',w,>0} \subset \dot{u} U^- B^+/B^+$. By Lemma 3.4 and Lemma 6.4, we have $\overline{{}^J \mathring{\mathcal{B}}'_{v'',w'',>0}} \bigcap {}^J \mathring{\mathcal{B}}_{v'',w} = {}^J c_{w,+} ({}^J \mathring{\mathcal{B}}'_{v'',w'',>0})$. Since $w \leqslant w''$, we have ${}^J \mathring{\mathcal{B}}'_{v'',w,>0} \subset \overline{{}^J \mathring{\mathcal{B}}'_{v'',w'',>0}} \bigcap {}^J \mathring{\mathcal{B}}_{v'',w}$. Since both sides are connected components of ${}^J \mathring{\mathcal{B}}_{v'',w}(\mathbb{R})$, we must have ${}^J \mathring{\mathcal{B}}'_{v'',w,>0} = {}^J c_{w,+} ({}^J \mathring{\mathcal{B}}'_{v'',w'',>0}) = \overline{{}^J \mathring{\mathcal{B}}'_{v'',w'',>0}} \bigcap {}^J \mathring{\mathcal{B}}_{v'',w}$. Now (a) follows by Lemma 3.2.

We then obtain $\overline{{}^J\mathring{\mathcal{B}}'_{v'',w'',>0}} \bigcap {}^J\mathring{\mathcal{B}}_{u,w} = \overline{{}^J\mathring{\mathcal{B}}'_{v'',w,>0}} \bigcap {}^J\mathring{\mathcal{B}}_{u,w} = {}^Jc_{u,-}({}^J\mathring{\mathcal{B}}'_{v'',w,>0})$. Since $v \leqslant v''$, we obtain that $\overline{{}^J\mathring{\mathcal{B}}'_{v,w,>0}} \bigcap {}^J\mathring{\mathcal{B}}_{u,w} \subset \overline{{}^J\mathring{\mathcal{B}}'_{v'',w,>0}} \bigcap {}^J\mathring{\mathcal{B}}_{u,w}$. Since both sides are connected components of ${}^J\mathring{\mathcal{B}}_{u,w}(\mathbb{R})$, we have

$$\overline{{}^J\mathring{\mathcal{B}}'_{v'',w'',>0}} \bigcap {}^J\mathring{\mathcal{B}}_{u,w} = {}^Jc_{u,-}({}^J\mathring{\mathcal{B}}'_{v'',w,>0}) = {}^Jc_{u,-}({}^J\mathring{\mathcal{B}}'_{v,w,>0}).$$

This finishes the proof. $\qquad\square$

Combining Lemma 6.5 with Lemmas 3.2 & 3.4, we have the following consequences.

Corollary 6.6 *Let $u \in W$ and $w \in {}^J W$ with $u \,{}^J\!\leqslant w$. Then*
(1) ${}^J\mathcal{B}_{u,w,>0} = \overline{{}^J\mathcal{B}_{v,w,>0}} \bigcap {}^J\mathring{\mathcal{B}}_{u,w}$ *is a connected component in ${}^J\mathring{\mathcal{B}}_{u,w}(\mathbb{R})$.*
(2) *For any $w' \in W$ with $u \,{}^J\!\leqslant w' \,{}^J\!\leqslant w$, ${}^J\mathcal{B}_{u,w',>0} = \overline{{}^J\mathcal{B}_{u,w,>0}} \bigcap {}^J\mathring{\mathcal{B}}_{u,w'}$.*

We remark that Proposition 6.1 (2) follows now.

6.4 Positivity

Lemma 6.7 *Let $u, w \in {}^J W$ with $u \,{}^J\!\leqslant w$. Then the isomorphism $\phi_{w,J} : {}^J\mathring{\mathcal{B}}_{u,w} \cong \mathring{\mathcal{B}}_{u,w}$ restricts to an isomorphism ${}^J\mathcal{B}_{u,w,>0} \cong \mathcal{B}_{u,w,<0}$. Moreover, ${}^J G_{u,\mathbf{w},>\mathbf{0}} \cdot B^+/B^+ = {}^J\mathcal{B}_{u,w,>0}$.*

Proof We first prove the statement for $u = 1$, that is, ${}^J\mathcal{B}_{1,w,>0} \cong \mathcal{B}_{1,w,<0}$. Let $h \in U^-_{1,w,<0}$. Then $hB^+/B^+ = g\dot{w}B^+/B^+$ for some $g \in U^+$. Since $w \in {}^J W$, we may further assume that $g \in U_{P_J^+}$. By definition, $\pi_J(h)^{-1}g \in {}^J B^+$ and $p_{+,J}(\pi_J(h)^{-1}g) = \pi_J(h)^{-1}$. Thus

$$\phi_{w,J}(\pi_J(h)^{-1}hB^+/B^+) = \phi_{w,J}(\pi_J(h)^{-1}g\dot{w}B^+/B^+)$$
$$= g\dot{w}B^+/B^+ = hB^+/B^+ \in \mathcal{B}_{1,w,<0}.$$

On the other hand, $\psi^{1,J}(hB^+/B^+) = \pi_J(h)^{-1}hB^+/B^+ \in {}^J U^-_{1,w,>0} \cdot B^+/B^+$. This finishes the proof.

Now we consider the general case. Since $\mathcal{B}_{u,w,<0} \subset \overline{\mathcal{B}_{1,w,<0}}$ and ${}^J\mathcal{B}_{u,w,>0} \subset \overline{{}^J\mathcal{B}_{1,w,>0}}$, we have $\phi^{-1}_{w,J}(\mathcal{B}_{u,w,<0}) \subset {}^J\mathcal{B}_{u,w,>0}$. Thanks to Corollary 6.6, ${}^J\mathcal{B}_{u,w,>0}$ is a connected component of ${}^J\mathring{\mathcal{B}}_{u,w}(\mathbb{R})$. Thanks to Theorem 5.3, $\mathcal{B}_{u,w,>0}$ is a connected component of $\mathring{\mathcal{B}}_{u,w}(\mathbb{R})$. Since isomorphism sends connected components to connected components, we must have $\phi^{-1}_{w,J}(\mathcal{B}_{u,w,<0}) = {}^J\mathcal{B}_{u,w,>0}$. $\qquad\square$

6.5 Proof of Proposition 6.1 (3)

By Lemma 6.7, ${}^J G_{{}^Ju,\mathbf{w},>\mathbf{0}} \cdot B^+/B^+ = {}^J\mathcal{B}_{{}^Ju,w,>0}$. Since ${}^J\mathcal{B}_{\geqslant 0}$ is stable under the action of $U^-_{J,\geqslant 0}$, we have

$${}^J G_{u,\mathbf{w},>\mathbf{0}} \cdot B^+/B^+ = U^-_{u_J,>0}\, {}^J G_{{}^Ju,\mathbf{w},>\mathbf{0}} \cdot B^+/B^+ \subset {}^J\mathcal{B}_{\geqslant 0} \bigcap {}^J\mathcal{B}_{u,w} = {}^J\mathcal{B}_{u,w,>0}.$$

Both sides are connected components of ${}^{J}\mathring{\mathcal{B}}_{u,w}(\mathbb{R})$ by Proposition 6.1 (1) and Corollary 6.6. Thus ${}^{J}G_{u,\mathbf{w},>\mathbf{0}} \cdot B^{+}/B^{+} = {}^{J}\mathcal{B}_{u,w,>0}$.

On the other hand, ${}^{J}G_{u,\mathbf{w},>\mathbf{0}} \cdot B^{+}/B^{+} \subset U_{J}^{-}B^{+}\dot{w}B^{+}/B^{+}$. Since $w \in {}^{J}W$, we have an isomorprhism $i : U_{J}^{-}B^{+}\dot{w}B^{+}/B^{+} \cong U_{J}^{-} \times B^{+}\dot{w}B^{+}/B^{+}$. By definition, for $h_1 \in U_{u_J,>0}^{-}$ and $h_2 \in G_{J u,\mathbf{w},<\mathbf{0}}$,

$$i(h_1 \pi_J (h_2{}^{J}\dot{u}^{-1})^{-1}h_2 \cdot B^{+}/B^{+}) = (h_1 \pi_J(h_2{}^{J}\dot{u}^{-1}), h_2 \cdot B^{+}/B^{+}).$$

Thus we may recover h_1 and h_2 from $h_1 \pi_J(h_2{}^{J}\dot{u}^{-1})^{-1}h_2 \cdot B^{+}/B^{+}$. This finishes the proof.

7 Basic J-Richardson varieties

7.1 Motivation

The totally nonnegative partial flag variety $\mathcal{P}_{K,\geqslant 0}$ has a cellular decomposition and each cell admits an explicit parametrization. However, $\mathcal{P}_{K,\geqslant 0}$ does not have an obvious product structure. We instead consider the Birkhoff-Bruhat atlas on $\mathcal{P}_K$ in §7.3, and translate the total positivity on $\mathcal{P}_K$ to the J-total positivity on the full flag variety of a larger Kac-Moody group. We then consider the product structure on the larger full flag variety and its compactibility with the J-total positivity.

In this section, we will introduce the basic J-Richardson varieties. This family of special J-Richardson varieties serves as models for both the projected Richardson varieties and the J-Richardson varieties. We will establish such connections in this section. Finally, in the last section, we will show that the totally positive part of the basic J-Richardson varieties are compatible with both the totally positive projective Richardson varieties and the general totally positive J-Richardson varieties. Such compatibility will allow us to bring together the information we have obtained on the totally positive projective Richardson varieties and the general totally positive J-Richardson varieties and finish the proof of our main results.

7.2 The larger Kac-Moody group

Let G' be a Kac-Moody group associated to the Kac-Moody root datum $(I', A', X', Y', \ldots)$. Let $K' \subset I'$. Following [3, §3], we associate a new Kac-Moody group $\tilde{G}'$ of adjoint type to G'. The Dynkin diagram of $\tilde{G}'$ is obtained by glueing two copies of the Dynkin diagram of G' along the subdiagram K'.

We denote by $\tilde{I}'$ the set of simple roots of $\tilde{G}'$. We denote by $(I')^{\flat}$ and $(I')^{\sharp}$ the two copies of I'. The elements in $(I')^{\flat}$ (resp. $(I')^{\sharp}$) are denoted by $i^{\flat}$ (resp. $i^{\sharp}$) for $i \in I'$. Then $\tilde{I}' = (I')^{\flat} \bigcup (I')^{\sharp}$ with $(I')^{\flat} \bigcap (I')^{\sharp} = \{k^{\flat} = k^{\sharp} | k \in K'\}$.

Let W' be the Weyl group of G' and $\tilde{W}'$ be the Weyl group of $\tilde{G}'$. We have natural identifications $W' \to \tilde{W}'_{(I')^{\flat}}$, $w \mapsto w^{\flat}$ and $W' \to \tilde{W}'_{(I')^{\sharp}}$, $w \mapsto w^{\sharp}$. For $w \in W'_{K'}$, $w^{\flat} = w^{\sharp}$. Similarly, we have natural maps $G' \to \tilde{L}_{(I')^{\flat}}$, $g \mapsto g^{\flat}$ and $G' \to \tilde{L}_{(I')^{\sharp}}$, $g \mapsto g^{\sharp}$. For $g \in L_{K'}$, $g^{\flat} = g^{\sharp}$.

Let $\tilde{\mathcal{B}}'$ be the flag variety of $\tilde{G}'$. Let $Q_{K'} = \{(v, w) \in W' \times (W')^{K'} | v \leqslant w\}$. Define

$$\tilde{\nu} : Q_{K'} \longrightarrow \tilde{W}', \qquad (v, w) \longmapsto (w)^{\flat}(v^{-1})^{\sharp}.$$

By [3, Proposition 4.2 (1)], $\tilde{\nu}$ is compatible with the partial order $\preceq$ on $Q_{K'}$ and the partial order $^{(I')^{\flat}}\leqslant$ on $\tilde{W}'$.

Definition 7.1 A J-Richardson variety $^{J}\mathring{\mathcal{B}}_{v,w}$ is called *basic* (with respect to G') if it is of the form $^{I'^{\flat}}\mathring{\tilde{\mathcal{B}}}'_{\tilde{\nu}(\alpha),\tilde{\nu}(\beta)}$ for some triple (K', α, β), where K' is a subset of the simple roots in G' and $\alpha \preceq \beta$ in $Q_{K'}$.

7.3 The Birkhoff-Bruhat atlas on $\mathcal{P}_K$

The technical definition of basic J-Richardson varieties arises from the Birkhoff-Bruhat atlas introduced in [3], which relates the projected Richardson varieties for G with the J-Richardson varieties for $\tilde{G}$.

Let $r \in W^K$. The isomorphism $\sigma_r : \dot{r}U^-\dot{r}^{-1} \to (\dot{r}U^-\dot{r}^{-1} \bigcap U^+) \times (\dot{r}U^-\dot{r}^{-1} \bigcap U^-)$ in (3.1) is compatible with Levi decompositions. The restriction of σ_r gives the isomorphism $\dot{r}U_{P_K^-}\dot{r}^{-1} \to (\dot{r}U_{P_K^-}\dot{r}^{-1} \bigcap U^+) \times (\dot{r}U_{P_K^-}\dot{r}^{-1} \bigcap U^-)$. We define

$$f_r : \dot{r}U^- P_K^+ \longrightarrow \tilde{G}, \quad g\dot{r}p \longmapsto (\sigma_{r,+}(g)\dot{r})^{\flat}((g\dot{r})^{\sharp})^{-1} \text{ for } g \in \dot{r}U_{P_K^-}\dot{r}^{-1}, p \in P_K^+.$$

The map f_r factors through $\dot{r}B^- P_K^+/P_K^+ \subset \mathcal{P}_K$ and induces a morphism

$$\tilde{c}_r : \dot{r}B^- P_K^+/P_K^+ \longrightarrow \tilde{\mathcal{B}}.$$

Let $(v, w) \in Q_K$ and $r \in W^K$ with $(r, r) \preceq (v, w)$ in Q_K. By [3, Theorem 3.2][3] we have the following commutative diagram

$$
\begin{array}{ccc}
\mathring{\mathcal{P}}_{K,(v,w)} \bigcap \dot{r}U^- P_K^+/P_K^+ & \xrightarrow[\cong]{\tilde{c}_r} & {}^{I^{\flat}}\mathring{\tilde{\mathcal{B}}}_{\tilde{\nu}(r,r),\tilde{\nu}(v,w)} \\
\Big\downarrow & & \Big\downarrow \\
\dot{r}B^- P_K^+/P_K^+ & \xhookrightarrow{\tilde{c}_r} & {}^{I^{\flat}}\tilde{\mathcal{B}}.
\end{array}
$$

In other words, $\tilde{c}_r$ gives an isomorphism from the stratified space

$$\dot{r}B^- P_K^+/P_K^+ = \bigsqcup_{(v,w)\in Q_K; (r,r)\preceq(v,w)} \left(\mathring{\mathcal{P}}_{K,(v,w)} \bigcap \dot{r}U^- P_K^+/P_K^+\right)$$

into its image in $^{I^{\flat}}\tilde{\mathcal{B}}$ stratified by the $I^{\flat}$-Richardson varieties.

---

[3] In loc.cit, we use $(\sigma_{r,+}(g)\dot{r})^{\flat}(\sigma_{r,-}(g)\dot{r}^{\sharp})^{-1}$ instead. However, it differs from $f(g\dot{r})$ by multiplying by an element in $(U^+)^{\sharp}$ on the right, so the induced maps to $\tilde{\mathcal{B}}$ coincide.

7.4 Basic J-Richardson varieties in $\mathcal{B}^{\spadesuit}$

Our next goal is to show any J-Richardson variety for the Kac-Moody group G can be realized as a basic one with respect to a different Kac-Moody group.
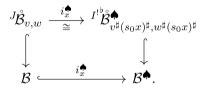
Set $I^{!} = I \bigsqcup \{0\}$. The generalized Cartan matrix $A^{!} = (a^{!}_{i,j})_{i,j \in I^{!}}$ is defined by

- for $i, j \in I$, $a^{!}_{i,j} = a_{i,j}$;
- for $i \in I$, $a^{!}_{i,0} = a^{!}_{0,i} = -2$;
- $a^{!}_{0,0} = 2$.

Let $W^{!}$ be the Weyl group associated to $(I^{!}, A^{!})$. Then we have the natural identification $W = W^{!}_{I}$. Moreover, for any $i \in I$, $s_0 s_i$ is of infinite order in $W^{!}$ and $s_0 x \in {}^{I}(W^{!})$ for all $x \in W$. Now for the triple $(I^{!}, A^{!}, J)$, we construct a triple $(\widetilde{I}^{!}, \widetilde{A}^{!}, I^{!\flat})$ following the construction in §7.2. Here $\widetilde{I}^{!}$ is obtained by glueing two copies of $I^{!}$ along the subset $J \subset I (\subset I^{!})$.

We write $I^{\spadesuit} = \widetilde{I}^{!}$ and $A^{\spadesuit} = \widetilde{A}^{!}$. Let $G^{\spadesuit}$ be the minimal Kac-Moody group of adjoint type associated to $(I^{\spadesuit}, A^{\spadesuit})$. Let $W^{\spadesuit}$ be the Weyl group associated to $G^{\spadesuit}$ and $\mathcal{B}^{\spadesuit}$ be the flag variety of $G^{\spadesuit}$.

Proposition 7.2 *Let $x \in W$. For any $g \in G$, define $i^{\spadesuit}_x(gB^{+}) = g^{\sharp}(\dot{s}_0 \dot{x})^{\sharp}(B^{\spadesuit})^{+}$. Then for any $v \overset{J}{\leqslant} w$, we have the following commutative diagram*

$$
\begin{array}{ccc}
{}^{J}\mathring{\mathcal{B}}_{v,w} & \xrightarrow[\cong]{\ i^{\spadesuit}_x\ } & {}^{I^{!\flat}}\mathring{\mathcal{B}}^{\spadesuit}_{v^{\sharp}(s_0 x)^{\sharp}, w^{\sharp}(s_0 x)^{\sharp}} \\
\downarrow & & \downarrow \\
\mathcal{B} & \xrightarrow[\quad i^{\spadesuit}_x \quad]{} & \mathcal{B}^{\spadesuit}.
\end{array}
$$

Remark 7.3 (1) In other words, $i^{\spadesuit}_x$ gives an isomorphism from the stratified space $\mathcal{B} = \bigsqcup_{v \overset{J}{\leqslant} w} {}^{J}\mathring{\mathcal{B}}_{v,w}$ into its image in $\mathcal{B}^{\spadesuit}$ stratified by the $I^{!\flat}$-Richardson varieties.

(2) Note that $v^{\sharp}(s_0 x)^{\sharp} = v^{\flat}_J ({}^{J} v \, s_0 x)^{\sharp}$ and $({}^{J} v \, s_0 x)^{\sharp} \in {}^{I^{!\flat}} W^{\spadesuit}$. Thus by definition, if $v_J, w_J \leqslant x^{-1}$, then ${}^{I^{!\flat}}\mathcal{B}^{\spadesuit}_{v^{\sharp}(s_0 x)^{\sharp}, w^{\sharp}(s_0 x)^{\sharp}}$ is a basic J-Richardson variety.

Proof Set $\diamond = \spadesuit - \{0^{\sharp}\}$. Let $\mathcal{B}^{\diamond}$ be the flag variety of $L^{\spadesuit}_{\diamond}$. We have

$$
{}^{I^{!\flat}}\mathring{\mathcal{B}}^{\spadesuit}_{v^{\sharp}(s_0 x)^{\sharp}, w^{\sharp}(s_0 x)^{\sharp}} \subset (P^{\spadesuit}_{\diamond})^{-}(\dot{s}_0 \dot{x})^{\sharp}(B^{\spadesuit})^{+}/(B^{\spadesuit})^{+} \bigcap (P^{\spadesuit}_{\diamond})^{+}(\dot{s}_0 \dot{x})^{\sharp}(B^{\spadesuit})^{+}/(B^{\spadesuit})^{+}
$$

$$
= L^{\spadesuit}_{\diamond}(\dot{s}_0 \dot{x})^{\sharp}(B^{\spadesuit})^{+}/(B^{\spadesuit})^{+}.
$$

The map $g \mapsto g(\dot{s}_0\dot{x})^\sharp(B^\spadesuit)^+$ for $g \in L_\diamond^\spadesuit$ induces the following Cartesian diagram

$$
\begin{array}{ccc}
I^{1^\flat}\mathring{\mathcal{B}}_{v^\sharp,w^\sharp}^\diamond & \xrightarrow{\;\cong\;} & I^{1^\flat}\mathring{\mathcal{B}}_{v^\sharp(s_0x)^\sharp,w^\sharp(s_0x)^\sharp}^\spadesuit \\
\downarrow & & \downarrow \\
L_\diamond^\spadesuit/(L_\diamond^\spadesuit\cap(B^\spadesuit)^+) & \xrightarrow{\;\cong\;} & L_\diamond^\spadesuit(\dot{s}_0\dot{x})^\sharp(B^\spadesuit)^+/(B^\spadesuit)^+.
\end{array}
$$

Moreover $I^{1^\flat}\mathring{\mathcal{B}}_{v^\sharp,w^\sharp}^\diamond \subset I^{1^\flat}(B^\diamond)^-(P_{I^\sharp}^\diamond)^+/(B^\diamond)^+ \cap I^{1^\flat}(B^\diamond)^+(P_{I^\sharp}^\diamond)^+/(B^\diamond)^+ = (P_{I^\sharp}^\diamond)^+/(B^\diamond)^+$. The isomoprhism $\mathcal{B} \to (P_{I^\sharp}^\diamond)^+/(B^\diamond)^+$, $gB^+ \mapsto g^\sharp(B^\diamond)^+$ induces an isomorphism $^J\mathring{\mathcal{B}}_{v,w} \cong I^{1^\flat}\mathring{\mathcal{B}}_{v^\sharp,w^\sharp}^\diamond$. The proposition is proved. $\qquad\square$

7.5 Some consequences on the J-Richardson varieties

We combine the results on the projected Richardson varieties and the J-Richardson varieties to prove the following result.

Proposition 7.4 *Let $v \overset{J}{\leqslant} w$. Then*
(1) *the J-Richardson variety $^J\mathring{\mathcal{B}}_{v,w}$ is irreducible of dimension $^J\ell(w) - {}^J\ell(v)$.*
(2) *the Zariski closure $^J\mathcal{B}_{v,w}$ of $^J\mathring{\mathcal{B}}_{v,w}$ equals*

$$
{}^J\mathcal{B}_w \bigcap {}^J\mathcal{B}^v = \bigsqcup_{v\overset{J}{\leqslant}v'\overset{J}{\leqslant}w'\overset{J}{\leqslant}w} {}^J\mathring{\mathcal{B}}_{v',w'}.
$$

Remark 7.5 In the case of reductive groups, one may deduce both statements for (ordinary) Richardson varieties easily from the transversal intersections of B^+-orbits and B^--orbits on $\mathcal{B}$. In the case of Kac-Moody groups, both statements for (ordinary) Richardson varieties are established recently in [21]. Our proof for J-Richardson varieties is based on [21].

Proof (1) Set $Q_J^! = \{(a,b) \in W^! \times (W^!)^J \,|\, a \leqslant b\}$. Let $v^! : Q_J^! \to W^\spadesuit$ be the map sending (a,b) to $a^\flat(b^{-1})^\sharp$. Let $x \in W$ with $v_J, w_J \leqslant x^{-1}$. Then we have

$$
(1,1) \preceq (v_J, x^{-1}s_0(^Jv)^{-1}) \preceq (w_J, x^{-1}s_0(^Jw)^{-1}).
$$

Moreover, $v^!(v_J, x^{-1}s_0(^Jv)^{-1}) = v^\sharp(s_0x)^\sharp$ and $v^!(w_J, x^{-1}s_0(^Jw)^{-1}) = w^\sharp(s_0x)^\sharp$. Thus

$$
1 \overset{(I^!)^\flat}{\leqslant} v^\sharp(s_0x)^\sharp \overset{(I^!)^\flat}{\leqslant} w^\sharp(s_0x)^\sharp. \tag{7.1}
$$

By [21, Proposition 6.6], $\mathring{\mathcal{B}}^!_{w_J,x^{-1}s_0(^Jw)^{-1}}$ is irreducible of dimension

$$
\ell(x^{-1}s_0(^Jw)^{-1}) - \ell(w_J) = \ell(x) + 1 + {}^J\ell(w).
$$

We have $\mathring{\mathcal{P}}^!_{J,(w_J,x^{-1}s_0(^Jw)^{-1})} \cong \mathring{\mathcal{B}}^!_{w_J,x^{-1}s_0(^Jw)^{-1}}$. Since $(1,1) \preceq (w_J, x^{-1}s_0(^Jw)^{-1})$, we have $\mathring{\mathcal{P}}^!_{J,(w_J,x^{-1}s_0(^Jw)^{-1})} \bigcap (U^!)^-(P^!_J)^+/(P^!_J)^+ \neq \emptyset$. By §7.3,

$$I^{!^\flat}\mathring{\mathcal{B}}^\spadesuit_{1,w^\sharp(s_0x)^\sharp} \cong \mathring{\mathcal{P}}^!_{J,(w_J,x^{-1}s_0(^Jw)^{-1})} \bigcap (U^!)^-(P^!_J)^+/(P^!_J)^+$$

is also irreducible of dimension $\ell(x) + 1 + {}^J\ell(w)$. Similarly, $I^{!^\flat}\mathring{\mathcal{B}}^\spadesuit_{1,v^\sharp(s_0x)^\sharp}$ is irreducible of dimension $\ell(x) + 1 + {}^J\ell(v)$.

By (7.1) and §3.3 (a), $I^{!^\flat}\mathring{\mathcal{B}}^\spadesuit_{1,w^\sharp(s_0x)^\sharp} \bigcap \dot{v}^\sharp(\dot{s}_0\dot{x})^\sharp(U^\spadesuit)^-(B^\spadesuit)^+/(B^\spadesuit)^+ \neq \emptyset$ and we have

$$I^{!^\flat}\mathring{\mathcal{B}}^\spadesuit_{1,w^\sharp(s_0x)^\sharp} \bigcap \dot{v}^\sharp(\dot{s}_0\dot{x})^\sharp(U^\spadesuit)^-(B^\spadesuit)^+/(B^\spadesuit)^+ \cong I^{!^\flat}\mathring{\mathcal{B}}^\spadesuit_{1,v^\sharp(s_0x)^\sharp} \times I^{!^\flat}\mathring{\mathcal{B}}^\spadesuit_{v^\sharp(s_0x)^\sharp,w^\sharp(s_0x)^\sharp}.$$

Since $I^{!^\flat}\mathring{\mathcal{B}}^\spadesuit_{1,v^\sharp(s_0x)^\sharp}$ is irreducible of dimension $\ell(x) + 1 + {}^J\ell(v)$, we have $I^{!^\flat}\mathring{\mathcal{B}}^\spadesuit_{v^\sharp(s_0x)^\sharp,w^\sharp(s_0x)^\sharp}$ is irreducible of dimension ${}^J\ell(w) - {}^J\ell(v)$. Now part (1) follows from Proposition 7.2.

(2) We have ${}^J\mathcal{B}_{v,w} \subset {}^J\mathcal{B}_w \bigcap {}^J\mathcal{B}^v$. By [4, Theorem 4],

$$ {}^J\mathcal{B}_w \bigcap {}^J\mathcal{B}^v = \bigsqcup_{v\,{}^J\!\leqslant v'\,{}^J\!\leqslant w'\,{}^J\!\leqslant w} {}^J\mathring{\mathcal{B}}_{v',w'}.$$

Let $u \in W$ with $v\,{}^J\!\leqslant u\,{}^J\!\leqslant w$. Set $Y = {}^J\mathring{\mathcal{B}}_{v,w} \bigcap (\dot{u}B^-B^+/B^+)$. By §3.3 (a), $Y \neq \emptyset$. Let $z \in Y$. Then $T \cdot z \subset {}^J\mathring{\mathcal{B}}_{v,w} \bigcap (\dot{u}B^-B^+/B^+)$. By the proof of Lemma 3.4, the closure of $T \cdot z$ contains $\dot{u}B^+/B^+$. Therefore, $\dot{u}B^+/B^+$ is contained in the Zariski closure of Y. We may apply Lemma 3.2 and the Remark 3.3 to the Zariski closure of Y. By (3.3) and the Remark 3.3, ${}^J\mathring{\mathcal{B}}_{v,u} = {}^Jc_{u,+}(Y)$ and ${}^J\mathring{\mathcal{B}}_{u,w} = {}^Jc_{u,-}(Y)$ are contained in the Zariski closure of Y, and hence in ${}^J\mathcal{B}_{v,w}$.

Now let $v', w' \in W$ with $v\,{}^J\!\leqslant v'\,{}^J\!\leqslant w'\,{}^J\!\leqslant w$. If $v' \neq v$, then we have ${}^J\mathring{\mathcal{B}}_{v',w} \subset {}^J\mathcal{B}_{u,w}$ and ${}^J\mathring{\mathcal{B}}_{v',w'} \subset {}^J\mathcal{B}_{v',w}$. So ${}^J\mathring{\mathcal{B}}_{v',w'} \subset {}^J\mathcal{B}_{u,w}$. If $v' = v$, then we have ${}^J\mathring{\mathcal{B}}_{v,w'} \subset {}^J\mathcal{B}_{v,w}$. Part (2) is proved. $\qquad\qquad\square$

Corollary 7.6 *Let $v\,{}^J\!\leqslant w$. Then ${}^J\mathcal{B}_{v,w,>0}$ is Zariski dense in ${}^J\mathring{\mathcal{B}}_{v,w}$.*

Proof We denote by $dim_{\mathbb{R}}(\cdot)$ the $\mathbb{R}$-dimension of a real semi-algebraic variety (see [6, §2.8]). We remark that all spaces considered here are semi-algebraic.

By Proposition 6.1 (3), we have a semi-algebraic homeomorphism ${}^J\mathcal{B}_{v,\,{}^Jw,>0} \cong \mathbb{R}_{>0}^{{}^J\ell({}^Jw)-{}^J\ell(v)}$. Therefore the Zariski closure of ${}^J\mathcal{B}_{v,\,{}^Jw,>0}$ in ${}^J\mathring{\mathcal{B}}_{v,\,{}^Jw}$ is irreducible and of dimension ${}^J\ell({}^Jw) - {}^J\ell(v)$. Thus by Proposition 7.4 (1), ${}^J\mathcal{B}_{v,\,{}^Jw,>0}$ is Zariski dense in ${}^J\mathring{\mathcal{B}}_{v,\,{}^Jw}$.

By definition, $v\,{}^J\!\leqslant w\,{}^J\!\leqslant {}^Jw$. By §3.3 (a), ${}^J\mathcal{B}_{v,\,{}^Jw,>0} \bigcap \dot{w}U^-B^+/B^+ \neq \emptyset$.

Set $X = {}^J\mathcal{B}_{v,\,{}^Jw,>0} \bigcap \dot{w}U^-B^+/B^+$. By (3.2), we have $X \cong {}^Jc_{w,+}(X) \times {}^Jc_{w,-}(X)$. By Lemma 3.2, ${}^Jc_{w,+}(X) = \overline{X} \bigcap {}^J\mathring{\mathcal{B}}_{v,w} \subset {}^J\mathcal{B}_{v,w,>0}$ and ${}^Jc_{w,-}(X) =$

$\overline{X} \bigcap {}^{J}\mathring{\mathcal{B}}_{w,{}^{J}w} \subset {}^{J}\mathcal{B}_{w,{}^{J}w,>0}$. In particular, $\dim_{\mathbb{R}}({}^{J}c_{w,-}(X)) \leqslant {}^{J}\ell({}^{J}w) - {}^{J}\ell(w)$. So

$$\dim_{\mathbb{R}}({}^{J}\mathcal{B}_{v,w,>0}) \geqslant \dim_{\mathbb{R}}({}^{J}c_{w,+}(X))$$
$$= \dim_{\mathbb{R}}(X) - \dim_{\mathbb{R}}({}^{J}c_{w,-}(X))) \geqslant {}^{J}\ell(w) - {}^{J}\ell(v).$$

By Proposition 7.4 (1), $\dim_{\mathbb{R}}({}^{J}\mathcal{B}_{v,w,>0}) = \dim_{\mathbb{R}}({}^{J}c_{w,+}(X)) = {}^{J}\ell(w) - {}^{J}\ell(v)$ and ${}^{J}\mathcal{B}_{v,w,>0}$ is Zariski dense in ${}^{J}\mathring{\mathcal{B}}_{v,w}$. □

8 Final part of the proofs

8.1 Compatibility of total positivities

In this subsection we show the compatibility among the total positivity on the projected Richardson varieties, the J-total positivity on the basic J-Richardson varieties, and the J-total positivity on arbitrary J-Richardson varieties.

Proposition 8.1 *Let* $(v, w) \in Q_K$ *and* $r \in W^K$ *with* $(r, r) \preceq (v, w)$ *in* Q_K. *Then the isomorphism* $\tilde{c}_r : \mathring{\mathcal{P}}_{K,(v,w)} \bigcap \dot{r}U^- P_K^+/P_K^+ \cong {}^{I^\flat}\mathring{\tilde{\mathcal{B}}}_{\tilde{v}(r,r),\tilde{v}(v,w)}$ *restricts to an isomorphism*

$$\tilde{c}_r : \mathcal{P}_{K,(v,w),>0} \to {}^{I^\flat}\tilde{\mathcal{B}}_{\tilde{v}(r,r),\tilde{v}(v,w),>0}.$$

Proof We claim it suffices to prove the statement for $(1, w')$ with sufficiently large w'.

Note that for any $(v, w) \in Q_K$, there exists $(1, w')$ with $(v, w) \preceq (1, w')$. By Proposition 5.5 (2), $\overline{\mathcal{P}_{K,(1,w'),>0}} \bigcap \mathring{\mathcal{P}}_{K,(v,w)} = \mathcal{P}_{K,(v,w),>0}$. By Proposition 6.1 (2),

$$\overline{{}^{I^\flat}\tilde{\mathcal{B}}_{\tilde{v}(r,r),\tilde{v}(1,w'),>0}} \bigcap {}^{I^\flat}\mathring{\tilde{\mathcal{B}}}_{\tilde{v}(r,r),\tilde{v}(v,w)} = {}^{I^\flat}\tilde{\mathcal{B}}_{\tilde{v}(r,r),\tilde{v}(v,w),>0}.$$

Suppose that the statement holds for $(1, w')$. Then the statement for (v, w) follows from the following commutative diagram.

$$
\begin{array}{ccccc}
\mathcal{P}_{K,(1,w'),>0} & \longrightarrow & \overline{\mathcal{P}_{K,(1,w'),>0}} & \longleftarrow & \overline{\mathcal{P}_{K,(1,w'),>0}} \bigcap \mathring{\mathcal{P}}_{K,(v,w)} \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
{}^{I^\flat}\tilde{\mathcal{B}}_{\tilde{v}(r,r),\tilde{v}(1,w'),>0} & \longrightarrow & \overline{{}^{I^\flat}\tilde{\mathcal{B}}_{\tilde{v}(r,r),\tilde{v}(1,w'),>0}} & \longleftarrow & {}^{I^\flat}\tilde{\mathcal{B}}_{\tilde{v}(r,r),\tilde{v}(1,w'),>0} \bigcap {}^{I^\flat}\mathring{\tilde{\mathcal{B}}}_{\tilde{v}(r,r),\tilde{v}(v,w)}.
\end{array}
$$

The statement for $(1, w)$ with sufficiently large w is proved by direct computation. Note that for any $w \in W$, $U_{w,>0}^- P_K^+ = U_{w^K,>0}^- P_K^+ \subset \dot{r}U^- P_K^+$ by Lemma 5.1. For computational purposes, we can further relax the condition and consider the image of $U_{w,>0}^- \subset \dot{r}U^- P_K^+$ under the map f_r for sufficiently large elements in W instead in W^K. Let $w \in W$ such that $\ell(r^{-1}w) = \ell(w) - \ell(r)$. We write $u = r^{-1}w$. The rest of the proof consists of direct computation of the map f_r.

Let $h \in U^-_{w,>0}$. By (6.2), we have

$$\dot{r}^{-1}h \in (U^- \bigcap \dot{r}^{-1}U^+\dot{r})h_1b_1 \quad \text{for some } h_1 \in U^-_{u^{-1},>0} \text{ and } b_1 \in B^+_{\geqslant 0}. \qquad (8.1)$$

We also have

$$h_1b_1 = b_2h_2, \quad \text{for some } h_2 \in U^-_{u^{-1},>0}, b_2 \in B^+_{\geqslant 0}. \qquad (8.2)$$

Set $g = hb_1^{-1}\pi_K(h_1)^{-1}\dot{r}^{-1}$. Since $r \in W^K$, we have $U^- \bigcap \dot{r}^{-1}U^+\dot{r} \subset U_{P_K^-}$ and thus $g \in \dot{r}U_{P_K^-}\dot{r}^{-1}$. We have $(g\dot{r})^{-1} \in \pi_K(h_1)b_1h^{-1}U^+ = \pi_K(h_1)h_1^{-1}\dot{r}^{-1}U^+$.

By Theorem 5.3 (1), $\sigma_{r,+}(g)\dot{r} \in h_3B^+$ for some $h_3 \in U^-_{r,>0}$. Since $h_1b_1 = b_2h_2$, we have $\pi_J(h_1)b_1 = b_2\pi_J(h_2)$. Then $\sigma_{r,+}(g)\dot{r} \in U^-g\dot{r} = U^-hb_1^{-1}\pi_K(h_1)^{-1} = U^-b_2^{-1}$. Thus $\sigma_{r,+}(g)\dot{r} = h_3b_2^{-1}$. We have

$$h_1\pi_K(b_1) = b_4h_4 \quad \text{for some } h_4 \in U^-_{u^{-1},>0} \text{ and } b_4 \in B^+_{K,\geqslant 0}. \qquad (8.3)$$

Then $\pi_K(h_1)\pi_K(b_1) = \pi_K(b_2)\pi_K(h_2) = \pi_K(b_4)\pi_K(h_4)$. So $\pi_K(h_2) = \pi_K(h_4)$ and

$$\pi_K(b_2)^{-1}\pi_K(h_1)h_1^{-1} = \pi_K(h_4)\pi_K(b_1)^{-1}h_1^{-1} \in \pi_K(h_4)h_4^{-1}b_4^{-1}.$$

Now we have

$$\begin{aligned}
(\sigma_{r,+}(g)\dot{r})^\flat(\dot{r}^{-1}\sigma_{r,-}(g)^{-1})^\sharp\tilde{U}^+ &= (h_3b_2^{-1})^\flat(\pi_K(h_1)h_1^{-1}\dot{r}^{-1})^\sharp\tilde{U}^+ \\
&= h_3^\flat(\pi_K(b_2)^{-1}\pi_K(h_1)h_1^{-1}\dot{r}^{-1})^\sharp\tilde{U}^+ \\
&= h_3^\flat(\pi_K(h_4)h_4^{-1}b_4^{-1}\dot{r}^{-1})^\sharp\tilde{U}^+. \qquad (8.4)
\end{aligned}$$

Hence $\tilde{c}_r(h \cdot P_K^+/P_K^+) = h_3^\flat(\pi_K(h_4)h_4^{-1}\dot{r}^{-1})^\sharp\tilde{B}^+ \in {}^{I^\flat}G_{\tilde{v}(r,r),\tilde{v}(1,w),>0} \cdot \tilde{B}^+$.

By Proposition 5.5 (3), $\mathcal{P}_{K,(v,w),>0}$ is a connected component of $\mathring{\mathcal{P}}_{K,(v,w)}(\mathbb{R}) \bigcap \dot{r}U^-P_K^+/P_K^+$. By Proposition 6.1, ${}^{I^\flat}G_{\tilde{v}(r,r),\tilde{v}(1,w),>0} \cdot \tilde{B}^+/\tilde{B}^+ = {}^{I^\flat}\mathring{\mathcal{B}}_{\tilde{v}(r,r),\tilde{v}(1,w),>0}$ is a connected component of ${}^{I^\flat}\mathring{\mathcal{B}}_{\tilde{v}(r,r),\tilde{v}(v,w)}(\mathbb{R})$.

Since the isomorphism $\tilde{c}_r$ sends the connected components of $\mathring{\mathcal{P}}_{K,(v,w)}(\mathbb{R}) \bigcap \dot{r}U^-(\mathbb{R})P_K^+/P_K^+$ to the connected components of ${}^{I^\flat}\mathring{\mathcal{B}}_{\tilde{v}(r,r),\tilde{v}(v,w)}(\mathbb{R})$, we have $\tilde{c}_r(\mathcal{P}_{K,(v,w),>0}) = {}^{I^\flat}\mathring{\mathcal{B}}_{\tilde{v}(r,r),\tilde{v}(v,w),>0}$. $\qquad \square$

Proposition 8.2 *Let $v {}^J\!\leqslant w$ and $x \in W$. Then the isomorphism $i_x^\spadesuit : {}^J\mathring{\mathcal{B}}_{v,w} \cong {}^{I^\flat}\mathring{\mathcal{B}}^\spadesuit_{v^\sharp(s_0x)^\sharp,w^\sharp(s_0x)^\sharp}$ restricts to an isomorphism*

$$i_x^\spadesuit : {}^J\mathcal{B}_{v,w,>0} \cong {}^{I^\flat}\mathcal{B}^\spadesuit_{v^\sharp(s_0x)^\sharp,w^\sharp(s_0x)^\sharp,>0}.$$

Proof We first consider the case where $w \in {}^JW$. Fix a reduced expression $\mathbf{w}$ of w. Fix a reduced expression of $\mathbf{s_0x}$ of s_0x. Then $\mathbf{w}^\sharp\mathbf{s_0^\sharp}\mathbf{x}^\sharp$ is a reduced expression of $w^\sharp(s_0x)^\sharp$.

In particular, we have

$$({}^J G_{v,\mathbf{w},>0})^\sharp \dot{s}_0^\sharp \dot{x}^\sharp = {}^{I^{1^\flat}}G_{v^\sharp (s_0 x)^\sharp, \mathbf{w}^\sharp \mathbf{s_0^\sharp x^\sharp}, >0}.$$

Now the statement follows from Proposition 6.1 (3).

We then consider the general case. Set $v' = v^\sharp (s_0 x)^\sharp$, $w' = w^\sharp (s_0 x)^\sharp$ and ${}^J w' = ({}^J w)^\sharp (s_0 x)^\sharp$. By Proposition 6.1 (2), we have $\overline{{}^J \mathcal{B}_{v, {}^J w, >0}} \bigcap {}^J \mathring{\mathcal{B}}_{v,w} = {}^J \mathcal{B}_{v,w,>0}$ and $\overline{{}^{I^{1^\flat}}\mathcal{B}^\spadesuit_{v', {}^J w', >0}} \bigcap {}^{I^{1^\flat}}\mathring{\mathcal{B}}^\spadesuit_{v',w'} = {}^{I^{1^\flat}}\mathring{\mathcal{B}}^\spadesuit_{v',w',>0}$. Now the statement follows from the following commutative diagram:

$$
\begin{array}{ccccc}
\overline{{}^J\mathcal{B}_{v,{}^Jw,>0}} & \longleftarrow & \overline{{}^J\mathcal{B}_{v,{}^Jw,>0}} \bigcap {}^J\mathring{\mathcal{B}}_{v,w} & \longrightarrow & {}^J\mathring{\mathcal{B}}_{v,w} \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
\overline{{}^{I^{1^\flat}}\mathcal{B}^\spadesuit_{v',{}^Jw',>0}} \bigcap \operatorname{Im}(i_x^\spadesuit) & \longleftarrow & \overline{{}^{I^{1^\flat}}\mathcal{B}^\spadesuit_{v',{}^Jw',>0}} \bigcap {}^{I^{1^\flat}}\mathring{\mathcal{B}}^\spadesuit_{v',w'} & \longrightarrow & {}^{I^{1^\flat}}\mathring{\mathcal{B}}^\spadesuit_{v',w'}.
\end{array}
$$
$\qquad \square$

8.2 Matrix coefficients and admissible functions

8.2.1 Admissible functions

Recall [27, §1.2] that a function $f : \mathbb{R}^m_{>0} \times \mathbb{R}^n_{>0} \to \mathbb{R}_{\geqslant 0}$ is called admissible if $f = f'/f''$, where $f', f'' \in \mathbb{Z}_{\geqslant 0}[t_1, \ldots, t_m, t'_1, \ldots, t'_n]$ with $f'' \neq 0$. Note that

(a) the value of an admissible function is either always 0 or never 0.

The following result proved in [12, Lemma 5.9] is useful to prove admissibility.

(b) Suppose that $f : \mathbb{R}^m_{>0} \times \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is continuous and the restriction $f : \mathbb{R}^m_{>0} \times \mathbb{R}^n_{>0} \to \mathbb{R}_{\geqslant 0}$ is an admissible function. Then the restriction $f : \mathbb{R}^m_{>0} \to \mathbb{R}_{\geqslant 0}$ is also admissible.

Let $v \leqslant w$ and $\mathbf{w}$ be a reduced expression of w. Then we have an isomorphism $\mathbb{R}^{\ell(w)-\ell(v)}_{>0} \to G_{\mathbf{v}_+, \mathbf{w}, >0}$. Let $v' \leqslant w'$ and $\mathbf{w}'$ be a reduced expression of w'. We call a map $G_{\mathbf{v}_+, \mathbf{w}, >0} \to G_{\mathbf{v}'_+, \mathbf{w}', >0}$ admissible if the composition $\mathbb{R}^{\ell(w)-\ell(v)}_{>0} \to G_{\mathbf{v}_+, \mathbf{w}, >0} \to G_{\mathbf{v}'_+, \mathbf{w}', >0} \to \mathbb{R}^{\ell(w')-\ell(v')}_{>0} \to \mathbb{R}_{>0}$ is admissible, where the last map is the projection to any coordinate. Thanks to [2, §6A], admissibility is independent of the choice of reduced expressions. We define the admissibility for maps $G_{\mathbf{v}_+, \mathbf{w}, >0} \to T_{>0}$ in the same way.

Let $w \in {}^J W$ and $\mathbf{w}$ be a reduced expression of w. Let $v \in W$ with ${}^J v \leqslant w$ and let $\mathbf{v_J}$ be a reduced expression of v_J. A map ${}^J G_{v,\mathbf{w},>\mathbf{0}} \to \mathbb{R}_{\geqslant 0}$ is called admissible if its composition with the map

$$\beta : \mathbb{R}^{{}^J\ell(w)-{}^J\ell(v)} \longrightarrow U^-_{\mathbf{v_J},>0} \times G_{{}^Jv,\mathbf{w},>0} \xrightarrow{(\mathrm{id},\iota)} U^-_{\mathbf{v_J},>0} \times G_{{}^Jv,\mathbf{w},<0} \longrightarrow {}^J G_{v,\mathbf{w},>\mathbf{0}}$$

is admissible. It follows from the previous discussion that the admissibility of a map ${}^J G_{v,\mathbf{w},>\mathbf{0}} \to \mathbb{R}_{\geqslant 0}$ is independent of the reduced expressions of w and v_J.
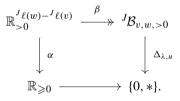
8.2.2 Matrix coefficients

We keep the notation in §4.1. For $\lambda \in X^{++}$, we consider the matrix coefficient

$$\Delta_\lambda : G \longrightarrow \mathbb{C}$$

mapping any $g \in G$ to the coefficient of $g\eta_\lambda$ in η_λ. We denote by $\{0, *\} = \mathbb{C}/\mathbb{C}^\times$ the set-theoretical quotient. The composition $\Delta_\lambda : G \to \mathbb{C} \to \{0, *\}$ factors through $\mathcal{B}$, which we still denote by Δ_λ. For any $u \in W$, we further define $\Delta_{\lambda,u} : \mathcal{B} \to \{0, *\}$, $gB^+/B^+ \mapsto \Delta_\lambda(\dot{u}^{-1}gB^+/B^+)$. Note that the image is independent of the choice of the representative of u in G. We then have $gB^+/B^+ \in \dot{u}U^-B^+/B^+$ if and only if $\Delta_{\lambda,u}(gB^+/B^+) = \{*\}$.

Our next goal is to show that $^J\mathcal{B}_{v,w,>0} \subset \dot{u}U^-B^+/B^+$ for $v^J \leqslant u^J \leqslant w$. Our strategy is as follows. By Corollary 7.6, $^J\mathcal{B}_{v,w,>0}$ is Zariski dense in $^J\mathring{\mathcal{B}}_{v,w}$. By §3.3 (a), $^J\mathring{\mathcal{B}}_{v,w} \cap \dot{u}U^-B^+/B^+ \neq \emptyset$. Thus $^J\mathcal{B}_{v,w,>0} \cap \dot{u}U^-B^+/B^+ \neq \emptyset$. In other words, $*$ is contained in $\Delta_{\lambda,u}(gB^+/B^+)$. We shall then construct an admissible map $\alpha : \mathbb{R}^n_{>0} \to \mathbb{R}_{\geqslant 0}$ so that the following diagram commutes

$$
\begin{array}{ccc}
\mathbb{R}^{^J\ell(w)-^J\ell(v)}_{>0} & \xrightarrow{\ \ \beta\ \ } & ^J\mathcal{B}_{v,w,>0} \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \Delta_{\lambda,u}} \\
\mathbb{R}_{\geqslant 0} & \longrightarrow & \{0, *\}.
\end{array}
$$

By §8.2.1 (a), $\Delta_{\lambda,u} : {}^J\mathcal{B}_{v,w,>0} \to \{0, *\}$ is constant. So $\Delta_{\lambda,u}(gB^+/B^+) = \{*\}$ and $^J\mathcal{B}_{v,w,>0} \subset \dot{u}U^-B^+/B^+$.

8.3 Some admissible functions

In this subsection, we consider some admissible functions arising from the group G.

Lemma 8.3 Let $v \leqslant w$. Define $f_{v,+} : U^-_{w,>0} \to U^-_{v,>0}$ by $c_{v,+}(gB^+/B^+) = f_{v,+}(g)B^+/B^+$ for any $g \in U^-_{w,>0}$. Then $f_{v,+}$ is admissible.

Proof Let $h \in U^-_{w,>0}$. By (6.2), we have $\dot{v}^{-1}h \in U^-gt$ for some $g \in U^+_{v^{-1},>0}$ and $t \in T_{>0}$. By (6.2) again, we have $\dot{v}g = g_1h_1t_1$ for some $g_1 \in U^+ \cap \dot{v}U^-\dot{v}^{-1}$, $h_1 \in U^-_{v,>0}$ and $t_1 \in T_{>0}$. Hence $ht^{-1}t_1^{-1}h_1^{-1}g_1^{-1} = ht^{-1}g^{-1}\dot{v}^{-1} \in \dot{v}U^-\dot{v}^{-1}$. So $t = t_1^{-1}$ and $g_1^{-1} = \sigma_{v,+}(ht^{-1}g^{-1}\dot{v}^{-1})$. Hence we conclude that

$$c_{v,+}(hB^+/B^+) = \sigma_{v,+}(ht^{-1}g^{-1}\dot{v}^{-1})\dot{v}B^+/B^+ = g_1^{-1}\dot{v}B^+/B^+$$
$$= h_1t_1g^{-1}B^+/B^+ = h_1B^+/B^+.$$

Thus $f_{v,+}(h) = h_1$. It is clear from the construction that $h \mapsto g \mapsto h_1$ is admissible. $\qquad\square$

Lemma 8.4 *Let $w_1, w_2 \in {}^J W$ and $v_1, v_2 \in W_J$ with $w_1 \leqslant w_2$ and $v_1 \leqslant v_2$. Fix a reduced expression $\mathbf{w_2}$ of w_2. Then ${}^J G_{v_2 w_1, \mathbf{w_2}, >0} \subset \dot{w}_1^{-1} \dot{v}_1^{-1} U^- B^+ / B^+$ and we have the following commutative diagram*

$$
\begin{array}{ccccc}
U^-_{v_2, >0} \times G_{w_1, w_2, <0} & \xrightarrow{\cong} & {}^J G_{v_2 w_1, \mathbf{w_2}, >0} & \xrightarrow{\cong} & {}^J \mathcal{B}_{v_2 w_1, w_2, >0} \\
{\scriptstyle (f_{v_1,+}, id)} \Big\downarrow & & & & \Big\downarrow {\scriptstyle {}^J c_{v_1 w_1, -}} \\
U^-_{v_1, >0} \times G_{w_1, w_2, <0} & \xrightarrow{\cong} & {}^J G_{v_1 w_1, \mathbf{w_2}, >0} & \xrightarrow{\cong} & {}^J \mathcal{B}_{v_1 w_1, w_2, >0}.
\end{array}
$$

Proof Let $g \in U^-_{v_2, >0}$ and $h \in G_{w_1, w_2, <0}$. By Lemma 8.3, $f_{v_1,+}(g) \in U^-_{v_1, >0}$. By definition, $f_{v_1,+}(g) \in (\dot{v}_1 U^-_J \dot{v}_1^{-1} \bigcap U^-_J) g$. Set $p = g \pi_J (h \dot{w}_1^{-1})^{-1} h B^+ / B^+ \in {}^J \mathcal{B}_{v_2 w_1, \mathbf{w_2}, >0}$ and $p' = c_{v_1,+}(g) \pi_J (h \dot{w}_1^{-1})^{-1} h B^+ / B^+ \in {}^J \mathcal{B}_{v_1 w_1, \mathbf{w_2}, >0}$. Then

$$
\begin{aligned}
p \in (\dot{v}_1 U^-_J \dot{v}_1^{-1} \bigcap U^-_J) p' & \subset (\dot{v}_1 U^-_J \dot{v}_1^{-1}) {}^J \mathring{\mathcal{B}}_{v_1 w_1, w_2} \\
& \subset (\dot{v}_1 U^-_J \dot{v}_1^{-1}) {}^J B^- \dot{v}_1 \dot{w}_1 B^+ / B^+ \subset (\dot{v}_1 U^-_J \dot{v}_1^{-1}) \dot{v}_1 \dot{w}_1 U^- B^+ / B^+ \\
& \subset \dot{v}_1 U^-_J \dot{w}_1 U^- B^+ / B^+ \subset \dot{v}_1 \dot{w}_1 U^- B^+ / B^+.
\end{aligned}
$$

Hence ${}^J c_{v_1 w_1, -}(p)$ is defined.

By definition, ${}^J c_{v_1 w_1, -}(p)$ is the unique element in

$$
{}^J \mathring{\mathcal{B}}_{v_1 w_1, w_2} \bigcap (\dot{v}_1 \dot{w}_1 U^- (\dot{v}_1 \dot{w}_1)^{-1} \bigcap {}^J B^+) p.
$$

Note that $\dot{v}_1 U^-_J \dot{v}_1^{-1} \subset \dot{v}_1 \dot{w}_1 U^- (\dot{v}_1 \dot{w}_1)^{-1}$. Thus

$$
\dot{v}_1 U^-_J \dot{v}_1^{-1} \bigcap U^-_J \subset \dot{v}_1 \dot{w}_1 U^- (\dot{v}_1 \dot{w}_1)^{-1} \bigcap {}^J B^+
$$

and ${}^J c_{v_1 w_1, -}(p) = p'$. $\qquad\square$

8.4 Further admissible functions related with $\tilde{G}$

In this section, we consider admissible functions arising from the group $\tilde{G}$ that are related with the morphism $\tilde{c}_r$ in §7.3.

For $\tilde{w} \in \tilde{W}$, we define

$$
\tilde{f}_{\tilde{w}} : \dot{\tilde{w}} \tilde{U}^- \tilde{B}^+ \to \tilde{G}, \quad \tilde{g} \dot{\tilde{w}} \tilde{b} \longmapsto {}^{I^\flat} \sigma_{\tilde{w}, -}(\tilde{g}) \text{ for } \tilde{g} \in \dot{\tilde{w}} \tilde{U}^- \dot{\tilde{w}}^{-1} \text{ and } \tilde{b} \in \tilde{B}^+.
$$

For $(s, r) \in Q_K$, we define the map

$$
f_{(s,r)} = \tilde{f}_{\dot{\tilde{v}}(s,r)} \circ f_r : \left(\dot{r} U^- P^+_K \right) \bigcap f_r^{-1} \left(\dot{\tilde{v}}(s, r) \tilde{U}^- \tilde{B}^+ \right) \to \tilde{G}.
$$

Here the map f_r is defined in §7.3. Note that $f_{(r,r)} = f_r$, since $f_r \left(\dot{r} U^- P^+_K \right) \subset \dot{\tilde{v}}(r, r) \tilde{U}^- \tilde{B}^+$.

Lemma 8.5 *Let $r \in W^K$ and $w \in W$ with $\ell(r^{-1}w) = \ell(w) - \ell(r)$. Then the map*

$$U^-_{u,>0} \times U^-_{w,>0} \longrightarrow \mathbb{R}, \qquad (g,h) \longmapsto \Delta_\lambda(\dot{t}^\sharp(\dot{u}^\flat)^{-1} g f_{(s,r)}(h))$$

is admissible for any $(u,t) \in Q_K$.

Remark 8.6 By Proposition 5.5 and Lemma 8.4, $U^-_{w,>0} \subset (\dot{r}U^- P^+_K) \cap f_r^{-1}(\dot{v}(s,r) \times \tilde{U}^- \tilde{B}^+)$. So the map $f_{(s,r)}$ is defined on $U^-_{w,>0}$.

Proof By the proof of Proposition 8.1 (in particular (8.4)), we have

$$f_r(h) \in h_3^\flat(\pi_K(h_4)h_4^{-1}b_4^{-1}\dot{r}^{-1})^\sharp \tilde{U}^+ = h_3^\flat(\pi_K(h_4)h_4^{-1}z_4\dot{r}^{-1})^\sharp \tilde{U}^+$$

for some $h_3 \in U^-_{r,>0}$, $h_4 \in U^-_{r^{-1}w,>0}$ and $b_4 \in B^+_{K,\geqslant 0}$. Here $z_4 \in T_{>0}$ with $b_4^{-1} \in z_4 U^+_K$. Moreover, by (8.1)–(8.4) in the proof of Proposition 8.1, all maps $h \mapsto h_3$, $h \mapsto h_4$, $h \mapsto z_4$ are admissible. By Lemma 8.4, $f_{(s,r)}(h) \in (c_{s,+}(h_3))^\flat(\pi_K(h_4) \times h_4^{-1}z_4\dot{r}^{-1})^\sharp \tilde{U}^+$. By Lemma 8.3, the map $h_3 \mapsto f_{s,+}(h_3)$ is admissible. By (6.2), we have $\dot{u}^{-1}(g\, f_{s,+}(h_3)) \in U^- c$ for some $c_1 \in B_{\geqslant 0}$. By §6.3 (a), we have $c^\flat(\pi_K(h_4)h_4^{-1}z_4\dot{r}^{-1})^\sharp \tilde{U}^+ = (\pi_K(h_5)h_5^{-1}z_5\dot{r}^{-1})^\sharp \tilde{U}^+$ for some $h_5 \in U^-_{u^{-1},>0}$ and $z_5 \in T_{>0}$.

Moreover, the maps $(g, f_{s,+}(h_3)) \mapsto c$, $(c, h_4) \mapsto h_5$ and $(c, h_4, z_4) \rightarrow z_5$ are all admissible. Since $t \in W^K$, we have $t^\sharp \in \tilde{W}^{I^\flat}$. Thus

$$\dot{t}^\sharp(\dot{u}^\flat)^{-1}\, g^\flat(c_{s,+}(h_3))^\flat(\pi_K(h_4)h_4^{-1}z_4\dot{r}^{-1})^\sharp \tilde{U}^+ \in \tilde{U}^- \dot{t}^\sharp(h_5^\sharp)^{-1}z_5^\sharp(\dot{r}^\sharp)^{-1}\tilde{U}^+.$$

So

$$\Delta_\lambda(\dot{t}^\sharp(\dot{u}^\flat)^{-1}g f_{(s,r)}(h)) = \Delta_\lambda(\dot{t}^\sharp(h_5^\sharp)^{-1}z_5^\sharp(\dot{r}^\sharp)^{-1}) = \Delta_\lambda(\iota(\dot{t}^\sharp(h_5^\sharp)^{-1}z_5^\sharp(\dot{r}^\sharp)^{-1}))$$
$$= \Delta_\lambda(\iota(\dot{t}^\sharp)\iota((h_5^\sharp)^{-1})z_5^\sharp\iota((\dot{r}^\sharp)^{-1})).$$

Note that $h_5 \mapsto \iota(h_5^{-1})$ is admissible, $\iota(\dot{t}) = \dot{x}^{-1}$ for $x = t^{-1} \in W$ and $\iota(\dot{r}^{-1}) = \dot{y}$ for $y = r^{-1} \in W$. By [12, Proposition 5.13], the map $(h_5, z_5) \mapsto \Delta_\lambda(\iota(\dot{t}^\sharp)\iota((h_5^\sharp)^{-1}) \times z_5^\sharp\iota((\dot{r}^\sharp)^{-1}))$ is admissible. The lemma follows now. $\qquad\square$
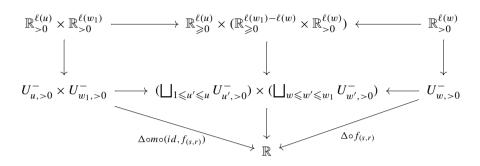
Lemma 8.7 *Let $(s,r) \in Q_K$ and $w \in W$ with $r \leqslant w$. Then the map*

$$U^-_{w,>0} \longrightarrow \mathbb{R}, \qquad h \longmapsto \Delta_\lambda(\dot{t}^\sharp(\dot{u}^\flat)^{-1} f_{(s,r)}(h))$$

is admissible for any $(u,t) \in Q_K$.

Proof We simply write Δ for $\Delta_\lambda(\dot{t}^\sharp(\dot{u}^\flat)^{-1} \cdot -)$. Let $w_1 \in W$ with $\ell(r^{-1}w_1) = \ell(w_1) - \ell(r)$ and $w \leqslant w_1$. We fix reduced expressions of u and w_1 (and thus the

positive subexpression for w). The statement is proved using the following commutative diagram.

$$
\begin{array}{ccccc}
\mathbb{R}_{>0}^{\ell(u)} \times \mathbb{R}_{>0}^{\ell(w_1)} & \longrightarrow & \mathbb{R}_{\geqslant 0}^{\ell(u)} \times (\mathbb{R}_{\geqslant 0}^{\ell(w_1)-\ell(w)} \times \mathbb{R}_{>0}^{\ell(w)}) & \longleftarrow & \mathbb{R}_{>0}^{\ell(w)} \\
\downarrow & & \downarrow & & \downarrow \\
U_{u,>0}^- \times U_{w_1,>0}^- & \longrightarrow & (\bigsqcup_{1 \leqslant u' \leqslant u} U_{u',>0}^-) \times (\bigsqcup_{w \leqslant w' \leqslant w_1} U_{w',>0}^-) & \longleftarrow & U_{w,>0}^-
\end{array}
$$

$$
\Delta \circ m \circ (id, f_{(s,r)}) \searrow \quad \downarrow \quad \swarrow \Delta \circ f_{(s,r)}
$$
$$
\mathbb{R}
$$

Now let us explain how the maps are defined. Let $\mathbf{u} = s_{i_1} \cdots s_{i_n}$ be the reduced expression we fixed in the beginning. The map $\mathbb{R}_{\geqslant 0}^{\ell(u)} \to U^-$ is defined by $(a_1, \ldots, a_n) \mapsto y_{i_1}(a_1) \cdots y_{i_n}(a_n)$. It is easy to see that the image is $\bigsqcup_{1 \leqslant u' \leqslant u} U_{u',>0}^-$. By a similar argument to [23, Proposition 4.2], $\bigsqcup_{1 \leqslant u' \leqslant u} U_{u',>0}^-$ is a closed subspace of G. Similarly, we have a continuous map $\mathbb{R}_{\geqslant 0}^{\ell(w_1)} \to \bigsqcup_{1 \leqslant w' \leqslant w_1} U_{w',>0}^-$. If we further require that all the coordinates associated to the positive subexpression of w are positive, then we obtain a continuous map $\mathbb{R}_{\geqslant 0}^{\ell(w_1)-\ell(w)} \times \mathbb{R}_{>0}^{\ell(w)} \to G$ and the image of this map is the locally closed subspace $\bigsqcup_{w \leqslant w' \leqslant w_1} U_{w',>0}^-$ of G.

Note that for any $w' \in W$ with $w \leqslant w'$, we have $r \leqslant w'$. Thus $U_{w',>0}^- \subset \dot{r}U^- P_K^+$. Therefore we have a continuous map

$$
\mathbb{R}_{\geqslant 0}^{\ell(u)} \times (\mathbb{R}_{\geqslant 0}^{\ell(w_1)-\ell(w)} \times \mathbb{R}_{>0}^{\ell(w)})
$$
$$
\longrightarrow (\bigsqcup_{1 \leqslant u' \leqslant u} U_{u',>0}^-) \times (\bigsqcup_{w \leqslant w' \leqslant w_1} U_{w',>0}^-) \xrightarrow{\Delta \circ m \circ (id, f_{(s,r)})} \mathbb{R}.
$$

By Lemma 8.5, the restriction to $\mathbb{R}_{>0}^{\ell(u)} \times \mathbb{R}_{>0}^{\ell(w_1)}$ is admissible. Hence by §8.2.1 (b), the map $\Delta \circ f_{(s,r)} : U_{w,>0}^- \to \mathbb{R}$ is admissible. $\qquad\square$

Lemma 8.8 *Let $r \in W^K$ and $v, w \in W$ with $v \leqslant r \leqslant w$. We fix a reduced expression $\mathbf{w}$ of w. Then the map*

$$
G_{\mathbf{v}_+,\mathbf{w},>0} \longrightarrow \mathbb{R}, \qquad h \longmapsto \Delta_\lambda (i^\sharp (\dot{u}^\flat)^{-1} f_{(r,r)}(h))
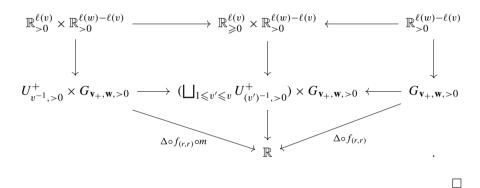$$

is admissible for any $(u, t) \in Q_K$.

Remark 8.9 By Proposition 5.5 that $G_{\mathbf{v}_+,\mathbf{w},>0} \subset \dot{r}U^- P_K^+$, so $f_{(r,r)}(G_{\mathbf{v}_+,\mathbf{w},>0})$ is defined. However, we have not proved $G_{\mathbf{v}_+,\mathbf{w},>0} \subset (\dot{r}U^- P_K^+) \cap f_r^{-1}(\dot{\tilde{v}}(s,r)\tilde{U}^- \tilde{B}^+)$ yet and thus we can not apply the general map $f_{(s,r)}$ to $G_{\mathbf{v}_+,\mathbf{w},>0}$. The general case will be handled after Lemma 8.8 and Corollary 8.10 are established.

Proof We simply write Δ for $\Delta_\lambda(\dot{t}^\sharp(\dot{u}^\flat)^{-1} \cdot -)$. By [2, Proposition 6.2], $U^+_{v^{-1},>0}\mathcal{B}_{v,w,>0} = \mathcal{B}_{1,w,>0}$ and the induced map $U^+_{v^{-1},>0} \times G_{\mathbf{v}_+,\mathbf{w},>0} \to U^-_{w,>0}$ is admissible. By Lemma 8.7, the map

$$U^+_{v^{-1},>0} \times G_{\mathbf{v}_+,\mathbf{w},>0} \longrightarrow U^-_{w,>0} \xrightarrow{\Delta} \mathbb{R}$$

is admissible. Moreover, for any $v' \leqslant v$, $U^+_{(v')^{-1},>0}\mathcal{B}_{v,w,>0} = \mathcal{B}_{(v')^{-1}\circ_l v,w,>0}$ and $(v')^{-1} \circ_l v \leqslant v \leqslant r$. Thus $U^+_{(v')^{-1},>0}G_{\mathbf{v}_+,\mathbf{w},>0} \subset \dot{r}U^-P^+_K$. Therefore the continuous map $\mathbb{R}^{\ell(u)}_{\geqslant 0} \times G_{\mathbf{v}_+,\mathbf{w},>0} \to (\bigsqcup_{1\leqslant v'\leqslant v} U^-_{(v')^{-1},>0}) \times G_{\mathbf{v}_+,\mathbf{w},>0} \xrightarrow{m} G$ has image inside $\dot{r}U^-P^+_K$. Thus we have a continuous map $\mathbb{R}^{\ell(u)}_{\geqslant 0} \times G_{\mathbf{v}_+,\mathbf{w},>0} \to \mathbb{R}$.

Now the statement follows from the following commutative diagram using the similar proof of Lemma 8.7:

$\square$

Corollary 8.10 *Let* $(r,r) \leqslant (s,r) \leqslant (v,w) \in Q_K$. *We have*

1. ${}^{I^\flat}\mathring{\tilde{\mathcal{B}}}_{\tilde{v}(r,r),\tilde{v}(v,w),>0} \subset \dot{\tilde{v}}(s,r)\tilde{U}^-\tilde{B}^+/\tilde{B}^+$;
2. $G_{\mathbf{v}_+,\mathbf{w},>0} \subset \left(\dot{r}U^-P^+_K\right) \bigcap f^{-1}_r\left(\dot{\tilde{v}}(s,r)\tilde{U}^-\tilde{B}^+\right)$.

Proof We have a commutative diagram

By Lemma 8.8, the map $G_{\mathbf{v}_+,\mathbf{w},>0} \to \mathbb{R}$, $h \mapsto \Delta_\lambda(\dot{r}^\sharp(\dot{s}^\flat)^{-1}f_{(r,r)}(h))$ is admissible. Hence the composition $G_{\mathbf{v}_+,\mathbf{w},>0} \to \{0,*\}$ is constant. By Proposition 8.1, all maps in the first row of the commutative diagram are surjective. Then the map $\Delta_{\lambda,\tilde{v}(s,r)} : {}^{I^\flat}\mathring{\tilde{\mathcal{B}}}_{\tilde{v}(r,r),\tilde{v}(v,w),>0} \to \{0,*\}$ is constant. Hence by §8.2 (a), we have ${}^{I^\flat}\mathring{\tilde{\mathcal{B}}}_{\tilde{v}(r,r),\tilde{v}(v,w),>0} \subset \dot{\tilde{v}}(s,r)\tilde{U}^-\tilde{B}^+/\tilde{B}^+$. Now the statements follow from §8.2.2. $\square$

The following lemma can be proved entirely similarly to Lemma 8.8, thanks to Corollary 8.10.

Lemma 8.11 *Let $(r, r) \preceq (s, r) \preceq (v, w) \in Q_K$. Fix a reduced expression $\mathbf{w}$ of w. Then the map*

$$G_{\mathbf{v}_+, \mathbf{w}, >0} \longrightarrow \mathbb{R}, \qquad h \longmapsto \Delta_\lambda(\dot{t}^\sharp(\dot{u}^\flat)^{-1} f_{(s,r)}(h))$$

is admissible for any $(u, t) \in Q_K$.

8.5 Proof of Proposition 5.9

A special case of part (1) has already been proved in Corollary 6.6. The general case follows by Theorem 3.5 and Corollary 3.7 once we have verified the assumptions in Theorem 3.5, that is, Proposition 5.9 part (2) & (3).

By Proposition 8.2, it suffices to prove part (2) & (3) for basic J-Richardson varieties. Namely,

(a) for $(s, r) \preceq (v, w)$ in Q_K, ${}^{I^\flat}\mathcal{B}_{\nu(s,r),\nu(v,w),>0}$ is a connected component of ${}^{I^\flat}\mathring{\mathcal{B}}_{\nu(s,r),\nu(v,w)}(\mathbb{R})$;

(b) for $(s, r) \preceq (u, t) \preceq (v, w)$ in Q_K, ${}^{I^\flat}\mathcal{B}_{\nu(s,r),\nu(v,w),>0} \subset \dot{\tilde{\nu}}(u,t)\tilde{U}^- \tilde{B}^+/\tilde{B}^+$.

We first show (a). The case where $s = r$ follows from Proposition 5.5 (3) and Proposition 8.1. The rest follows from Lemma 3.4 (3) thanks to Corollary 8.10. We now show part (b). We have a commutative diagram

$$
\begin{array}{ccccccc}
G_{\mathbf{v}_+, \mathbf{w}, >0} & \longrightarrow & \mathcal{P}_{K,(v,w),>0} & \longrightarrow & {}^{I^\flat}\mathring{\tilde{\mathcal{B}}}_{\tilde{\nu}(r,r),\tilde{\nu}(v,w),>0} & \longrightarrow & {}^{I^\flat}\mathring{\tilde{\mathcal{B}}}_{\tilde{\nu}(s,r),\tilde{\nu}(v,w),>0} \\
\downarrow & & & & & & \downarrow {\scriptstyle \Delta_{\lambda,\tilde{\nu}(u,t)}} \\
\mathbb{R} & & \longrightarrow & & & & \{0, *\}.
\end{array}
$$

By Lemma 8.11, the map $G_{\mathbf{v}_+, \mathbf{w}, >0} \to \mathbb{R}$, $h \mapsto \Delta_\lambda(\dot{t}^\sharp(\dot{u}^\flat)^{-1} f_{(s,r)}(h))$, is admissible. So the map $G_{\mathbf{v}_+, \mathbf{w}, >0} \to \{0, *\}$ is constant.

By (5.1) and Proposition 5.5 (1), the map $G_{\mathbf{v}_+, \mathbf{w}, >0} \to \mathcal{P}_{K,(v,w),>0}$ is surjective. By Proposition 8.1, the map $\mathcal{P}_{K,(v,w),>0} \to {}^{I^\flat}\mathring{\tilde{\mathcal{B}}}_{\tilde{\nu}(r,r),\tilde{\nu}(v,w),>0}$ is surjective. By part (a) and Lemma 3.4, the map ${}^{I^\flat}\mathring{\tilde{\mathcal{B}}}_{\tilde{\nu}(r,r),\tilde{\nu}(v,w),>0} \to {}^{I^\flat}\mathring{\tilde{\mathcal{B}}}_{\tilde{\nu}(s,r),\tilde{\nu}(v,w),>0}$ is surjective. Hence the map $\Delta_{\lambda,\tilde{\nu}(u,t)} : {}^{I^\flat}\mathring{\tilde{\mathcal{B}}}_{\tilde{\nu}(s,r),\tilde{\nu}(v,w),>0} \to \{0, *\}$ is constant. By §8.2.2, we have ${}^{I^\flat}\mathring{\tilde{\mathcal{B}}}_{\tilde{\nu}(s,r),\tilde{\nu}(v,w),>0} \subset \dot{\tilde{\nu}}(u,t)\tilde{U}^- \tilde{B}^+/\tilde{B}^+$.

8.6 Proof of Proposition 5.6

For any $r \in W^K$ with $(r, r) \preceq (v, w)$, we set

$$Y_r = \overline{\mathcal{P}_{K,(v,w),>0}} \bigcap \dot{r} U^- P_K^+/P_K^+.$$

By Proposition 5.5 (4), $\mathcal{P}_{K,(v',w'),>0} \subset \dot{r}U^{-}P_{K}^{+}/P_{K}^{+}$ for any $(r,r) \preceq (v',w') \preceq (v,w)$. Recall that the embedding $\tilde{c}_r$ in §7.3 is stratified. Then by Proposition 8.1, the embedding $\tilde{c}_r$ maps $\mathcal{P}_{K,(v',w'),>0}$ to $^{I^{\flat}}\tilde{\mathcal{B}}_{\tilde{v}(r,r),\tilde{v}(v',w'),>0}$.

Note that Theorem 5.10 has been fully established now. It follows that $^{I^{\flat}}\tilde{\mathcal{B}}_{\tilde{v}(r,r),\tilde{v}(v,w),>0}$ is a topological manifold with boundary

$$\partial\left(\overline{^{I^{\flat}}\tilde{\mathcal{B}}_{\tilde{v}(r,r),\tilde{v}(v,w),>0}}\right) = \bigsqcup_{\substack{\tilde{v}(r,r)^{I^{\flat}}\leqslant\tilde{w}_1{}^{I^{\flat}}\leqslant\tilde{w}_2{}^{I^{\flat}}\leqslant\tilde{v}(v,w), \\ (\tilde{w}_1,\tilde{w}_2)\neq(\tilde{v}(r,r),\tilde{v}(v,w))}} {}^{I^{\flat}}\mathring{\tilde{\mathcal{B}}}_{\tilde{w}_1,\tilde{w}_2,>0}.$$

Hence the boundary of the Hausdorff closure of $^{I^{\flat}}\tilde{\mathcal{B}}_{\tilde{v}(r,r),\tilde{v}(v,w),>0}$ in $^{I^{\flat}}\mathring{\tilde{\mathcal{B}}}^{\tilde{v}(r,r)}$ is

$$\bigsqcup_{\tilde{v}(r,r)^{I^{\flat}}\leqslant\tilde{w}_1{}^{I^{\flat}}<\tilde{v}(v,w)} {}^{I^{\flat}}\mathring{\tilde{\mathcal{B}}}_{\tilde{v}(r,r),\tilde{w}_1,>0}.$$

By [3, Proposition 4.2 (1) & (3)], the map $\tilde{v}$ gives a bijection

$$(\{(u,t)\in Q_K; (r,r)\preceq(u,t)\preceq(v,w)\},\preceq)$$

$$\longleftrightarrow \{\tilde{w}\in\tilde{W}; \tilde{v}(r,r)\,^{I^{\flat}}\leqslant\tilde{w}\,^{I^{\flat}}\leqslant\tilde{v}(v,w),\,^{I^{\flat}}\leqslant\}.$$

Hence Y_r is a topological manifold with boundary

$$\partial Y_r = \bigsqcup_{(v',w')\in Q_K; (r,r)\preceq(v',w')\lneq(v,w)} \mathcal{P}_{K,(v',w'),>0}.$$

By Proposition 5.5 (4), $\overline{\mathcal{P}_{K,(v,w),>0}} = \bigcup_{r\in W^K;(r,r)\preceq(v,w)} Y_r$ is an open covering. In particular, $\overline{\mathcal{P}_{K,(v,w),>0}}$ is a topological manifold with boundary

$$\bigcup_{r\in W^K;(r,r)\preceq(v,w)} Y_r = \bigsqcup_{(v',w')\in Q_K;(v',w')\preceq(v,w) \text{ and } (r,r)\preceq(v',w') \text{ for some } r\in W^K} \mathcal{P}_{K,(v',w'),>0}.$$

By definition, $(w',w')\preceq(v',w')$ for any $(v',w')\in Q_K$. In other words, there always exists $r\in W^K$ with $(r,r)\preceq(v',w')$. This finishes the proof.

References

1. Arkani-Hamed, N., Bourjaily, J., Cachazo, F., Goncharov, A., Postnikov, A., Trnka, J.: Grassmannian Geometry of Scattering Amplitudes. Cambridge University Press, Cambridge (2016)
2. Bao, H., He, X.: Flag manifolds over semifields. Algebra Number Theory **15**(8), 2037–2069 (2021)
3. Bao, H., He, X.: A Birkhoff-Bruhat atlas for partial flag varieties. Indag. Math. **32**, 1152–1173 (2021)
4. Billig, Y., Dyer, M.J.: Decompositions of Bruhat type for Kac-Moody groups. Nova J. Algebra Geom. **3**(1), 11–39 (1994)
5. Bjorner, A.: Posets, regular CW complexes and Bruhat order. Eur. J. Comb. **5**, 7–16 (1984)
6. Bochnak, J., Coste, M., Roy, M.-F.: Real algebraic geometry. In: Ergeb. Math. Grenzgeb. (3), vol. 36. Springer, Berlin (1998)
7. Dyer, M.J.: Hecke algebras and shellings of Bruhat intervals. II. Twisted Bruhat orders. In: Kazhdan-Lusztig Theory and Related Topics, Chicago, IL, 1989. Contemp. Math., vol. 139, pp. 141–165. Am. Math. Soc., Providence (1992)
8. Fomin, S., Shapiro, M.: Stratified spaces formed by totally positive varieties. Mich. Math. J. **48**, 253–270 (2000)
9. Fomin, S., Zelevinksy, A.: Cluster algebras. I. Foundations. J. Am. Math. Soc. **15**(2), 497–529 (2002)
10. Freedman, M.H.: The topology of four-dimensional manifolds. J. Differ. Geom. **17**, 357–453 (1982)
11. Galashin, P., Karp, S., Lam, T.: The totally nonnegative part of G/P is a ball. Adv. Math. **351**, 614–620 (2019)
12. Galashin, P., Karp, S., Lam, T.: Regularity theorem for totally nonnegative flag varieties. J. Am. Math. Soc. **35**(2), 513–579 (2021)
13. Galashin, P., Karp, S., Lam, T.: The totally nonnegative Grassmannian is a ball. Adv. Math. **397**, 108123 (2022)
14. He, X.: A subalgebra of 0-Hecke algebra. J. Algebra **322**, 4030–4039 (2009)
15. Hersh, P.: Regular cell complexes in total positivity. Invent. Math. **197**(1), 57–114 (2014)
16. Kac, V.G.: Infinite-Dimensional Lie Algebras, 3rd edn. Cambridge University Press, Cambridge (1990)
17. Kazhdan, D., Lusztig, G.: Representation of Coxeter groups and Hecke algebras. Invent. Math. **53**, 165–184 (1979)
18. Knutson, A., Woo, A., Yong, A.: Singularities of Richardson varieties. Math. Res. Lett. **20**(02), 391–400 (2013)
19. Knutson, A., Lam, T., Speyer, D.E.: Projections of Richardson varieties. J. Reine Angew. Math. **687**, 133–157 (2014)
20. Kumar, S.: Kac-Moody Groups, Their Flag Varieties and Representation Theory. Progress in Mathematics, vol. 204. Birkhäuser, Boston (2002)
21. Kumar, S.: Positivity in T-equivariant K-theory of flag varieties associated to Kac-Moody groups (with an appendix by M. Kashiwara). J. Eur. Math. Soc. **19**, 2469–2519 (2017)
22. Lam, T.: Totally nonnegative Grassmannian and Grassmann polytopes. In: Current Developments in Mathematics 2014, pp. 51–152. International Press, Somerville (2016)
23. Lusztig, G.: Total positivity in reductive groups. In: Lie Theory and Geometry. Progr. Math., vol. 123, pp. 531–568. Birkhäuser, Boston (1994)
24. Lusztig, G.: Total positivity in partial flag manifolds. Represent. Theory **2**, 70–78 (1998)
25. Lusztig, G.: Introduction to Quantum Groups, Modern Birkhäuser Classics, Reprint of Theition, 1993th edn. Birkhäuser, Boston (2010)
26. Lusztig, G.: Introduction to Total Positivity, Positivity in Lie Theory: Open Problems, pp. 133–146. de Gruyter, Berlin (2011)
27. Lusztig, G.: Total positivity in reductive groups, II. Bull. Inst. Math. Acad. Sin. (N.S.) **14**, 403–460 (2019)
28. Lusztig, G.: Positive structures in Lie theory. ICCM Not. **8**(1), 50–54 (2020)
29. Lusztig, G.: The flag manifold over the semifield $\mathbb{Z}$. Bull. Inst. Math. Acad. Sin. (N.S.) **15**(1), 63–92 (2020)
30. Lusztig, G.: Partial flag manifolds over a semifield. Represent. Theory **24**, 397–402 (2020)
31. Lusztig, G.: Total positivity in Springer fibres. Q. J. Math. **72**(1–2), 31–49 (2021)
32. Marquis, T.: An Introduction to Kac-Moody Groups over Fields. EMS Textbooks in Mathematics. Eur. Math. Soc., Zurich (2018)
33. Marsh, B.R., Rietsch, K.: Parametrizations of flag varieties. Represent. Theory **8**, 212–242 (2004)
34. Perelman, G.: The entropy formula for the Ricci flow and its geometric applications (2002). arXiv: math/0211159

35. Postnikov, A.: Total positivity, Grassmannians, and networks (2007). http://math.mit.edu/~apost/papers/tpgrass.pdf
36. Postnikov, A., Speyer, D., Williams, L.: Matching polytopes, toric geometry, and the totally nonnegative Grassmannian. J. Algebraic Comb. **30**(2), 173–191 (2009)
37. Rietsch, K.: Intersections of Bruhat cells in real flag varieties. Int. Math. Res. Not. **13**, 623–640 (1997)
38. Rietsch, K.: An algebraic cell decomposition of the nonnegative part of a flag variety. J. Algebra **213**(1), 144–154 (1999)
39. Rietsch, K.: Closure relations for totally nonnegative cells in G/P. Math. Res. Lett. **13**(5–6), 775–786 (2006)
40. Rietsch, K., Williams, L.: The totally nonnegative part of G/P is a CW complex. Transform. Groups **13**, 839–853 (2008)
41. Rietsch, K., Williams, L.: Discrete Morse theory for totally non-negative flag varieties. Adv. Math. **223**(6), 1855–1884 (2010)
42. Smale, S.: Generalized Poincaré's conjecture in dimensions greater than four. Ann. Math. **74**, 391–406 (1961)
43. Tits, J.: Uniqueness and presentation of Kac-Moody groups over fields. J. Algebra **105**(2), 542–573 (1987)
44. Wiliams, L.K.: Shelling totally nonnegative flag varieties. J. Reine Angew. Math. **609**, 1–21 (2007)