



Persistence of heterodimensional cycles

Dongchen Li¹ · Dmitry Turaev¹

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Abstract

A heterodimensional cycle is an invariant set of a dynamical system consisting of two hyperbolic periodic orbits with different dimensions of their unstable manifolds and a pair of orbits that connect them. For systems which are at least C^2 , we show that bifurcations of a coindex-1 heterodimensional cycle within a generic 2-parameter family create robust heterodimensional dynamics, i.e., a pair of non-trivial hyperbolic basic sets with different numbers of positive Lyapunov exponents, such that the unstable manifold of each of the sets intersects the stable manifold of the second set and these intersections persist for an open set of parameter values. We also give a solution to the so-called local stabilization problem of coindex-1 heterodimensional cycles in any regularity class $r = 2, \dots, \infty, \omega$. The results are based on the observation that arithmetic properties of moduli of topological conjugacy of systems with heterodimensional cycles determine the emergence of Bonatti-Díaz blenders.

Contents

1	Introduction	
1.1	Stabilization of heterodimensional cycles	
1.2	Heterodimensionality vs. equidimensionality	
1.3	Applications of Theorem B	
1.4	Blenders	
2	Robust heterodimensional dynamics in finite-parameter families	
2.1	Local maps near periodic orbits	
2.2	Transition maps and geometric non-degeneracy conditions	
2.3	Local partial linearization and the fourth non-degeneracy condition	
2.4	Finite-parameter unfoldings	
2.5	Three types of heterodimensional cycles in the saddle case	
2.6	Main results for the saddle case	
2.7	The case of nonreal central multipliers	
2.8	Local stabilization of heterodimensional cycles	

✉ D. Li

¹ Department of Mathematics, Imperial College London, London, UK

3	First-return maps in the saddle case
3.1	Local maps
3.2	Transition maps
3.3	First-return maps and cone field lemma
4	Blenders near type-I heterodimensional cycles. Proof of Theorem 1
4.1	Blenders with activating pairs
4.2	Finding cs-blenders when $ \alpha < 1$. Proof of Proposition 4.4
4.3	Finding cu-blenders when $ \alpha > 1$. Proof of Proposition 4.5
4.4	A criterion for the existence of blenders
5	Local stabilization of heterodimensional cycles in the saddle case
5.1	Unfolding type-I cycles. Proof of Theorem 2
5.2	Nonstabilizability of type-I cycles. Proof of Theorem 3
5.3	Unfolding a tied pair of type-I and -II cycles. Proof of Theorem 4
5.4	Creating a tied pair of type-I and -II cycles. Proofs of Theorem 5 and Corollary 2
5.5	Heterodimensional cycles with rational θ . Proof of Theorem 6
6	Saddle-focus and double-focus cases. Proof of Theorem 7
6.1	First-return maps in the saddle-focus case
6.2	Coexistence of mutually activating blenders
6.3	Local stabilization of heterodimensional cycles in the saddle-focus case
6.4	Double-focus case
	Appendix: Standard blenders
	Acknowledgements
	References

1 Introduction

In this paper, we solve the C^r -persistence problem for heterodimensional cycles of coindex 1 in any regularity class $r = 2, \dots, \infty, \omega$. The result gives a heterodimensional counterpart to the Newhouse theorem on the C^r -persistence of non-transverse equidimensional cycles (homoclinic tangencies) [49]. It implies the ubiquity of heterodimensional dynamics, which is, in our opinion, one of the most basic properties of non-hyperbolic multi-dimensional dynamical systems with chaotic behavior.

For uniformly hyperbolic systems, all orbits within the same chain-recurrent class have the same number of positive Lyapunov exponents and the same number of negative ones. However, chaotic dynamics are often not hyperbolic, and then one can expect that orbits with different numbers of positive Lyapunov exponents coexist and are, in a sense, inseparable from each other. The first example of such sort was given by Abraham and Smale in [1]. As an example of the non-density of hyperbolicity in the space of dynamical systems, they described an open region in the space of C^1 -diffeomorphisms where each diffeomorphism has hyperbolic periodic orbits with different dimensions of unstable manifolds within the same transitive set. More examples followed, see e.g. [21, 22, 26, 36, 45, 61, 62], with a general construction for robust heterodimensionality developed by Bonatti and Díaz in [13, 14].

We study both the discrete-time and continuous-time cases (our approach allows for a simultaneous consideration of both cases, as explained in Sect. 2). Unless otherwise stated, by a dynamical system, we always mean a diffeomorphism of a manifold of dimension 3 or higher,¹ or a flow on a manifold of dimension 4 or higher. We use

Definition 1.1 (Heterodimensional dynamics) Let a dynamical system f have two compact, transitive, and uniformly hyperbolic invariant sets Λ_1 and Λ_2 . Let $\text{ind}(\Lambda)$ (the index of a transitive hyperbolic set Λ) denote the dimension of the unstable manifold of any of its orbits.² We say that f has *heterodimensional dynamics involving Λ_1 and Λ_2* if

- $\text{ind}(\Lambda_1) \neq \text{ind}(\Lambda_2)$; and
- the unstable sets $W^u(\Lambda_1)$ and $W^u(\Lambda_2)$ intersect the stable sets $W^s(\Lambda_2)$ and, respectively, $W^s(\Lambda_1)$.

The difference $|\text{ind}(\Lambda_1) - \text{ind}(\Lambda_2)|$ is called the coindex of the heterodimensional dynamics.

Often, the term “heterodimensional cycle” is used for what we call the heterodimensional dynamics [14, 18]. We, however, reserve the term for the most basic case, where Λ_1 and Λ_2 are trivial.

Definition 1.2 (Heterodimensional cycles) A *heterodimensional cycle* is a closed invariant set consisting of four orbits: two hyperbolic periodic orbits L_1 and L_2 , with $\text{ind}(L_1) \neq \text{ind}(L_2)$, and two heteroclinic orbits, one from $W^s(L_1) \cap W^u(L_2)$ and the other from $W^u(L_1) \cap W^s(L_2)$.

See Fig. 1 for an illustration. Most of this paper is focused on bifurcations in this particular case. This does not lead to a loss in generality, because whenever we have heterodimensional dynamics, a heterodimensional cycle with periodic orbits can be created by a C^r -small perturbation (see discussion above Corollary A).

Due to the difference between the dimensions of the unstable manifolds, some heterodimensional intersections can be fragile. Indeed, consider a diffeomorphism of a d -dimensional manifold with a heterodimensional cycle involving periodic orbits L_1 and L_2 . Let $\dim W^u(L_1) = d_1$ and $\dim W^s(L_2) = d - d_2$, with $d_1 < d_2$. The space spanned by the tangents to $W^u(L_1)$ and $W^s(L_2)$ at any of their intersection points has dimension less than the dimension d of the full space. This means that every particular heteroclinic intersection of $W^u(L_1)$ and $W^s(L_2)$ is non-transverse and, by Kupka-Smale theorem, can be removed by an arbitrarily small perturbation.

However, the situation changes when we have heterodimensional dynamics involving two non-trivial hyperbolic sets Λ_1 and Λ_2 . For example, when $\text{ind}(\Lambda_2) > \text{ind}(\Lambda_1)$, it may happen that when a non-transverse intersection of $W^u(\Lambda_1)$ with

¹Heterodimensional cycles can also be defined for endomorphisms on two-dimensional manifolds, see e.g. [10].

²When a hyperbolic set is transitive, i.e., when some of its orbits is dense in it, all of its orbits have unstable manifolds of the same dimension.

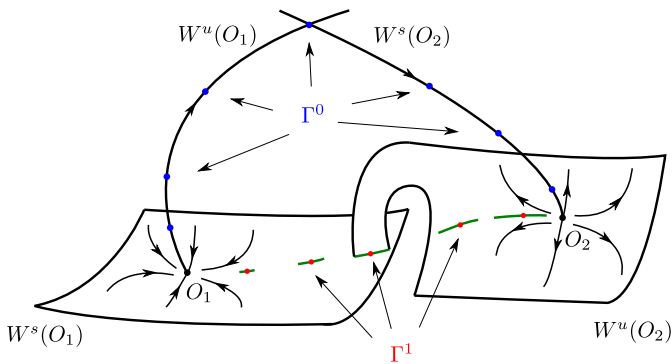


Fig. 1 A heterodimensional cycle involving two hyperbolic fixed points O_1 and O_2 (black dots) of a three-dimensional diffeomorphism. The cycle consists of the two fixed points, a fragile heteroclinic orbit Γ^0 (blue dots) in the non-transverse intersection of the one-dimensional invariant manifolds, and a robust heteroclinic orbit Γ^1 (red dots) in the transverse intersection (green curves) of the two-dimensional invariant manifolds. See Sect. 2 for details (Color figure online)

$W^s(\Lambda_2)$ at the points of some orbits is destroyed, a new one arises. In this case the heterodimensional dynamics are called *robust*.

Recall that basic (i.e., compact, transitive, and locally maximal) hyperbolic invariant sets continue uniquely when the dynamical system varies continuously (in the C^1 topology).

Definition 1.3 (Robust heterodimensional dynamics) We say that a system exhibits C^1 -robust heterodimensional dynamics if it has heterodimensional dynamics involving two hyperbolic basic sets Λ_1 and Λ_2 , where at least one of them is non-trivial, and there exists a C^1 -neighborhood \mathcal{U} of the original dynamical system such that every system from \mathcal{U} has heterodimensional dynamics involving the hyperbolic continuations of Λ_1 and Λ_2 .

This was the case in the original Abraham-Smale example and in the other examples we mentioned. Moreover, Bonatti and Díaz proved in [14] that any diffeomorphism with a heterodimensional cycle of coindex 1 can be arbitrarily well approximated, in the C^1 topology, by a diffeomorphism with C^1 -robust heterodimensional dynamics. The result gave a nice topological characterization of the set of systems with heterodimensional dynamics of coindex 1: this set is the C^1 -closure of its C^1 -interior. However, the perturbation techniques used in [14] (in an essential way) can only be valid in the C^1 topology. As a result, the C^1 -small perturbations proposed in [14] are *large* in C^r for any $r > 1$.

This leads to the question whether the Bonatti-Díaz result survives in higher regularity.³ In this paper, we close the question with a positive answer.

³That the C^1 bifurcation theory of heterodimensional cycles cannot be straightforwardly translated to the C^r -case is illustrated by the result of [5, 6] which shows that the dynamics created by C^2 -small or C^3 -small perturbations of partially-hyperbolic heterodimensional cycles of coindex 1 can be very much different from what can be achieved by C^1 -small perturbations. The reason is that the dynamics in the

Theorem A *Any dynamical system of class C^r ($r = 1, \dots, \infty, \omega$) having a heterodimensional cycle of coindex 1 can be C^r -approximated by a system which has C^1 -robust heterodimensional dynamics.*

Remark 1.4 A partial case of Theorem A can be derived from the result in [24] about renormalization near heterodimensional cycles of three-dimensional diffeomorphisms with two saddle-foci (see also [25]). For two-dimensional endomorphisms under the partial hyperbolicity condition, the result of Theorem A is Theorem B in [10].

It is well-known that every point in a transitive, uniformly hyperbolic set Λ is a limit point of hyperbolic periodic points (with the same dimension of the unstable manifolds equal to $\text{ind}(\Lambda)$), and the C^r -closure of invariant manifolds of these periodic points contains the stable and unstable sets of Λ , see e.g. [39, Theorem 6.4.15]. Hence, whenever we have heterodimensional dynamics involving two such sets, we can always obtain, by an arbitrarily small perturbation, a heterodimensional cycle associated with some hyperbolic periodic orbits. Thus, Theorem A implies

Corollary A *Any dynamical system of class C^r ($r = 1, \dots, \infty, \omega$) having heterodimensional dynamics of coindex 1 can be C^r -approximated by a system which has C^1 -robust heterodimensional dynamics.*

We stress that the results hold true, in particular, in the real-analytic case ($r = \omega$): given a real-analytic dynamical system on a real-analytic manifold we consider any complex neighborhood \mathcal{M} of this manifold such that the system admits a holomorphic extension on \mathcal{M} ; then the C^ω -topology in Theorem A and Corollary A is the topology of uniform convergence on compacta in \mathcal{M} .

In fact, we obtain Theorem A from its “constructive version”, Theorem B below. Namely, to obtain the result of Theorem A, one proceeds as follows. First, bring a given heterodimensional cycle into a general position (so that it satisfies the non-degeneracy conditions defined in Sects. 2.2 and 2.3) – this can be done by an arbitrarily small C^r -perturbation of any heterodimensional cycle. Then, we embed our system f into a finite-parameter family of perturbations f_ε with at least 2 parameters. We formulate certain explicit conditions in Sect. 2.4 which define an open and dense set in the space of C^r -families f_ε such that $f_0 = f$. An arbitrary family from this set is called a *proper unfolding* of f .

Theorem B *Let f be of class C^r ($r = 2, \dots, \infty, \omega$) and have a non-degenerate heterodimensional cycle of coindex 1, and let f_ε be a proper unfolding of f . Then, arbitrarily close to $\varepsilon = 0$ in the space of parameters there exist open regions where f_ε has C^1 -robust heterodimensional dynamics.*

central direction are restricted by the signs of the second derivative and the Schwarzian derivative of the one-dimensional transition map; this signature can be changed by C^1 -small perturbations but not by C^3 -small perturbations.

Remark 1.5 As the family f_ε is of class C^r , small ε correspond to C^r -small perturbations of f . Thus, this theorem implies Theorem A indeed. It should be stressed, however, that a result similar to Theorem B does not necessarily hold for one-parameter families, see Theorem 6.

Remark 1.6 The proof of Theorem B (and all remaining results in this paper) makes use of the existence of certain C^2 coordinate transformations (local partial linearization, see Sect. 2.3). We do not know, therefore, if these results hold for C^1 systems (except for Theorem C, whose C^1 -analogue is proven in [18]). Thus, when dealing with systems of class C^1 , in order to derive Theorem A and Corollary A from Theorem B, one can first perturb the system to make it C^2 and only then apply Theorem B.

The non-degeneracy/propriety conditions of Theorem B are explicit, and checking them requires only a finite amount of computations with a finite number of periodic and heteroclinic orbits. Thus, Theorem B provides a universal tool for detecting and demonstrating the robust codimension-1 heterodimensional dynamics in multi-dimensional systems. This theorem can be used for showing that robust heterodimensional dynamics exist in specific restrictive settings, for example for polynomial perturbations, perturbations which keep various sorts of symmetry, etc., and can be directly applied to dynamical systems coming from scientific applications, which usually have a form of finite-parameter families of differential equations or maps.

1.1 Stabilization of heterodimensional cycles

The hyperbolic basic sets involved in the robust heterodimensional dynamics described in Theorem B are not necessarily homoclinically related⁴ to the continuations of the periodic orbits from the original heterodimensional cycle. This means that even though parameter values corresponding to the existence of heterodimensional cycles are dense in the open regions of robust heterodimensional dynamics given by Theorem B, it may happen that these heterodimensional cycles do not contain the continuations of the periodic orbits of the original cycle.

To address this question, we employ the following adaptation of a definition from [15, 18]. Recall that a heterodimensional cycle in our terminology refers to an invariant set consisting of only four orbits (two periodic and two connecting ones), see Definition 1.2.

Definition 1.7 (Stabilization of heterodimensional cycles) Let a dynamical system f of class C^r have a heterodimensional cycle involving two hyperbolic periodic orbits L_1 and L_2 . The cycle is called C^r -stabilizable if arbitrarily close, in C^r , to f there exists a dynamical system g , which exhibits C^1 -robust heterodimensional dynamics involving non-trivial hyperbolic basic sets Λ_1 and Λ_2 that contain the continuations of L_1 and L_2 , respectively.

⁴Two hyperbolic basic sets of the same index are homoclinically related if their stable and unstable manifolds have transverse intersections.

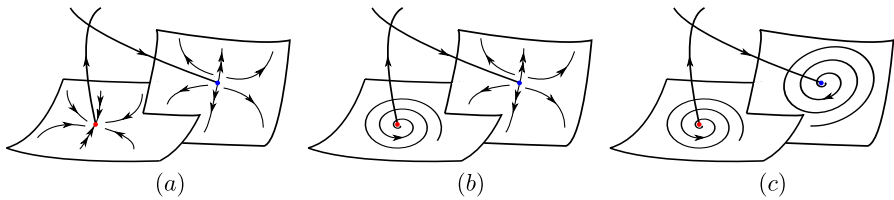


Fig. 2 Three cases of a heterodimensional cycle with two hyperbolic fixed points of a three-dimensional diffeomorphism. The central multipliers – corresponding to the weakest contraction rate at the left fixed point and the weakest expansion rate at the right fixed point – are both real in the saddle case (a), one real and one complex in the saddle-focus case (b), and both complex in the double-focus case (c)

In particular, since the stable and unstable manifolds of L_j are dense in the stable and, respectively, unstable sets of the basic set Λ_j , $j = 1, 2$, it follows that systems with heterodimensional cycles involving the continuations of L_1 and L_2 accumulate on g in the C^r -topology (and, more generally, on any C^r -system which is sufficiently close to g in C^1).

Definition 1.8 (Local stabilization of heterodimensional cycles) A heterodimensional cycle is *locally C^r -stabilizable*, if, given any neighborhood U of the cycle, the C^r -close to f system g of Definition 1.7 can be chosen such that the sets $\Lambda_{1,2}$ and the corresponding new heterodimensional cycles involving the continuations of L_1 and L_2 all lie in U .

Bonatti and Díaz constructed in [15] an example of diffeomorphisms with heterodimensional cycles which cannot be C^1 -stabilized (hence they cannot be C^r -stabilized). This work motivated the paper [18] by Bonatti, Díaz and Kiriki, showing that all heterodimensional cycles except for the so-called twisted ones (this class contains the cycles from the example in [15]) can be locally C^1 -stabilized. In the above definitions, we require a higher regularity of the stabilizing perturbations. We solve the question of local C^r -stabilization in Theorem C below.

We distinguish three main cases of heterodimensional cycles, as depicted in Fig. 2: saddle, saddle-focus, and double-focus, depending on whether the central multipliers are real or not (see Sect. 2.1 for the precise definition). We show that in the saddle-focus and double-focus cases, the robust heterodimensional dynamics given by Theorem B are always associated with hyperbolic basic sets which are homoclinically related to the continuations of the periodic orbits in the original heterodimensional cycle, see Theorem 7. However, in the saddle case, whether this homoclinic relation holds or not, this depends on whether the heterodimensional cycle is of type I or type II, as described in Sect. 2.5. Thus, we have

Theorem C Given any $r = 2, \dots, \infty, \omega$, a non-degenerate heterodimensional cycle of coindex 1 in the saddle-focus and double-focus cases is locally C^r -stabilizable. In the saddle case, the cycle is locally C^r -stabilizable if and only if it is not of type I.

Up to technical details, the type-I cycles correspond to twisted cycles from [18]. So, this theorem is the high regularity counterpart of the main result of [18]. Similarly

to [18], the type-I cycles become C^r -stabilizable (though not locally) when at least one of the periodic orbits in the cycle has a transverse homoclinic, see Corollary 3.

1.2 Heterodimensionality vs. equidimensionality

In the simplest setting, a *cycle* is an invariant set which consists of a cyclically ordered finite collection of periodic orbits and orbits that connect them (heteroclinic orbits), such that for each periodic orbit in the cycle there is exactly one heteroclinic orbit that tends to this periodic orbit in backward time and to the next periodic orbit in forward time. If all periodic orbits in the cycle have the same dimension of the unstable manifold and the same dimension of the stable manifold, the cycle is called *equidimensional*, and it is *heterodimensional* otherwise, see [52].

In this paper we consider the simplest case of heterodimensional cycles: coindex-1, and with 2 periodic orbits. The simplest case of an equidimensional cycle has just one periodic orbit and one homoclinic (the orbit of an intersection of the stable and unstable manifolds of the periodic orbit).

For a uniformly hyperbolic system, any cycle is equidimensional and the connecting orbits correspond to transverse intersections of the stable and unstable manifolds. Therefore, heterodimensional cycles and equidimensional cycles with non-transverse intersections (*homoclinic* or *heteroclinic tangencies*) are two obvious obstructions to the hyperbolicity.

A non-trivial fact is that both heterodimensional cycles and non-transverse equidimensional cycles can be robust. Thus, any hope that uniformly hyperbolic systems could be dense in the space of dynamical systems was destroyed with the above mentioned example by Abraham and Smale [1] of a C^1 -open region in the space of 4-dimensional diffeomorphisms where diffeomorphisms with heterodimensional cycles are dense and the example by Newhouse [49] of a C^2 -open region in the space of 2-dimensional diffeomorphisms where diffeomorphisms with non-transverse equidimensional cycles are dense.

Newhouse built a theory of *thickness* of hyperbolic sets (for two-dimensional C^2 -diffeomorphisms) and introduced a concept of a *wild* hyperbolic set: a non-trivial hyperbolic basic set whose stable and unstable sets have a tangency, for the system itself and for every C^2 -close system. Essentially, if a hyperbolic set is “thick enough” and its stable and unstable sets have a tangency, then the tangency is, typically, C^2 -robust, i.e., the hyperbolic set is wild. Based on this theory, Newhouse proved in [51] the C^2 -*persistence of homoclinic tangencies*: for any generic one-parameter family of C^r surface diffeomorphisms ($r \geq 2$) which unfolds a quadratic homoclinic tangency or an equidimensional cycle with a quadratic heteroclinic tangency, there exist open intervals of parameter values for which a wild hyperbolic set exists and parameter values corresponding to quadratic homoclinic tangencies are dense in these intervals (a generalization to multi-dimensional systems was done in [31, 55, 56]).

Although many examples of robust heterodimensional cycles appeared after the Abraham-Smale construction, a heterodimensional analogue of the Newhouse theory was missing until the discovery of *blender* by Bonatti and Díaz [13, 14]. A blender is a hyperbolic basic set whose projection along strong-stable or strong-unstable directions contains an open set, see Sect. 1.4 and Appendix for the precise definition. This

“openness in projection” property make the heterodimensional dynamics that involve a blender robust. Thus, blenders play the same role in the creation of robust heterodimensional dynamics as Newhouse thick horseshoes do for persistent homoclinic tangencies. In particular, finding a blender near a perturbed heterodimensional cycle is the essence of the Bonatti-Díaz C^1 -persistence result [14] and our C^r -persistence results (Theorems A and B).

Note that Newhouse “thick horseshoe” construction of the robust non-transverse intersections of stable and unstable manifolds is different from the Abraham-Smale construction and the later Bonatti-Díaz blender construction – in particular, the homoclinic tangencies in the Newhouse construction are C^2 -persistent, but not necessarily C^1 -persistent.⁵ Indeed, Ures [68] discovered that for C^1 surface diffeomorphisms, the thickness of a horseshoe loses continuous dependence on the system, which is a crucial condition for the Newhouse construction; and later Moreira [47] built a more general theory and proved the non-existence of persistent homoclinic tangencies for C^1 surface diffeomorphisms. However, there are several examples of C^1 -persistent homoclinic tangencies in higher dimensions, see [4, 15, 44, 63], which use blenders, or their variants, as supporting structures.

It is important to mention that there is a strong link between the homoclinic tangencies and the “heterodimensionality”. The simplest manifestation of this is that bifurcations of homoclinic tangencies can lead to the birth of coexisting sinks and saddles [27, 28]. Moreover, Newhouse showed in [50, 51] that a wild hyperbolic set of a generic area-contracting surface diffeomorphism is in the closure of the set of sinks, i.e., the periodic orbits with different indices (here – saddles and sinks) are generically inseparable from each other. Without the contraction of areas, one has coexisting sets of sinks, saddles, and sources [32], and, in the multi-dimensional case, coexisting saddles of different indices [34, 56]. In [44], we use the results of the present paper to give conditions under which the saddles of different indices that are born out of a homoclinic tangency get involved into the C^1 -robust heterodimensional dynamics, as in Definition 1.3.

1.3 Applications of Theorem B

A commonly shared conjecture is that any C^r -diffeomorphism is either uniformly hyperbolic (i.e., every chain-recurrent class is uniformly hyperbolic) or it is arbitrarily close in C^r to a diffeomorphism with wild hyperbolic sets (persistent homoclinic tangencies) or with robust heterodimensional dynamics, or both, see [12, 54] (for flows, one should also add a possibility of robust “Lorenz-like” dynamics [3, 38, 46]). Irrespective of whether this conjecture is true or not, it is an empirical fact that homoclinic tangencies easily appear in many non-hyperbolic systems; we expect the same should be true for heterodimensional cycles.

In support of such claim, we have shown in a series of papers [40, 42–44] that heterodimensional cycles emerge due to several types of homoclinic bifurcations. In fact, in the spirit of [37, 64, 65], one can conjecture that coindex-1 heterodimensional

⁵We have also learnt that Crovisier and Gourmelon obtained results on the $C^{1+\varepsilon}$ -persistence of homoclinic tangencies, in a work under preparation.

cycles can appear, with very few exceptions, in any homoclinic/heteroclinic bifurcation whose effective dimension allows it, i.e., when the dynamics of the map under consideration are not reduced to a two-dimensional invariant manifold and the map is not sectionally-dissipative (i.e., not area-contracting).⁶

We believe this conjecture is true, so Theorem B allows for establishing the presence of robust heterodimensional dynamics whenever a non-hyperbolic chaotic behavior with more than one positive Lyapunov exponent (the “hyperchaos” in the terminology of [57]) is observed. In particular, it was also shown in [40, 42, 43] that the coindex-1 heterodimensional cycles can be a part of a pseudohyperbolic chain-transitive attractor which appears in systems with Shilnikov loops [35, 59, 66] or after a periodic perturbation of the Lorenz attractor [67]. It, thus, follows from Theorems A and B that the attractor in such systems remains heterodimensional for an open set of parameter values.

An important feature of robust heterodimensional dynamics is the robust presence of orbits with a zero Lyapunov exponent. In particular, the result of [23] implies, for parametric families described by our Theorem B, the existence of open regions of parameter values where a generic system has an ergodic invariant measure with at least one zero Lyapunov exponent, i.e., the dynamics for such parameter values are manifestly non-hyperbolic.

For a dense set of parameter values from such regions the system has a non-hyperbolic periodic orbit. Bifurcations of such periodic orbits depend on the coefficients of the nonlinear terms of the Taylor expansion of the first-return map restricted to a center manifold. The degeneracy in the nonlinear terms increases complexity of the bifurcations. It follows from [5, 6] that once the so-called “sign conditions” are imposed on a heterodimensional cycle, the regions of robust heterodimensional dynamics given by Theorem A contain a C^r -dense set of systems having infinitely degenerate (flat) non-hyperbolic periodic orbits. This fact also leads to the C^∞ -genericity (for systems from these regions) of a superexponential growth of the number of periodic orbits and the so-called C^r -universal dynamics, see [5, 6].

1.4 Blenders

As we mentioned, the main object responsible for the robustness of heterodimensional dynamics is a particular type of a hyperbolic set, a blender, introduced by Bonatti and Díaz in [13]. It can be defined in many ways [7, 11, 16, 17, 19, 48]. For instance, the “operational definition” as in [17, Definition 6.11] can be rephrased as follows. Let f be a dynamical system on a smooth manifold \mathcal{M} with $\dim(\mathcal{M}) \geq 3$ if f is a diffeomorphism, or with $\dim(\mathcal{M}) \geq 4$ if f is a flow.

Definition 1.9 (Blenders) A basic hyperbolic set Λ of f is called a center-stable (cs) blender if there exists a C^1 -open set \mathcal{D}^{ss} of d^{ss} -dimensional discs (embedded copies of $\mathbb{R}^{d^{ss}}$) with d^{ss} strictly less than the dimension of the stable manifolds of the orbits

⁶To have a heterodimensional cycle, we need saddles with different dimensions of unstable manifolds, and these conditions are obviously necessary. The conjecture is that they should also be sufficient for the birth of a heterodimensional cycle in most situations.

of Λ), such that for every system g which is C^1 -close to f , for the hyperbolic continuation Λ_g of the basic set Λ , the set $W^u(\Lambda_g)$ intersects every element from \mathcal{D}^{ss} ; a center-unstable (cu) blender is a cs-blender for the dynamical system obtained from f by the time-reversal.

It is immediate from this definition that the existence of heterodimensional dynamics involving a blender is, essentially, a reformulation of the existence of the robust heterodimensional dynamics. Namely, for a system having two hyperbolic sets $\Lambda_{1,2}$ where $ind(\Lambda_2) > ind(\Lambda_1)$ and Λ_1 is a cs-blender, if there exists a transverse intersection of $W^u(\Lambda_2)$ with $W^s(\Lambda_1)$, and $W^s(\Lambda_2)$ contains a disc from \mathcal{D}^{ss} defined as in the previous section (so there is an intersection of $W^u(\Lambda_1)$ with this piece of $W^s(\Lambda_2)$), then the system exhibits C^1 -robust heterodimensional dynamics: the intersection of $W^u(\Lambda_2)$ with $W^s(\Lambda_1)$ survives small perturbations of the system because of the transversality, and the non-transverse intersection of $W^u(\Lambda_1)$ with $W^s(\Lambda_2)$ survives simply by the definition of the blender, as $W^s(\Lambda_2)$ varies continuously as the system varies.

It should be noted that the blenders obtained in this paper have additional dynamical properties (partial hyperbolicity, the existence of a special Markov partition) which are not included in Definition 1.9. This additional structure is important for the actual construction of blenders and it is also crucial in applications, for example for proving the C^1 -persistence of a certain type of homoclinic tangencies [16, 44]. We describe such blenders in Definition A.1, and call them *standard*. A standard blender is a version of the *blender-horseshoe* defined in [16]. The main difference is that the latter has a Markov partition of exactly two elements, whereas we take the Markov partition with a large number of elements, like in [17, Sect. 6.2]. Since Definition 1.9 suffices for discussing the persistence problem of heterodimensional cycles, the main goal of this paper, we do not make a digress to define standard blenders here. Instead, we detail the construction in the Appendix, see Definition A.1. In Proposition A.4, we prove that the blenders we construct in this paper are indeed standard blenders.

The central result (Theorem D below) of the current paper is that we identify a class of heterodimensional cycles for which standard blenders exist in an arbitrarily small neighborhood of the cycle. In other words, any such cycle is *a limit of an infinite sequence of standard blenders*. We infer Theorem B from this result by showing that a proper unfolding of a non-degenerate heterodimensional cycle creates heterodimensional cycles of the “blender-producing” class described in Theorem D, thus proving creation of blenders by a generic perturbation of an arbitrary heterodimensional cycle of coindex 1.

We always enumerate the periodic orbits $L_{1,2}$ in the heterodimensional cycle such that $ind(L_1) < ind(L_2)$, so the intersection $W^u(L_1)$ with $W^s(L_2)$ is fragile, whereas the intersection $W^u(L_2) \cap W^s(L_1)$ is transverse. By a multiplier of a periodic orbit, we mean an eigenvalue of the derivative matrix of the return map at the corresponding fixed point, see Sect. 2.1. The central multipliers $\lambda_{1,1}$ and $\gamma_{2,1}$ are the nearest to the unit circle among those multipliers of L_1 whose absolute value is smaller than 1 and, respectively, the nearest to the unit circle among those multipliers of L_2 whose absolute value is greater than 1. Recall that we distinguish three cases of non-degenerate heterodimensional cycles of coindex 1: saddle, saddle-focus, and double-focus, depending on whether $\lambda_{1,1}$ and $\gamma_{2,1}$ are real or not (in the saddle-focus case, we assume

that $\lambda_{1,1}$ is complex and $\gamma_{1,1}$ is real, with no loss of generality). In the saddle case (both $\lambda_{1,1}$ and $\gamma_{2,1}$ are real) we also have cycles of type I and type II (see Sect. 2.5). It is well-known [53, 69] that the values of

$$\theta = -\frac{\ln |\lambda_{1,1}|}{\ln |\gamma_{2,1}|}$$

and of

$$\omega_1 = \text{Arg}(\lambda_{1,1}), \quad \omega_2 = \text{Arg}(\gamma_{2,1})$$

(when $\lambda_{1,1}$ or $\gamma_{2,1}$ are complex) are *moduli*, i.e., invariants of topological equivalence, for systems with heterodimensional cycle.

Theorem D *Let f be of class C^r ($r = 2, \dots, \infty, \omega$) and have a non-degenerate heterodimensional cycle of coindex-1, and let U be any neighborhood of the cycle.*

- *In the saddle case, if the cycle is of type I and θ is irrational, then a standard blender exists in U .*
- *In the saddle-focus case when θ , $\frac{1}{2\pi}\omega_1$ and 1 are rationally independent, and in the double-focus case when θ , $\frac{1}{2\pi}\omega_1$, $\frac{1}{2\pi}\theta\omega_2$ and 1 are rationally independent, the system has simultaneously a standard cs-blender and a standard cu-blender in U . The blenders have different indices and their stable and unstable sets intersect C^1 -robustly.*

The result is the summary of Theorems 1 and 7 of Sect. 2. It should be stressed that Theorem D is a non-perturbative result; this is the major difference with other works, where heterodimensional cycles are unfolded to obtain blenders, see [10, 14, 24]. Note that by Theorem 6 of Sect. 2, no blenders exist in a sufficiently small neighborhood of the heterodimensional cycle in the saddle case when θ is rational; in the case of complex central multipliers a similar result can also be derived [41]. Therefore, we conclude that when the values of the moduli change, *new blenders are ceaselessly produced by the heterodimensional cycle*. One can see here a parallel to Gonchenko's theory of a homoclinic tangency, which relates dynamics near a homoclinic tangency – the structure of hyperbolic sets, the existence of infinitely many sinks – to arithmetic properties of moduli of topological and Ω -conjugacy [29, 30].

The technique we use to establish the blender is based on the approximation of the first-return map near a heterodimensional cycle by an iterated function system (IFS) which is composed of a collection of affine maps of an interval, see formula (4.23). This is similar to the approach used in many other works, see e.g. [7, 11, 16, 17, 48]. Note also that since the maps in our IFS are nearly affine, we can expect that the parablenders, introduced by Berger, can be implemented in our case too, cf. [8–10].

Throughout the rest of the paper, by heterodimensional cycles/dynamics we always mean those of coindex 1, and the blenders we find/construct are always standard blenders as defined in the Appendix (see Proposition A.4). In Sect. 2, we give precise definitions of the notions used in the paper, introduce non-degeneracy conditions for heterodimensional cycles, define the proper unfolding families, and give a complete formulation of the results. We start the proofs for the saddle case in Sect. 3, where a

computation for the first-return maps is carried out, and we prove the partial hyperbolicity of these maps. After that, we find blenders near type-I cycles in Sect. 4. Next, in Sect. 5, we investigate the bifurcations of heterodimensional cycles in the saddle case and construct robust heterodimensional dynamics using the previously obtained blenders. Finally, we deal with the saddle-focus and double-focus cases in Sect. 6.

2 Robust heterodimensional dynamics in finite-parameter families

In this section, we give a precise formulation of the results, which, in particular, imply Theorems B, C and D. We consider the discrete and continuous-time cases. For both cases we define local maps and transition maps near the heterodimensional cycle (see Sects. 2.1 and 2.2). After that the proofs are solely based on the analysis of these maps and hence hold for both cases simultaneously.

We start with a more precise description of a heterodimensional cycle. Let f be a C^r -diffeomorphism of a d -dimensional manifold or a C^r -flow of a $(d + 1)$ -dimensional manifold, where $d \geq 3$ and $r = 2, \dots, \infty, \omega$. Let f have a heterodimensional cycle Γ of coindex 1 associated with two hyperbolic periodic orbits L_1 and L_2 with $\dim W^u(L_2) = \dim W^u(L_1) + 1$. Along with the orbits L_1 and L_2 , the heterodimensional cycle Γ consists of two heteroclinic orbits $\Gamma^0 \in W^u(L_1) \cap W^s(L_2)$ and $\Gamma^1 \in W^u(L_2) \cap W^s(L_1)$. Due to the difference in the dimensions of $W^u(L_1)$ and $W^u(L_2)$, the intersection $W^u(L_1) \cap W^s(L_2)$ is non-transverse and can be removed by a small perturbation. We call the orbit Γ^0 a *fragile heteroclinic orbit*. On the other hand, the intersection $W^s(L_1) \cap W^u(L_2)$ at the points of the orbit Γ^1 is assumed to be transverse and it gives a smooth one-parameter family of heteroclinic orbits. We call them *robust heteroclinic orbits*. See Fig. 1 for an illustration.

The two orbits Γ^0 and Γ^1 will be required to satisfy certain non-degeneracy conditions, introduced in Sects. 2.2 and 2.3. Our goal is to show how C^1 -robust heterodimensional dynamics emerge in a small neighborhood of the cycle Γ after C^r -small perturbations. The mechanisms for that depend on the type of the heterodimensional cycle, as described in detail below.

2.1 Local maps near periodic orbits

In the discrete-time case, f is a diffeomorphism. Let O_1 and O_2 be some points of the orbits L_1 and L_2 . We take a small neighborhood U_{O_j} of the point O_j , $j = 1, 2$, and consider the first-return map F_j in this neighborhood: $F_j = f^{\tau_j}$ where τ_j is the period of O_j (see Fig. 3).

In the continuous-time case, the system f is a flow generated by some smooth vector field. In this case we take some points $O_1 \in L_1$ and $O_2 \in L_2$ and let U_{O_j} ($j = 1, 2$) be small d -dimensional (i.e., of codimension 1) cross-sections transverse at O_j to the vector field of f . Let F_j be the first-return map (the Poincaré map) to the cross-section U_{O_j} (see Fig. 3).

In both cases O_j is a hyperbolic fixed point of F_j : $F_j(O_j) = O_j$. The *multipliers* of O_j are defined as the eigenvalues of the derivative of F_j at O_j . The hyperbolicity means that no multipliers are equal to 1 in the absolute value; we assume that $d_j < d$

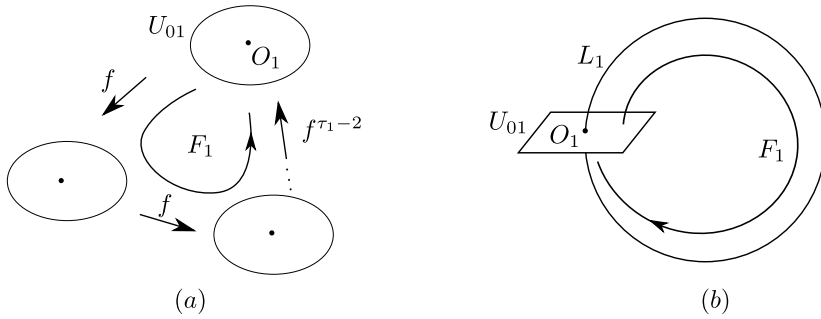


Fig. 3 The local maps near O_1 in the discrete-time case (a) and the continuous-time case (b)

multipliers of O_j lie outside the unit circle and $(d - d_j)$ multipliers lie inside. By our “coindex-1 assumption” $\dim W^u(L_2) = \dim W^u(L_1) + 1$, we have

$$d_2 = d_1 + 1.$$

We denote the multipliers of O_j , $j = 1, 2$, as $\lambda_{j,d-d_j}, \dots, \lambda_{j,1}, \gamma_{j,1}, \dots, \gamma_{j,d_j}$ and order them as follows:

$$|\lambda_{j,d-d_j}| \leq \dots \leq |\lambda_{j,2}| \leq |\lambda_{j,1}| < 1 < |\gamma_{j,1}| \leq |\gamma_{j,2}| \leq \dots \leq |\gamma_{j,d_j}|. \quad (2.1)$$

We call the largest in the absolute value multipliers inside the unit circle *center-stable* multipliers and those nearest to the unit circle from the outside are called the *center-unstable* multipliers. The rest of the multipliers λ and γ are called strong-stable and, respectively, strong-unstable.

It is important whether the center-stable multipliers of O_1 and center-unstable multipliers of O_2 are real or complex.⁷ Note that by adding an arbitrarily small perturbation, if necessary, we can always bring the multipliers into a general position. In our situation, this means that we can assume that O_1 has only one center-stable multiplier $\lambda_{1,1}$ which is real and simple, or a pair of simple complex conjugate center-stable multipliers $\lambda_{1,1} = \lambda_{1,2}^*$; we also can assume that O_2 has either only one center-unstable multiplier $\gamma_{2,1}$ which is real and simple, or a pair of simple complex conjugate center-unstable multipliers $\gamma_{2,1} = \gamma_{2,2}^*$.

Accordingly, we distinguish three main cases.

- Saddle case: here $\lambda_{1,1}$ and $\gamma_{2,1}$ are real and simple, i.e., we have $|\lambda_{1,2}| < |\lambda_{1,1}|$ and $|\gamma_{2,1}| < |\gamma_{2,2}|$.
- Saddle-focus case: here either

$$\lambda_{1,1} = \lambda_{1,2}^* = \lambda e^{i\omega}, \quad \omega \in (0, \pi), \quad \text{and} \quad \gamma_{2,1} \text{ is real,}$$

where $\lambda > |\lambda_{1,3}|$ and $|\gamma_{2,1}| < |\gamma_{2,2}|$, or

$$\gamma_{2,1} = \gamma_{2,2}^* = \gamma e^{i\omega}, \quad \omega \in (0, \pi), \quad \text{and} \quad \lambda_{1,1} \text{ is real,}$$

⁷It is well-known that the dynamics are quite different in these cases: for real multipliers, one has partially hyperbolic dynamics with one-dimensional center typically, while nonreal multipliers mean higher-dimensional dynamics, e.g. can lead to the emergence of homoclinic tangencies, see e.g. [26].

where $0 < \gamma < |\gamma_{2,3}|$ and $|\lambda_{1,2}| < |\lambda_{1,1}|$. Note that the second option is reduced to the first one by the inversion of time. Therefore, we assume below that in the saddle-focus case $\lambda_{1,1}$ is complex and $\gamma_{2,1}$ is real.

- Double-focus case: here

$$\lambda_{1,1} = \lambda_{1,2}^* = \lambda e^{i\omega_1}, \quad \omega_1 \in (0, \pi), \quad \text{and} \quad \gamma_{2,1} = \gamma_{2,2}^* = \gamma e^{i\omega_2}, \quad \omega_2 \in (0, \pi),$$

where $\lambda > |\lambda_{1,3}|, 0 < \gamma < |\gamma_{2,3}|$.

Below, we denote $\lambda_{1,1}$ and $\gamma_{2,1}$ by λ and γ if they are real. Let d_{cs} denote the number of the center-stable multipliers of O_1 and d_{cu} be the number of the center-unstable multipliers of O_2 . We have $d_{cs} = d_{cu} = 1$ in the saddle case, $d_{cs} = 2, d_{cu} = 1$ in the saddle-focus case, and $d_{cs} = d_{cu} = 2$ in the double-focus case.

Recall (see e.g. [34, 60]) that the first-return map F_1 near the point O_1 has a d_1 -dimensional local unstable manifold $W_{loc}^u(O_1)$ which is tangent at O_1 to the eigenspace corresponding to the multipliers $\gamma_{1,1}, \dots, \gamma_{1,d_1}$ and $(d - d_1)$ -dimensional local stable manifold $W_{loc}^s(O_1)$ which is tangent at O_1 to the eigenspace corresponding to the multipliers $\lambda_{1,1}, \dots, \lambda_{1,d-d_1}$. In $W_{loc}^s(O_1)$ there is a $(d - d_1 - d_{cs})$ -dimensional strong-stable C^r -smooth invariant manifold $W_{loc}^{ss}(O_1)$ which is tangent at O_1 to the eigenspace corresponding to the strong-stable multipliers $\{\lambda_{1,d-d_1}, \dots, \lambda_{1,d-d_1-d_{cs}}\}$. This manifold is a leaf of the strong-stable C^r -smooth foliation \mathcal{F}^{ss} of $W_{loc}^s(O_1)$. There also exists a $(d_1 + d_{cs})$ -dimensional extended-unstable invariant manifold $W_{loc}^{uE}(O_1)$ corresponding to the center-stable multipliers and the multipliers $\gamma_{1,1}, \dots, \gamma_{1,d_1}$. Such manifold is not unique but all of them contain $W_{loc}^u(O_1)$ and are tangent to each other at the points of $W_{loc}^u(O_1)$ (see Fig. 4).

Similarly, the first-return map F_2 near the point O_2 has a d_2 -dimensional local unstable manifold $W_{loc}^u(O_2)$ and $(d - d_2)$ -dimensional local stable manifold $W_{loc}^s(O_2)$. In $W_{loc}^u(O_2)$ there is a $(d_2 - d_{cu})$ -dimensional strong-unstable invariant manifold $W_{loc}^{uu}(O_2)$ which is tangent at O_1 to the eigenspace corresponding to the strong-unstable multipliers $\{\gamma_{2,d_{cu}+1}, \dots, \gamma_{2,d_2}\}$. This manifold is a leaf of the strong-unstable C^r -smooth foliation \mathcal{F}^{uu} of $W_{loc}^u(O_2)$. There also exists a $(d - d_2 + d_{cu})$ -dimensional extended-stable invariant manifold $W_{loc}^{sE}(O_2)$ corresponding to the center-unstable multipliers and the multipliers $\lambda_{2,1}, \dots, \lambda_{2,d-d_2}$. Any two such manifolds contain $W_{loc}^s(O_2)$ and are tangent to each other at the points of $W_{loc}^s(O_2)$.

2.2 Transition maps and geometric non-degeneracy conditions

For each of the heteroclinic orbits Γ^0 and Γ^1 , a transition map between neighborhoods of O_1 and O_2 is defined, as follows.

Consider, first, the case of discrete time, i.e., let f , be a diffeomorphism. Take four points $M_1^+ \in \Gamma^1 \cap W_{loc}^s(O_1), M_1^- \in \Gamma^0 \cap W_{loc}^u(O_1), M_2^+ \in \Gamma^0 \cap W_{loc}^s(O_2),$ and $M_2^- \in \Gamma^1 \cap W_{loc}^u(O_2)$. Note that M_2^- and M_1^+ belong to the same robust heteroclinic orbit Γ^1 and M_1^- and M_2^+ belong to the same fragile heteroclinic orbit Γ^0 . Thus, there exist positive integers n_1 and n_2 such that $f^{n_1}(M_1^-) = M_2^+$ and $f^{n_2}(M_2^-) = M_1^+$. We define the *transition maps* from a small neighborhood of M_1^- to a small neighborhood of M_2^+ and from a small neighborhood of M_2^- to a small neighborhood of M_1^+ as, respectively, $F_{12} = f^{n_1}$ and $F_{21} = f^{n_2}$ (see Fig. 4).

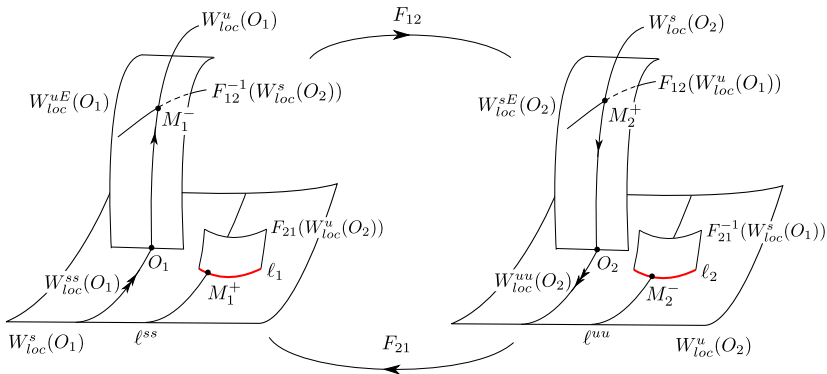


Fig. 4 A heterodimensional cycle of codimension 1 satisfying conditions C1 - C3, which consists of two periodic orbits containing O_1 and O_2 , a fragile heteroclinic orbit containing M_1^- and M_2^+ , and a robust heteroclinic orbit containing M_2^- and M_1^+ . Here ℓ^{ss} is a strong-stable leaf through M_1^+ , ℓ^{uu} is a strong-unstable leaf through M_2^- , $\ell_1 = W_{loc}^s(O_1) \cap F_{21}(W_{loc}^u(O_2))$ and $\ell_2 = F_{21}^{-1}\ell_1$

In the continuous-time case, when f is a flow, we take the points $M_1^+ \in \Gamma^1 \cap W_{loc}^s(O_1)$ and $M_1^- \in \Gamma^0 \cap W_{loc}^u(O_1)$ on the cross-section U_{O1} and the points $M_2^+ \in \Gamma^0 \cap W_{loc}^s(O_2)$ and $M_2^- \in \Gamma^1 \cap W_{loc}^u(O_2)$ on the cross-section U_{O2} . Then the transition map F_{12} is defined as the map by the orbits of the flow which start in the cross-section U_{O1} near M_1^- and hit the cross-section U_{O2} near the point M_2^+ , and the transition map F_{21} is defined as the map by the orbits of the flow which start in the cross-section U_{O2} near M_2^- and hit the cross-section U_{O1} near the point M_1^+ . By the definition, $F_{12}(M_1^-) = M_2^+$ and $F_{21}(M_2^-) = M_1^+$.

We can now precisely describe the non-degeneracy conditions which we impose on the heterodimensional cycle Γ .

- C1.** Simplicity of the fragile heteroclinic: $F_{12}^{-1}(W_{loc}^{sE}(O_2))$ intersects $W_{loc}^u(O_1)$ transversely at the point M_1^- , and $F_{12}(W_{loc}^{uE}(O_1))$ intersects $W_{loc}^s(O_2)$ transversely at M_2^+ ;
- C2.** Simplicity of the robust heteroclinic: the leaf of \mathcal{F}^{uu} at the point M_2^- is not tangent to $F_{21}^{-1}(W_{loc}^s(O_1))$ and the leaf of \mathcal{F}^{ss} at the point M_1^+ is not tangent to $F_{21}(W_{loc}^u(O_2))$; and
- C3.** $\Gamma^1 \cap (W^{ss}(O_1) \cup W^{uu}(O_2)) = \emptyset$, i.e., $M_1^+ \notin W^{ss}(O_1)$ and $M_2^- \notin W^{uu}(O_2)$.

Figure 4 provides an illustration of these conditions. Note that condition C1 does not depend on the choice of $W_{loc}^{uE}(O_1)$ and $W_{loc}^{sE}(O_2)$, as any two extended stable/unstable manifolds are tangent to each other at the points of the stable or, respectively, unstable manifold, see the discussion in the end of Sect. 2.1. Moreover, the corresponding requirement in C1 is automatically satisfied if O_2 has no non-center stable multipliers or O_1 has no non-center unstable multipliers, that is, in the case where O_2 has a pair of complex conjugate center-unstable multipliers and $\dim W^u(O_2) = d_2 = 2$, or O_1 has a pair of complex conjugate center-stable multipliers and $W^s(O_1) = d - d_1 = 2$. Similarly, the corresponding requirements of condition C2 hold automatically in these cases.

The manifolds involved in these conditions depend continuously (as C^1 -manifolds) on f in the C^r -topology. This implies that conditions C2 and C3 are C^r -open, and condition C1 is C^r -open in the class of systems with the heterodimensional cycle. It is also quite standard that one can always achieve the fulfillment of C1 and C2 by adding an arbitrarily C^r -small perturbation to f (in the smooth case one adds a local perturbation to f ; in the analytic case one uses the scheme described in [20, 33]). In the case where condition C3 is not fulfilled, we do not need to perturb the system: for the same system f we can always find, close to Γ^1 , another robust heteroclinic orbit which satisfies C3. To see this, note that condition C2 ensures that the line $\ell_1 = W_{loc}^s(O_1) \cap F_{21}(W_{loc}^u(O_2))$ (corresponding to robust heteroclinics, see Fig. 4) is not tangent to $W^{ss}(O_1) \cup F_{21}(W_{loc}^{uu}(O_2))$, so we can always shift the position of the point M_1^+ on this line and hence the position of $M_2^- = F_{21}^{-1}(M_1^+)$.

2.3 Local partial linearization and the fourth non-degeneracy condition

There is one last non-degeneracy condition, which is different for the saddle case and the other cases. To state it precisely, let us introduce C^r -coordinates $(x, y, z) \in \mathbb{R}^{d_{cs}} \times \mathbb{R}^{d_1} \times \mathbb{R}^{d-d_1-d_{cs}}$ in U_{O_1} such that the local stable and unstable manifolds get straightened near O_1 :

$$W_{loc}^s(O_1) = \{y = 0\}, \quad W_{loc}^u(O_1) = \{x = 0, z = 0\},$$

and the extended-unstable manifold $W_{loc}^{uE}(O_1)$ is tangent to $\{z = 0\}$ when $x = 0, z = 0$ (see Sect. 3). Moreover, the leaves of the foliation \mathcal{F}^{ss} are also straightened and are given by $\{x = const, y = 0\}$. In particular, we have

$$W_{loc}^{ss}(O_1) = \{x = 0, y = 0\}.$$

We also introduce C^r -coordinates $(u, v, w) \in \mathbb{R}^{d_{cu}} \times \mathbb{R}^{d-d_2} \times \mathbb{R}^{d_2-d_{cu}}$ in U_{O_2} such that the local stable and unstable manifolds are straightened near O_2 :

$$W_{loc}^s(O_2) = \{u = 0, w = 0\}, \quad W_{loc}^u(O_2) = \{v = 0\},$$

the extended-sable manifold $W_{loc}^{sE}(O_2)$ is tangent to $\{w = 0\}$ when $u = 0, v = 0$, and the leaves of the foliation \mathcal{F}^{uu} are also straightened and given by $\{u = const, v = 0\}$. We have

$$W_{loc}^{uu}(O_2) = \{u = 0, v = 0\}.$$

We restrict the choice of the coordinates by a further requirement (which can always be fulfilled, see e.g. [34]) that the first-return maps F_1 and F_2 act linearly on center-stable and, respectively, center-unstable coordinates. Namely, if we restrict these maps on $W_{loc}^s(O_1) = \{y = 0\}$ and, respectively, $W_{loc}^u(O_2) = \{v = 0\}$ and use the notation $F_1|_{W_{loc}^s(O_1)} : (x, z) \mapsto (\bar{x}, \bar{z})$ and $F_2|_{W_{loc}^u(O_2)} : (u, w) \mapsto (\bar{u}, \bar{w})$, then we

have

$$\begin{aligned}
 \text{Saddle case: } & \bar{x} = \lambda x \quad \text{and} \quad \bar{u} = \gamma u; \\
 \text{Saddle-focus case: } & \bar{x} = \lambda \begin{pmatrix} \cos \omega & \sin \omega \\ -\sin \omega & \cos \omega \end{pmatrix} x \quad \text{and} \quad \bar{u} = \gamma u; \\
 \text{Double-focus case: } & \bar{x} = \lambda \begin{pmatrix} \cos \omega_1 & \sin \omega_1 \\ -\sin \omega_1 & \cos \omega_1 \end{pmatrix} x \quad \text{and} \quad \bar{u} = \gamma \begin{pmatrix} \cos \omega_2 & \sin \omega_2 \\ -\sin \omega_2 & \cos \omega_2 \end{pmatrix} u.
 \end{aligned}
 \tag{2.2}$$

We denote, in these coordinates, $M_1^+ = (x^+, 0, z^+)$ and $M_2^- = (u^-, 0, w^-)$ (so condition C3 reads $x^+ \neq 0$ and $u^- \neq 0$).

Recall that F_{21} takes a point with coordinates (u, v, w) to a point with coordinates (x, y, z) , where x and u are the center-stable and center-unstable coordinates near the points O_1 and O_2 , respectively. By condition C2, the line ℓ_1 is not tangent to the foliation \mathcal{F}^{ss} and the line $\ell_2 = F_{21}^{-1}\ell_1$ is not tangent to the foliation \mathcal{F}^{uu} . In the saddle case this means that these curves are transverse to these foliations (see Fig. 4), so they are parametrized by the coordinates x (the line ℓ_1) and u (the line ℓ_2). Therefore, as $F_{21}|_{\ell_2}$ acts as a diffeomorphism $\ell_2 \rightarrow \ell_1$, we have

$$b = \left. \frac{\partial x}{\partial u} \right|_{M_2^-} \neq 0.
 \tag{2.3}$$

C4.1 (saddle case). The quantity $\alpha := bu^-/x^+$ satisfies

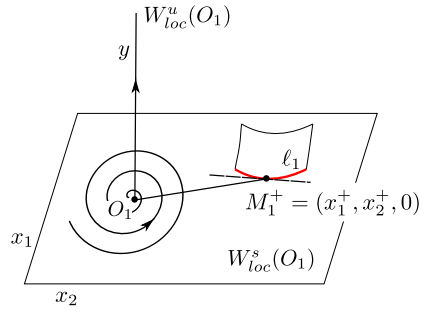
$$|\alpha| \neq 1.
 \tag{2.4}$$

Note that in the saddle case conditions C1 and C2 are equivalent (see [64]) to the requirement that the heteroclinic cycle Γ is a partially hyperbolic set with the 1-dimensional central direction field which includes the center-stable eigenvector at O_1 and the center-unstable eigenvector at O_2 . As we show, α determines the behavior in the central direction: the first-return maps near Γ are contracting in the central direction when $|\alpha| < 1$ and expanding when $|\alpha| > 1$ (see Lemma 3.1). Note that α is an invariant of smooth coordinate transformations which keep the foliations \mathcal{F}^{ss} and \mathcal{F}^{uu} locally straightened and keep the action of the local maps F_1 and F_2 in the central direction linear, as in (2.2). Indeed, any such transformation is linear in the central directions in a small neighborhoods U_{01} and U_{02} of the points O_1 and O_2 , i.e., the coordinates x and u are only multiplied to some constants c_x and c_u . As a result, the coefficient b is replaced by bc_x/c_u , and x^+ and u^- are replaced by $c_x x^+$ and $c_u u^-$, so α remains unchanged. Similarly, the invariant α does not depend on the choice of the points M_1^+ and M_2^- on the given heteroclinic orbit Γ^1 .

In the saddle-focus and double-focus cases, the partial hyperbolicity is not assumed, and no condition similar to C4.1 is needed. However, we need another condition:

C4.2 (saddle-focus and double-focus cases). When the center-stable multipliers $\lambda_{1,1}$ and $\lambda_{1,2}$ are complex and $x \in \mathbb{R}^2$, the x -vector component of the tangent to ℓ_1 at the point M_1^+ is not parallel to the vector x^+ (see Fig. 5). When the center-unstable

Fig. 5 An illustration of condition C4.2. The vector (x_1^+, x_2^+) is not parallel to the x -vector component (the dashed line) of the tangent to ℓ_1



multipliers $\gamma_{2,1}$ and $\gamma_{2,2}$ are complex and $u \in \mathbb{R}^2$, the u -vector component of the tangent to ℓ_2 at the point M_2^- is not parallel to the vector u^+ .

Like in condition C4.1, coordinate transformations that keep the action of the local maps F_1 and F_2 in x and u linear are also linear in x and u , respectively. This immediately implies that condition C4.2 is invariant with respect to the choice of the linearizing coordinates. It also does not depend on the choice of the points M_1^+ and M_2^- .

The heterodimensional cycles satisfying conditions C1-C4.1,2 will be further called *non-degenerate*.

2.4 Finite-parameter unfoldings

The perturbations we use to prove Theorem A are done within finite-parameter families f_ε which we assume to be of class C^r ($r = 2, \dots, \infty, \omega$) jointly with respect to coordinates and parameters ε .

Let $f_0 = f$; for any sufficiently small ε the hyperbolic points O_1 and O_2 exist and depend smoothly on ε . The corresponding multipliers also depend smoothly (C^{r-1}) on ε . We define

$$\theta(\varepsilon) = -\frac{\ln |\lambda|}{\ln |\gamma|}. \tag{2.5}$$

In the saddle-focus and double-focus cases, an important role is also played by the frequencies $\omega(\varepsilon)$ and, respectively, $\omega_{1,2}(\varepsilon)$. The values of θ as well as $\omega_{1,2}$ are moduli of topological conjugacy of diffeomorphisms with non-degenerate heterodimensional cycles (see [53, 69]).

The local stable and unstable manifolds of $O_{1,2}$, as well as their images by the transition maps F_{12} and F_{21} , also depend smoothly on ε . The fragile heteroclinic Γ^0 is not, in general, preserved when ε changes. To determine whether the fragile heteroclinic disappears or not, one introduces a *splitting parameter* μ , a continuous functional such that for any system g from a small C^r -neighborhood of f the absolute value of $\mu(g)$ equals to the distance between $W_{loc}^s(O_2)$ and $F_{12}(W_{loc}^u(O_1))$; the fragile heteroclinic persists for those g for which $\mu(g) = 0$. The codimension-1 manifold $\mu = 0$ separates the neighborhood of the system f into two connected components; we define μ such that it changes sign when going from one component to the other.

A one-parameter family f_ε is called a *generic one-parameter unfolding* of f if $\mu(f_\varepsilon)$ depends on ε smoothly and $\frac{d\mu}{d\varepsilon} \neq 0$. This means that we can make $\mu(f_\varepsilon) = \varepsilon$ by a smooth change of parameters.

We also need to consider families depending on two or more parameters, i.e., $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots)$. We call the family f_ε a *proper unfolding*, if $\frac{d\mu}{d\varepsilon} \neq 0$ (so the set $\mu(\varepsilon) = 0$ forms a smooth codimension-1 manifold \mathcal{H}_0 in the space of parameters ε) and, the following conditions hold for the subfamily corresponding to $\varepsilon \in \mathcal{H}_0$:

- in the saddle case, $\frac{d\theta}{d\varepsilon} \neq 0$, where the derivative is taken over $\varepsilon \in \mathcal{H}_0$ (this implies that we can make a smooth change of parameters in the family f_ε such that $\mu(\varepsilon) = \varepsilon_1$ and $\theta(\varepsilon) = \varepsilon_2$);
- in the saddle-focus case, the condition is that the functions $\theta(\varepsilon)$ and $\frac{1}{2\pi}\omega(\varepsilon)$ and 1 are linearly independent in a neighborhood of $\varepsilon = 0$ on \mathcal{H}_0 ;
- in the double-focus case, the condition is the linear independence of $\theta(\varepsilon)$, $\frac{1}{2\pi}\omega_1(\varepsilon)$, $\frac{1}{2\pi}\omega_2(\varepsilon)\theta(\varepsilon)$ and 1 in a neighborhood of $\varepsilon = 0$ on \mathcal{H}_0 .⁸

Note that the linear independence conditions for the saddle-focus and double-focus case are only used to ensure that the corresponding quantities can be made *rationally independent* by an arbitrarily small change of ε . However, we formulate the propriety conditions in this way in order to make the class of proper families open.

With the above definitions, the formulation of our main result, Theorem B as given in Sect. 1, is now complete. The proof goes differently in different cases: for the saddle case the theorem follows from the results described in Sects. 2.5 and 2.6, and in the saddle-focus and double focus case it follows from the results of Sect. 2.7.

2.5 Three types of heterodimensional cycles in the saddle case

In the saddle case, the proof of Theorem B is most involved: not because of technicalities, but because the dynamics emerging at perturbations of the non-degenerate heterodimensional cycles depend, in the saddle case, very essentially on the type of the cycle. According to that, we introduce three types of the heterodimensional cycles in the saddle case, as follows.

First, note (by counting dimensions) that in the saddle case condition C1 implies that the intersection of $F_{12}^{-1}(W_{loc}^{SE}(O_2))$ and $W_{loc}^{uE}(O_1)$ is a smooth curve, which we denote as ℓ^0 (see Fig. 6). At $\varepsilon = 0$, this curve goes through the point M_1^- and its image $F_{12}(\ell^0)$ goes through the point M_2^+ . The tangent space $\mathcal{T}_{M_1^-}\ell^0$ lies in $\mathcal{T}_{M_1^-}W^{uE}(O_1) = \{z = 0\}$ and, by C1, $\mathcal{T}_{M_1^-}\ell^0 \not\subset \mathcal{T}_{M_1^-}W_{loc}^u(O_1) = \{x = 0, z = 0\}$, which implies that $\mathcal{T}_{M_1^-}\ell^0$ has a non-zero projection to the x -axis. Thus, the curve ℓ^0 is parametrized by the coordinate x . Similarly, the curve $F_{12}(\ell^0)$ is parametrized

⁸Note that if this condition holds, it also holds for the system obtained after a time reversal. Indeed, this operation interchanges O_1 and O_2 , and hence the propriety condition for the resulting system becomes the linear independence of $\theta^{-1}(\varepsilon)$, $\frac{1}{2\pi}\omega_2(\varepsilon)$, $\frac{1}{2\pi}\omega_1(\varepsilon)\theta^{-1}(\varepsilon)$ and 1, which is equivalent to the linear independence of $\theta(\varepsilon)$, $\frac{1}{2\pi}\omega_1(\varepsilon)$, $\frac{1}{2\pi}\omega_2(\varepsilon)\theta(\varepsilon)$ and 1.

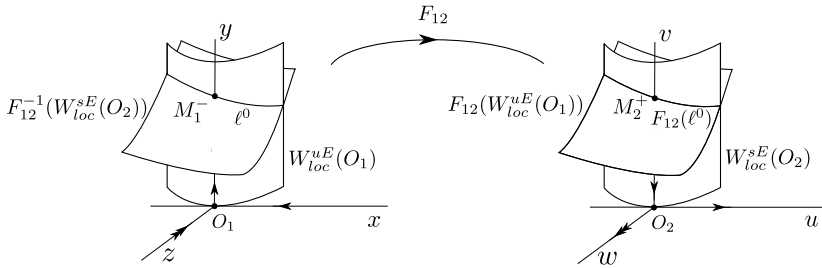


Fig. 6 Condition C1 in straightened coordinates

by coordinate u . The restriction of F_{12} to ℓ^0 is a diffeomorphism, so

$$a = \left. \frac{du}{dx} \right|_{M_1^-} \neq 0. \tag{2.6}$$

We say that a heterodimensional cycle Γ in the saddle case is of

- type I,⁹ if there exist points $M_1^+(x^+, 0, z^+) \in \Gamma^1 \cap U_{O1}$ and $M_2^-(u^-, 0, w^-) \in \Gamma^1 \cap U_{O2}$ such that $ax^+u^- > 0$;
- type II, if there exist points $M_1^+(x^+, 0, z^+) \in \Gamma^1 \cap U_{O1}$ and $M_2^-(u^-, 0, w^-) \in \Gamma^1 \cap U_{O2}$ such that $ax^+u^- < 0$;
- type III, if there exist points $M_1^+ \in \Gamma^1 \cap U_{O1}$ and $M_2^- \in \Gamma^1 \cap U_{O2}$ for which $ax^+u^- > 0$ and another pair of points $M_1^- \in \Gamma^1 \cap U_{O1}$ and $M_2^- \in \Gamma^1 \cap U_{O2}$ for which $ax^+u^- < 0$.

The cycle of type III is, by definition, a cycle which is simultaneously of type I and type II. Like in condition C4.1, one shows that the sign of ax^+u^- is independent of the choice of coordinates which keep the action of the local maps F_1 in the neighborhood U_{O1} of O_1 and F_2 in the neighborhood U_{O2} of O_2 linear in the central coordinates x and u . Thus, the above definition is coordinate-independent.

Notice that a is determined by a pair of points M_1^- and M_2^+ on the fragile heteroclinic Γ^0 , while x^+ and u^- are coordinates of points on the robust heteroclinic Γ^1 . By (2.2) (the saddle case), if the central multipliers λ and γ are positive, the local maps F_1 and F_2 multiply x^+ and u^- to positive factors, so the sign of ax^+u^- is independent of the choice of the points M_1^+ and M_2^- on Γ^1 in this case. Similarly, it does not depend on the choice of the points M_1^- and M_2^+ on Γ^0 . On the other hand, if at least one of the central multipliers is negative, the sign of x^+u^- changes when one replaces the pair (M_1^+, M_2^-) by the points $(F_1(M_1^+), M_2^-)$ or $(M_1^+, F_2^{-1}(M_2^-))$ on the same orbit. Thus, a non-degenerate heterodimensional cycle has either type I or type II, and not type III, if and only if both central multipliers are positive, and it has type III if and only if at least one of the central multipliers is negative.

⁹Type-I and type-II cycles correspond to twisted, and, respectively, non-twisted cycles in [18]. Strictly speaking, our definition is somewhat more general, as the notion of twisted and untwisted cycles was introduced in [18] only in the case when the local maps and transition maps satisfy certain additional restrictions.

2.6 Main results for the saddle case

The key observation in our proof of Theorem B in the saddle case and the fundamental reason behind the emergence of robust heterodimensional dynamics is given by the following result proven in Sect. 4.

Theorem 1 *In the saddle case, in any neighborhood of a non-degenerate heterodimensional cycle Γ of type I (including type III) for which the value of $\theta = -\frac{\ln|\lambda|}{\ln|\gamma|}$ is irrational, there exists a standard blender, center-stable with index d_1 if $|\alpha| < 1$ or center-unstable with index d_2 if $|\alpha| > 1$.*

The result holds true for systems f of class at least C^2 . The blender is not the one constructed in [14] by means of a C^1 -small but not C^2 -small perturbation of f . We do not perturb f , but give explicit conditions for the existence of the blender. Moreover, the (at least) C^2 regularity is important for the proof, and it is not clear whether Theorem 1 holds when f is only C^1 . Namely, it is a priori possible that there could exist C^1 systems for which a neighborhood of a heterodimensional cycle of type I does not contain a blender even when θ is irrational.

Next theorem tells us when the blender of Theorem 1 is *activated*, implying that it gets involved in robust heterodimensional dynamics. Recall that by definition the blender exists for any system C^1 -close to f .

Theorem 2 *Let Γ be a non-degenerate type-I cycle and let θ be irrational. Consider a sufficiently small C^1 -neighborhood \mathcal{V} of f such that the blender given by Theorem 1 persists for any system $g \in \mathcal{V}$. Let μ be the splitting functional.*

- *In the case $|\alpha| < 1$, there exist constants $C_1 < C_2$ such that for all sufficiently large $m \in \mathbb{N}$ any system $g \in \mathcal{V}$ which satisfies $\mu\gamma^m \in [C_1, C_2]$ has C^1 -robust heterodimensional dynamics involving the index- d_1 cs-blender Λ^{cs} of Theorem 1 and a non-trivial, index- d_2 hyperbolic basic set containing O_2 .*
- *In the case $|\alpha| > 1$, there exist constants $C_1 < C_2$ such that for all sufficiently large $k \in \mathbb{N}$ any system $g \in \mathcal{V}$ which satisfies $\mu\lambda^{-k} \in [C_1, C_2]$ has C^1 -robust heterodimensional dynamics involving the index- d_2 cu-blender Λ^{cu} of Theorem 1 and a non-trivial, index- d_1 hyperbolic basic set containing O_1 .*

The theorem is proven in Sect. 5 (see Proposition 5.2); Theorems 3 – 5 below are proven there as well. Note that the cases $|\alpha| < 1$ and $|\alpha| > 1$ are reduced to each other by the reversion of time and the interchange of the points O_1 and O_2 . Theorem 2 immediately implies Theorem B in the case of type-I cycles. Indeed, in a proper unfolding of f we can, by an arbitrarily small increment, make θ irrational while keeping $\mu = 0$, and then put μ to an interval corresponding to the C^1 -robust heterodimensional dynamics.

We also show (see Proposition 5.2) that *there exist intervals of μ for which the blender Λ^{cs} is homoclinically related to O_1 if $|\alpha| < 1$, and the blender Λ^{cu} is homoclinically related to O_2 if $|\alpha| > 1$* . Recall that a hyperbolic point is homoclinically related to a hyperbolic basic set of the same index if their stable and unstable manifolds intersect transversely. If the blender is homoclinically related to a saddle O_1 or

O_2 and is, simultaneously, involved in robust heterodimensional dynamics with the other saddle, this would give robust heterodimensional dynamics involving both these saddles. However, the following result shows that if the central multipliers λ and γ are both positive, this does not happen within a small neighborhood of the cycle Γ under consideration.

Let f have a non-degenerate heterodimensional cycle Γ of type I (we do not insist now that θ is irrational). Let $\lambda > 0$ and $\gamma > 0$, i.e., Γ is not type-III. Let U be a small neighborhood of Γ .

Theorem 3 *One can choose the sign of the splitting functional μ such that for every system g from a small C^r -neighborhood of f*

- *in the case $|\alpha| < 1$, for $\mu(g) > 0$, the set of all points whose orbits lie entirely in U consists of a hyperbolic set Λ of index d_1 (this set includes the cs-blender of Theorem 1 and the orbit L_1 of O_1), the orbit L_2 of the periodic point O_2 , and heteroclinic orbits corresponding to the transverse intersection of $W^u(L_2)$ with $W^s(\Lambda)$, so there are no heterodimensional dynamics in U ; for $\mu(g) \leq 0$, no orbit in $W^u(L_1)$ stays entirely in U , except for L_1 itself and, at $\mu(g) = 0$, the fragile heteroclinic Γ^0 , so L_1 cannot be a part of any heterodimensional cycle in U when $\mu(g) < 0$;*
- *in the case $|\alpha| > 1$, for $\mu(g) < 0$, the set of all points whose orbits lie entirely in U consists of a hyperbolic set Λ of index d_2 (this set includes the cu-blender of Theorem 1 and the orbit L_2 of O_2), the orbit L_1 of the periodic point O_1 , and heteroclinic orbits corresponding to the transverse intersection of $W^s(L_1)$ with $W^u(\Lambda)$, so there are no heterodimensional dynamics in U ; for $\mu(g) \geq 0$, no orbit in $W^s(L_2)$ stays entirely in U , except for L_2 itself and, at $\mu(g) = 0$, the fragile heteroclinic Γ^0 , so L_2 cannot be a part of any heterodimensional cycle in U when $\mu(g) > 0$.*

This situation changes if the type-I cycle is accompanied by a type-II cycle in the following sense.

Definition 2.1 (Tied cycles) We say that two non-degenerate heterodimensional cycles associated with O_1 and O_2 are *tied* if they share the same fragile heteroclinic, and the robust heteroclinic orbits Γ^1 and $\tilde{\Gamma}^1$ belonging to the corresponding cycles Γ and $\tilde{\Gamma}$ intersect the same leaf of the foliation \mathcal{F}^{ss} or the same leaf of the foliation \mathcal{F}^{uu} . Specifically, there exists a pair of points $M_1^+ = (x^+, 0, z^+) \in \Gamma^1 \cap W_{loc}^s(O_1)$ and $\tilde{M}_1^+ = (\tilde{x}^+, 0, \tilde{z}^+) \in \tilde{\Gamma}^1 \cap W_{loc}^s(O_1)$ such that $x^+ = \tilde{x}^+$ or a pair of points $M_2^- = (u^-, 0, w^-) \in \Gamma^1 \cap W_{loc}^u(O_2)$ and $\tilde{M}_2^- = (\tilde{u}^-, 0, \tilde{w}^-) \in \tilde{\Gamma}^1 \cap W_{loc}^u(O_2)$ such that $u^- = \tilde{u}^-$.

See Fig. 7 for an illustration of tied cycles. The existence of tied cycles is a C^r -open property in the set of systems for which the fragile heteroclinic is preserved. Indeed, if for a system f we have two points $M_1^+ \in \Gamma^1$ and $\tilde{M}_1^+ \in \tilde{\Gamma}^1$ lying in a common leaf l^{ss} of the foliation \mathcal{F}^{ss} , then there are curves ℓ_1 and $\tilde{\ell}_1$ containing these points, which correspond to the transverse intersection of $W^u(O_2)$ and $W_{loc}^s(O_1)$

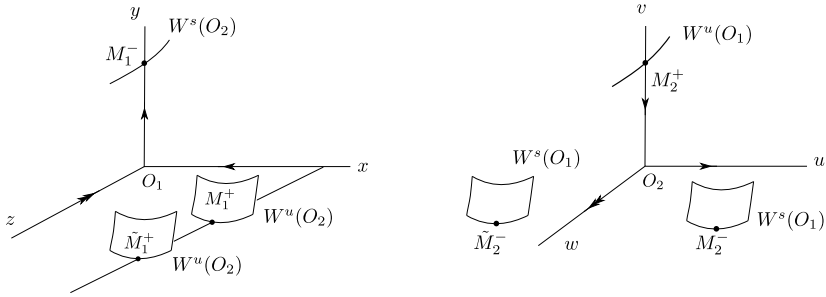


Fig. 7 A pair of tied cycles intersecting the same strong-stable leaf

and which, by condition C2, are transverse to the leaf l^{ss} of \mathcal{F}^{ss} . The transversality implies that a C^r -small perturbation of f does not destroy this double intersection in l^{ss} . The same is true if we have a double intersection with a leaf of \mathcal{F}^{uu} .

Theorem 4 *Let a C^r ($r \geq 2$) system f have a non-degenerate type-I cycle Γ tied with a non-degenerate type-II cycle $\tilde{\Gamma}$. Assume that θ is irrational. Then, for any generic one-parameter unfolding f_μ there exists a converging to $\mu = 0$ sequence of intervals I_j such that f_μ at $\mu \in I_j$ has C^1 -robust heterodimensional dynamics involving the blender given by Theorem 1 and a non-trivial hyperbolic basic set; of these two hyperbolic sets, the one with index d_1 is homoclinically related to $O_1(\mu)$ and the one with index d_2 is homoclinically related to $O_2(\mu)$.*

Observe that a type-III cycle is, by definition, a cycle of type I and II, and, obviously, it is tied with itself. Hence, applying the above theorem, we obtain

Corollary 1 *Let Γ be a non-degenerate cycle with real central multipliers λ and γ , at least one of which is negative. If $\theta = -\frac{\ln|\lambda|}{\ln|\gamma|}$ is irrational, then for any generic one-parameter unfolding f_μ there exist converging to zero intervals of μ corresponding to C^1 -robust heterodimensional dynamics involving non-trivial hyperbolic basic sets, one of which contains $O_1(\mu)$ and the other contains $O_2(\mu)$.*

Remark 2.2 Tied cycles also occur when O_1 or O_2 have a transverse homoclinic. For example, let us have a non-degenerate heterodimensional cycle Γ with a fragile heteroclinic Γ^0 and a robust heteroclinic Γ^1 . Assume the central multipliers are real, and let $M' \in W_{loc}^u(O_2)$ be a point of transverse intersection of the stable and unstable manifolds of O_2 . If we take a small piece of $W_{loc}^u(O_2)$ around M' , its forward images converge to the entire unstable manifold of O_2 . Therefore, some of them must intersect transversely the strong-stable leaf of the point $M_+^1 \in \Gamma^1 \cap W_{loc}^s(O_1)$ (as this leaf intersects $W^u(O_2)$ transversely at the point M_+^1 by condition C2), see Fig. 8. The orbit of the intersection point is a robust heteroclinic $\tilde{\Gamma}^1$, and the corresponding cycle $\tilde{\Gamma}$ is tied with Γ . By construction, the orbit $\tilde{\Gamma}^1$ has a point \tilde{M} in $W_{loc}^u(O_2)$ close to the homoclinic point M' . Therefore, its u -coordinate is close to the u -coordinate u' of M' . Therefore, if $u^-u' < 0$, i.e., the homoclinic point M' and the point M_2^- of Γ^1 lie in $W_{loc}^u(O_2)$ on opposite sides from $W_{loc}^{uu}(O_2)$, then the tied cycles Γ and $\tilde{\Gamma}$ have different types, and Theorem 4 is applicable.



Fig. 8 Creation of a new cycle that is tied with the original one by the strong-stable leaf ℓ^{ss} passing through M_1^+ (Color figure online)

Theorem B for type-II cycles is inferred from Theorem 4 by means of the following result.

Theorem 5 *Let f have a non-degenerate type-II cycle Γ with irrational θ . For any generic one-parameter unfolding f_μ there exists a sequence $\mu_j \rightarrow 0$ such that f_μ at $\mu = \mu_j$ has a pair of tied heterodimensional cycles $\Gamma_{j,I}$ and $\Gamma_{j,II}$ of type I and type II, which are associated*

- with $O_1(\mu)$ and an index- d_2 saddle $O'_2(\mu)$ which is homoclinically related to $O_2(\mu)$ if $|\alpha| < 1$; or
- with $O_2(\mu)$ and an index- d_1 saddle $O'_1(\mu)$ which is homoclinically related to $O_1(\mu)$ if $|\alpha| > 1$.

In order to apply Theorem 4 to the tied cycles $\Gamma_{j,I}$ and $\Gamma_{j,II}$ obtained in Theorem 5, we extend f_μ to a proper, at least two-parameter unfolding f_ε . In Sect. 5.4, we make the following observation: *when $r \geq 3$, the same family f_ε gives a proper unfolding for the cycles $\Gamma_{j,I}$ and $\Gamma_{j,II}$, see Lemma 5.4 and equation (5.31).* Since f_ε is proper for the cycles $\Gamma_{j,I}$ and $\Gamma_{j,II}$, one can always find the values of ε for which the value of θ for these cycles is irrational. Hence, applying Theorem 4, we obtain the result of Theorem B when f_ε is at least C^3 .

In the C^2 -case, the reduction of Theorem B (for type-II cycles) to Theorem 5 requires a revision of Theorem 4, as described in Remark 5.6. The difficulty is that we use, for every parameter value, the coordinates which linearize the action of the local maps F_1 and F_2 in the central direction. It is known that in the C^2 -case the linearizing coordinate transformation is, in general, not smooth with respect to parameters, so our technique does not allow to compute derivatives with respect to ε which enter the definition of a proper unfolding. Instead, we use continuity arguments to show in the C^2 -case that, still, for the tied cycles $\Gamma_{j,I}$ and $\Gamma_{j,II}$ the value of θ can be made irrational and the splitting parameter for these cycles can be pushed, by a small change of ε , inside the open regions described by Remark 5.6 – analogues of intervals I_j described by Theorem 4.

Altogether, we prove in Sect. 5.4 Theorem B for type-II cycles in the following form.

Corollary 2 *Let Γ be a non-degenerate type-II cycle, and let f_ε be a proper, at least two-parameter unfolding. Arbitrarily close to $\varepsilon = 0$ there exist open regions in the parameter space for which the corresponding system f_ε has C^1 -robust heterodimensional dynamics involving a standard blender and a non-trivial hyperbolic basic set;*

one of these sets is homoclinically related to O_1 and the other is homoclinically related to O_2 .

As we see, the emergence of heterodimensional dynamics depends strongly on the arithmetic properties of θ , the modulus of topological equivalence. The following result shows that, in the saddle case, we have a clear dichotomy: for irrational θ we have highly non-trivial dynamics and bifurcations in any neighborhood of the heterodimensional cycle Γ , and for rational θ the dynamics in a small neighborhood of Γ are quite simple, in general.

Theorem 6 *Let a C^r ($r \geq 2$) system f have a non-degenerate heterodimensional cycle Γ , and let the central multipliers be real, $|\lambda| < 1$ and $|\gamma| > 1$. Let $\theta = -\frac{\ln|\lambda|}{\ln|\gamma|}$ be rational, i.e., $|\gamma| = |\lambda|^{-\frac{p}{q}}$ for some coprime integers $p > 0, q > 0$. Suppose the following conditions are fulfilled:*

$$|ab| \neq |\gamma|^{\frac{s}{q}} \quad \text{for } s \in \mathbb{Z}, \tag{2.7}$$

$$\left| \frac{u^-}{ax^+} \right| \notin cl \left\{ |\gamma|^{\frac{s}{q}} \frac{1 - \lambda^l}{1 - \gamma^{-n}} \right\}_{s \in \mathbb{Z}, l \in \mathbb{N}, n \in \mathbb{N}}. \tag{2.8}$$

Let U be a sufficiently small neighborhood of Γ and let \mathcal{N} be the set of all orbits that lie entirely in U . Then, at $\mu = 0$, the set \mathcal{N} is the union of L_1, L_2, Γ^0 , and the orbits of transverse intersection of $W^u(L_2)$ with $W^s(L_1)$ near Γ^1 .

For any generic one-parameter unfolding f_μ , for any small $\mu \neq 0$, either

- \mathcal{N} is comprised by L_2 , an index- d_1 uniformly-hyperbolic compact set Λ_1 containing L_1 , and transverse heteroclinic connections between $W^u(L_2)$ and $W^s(\Lambda_1)$, while no heteroclinic connection between $W^u(\Lambda_1)$ and $W^s(L_2)$ exists, or
- \mathcal{N} is comprised by L_1 , an index- d_2 uniformly-hyperbolic compact set Λ_2 containing L_2 , and transverse heteroclinic connections between $W^u(\Lambda_2)$ and $W^s(L_1)$, while no heteroclinic connection between $W^u(L_1)$ and $W^s(\Lambda_2)$ exists.

The proof of this theorem is given in Sect. 5.5. Notice that, for fixed values of λ and γ , conditions (2.7) and (2.8) are fulfilled for all ab and ax^+/u^- except for a countable, nowhere dense set of values. Thus, the simplicity of dynamics at rational θ is indeed quite generic. It also follows from this theorem and Theorem 1 that whenever we have a heterodimensional cycle Γ of type I, if we change θ without destroying Γ , the blender that forms at irrational θ 's immediately departs from Γ , so that for each rational θ a sufficiently small ‘‘blender-free’’ neighborhood of Γ emerges.

2.7 The case of nonreal central multipliers

In the remaining saddle-focus and double-focus cases, we obtain Theorem B from

Theorem 7 *Let a C^r ($r \geq 2$) system f have a heterodimensional cycle Γ and let central multipliers be $\lambda_{1,1} = \lambda_{1,2}^* = \lambda e^{i\omega}$ and real $\gamma_{2,1} = \gamma$ (the saddle-focus case) or $\lambda_{1,1} = \lambda_{1,2}^* = \lambda e^{i\omega_1}$ and $\gamma_{2,1} = \gamma_{2,2}^* = \gamma e^{i\omega_2}$ (the double-focus case). Assume,*

in the saddle-focus case, that the numbers $\theta = -\frac{\ln \lambda}{\ln |\gamma|}$, $\frac{1}{2\pi}\omega$ and 1 are rationally independent. In the double-focus case, assume that θ , $\frac{1}{2\pi}\omega_1$, $\frac{1}{2\pi}\theta\omega_2$, and 1 are rationally independent. Then, in any neighborhood of Γ , the system f has C^1 -robust heterodimensional dynamics associated with a standard cs-blender and a standard cu-blender.

Moreover, in the double-focus case, the point O_1 is homoclinically related to the cs-blender and the point O_2 is homoclinically related to the cu-blender, and the same holds true for any C^r -close system g . In the saddle-focus case, the point O_2 is, for the system f and for any C^r -close system g , homoclinically related to the cu-blender, and there exist intervals I_j converging to $\mu = 0$ such that if $\mu(g) \in I_j$, then the point O_1 is homoclinically related to the cs-blender.

Remark 2.3 By the reversion of time, we deduce a similar result for the case where $\gamma_{2,1} = \gamma_{2,2}^* = \gamma e^{i\omega}$ and $\lambda_{1,1} = \lambda$ is real. Namely, we obtain that the point O_1 is homoclinically related to a cs-blender for the system f and for any C^r -close system g , and that there exist intervals I_j converging to $\mu = 0$ such that if $\mu(g) \in I_j$, then the point O_2 is homoclinically related to the cu-blender.

This theorem is proved in Sect. 6. In particular, the coexistence of a cs-blender and a cu-blender for the non-perturbed system f is obtained in Propositions 6.2 and 6.7. Since in a proper unfolding of f the rational independence conditions of Theorem 7 are achieved by arbitrarily small changes of parameters, the claim of Theorem B follows immediately.

2.8 Local stabilization of heterodimensional cycles

Corollary 1, Corollary 2, and Theorem 7 immediately imply that if the heterodimensional cycle is not type-I, then it is locally C^r -stabilized as claimed in Theorem C (in the C^1 case, this result is obtained in [18], but certain perturbations essential for their construction are large in the C^2 -topology). On the other hand, Theorem 3 shows that any type-I heterodimensional cycle cannot be locally C^r -stabilized, concluding Theorem C. A similar result on the impossibility of the local C^1 -stabilization for type-I cycles can be inferred from [15].

If we replace at least one of the periodic orbits in the heterodimensional cycle by a non-trivial, transitive, compact hyperbolic set Λ , then even if all periodic orbits in Λ have central multipliers real and positive, one can, in the generic situation, find a periodic orbit in Λ such that its homoclinic points accumulate to it from both sides in the central direction. Then, by Remark 2.2 and Theorem 4 we obtain

Corollary 3 *If at least one of the periodic orbits in a heterodimensional cycle belongs to a non-trivial, transitive, compact hyperbolic set, then the cycle is C^r -stabilizable for any $r = 2, \dots, \infty, \omega$*

3 First-return maps in the saddle case

Recall that we consider the behavior in a small neighborhood of the heterodimensional cycle Γ . We take periodic points O_1 and O_2 in Γ and consider, if f is a

discrete dynamical system, small neighborhoods U_{01} and U_{02} of O_1 and O_2 . If f is a flow, then U_{01} and U_{02} are small codimension-1 cross-section to the flow through the points O_1 and O_2 . In both cases, the local maps F_1 and F_2 act on U_{01} and U_{02} , which are defined by the orbits of the system f near the orbits L_1 and L_2 of the points O_1 and O_2 . We take a pair of points $M_1^- \in U_{01}$ and $M_2^+ \in U_{02}$ on the fragile heteroclinic Γ^0 and a pair of points $M_2^- \in U_{02}$ and $M_1^+ \in U_{01}$ on the robust heteroclinic Γ^1 . The orbits near the fragile heteroclinic define the transition map F_{12} from a small neighborhood of M_1^- in U_{01} to a small neighborhood of M_2^+ in U_{02} , and the orbits near the robust heteroclinic define the transition map F_{21} from a small neighborhood of M_2^- in U_{02} to a small neighborhood of M_1^+ in U_{01} .

Our immediate goal is to study the first-return maps near the heterodimensional cycle. Such map is a composition $T_{k,m} := F_{21} \circ F_2^m \circ F_{12} \circ F_1^k$, which takes points from a small neighborhood M_1^+ in U_{01} back to a vicinity of M_1^+ . Here, k and m are sufficiently large positive integers such that F_1^k takes points from a small neighborhood of M_1^+ in U_{01} to a small neighborhood of M_1^- in U_{01} and F_2^m takes points from a small neighborhood of M_2^+ in U_{02} to a small neighborhood of M_2^- in U_{02} .

First, we use results in [34] to obtain formulas for the iterations F_1^k and F_2^m . The formulas are written in the so-called ‘‘cross-form’’, see (3.5) and (3.7). Next, we use the transversality conditions C1 and C2 and write the transition maps F_{12} and F_{21} also in the cross-form, see (3.10) and (3.13). Then, combining the cross-form formulas for these four maps, we obtain a formula for the map $T_{k,m}$ in some rescaled variables and show that it is indeed a return map to U_{01} for suitable values of k and m , see (3.26). Finally, we conclude this section by proving the partial hyperbolicity of $T_{k,m}$ in Lemma 3.1.

3.1 Local maps

First, we discuss necessary estimates for F_1^k and F_2^m . By [34, Lemmas 5 and 6], one can choose local coordinates $(x, y, z) \in \mathbb{R} \times \mathbb{R}^{d_1} \times \mathbb{R}^{d-d_1-1}$ in U_{01} such that O_1 is at the origin and the map F_1 takes the form

$$\begin{aligned} \bar{x} &= \lambda x + g_1(x, y, z), \\ \bar{y} &= P_1 y + g_2(x, y, z), \\ \bar{z} &= P_2 z + g_3(x, y, z), \end{aligned} \tag{3.1}$$

where $\lambda = \lambda_{1,1}$, and the eigenvalues of the matrices P_1 and P_2 are $\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{1,d_1}$ and $\lambda_{1,2} \dots \lambda_{1,d-d_1}$, respectively (see (2.1)). The functions $g_{1,2,3}$ vanish along with their first derivatives at the origin and satisfy the identities

$$g_{1,3}(0, y, 0) = 0, \quad g_2(x, 0, z) = 0, \quad g_1(x, 0, z) = 0, \quad \frac{\partial g_{1,3}}{\partial x}(0, y, 0) = 0, \tag{3.2}$$

for all sufficiently small x, y , and z . As discussed in Sect. 2, the first two identities in (3.2) imply that the local manifolds $W_{loc}^s(O_1)$ and $W_{loc}^u(O_1)$ are straightened and given by the equations $\{y = 0\}$ and $\{x = 0, z = 0\}$, respectively. The third identity

shows that the strong-stable foliation \mathcal{F}^{ss} (which enters condition C2) is straightened, its leaves are given by equations $\{x = const, y = 0\}$, and the map F_1 restricted to $W_{loc}^s(O_1)$ is linear in x . The fourth identity implies that the extended unstable manifold $W_{loc}^{uE}(O_1)$ (which is in condition C1) is tangent to $z = 0$ at the points of $W_{loc}^u(O_1)$.

Similarly, we introduce coordinates $(u, v, w) \in \mathbb{R} \times \mathbb{R}^{d-d_1-1} \times \mathbb{R}^{d_1}$ in U_{O_2} with O_2 at the origin such that F_2 takes the form

$$\begin{aligned} \bar{u} &= \gamma u + \hat{g}_1(u, v, w), \\ \bar{v} &= Q_1 v + \hat{g}_2(u, v, w), \\ \bar{w} &= Q_2 w + \hat{g}_3(u, v, w), \end{aligned} \tag{3.3}$$

where $\gamma = \gamma_{2,1}$, and the eigenvalues of the matrices Q_1 and Q_2 are $\lambda_{2,2}, \lambda_{2,2}, \dots, \lambda_{2,d-d_1-1}$ and $\gamma_{2,2}, \dots, \gamma_{2,d_1+1}$, respectively (see (2.1)). The functions $\hat{g}_{1,2,3}$ vanish along with their first derivatives at the origin and satisfy

$$\hat{g}_{1,3}(0, v, 0) = 0, \quad \hat{g}_2(u, 0, w) = 0, \quad \hat{g}_1(u, 0, w) = 0, \quad \frac{\partial \hat{g}_{1,3}}{\partial u}(0, v, 0) = 0, \tag{3.4}$$

for all sufficiently small u, v and w .

It is shown in [34] (see remarks after Lemma 6 there), that the transformations that bring F_1 to form (3.1) and F_2 to form (3.3) are of class C^r , so we do not lose regularity when using these coordinates. Moreover, these coordinate transformations depend uniformly-continuously on the system f . When we consider parametric families f_ε , and f is analytic or C^∞ with respect to coordinates and parameters, the coordinate transformations are also analytic or C^∞ with respect to ε . In the case of finite smoothness r , we lose, in general, two derivatives with respect to ε . Namely, the second derivative of the transformation is C^{r-2} -smooth with respect to both the coordinates and parameters. Therefore, the matrices $P_{1,2}$ and $Q_{1,2}$ in (3.1) and (3.3) are C^{r-2} -functions of ε , and the functions $g_{1,2,3}$ and $\hat{g}_{1,2,3}$, as well as their derivatives with respect to (x, y, z) or (u, v, w) up to order 2, are C^{r-2} -functions of the coordinates and ε . If $r \geq 3$, this gives us at least 1 continuous derivative with respect to ε . In the C^2 case, we can only assume continuity of $P_{1,2}, Q_{1,2}$ with respect to ε , the same goes for $g_{1,2,3}, \hat{g}_{1,2,3}$, and their derivatives with respect to the coordinates. The eigenvalues λ and γ do not depend on coordinate transformations; as they are the eigenvalues of the first derivative of F_1 or F_2 (at O_1 and, respectively, O_2), they are at least C^1 with respect to ε in any case.

Take any point (x, y, z) in U_{O_1} and let $(\tilde{x}, \tilde{y}, \tilde{z}) = F_1^k(x, y, z)$. It is known (see e.g. [2, 58]) that the value of $(\tilde{x}, \tilde{y}, \tilde{z})$ is uniquely defined by (x, \tilde{y}, z) for all $k \geq 0$. By [34, Lemma 7], when identities (3.2) are fulfilled, the relation between the coordinates can be written as

$$\begin{aligned} \tilde{x} &= \lambda^k x + p_1(x, \tilde{y}, z), \\ y &= p_2(x, \tilde{y}, z), \quad \tilde{z} = p_3(x, \tilde{y}, z), \end{aligned} \tag{3.5}$$

where

$$\|p_{1,3}\|_{C^1} = o(\lambda^k), \quad \|p_2\|_{C^1} = o(\gamma^{-k}), \tag{3.6}$$

for some constant $\hat{\gamma} \in (1, |\gamma_{1,1}|)$. These estimates are uniform for all systems C^2 -close to f ; when we consider parametric families f_ε , the functions $p_{1,2,3}$ depend on ε uniformly-continuously, along with their first derivatives with respect to (x, \tilde{y}, z) . In the case $r \geq 3$, we have the same $o(\lambda^k)$ and $o(\hat{\gamma}^{-k})$ estimates for the derivatives of $p_{1,2}$ and, respectively, p_3 with respect to parameters ε , see [34, Lemma 7] for a detailed discussion.

Likewise, for any $(u, v, w) \in U_{02}$ we have $(\tilde{u}, \tilde{v}, \tilde{w}) = F_2^m(u, v, w)$ if and only if

$$\begin{aligned} u &= \gamma^{-m}\tilde{u} + q_1(\tilde{u}, v, \tilde{w}), \\ \tilde{v} &= q_2(\tilde{u}, v, \tilde{w}), \quad w = q_3(\tilde{u}, v, \tilde{w}), \end{aligned} \tag{3.7}$$

where

$$\|q_{1,3}\|_{C^1} = o(\gamma^{-m}), \quad \|q_2\|_{C^1} = o(\hat{\lambda}^m), \tag{3.8}$$

for some constant $\hat{\lambda} \in (|\lambda_{2,1}|, 1)$; the functions $q_{1,2,3}$ depend uniformly-continuously on the system f , and the estimates (3.8) hold uniformly for all systems C^2 -close to f . When $r \geq 3$, estimates (3.8) also hold for the derivatives with respect to parameters ε .

3.2 Transition maps

Next, we consider the transition maps F_{12} and F_{21} . We use the following notation for the coordinates of the points $M_{1,2}^\pm$:

$$M_1^+ = (x^+, 0, z^+), \quad M_1^- = (0, y^-, 0), \quad M_2^+ = (0, v^+, 0), \quad M_2^- = (u^-, 0, w^-).$$

We can write the Taylor expansion of the transition map $F_{12} : (\tilde{x}, \tilde{y}, \tilde{z}) \mapsto (u, v, w)$ near M_1^- as

$$\begin{aligned} u &= a'_1 + a'_{11}\tilde{x} + a'_{12}(\tilde{y} - y^-) + a'_{13}\tilde{z} + O(\tilde{x}^2 + (\tilde{y} - y^-)^2 + \tilde{z}^2), \\ v - v^+ &= a'_2 + a'_{21}\tilde{x} + a'_{22}(\tilde{y} - y^-) + a'_{23}\tilde{z} + O(\tilde{x}^2 + (\tilde{y} - y^-)^2 + \tilde{z}^2), \\ w &= a'_3 + a'_{31}\tilde{x} + a'_{32}(\tilde{y} - y^-) + a'_{33}\tilde{z} + O(\tilde{x}^2 + (\tilde{y} - y^-)^2 + \tilde{z}^2), \end{aligned} \tag{3.9}$$

where a'_i and a'_{ij} ($i, j = 1, 2, 3$) are some constants. Since $F_{12}(M_1^-) = M_2^+$ for the system f , it follows that $a'_{1,2,3}$ vanish, but when we perturb f , these coefficients can become non-zero (though small).

Recall that $W_{loc}^{sE}(O_2)$ is tangent at M_2^+ to $w = 0$ and $W_{loc}^u(O_1)$ is given by $(\tilde{x} = 0, \tilde{z} = 0)$. Thus, the transversality condition C1 writes as the uniqueness of the (trivial) solution to the system

$$u = a'_{12}(\tilde{y} - y^-), \quad v - v^+ = a'_{22}(\tilde{y} - y^-), \quad 0 = a'_{32}(\tilde{y} - y^-),$$

i.e., a'_{32} is invertible. It follows that $\tilde{y} - y^-$ can be expressed, from the last equation of (3.9), as a smooth function of $(w, \tilde{x}, \tilde{z})$, so the map F_{12} can be written in the

cross-form as

$$\begin{aligned}
 u &= \hat{\mu} + a\tilde{x} + a_{12}w + a_{13}\tilde{z} + O(\tilde{x}^2 + w^2 + \tilde{z}^2), \\
 v - \hat{v}^+ &= a_{21}\tilde{x} + a_{22}w + a_{23}\tilde{z} + O(\tilde{x}^2 + w^2 + \tilde{z}^2), \\
 \tilde{y} - \hat{y}^- &= a_{31}\tilde{x} + a_{32}w + a_{33}\tilde{z} + O(\tilde{x}^2 + w^2 + \tilde{z}^2),
 \end{aligned}
 \tag{3.10}$$

where the coefficients $\hat{v}^+, \hat{y}^-, \hat{\mu}, a$, and a_{ij} change uniformly-continuously when the system f is perturbed, and for the original system f , we have $\hat{v}^+ = v^+, \hat{y}^- = y^-$, and $\hat{\mu} = 0$. Since it does not cause ambiguity, in further references to (3.10) we use v^+ and y^- instead of \hat{v}^+ and \hat{y}^- . Note that the coefficient $a \neq 0$ is exactly the derivative defined in (2.6).

Since $W_{loc}^u(O_1)$ is given by $\{\tilde{x} = 0, \tilde{z} = 0\}$ and $W_{loc}^s(O_2)$ is given by $\{u = 0, w = 0\}$, a fragile heteroclinic corresponding to the intersection of $F_{12}(W_{loc}^u(O_1))$ and $W_{loc}^s(O_2)$ persists for a perturbation of f if and only if $\hat{\mu} = 0$. More precisely, $\hat{\mu}$ is the u -coordinate of the point of the intersection of $F_{12}(W_{loc}^u(O_1))$ with $\{w = 0\}$. This intersection is transverse by condition C1, from which one infers that the ratio of $|\hat{\mu}|$ to the distance between $F_{12}(W_{loc}^u(O_1))$ and $W_{loc}^s(O_2)$ tends (uniformly in some C^2 -neighborhood of f) to a finite non-zero limit value when $\hat{\mu} \rightarrow 0$. Recall that we defined the splitting parameter $\mu(f)$, whose absolute value equal the distance between $F_{12}(W_{loc}^u(O_1))$ and $W_{loc}^s(O_2)$ and which enters Theorems 2 – 6. By scaling the variable u , we can always obtain

$$\lim_{\mu \rightarrow 0} \frac{\hat{\mu}}{\mu(f)} = 1, \quad \text{or} \quad \hat{\mu} = \mu + o(\mu).
 \tag{3.11}$$

If $r \geq 3$, then when we consider parametric families f_ε , the coefficients of (3.10) are at least C^1 with respect to ε . When the family is generic or proper, $\mu = \mu(f_\varepsilon)$ is one of the parameters, and we have (for $r \geq 3$)

$$\left. \frac{\partial \hat{\mu}}{\partial \mu} \right|_{\mu=0} = 1.$$

It follows that in this case we can make a smooth change of parameters such that $\hat{\mu} = \mu$.

The Taylor expansion of the other transition map $F_{21} : (\tilde{u}, \tilde{v}, \tilde{w}) \mapsto (x, y, z)$, which is defined for $(\tilde{u}, \tilde{v}, \tilde{w})$ near $M_2^- = (u^-, 0, w^-)$ and takes values (x, y, z) near $M_1^+ = (x^+, 0, z^+)$, is given by

$$\begin{aligned}
 x - x^+ &= b'_{11}(\tilde{u} - u^-) + b'_{12}\tilde{v} + b'_{13}(\tilde{w} - w^-) + O((\tilde{u} - u^-)^2 + \tilde{v}^2 + (\tilde{w} - w^-)^2), \\
 y &= b'_{21}(\tilde{u} - u^-) + b'_{22}\tilde{v} + b'_{23}(\tilde{w} - w^-) + O((\tilde{u} - u^-)^2 + \tilde{v}^2 + (\tilde{w} - w^-)^2), \\
 z - z^+ &= b'_{31}(\tilde{u} - u^-) + b'_{32}\tilde{v} + b'_{33}(\tilde{w} - w^-) + O((\tilde{u} - u^-)^2 + \tilde{v}^2 + (\tilde{w} - w^-)^2),
 \end{aligned}
 \tag{3.12}$$

where b'_{ij} are some constants. Arguing as for the map F_{12} above, one can use the assumption that $F_{21}^{-1}(W_{loc}^s(O_1)) \cap \mathcal{F}^{uu} \neq \emptyset$ from condition C2 to deduce that $\det b'_{23} \neq$

0 (note that the leaf of the foliation \mathcal{F}^{uu} through M_2^- is given by $(\tilde{u} = u^-, \tilde{v} = 0)$ and $W_{loc}^s(O_1)$ is given by $\{y = 0\}$ here). Consequently, F_{21} can be written in the following cross-form:

$$\begin{aligned} x - x^+ &= b(\tilde{u} - u^-) + b_{12}\tilde{v} + b_{13}y + O((\tilde{u} - u^-)^2 + \tilde{v}^2 + y^2), \\ \tilde{w} - w^- &= b_{21}(\tilde{u} - u^-) + b_{22}\tilde{v} + b_{23}y + O((\tilde{u} - u^-)^2 + \tilde{v}^2 + y^2), \\ z - z^+ &= b_{31}(\tilde{u} - u^-) + b_{32}\tilde{v} + b_{33}y + O((\tilde{u} - u^-)^2 + \tilde{v}^2 + y^2), \end{aligned} \tag{3.13}$$

where b is the derivative defined by (2.3). All the coefficients in (3.13) change uniformly-continuously when the system f is perturbed; if $r \geq 3$, then they are at least C^1 with respect to the perturbation parameters ε . Note that since $b \neq 0$, we can express $\tilde{u} - u^-$ as a function of $(x - x^+, \tilde{v}, y)$ and rewrite (3.13) as

$$\begin{aligned} \tilde{u} - u^- &= b^{-1}(x - x^+ - b_{13}y) + O(\|\tilde{v}\| + (x - x^+)^2 + y^2), \\ \tilde{w} - w^- &= O(|x - x^+| + \|\tilde{v}\| + \|y\|), \\ z - z^+ &= b_{31}b^{-1}(x - x^+ - b_{13}y) + b_{33}y + O(\|\tilde{v}\| + (x - x^+)^2 + y^2). \end{aligned} \tag{3.14}$$

3.3 First-return maps and cone field lemma

We can now find a formula for the first-return map $T_{k,m} = F_{21} \circ F_2^m \circ F_{12} \circ F_1^k$. Combining (3.5) and (3.10), we obtain a formula for the map $F_{12} \circ F_1^k$. Namely, by substituting the last equation of (3.10) (\tilde{y} as a function of \tilde{x}, w , and \tilde{z}) into the first and the last equations of (3.5), we can express \tilde{x} and \tilde{z} as functions of (x, z, w) :

$$\tilde{x} = \lambda^k x + o(\lambda^k), \quad \tilde{z} = o(\lambda^k).$$

After that, we substitute these formulas into the rest equations in (3.5) and (3.10), and find that there exist smooth functions

$$\begin{aligned} \tilde{h}_1(x, z, w) &= O(\|w\|) + o(\lambda^k), & \tilde{h}_2(x, z, w) &= o(\hat{\gamma}^{-k}), \\ \tilde{h}_3(x, z, w) &= O(\|w\| + |\lambda|^k), \end{aligned}$$

such that, for sufficiently large k , a point (x, y, z) from a small neighborhood of M_1^+ is taken by the map $F_{12} \circ F_1^k$ to a point (u, v, w) in a small neighborhood of M_2^+ if and only if

$$\begin{aligned} u &= \hat{\mu} + a\lambda^k x + \tilde{h}_1(x, z, w) \\ v &= \tilde{h}_2(x, z, w), & v - v^+ &= \tilde{h}_3(x, z, w). \end{aligned} \tag{3.15}$$

Similarly, combining (3.7) and (3.14) yields that there exist smooth functions

$$\begin{aligned} \hat{h}_{0i}(x - x^+, \bar{y}) &= O((x - x^+)^2 + y^2) \quad (i = 1, 2), \\ \hat{h}_{1,3}(x, v, y) &= o(\gamma^{-m}), & \hat{h}_2(x, v, y) &= o(\hat{\lambda}^m), \end{aligned} \tag{3.16}$$

such that, for sufficiently large m , a point (u, v, w) from a small neighborhood of M_2^+ is taken by the map $F_{21} \circ F_2^m$ to a point (x, y, z) in a small neighborhood of M_1^+ if and only if

$$\begin{aligned} u &= \gamma^{-m}(u^- + b^{-1}(x - x^+ - b_{13}y + \hat{h}_{01}(x - x^+, y))) + \hat{h}_1(x, v, y), \\ z - z^+ &= b_{31}b^{-1}(x - x^+ - b_{13}y) + b_{33}y + \hat{h}_{02}(x - x^+, y) + \hat{h}_2(x, v, y), \quad (3.17) \\ w &= \hat{h}_3(x, v, y). \end{aligned}$$

Let us now consider the first-return map $T_{k,m} = F_{21} \circ F_2^m \circ F_{12} \circ F_1^k$ for sufficiently large positive integers k and m that takes a point (x, y, z) from a small neighborhood of M_1^+ to its image $(\bar{x}, \bar{y}, \bar{z})$. To obtain a formula for this map, we replace (x, y, z) in (3.17) by $(\bar{x}, \bar{y}, \bar{z})$ and combine it with (3.15). More specifically, from the system comprised by the last equations of (3.15) (for v) and (3.17) (for w) we can express v and w as smooth functions of (x, \bar{x}, z, \bar{y}) . By substituting these functions into the rest of the equations (3.15) and (3.17), we find that there exist functions

$$\begin{aligned} h_1(x, \bar{x}, z, \bar{y}) &= o(\lambda^k) + o(\gamma^{-m}), & h_2(x, \bar{x}, z, \bar{y}) &= o(\hat{\lambda}^m), \\ h_3(x, \bar{x}, z, \bar{y}) &= o(\hat{\gamma}^{-k}), \end{aligned}$$

such that $T_{k,m}(x, y, z) = (\bar{x}, \bar{y}, \bar{z})$ if and only if

$$\begin{aligned} \bar{x} - x^+ &= b\gamma^m\hat{\mu} - bu^- + ab\lambda^k\gamma^m x + b_{13}\bar{y} - \hat{h}_{01}(\bar{x} - x^+, \bar{y}) \\ &\quad + \gamma^m h_1(x, \bar{x}, z, \bar{y}), \\ \bar{z} - z^+ &= b_{31}b^{-1}(x - x^+ - b_{13}\bar{y}) + b_{33}\bar{y} + \hat{h}_{02}(\bar{x} - x^+, \bar{y}) \\ &\quad + h_2(x, \bar{x}, z, \bar{y}), \\ y &= h_3(x, \bar{x}, z, \bar{y}). \end{aligned} \quad (3.18)$$

After the coordinate transformation

$$\begin{aligned} X &= x - x^+ - b_{13}y, & Y &= y, \\ Z &= z - z^+ - b_{31}b^{-1}(x - x^+ - b_{13}y) - b_{33}y - \hat{h}_{02}(x - x^+, y) \end{aligned} \quad (3.19)$$

the map $T_{k,m}$ assumes the form

$$\begin{aligned} \bar{X} &= b\gamma^m\hat{\mu} + ab\lambda^k\gamma^m x^+ - bu^- + ab\lambda^k\gamma^m X + \hat{\phi}_0(\bar{X}, \bar{Y}) \\ &\quad + \gamma^m\hat{\phi}_1(X, \bar{X}, \bar{Y}, Z), \\ Y &= \hat{\phi}_2(X, \bar{X}, \bar{Y}, Z), & \bar{Z} &= \hat{\phi}_3(X, \bar{X}, \bar{Y}, Z), \end{aligned} \quad (3.20)$$

where

$$\hat{\phi}_0 = O(\bar{X}^2 + \bar{Y}^2), \quad (3.21)$$

i.e., $\hat{\phi}_0$ vanishes at $(\bar{X}, \bar{Y}) = 0$ along with the first derivatives, and

$$\|\hat{\phi}_1\|_{C^1} = o(\lambda^k) + o(\gamma^{-m}), \quad \|\hat{\phi}_2\|_{C^1} = o(\hat{\gamma}^{-k}), \quad \|\hat{\phi}_3\|_{C^1} = o(\hat{\lambda}^m). \quad (3.22)$$

By construction, these estimates are uniform for all systems C^2 -close to f , and when the system is at least C^3 -smooth, the same estimates are also true for the first derivatives with respect to parameters ε . We note also that in the coordinates (3.19) we have the local stable manifold of O_1 and the image of the local unstable manifold of O_2 straightened:

$$W_{loc}^s(O_1) : \{Y = 0\} \quad \text{and} \quad F_{21}(W_{loc}^u(O_2)) : \{Z = 0\} \quad (3.23)$$

(one can see this from that the equation for $F_{21}(W_{loc}^u(O_2))$ in the (x, y, z) coordinates can be obtained by taking the limit $m \rightarrow \infty$ in (3.17)).

We stress that the first-return map must take points from a small neighborhood of M_1^+ in U_{01} to a small neighborhood of M_1^+ , which corresponds to small values of (X, Y, Z) and $(\bar{X}, \bar{Y}, \bar{Z})$. We further denote this neighborhood where we want the first-return maps to be defined by

$$\Pi = [-\delta, \delta] \times [-\delta, \delta]^{d_1} \times [-\delta, \delta]^{d-d_1-1}. \quad (3.24)$$

For $(X, \bar{Y}, Z) \in \Pi$, we can assure that Y and \bar{Z} are small in (3.20) by taking k and m sufficiently large. However, to have \bar{X} small (i.e. $\bar{X} = O(\delta)$) one needs some additional restriction on possible values of k and m .

In the next section we will consider the first-return maps near the unperturbed cycles Γ , which corresponds to $\mu = \hat{\mu} = 0$ in (3.20) (see (3.11)). In this case, the $O(\delta)$ smallness of both \bar{X} and X in the first equation of (3.20) implies that the map $T_{k,m}$ at $\mu = 0$ acts from Π to an $O(\delta)$ -neighborhood of M_1^+ in U_{01} only when there is a certain balance between k and m , namely

$$ab\lambda^k\gamma^m = \alpha + O(\delta), \quad (3.25)$$

where $\alpha = bu^-/x^+ \neq 0$ is the quantity introduced in condition (2.4).

Thus, $\lambda^k\gamma^m$ must be uniformly bounded in this case, and hence $\hat{\phi}_1 = o(\gamma^{-m})$ in (3.22). Consequently, the derivative of the right-hand side of the first equation of (3.20) with respect to \bar{X} is of order $O(\delta) + o(1)_{k,m \rightarrow \infty}$, so \bar{X} can be expressed as a function of (X, \bar{Y}, Z) . Therefore, for sufficiently large k, m such that (3.25) is satisfied, formula (3.20) for $T_{k,m}$ at $\mu = 0$ implies that for a point $(X, Y, Z) \in \Pi$ we have $(\bar{X}, \bar{Y}, \bar{Z}) = T_{k,m}(X, Y, Z)$ if and only if the points are related by the cross-map $T_{k,m}^x : (X, \bar{Y}, Z) \mapsto (\bar{X}, Y, \bar{Z})$ given by

$$\begin{aligned} \bar{X} &= ab\lambda^k\gamma^m x^+ - bu^- + ab\lambda^k\gamma^m X + \phi_1(X, \bar{Y}, Z), \\ &= ab\lambda^k\gamma^m x^+ - bu^- + (\alpha + O(\delta))X + \phi_1(X, \bar{Y}, Z), \\ Y &= \phi_2(X, \bar{Y}, Z), \quad \bar{Z} = \phi_3(X, \bar{Y}, Z), \end{aligned} \quad (3.26)$$

where

$$\begin{aligned} \phi_1 &= O(\delta^2) + o(1)_{k,m \rightarrow \infty}, & \frac{\partial \phi_1}{\partial (X, \bar{Y}, Z)} &= O(\delta) + o(1)_{k,m \rightarrow \infty}, \\ \|\phi_2\|_{C^1} &= o(\hat{\gamma}^{-k}), & \|\phi_3\|_{C^1} &= o(\hat{\lambda}^m). \end{aligned} \tag{3.27}$$

The following result characterizes the action of the derivative $DT_{k,m}$ of the maps $T_{k,m}$ at $\mu = 0$. We will further use the notation $(\Delta X, \Delta Y, \Delta Z)$ for vectors in the tangent space to Π .

Lemma 3.1 *Let $\mu = 0$. Given any positive $K < 1$, for all sufficiently small δ and large (k, m) satisfying (3.25), the cone fields on Π*

$$C^{cu} = \{(\Delta X, \Delta Y, \Delta Z) : \|\Delta Z\| \leq K(|\Delta X| + \|\Delta Y\|)\}, \tag{3.28}$$

$$C^{uu} = \{(\Delta X, \Delta Y, \Delta Z) : \max\{|\Delta X|, \|\Delta Z\|\} \leq K\|\Delta Y\|\}, \tag{3.29}$$

are forward-invariant in the sense that if a point $M \in \Pi$ has its image $\bar{M} = T_{k,m}(M)$ in Π , then the cone at M is mapped into the cone at \bar{M} by $DT_{k,m}$; and the cone fields

$$C^{cs} = \{(\Delta X, \Delta Y, \Delta Z) : \|\Delta Y\| \leq K(|\Delta X| + \|\Delta Z\|)\}, \tag{3.30}$$

$$C^{ss} = \{(\Delta X, \Delta Y, \Delta Z) : \max\{|\Delta X|, \|\Delta Y\|\} \leq K\|\Delta Z\|\}, \tag{3.31}$$

are backward-invariant in the sense that if a point $\bar{M} \in \Pi$ has its preimage $M = T_{k,m}^{-1}(\bar{M})$ in Π , then the cone at \bar{M} is mapped into the cone at M by $DT_{k,m}^{-1}$. Moreover, vectors in C^{uu} and, if $|\alpha| > 1$, also in C^{cu} are expanded by $DT_{k,m}$; vectors in C^{ss} and, if $|\alpha| < 1$, also in C^{cs} are contracted by $DT_{k,m}$.

Proof Let us establish the backward invariance of C^{ss} and C^{cs} . Take $\bar{M} = (\bar{X}, \bar{Y}, \bar{Z}) \in \Pi$ such that $T_{k,m}^{-1}(\bar{M}) = M \in \Pi$ for some (k, m) . Take a vector $(\Delta \bar{X}, \Delta \bar{Y}, \Delta \bar{Z})$ in the tangent space at the point \bar{M} . Let $(\Delta X, \Delta Y, \Delta Z) = DT_{k,m}^{-1}(\Delta \bar{X}, \Delta \bar{Y}, \Delta \bar{Z})$.

It follows from (3.26) and (3.27) that

$$\begin{aligned} \Delta \bar{X} &= (\alpha + O(\delta) + o(1)_{k,m \rightarrow \infty})\Delta X + (O(\delta) + o(1)_{k,m \rightarrow \infty})\Delta \bar{Y} \\ &\quad + (O(\delta) + o(1)_{k,m \rightarrow \infty})\Delta Z, \\ \Delta Y &= o(\hat{\gamma}^{-k})\Delta X + o(\hat{\gamma}^{-k})\Delta \bar{Y} + o(\hat{\gamma}^{-k})\Delta Z, \\ \Delta \bar{Z} &= o(\hat{\lambda}^m)\Delta X + o(\hat{\lambda}^m)\Delta \bar{Y} + o(\hat{\lambda}^m)\Delta Z. \end{aligned} \tag{3.32}$$

Thus, there exists a constant C such that if k and m are large enough, then

$$(|\alpha| - C\delta)|\Delta X| \leq |\Delta \bar{X}| + C\delta\|\Delta \bar{Y}\| + C\delta\|\Delta Z\|, \tag{3.33}$$

$$|\Delta \bar{X}| \leq (|\alpha| + C\delta)|\Delta X| + C\delta\|\Delta \bar{Y}\| + C\delta\|\Delta Z\|, \tag{3.34}$$

$$\|\Delta Y\| = o(\hat{\gamma}^{-k})(|\Delta X| + \|\Delta \bar{Y}\| + \|\Delta Z\|), \tag{3.35}$$

$$\|\Delta \bar{Z}\| = o(\hat{\lambda}^m)(|\Delta X| + \|\Delta \bar{Y}\| + \|\Delta Z\|). \tag{3.36}$$

Let $(\Delta \bar{X}, \Delta \bar{Y}, \Delta \bar{Z}) \in \mathcal{C}^{ss}$. Since $K < 1$, we have

$$\max\{|\Delta \bar{X}|, \|\Delta \bar{Y}\|\} \leq \|\Delta \bar{Z}\|.$$

Now, it follows from (3.36) that

$$\|\Delta \bar{Z}\| = o(\hat{\lambda}^m)(|\Delta X| + \|\Delta Z\|) \quad (3.37)$$

and, hence,

$$|\Delta \bar{X}| + \|\Delta \bar{Y}\| = o(\hat{\lambda}^m)(|\Delta X| + \|\Delta Z\|).$$

We substitute these estimates into (3.33) and (3.35) and obtain

$$|\Delta X| = O(\delta)\|\Delta Z\| \quad \text{and} \quad \|\Delta Y\| = o(\hat{\gamma}^{-k})\|\Delta Z\|,$$

i.e., for any fixed choice of the constant K , if k and m are large enough and δ is small enough, the vector $(\Delta X, \Delta Y, \Delta Z)$ lies in \mathcal{C}^{ss} at the point M , as required. Equation (3.37) implies the contraction in \mathcal{C}^{ss} if m is large enough.

Similar arguments are applied when $(\Delta \bar{X}, \Delta \bar{Y}, \Delta \bar{Z}) \in \mathcal{C}^{cs}$. Here, we have

$$\|\Delta \bar{Y}\| \leq |\Delta \bar{X}| + \|\Delta \bar{Z}\|.$$

Substituting this into (3.34) and (3.36) gives

$$\begin{aligned} |\Delta \bar{X}| &\leq (|\alpha| + O(\delta))|\Delta X| + O(\delta)\|\Delta Z\|, \\ \|\Delta \bar{Z}\| &= o(\hat{\lambda}^m)(|\Delta X| + \|\Delta Z\|), \end{aligned} \quad (3.38)$$

for sufficiently small δ and sufficiently large m , and, hence,

$$\|\Delta \bar{Y}\| \leq (|\alpha| + 1)|\Delta X| + \|\Delta Z\|.$$

We substitute the last estimate into (3.35) and obtain

$$\|\Delta Y\| = o(\hat{\gamma}^{-k})(|\Delta X| + \|\Delta Z\|),$$

i.e., for any fixed choice of the constant K , if k is large enough, the vector $(\Delta X, \Delta Y, \Delta Z)$ lies in \mathcal{C}^{cs} at the point M , as required. Equation (3.38) implies the contraction in \mathcal{C}^{cs} if $|\alpha| < 1$ and m is large enough and δ is small enough.

The proof of the forward invariance of \mathcal{C}^{uu} and \mathcal{C}^{cu} is done in the same way, as everything is symmetric here with respect to the change of $T_{k,m}$ to $T_{k,m}^{-1}$. \square

4 Blenders near type-I heterodimensional cycles. Proof of Theorem 1

In this section we obtain a detailed description of the blenders that may appear near a type-I heterodimensional cycle, which is summarized in Propositions 4.4 and 4.5 below. These two propositions together imply Theorem 1. Throughout this section, we do not perturb the system f in this theorem, i.e., the fragile heteroclinic is not split, so $\hat{\mu} = 0$ in (3.15). We consider non-degenerate type-I cycles, which means that $|\alpha| = |bu^-/x^+| \neq 1$ and $ax^+u^- > 0$ in (3.20). We also assume that $\theta = -\ln|\lambda|/\ln|\gamma|$ is irrational.

4.1 Blenders with activating pairs

Definition 4.1 (Proper crossing) Consider a cube $Q = \{(x_1, \dots, x_d) \mid x_i \in I_i\} \subset \mathbb{R}^d$, where I_i are closed intervals in \mathbb{R} . A k -dimensional disc S is said to *cross* Q if the intersection $S \cap Q$ is given by $(x_{i_{k+1}}, \dots, x_{i_d}) = s(x_{i_1}, \dots, x_{i_k})$, where (i_1, \dots, i_d) is some permutation of $(1, \dots, d)$, and s is a smooth function defined on $I_{i_1} \times \dots \times I_{i_k}$. It crosses Q *properly* with respect to a cone field \mathcal{C} defined on Q if the tangent spaces of $S \cap Q$ lie in \mathcal{C} .

For example, a disc S crossing Π properly with respect to \mathcal{C}^{ss} in Lemma 3.1 is the graph of some smooth function $(X, Y) = s(Z)$, which is defined on $[-\delta, \delta]^{d-d_1-1}$ and whose tangent spaces lie in \mathcal{C}^{ss} .

For an invariant cone field \mathcal{C} , denote by $\dim(\mathcal{C})$ the largest possible dimension of a linear subspace of the tangent space that can be contained in \mathcal{C} (at each point where \mathcal{C} is defined).

Definition 4.2 (Activating pair) A pair (Q, \mathcal{C}) consisting of a cube and a cone field is called an *activating pair* for a cs-blender Λ if $\dim(\mathcal{C}) = \dim W^s(\Lambda) - 1$ and if any disc S crossing Q properly with respect to \mathcal{C} intersects $W^u(\Lambda)$. Similarly, a pair (Q, \mathcal{C}) is an activating pair for a cu-blender Λ if $\dim(\mathcal{C}) = \dim W^u(\Lambda) - 1$ and any disc S crossing Q properly with respect to \mathcal{C} intersects $W^s(\Lambda)$. The cube Q is called an *activating domain*.

Remark 4.3 Every disc crossing Q properly with respect to \mathcal{C} belongs to the set \mathcal{D}^{ss} of Definition 1.9 in the case of a cs-blender, or to the set \mathcal{D}^{uu} in the case of a cu-blender. Thus, the activating pair specifies the position of robust non-transverse intersections. The cone field \mathcal{C} is the field \mathcal{C}^{ss} (for the cs-blender) or \mathcal{C}^{uu} (for the cu-blender) from the definition of the standard blender in the Appendix.

Note that any disc C^1 -close to S in the above definition is also a properly crossing disc, and hence intersects the invariant manifold of the blender. In particular, if Λ is a cs-blender and S is a piece of the stable manifold $W^s(\Lambda')$ of some hyperbolic basic set Λ' , then we obtain robust non-transverse intersection of $W^u(\Lambda)$ with $W^s(\Lambda')$. If at the same time we have $W^s(\Lambda) \cap W^u(\Lambda') \neq \emptyset$, then robust heterodimensional dynamics emerge. Similar arguments hold in the case of a cu-blender.

Recall that Π defined in (3.24) is a cube around the robust heteroclinic point M_1^+ .

Proposition 4.4 *Let Γ be a non-degenerate cycle of type I, with irrational θ . If $|\alpha| < 1$, then there exists arbitrarily close to Γ an index- d_1 cs-blender Λ^{cs} with an activating pair (Π', \mathcal{C}^{ss}) , where Π' is a subcube of Π with the center at M_1^+ , and*

$$\mathcal{C}^{ss} = \{(\Delta X, \Delta Y, \Delta Z) : \max(|\Delta X|, \|\Delta Y\|) < \frac{q|\alpha|}{4} \|\Delta Z\|\}. \tag{4.1}$$

When $|\alpha| > 1$, one immediately gets a cu-blender of index $d_1 + 1$ by applying the above proposition due to the system obtained by the time reversal. However, the corresponding activating domain would lie near O_2 but not O_1 (the time-reversal also

interchanges O_1 and O_2). This would not match with the generalization, Proposition 4.12, which is the main tool used for showing the existence of blenders in the saddle-focus and double-focus cases. So, we prove the following proposition about the cu-blender with an activating domain Π' near O_1 as in the case $|\alpha| < 1$.

Proposition 4.5 *Let Γ be a non-degenerate cycle of type I, with irrational θ . If $|\alpha| > 1$, then there exists, arbitrarily close to Γ , an index- $(d_1 + 1)$ cu-blender Δ^{cu} with an activating pair (Π', \mathcal{C}^{uu}) , where Π' is a subcube of Π with the center at M_1^+ , and*

$$\mathcal{C}^{uu} = \{(\Delta X, \Delta Y, \Delta Z) : \max\{|\Delta X|, \|\Delta Z\|\} \leq \frac{q|\alpha|^{-1}}{4} \|\Delta Y\|\}. \tag{4.2}$$

Below, in the proof of these propositions, we establish the existence of blenders in the sense of the ‘‘operational’’ Definition 1.9. However, the construction is explicit, and the obtained blenders are standard in the sense of Definition A.1, as we show in Proposition prop:stanble.

4.2 Finding cs-blenders when $|\alpha| < 1$. Proof of Proposition 4.4

Recall that the cube Π defined in (3.24) is the region for which we consider the first-return maps $T_{k,m}$. We start with finding hyperbolic sets for the collection of the return maps in Π , which can be candidates for blenders. Consider the set of pairs (k, m) of sufficiently large integers:

$$\mathcal{P}_N = \{(k, m) : k > N, m > N \text{ and } |ab\lambda^k \gamma^m x^+ - bu^-| \leq \frac{2}{3}(1 - |\alpha|)\delta\}. \tag{4.3}$$

Note that this set is non-empty for any N and $\delta > 0$: because $\theta = -\ln|\lambda|/\ln|\gamma|$ is irrational, the set $\lambda^k \gamma^m$ is dense among positive reals, hence, since abx^+ and bu^- have the same sign by the assumption of the theorem (we consider the cycles of type I), $ab\lambda^k \gamma^m x^+$ can be made as close as we want to bu^- for arbitrarily large k and m .

By construction, the estimate (3.25) is satisfied for every $(k, m) \in \mathcal{P}_N$. Therefore, for every such (k, m) the relation between the coordinates $(X, Y, Z) \in \Pi$ of a point in the domain of definition of the first-return map $T_{k,m}$ and the coordinates $(\bar{X}, \bar{Y}, \bar{Z})$ of its image by $T_{k,m}$ is given by (3.26).

Let us denote by $T_{k,m}^\times : (X, \bar{Y}, Z) \mapsto (\bar{X}, Y, \bar{Z})$ the cross-map (3.26). Since $|\alpha| < 1$, comparing (4.3) with the X -equation in (3.26), one sees that for every $(k, m) \in \mathcal{P}_N$, with N sufficiently large and δ sufficiently small, we have $\bar{X} \in [-\delta, \delta]$. On the other hand, it is obvious that (Y, \bar{Z}) -coordinates lie in $[-\delta, \delta]^{d-1}$ for all sufficiently large k and m , due to the strong contraction given by (3.26). Thus, for any $(k, m) \in \mathcal{P}_N$ with sufficiently large N and m , the cross-map $T_{k,m}^\times$ satisfies

$$T_{k,m}^\times(\Pi) \subset \Pi.$$

It also follows from $|\alpha| < 1$ and (3.26) that the cross-map is contracting on Π , i.e.,

$$\left\| \frac{\partial(\bar{X}, Y, \bar{Z})}{\partial(X, \bar{Y}, Z)} \right\| < 1.$$

By Shilnikov lemma on the fixed point in a direct product of metric spaces (Theorem 6.2 in [58]), these two facts immediately imply

Lemma 4.6 *For any sequence $\{(k_s, m_s)\}_{s \in \mathbb{Z}}$ of pairs (k, m) from \mathcal{P}_N , there exists a unique sequence of points $\{M_s = (X_s, Y_s, Z_s)\}_{s \in \mathbb{Z}}$ in Π such that $M_{s+1} = T_{k_s, m_s} M_s$ for every $s \in \mathbb{Z}$.*

Proof By definition of the cross-map, a sequence of points $\{M_s = (X_s, Y_s, Z_s)\}$ in Π satisfies

$$M_{s+1} = T_{k_s, m_s} M_s \tag{4.4}$$

if and only if

$$(X_{s+1}, Y_s, Z_{s+1}) = T_{k_s, m_s}^\times (X_s, Y_{s+1}, Z_s).$$

Thus, the sought sequence $\{(X_s, Y_s, Z_s)\}_{s \in \mathbb{Z}}$ is a fixed point of the map $\{(X_s, Y_s, Z_s)\}_{s \in \mathbb{Z}} \mapsto \{(\tilde{X}_s, \tilde{Y}_s, \tilde{Z}_s)\}_{s \in \mathbb{Z}}$ (acting on the space of sequences of points in Π) which is defined by the rule

$$(\tilde{X}_{s+1}, \tilde{Y}_s, \tilde{Z}_{s+1}) = T_{k_s, m_s}^\times (X_s, Y_{s+1}, Z_s).$$

Obviously, this map is a contraction (because each of the maps T_{k_s, m_s}^\times is a contraction on Π), so the fixed point indeed exists and is unique. \square

We call the sequence $\{(k_s, m_s)\}_{s \in \mathbb{Z}}$ the *coding* of the point M_0 . Lemma 4.6 establishes the existence of an invariant set Λ whose intersection with Π is in *one-to-one correspondence* with the set of all codings formed from the pairs $(k, m) \in \mathcal{P}_N$. Namely, a point M_0 lies in $\Lambda \cap \Pi$ if and only if there is a coding $\{(k_s, m_s) \in \mathcal{P}_N\}_{s \in \mathbb{Z}}$ such that the points defined by

$$M_s = \begin{cases} T_{k_{s-1}, m_{s-1}} \circ \dots \circ T_{k_0, m_0} M_0 & \text{for } s > 0 \\ T_{k_{-s}, m_{-s}}^{-1} \circ \dots \circ T_{k_{-1}, m_{-1}}^{-1} M_0 & \text{for } s < 0 \end{cases}$$

all lie in Π . The set Λ is the union of the orbits of all such points $M_0 \in \Pi$ by the system f .

We are interested in a certain closed subset $\Lambda_{\mathcal{J}} \subset \Lambda$ which corresponds to the set of codings with pairs (k, m) taken from some finite subset $\mathcal{J} \subset \mathcal{P}_N$ (so the dynamics on $\Lambda_{\mathcal{J}}$ correspond to the full shift with this finite set of symbols). It follows from Lemma 3.1 that this set is a locally maximal, transitive hyperbolic set and, also, partially-hyperbolic with a 1-dimensional central (weakly-hyperbolic) direction. In summary, we have

Lemma 4.7 *Let $|\alpha| < 1$ and \mathcal{J} be any finite subset of \mathcal{P}_N . Then, there exists a hyperbolic basic set $\Lambda_{\mathcal{J}}$ of index d_1 near the heterodimensional cycle Γ such that it is in one-to-one correspondence with the set of codings $\{(k_s, m_s) \in \mathcal{J}\}_{s \in \mathbb{Z}}$.*

Remark 4.8 The intersection $\Lambda_{\mathcal{J}} \cap \Pi$ is located in a finite union of “horizontal strips” in Π . Indeed, one sees from (3.26) that the domains of $T_{k,m}$ with $(k, m) \in \mathcal{P}_N$ are given by the strips:

$$\sigma_{k,m} = \{(X, Y, Z) \mid X \in [-\delta, \delta], Y \in \phi_2(X, [-\delta, \delta]^{d_1}, Z), Z \in [-\delta, \delta]^{d-d_1-1}\}. \tag{4.5}$$

Since $\phi_2 = o(\hat{\gamma}^{-k})$, these strips accumulate on $\{Y = 0\}$ and strips corresponding to different (k, m) are disjoint. By construction, we have

$$\Lambda_{\mathcal{J}} \cap \Pi \subset \bigcup_{(k,m) \in \mathcal{J}} \sigma_{k,m} =: \Sigma_{\mathcal{J}}. \tag{4.6}$$

Every point whose backward orbit stays in $\Sigma_{\mathcal{J}}$ lies in $W^u(\Lambda_{\mathcal{J}})$, and every point whose forward orbit stays in $\Sigma_{\mathcal{J}}$ lies in $W^s(\Lambda_{\mathcal{J}})$. This property holds for any C^1 -close system since the hyperbolic set persists under C^1 -small perturbations.

Remark 4.9 For every point $M \in \Lambda_{\mathcal{J}}$, we can define its local stable manifold as a connected piece of $W^s(\Lambda_{\mathcal{J}}) \cap \Pi$ through M , and the local unstable manifold as a connected piece of $W^u(\Lambda_{\mathcal{J}}) \cap \Pi$ through M . If $M \in \sigma_{k,m}$, then $W^s_{loc}(M) \subset \sigma_{k,m}$. Moreover, since the tangent to $W^s_{loc}(M)$ at any point lies in the stable cone \mathcal{C}^{cs} , it follows that $W^s_{loc}(M)$ is a “horizontal disc” of the form $Y = \xi^s_M(X, Z)$, where the smooth function ξ^s_M is defined for all $X \in [-\delta, \delta], Z \in [-\delta, \delta]^{d-d_1-1}$. Similarly, $W^u_{loc}(M)$ is a “vertical disc” of the form $(X, Z) = \xi^u_M(Y)$, where ξ^u_M is defined for all $Y \in [-\delta, \delta]^{d_1}$. In particular, it follows from (3.23) that for each M there exist transverse intersections

$$W^u_{loc}(M) \cap W^s_{loc}(O_1) \neq \emptyset \quad \text{and} \quad W^s_{loc}(M) \cap F_{21}(W^u_{loc}(O_2)) \neq \emptyset.$$

Now let us find a subset \mathcal{J} such that the corresponding hyperbolic set $\Lambda_{\mathcal{J}}$ is a cs-blender. We will define a cube $\Pi' \subset \Pi$ and take discs which cross Π' properly with respect to the cone field \mathcal{C}^{ss} (see Definition 4.1) given by Lemma 3.1 (with an appropriate K in its definition to be determined). We will show that any such disc S intersects the unstable manifold $W^u(\Lambda_{\mathcal{J}})$ (and this property persists at C^1 -small perturbations of the system).

Recall that by Remark 4.8, any point whose entire backward orbit by $T_{k,m}$ stays in $\Sigma_{\mathcal{J}}$ belongs to $W^u(\Lambda_{\mathcal{J}})$. Therefore, we will have the desired property that $S \cap W^u(\Lambda_{\mathcal{J}}) \neq \emptyset$ by showing below that any disc S crossing Π' properly with respect to \mathcal{C}^{ss} contains a points whose entire backward orbit stays in $\Sigma_{\mathcal{J}}$.

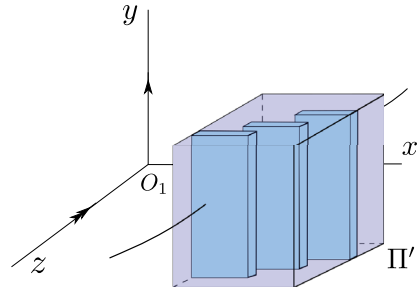
Take any positive $q < (1 - |\alpha|)/2$, and define

$$\delta' = q\delta. \tag{4.7}$$

Lemma 4.10 *There exists a finite subset $\mathcal{J}_{\delta'} \subset \mathcal{P}_N$ such that, if a $(d - d_1 - 1)$ -dimensional disc S crosses*

$$\Pi' := \Pi \cap \{-\delta' \leq X \leq \delta'\} \tag{4.8}$$

Fig. 9 The blue regions depict intersections $\Pi' \cap T_{k,m}(\Pi')$ with $(k, m) \in \mathcal{J}_{\delta'}$. For any disc (or curve in this three dimensional figure) crossing Π' properly, it must intersect at least one of the three blue regions. Hence its preimage contains a piece crossing Π' (Color figure online)



properly with respect to the cone field \mathcal{C}^{ss} defined in (4.1), then one can find a pair $(k, m) \in \mathcal{J}_{\delta'}$ such that the preimage $T_{k,m}^{-1}(S)$ contains a disc crossing Π' properly with respect to \mathcal{C}^{ss} .

The proof of the lemma is postponed until the end of this section. See Fig. 9 for an illustration. The lemma shows that given any properly crossing disc S , there is a sequence of discs defined by $S_{i+1} = T_{k_i, m_i}^{-1}(S_i) \cap \Pi'$ with $S_0 := S$ and some sequence $(k_i, m_i) \in \mathcal{J}_{\delta'}$. By construction, this gives a sequence of nested closed sets

$$\hat{S}_i := T_{k_0, m_0} \circ \dots \circ T_{k_{i-1}, m_{i-1}}(S_i) \subset S.$$

The intersection of all \hat{S}_i contains a point M whose backward orbit stays in $\Sigma_{\mathcal{J}_{\delta'}}$ defined by (4.6) (in fact, this point is unique due to the contraction in Z -directions.) Hence, we have $M \in W^u(\Lambda_{\mathcal{J}_{\delta'}})$, where $\Lambda_{\mathcal{J}_{\delta'}}$ is the hyperbolic set given by Lemma 4.7 associated with the set $\mathcal{J}_{\delta'}$. Obviously, Lemma 4.10 holds for any C^1 -close system for the same set $\mathcal{J}_{\delta'}$. Therefore, we obtain a cs-blender of Definition 1.9.

Since the cone field \mathcal{C}^{ss} is backward-invariant by Lemma 3.1, the preimage of any disc proper with respect to \mathcal{C}^{ss} is still proper. Hence, in order to prove Lemma 4.10 we only need to show that any properly crossing disc has a preimage which crosses Π' . The key procedure in controlling the preimage of the disc S is to show that the X -coordinates of its points always lie in the domain $\sigma_{k,m}$ of some return map $T_{k,m}$ (the so-called “covering property”, as described in the definition of a standard blender in the Appendix, see more discussion in [7, 17, 48]). This amounts to prove the Lemma 4.11, which we formulate below.

Denote the length of an interval E by $|E|$, and the affine part of the X -equation in (3.26) by

$$R_{k,m}(X) := ab\lambda^k \gamma^m x^+ - bu^- + ab\lambda^k \gamma^m X. \tag{4.9}$$

Recall that δ is the size of the domain of the return maps $T_{k,m}$ as in (4.5), and δ' is the size of the smaller cube $\Pi' \subset \Pi$, which depends on δ as given by (4.7).

Lemma 4.11 *Let $|\alpha| < 1$. There exists, for all sufficiently large N and sufficiently small δ , a finite subset $\mathcal{J}_{\delta'} = \{(k_j, m_j)\}_{j=1}^n \subset \mathcal{P}_N$ such that the intervals $E_j :=$*

$R_{k_j, m_j}([-\delta', \delta'])$ satisfy

$$\bigcup_{j=1}^n \text{int}(E_j) \supset [-\delta', \delta'] \tag{4.10}$$

and

$$|E_j \cap E_{j+1}| > \frac{\delta'|\alpha|}{2}, \quad j = 1, \dots, n - 1. \tag{4.11}$$

Proof One notes that since θ is irrational, the set of numbers $k\theta - m$ where k and m are even positive integers is dense in \mathbb{R} . Thus, given any $\rho \in \mathbb{R}$, we can find a sequence $\{(k_i(\rho), m_i(\rho))\}_{i \in \mathbb{N}}$ of pairs of positive even integers such that $k_i(\rho), m_i(\rho) \rightarrow \infty$ and

$$k_i(\rho)\theta - m_i(\rho) \rightarrow -\ln\left(\frac{bu^- + \delta\rho}{abx^+}\right) \ln^{-1} |\gamma|$$

as $i \rightarrow \infty$ (the logarithm is defined for small δ since $ax^+u^- > 0$ and $b \neq 0$ by our assumptions). Since k_i and m_i are even, we have $|\lambda|^{k_i} = \lambda^{k_i}$ and $|\gamma|^{m_i} = \gamma^{m_i}$ even when λ or γ are negative, so we obtain that

$$\frac{1}{\delta}(ab\lambda^{k_i(\rho)}\gamma^{m_i(\rho)}x^+ - bu^-) \rightarrow \rho \tag{4.12}$$

as $i \rightarrow \infty$. It follows from (4.3) that for any

$$\rho \in \left(-\frac{2}{3}(1 - |\alpha|), \frac{2}{3}(1 - |\alpha|)\right) =: I_{hyp}, \tag{4.13}$$

we have

$$\mathcal{P}_N(\rho) := \{(k_i(\rho), m_i(\rho)) : k_i(\rho), m_i(\rho) > N\} \subset \mathcal{P}_N. \tag{4.14}$$

In what follows we construct $\mathcal{J}_{\delta'}$ by taking pairs (k, m) from $\mathcal{P}_N(\rho)$ for a finite set of values of ρ .

Recall $\alpha = bu^-/x^+$. Note from (4.9) and (4.12) that, for any $(k_i(\rho), m_i(\rho)) \in \mathcal{P}_N(\rho)$, we have

$$R_{k_i(\rho), m_i(\rho)}(X) = \rho\delta + \left(\alpha + \frac{\rho\delta}{x^+}\right)X + o(1)_{i \rightarrow \infty}.$$

Consequently, the end points of $E_{k_i(\rho), m_i(\rho)} = R_{k_i(\rho), m_i(\rho)}([-\delta', \delta])$ are

$$a_i(\rho) = \rho\delta - \delta' \left(\alpha + \frac{\rho\delta}{x^+}\right) + o(1)_{i \rightarrow \infty} \quad \text{and}$$

$$b_i(\rho) = \rho\delta + \delta' \left(\alpha + \frac{\rho\delta}{x^+}\right) + o(1)_{i \rightarrow \infty}.$$

The length of each $E_{k_i(\rho), m_i(\rho)}$ satisfies

$$|E_{k_i(\rho), m_i(\rho)}| = |b_i - a_i| = 2\delta' \left| \alpha + \frac{\rho\delta}{x^+} \right| + o(1)_{i \rightarrow \infty} > \delta'|\alpha| \tag{4.15}$$

for all sufficiently large i and sufficiently small δ .

We can now construct the desired set $\mathcal{J}_{\delta'}$ of pairs. Assume $\alpha > 0$ for certainty. The case of negative α is completely parallel.

Let n be the smallest integer not less than $4/|\alpha| + 1$. Take

$$\rho_j = -\frac{\delta'}{\delta} + \frac{(j-1)2\delta'}{(n-1)\delta} \quad j = 1, \dots, n.$$

The length of the intersections between two consecutive intervals $E_{k_i(\rho_j), m_i(\rho_j)}$ and $E_{k_i(\rho_{j+1}), m_i(\rho_{j+1})}$ are computed as

$$\begin{aligned} b_i(\rho_j) - a_i(\rho_{j+1}) &= (\rho_j - \rho_{j+1})\delta + \delta' \left(2\alpha + \frac{(\rho_j + \rho_{j+1})\delta}{x^+} \right) + o(1)_{i \rightarrow \infty} \\ &= -\frac{2\delta'}{n-1} + \delta'(2\alpha + O(\delta')) + c_i, \end{aligned}$$

where $c_i = o(1)_{i \rightarrow \infty}$. Let us fix a large $i = i^*$ such that for all $j = 1, \dots, n-1$ and for all sufficiently small δ (hence sufficiently small δ' by (4.7)), the last two terms in the above equation satisfy $\delta'(2\alpha + O(\delta')) + c_{i^*} > \delta'\alpha$. By the choice of n , this implies

$$b_{i^*}(\rho_j) - a_{i^*}(\rho_{j+1}) > \frac{\delta'\alpha}{2}.$$

Since by construction one has $-\delta' \in \text{int}(E_{k_{i^*}(\rho_1), m_{i^*}(\rho_1)})$ and $-\delta' \in \text{int}(E_{k_{i^*}(\rho_n), m_{i^*}(\rho_n)})$, it follows that $\mathcal{J}_{\delta'} = \{(k_j, m_j) := (k_{i^*}(\rho_j), m_{i^*}(\rho_j))\}_{j=1}^n$. \square

Proof of Lemma 4.10 As we explained before Lemma 4.11, it suffices to show that for any proper disc S there exists a pair $(k, m) \in \mathcal{J}_{\delta'}$ such that $T_{k,m}^{-1}(S)$ crosses Π' . As S crosses Π' , it is a graph of a smooth function $s = (s_X, s_Y) : [-\delta, \delta]^{d-d_1-1} \rightarrow [-\delta', \delta'] \times [-\delta, \delta]^{d_1}$. We need to find some $(k, m) \in \mathcal{J}_{\delta'}$ such that, for any $Z \in [-\delta, \delta]^{d-d_1-1}$, there exist $X \in [-\delta', \delta']$, $Y \in [-\delta, \delta]^{d_1}$ and $\bar{Z} \in [-\delta, \delta]^{d-d_1-1}$ which satisfy $(X, Y, Z) = T_{k,m}^{-1}(s_X(\bar{Z}), s_Y(\bar{Z}), \bar{Z})$. By formulas (3.26) and (4.9), this is equivalent to solving the system of equations

$$s_X(\bar{Z}) = R_{k,m}(X) + \phi_1(X, s_Y(\bar{Z}), Z), \tag{4.16}$$

$$Y = \phi_2(X, s_Y(\bar{Z}), Z), \tag{4.17}$$

$$\bar{Z} = \phi_3(X, s_Y(\bar{Z}), Z). \tag{4.18}$$

According to the estimates (3.27), Y lies in $[-\delta, \delta]^{d_1}$ for all sufficiently large k as required; also, \bar{Z} can be expressed from the last equation as a function of (X, Z) :

$$\bar{Z} = \tilde{\phi}_3(X, Z)$$

where $\|\tilde{\phi}_3\|_{C^1} = o(\hat{\lambda}^m)$. Hence, to solve the above system it suffices to find $X \in [-\delta', \delta']$ satisfying

$$R_{k,m}(X) - s_X(\tilde{\phi}_3(X, Z)) + \tilde{\phi}_1(X, Z) = 0, \tag{4.19}$$

where

$$\tilde{\phi}_1(X, Z) = \phi_1(X, s_Y(\tilde{\phi}_3(X, Z))), Z) = O(\delta^2) + o(1)_{k,m \rightarrow \infty}. \tag{4.20}$$

We claim that one can choose $(k, m) \in \mathcal{J}_{\delta'}$ such that for any fixed $Z \in [-\delta, \delta]^{d-d_1-1}$, the left-hand side of (4.19) takes both positive and negative values when X runs over $[-\delta', \delta']$. The lemma then follows by the intermediate value theorem.

Let us prove this claim. On one hand, the proper crossing with respect to \mathcal{C}^{ss} means that the range of s_X lies in $[-\delta', \delta']$, and, according to (4.1), the total change of s_X is bounded by

$$\max_{Z_1, Z_2 \in [-\delta, \delta]^{d-d_1-1}} \{|s_X(Z_1) - s_X(Z_2)|\} < \frac{\delta'|\alpha|}{4}.$$

Since $\tilde{\phi}_3([-\delta', \delta'], Z) \subset [-\delta, \delta]^{d-d_1-1}$ for large m , we obtain that $s_X(\tilde{\phi}_3([-\delta', \delta'], Z))$ lies in some interval in $[-\delta', \delta']$ of length less than $\delta'|\alpha|/4$.

On the other hand, Lemma 4.11 shows that the interiors of the intervals $R_{k,m}([-\delta', \delta'])$ with $(k, m) \in \mathcal{J}_{\delta'}$ cover the interval $[-\delta', \delta']$ with overlaps larger than $\delta'|\alpha|/2$. It follows that there exist pairs $(k, m) \in \mathcal{J}_{\delta'}$ with arbitrarily large k, m such that the $\delta'|\alpha|/8$ -neighbourhood of $s_X(\tilde{\phi}_3([-\delta', \delta'], Z))$ lies in $\text{int}(R_{k,m}([-\delta', \delta']))$ for all $Z \in [-\delta, \delta]^{d-d_1-1}$. Thus, the difference $R_{k,m}(X) - s_X(\tilde{\phi}_3(X, Z))$ changes its sign when X runs the interval $[-\delta', \delta']$. In fact, there exists $C > 0$ such that $R_{k,m} - s_X$ runs from $C\delta'$ to $-C\delta'$. Therefore, by (4.20), the claim is proved when δ is sufficiently small and k, m are sufficiently large. \square

The preceding lemma gives us a cs-blender of index d_1 with an activating pair (Π', \mathcal{C}^{ss}) , and this blender can be taken arbitrarily close to the heterodimensional cycle by taking N of \mathcal{P}_N sufficiently large. Proposition 4.4 follows immediately.

4.3 Finding cu-blenders when $|\alpha| > 1$. Proof of Proposition 4.5

As mentioned before, instead of using the symmetry of the problem to obtain cu-blenders in the $|\alpha| > 1$ case, we make computations directly for the maps $T_{k,m}^{-1}$ in order to get an activating pair with the same activating domain Π' defined in (4.8).

Proof of Proposition 4.5 Inverting the first equation in (3.26), we obtain the following formula for the map $T_{k,m}^{-1}$:

$$\begin{aligned} X &= (\alpha^{-1} + O(\delta))\bar{X} + (a\lambda^k\gamma^m)^{-1}u^- - x^+ + \psi_1(\bar{X}, \bar{Y}, Z), \\ Y &= \psi_2(\bar{X}, \bar{Y}, Z), \\ \bar{Z} &= \psi_3(\bar{X}, \bar{Y}, Z), \end{aligned} \tag{4.21}$$

where

$$\begin{aligned} \psi_1 &= O(\delta^2) + o(1)_{k,m \rightarrow \infty}, & \frac{\partial \psi_1}{\partial (X, \bar{Y}, Z)} &= O(\delta) + o(1)_{k,m \rightarrow \infty}, \\ \|\psi_2\|_{C^1} &= o(\hat{\gamma}^{-k}), & \|\psi_3\|_{C^1} &= o(\hat{\lambda}^m). \end{aligned} \tag{4.22}$$

Observe that this formula has the same form as (3.26), with the replacement of α by α^{-1} and the term $(ab\lambda^k\gamma^m x^+ - bu^-)$ by the term $((a\lambda^k\gamma^m)^{-1}u^- - x^+)$. So, since $|\alpha^{-1}| < 1$, we obtain the result by repeating the same arguments we used for the case $|\alpha| < 1$ – we only need to replace the cone field C^{ss} by C^{uu} , as we work with the inverse map $T_{k,m}^{-1}$. \square

4.4 A criterion for the existence of blenders

Note that in proving Propositions 4.4 and 4.5, we did not use the full strength of the estimates in (3.27). We essentially used the fact that the functions ϕ along with their first derivatives go to 0 as $\delta \rightarrow 0$ and $k, m \rightarrow \infty$. Additionally, we used the fact that $\phi_1 = o(\delta)$ (hence $\tilde{\phi}_1 = o(\delta)$ in (4.20)) in the last line in the proof of Lemma 4.10. With the above observation, we finish this section by the following summary which is used later for the saddle-focus and double-focus cases.

Take small $\delta > 0$ and let $\Pi = [-\delta, \delta] \times [-\delta, \delta]^{d_Y} \times [-\delta, \delta]^{d_Z}$ be a cube in \mathbb{R}^d for some $d_Y, d_Z \in \mathbb{N}$ satisfying $1 + d_Y + d_Z = d$. Let $\{T_n\}_{n=1}^\infty$ be a sequence of maps whose domains of definition σ_n lie in Π and which satisfy

$$\sigma_i \cap \sigma_j = \emptyset \quad \text{and} \quad T_i(\sigma_i) \cap T_j(\sigma_j) \cap \Pi = \emptyset, \quad i \neq j.$$

Consider the map $T : \bigcup_n \sigma_n \rightarrow \mathbb{R}^d$ defined by $T(X, Y, Z) = T_n(X, Y, Z)$ if $(X, Y, Z) \in \sigma_n$.

Proposition 4.12 *Suppose for every n and every $(X, Y, Z) \in \Pi$ we have $(\bar{X}, \bar{Y}, \bar{Z}) = T_n(X, Y, Z) \in \Pi$ if and only if*

$$\begin{aligned} \bar{X} &= A_n X + B_n + \phi_1(X, \bar{Y}, Z; n), \\ Y &= \phi_2(X, \bar{Y}, Z; n), \\ \bar{Z} &= \phi_3(X, \bar{Y}, Z; n), \end{aligned} \tag{4.23}$$

where the coefficients A_n and B_n and the functions ϕ defined on Π may depend on δ . Let $\|\phi_{2,3}\|_{C^1} = o(1)_{\delta \rightarrow 0} + o(1)_{n \rightarrow \infty}$, whereas $\phi_1 = o(\delta) + o(1)_{n \rightarrow \infty}$ and the derivative of ϕ_1 is $o(1)_{\delta \rightarrow 0} + o(1)_{n \rightarrow \infty}$.

Assume that there exists a neighborhood Δ of zero independent of δ such that the set $\{B_n(\delta)\}$ is dense in Δ for each fixed small δ . Assume also that the set $\{A_n(\delta)\}$ stays bounded away from 0, ± 1 , and $\pm \infty$ for all sufficiently small δ . Then, there exists $q \in (0, 1)$ such that, for all sufficiently small δ , the map T in the cube $\Pi' = [-q\delta, q\delta] \times [-\delta, \delta]^{d_Y} \times [-\delta, \delta]^{d_Z} \subset \Pi$ has

- a cs-blender of index d_Y if $|A_n| < 1$ for all n , or
- a cu-blender of index $(d_Y + 1)$ if $|A_n| > 1$ for all n .

The blender has an activating pair (Π', C) , where C is a field of cones around Z -coordinates if the blender is center-stable, or around Y -coordinates if it is center-unstable. The cone field C can be chosen independent of δ .

Note that in this paper, the maps T_n are the first-return maps $T_{k,m}$. So, they are either iterations of a diffeomorphism or flow maps, depending on whether the system is discrete-time or continuous-time. The original system has a blender equal to the orbit of the blender of T . Like it is shown in [Appendix](#) for the blender of Proposition 4.4, the blenders of Proposition 4.12 are standard in the sense of Definition A.1.

5 Local stabilization of heterodimensional cycles in the saddle case

In this section we prove Theorems 2–5 and Corollary 2. We conduct the proofs only for the $|\alpha| < 1$ case – the $|\alpha| > 1$ case is dealt with by using the symmetry argument (i.e., by considering the system obtained by the reversion of time) except for Theorem 2, where Proposition 4.5 is additionally used. Depending on the situations, we embed f into one- or two-parameter families that generically and properly unfold the heterodimensional cycle of f . In the remaining part of this paper, we denote the continuations of hyperbolic objects (e.g. $O_{1,2}$ and Λ^{cs}) after a small perturbation by the same letters and omit the term ‘continuation’.

5.1 Unfolding type-I cycles. Proof of Theorem 2

Recall that δ' is the size of the subcube $\Pi' \subset \Pi$ and it is related to the size δ of Π via (4.7). Let us define two sequences of intervals

$$I_m^u = \left(\gamma^{-m} u^- - \frac{1}{2} |b^{-1} \gamma^{-m} \delta', \gamma^{-m} u^- + \frac{1}{2} |b^{-1} \gamma^{-m} \delta' \right), \quad (5.1)$$

and

$$I_k^s = \left(-a \lambda^k x^+ - \frac{1}{2} |a \lambda^k \delta', -a \lambda^k x^+ + \frac{1}{2} |a \lambda^k \delta' \right). \quad (5.2)$$

The goal of this section is to prove the following general result for all saddle heterodimensional cycles, which not only implies Theorem 2 but also will be used in the proof of Theorem 4 in Sect. 5.3.

Proposition 5.1 *Let the system f have a non-degenerate saddle heterodimensional cycle (of type I or II).*

- Suppose f has an index- d_1 cs-blender Λ^{cs} with the activating pair (Π', C^{ss}) defined in (4.8) and (4.1). Moreover, let the properties described in Remark 4.9 hold for Λ^{cs} , namely, $W_{loc}^s(\Lambda^{cs})$ intersects transversely any vertical disc that crosses Π properly with respect to C^{uu} , and $W_{loc}^u(\Lambda^{cs})$ intersects transversely $W^s(O_1)$. Then,
 - when $\hat{u} \in I_m^u$, the index- d_1 saddle O_1 is homoclinically related to Λ^{cs} , and

- when $\hat{\mu} \in I_k^s$, there exist robust heterodimensional dynamics involving Λ^{cs} and a non-trivial hyperbolic basic set containing O_2 .
- Suppose f has an index- d_2 cu-blender Λ^{cu} with the activating pair (Π', \mathcal{C}^{uu}) , where \mathcal{C}^{uu} is defined in (4.2). Moreover, let $W_{loc}^u(\Lambda^{cu})$ intersect transversely any horizontal disc that crosses Π properly with respect to \mathcal{C}^{ss} , and $W_{loc}^s(\Lambda^{cu})$ intersect transversely $W^u(O_2)$. Then,
 - when $\hat{\mu} \in I_k^s$, the saddle O_2 is homoclinically related to Λ^{cu} , and
 - when $\hat{\mu} \in I_m^u$, there exist robust heterodimensional dynamics involving Λ^{cu} and a non-trivial hyperbolic basic set containing O_1 .

For type-I cycles, Propositions 4.4 and 4.5 guarantee the existence of a cs-blender Λ^{cs} when $|\alpha| < 1$, and a cu-blender Λ^{cu} when $|\alpha| > 1$. Thus, Proposition 5.1 gives

Proposition 5.2 *Let the cycle Γ have type I and let θ be irrational.*

- Let $|\alpha| < 1$. Then the blender Λ^{cs} has the same index as O_1 and, when $\hat{\mu} \in I_m^u$, the saddle O_1 is homoclinically related to Λ^{cs} . If $\hat{\mu} \in I_k^s$, then there exists a persistent non-transverse heteroclinic connection between $W^u(\Lambda^{cs})$ and $W^s(O_2)$ and a transverse heteroclinic intersection of $W^u(O_2)$ and $W^s(\Lambda^{cs})$, i.e., there exist robust heterodimensional dynamics involving Λ^{cs} and a non-trivial hyperbolic basic set containing O_2 .
- Let $|\alpha| > 1$. Then the blender Λ^{cu} has the same index as O_2 and, when $\hat{\mu} \in I_k^s$, the saddle O_2 is homoclinically related to Λ^{cu} . If $\hat{\mu} \in I_m^u$, then there exists a persistent non-transverse heteroclinic connection between $W^u(O_1)$ and $W^s(\Lambda^{cu})$ and a transverse heteroclinic intersection of $W^u(\Lambda^{cu})$ and $W^s(O_1)$, i.e., there exist robust heterodimensional dynamics involving Λ^{cu} and a non-trivial hyperbolic basic set containing O_1 .

This implies Theorem 2 as follows.

Proof of Theorem 2 Recall that $\hat{\mu} \sim \mu$ by (3.11), and all coefficients in formulas (5.1) and (5.2) for the intervals I_m^u and I_k^s depend continuously on the system. Thus, there exists $\kappa > 0$ such that for any system g in a small neighborhood of f , we have $\hat{\mu} \in I_m^u$ if

$$\mu(g)\gamma(g)^m \in [u_0^- - \frac{1}{2}b_0^{-1}\delta' + \kappa, u_0^- + \frac{1}{2}b_0^{-1}\delta' - \kappa]$$

and $\hat{\mu} \in I_m^u$ if

$$\mu(g)\lambda(g)^k \in [-a_0x_0^+ - \frac{1}{2}a_0\delta' + \kappa, -a_0x_0^+ + \frac{1}{2}a_0\delta' - \kappa],$$

where a_0, b_0, x_0^+ , and u_0^- are the values of the corresponding coefficients for f . So, Proposition 5.2 implies Theorem 2 immediately. □

Let us now prove Proposition 5.1. The first step is to investigate the iterations of $W_{loc}^u(O_1)$ and $W_{loc}^s(O_2)$. Whether their iterates intersect the region ‘reserved’ for the emergence of blenders is crucial for the stabilization of heterodimensional cycles.

Lemma 5.3 *For all sufficiently large k and m , and all sufficiently small δ (and hence small δ' by (4.7)), the follow results hold:*

- When $\hat{\mu} \in I_m^u$, the image $S_m^u := F_{21} \circ F_2 \circ F_{12}(W_1^u)$ of

$$W_1^u := W_{loc}^u(O_1) \cap \{\|y - y^-\| \leq \delta\}$$

is a ‘vertical’ disc of the form $(X, Z) = s(Y)$ for some smooth function s . The disc S_m^u crosses the cube Π' (defined in (4.8)) properly with respect to the cone field C^{uu} defined in (4.2). In particular, S_m^u intersects $W_{loc}^s(O_1) : \{Y = 0\}$ transversely, i.e., O_1 has a transverse homoclinic orbit when $\hat{\mu} \in I_m^u$, and hence is contained in a non-trivial hyperbolic basic set.

- When $\hat{\mu} \in I_k^s$, the preimage $S_k^s := F_1^{-k} \circ F_{12}^{-1}(W_2^s)$ of

$$W_2^s := W_{loc}^s(O_2) \cap \{\|v - v^+\| \leq \delta\}$$

is a ‘horizontal’ disc of the form $(X, Y) = s(Z)$ for some smooth function s . The disc S_k^s crosses Π' properly with respect to the cone field C^{ss} defined in (4.1). In particular, S_k^s intersects $F_{21}(W_{loc}^u(O_2)) : \{Z = 0\}$ transversely (see (3.23)), i.e., O_2 has a transverse homoclinic orbit when $\hat{\mu} \in I_k^s$, and hence is contained in a non-trivial hyperbolic basic set.

Proof (The first statement.) By formulas (3.10) and (3.17), for any $(0, y, 0) \in W_1^u$, we have $F_{21} \circ F_2^m \circ F_{12}(0, y, 0) = (\bar{x}, \bar{y}, \bar{z})$ if and only if

$$\begin{aligned} \bar{x} - x^+ &= b\gamma^m \hat{\mu} - bu^- + b_{13}\bar{y} + O((b\gamma^m \hat{\mu} - bu^-)^2 + \bar{y}^2) + o(1)_{m \rightarrow \infty}, \\ \bar{z} - z^+ &= \hat{b}_{31}\gamma^m \hat{\mu} - b_{31}u^- + b_{33}\bar{y} + O((b\gamma^m \hat{\mu} - bu^-)^2 + \bar{y}^2) + o(1)_{m \rightarrow \infty}, \end{aligned}$$

which after the coordinate transformation (3.19) recasts as

$$\begin{aligned} \bar{X} &= b\gamma^m \hat{\mu} - bu^- + O((b\gamma^m \hat{\mu} - bu^-)^2 + \bar{Y}^2) + o(1)_{m \rightarrow \infty}, \\ \bar{Z} &= O((b\gamma^m \hat{\mu} - bu^-)^2 + \bar{Y}^2) + o(1)_{m \rightarrow \infty}. \end{aligned} \tag{5.3}$$

By (5.1) we have

$$|b\gamma^m \hat{\mu} - bu^-| < \frac{\delta'}{2},$$

which for $\|\bar{Y}\| \leq \delta$ implies

$$|\bar{X}| < \frac{\delta'}{2} + O(\delta^2) + o(1)_{m \rightarrow \infty} < \delta' \quad \text{and} \quad \|\bar{Z}\| = O(\delta^2) + o(1)_{m \rightarrow \infty} < \delta.$$

This means that S_m^u crosses Π' . One also finds from (5.3) that

$$\frac{\partial(\bar{X}, \bar{Z})}{\partial \bar{Y}} = O(\delta) + o(1)_{m \rightarrow \infty},$$

which can be made sufficiently small so that the tangent spaces of $S_m^u \cap \Pi$ lie in C^{uu} . So, the crossing is also proper with respect to C^{uu} . This completes the proof of the first statement.

(The second statement.) According to (3.10), the preimage $F_{12}^{-1}(W_2^s)$ is given by

$$\begin{aligned} \tilde{x} &= -a^{-1}\hat{\mu} - a_{13}a_{11}^{-1}\tilde{z} + O(\tilde{z}^2 + \hat{\mu}^2), \\ \tilde{y} &= y^- - a_{31}a^{-1}\hat{\mu} + (a_{33} - a_{13}a_{31}a^{-1})\tilde{z} + O(\tilde{z}^2 + \hat{\mu}^2), \end{aligned}$$

By (3.5), we find in coordinates (3.19) the disc $F_1^{-k} \circ F_{12}^{-1}(W_2^s) \cap \Pi$ as

$$\begin{aligned} X &= -a^{-1}\lambda^{-k}(\hat{\mu} + O(\hat{\mu}^2)) - x^+ + o(1)_{k \rightarrow \infty}, \\ Y &= o(\hat{\gamma}^{-k}), \end{aligned} \tag{5.4}$$

where $o(\cdot)$ and $O(\cdot)$ terms are functions of $(\hat{\mu}, Z)$, and the first derivatives satisfy

$$\frac{\partial(X, Y)}{\partial Z} = o(1)_{k \rightarrow \infty}. \tag{5.5}$$

This disc crosses Π' if $|X| < \delta'$ and $\|Y\| < \delta$. One readily finds from (5.4) that this happens when k is sufficiently large and

$$|-a^{-1}\lambda^{-k}\hat{\mu} - x^+| < \frac{\delta'}{2},$$

which gives the intervals (5.2). By (5.5), the crossing is proper with respect to C^{ss} for large k . □

Proof of Proposition 5.1 We only prove for the cs-blender Λ^{cs} . In the case of the cu-blender, the arguments are completely analogous. By assumption, the local stable

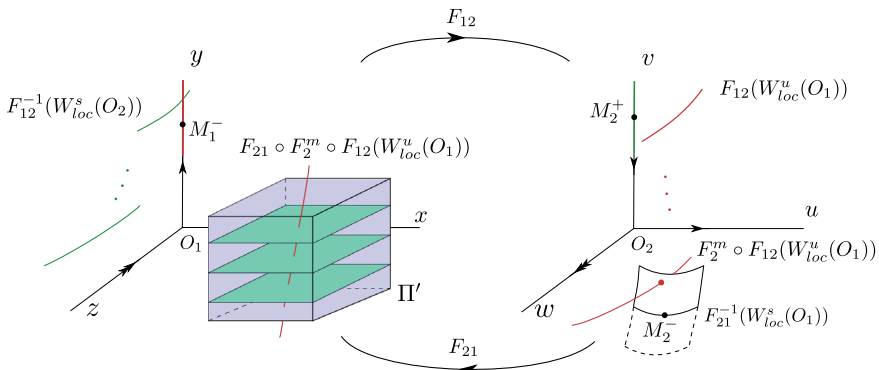


Fig. 10 For a cs-blender Λ^{cs} satisfying the properties in Remark 4.9, there exist positive $\hat{\mu}$ values such that some forward iterate of $W_{loc}^u(O_1)$ intersects $W^s(\Lambda^{cs})$ while the backward iterates of $W_{loc}^s(O_2)$ leave the neighborhood of the cycle. Here the green planes in Π' represent the local stable manifolds of points in the blender Λ^{cs} (Color figure online)

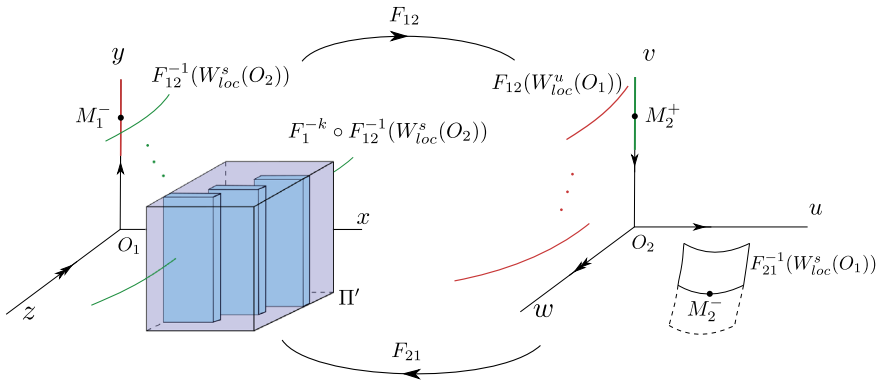


Fig. 11 For a cs-blender Λ^{cs} satisfying the properties in Remark 4.9, there exist negative $\hat{\mu}$ values such that some backward iterate of $W_{loc}^s(O_2)$ intersects $W^u(\Lambda^{cs})$ and the forward iterates of $W_{loc}^u(O_1)$ leave the neighborhood of the cycle (Color figure online)

manifold of Λ^{cs} intersects transversely (for $\mu = 0$, hence for all small μ ; see Fig. 10) any vertical disc that crosses Π properly with respect to C^{uu} , so it intersects transversely $F_{21} \circ F_2^m \circ F_{12}(W_{loc}^u(O_1))$ when $\hat{\mu} \in I_m^u$ (by the first item of Lemma 5.3). Our assumption on Λ^{cs} also gives us the existence of a transverse intersection of $W_{loc}^u(\Lambda^{cs})$ and $W_{loc}^s(O_1)$, which proves that O_1 and Λ^{cs} are homoclinically related for $\hat{\mu} \in I_m^u$.

Now, let $\hat{\mu} \in I_k^s$. With the formula for $F_{21}(W_{loc}^u(O_2))$ in (3.23), the existence of a non-trivial hyperbolic basic set containing O_2 is given by the second item of Lemma 5.3 (the existence of a transverse homoclinic to O_2). Also by the same result, we immediately find that O_2 activates Λ^{cs} , meaning the existence of a persistent non-transverse heteroclinic connection between $W^s(O_2)$ and $W^u(\Lambda^{cs})$ (see Fig. 11). It remains to note that the non-empty transverse intersection of $W^s(\Lambda^{cs})$ with $W^u(O_2)$ is given by the assumption on Λ^{cs} . □

5.2 Nonstabilizability of type-I cycles. Proof of Theorem 3

By the assumption that the cycle is of type-I, we have $\lambda > 0$, $\gamma > 0$ and $ax^+u^- > 0$ at $\mu = 0$, and hence at all sufficiently small μ . We further assume that $u^- > 0$ at $\mu = 0$, which means that $ax^+ > 0$ too. It is clear from the proof that when $u^- < 0$, all results remain true after reversing the sign of μ . As mentioned before, we only consider the $|\alpha| < 1$ case.

We first prove the second part of the statement describing the situation at $\mu \leq 0$. It suffices to show that, at $\mu \leq 0$, the image $F^m \circ F_{12}(W_{loc}^u(O_1))$ does not enter the domain of F_{21} , i.e., it is outside a small neighborhood of $M_2^- = (u^-, 0, w^-)$ for any sufficiently large m . Since $W_{loc}^u(O_1)$ is given by the equation $(x = 0, z = 0)$, this is equivalent (see (3.10) and (3.7)) to showing that the system of equations

$$\begin{aligned} u &= \hat{\mu} + a_{12}w + O(w^2), & u &= \gamma^{-m}\tilde{u} + o(\gamma^m), \\ v &= a_{22}w + O(w^2), & \tilde{v} &= o(\hat{\lambda}^m), \\ y - y^- &= a_{32}w + O(w^2), & w &= o(\gamma^m), \end{aligned}$$

does not have a solution $(\tilde{u}, \tilde{v}, \tilde{w})$ near $(u^-, 0, w^-)$ for small $(y - y^-)$ and w , for any sufficiently large m . A straightforward computation reduces the above system to

$$\tilde{u} = \gamma^m \hat{\mu} + o(1)_{m \rightarrow \infty}.$$

Since we have here $\gamma > 0$, the above equation together with the relation (3.11) implies that $\tilde{u} \leq 0$ for $\mu \leq 0$ and sufficiently large m . Thus, by the assumption $u^- > 0$, the point $(\tilde{u}, \tilde{v}, \tilde{w})$ never enters a small neighborhood of $(u^-, 0, w^-)$ indeed.

Now we switch to the case $\mu > 0$. First, note that like above, one can easily see (e.g. from equation (5.4) for $F^{-k} \circ F_{12}^{-1}(W_{loc}^s(O_2))$) that no point of $W_{loc}^s(O_2)$ has an orbit that lies entirely in U for $\mu > 0$. In particular, no heteroclinic orbits of intersection of $W^u(O_1)$ and $W^s(O_2)$ can lie entirely in U . Thus, given an orbit \mathcal{O} that lies entirely in U , we have two possibilities: either \mathcal{O} is one of the orbits $L_{1,2}$ of the points $O_{1,2}$, or it intersects the δ -neighborhood Π of the heteroclinic point M_1^+ in U_{01} . Let M_s be the consecutive points of intersection of \mathcal{O} with Π ; we have $M_{s+1} = T_{k_s, m_s} M_s$, where $T_{k, m}$ is the first-return map given by (3.20) and (k_s, m_s) are positive integers. The sequence M_s can be infinite in both directions, infinite in one direction, or finite; let us consider the case where the sequence $\{M_s\}_{s \in \mathbb{Z}}$ is infinite in both directions first.

Let us show that the set $\hat{\Lambda}$ comprised by such orbits \mathcal{O} is indeed a hyperbolic set with index- d_1 when $\mu > 0$ (equivalently, when $\hat{\mu} > 0$, see (3.11)). It follows from (3.20) and $M_{s+1} \in \Pi$ that

$$\gamma^{m_s} \hat{\mu} + ax^+ \lambda^{k_s} \gamma^{m_s} = u^- + O(\delta).$$

Recall that we assumed $\gamma > 0, \lambda > 0, ax^+ > 0, u^- > 0$, and $\hat{\mu} > 0$, so we obtain

$$0 < ax^+ \lambda^{k_s} \gamma^{m_s} < u^- + O(\delta). \tag{5.6}$$

In particular, the numbers (k_s, m_s) are always such that $\lambda^{k_s} \gamma^{m_s}$ is uniformly bounded. Therefore, by the same argument we used to deduce formula (3.26) from (3.20) at $\hat{\mu} = 0$, we obtain for our case the following formula for $T_{k_s, m_s} : (X, Y, Z) \mapsto (\bar{X}, \bar{Y}, \bar{Z})$ (where we use coordinates (3.19)):

$$\begin{aligned} \bar{X} &= b\gamma^{m_s} \hat{\mu} + ab\lambda^{k_s} \gamma^{m_s} x^+ - bu^- + ab\lambda^{k_s} \gamma^{m_s} X + \tilde{\phi}_1(X, \bar{Y}, Z), \\ Y &= \tilde{\phi}_2(X, \bar{Y}, Z), \quad \bar{Z} = \tilde{\phi}_3(X, \bar{Y}, Z), \end{aligned} \tag{5.7}$$

where

$$\begin{aligned} \tilde{\phi}_1 &= O(\delta^2) + o(1)_{k, m \rightarrow \infty}, & \frac{\partial \tilde{\phi}_1}{\partial (X, \bar{Y}, Z)} &= O(\delta) + o(1)_{k, m \rightarrow \infty}, \\ \|\tilde{\phi}_2\|_{C^1} &= o(\hat{\gamma}^{-k}), & \|\tilde{\phi}_3\|_{C^1} &= o(\hat{\lambda}^m). \end{aligned}$$

Multiplying $|b/x^+|$ to each term in (5.6) yields

$$|ab\lambda^{k_s} \gamma^{m_s}| < |\alpha| + O(\delta) < 1 \tag{5.8}$$

for all sufficiently small δ . So, we have exactly the same formula (3.32) for the derivative of T_{k_s, m_s} (one should only replace α by $ab\lambda^{k_s} \gamma^{m_s}$). Also note that by shrinking

the size of U , the values k_s, m_s can be made arbitrarily large. Now, we have the result of Lemma 3.1, which along with (5.8), gives us the existence of invariant cone fields \mathcal{C}^{cs} and \mathcal{C}^{uu} , which immediately implies that $\hat{\Lambda}$ is a hyperbolic set with index- d_1 (all maps T_{k_s, m_s} contract in (X, Z) and expand in Y).

The results analogous to Remarks 4.8 and 4.9 also hold. In particular, every forward orbit $\{M_s\}$ that intersects Π infinitely many times must belong to $W_{loc}^s(\hat{\Lambda} \cap \Pi)$, while every backward orbit that intersects Π infinitely many times must belong to $W_{loc}^u(\hat{\Lambda} \cap \Pi)$. Also, $W_{loc}^u(\hat{\Lambda} \cap \Pi)$ intersects $W_{loc}^s(O_1)$ transversely, while $W_{loc}^s(\hat{\Lambda} \cap \Pi)$ intersects $F_{21}(W_{loc}^u(O_2))$ transversely.

We can now go through various cases where the sequence M_s of consecutive intersections of \mathcal{O} with Π is not infinite in both directions.

1. $\{M_s\}_{s=-\infty}^0$ is infinite backwards. As we just explained, this may happen only when $M_0 \in W_{loc}^u(\tilde{\Lambda} \cap \Pi) \cap W_{loc}^s(O_1)$ or $M_0 \in W_{loc}^u(\tilde{\Lambda} \cap \Pi) \cap W^s(O_2)$. The latter case does not happen since no point of $W_{loc}^s(O_2)$ has an orbit entirely contained in U for $\mu > 0$, as we mentioned. In the former case, M_0 is a hyperbolic point of index d_1 (since O and $\hat{\Lambda}$ have the same index d_1 and the intersection of $W_{loc}^u(\hat{\Lambda} \cap \Pi)$ and $W_{loc}^s(O_1)$ is transverse). We include all such orbits into the set Λ of Theorem 3, along with the set $\tilde{\Lambda}$ and the orbit L_1 (and orbits in $W^u(L_1)$ which we consider below).
2. $\{M_s\}_{s=0}^\infty$ is infinite forwards. This happens when $M_0 \in W_{loc}^s(\tilde{\Lambda} \cap \Pi) \cap F_{21} \circ F_2^m \circ F_{12}(W_{loc}^u(O_1))$ for some large m or when $M_0 \in W_{loc}^s(\tilde{\Lambda} \cap \Pi) \cap F_{21}(W_{loc}^u(O_2))$. The latter case, as the intersection is transverse, is in the complete agreement with the statement of the Theorem. In the former case, M_0 is a hyperbolic point of index d_1 because the intersection of $W_{loc}^s(\tilde{\Lambda} \cap \Pi)$ and $F_{21} \circ F_2^m \circ F_{12}(W_{loc}^u(O_1))$ is transverse. Indeed, the disc $F_{21} \circ F_2^m \circ F_{12}(W_{loc}^u(O_1))$ is given by equation (5.3) where we have

$$b\gamma^m \hat{\mu} - bu^- = O(\delta)$$

because $M_0 \in \Pi$. From that, exactly like in Lemma 5.3, one obtains that the tangent spaces of $F_{21} \circ F_2 \circ F_{12}(W_{loc}^u(O_1))$ lie in \mathcal{C}^{uu} , hence the transversality with $W_{loc}^s(\tilde{\Lambda} \cap \Pi)$.

3. The sequence $\{M_s\}$ is finite. This happens when $M_0 \in W^s(O_1) \cap F_{21}(W_{loc}^u(O_2))$ or $M_0 \in W^s(O_1) \cap F_{21} \circ F_2 \circ F_{12}(W_{loc}^u(O_1))$. Since the tangents to $W_{loc}^s(O_1) : \{Y = 0\}$ lie in the backward-invariant cone field \mathcal{C}^{cs} , all its preimages by T_{k_s, m_s} also have tangents in \mathcal{C}^{cs} , hence the intersections of $W^s(O_1)$ with $F_{21}(W_{loc}^u(O_2))$ and $F_{21} \circ F_2 \circ F_{12}(W_{loc}^u(O_1))$ are transverse.

In all cases we have the agreement with the statement of the theorem. □

5.3 Unfolding a tied pair of type-I and -II cycles. Proof of Theorem 4

As can be seen from the proof of Theorem 3, the main reason that prevents the blender from having homoclinic connections to both O_1 and O_2 is that, when one saddle connects to the blender, the iterates of the local stable or unstable manifold of the other leave the small neighborhood U of Γ (see Figs. 10 and 11). In what follows, we

show that, if there exists a type-II cycle tied with Γ , then the leaving manifold can return to U by following the robust heteroclinic orbit of the type-II cycle.

Let $\tilde{\Gamma} = L_1 \cup L_2 \cup \Gamma^0 \cup \tilde{\Gamma}^1$ be a heterodimensional cycle sharing the fragile heteroclinic orbit Γ^0 with Γ . By definition, any generic one-parameter unfolding of Γ is also a generic one for $\tilde{\Gamma}$. These two cycles have the same transition map F_{12} , and the local maps F_1 and F_2 are the same. Denote by \tilde{F}_{21} the transition map from a neighborhood of $\tilde{M}_2^- = (\tilde{u}^-, 0, \tilde{w}^-) \in \tilde{\Gamma}^1$ to a neighborhood of $\tilde{M}_1^+ = (\tilde{x}^+, 0, \tilde{z}^+) \in \tilde{\Gamma}^1$. The map \tilde{F}_{21} has the same form as (3.13), just one needs to replace coefficients b_{ij} by some \tilde{b}_{ij} , b by $\tilde{b} \neq 0$ and also x^+, z^+, u^- by $\tilde{x}^+, \tilde{z}^+, \tilde{u}^-$.

Recall that the coefficients in (3.13) depend continuously on μ ; they are smooth in μ when the smoothness class of f_μ is at least C^3 . The eigenvalues λ and γ remain smooth in μ in the C^2 -case too.

Let us mark the values of the coefficients $a, \tilde{b}, \lambda, \gamma, \tilde{u}^-, \tilde{x}^+$ at $\mu = 0$ by the subscript “0”

Lemma 5.4 *Let $\tilde{\Gamma}$ be a non-degenerate type-II cycle (i.e., $a_0\tilde{x}_0^+\tilde{u}_0^- < 0$ and $|\tilde{b}_0\tilde{u}_0^-/\tilde{x}_0^+| \neq 1$). Consider a generic one-parameter unfolding $\{f_\mu\}$ of $\tilde{\Gamma}$. Assume θ_0 is irrational and take a sequence $\{(k_j, m_j)\}$ of pairs of positive even integers satisfying $k_j, m_j \rightarrow \infty$ and*

$$a_0\tilde{b}_0\lambda_0^{k_j}\gamma_0^{m_j} \rightarrow -\tilde{\alpha}_0 = -\frac{\tilde{b}_0\tilde{u}_0^-}{\tilde{x}_0^+} \tag{5.9}$$

as $j \rightarrow \infty$. There exists a sequence of values $\{\mu_j\}$ satisfying

$$\mu_j = -a_0\tilde{x}_0^+\lambda_0^{k_j} + o(\lambda_0^{k_j}) = \tilde{u}_0^-\gamma_0^{-m_j} + o(\gamma_0^{-m_j}), \tag{5.10}$$

such that the system f_{μ_j} has a new orbit $\Gamma_j^{0,new}$ of heteroclinic intersection of $W^u(O_1)$ with $W^s(O_2)$, and hence a new heterodimensional cycle $\Gamma_j^{new} = L_1 \cup L_2 \cup \Gamma_j^{0,new} \cup \Gamma^1$. The heteroclinic connection $\Gamma_j^{0,new}$ splits when μ varies in an $o(\mu_j)$ -interval around μ_j . It splits with non-zero velocity if f_μ is of class C^r with $r \geq 3$.

Remark 5.5 The newly created heterodimensional cycle Γ_j^{new} is non-degenerate. Indeed, one only needs to check that $\Gamma_j^{0,new}$ satisfies condition C1 (the other non-degeneracy conditions hold because of the non-degeneracy of the cycle Γ). Condition C1 holds for $\Gamma_j^{0,new}$ because Γ_j^{new} is a partially-hyperbolic set with a one-dimensional center direction. This is true because $cl(\Gamma_j^{0,new})$ lies in a small neighborhood of $\tilde{\Gamma}$ which is a compact partially-hyperbolic set with a one-dimensional center direction (by the non-degeneracy assumption), and the partial hyperbolicity of a compact invariant set is inherited by every closed invariant set in its neighborhood.

Proof of Lemma 5.4 We create the new (fragile) heteroclinic orbit $\Gamma_j^{0,new}$ by finding values of μ at which $W_{loc}^s(O_2)$ intersects $\tilde{W}_1^\mu := F_{12} \circ F_1^k \circ \tilde{F}_{21} \circ F_2^m \circ F_{12}(W_{loc}^u(O_1))$.

By putting $\tilde{x} = 0, \tilde{z} = 0$ in (3.10), we find that the image $F_{12}(W_{loc}^u(O_1))$ near the point M_2^+ is given by

$$u = \hat{\mu} + O(w), \quad v = v^+ + O(w). \tag{5.11}$$

Let us now find an equation for $\tilde{F}_{21} \circ F_2^m \circ F_{12}(W_{loc}^u(O_1))$. We use an analogue of formula (3.17) for the map $\tilde{F}_{21} \circ F_2^m$ from a small neighborhood of M_2^+ to a small neighborhood of \tilde{M}_1^+ , which now reads

$$\begin{aligned} u &= \gamma^{-m}(u^- + \tilde{b}^{-1}(x - \tilde{x}^+ - \tilde{b}_{13}y + \hat{h}_{01}(x - \tilde{x}^+, y))) + \hat{h}_1(x, v, y), \\ z - \tilde{z}^+ &= \tilde{b}_{31}\tilde{b}^{-1}(x - \tilde{x}^+ - \tilde{b}_{13}y) + \tilde{b}_{33}y + \hat{h}_{02}(x - \tilde{x}^+, y) + \hat{h}_2(x, v, y) \end{aligned}$$

with the modified functions \hat{h} satisfying the same estimates (3.16). By substituting equation (5.11) into this formula, we obtain that $\tilde{F}_{21} \circ F_2^m \circ F_{12}(W_{loc}^u(O_1))$ is given by

$$\begin{aligned} x &= \tilde{x}^+ + \tilde{b}(\gamma^m \hat{\mu} - \tilde{u}^-) + O(\|y\| + (\gamma^m \hat{\mu} - \tilde{u}^-)^2) + o(1)_{m \rightarrow \infty}, \\ z &= \tilde{z}^+ + O(\|y\| + |\gamma^m \hat{\mu} - \tilde{u}^-|) + o(1)_{m \rightarrow \infty}, \end{aligned}$$

where m must be such that $\gamma^m \hat{\mu}$ is sufficiently close to \tilde{u}^- .

Finally, substituting the above two equations into (3.15) yields the equation for $\bar{W}_1^u := F_{12} \circ F_1^k \circ \tilde{F}_{21} \circ F_2^m \circ F_{12}(W_{loc}^u(O_1))$:

$$\begin{aligned} u &= \hat{\mu} + a\tilde{b}\lambda^k \gamma^m \hat{\mu} + a\lambda^k \tilde{x}^+ - a\tilde{b}\lambda^k \tilde{u}^- + O(w) + \lambda^k O((\gamma^m \hat{\mu} - \tilde{u}^-)^2) + o(\lambda^k), \\ v &= v^+ + O(w) + O(\lambda^k). \end{aligned}$$

The heteroclinic orbit $\Gamma^{0,new}$ corresponds to an intersection of \bar{W}_1^u with $W_{loc}^s(O_2)$, which corresponds to letting $(u, w) = 0$ in the above two equations, i.e.,

$$0 = \hat{\mu} + a\tilde{b}\lambda^k \gamma^m \hat{\mu} + a\lambda^k \tilde{x}^+ - a\tilde{b}\lambda^k \tilde{u}^- + \lambda^k O((\gamma^m \hat{\mu} - \tilde{u}^-)^2) + o(\lambda^k). \tag{5.12}$$

We will look for solutions $\mu = O(\lambda^k)$ for large enough and even (k, m) from the sequence $\{k_j, m_j\}$ satisfying (5.9), i.e.,

$$m - k\theta_0 \rightarrow \frac{1}{\ln |\gamma_0|} \ln \left| \frac{\tilde{u}_0^-}{a_0 \tilde{x}_0^+} \right|, \tag{5.13}$$

where the 0 subscript stands for the value at $\mu = 0$. Since λ and γ depend smoothly on μ , we have, in particular, *Lipshitz dependence of $\theta = -\ln |\lambda| / \ln |\gamma|$ on μ* , implying

$$\theta = \theta_0 + O(\mu) = \theta_0 + O(\lambda^k). \tag{5.14}$$

Since $a_0\tilde{x}_0^+\tilde{u}_0^- < 0$ by assumption, and k, m are even, the above equation along with (5.13) implies

$$\begin{aligned} \lambda^k \gamma^m &= |\gamma|^{m-k\theta} = |\gamma_0|^{m-k\theta_0+O(k\lambda^k)} \left| \frac{\gamma}{\gamma_0} \right|^{-k\theta_0+O(k\lambda^k)} \\ &= -\frac{\tilde{u}_0^-}{a_0\tilde{x}_0^+} + o(1)_{k,m \rightarrow \infty}. \end{aligned} \tag{5.15}$$

Note that we used here Lipschitz dependence of θ on μ , but we only used continuity of γ 's dependence on μ .

Now, we rewrite (5.12) as

$$\begin{aligned} 0 &= (\gamma^m \hat{\mu} - \tilde{u}^-)(1 + a\tilde{b}\lambda^k \gamma^m) + \tilde{u}^- + a\lambda^k \gamma^m \tilde{x}^+ + \lambda^k \gamma^m O((\gamma^m \hat{\mu} - \tilde{u}^-)^2) \\ &\quad + o(\lambda^k \gamma^m), \end{aligned}$$

or, by (5.15) and continuous dependence of coefficients on μ ,

$$0 = (\gamma^m \hat{\mu} - \tilde{u}^-) \left(1 - \frac{\tilde{b}\tilde{u}_0^-}{\tilde{x}_0^+} + o(1)_{k,m \rightarrow \infty}\right) + O((\gamma^m \hat{\mu} - \tilde{u}^-)^2) + o(1)_{k,m \rightarrow \infty}.$$

Since $\tilde{b}\tilde{u}_0^-/\tilde{x}_0^+ = \tilde{\alpha}_0 \neq 1$ (as given by the non-degeneracy assumption C4.1 for the cycle $\tilde{\Gamma}$), we find that the new heteroclinic connection $\Gamma_j^{0,new}$ exists when

$$\hat{\mu}_j = \gamma^{-mj} (\tilde{u}_0^- + o(1)_{j \rightarrow \infty}) = -\lambda^{kj} (a_0\tilde{x}_0^+ + o(1)_{j \rightarrow \infty}) \tag{5.16}$$

for large enough even (k_j, m_j) defined by (5.9). By (3.11), this can be rewritten as

$$\mu_j = \gamma^{-mj} (\tilde{u}_0^- + o(1)_{j \rightarrow \infty}) = -\lambda^{kj} (a_0\tilde{x}_0^+ + o(1)_{j \rightarrow \infty}). \tag{5.17}$$

Note that γ and λ in this formula (as well as the $o(1)$ terms) depend on μ , so this is an implicit relation on μ_j .

If f_μ is of class C^3 with respect to variables and μ , then the right-hand side of (5.17) is smooth with respect to μ , so we obtain a unique solution μ_j for each sufficiently large j by the Implicit Function Theorem. It also follows that the corresponding heteroclinic connection $\Gamma_j^{0,new}$ splits with non-zero velocity as μ varies across μ_j .

If f_μ is only C^2 , we can only guarantee a continuous dependence of the right-hand side of (5.17) on μ . Still, it is obvious that the continuity gives the existence (though not necessarily uniqueness) of the solution μ_j for all sufficiently large j . Thus, we have the existence of the sought fragile heteroclinic connection $\Gamma_j^{0,new}$ for some $\mu_j \in \text{int}(\Delta_j)$ where the interval Δ_j corresponds to

$$|\mu_j - \gamma^{-mj} \tilde{u}_0^-| < \kappa_j \gamma^{-mj}, \quad |\mu_j + a_0\tilde{x}_0^+ \lambda^{kj}| < \kappa_j \lambda^{kj}, \tag{5.18}$$

for some $\kappa_j \rightarrow 0$. Note that we can choose κ_j going to 0 sufficiently slowly, so that $\Gamma_j^{0,new}$ necessarily splits when μ gets out of Δ_j .

We finish the proof of the lemma by recalling that λ and γ depend smoothly on μ , hence

$$\lambda = \lambda_0 + O(\mu), \quad \gamma = \gamma_0 + O(\mu),$$

which, upon substitution into (5.17), gives (5.10). □

Since the cycle $\tilde{\Gamma}$ is tied with Γ , we have $x^+ = \tilde{x}^+$ or $u^- = \tilde{u}^-$. It follows immediately from formula (5.16) that if $x^+ = \tilde{x}^+$, then $\hat{\mu}_j$ lies in the interval I_k^s defined in (5.2), and if $u^- = \tilde{u}^-$, then $\hat{\mu}_j$ lies in the interval I_m^u defined in (5.1). In either case, Proposition 5.2 implies that the blender which exists by Theorem 1 near the type-I cycle Γ is robustly connected¹⁰ to one of the saddles $O_{1,2}$ when the system has the newly created fragile heteroclinic orbit $\Gamma_j^{0,new}$ of Lemma 5.4.

Theorem 1 gives us the blender at $\mu = 0$, which persists for all small μ , and satisfies conditions of Proposition 5.1. The robust connection to the saddle O_1 or O_2 also persists for all μ corresponding to the interval I_m^u or I_s^k , i.e., it persists for all μ from the intervals Δ_j defined by (5.18). Let us show that there are sub-intervals inside Δ_j where the blender has a robust connection to the other saddle too.

Indeed, when μ varies from one end of Δ_j to the other, the heteroclinic orbit $\Gamma_j^{0,new}$ splits. Therefore, we can apply Proposition 5.1 to the heterodimensional cycle $\Gamma_j^{new} = L_1 \cup L_2 \cup \Gamma_j^{0,new} \cup \Gamma^1$ at $\mu = \mu_j$ (that is, system f in Proposition 5.1 is our system f_{μ_j} ; the coefficient $\hat{\mu}$ in Proposition 5.1 measures the distance between \bar{W}_1^u with $W_{loc}^s(O_2)$ when $\Gamma_j^{0,new}$ splits; note that Proposition 5.1 is applicable because Γ_j^{new} is non-degenerate, see Remark 5.5). As a result, we find that there are intervals of μ values inside Δ_j which correspond to a robust connection between the blender and O_1 , as well as intervals which correspond to a robust connection between the blender and O_2 (the connections can be transverse, or persistent heterodimensional). In any case, we have found intervals of μ values for which the blender is connected to both saddles O_1 and O_2 ; see Fig. 12. This completes the proof of Theorem 4. □

Remark 5.6 In the proof of this theorem we only used the smoothness with respect to μ in two places: when we claimed the Lipschitz dependence of θ on μ (see (5.14)), and when we inferred (5.10) from (5.17). However, we did not use formula (5.10) for μ_j when we derived Theorem 4 from Lemma 5.4, and used only relation (5.17) for μ_j . The existence of such values μ_j is established for any continuous family in which Γ^0 splits, i.e., for which the splitting functional is not constantly zero and changes sign. Therefore, Theorem 4 remains intact, for example, for any one-parameter family going through f such that

- the splitting functional μ changes sign, and
- θ stays equal to the same irrational constant.

Even more, the result still holds if we replace the one-parameter unfolding by any connected set of systems which contains f and satisfies the above conditions – the

¹⁰Here by robust connection between a blender and a saddle we mean that they are homoclinically related if they have the same index, or they are involved in robust heterodimensional dynamics if they have different indices.

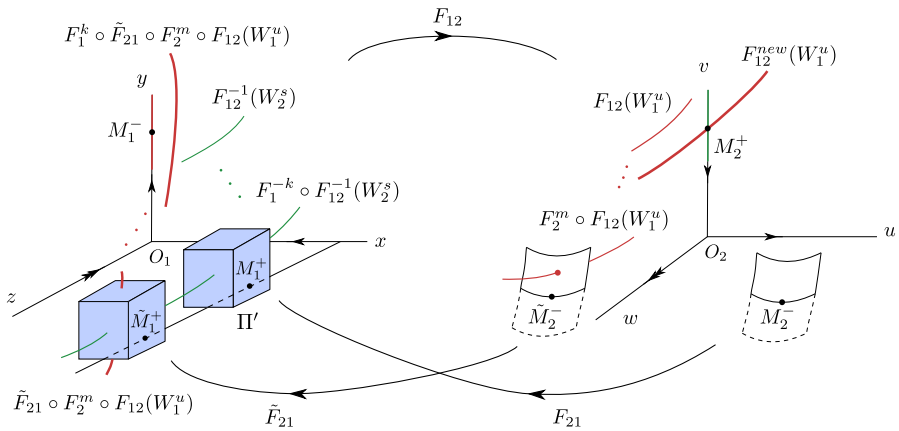


Fig. 12 A pair of tied heterodimensional cycles of type I and II with M_1^+ and \tilde{M}_1^+ lying in the same strong-stable leaf and with $x^+ > 0, u^- > 0$. There exist negative μ values such that some backward iterate of $W_{loc}^s(O_2)$ intersects $W^u(\Lambda^{cs})$, and the forward iterate $F_{12} \circ F_1^k \circ \tilde{F}_{21} \circ F_2^m \circ F_{12}(W_{loc}^u(O_1)) =: F_{12}^{new}(W_{loc}^u(O_1))$ intersects $W_{loc}^s(O_2)$ producing a new heterodimensional cycle. After a further change in μ , so that the newly obtained fragile heteroclinics splits to the right (i.e., $F_{12}^{new}(W_1^u)$ lies in $\{u > 0\}$), one can find m' satisfying $F_2^{m'} \circ F_{12}^{new}(W_1^u) \cap F_{21}^{-1}(W_{loc}^s(O_1)) \neq \emptyset$ such that the iterate $F_{21} \circ F_2^{m'} \circ F_{12}^{new}(W_1^u)$ crosses Π' vertically, and, therefore, it intersects $W^s(\Lambda^{cs})$. As a result, the two saddles O_1 and O_2 get connected to the blender at the same time (Color figure online)

intervals I_j in the formulation of Theorem 4 should be replaced by open subsets I_j converging to f .

5.4 Creating a tied pair of type-I and -II cycles. Proofs of Theorem 5 and Corollary 2

Theorem 4 immediately shows that cycles of type III can be stabilized. In this section we prove the same for type-II cycles in the following way. We let $|\alpha| < 1$ (the case $|\alpha| > 1$ is reduced to this one by the time-reversal). Then, we show that a generic one-parameter unfolding of a type-II cycle gives rise to a pair of tied cycles of type I and type II associated with O_1 and to a new saddle periodic point O_2' of index $d_1 + 1$, see Fig. 13. Next, we compute the value of the modulus θ for the new cycles: $\theta' := -\ln|\lambda|/\ln|\gamma'|$, where γ' is the center-unstable multiplier of O_2' , and show that by taking θ of the original cycle as a second parameter and varying it, we can make θ' irrational, so that Theorem 4 becomes applicable.

Proof of Theorem 5 Assume $|\alpha| < 1$. Since Γ is a cycle of type II, Lemma 5.4 is applicable to it (just replace $\tilde{\Gamma}$ by Γ in the formulation of the lemma, and remove the tildes from the coefficients in (5.9) and (5.10)). This gives that for a sequence $\{(k_j, m_j)\}$ of pairs of positive even integers satisfying $k_j, m_j \rightarrow \infty$ and

$$a_0 b_0 \lambda_0^{k_j} \gamma_0^{m_j} \rightarrow -\frac{b_0 u_0^-}{x_0^+} \tag{5.19}$$

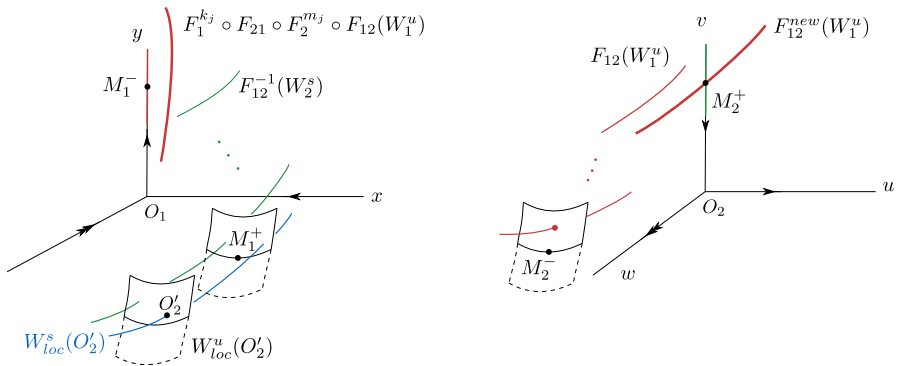


Fig. 13 At the moment of a new cycle given by Lemma 5.4 at $\mu = \mu_j$, we find an index- $(d_1 + 1)$ point O_2' near M_1^+ , homoclinically related to O_2 . After a small perturbation, we can transfer the fragile heteroclinic intersection of $W^s(O_2)$ with $W^u(O_1)$ to $W^s(O_2')$. We will show that there are two robust heteroclinics between O_1 and O_2' , giving rise to two heterodimensional cycles of type I and II

as $j \rightarrow \infty$, there exists a sequence of values $\{\mu'_j\}$

$$\mu'_j = -a_0 x_0^+ \lambda_0^{kj} + o(\lambda_0^{kj}) \tag{5.20}$$

such that the system $f_{\mu'_j}$ has a new heterodimensional cycle associated with O_1 and O_2 . By (3.11), we also have

$$\hat{\mu}'_j := \hat{\mu}(\mu'_j) = -a_0 x_0^+ \lambda_0^{kj} + o(\lambda_0^{kj}) \tag{5.21}$$

for the coefficient $\hat{\mu}$ in formula (3.20) for the first-return map near Γ .

Denote $m'_j = m_j + m^*$ for some fixed even integer m^* (to be determined below). In what follows, we find for each sufficiently large j a fixed point of the first-return map T_{k_j, m'_j} for the values of μ which are $o(\mu'_j)$ -close to μ'_j . Note first that due to the smooth dependence of λ and γ on the parameter μ , we obtain from (5.20) and (5.19) that

$$\begin{aligned} \lambda(\mu_j)^{k_j} &= (\lambda_0 + O(\mu_j))^{k_j} = \lambda_0^{k_j} + O(k_j \lambda_0^{k_j} \mu_j) = \lambda_0^{k_j} (1 + O(k_j \lambda_0^{k_j} \mu_j)), \\ \gamma(\mu_j)^{m_j} &= (\gamma_0 + O(\mu_j))^{m_j} = \gamma_0^{m_j} + O(m_j \gamma_0^{m_j} \mu_j) \\ &= \gamma_0^{m_j} (1 + O(m_j \gamma_0^{-m_j} \mu_j)). \end{aligned} \tag{5.22}$$

Thus,

$$\lambda^{k_j} \gamma^{m'_j} \rightarrow -\frac{u_0^- \gamma^{m^*}}{a_0 x_0^+}, \tag{5.23}$$

so, in particular, $\lambda^{k_j} \gamma^{m'_j}$ stays uniformly bounded as $j \rightarrow \infty$.

With this, one notices that the term $\gamma^m \hat{\phi}_1$ in formula (3.20) for the first-return map $T_{k,m}$ is $o(1)_{j \rightarrow \infty}$ when $k = k_j$ and $m = m'_j$. Hence, with the estimates in (3.21) and

(3.22), we can write the map T_{k_j, m'_j} as

$$\begin{aligned} \bar{X} &= b\gamma^{m'_j}\hat{\mu} + ab\lambda^{k_j}\gamma^{m'_j}x^+ - bu^- + ab\lambda^{k_j}\gamma^{m'_j}X + O(\bar{X}^2 + \bar{Y}^2) \\ &\quad + o(1)_{j \rightarrow \infty}, \\ Y &= o(\hat{\gamma}^{-k_j}), \quad \bar{Z} = o(\hat{\lambda}^{m'_j}). \end{aligned} \tag{5.24}$$

It follows that the fixed point (X_j, Y_j, Z_j) of T_{k_j, m'_j} satisfies

$$X_j = \frac{-(a\lambda^{k_j})^{-1}\hat{\mu}'_j + (a\lambda^{k_j}\gamma^{m'_j})^{-1}u^- - x^+ + O(X_j^2) + o(1)_{k_j, m'_j \rightarrow \infty}}{1 - (ab\lambda^{k_j}\gamma^{m'_j})^{-1}}, \tag{5.25}$$

$$Y_j = O(\hat{\gamma}^{-k_j}), \quad Z_j = O(\hat{\lambda}^{m'_j});$$

note that $1 - (ab\lambda^{k_j}\gamma^{m'_j})^{-1} \neq 0$, as follows from (5.23) if m^* is sufficiently large.

We need to verify that $(X_j, Y_j, Z_j) \in \Pi$. It suffices to show that $|X_j| < \delta$ for all large k_j and m'_j . Substituting (5.21), (5.22), and (5.23) into the first equation of (5.25) yields

$$X_j = x_\infty + o(1)_{j \rightarrow \infty}, \tag{5.26}$$

where

$$x_\infty = -x_0^+ \gamma_0^{-m^*} + O(\gamma_0^{-2m^*}). \tag{5.27}$$

Thus, by taking m^* sufficiently large and δ sufficiently small, we obtain that $|X_j| < \delta$ for all sufficiently large j , as required.

We have shown that at $\mu = \mu'_j$ the system has a non-degenerate heterodimensional cycle associated with O_1 and O_2 and the first-return map T_{k_j, m'_j} has a fixed point in Π given by (5.26) for any sufficiently large m^* . We denote this point as O'_2 . By (5.24), the map T_{k_j, m'_j} strongly contracts in Z and strongly expands in Y . Also, since

$$|ab\lambda^{k_j}\gamma^{m'_j}| = \left| \frac{b_0 u_0^- \gamma_0^{m^*}}{x_0^+} \right| + o(1)_{j \rightarrow \infty} \gg 1,$$

the map is expanding in X if m^* is taken large enough. Thus, the point O'_2 is a saddle of index $d_1 + 1$.

Arguing as in the proof of Lemma 3.1, one finds, on the set of points whose images under T_{k_j, m'_j} belong to Π , a forward-invariant unstable cone field around the (X, Y) -space and a forward-invariant strong-unstable cone-field around the Y -space, and also, on the set of points whose preimages belong to Π , a backward-invariant stable cone field around the Z -space. This implies that $W_{loc}^u(O'_2)$ and $W_{loc}^s(O'_2)$ are given by $Z = Z_j + w_j^u(X, Y)$ and $(X, Y) = (X_j, Y_j) + w_j^s(Z)$, respectively, for a smooth function w_j^u defined for $(X, Y) \in [-\delta, \delta] \times [-\delta, \delta]^{d_1}$ and a smooth function w_j^s defined for $Z \in [-\delta, \delta]^{d-d_1-1}$. Also, $W_{loc}^u(O'_2)$ contains the strong-unstable

$o(1)_{j \rightarrow \infty}$. Hence, the X -coordinate of the homoclinic point M' can be made as close to zero as we want, i.e., it is bounded away, as claimed, from x_∞ (which is non-zero by (5.27)), see Fig. 14.

Now, let us construct a heteroclinic intersection of $W^u(O_1)$ with $W^s(O'_2)$. The newly created heterodimensional cycle at $\mu = \mu'_j$ includes an orbit of a non-transverse intersection between $W^u(O_1)$ and $W^s(O_2)$. This orbit splits as μ varies in an interval of size $o(\mu'_j)$ (by Lemma 5.4). Since O_2 and O'_2 are homoclinically related, the invariant manifold $W^s(O'_2)$ accumulates on $W^s(O_2)$, which means that when μ varies, an orbit of a non-transverse intersection of $W^u(O_1)$ and $W^s(O'_2)$ emerges. Let it happen at $\mu = \mu_j$; it is $o(\mu'_j)$ -close to μ'_j , and by (5.20) we have

$$\mu_j = -a_0 x_0^+ \lambda_0^{k_j} + o(\lambda_0^{k_j}). \tag{5.29}$$

Note that the orbit Γ_j^0 of the fragile heteroclinic intersection of $W^u(O_1)$ and $W^s(O'_2)$ at $\mu = \mu_j$ satisfies condition C1 by the partial-hyperbolicity argument as in Remark 5.5.

To finish the proof, we need to find a pair of orbits $\Gamma_{j,I}^1$ and $\Gamma_{j,II}^1$ of a transverse intersection of $W^u(O'_2)$ and $W^s(O_1)$ such that the corresponding heterodimensional cycles $\Gamma_{j,I} = L_1 \cup L'_2 \cup \Gamma_j^0 \cup \Gamma_{j,I}^1$ and $\Gamma_{j,II} = L_1 \cup L'_2 \cup \Gamma_j^0 \cup \Gamma_{j,II}^1$ (where L'_2 is the orbit of the periodic point O'_2) satisfy the remaining non-degenerate conditions C2–C4.1, are tied to each other, and have different types (I and II).

We, first, notice that $W_{loc}^s(O_1) : \{Y = 0\}$ transversely intersects the local strong-unstable manifold $W_{loc}^{uu}(O'_2)$ (indeed, by (5.28), $W_{loc}^{uu}(O'_2)$ is C^1 -close to $\{X = x_\infty, Z = 0, Y \in [-\delta, \delta]^{d_1}\}$). Let M be the point of intersection. The local unstable manifold $W_{loc}^u(O'_2)$ is divided by $W_{loc}^{uu}(O'_2)$ into two connected components. It follows from the transversality of the intersection of $W_{loc}^{uu}(O'_2)$ with $W_{loc}^s(O_1)$ that the intersection of $W_{loc}^s(O_1)$ with $W_{loc}^u(O'_2)$ near M is a curve that goes from one component to another while crossing $W_{loc}^{uu}(O'_2)$ at M . Choose a point \tilde{M} on this curve such that \tilde{M} is close to M and \tilde{M} lies in a different component of $W_{loc}^u(O'_2) \setminus W_{loc}^{uu}(O'_2)$ from the homoclinic point $M' \in W_{loc}^u(O'_2) \cap W^s(O'_2)$.

The orbit of \tilde{M} is an orbit of transverse intersection of $W^u(L'_2)$ and $W^s(L_1)$. The non-degeneracy condition C2 is satisfied by this orbit due to partial hyperbolicity of every orbit lying in a small neighborhood of the original non-degenerate cycle Γ (see Remark 5.5). Condition C3 is satisfied because we take $\tilde{M} \neq M$, i.e., \tilde{M} is not in $W_{loc}^{uu}(O'_2)$. However, we take \tilde{M} sufficiently close to $W_{loc}^{uu}(O'_2)$, so the u -coordinates of \tilde{M} is sufficiently close to zero, making the non-degeneracy condition C4.1 fulfilled too. The non-degenerate heterodimensional cycle Γ^* comprised by the orbit of \tilde{M} , the periodic orbits L_1 and L'_2 and the fragile heteroclinic Γ_j^0 at $\mu = \mu_j$ is the sought cycle $\Gamma_{j,I}$ or $\Gamma_{j,II}$ (depending on whether it is type I or type II).

By Remark 2.2, the fact that \tilde{M} and the homoclinic point M' belong to different components of $W_{loc}^u(O'_2) \setminus W_{loc}^{uu}(O'_2)$ implies that the cycle Γ^* is tied with a cycle Γ^{**} of a different type. So, we have the sought pair of heterodimensional cycles $\Gamma_{j,I}$ and $\Gamma_{j,II}$, and hence the result of the theorem, once we show the non-degeneracy of Γ^{**} below.

The orbit of transverse intersection of $W^s(O_1)$ and $W^u(O'_2)$ given by Remark 2.2 (i.e., the robust heteroclinic of Γ^{**}) intersects $W_{loc}^u(O'_2)$ at some point M'' close to the homoclinic point M' . By the partial hyperbolicity argument, we have the non-degeneracy condition C2 for this orbit; condition C3 holds because $M' \notin W_{loc}^{uu}(O'_2)$. Let us establish the last non-degeneracy condition C4.1. The orbit of the point M'' goes close to the orbit of M' and gets back to a small neighborhood of O'_2 (since M' is homoclinic). After that the orbit spends a long time near L'_2 , which corresponds to a large number of iterations of the first-return map $T_{k_j, m_j + m^*}$ near O_2 , before getting to $W_{loc}^s(O_1)$. The iterations near O_2 create a very large expansion in the central direction, i.e., the expansion factor b in (2.3) gets very large for this orbit, making $|\alpha| \gg 1$ in (2.4). Thus, the non-degeneracy condition C4.1 is fulfilled for this orbit, provided M'' is chosen close enough to M' . \square

We can now finish the proof of Theorem B for the case of real central multipliers.

Proof of Corollary 2 In a proper unfolding f_ε of a heterodimensional cycle of type II, we fix the values of all parameters except for μ and θ and show that an arbitrary small change of μ and θ can lead the system into the region of robust heterodimensional dynamics involving O_1 and O_2 . So, we may from the very beginning consider a two-parameter family $f_{\mu, \theta}$ where μ varies in a small neighborhood of zero and θ varies in a neighborhood of some irrational $\theta = c_0$. By Theorem 5, on the line $\theta = c_0$ we have (see (5.29)) a sequence of values

$$\mu_j = -a_0 x_0^+ \lambda_0^{k_j} (1 + o(1))_{j \rightarrow \infty}$$

for which the system has a tied pair of cycles of different types involving O_1 and a periodic point O'_2 , homoclinically related to O_2 . The coefficients a_0, x_0^+, λ_0 correspond to $\mu = 0$ and depend continuously on the parameter θ ; the $o(1)$ term also depends continuously on θ and the rate at which it tends to zero is uniform with respect to θ .

Moreover, as proved in Lemma 5.4, when μ varies (on the line $\theta = c_0$) within a centered at $\mu = \mu_j$ interval Δ_j of size $o(\mu_j)$, the fragile heteroclinic connection Γ^0 shared by the tied cycles splits. Namely, there exist $\mu_j^- = \lambda_0^{k_j} (-a_0 x_0^+ - \kappa)$ and $\mu_j^+ = \lambda_0^{k_j} (-a_0 x_0^+ + \kappa)$ with some small $\kappa > 0$ such that Γ^0 is split at $(\mu = \mu_j^\pm, \theta = c_0)$ in opposite directions. The same holds true at $\mu = \mu_j^\pm$ for any value of θ close to c_0 . It follows that for any connected set \mathcal{L} sufficiently close to the line $\theta = c_0$ in the (μ, θ) -plane such that \mathcal{L} has a point at the line $\mu = \mu_j^-$ and another point at the line $\mu = \mu_j^+$, there is a point in \mathcal{L} corresponding to the existence of Γ^0 , i.e., to the tied pair of cycles involving O_1 and O'_2 . Moreover, Γ^0 splits when we move within \mathcal{L} from $\mu = \mu_j^-$ to $\mu = \mu_j^+$. Therefore, we immediately get the result (the existence of robust heterodimensional dynamics involving O_1 and O'_2 – hence, O_2) by applying the version of Theorem 4 given in Remark 5.6, if we can choose the connected set \mathcal{L} such that the modulus $\theta' = -\ln |\lambda| / \ln |\gamma'|$ stays equal to a certain irrational constant everywhere on \mathcal{L} (recall that we denote by γ' the central multiplier of the point O'_2).

For that, we just need to show that arbitrarily close to c_0 there exist constants c_- and c_+ such that, if j is large enough,

$$\sup_{\mu \in [\mu_j^-, \mu_j^+]} \theta'(\mu, \theta = c^-) < \inf_{\mu \in [\mu_j^-, \mu_j^+]} \theta'(\mu, \theta = c^+). \tag{5.30}$$

Let us compute the central multiplier γ' of O'_2 . It is, up to $o(1)_{j \rightarrow \infty}$ terms, the derivative $d\bar{X}/dx$ of the map (5.24) at O'_2 . This gives us

$$\gamma' = ab\lambda^{k_j} \gamma^{m'_j} (1 + O(\delta)) + o(1)_{j \rightarrow \infty},$$

or, according to (5.23),

$$\gamma' = -\frac{b_0 u_0^-}{x_0^+} \gamma^{m^*} (1 + O(\delta)) + o(1)_{j \rightarrow \infty}.$$

This gives

$$\theta' = \frac{1}{m^*} \theta + O((m^*)^{-2}), \tag{5.31}$$

so, by taking m^* sufficiently large, we obtain (5.30). □

5.5 Heterodimensional cycles with rational θ . Proof of Theorem 6

Recall that by a heterodimensional cycle we mean the set $\Gamma = L_1 \cup L_2 \cup \Gamma^0 \cup \Gamma^1$, where $L_{1,2}$ are the orbits of $O_{1,2}$, and Γ^0 and Γ^1 are the fragile and robust heteroclinic orbits, respectively. In this section we prove that the complexity of dynamics in a sufficiently small neighborhood of a heterodimensional cycle Γ in the saddle case depends essentially on whether θ is rational or not. That is, in contrast to the appearance of blenders for irrational θ , we show the persistence of simple dynamics near Γ for generic one-parameter unfoldings when the θ value (for the unperturbed system) is rational. This is done by investigating the structure of the set consisting of points whose entire orbits lie in a small neighborhood U of Γ .

Note that such orbits, except for those in the stable and unstable manifolds of L_1 and L_2 , must intersect the neighborhood Π of the heteroclinic point M_1^+ in U_{O_1} (the small neighborhood of O_1 contained in U) infinitely many times both forwards and backwards in time. If such orbit exists, then for any two consecutive intersection points of the orbit with Π , the second one is the image of the first one under the map $T_{k,m}$ given by (3.20) for some pair of integers (k, m) . Moreover, k and m have to be large enough if U is small. Recall that for a saddle heterodimensional cycle, we use λ and γ to denote the center-stable multiplier of O_1 and, respectively, center-unstable multiplier of O_2 .

Lemma 5.7 *Let f_μ be a generic one-parameter unfolding of the system f_0 with a non-degenerate heterodimensional cycle involving two saddles, such that $\theta(f_0) = -\frac{\ln|\lambda|}{\ln|\gamma|} = \frac{p}{q}$ is rational. Assume condition (2.8) is satisfied. Let $\{(k_s, m_s)\}$ be a sequence (finite or infinite) of different pairs of integers such that, for some small μ and*

some points $M_i = (X_i, Y_i, Z_i) \in \Pi$, we have $T_{k_i, m_i}(M_i) = (\bar{X}_i, \bar{Y}_i, \bar{Z}_i) =: \bar{M}_i \in \Pi$ for all i . Then, for all sufficiently small δ , either all m_i are equal to each other, $m_i = m$, and

$$\lambda^{k_i} = O(\delta)\gamma^{-m} \tag{5.32}$$

for all i , or all k_i are equal to each other, $k_i = k$, and

$$\gamma^{-m_i} = O(\delta)\lambda^k \tag{5.33}$$

for all i .

Remark 5.8 Note that in this lemma we also allow k_i or m_i to be infinite (with the convention $\gamma^{-\infty} = 0, \lambda^\infty = 0$). Here, $k_i = \infty$ corresponds to $M_i \in W_{loc}^s(O_1)$ and \bar{M}_i being a point of intersection of $W^u(L_1)$ with Π ; more specifically, $\bar{M}_i \in F_{21} \circ F_2^{m_i} \circ F_{12}(W_{loc}^u(O_1))$. The case $m_i = \infty$ corresponds to M_i being a point of intersection of $W^s(L_2)$ with Π , i.e., $M_i \in F_1^{-k_i} \circ F_{12}^{-1}(W_{loc}^s(O_2))$ and $\bar{M}_i \in F_{21}(W_{loc}^u(O_2))$.

Proof of Lemma 5.7 Consider, first, the case where we have only 2 pairs in the sequence (k_i, m_i) . By (3.20), conditions $T_{k_1, m_1}(M_1) = \bar{M}_1$ and $T_{k_2, m_2}(M_2) = \bar{M}_2$ give

$$\begin{aligned} \hat{\mu} + a\lambda^{k_1}X_1 + a\lambda^{k_1}x^+ - \gamma^{-m_1}u^- + \gamma^{-m_1}O((\bar{X}_1)^2 + (\bar{Y}_1)^2) \\ + o(\lambda^{k_1}) + o(\gamma^{-m_1}) = \gamma^{-m_1}b^{-1}\bar{X}_1, \\ \hat{\mu} + a\lambda^{k_2}X_2 + a\lambda^{k_2}x^+ - \gamma^{-m_2}u^- + \gamma^{-m_2}O((\bar{X}_2)^2 + (\bar{Y}_2)^2) \\ + o(\lambda^{k_2}) + o(\gamma^{-m_2}) = \gamma^{-m_2}b^{-1}\bar{X}_2. \end{aligned} \tag{5.34}$$

Assumptions that $M_{1,2} \in \Pi$ and $\bar{M}_{1,2} \in \Pi$ imply $|X_{1,2}| < \delta, |\bar{X}_{1,2}| < \delta, \|\bar{Y}_{1,2}\| < \delta$. Thus, it follows from (5.34) that the system

$$\begin{aligned} \hat{\mu} + a\lambda^{k_1}(x^+ + K_1\delta) - \gamma^{-m_1}u^- = \gamma^{-m_1}b^{-1}C_1\delta, \\ \hat{\mu} + a\lambda^{k_2}(x^+ + K_2\delta) - \gamma^{-m_2}u^- = \gamma^{-m_2}b^{-1}C_2\delta, \end{aligned} \tag{5.35}$$

must have a solution $(\hat{\mu}, k_1, m_1, K_1, C_1, k_2, m_2, K_2, C_2)$ with $|K_{1,2}| < 1, |C_{1,2}| < 1$. Subtracting the second equation of (5.35) from the first one, yields

$$\lambda^{k_1} - \lambda^{k_2} \frac{x^+ + K_2\delta}{x^+ + K_1\delta} = \gamma^{-m_1} \frac{C_1\delta + bu^-}{ab(x^+ + K_1\delta)} - \gamma^{-m_2} \frac{C_2\delta + bu^-}{ab(x^+ + K_1\delta)}. \tag{5.36}$$

Recall that $\lambda, \gamma, a, b, x^+$, and u^- depend on μ . Let us indicate their values at $\mu = 0$ by the subscript “0”. Since the multipliers λ and γ depend smoothly on μ , we have

$$\begin{aligned} \lambda^k &= (\lambda_0 + O(\mu))^k = \lambda_0^k + O(k\lambda_0^{k-1}\mu), \\ \gamma^{-m} &= (\gamma_0 + O(\mu))^{-m} = \gamma_0^{-m} + O(m\gamma_0^{-m-1}\mu). \end{aligned} \tag{5.37}$$

Using (3.11) and (5.37), one can estimate μ from (5.35) as

$$\mu = O(|\lambda_0|^{k_1} + |\gamma_0|^{-m_1}) \quad \text{and} \quad \mu = O(|\lambda_0|^{k_2} + |\gamma_0|^{-m_2}).$$

Substituting this into (5.37), and also using the continuous dependence of all the coefficients on μ , we rewrite (5.36) as

$$\begin{aligned} & \lambda_0^{k_1} - \lambda_0^{k_2} \frac{x_0^+ + K_2\delta}{x_0^+ + K_1\delta} (1 + o(1)) \\ &= \gamma_0^{-m_1} \frac{C_1\delta + b_0u_0^-}{a_0b_0(x_0^+ + K_1\delta)} (1 + o(1)) - \gamma_0^{-m_2} \frac{C_2\delta + b_0u_0^-}{a_0b_0(x_0^+ + K_1\delta)} (1 + o(1)), \end{aligned}$$

where $o(1)$ denotes terms that tend to zero as $k_{1,2} \rightarrow \infty, m_{1,2} \rightarrow \infty$. Note that these estimates make sense since when δ is sufficiently small, the k and m satisfying the assumption must be sufficiently large.

It is obvious that this equation is not solvable for sufficiently small δ and sufficiently large $k_1 \neq k_2$ and $m_1 \neq m_2$, unless the quantity $u_0^- / (a_0x_0^+)$ is a limit point of the set

$$\left\{ \frac{\lambda_0^{k_1} - \lambda_0^{k_2}}{\gamma_0^{-m_1} - \gamma_0^{-m_2}} : k_{1,2} \in \mathbb{N}, m_{1,2} \in \mathbb{N}, k_1 \neq k_2, m_1 \neq m_2 \right\}. \tag{5.38}$$

Since

$$|\lambda_0| = |\gamma_0|^{-\theta_0}, \tag{5.39}$$

and θ_0 is a rational number p/q (with p and q coprime), we obtain, assuming $k_1 > k_2$ and $m_1 > m_2$, that

$$\left| \frac{\lambda_0^{k_1} - \lambda_0^{k_2}}{\gamma_0^{-m_1} - \gamma_0^{-m_2}} \right| = \frac{|\lambda_0|^{k_2}}{|\gamma_0|^{-m_2}} \cdot \left| \frac{\gamma_0^{-(k_1-k_2)\theta} - 1}{\gamma_0^{-(m_1-m_2)\theta} - 1} \right| = |\gamma_0|^{\frac{-k_2p+m_2q}{q}} \frac{1 - \lambda_0^{k_1-k_2}}{1 - \gamma_0^{-(m_1-m_2)\theta}}.$$

This implies that the absolute values of the limit points of the set (5.38) form the set

$$\text{cl} \left\{ |\gamma_0|^{\frac{s}{q}} \frac{1 - \lambda_0^l}{1 - \gamma_0^{-n}} \right\}_{s \in \mathbb{Z}, l \in \mathbb{N}, n \in \mathbb{N}}$$

and, therefore, by (2.8), we find that $u_0^- / (a_0x_0^+)$ is not a limit point of the set (5.38). Thus, the system (5.35) can have a solution (for sufficiently small δ and large $k_{1,2}, m_{1,2}$) only if $k_1 = k_2$ or $m_1 = m_2$.

If $m_1 = m_2 = m$, then we have

$$\lambda^{k_1} - \lambda^{k_2} \frac{x^+ + K_2\delta}{x^+ + K_1\delta} = O(\delta)\gamma^{-m}$$

from (5.36), which implies (5.32) when $k_1 \neq k_2$. If $k_1 = k_2 = k$, then we have

$$\gamma^{-m_1} \frac{C_1\delta + bu^-}{ab(x^+ + K_1\delta)} - \gamma^{-m_2} \frac{C_2\delta + bu^-}{ab(x^+ + K_1\delta)} = O(\delta)\lambda^k$$

from (5.36), which implies (5.33) when $m_1 \neq m_2$.

This proves the lemma in the particular case of two pairs (k_i, m_i) . If we have more pairs, take the first two in the sequence. As we just proved, we either have $m_1 = m_2 = m$, or $k_1 = k_2 = k$. The arguments are the same in both cases, so we consider only the case $m_1 = m_2 = m$; note that we then have

$$\lambda^{k_1} \ll \gamma^{-m_1} \tag{5.40}$$

by (5.32). We want to prove that $m_i = m$ for all i in this case, so assume, to the contrary, that $m_i \neq m_1 = m$ for some i . Then, as we just proved, $k_i = k_1$, and

$$\gamma^{-m_1} \ll \lambda^{k_1}$$

by (5.33). This is a contradiction with (5.40), which proves the claim. □

Proof of Theorem 6 The case $k_1 = m_1 = \infty$ in Lemma 5.7 corresponds (see Remark 5.8) to the orbit of the fragile heteroclinic intersection of $F_{12}(W_{loc}^u(O_1))$ with $W_{loc}^s(O_2)$ (so we may think of $M_1 \in W_{loc}^s(O_1)$ and $\bar{M}_1 \in F_{21}(W_{loc}^u(O_2))$). Such intersection exists at $\mu = 0$. Note that in this case we also have $k_2 = m_2 = \infty$. Indeed, if we assume $m_2 \neq m_1$, then $k_2 = k_1 = \infty$ and, by (5.33), we get $\gamma^{-m_2} = 0$, i.e., $m_2 = \infty$, and if we assume $k_2 \neq k_1$, then $m_2 = m_1 = \infty$ and, by (5.32), $\lambda^{k_2} = 0$, i.e., $k_2 = \infty$, too. This means that no other orbits in U can intersect Π in this case except for the orbits corresponding to the transverse intersection of $F_{21}(W_{loc}^u(O_2))$ with $W_{loc}^s(O_1)$. This gives us the result of Theorem 6 at $\mu = 0$.

At $\mu \neq 0$, let there exist an orbit L in U , different from L_1, L_2 and from heteroclinic orbits corresponding to the intersection of $F_{21}(W_{loc}^u(O_2))$ with $W_{loc}^s(O_1)$ (we call L a non-exceptional orbit). Then, L intersects Π in a sequence of points M_s such that $M_{s+1} = T_{k_s, m_s}(M_s)$. If this sequence is infinite, then all (k_s, m_s) consist of finite positive integers. If the sequence is finite from the left, then either the most left point $M_{s_l} \in F_{21}(W_{loc}^u(O_2)) \cap \Pi$, or $M_{s_l} \in F_{21} \circ F_{21}^m \circ F_{12}(W_{loc}^u(O_1)) \cap \Pi$ for some finite m – in this case we define $k_{s_l-1} = \infty, m_{s_l-1} = m$. Similarly, if this sequence is finite from the right, then either the most right point $M_{s_r} \in W_{loc}^s(O_1) \cap \Pi$, or $M_{s_r} \in F_1^{-k} \circ F_{12}^{-1}(W_{loc}^s(O_2)) \cap \Pi$ for some finite k – in this case we define $m_{s_r} = \infty, k_{s_r} = k$.

By Lemma 5.7, we have three possibilities.

- The first possibility is that k_s and m_s are finite and the same for all s and all non-exceptional orbits, that is, $k_s = k$ and $m_s = m$ for some integers k and m . In this case $T_{k,m}(\Pi)$ intersects Π , which implies, by the first equation of (3.20), that $\mu = O(|\lambda|^k + |\gamma|^{-m})$. By (5.37), this implies $\mu = O(|\lambda_0|^k + |\gamma_0|^{-m})$, hence

$$\begin{aligned} \lambda^k \gamma^m &= \lambda_0^k (1 + O(k|\lambda_0|^k) + O(k|\gamma_0|^{-m})) \gamma_0^m ((1 + O(m|\gamma_0|^{-m}) + O(m|\lambda_0|^k)) \\ &= \lambda_0^k \gamma_0^m (1 + o(1)_{k,m \rightarrow \infty}) + o(1)_{k,m \rightarrow \infty}. \end{aligned}$$

Thus, using (5.39), we obtain

$$\begin{aligned} ab \lambda^k \gamma^m &= a_0 b_0 \lambda_0^k \gamma_0^m (1 + o(1)_{k,m \rightarrow \infty}) + o(1)_{k,m \rightarrow \infty} \\ &= a_0 b_0 \gamma_0^{\frac{mq-kp}{q}} (1 + o(1)_{k,m \rightarrow \infty}) + o(1)_{k,m \rightarrow \infty}. \end{aligned}$$

By (2.7), we obtain that $|ab\lambda^k\gamma^m|$ stays bounded away from 1. Since this is, up to small corrections, the derivative $d\bar{X}/dX$ in (3.20), and since we have a strong contraction in Z and a strong expansion in Y , the hyperbolicity of $T_{k,m}$ follows, if k, m are sufficiently large and the neighborhood U is sufficiently small.

By the hyperbolicity of $T_{k,m}$, it can have only one fixed point and it is the only orbit of $T_{k,m}$ that never leaves Π . Thus, in the case under consideration, we have that $M_s = M_0$ for all s and the orbit L of M_0 is a hyperbolic periodic orbit. Any other orbits in U must lie in the stable or unstable manifold of $L_{1,2}$, which includes the orbits L_1 and L_2 themselves, as well as orbits of transverse intersections of $W^u(L_2)$ with $W^s(L_1)$, of $W^u(L_2)$ with $W^s(L)$ if L has index d_1 , and of $W^s(L_1)$ with $W^u(L)$ if L has index d_2 . At the same time, no orbits from $W^u(L_1) \setminus L_1$ or $W^s(L_2) \setminus L_2$ can lie entirely in U in this case (as such orbits would correspond to infinite k or m). This is in a complete agreement with the statement of the theorem: if L is of index d_1 , then the hyperbolic set Λ_1 from the statement of the theorem is the union of L_1, L , and the heteroclinic orbits corresponding to the intersection of $W^u(L)$ with $W^s(L_1)$; otherwise, we have the set Λ_2 comprised by L_2, L , and the heteroclinic orbits corresponding to the intersection of $W^u(L_2)$ with $W^s(L)$.

- The second possibility is that m_s are finite and the same for all s and all non-exceptional orbits, that is, we have $m_s = m$ while some k_s are different. In this case, $|ab\lambda^k\gamma^m| < 1$ by (5.32), where k is the minimal value of k_s taken over all non-exceptional orbits. Thus, the derivative $d\bar{X}/dX$ in (3.20) is small. This means that the maps $T_{k_s, m_s = m}$ are all hyperbolic, with contraction in (X, Z) and expansion in Y , and the set of all non-exceptional orbits which are not in $W^u(L_2)$ is uniformly hyperbolic of index d_1 . The union of this set and L_1 is the hyperbolic set Λ_1 from the statement of the theorem. Note also that the finiteness of m implies that no orbit in $W^s(L_2) \setminus L_2$ can lie entirely in U in this case.

- The last possibility is that k_s are finite and the same for all s and all non-exceptional orbits, that is, we have $k_s = k$ while some m_s are different. In this case, $|ab\lambda^k\gamma^m| > 1$ by (5.33), where m is the minimal value of m_s taken over all non-exceptional orbits. Thus, the derivative $d\bar{X}/dX$ in (3.20) is very large, so all the maps $T_{k_s = k, m_s}$ are hyperbolic, with expansion in (X, Y) and contraction in Z . The set of all non-exceptional orbits which are not in $W^s(L_1)$ is uniformly hyperbolic of index d_2 , and stays at a non-zero distance from L_1 (the union of this set with L_2 is the set Λ_2 from the statement of the theorem). The finiteness of k implies that no orbit in $W^u(L_1) \setminus L_1$ can lie entirely in U in this case.

In all three cases we have a complete agreement with the statement of the theorem. \square

6 Saddle-focus and double-focus cases. Proof of Theorem 7

In this section, we consider the case where at least one of the central multipliers $\lambda_{1,1}$ and $\gamma_{2,1}$ is complex (nonreal), and prove Theorem 7. The strategy is to detect the dynamics similar to those near type-I heterodimensional cycles; namely, we check that the rational independence conditions of Theorem 7 imply that certain collections of the first-return maps satisfy conditions of Proposition 4.12.

It should be pointed out that we do not create type-I cycles via bifurcations; instead, we use the rotation brought by the complex central multipliers in order to see that conditions of Proposition 4.12 are satisfied, both for the creation of cs-blenders and cu-blenders. The only place where we study bifurcations is the result on the stabilization of cycles in the saddle-focus case.

Let us start with the saddle-focus case. Like in Sect. 3, we derive formulas for first-return maps $T_{k,m}$ near a saddle-focus heterodimensional cycle, see Sect. 6.1. An immediate observation is that the formula (6.18) for these maps has almost the same form as the formula (3.26) in the saddle case. The only difference is that the coefficient governing the contraction/expansion in the central coordinate now contains trigonometric functions, which gives the possibility to obtain partially hyperbolic sets with different types of central dynamics by choosing appropriate k and m . This fact is summarized in Lemma 6.1, a counterpart of Lemma 3.1 in the saddle case. Then, we prove in Proposition 6.2 that, without destroying the original cycle, the simultaneous existence of both cs- and cu-blenders follows from Proposition 4.12, provided that θ , $\frac{\omega}{2\pi}$ and 1 are rationally independent (θ is defined in (2.5) and ω is the argument of the nonreal central multiplier).

Next, we study the local stabilization of saddle-focus cycles in Sect. 6.3. The key is to find a replacement of Lemma 5.3, which tells us when the iterates $F_{21} \circ F_2^m \circ F_{12}(W_{loc}^u(O_1))$ and $F_1^{-k} \circ F_{12}^{-1}(W_{loc}^s(O_2))$ enter the activating domain Π' of the blenders. Thanks to the rotation, some of the iterates automatically cross Π' properly, depending on which central multiplier is nonreal, i.e., where the rotation happens, in F_1 or F_2 . The other manifold can also cross Π' after unfolding the heterodimensional cycle, as in the saddle case (this is the only place where we unfold the original cycle). These facts are proven in Lemmas 6.4 and 6.5.

Finally, double-focus heterodimensional cycles are dealt with in Sect. 6.4. The main effort in this case is to obtain the necessary formula for the first-return maps. After that, one immediately finds that it has the same form as in the saddle-case. Consequently, a counterpart to Proposition 6.2 stating the coexistence of blenders follows, see Proposition 6.7. The main difference is that the required arithmetic condition becomes the rational independence among θ , $\frac{\omega_1}{2\pi}$, $\theta \frac{\omega_2}{2\pi}$ and 1, where $\omega_{1,2}$ are the arguments of the nonreal central multipliers. The stabilization part is also similar to the saddle-focus case but now both $W^u(O_1)$ and $W^s(O_2)$ enter Π' automatically due to the simultaneous rotations near O_1 and O_2 , see Lemmas 6.8 and 6.9.

6.1 First-return maps in the saddle-focus case

We assume that

$$\lambda_{1,1} = \lambda_{1,2}^* = \lambda e^{i\omega}, \quad \omega \in (0, \pi), \quad \text{and} \quad \gamma := \gamma_{2,1} \quad \text{is real,}$$

where $\lambda > |\lambda_{1,3}|$ and $|\gamma_{2,1}| < |\gamma_{2,2}|$. As mentioned in the introduction, the other case (where $\lambda_{1,1}$ is real while $\gamma_{2,1}$ is complex) can be reduced to this one by the time reversal. The main goal of this section is to obtain a formula for the first-return maps and prove its partial hyperbolicity in Lemma 6.1. As mentioned before, the way to find the first-return maps is the same as in the saddle case.

6.1.1 Local maps

We use the same coordinates near O_2 as in the saddle case so that the formulas for the local map F_2 remains the same (see (3.7)). Let us now introduce coordinates $(x_1, x_2, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d_1} \times \mathbb{R}^{d-d_1-2}$ near O_1 such that the local map F_1 takes the form (see [34, Lemmas 5 and 6])

$$\begin{aligned} \bar{x}_1 &= \lambda x_1 \cos k\omega + \lambda x_2 \sin k\omega + g_1(x_1, x_2, y, z), \\ \bar{x}_2 &= -\lambda x_1 \sin k\omega + \lambda x_2 \cos k\omega + g_2(x_1, x_2, y, z), \\ \bar{y} &= P_1 y + g_3(x_1, x_2, y, z), \quad \bar{z} = P_2 z + g_4(x_1, x_2, y, z), \end{aligned} \tag{6.1}$$

where we do not indicate the dependence on parameters for simplicity. The eigenvalues of the matrices P_1 and P_2 are $\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{1,d_1}$ and $\lambda_{1,3} \dots \lambda_{1,d-d_1}$, respectively. The functions g vanish along their first derivatives at the origin, and satisfy

$$\begin{aligned} g_{1,2,4}(0, 0, y, 0) &= 0, \quad g_3(x_1, x_2, 0, z) = 0, \quad g_{1,2}(x_1, x_2, 0, z) = 0, \\ \frac{\partial g_{1,2,4}}{\partial(x_1, x_2)}(0, 0, y, 0) &= 0, \end{aligned}$$

for all sufficiently small x_1, x_2, y and z . Similar to the saddle case, in these coordinates the local manifolds $W_{loc}^s(O_1)$ and $W_{loc}^u(O_1)$ are given by $\{y = 0\}$ and $\{x_1 = 0, x_2 = 0, z = 0\}$; the leaves of the strong-stable foliation have the form $\{(x_1, x_2) = const, y = 0\}$; the restriction of the map to $W_{loc}^s(O_1)$ is linear in x . This is the same coordinate system as described in Sect. 2.

By Lemma 7 of [34], for any point (x_1, x_2, y, z) in U_{01} , we have $(\tilde{x}_1, \tilde{x}_2, \tilde{y}, \tilde{z}) = F_1^k(x_1, x_2, y, z)$ if and only if

$$\begin{aligned} \tilde{x}_1 &= \lambda^k x_1 \cos k\omega + \lambda^k x_2 \sin k\omega + p_1(x_1, x_2, \tilde{y}, z), \\ \tilde{x}_2 &= -\lambda^k x_1 \sin k\omega + \lambda^k x_2 \cos k\omega + p_2(x_1, x_2, \tilde{y}, z), \\ y &= p_3(x_1, x_2, \tilde{y}, z), \quad \tilde{z} = p_4(x_1, x_2, \tilde{y}, z), \end{aligned} \tag{6.2}$$

where

$$\|p_{1,2,4}\|_{C^1} = o(\lambda^k), \quad \|p_3\|_{C^1} = o(\hat{\gamma}^{-k}),$$

for some constant $\hat{\gamma} \in (1, |\gamma_{1,1}|)$, and these estimates are uniform for all systems C^2 -close to f .

6.1.2 Transition maps

Let us now define the transition maps F_{12} and F_{21} from a neighborhood of $M_1^- = (0, 0, y^-, 0)$ to a neighborhood of $M_2^+ = (0, v^+, 0)$ and, respectively, from a neighborhood of $M_2^- = (u^-, 0, w^-)$ to a neighborhood of $M_1^+ = (x_1^+, x_2^+, 0, z^+)$. Arguing like in the saddle case, by the first part of condition C1 (that $F_{12}^{-1}(W_{loc}^{SE}(O_2)) \cap$

$W_{loc}^u(O_1)$ at M_1^-), the transition map $F_{12} : (\tilde{x}_1, \tilde{x}_2, \tilde{y}, \tilde{z}) \mapsto (u, v, w)$ can be written as follows (the dots refer to the second and higher order terms in the Taylor expansion):

$$\begin{aligned} u &= \hat{\mu} + a_{11}\tilde{x}_1 + a_{12}\tilde{x}_2 + a_{13}\tilde{z} + a_{14}w + \dots, \\ v - v^+ &= a_{21}\tilde{x}_1 + a_{22}\tilde{x}_2 + a_{23}\tilde{z} + a_{24}w + \dots, \\ \tilde{y} - y^- &= a_{31}\tilde{x}_1 + a_{32}\tilde{x}_2 + a_{33}\tilde{z} + a_{34}w + \dots, \end{aligned} \tag{6.3}$$

where the relation between $\hat{\mu}$ and μ is the same as in the saddle case and is given by (3.11).

Recall the second part of C1 that $F_{12}(W_{loc}^{uE}(O_1))$ intersects $W_{loc}^s(O_2)$ transversely at M_2^+ . When the dimension of the map F_{12} is large than three (i.e., when the system has dimension higher than three if it is a diffeomorphism or higher than four if it is a flow), this condition means that the common directions shared by $DF_{12}(\mathcal{T}_{M_1^-}W_{loc}^{uE}(O_1)) \cap \mathcal{T}_{M_2^+}W_{loc}^s(O_2)$ is at most one-dimensional, which happens only if

$$a_{11}^2 + a_{12}^2 \neq 0. \tag{6.4}$$

When the dimension of F_{12} equals three, the strong-stable coordinates z are absent. In this case the above inequality holds automatically since F_{12} is a diffeomorphism.

Similarly, condition C2 shows that the other transition map $F_{21} : (\tilde{u}, \tilde{v}, \tilde{w}) \mapsto (x_1, x_2, y, z)$ can be written as

$$\begin{aligned} x_1 - x_1^+ &= b_{11}(\tilde{u} - u^-) + b_{12}\tilde{v} + b_{13}y + \dots, \\ x_2 - x_2^+ &= b_{21}(\tilde{u} - u^-) + b_{22}\tilde{v} + b_{23}y + \dots, \\ \tilde{w} - w^- &= b_{31}(\tilde{u} - u^-) + b_{32}\tilde{v} + b_{33}y + \dots, \\ z - z^+ &= b_{41}(\tilde{u} - u^-) + b_{42}\tilde{v} + b_{43}y + \dots, \end{aligned} \tag{6.5}$$

with $b_{11}^2 + b_{21}^2 \neq 0$. By rotating the x -coordinates to a small angle, if necessary, we can assume

$$b_{11} \neq 0. \tag{6.6}$$

Thus, the above formula can be rewritten as

$$\begin{aligned} \tilde{u} - u^- &= b_{11}^{-1}(x_1 - x_1^+ - b_{13}y) + O(\|\tilde{v}\| + (x_1 - x_1^+)^2 + y^2), \\ x_2 - x_2^+ &= b_{21}b_{11}^{-1}(x - x_1^+ - b_{13}y) + b_{23}y + O(\|\tilde{v}\| + (x_1 - x_1^+)^2 + y^2), \\ \tilde{w} - w^- &= O(|x_1 - x_1^+| + \|\tilde{v}\| + \|y\|), \\ z - z^+ &= b_{41}b_{11}^{-1}(x - x_1^+ - b_{13}y) + b_{43}y + O(\|\tilde{v}\| + (x_1 - x_1^+)^2 + y^2). \end{aligned} \tag{6.7}$$

6.1.3 First-return maps and cone field lemma

Now we can find a formula for the first-return map in the same way as in the saddle case. Namely, combining (6.2) and (6.3) yields the analogue of (3.15):

$$\begin{aligned}
 u &= \hat{\mu} + a_{11}\lambda^k x_1 \cos k\omega + a_{11}\lambda^k x_2 \sin k\omega - a_{12}\lambda^k x_1 \sin k\omega \\
 &\quad + a_{12}\lambda^k x_2 \cos k\omega + \tilde{h}_1(x_1, x_2, z, w), \\
 y &= \tilde{h}_2(x_1, x_2, z, w), \quad v - v^+ = \tilde{h}_3(x_1, x_2, z, w),
 \end{aligned}
 \tag{6.8}$$

where

$$\begin{aligned}
 \tilde{h}_1(x_1, x_2, z, w) &= O(\|w\|) + o(\lambda^k), & \tilde{h}_2(x_1, x_2, z, w) &= o(\hat{\gamma}^{-k}), \\
 \tilde{h}_3(x_1, x_2, z, w) &= O(\|w\| + |\lambda|^k).
 \end{aligned}$$

Combining (3.7) and (6.7) yields the analogue of (3.17)

$$\begin{aligned}
 u &= \gamma^{-m}(u^- + b_{11}^{-1}(x_1 - x_1^+ - b_{13}y + \hat{h}_{01}(x_1 - x_1^+, y))) \\
 &\quad + \hat{h}_1(x_1, v, y), \\
 x_2 - x_2^+ &= b_{21}b_{11}^{-1}(x_1 - x_1^+ - b_{13}y) + b_{23}y + \hat{h}_{02}(x_1 - x_1^+, y) \\
 &\quad + \hat{h}_2(x_1, v, y), \\
 z - z^+ &= b_{41}b_{11}^{-1}(x_1 - x_1^+ - b_{13}y) + b_{43}y + \hat{h}_{03}(x_1 - x_1^+, y) \\
 &\quad + \hat{h}_3(x_1, v, y), \\
 w &= \hat{h}_4(x_1, v, y),
 \end{aligned}
 \tag{6.9}$$

where

$$\begin{aligned}
 \hat{h}_{0i}(x_1 - x_1^+, \bar{y}) &= O((x_1 - x_1^+)^2 + y^2) \quad (i = 1, 2, 3), \\
 \hat{h}_{1,4}(x_1, v, y) &= o(\gamma^{-m}), \quad \hat{h}_{2,3}(x_1, v, y) = o(\hat{\lambda}^m).
 \end{aligned}$$

Finally, combining (6.8) and (6.9) with renaming x_1, x_2, y, z by $\bar{x}_1, \bar{x}_2, \bar{y}, \bar{z}$ in (6.9), we obtain the following form for the first-return map $T_{k,m} := F_{21} \circ F_2^m \circ F_{12} \circ F_1^k : (x_1, x_2, y, z) \mapsto (\bar{x}_1, \bar{x}_2, \bar{y}, \bar{z})$:

$$\begin{aligned}
 \bar{x}_1 - x_1^+ &= b_{11}\gamma^m \hat{\mu} + b_{11}\lambda^k \gamma^m ((a_{11}x_1 + a_{12}x_2) \cos k\omega \\
 &\quad + (a_{11}x_2 - a_{12}x_1) \sin k\omega) \\
 &\quad - b_{11}u^- + b_{13}\bar{y} - \hat{h}_{01}(\bar{x}_1 - x_1^+, \bar{y}) + \gamma^m h_1(x_1, \bar{x}_1, x_2, \bar{y}, z), \\
 \bar{x}_2 - x_2^+ &= b_{21}b_{11}^{-1}(\bar{x}_1 - x_1^+ - b_{13}\bar{y}) + b_{23}\bar{y} + \hat{h}_{02}(\bar{x}_1 - x_1^+, y) \\
 &\quad + h_2(x_1, \bar{x}_1, x_2, \bar{y}, z), \\
 \bar{z} - z^+ &= b_{41}b_{11}^{-1}(\bar{x}_1 - x_1^+ - b_{13}\bar{y}) + b_{43}\bar{y} + \hat{h}_{03}(\bar{x}_1 - x_1^+, \bar{y})
 \end{aligned}
 \tag{6.10}$$

$$\begin{aligned}
 &+ h_3(x_1, \bar{x}_1, x_2, \bar{y}, z) \\
 y &= h_4(x_1, \bar{x}_1, x_2, \bar{y}, z),
 \end{aligned}$$

where

$$h_1 = o(\lambda^k) + o(\gamma^{-m}), \quad h_{2,3} = o(\hat{\lambda}^m), \quad h_4 = o(\hat{\gamma}^{-k}).$$

We further do computations with the first-return map $T_{k,m}$ only at $\mu = 0$, so we will omit the term $b\gamma^m \hat{\mu}$ in formula (6.10). Make the coordinate transformation

$$\begin{aligned}
 X_1 &= x_1 - x_1^+ - b_{13}y, & Y &= y, \\
 X_2 &= \delta^{-\frac{1}{2}}(x_2 - x_2^+ - b_{21}b_{11}^{-1}(x_1 - x_1^+ - b_{13}y) - b_{23}y \\
 &\quad - \hat{h}_{02}(x_1 - x_1^+, y)), \\
 Z &= z - z^+ - b_{41}b_{11}^{-1}(x - x_1^+ - b_{13}y) - b_{43}y - \hat{h}_{03}(x_1 - x_1^+, y).
 \end{aligned} \tag{6.11}$$

The first-return map acquires the form

$$\begin{aligned}
 \bar{X}_1 &= A_{k,m}X_1 + B_{k,m} + \delta^{\frac{1}{2}}O(\lambda^k\gamma^m)X_2 + \hat{\phi}_{01}(\bar{X}_1, \bar{Y}) \\
 &\quad + \gamma^m\hat{\phi}_1(X_1, \bar{X}_1, X_2, \bar{Y}, Z), \\
 \bar{X}_2 &= \delta^{-\frac{1}{2}}\hat{\phi}_2(X_1, \bar{X}_1, X_2, \bar{Y}, Z), \\
 Y &= \hat{\phi}_3(X_1, \bar{X}_1, X_2, \bar{Y}, Z), & \bar{Z} &= \hat{\phi}_4(X_1, \bar{X}_1, X_2, \bar{Y}, Z),
 \end{aligned} \tag{6.12}$$

where

$$\begin{aligned}
 \hat{\phi}_{01} &= O(\bar{X}_1^2 + \bar{Y}^2), & \|\hat{\phi}_1\|_{C^1} &= o(\lambda^k) + o(\gamma^{-m}), \\
 \|\hat{\phi}_{2,4}\|_{C^1} &= o(\hat{\lambda}^m), & \|\hat{\phi}_3\|_{C^1} &= o(\hat{\gamma}^{-k}),
 \end{aligned} \tag{6.13}$$

and

$$\begin{aligned}
 A_{k,m} &= \lambda^k\gamma^m((a_{11}b_{21} - a_{12}b_{11})\sin k\omega + (a_{11}b_{11} + a_{12}b_{21})\cos k\omega) \\
 &= \lambda^k\gamma^m A \sin(k\omega + \eta_1), \\
 B_{k,m} &= \lambda^k\gamma^m b_{11}(a_{11}x_2^+ - a_{12}x_1^+)\sin k\omega + b_{11}(a_{11}x_1^+ + a_{12}x_2^+)\cos k\omega \\
 &\quad - b_{11}u^- \\
 &= \lambda^k\gamma^m B \sin(k\omega + \eta_2) - b_{11}u^-.
 \end{aligned} \tag{6.14}$$

Note the factor $\delta^{\frac{1}{2}}$ in front of X_2 in (6.12) which appears because we scale x_2 to $\delta^{\frac{1}{2}}$ in (6.11). Note also that the coefficients

$$A = \sqrt{(a_{11}^2 + a_{12}^2)(b_{11}^2 + b_{21}^2)} \quad \text{and} \quad B = |b_{11}|\|x^+\|\sqrt{a_{11}^2 + a_{12}^2}$$

are non-zero by (6.4), (6.6) and because $x^+ = (x_1^+, x_2^+) \neq 0$ due to the non-degeneracy condition C3. The phases $\eta_{1,2}$ are defined by

$$\begin{aligned}
 A \sin \eta_1 &= a_{11}b_{11} + a_{12}b_{21}, & A \cos \eta_1 &= a_{11}b_{21} - a_{12}b_{11}, \\
 \frac{B}{b_{11}} \sin \eta_2 &= a_{11}x_1^+ + a_{12}x_2^+, & \frac{B}{b_{11}} \cos \eta_2 &= a_{11}x_2^+ - a_{12}x_1^+,
 \end{aligned}$$

which, together with $b_{11}/b_{21} \neq x_1^+/x_2^+$ due to the non-degeneracy condition C4.2 (see (6.5)), imply that

$$\tan \eta_1 \neq \tan \eta_2. \tag{6.15}$$

Similarly to the saddle case, we consider pairs (k, m) such that the maps $T_{k,m}$ take

$$\Pi = [-\delta, \delta] \times [-\delta, \delta] \times [-\delta, \delta]^{d_1} \times [-\delta, \delta]^{d-d_1-1} \tag{6.16}$$

into itself, which implies that

$$\lambda^k \gamma^m B \sin(k\omega + \eta_2) = b_{11}u^- + O(\delta). \tag{6.17}$$

We will consider only such k for which $\sin(k\omega + \eta_2)$ stays bounded away from 0. This, along with (6.17), implies that $\lambda^k \gamma^m$ is uniformly bounded. In particular, the term $\gamma^m \hat{\phi}_1$ in (6.12) tends to zero as $k, m \rightarrow \infty$. This allows to express \bar{X}_1 as a function of (X_1, X_2, Z, \bar{Y}) from the first equation of (6.12), and thus get rid of the dependence on \bar{X}_1 in the right-hand side of (6.12). Thus, we can rewrite formula (6.12) for $T_{k,m}$ as

$$\begin{aligned}
 \bar{X}_1 &= A_{k,m}X_1 + B_{k,m} + \phi_1(X_1, X_2, Z, \bar{Y}), \\
 \bar{X}_2 &= \phi_2(X_1, X_2, Z, \bar{Y}), & \bar{Z} &= \phi_4(X_1, X_2, Z, \bar{Y}), \\
 Y &= \phi_3(X_1, X_2, Z, \bar{Y}),
 \end{aligned} \tag{6.18}$$

where

$$\begin{aligned}
 \phi_1 &= O(\delta^{\frac{3}{2}}) + o(1)_{k,m \rightarrow \infty}, & \frac{\partial \phi_1}{\partial (X_1, X_2, \bar{Y}, Z)} &= O(\delta^{\frac{1}{2}}) + o(1)_{k,m \rightarrow \infty}, \\
 \|\phi_{2,4}\|_{C^1} &= o(\hat{\lambda}^m), & \|\phi_3\|_{C^1} &= o(\hat{\gamma}^{-k}).
 \end{aligned} \tag{6.19}$$

We have the above estimate for ϕ_2 because in the \bar{X}_2 -equation in (6.12) the coefficient $\delta^{-\frac{1}{2}}$ can be absorbed by $\hat{\phi}_2$. Indeed, we always first take δ sufficiently small and then take k, m sufficiently large.

Formula (6.18) represents the first-return maps in the form used in Proposition 4.12 (where one should choose (X_2, Z) as a new Z -variable). We will check in Sect. 6.2 that these maps indeed satisfy conditions of this proposition, thus establishing the existence of the blenders. Before doing that, we further restrict the choice of (k, m) by the requirement that $\sin(k\omega + \eta_1)$ stays bounded away from zero. Then, the constant $A_{k,m}$ in (6.18) stays bounded away from zero, and we notice that the map (6.18) assumes, upon setting $X = X_1$ and $Z^{new} = (X_2, Z)$, the same form as

(3.26) (with the only difference being a slightly worse estimate for ϕ_1 , which does not affect the further results). This gives us the following analogue of Lemma 3.1 for the saddle-focus case:

Lemma 6.1 *Let $\mu = 0$. Given any $K > 0$, one can choose δ sufficiently small, such that for all sufficiently large (k, m) such that $\sin(k\omega + \eta_1)$ and $\sin(k\omega + \eta_2)$ stay bounded away from zero, the cone fields on Π*

$$C^{cu} = \{(\Delta X_1, \Delta X_2, \Delta Y, \Delta Z) : \|(\Delta X_2, \Delta Z)\| \leq K(|\Delta X_1| + \|\Delta Y\|)\}, \tag{6.20}$$

$$C^{uu} = \{(\Delta X_1, \Delta X_2, \Delta Y, \Delta Z) : \max\{|\Delta X_1|, \|(\Delta X_2, \Delta Z)\|\} \leq K\|\Delta Y\|\}, \tag{6.21}$$

are forward-invariant in the sense that if a point $M \in \Pi$ has its image $\bar{M} = T_{k,m}(M)$ in Π , then the cone at M is mapped into the cone at \bar{M} by $DT_{k,m}$; the cone fields

$$C^{cs} = \{(\Delta X_1, \Delta X_2, \Delta Y, \Delta Z) : \|\Delta Y\| \leq K(|\Delta X_1| + \|(\Delta X_2, \Delta Z)\|)\}, \tag{6.22}$$

$$C^{ss} = \{(\Delta X_1, \Delta X_2, \Delta Y, \Delta Z) : \max\{|\Delta X_1|, \|\Delta Y\|\} \leq K\|(\Delta X_2, \Delta Z)\|\}, \tag{6.23}$$

are backward-invariant in the sense that if a point $\bar{M} \in \Pi$ has its preimage $M = T_{k,m}^{-1}(\bar{M})$ in Π , then the cone at \bar{M} is mapped into the cone at M by $DT_{k,m}^{-1}$. Moreover, vectors in C^{uu} and, if $|A_{k,m}| > 1$, also in C^{cu} are expanded by $DT_{k,m}$; vectors in C^{ss} and, if $|A_{k,m}| < 1$, also in C^{cs} are contracted by $DT_{k,m}$.

6.2 Coexistence of mutually activating blenders

Here we prove the first part of Theorem 7 for the saddle-focus case.

Proposition 6.2 *If $\theta, \frac{\omega}{2\pi}$ and 1 are rationally independent, then, at $\mu = 0$, there exist, arbitrarily close to the heterodimensional cycle Γ , a cs-blender Λ^{cs} with an activating pair (Π', C^{ss}) and a cu-blender Λ^{cu} with an activating pair (Π', C^{uu}) such that $W^s(\Lambda^{cs}) \cap W^u(\Lambda^{cu}) \neq \emptyset$ and the two blenders activate each other, namely, $W^u(\Lambda^{cs})$ contains a piece crossing Π' properly with respect to C^{uu} and $W^s(\Lambda^{cu})$ contains a piece crossing Π' properly with respect to C^{ss} . Here*

$$\Pi' = [-q\delta, q\delta] \times [-\delta, \delta] \times [-\delta, \delta]^{d_1} \times [-\delta, \delta]^{d-d_1-2} \tag{6.24}$$

for some fixed $q \in (0, 1)$, and C^{ss}, C^{uu} are given by Lemma 6.1 with some sufficiently small K .

This proposition immediately leads to robust heterodimensional dynamics involving the two blenders, see the discussion after Definition 4.2.

Proof The rational independence condition implies that the set

$$\{(-k\theta + m, k\frac{\omega}{2\pi} - p)\}_{k,m \in \mathbb{N}, p \in \mathbb{Z}}$$

is dense in \mathbb{R}^2 . So, given any $(s, t) \in \mathbb{R}^2$, one can find a sequence $\{(k_n, m_n, p_n)\}$ with $k_n, m_n \rightarrow \infty$ such that m_n are even and

$$-k_n\theta + m_n \rightarrow t \quad \text{and} \quad k_n \frac{\omega}{2\pi} - p_n \rightarrow s. \tag{6.25}$$

In fact, we fix a sufficiently large value of t and let s depend on n so that

$$k_n \frac{\omega}{2\pi} - p_n - s_n \rightarrow 0.$$

We take the sequence $\{s_n\}$ dense in a sufficiently small closed interval Δ such that the values

$$|\gamma|^t B \sin(2\pi s_n + \eta_2) - b_{11}u^- \tag{6.26}$$

are dense in a small interval around zero. In particular, this difference is $O(\delta)$. Moreover, by (6.15), we can always choose Δ such that, for some constants C_1 and C_2 ,

$$0 < C_1 < \left| \frac{b_{11}u^- A \sin(2\pi s + \eta_1)}{B \sin(2\pi s + \eta_2)} \right| < C_2 < 1 \tag{6.27}$$

for all $s \in \Delta$, or such that, for some constants C_3 and C_4 ,

$$1 < C_3 < \left| \frac{b_{11}u^- A \sin(2\pi s + \eta_1)}{B \sin(2\pi s + \eta_2)} \right| < C_4 < \infty \tag{6.28}$$

for all $s \in \Delta$.

For the corresponding sequences $\{(k_n, m_n, p_n)\}$, the values of $\sin(k_n\omega + \eta_1)$ and $\sin(k_n\omega + \eta_2)$ get bounded away from zero for all sufficiently large n . Since m_n are even, we have $\gamma^{m_n} = |\gamma|^{m_n}$ and, since $\lambda > 0$, we have $\lambda^{k_n} \gamma^{m_n} = |\gamma|^{m_n - k_n\theta} \rightarrow |\gamma|^t$. It then follows from (6.26) that condition (6.17) is satisfied, and hence the first-return maps $T_n := T_{k_n, m_n}$ can be represented in the form (6.18) with estimates in (6.19), and the cone lemma (Lemma 6.1) holds.

The coefficients B_{k_n, m_n} given by (6.14) are dense in a small interval around zero, and, by passing to a subsequence if necessary, we have that B_{k_n, m_n} lie in this interval for all n . Since by (6.14) one can write

$$A_{k_n, m_n} = \frac{(B_{k_n, m_n} + b_{11}u^-) A \sin(k_n\omega + \eta_1)}{B \sin(k_n\omega + \eta_2)},$$

inequalities (6.27) and (6.28) imply that $C_1 < |A_{k_n, m_n}| < C_2$ or $C_3 < |A_{k_n, m_n}| < C_4$, depending on the choice of the interval Δ .

Then, by setting $X = X_1$ and $Z^{new} = (X_2, Z)$, one immediately sees that conditions of Proposition 4.12 are satisfied by the maps T_n , which gives us both the cu- and cs- blender (by appropriate choices of the interval Δ).

What remains is to show the homoclinic connection between the two blenders. Similarly to Remark 4.9, for every point $M_1 \in \Lambda^{cs}$, we define its local stable manifold as a connected piece of $W^s(\Lambda^{cs}) \cap \Pi$ through M_1 , and the local unstable manifold as a connected piece of $W^u(\Lambda^{cs}) \cap \Pi$ through M_1 . By Lemma 6.1, the tangent space

of $W_{loc}^s(M_1)$ at any point lies in the stable cone C^{cs} , so $W_{loc}^s(M_1)$ is given by an equation $Y = \xi_{M_1}^s(X_1, X_2, Z)$ where $\xi_{M_1}^s$ has its derivative small and is defined on $[-\delta, \delta] \times [-\delta, \delta] \times [-\delta, \delta]^{d-d_1-2}$. Similarly, the manifold $W_{loc}^u(M_1)$ is the graph of $(X_1, X_2, Z) = \xi_{M_1}^u(Y)$ defined for $Y \in [-\delta, \delta]^{d_1}$ and the tangent space of $W_{loc}^u(M_1)$ at any point lies in C^{uu} . This means that $W_{loc}^u(M_1)$ crosses Π' properly with respect to C^{uu} , i.e., Λ^{cs} activates Λ^{cu} .

For every point $M_2 \in \Lambda^{cu}$, the manifold $W_{loc}^u(M_2)$ is the graph of $(X_2, Z) = \xi_{M_2}^u(X_1, Y)$ defined for $(X_1, Y) \in [-\delta, \delta] \times [-\delta, \delta]^d$ and its tangents lie in C^{cu} . It immediately follows that it has a non-empty transverse intersection with $W_{loc}^s(M_1)$ for any point $M_1 \in \Lambda^{cs}$, so we have $W^s(\Lambda^{cs}) \cap W^u(\Lambda^{cu}) \neq \emptyset$. The manifold $W_{loc}^s(M_2)$ is the graph of $(X_1, Y) = \xi_{M_2}^s(X_2, Z)$ defined for $(X_2, Z) \in [\delta, \delta] \times [-\delta, \delta]^{d-d_1-2}$, and its tangents lie in C^{ss} . So, it crosses Π' properly with respect to C^{ss} , which means that Λ^{cu} activates Λ^{cs} . □

Remark 6.3 It follows from the arguments at the end of this proof that $W^u(\Lambda^{cs})$ intersects transversely $W_{loc}^s(O_1)$ (i.e., the manifold $\{Y = 0\}$), while $W^s(\Lambda^{cs})$ intersects transversely any manifold which crosses Π properly with respect to C^{uu} . Similarly, $W^s(\Lambda^{cu})$ intersects transversely $F_{21}(W_{loc}^u(O_2))$ (i.e., the manifold $\{X_2 = 0, Z = 0\}$ obtained by taking $m \rightarrow \infty$ in (6.9)), and $W^u(\Lambda^{cu})$ intersects transversely any manifold which crosses Π properly with respect to C^{ss} .

6.3 Local stabilization of heterodimensional cycles in the saddle-focus case

In this section we prove the second part of Theorem 7 for the saddle-focus case. As in the saddle case, we investigate the iterates of the local invariant manifolds $W_{loc}^u(O_1)$ and $W_{loc}^s(O_2)$.

Lemma 6.4 *Let q, Π', C^{uu} be given by Proposition 6.2. Define the intervals*

$$I_m^u = \left(\gamma^{-m}u^- - \frac{1}{2}|b_{11}^{-1}\gamma^{-m}|q\delta, \gamma^{-m}u^- + \frac{1}{2}|b_{11}^{-1}\gamma^{-m}|q\delta \right).$$

Take sufficiently small δ . For every sufficiently large m , if $\hat{\mu} \in I_m^u$, then the image $S_m^u := F_{21} \circ F_2 \circ F_{12}(W_{loc}^u(O_1))$ is a disc of the form $(X_1, X_2, Z) = s(Y)$ for some smooth function s . The disc S_m^u crosses the cube Π' properly with respect to the cone field C^{uu} .

Proof Since $W_{loc}^u(O_1)$ near M_1^- has equation $(\tilde{x}, \tilde{z}) = 0$, it follows from formulas (6.3), (3.7) and (6.5), that the image S_m^u is given by

$$\begin{aligned} x_1 - x_1^+ &= b_{11}\gamma^m\hat{\mu} - b_{11}u^- + b_{13}y + O((\gamma^m\hat{\mu} - u^-)^2 + y^2) + o(1)_{m \rightarrow \infty}, \\ x_2 - x_2^+ &= b_{21}\gamma^m\hat{\mu} - b_{21}u^- + b_{23}y + O((\gamma^m\hat{\mu} - u^-)^2 + y^2) + o(1)_{m \rightarrow \infty}, \\ z - z^+ &= b_{41}\gamma^m\hat{\mu} - b_{41}u^- + b_{43}y + O((\gamma^m\hat{\mu} - u^-)^2 + y^2) + o(1)_{m \rightarrow \infty}, \end{aligned}$$

which, after the transformation (6.11), recasts as

$$\begin{aligned} X_1 &= b_{11}\gamma^m \hat{\mu} - b_{11}u^- + O((\gamma^m \hat{\mu} - u^-)^2 + Y^2) + o(1)_{m \rightarrow \infty}, \\ X_2 &= \delta^{-\frac{1}{2}} O((\gamma^m \hat{\mu} - u^-)^2 + Y^2) + \delta^{-\frac{1}{2}} o(1)_{m \rightarrow \infty}, \\ Z &= O((\gamma^m \hat{\mu} - u^-)^2 + Y^2) + o(1)_{m \rightarrow \infty} \end{aligned} \tag{6.29}$$

Since $\hat{\mu} \in I_m^u$, we have

$$|b_{11}\gamma^m \hat{\mu} - b_{11}u^-| < \frac{q\delta}{2},$$

which for $Y \in [-\delta, \delta]^{d_1}$ implies

$$\begin{aligned} |X_1| &< \frac{q\delta}{2} + O(\delta^2) + o(1)_{m \rightarrow \infty} < q\delta \quad \text{and} \\ \|(X_2, Z)\| &= O(\delta^{\frac{3}{2}}) + \delta^{-\frac{1}{2}} o(1)_{m \rightarrow \infty} < \delta, \end{aligned}$$

where we first take δ sufficiently small and then take m sufficiently large (and we do the same whenever terms like $\delta^{-\frac{1}{2}} o(1)_{m \rightarrow \infty}$ appear). This means that S_m^u crosses Π' . One also finds from (6.29) that

$$\frac{\partial(X_1, X_2, Z)}{\partial Y} = O(\delta^{\frac{1}{2}}) + \delta^{-\frac{1}{2}} o(1)_{m \rightarrow \infty},$$

which can be made sufficiently small so that the tangent spaces of $S_m^u \cap \Pi$ lie in C^{uu} . So, the crossing is also proper with respect to C^{uu} . \square

Lemma 6.5 *Let Π' and C^{ss} be given by Proposition 6.2. There exists a sequence $\{k_j\} \rightarrow \infty$ such that, at $\mu = 0$, the preimage $S_j^s := F_1^{-k_j} \circ F_{12}^{-1}(W_{loc}^s(O_2))$ is a disc of the form $(X_1, Y) = s(X_2, Z)$ for some smooth function s . The disc S_j^s crosses Π' properly with respect to the cone field C^{ss} .*

Proof Since $W_{loc}^s(O_2)$ near the point M_2^+ is given by $\{(u, w) = 0\}$, the preimage $F_1^{-k} \circ F_{12}^{-1}(W_{loc}^s(O_2))$ can be found from (6.8) by setting $\hat{\mu} = 0$ and $u = w = 0$, which in coordinates (6.11) is given by

$$\begin{aligned} X_1 &= -\frac{B \sin(k\omega + \eta_2)}{A \sin(k\omega + \eta_1)} + O(\delta^{\frac{3}{2}}) + o(1)_{k \rightarrow \infty}, \\ Y &= o(\hat{\gamma}^{-k}), \end{aligned} \tag{6.30}$$

where the right-hand sides are functions of (X_2, Z) .

The assumption that $\theta, \omega/2\pi$, and 1 are rationally independent implies that $\omega/2\pi$ is irrational. As a result, one can find a sequence $\{k_j\}$ of positive integers with $k_j \rightarrow \infty$ such that

$$\left| \frac{B \sin(k_j \omega + \eta_2)}{A \sin(k_j \omega + \eta_1)} \right| < \frac{q\delta}{2}$$

for all sufficiently large k_j . It follows that for all sufficiently small δ and sufficiently large k_j , we have

$$|X_1| < q\delta, \quad \|Y\| < \delta, \quad \frac{\partial(X_1, Y)}{\partial(X_2, Z)} = O(\delta^{\frac{3}{2}}) + o(1)_{k \rightarrow \infty}$$

in (6.30), which completes the proof of the lemma. □

We can now finish the proof of Theorem 7 for the saddle-focus case. Let Λ^{cs} and Λ^{cu} be the cs- and cu-blenders of Proposition 6.2. By Remark 6.3, Lemma 6.4 implies that Λ^{cs} is homoclinically related to O_1 when $\hat{\mu} \in I_m^u$; and Lemma 6.5 implies that Λ^{cu} is homoclinically related to O_2 at $\mu = 0$, hence at all small μ , e.g. when $\hat{\mu} \in I_m^u$. □

6.4 Double-focus case

We finish the paper by considering the case

$$\lambda_{1,1} = \lambda_{1,2}^* = \lambda e^{i\omega_1}, \quad \omega_1 \in (0, \pi), \quad \text{and} \quad \gamma_{2,1} = \gamma_{2,2}^* = \gamma e^{i\omega_2}, \quad \omega_2 \in (0, \pi),$$

where $\lambda > |\lambda_{1,3}|$, $\gamma < |\gamma_{2,3}|$. We no longer need to split the heterodimensional cycle to get robust heterodimensional dynamics, so below we write formulas only for the unperturbed system (e.g. $\mu = 0$).

6.4.1 Local maps

We use the same coordinates near O_1 as in the saddle-focus case, so the local map F_1 is given by (6.1) with replacing ω by ω_1 . Near O_2 we introduce coordinates $(u_1, u_2, v, w) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-d_1-1} \times \mathbb{R}^{d_1-1}$ such that the local map F_2 assumes the form (see [34, Lemmas 5 and 6])

$$\begin{aligned} \tilde{u}_1 &= \gamma u_1 \cos k\omega_2 + \gamma u_2 \sin k\omega_2 + \hat{g}_1(u_1, u_2, v, w), \\ \tilde{u}_2 &= -\gamma u_1 \sin k\omega_2 + \gamma u_2 \cos k\omega_2 + \hat{g}_2(u_1, u_2, v, w), \\ \tilde{v} &= Q_1 v + \hat{g}_3(u_1, u_2, v, w), \\ \tilde{w} &= Q_2 w + \hat{g}_4(u_1, u_2, v, w), \end{aligned}$$

where the eigenvalues of the matrices Q_1 and Q_2 are $\lambda_{2,1}, \dots, \lambda_{2,d-d_1-1}$ and $\gamma_{2,3} \dots \gamma_{1,d_1+1}$, respectively. Here the functions \hat{g} satisfy

$$\begin{aligned} \hat{g}_{1,2,4}(0, 0, v, 0) &= 0, \quad \hat{g}_3(u_1, u_2, 0, w) = 0, \quad \hat{g}_{1,2}(u_1, u_2, 0, w) = 0, \\ \frac{\partial \hat{g}_{1,2,4}}{\partial(u_1, u_2)}(0, 0, v, 0) &= 0 \end{aligned}$$

for all sufficiently small u_1, u_2, v and w . Similar to the saddle case, in these coordinates the local manifolds $W_{loc}^u(O_2)$ and $W_{loc}^s(O_2)$ are given by $\{v = 0\}$ and $\{u_1 = 0, u_2 = 0, w = 0\}$; the leaves of the strong-unstable foliation have the form

$\{(u_1, u_2) = const, v = 0\}$; and the restriction of the map to $W_{loc}^u(O_2)$ is linear in u . This is the same coordinate system as discussed in Sect. 2.

By Lemma 7 of [34], the above coordinates can be chosen such that for any point $(u_1, u_2, v, w) \in U_{O_2}$, we have $(\tilde{u}_1, \tilde{u}_2, \tilde{v}, \tilde{w}) = F_2^m(u_1, u_2, v, w)$ if and only if

$$\begin{aligned} u_1 &= \gamma^{-m} \tilde{u}_1 \cos m\omega_2 + \gamma^{-m} \tilde{u}_2 \sin m\omega_2 + q_1(\tilde{u}_1, \tilde{u}_2, v, \tilde{w}), \\ u_2 &= -\gamma^{-m} \tilde{u}_1 \sin m\omega_2 + \gamma^{-m} \tilde{u}_2 \cos m\omega_2 + q_2(\tilde{u}_1, \tilde{u}_2, v, \tilde{w}), \\ \tilde{v} &= q_3(\tilde{u}_1, \tilde{u}_2, v, \tilde{w}), \\ w &= q_4(\tilde{u}_1, \tilde{u}_2, v, \tilde{w}), \end{aligned} \tag{6.31}$$

where

$$\|q_{1,2,4}\|_{C^1} = o(\gamma^{-m}), \quad \|q_3\|_{C^1} = o(\hat{\lambda}^m),$$

for some constant $\hat{\lambda} \in (1, |\lambda_{2,1}|)$, and these estimates are uniform for all systems C^2 -close to f .

6.4.2 Transition maps

We now take the heteroclinic points

$$\begin{aligned} M_1^+ &= (x_1^+, x_2^+, 0, z^+), & M_1^- &= (0, 0, y^-, 0), & M_2^+ &= (0, 0, v^+, 0), \\ M_2^- &= (u_1^-, u_2^-, 0, w^-). \end{aligned}$$

By the non-degeneracy condition C3, we have $x^+ \neq 0$ and $u^- \neq 0$. Up to a linear rotation of the coordinates u , we can always achieve

$$u_1^- \neq 0,$$

which will be our standing assumption.

The transition map $F_{12} : (\tilde{x}_1, \tilde{x}_2, \tilde{y}, \tilde{z}) \mapsto (u_1, u_2, v, w)$ from a neighborhood of M_1^- to a neighborhood of M_2^+ is given by

$$\begin{aligned} u_1 &= a'_{11} \tilde{x}_1 + a'_{12} \tilde{x}_2 + a'_{13} (\tilde{y} - y^-) + a'_{14} \tilde{z} + \dots, \\ u_2 &= a'_{21} \tilde{x}_1 + a'_{22} \tilde{x}_2 + a'_{23} (\tilde{y} - y^-) + a'_{24} \tilde{z} + \dots, \\ v - v^+ &= a'_{31} \tilde{x}_1 + a'_{32} \tilde{x}_2 + a'_{33} (\tilde{y} - y^-) + a'_{34} \tilde{z} + \dots, \\ w &= a'_{41} \tilde{x}_1 + a'_{42} \tilde{x}_2 + a'_{43} (\tilde{y} - y^-) + a'_{44} \tilde{z} + \dots, \end{aligned}$$

where dots denote the second and higher order terms in the Taylor expansion. Note that here $\dim y = \dim w + \dim u_2$ and condition C1 means that $\det(a'_{23}, a'_{43}) \neq 0$. So, we can rewrite the formula as

$$\begin{aligned} u_1 &= a_{11} \tilde{x}_1 + a_{12} \tilde{x}_2 + a_{13} \tilde{z} + a_{14} u_2 + a_{15} w + \dots, \\ v - v^+ &= a_{21} \tilde{x}_1 + a_{22} \tilde{x}_2 + a_{23} \tilde{z} + a_{24} u_2 + a_{25} w + \dots, \\ \tilde{y} - y^- &= a_{31} \tilde{x}_1 + a_{32} \tilde{x}_2 + a_{33} \tilde{z} + a_{34} u_2 + a_{35} w + \dots, \end{aligned} \tag{6.32}$$

where $a_{11}^2 + a_{12}^2 \neq 0$ by the second part of C1. Similarly, it follows from condition C2 that the transition map $F_{21} : (\tilde{u}_1, \tilde{u}_2, \tilde{v}, \tilde{w}) \mapsto (x_1, x_2, y, z)$ from a neighborhood of M_2^- to a neighborhood of M_1^+ is given by

$$\begin{aligned}x_1 - x_1^+ &= b_{11}(\tilde{u}_1 - u_1^-) + b_{12}\tilde{v} + b_{13}y + \cdots, \\x_2 - x_2^+ &= b_{21}(\tilde{u}_1 - u_1^-) + b_{22}\tilde{v} + b_{23}y + \cdots, \\z - z^+ &= b_{31}(\tilde{u}_1 - u_1^-) + b_{32}\tilde{v} + b_{33}y + \cdots, \\ \tilde{u}_2 - u_2^- &= b_{41}(\tilde{u}_1 - u_1^-) + b_{42}\tilde{v} + b_{43}y + \cdots, \\ \tilde{w} - w^- &= b_{51}(\tilde{u}_1 - u_1^-) + b_{52}\tilde{v} + b_{53}y + \cdots,\end{aligned}\tag{6.33}$$

where $b_{21}^2 + b_{12}^2 \neq 0$.

6.4.3 First-return maps

Slightly different from the previous cases, here we work with the first-return maps

$$F_{k,m} := F_2^m \circ F_{12} \circ F_1^k \circ F_{21} : (\tilde{u}_1, \tilde{u}_2, \tilde{v}, \tilde{w}) \mapsto (\bar{u}_1, \bar{u}_2, \bar{v}, \bar{w})$$

defined in a small neighbourhood of M_2^- .

Lemma 6.6 *Suppose that $\cos m\omega_2$ and $(\cos m\omega_2 + a_{14} \sin m\omega_2)$ are both bounded away from zero. Then, for a point $(\tilde{u}_1, \tilde{u}_2, \tilde{v}, \tilde{w})$ in a small neighborhood of M_2^- , we have $(\bar{u}_1, \bar{u}_2, \bar{v}, \bar{w}) = F_{k,m}(\tilde{u}_1, \tilde{u}_2, \tilde{v}, \tilde{w})$ if and only if*

$$\begin{aligned}(\cos m\omega_2 + a_{14} \sin m\omega_2)\bar{u}_1 &= \lambda^k \gamma^m (C_k(\tilde{u}_1 - u_1^-) + D_k) \\ &\quad + (a_{14} \cos m\omega_2 - \sin m\omega_2)\bar{u}_2 \\ &\quad + \lambda^k \gamma^m h_{01}(\tilde{u}_1 - u_1^-, \tilde{v}) \\ &\quad + \gamma^m h_1(\tilde{u}_1, \bar{u}_2, \tilde{v}, \bar{w}), \\ \bar{u}_2 - u_2^- &= b_{41}(\tilde{u}_1 - u_1^-) + b_{42}\tilde{v} + h_{02}(\tilde{u}_1 - u_1^-, \tilde{v}) \\ &\quad + h_2(\tilde{u}_1, \bar{u}_2, \tilde{v}, \bar{w}), \\ \bar{w} - w_2^- &= b_{51}(\tilde{u}_1 - u_1^-) + b_{52}\tilde{v} + h_{03}(\tilde{u}_1 - u_1^-, \tilde{v}) \\ &\quad + h_3(\tilde{u}_1, \bar{u}_2, \tilde{v}, \bar{w}), \\ \bar{v} &= h_4(\tilde{u}_1, \bar{u}_2, \tilde{v}, \bar{w}),\end{aligned}\tag{6.34}$$

where

$$\begin{aligned}C_k &= (a_{11}b_{11} + a_{12}b_{21}) \cos k\omega_1 + (a_{11}b_{21} - a_{12}b_{11}) \sin k\omega_1, \\ D_k &= (a_{11}x_1^+ + a_{12}x_2^+) \cos k\omega_1 + (a_{11}x_2^+ - a_{12}x_1^+) \sin k\omega_1,\end{aligned}\tag{6.35}$$

and

$$\begin{aligned}
 h_{01} &= O((\tilde{u}_1 - u_1^-)^2 + \|\tilde{v}\|), & h_{0i} &= O((\tilde{u}_1 - u_1^-)^2 + \tilde{v}^2) \quad (i = 2, 3), \\
 h_1 &= o(\lambda^k) + o(\gamma^{-m}), & h_{2,3} &= o(\hat{\gamma}^{-k}), & h_4 &= o(\hat{\lambda}^m).
 \end{aligned}$$

Proof Consider first the composition $F_1^k \circ F_{21}$. Substituting the y -equation of (6.2) into the first three equations in (6.33), one expresses x_1, x_2, z as functions of $\tilde{u}_1, \tilde{v}, \tilde{y}$. Substituting these expressions together with the y -equation of (6.2) into the remaining equations in (6.33) leads to \tilde{u}_2, \tilde{w} as functions of $\tilde{u}_1, \tilde{v}, \tilde{y}$. Then, combining (6.33) with the newly obtained equations for $x_1, x_2, z, \tilde{u}_2, \tilde{w}$ yields that for a point $(\tilde{u}_1, \tilde{u}_2, \tilde{v}, \tilde{w})$ in a small neighborhood of M_2^- we have $(\tilde{x}_1, \tilde{x}_2, \tilde{y}, \tilde{z}) = F_1^k \circ F_{21}(\tilde{u}_1, \tilde{u}_2, \tilde{v}, \tilde{w})$ if and only if

$$\begin{aligned}
 \tilde{x}_1 &= \lambda^k(x_1^+ + b_{11}(\tilde{u}_1 - u_1^-) + b_{12}\tilde{v}) \cos k\omega_1 \\
 &\quad + \lambda^k(x_2^+ + b_{21}(\tilde{u}_1 - u_1^-) + b_{22}\tilde{v}) \sin k\omega_1 \\
 &\quad + \lambda^k O((\tilde{u}_1 - u_1^-)^2 + \tilde{v}^2) + \tilde{h}_1(\tilde{u}_1, \tilde{v}, \tilde{y}), \\
 \tilde{x}_2 &= -\lambda^k(x_1^+ + b_{11}(\tilde{u}_1 - u_1^-) + b_{12}\tilde{v}) \sin k\omega_1 \\
 &\quad + \lambda^k(x_2^+ + b_{21}(\tilde{u}_1 - u_1^-) + b_{22}\tilde{v}) \cos k\omega_1 \\
 &\quad + \lambda^k O((\tilde{u}_1 - u_1^-)^2 + \tilde{v}^2) + \tilde{h}_2(\tilde{u}_1, \tilde{v}, \tilde{y}), \\
 \tilde{u}_2 - u_2^- &= b_{41}(\tilde{u}_1 - u_1^-) + b_{42}\tilde{v} + O((\tilde{u}_1 - u_1^-)^2 + \tilde{v}^2) + \tilde{h}_3(\tilde{u}_1, \tilde{v}, \tilde{y}), \\
 \tilde{w} - w_2^- &= b_{51}(\tilde{u}_1 - u_1^-) + b_{52}\tilde{v} + O((\tilde{u}_1 - u_1^-)^2 + \tilde{v}^2) + \tilde{h}_4(\tilde{u}_1, \tilde{v}, \tilde{y}), \\
 \tilde{z} &= \tilde{h}_5(\tilde{u}_1, \tilde{v}, \tilde{y}),
 \end{aligned} \tag{6.36}$$

where $\|\tilde{h}_{1,2,5}\|_{C^1} = o(\lambda^k)$ and $\|\tilde{h}_{3,4}\|_{C^1} = o(\hat{\gamma}^{-k})$.

We proceed to find a formula for $F_2^m \circ F_{12}$. Substituting the v -equation in (6.32) into the u_2 - and w -equations in (6.31), one obtains u_2 and w as functions of $\tilde{u}_1, \tilde{u}_2, \tilde{w}, \tilde{x}_1, \tilde{x}_2, \tilde{z}$. Substituting these into the remaining equations, leads to u_1 and \tilde{v} as functions of $\tilde{u}_1, \tilde{u}_2, \tilde{w}, \tilde{x}_1, \tilde{x}_2, \tilde{z}$. So we have the following relations:

$$\begin{aligned}
 u_1 &= \gamma^{-m}\tilde{u}_1 \cos m\omega_2 + \gamma^{-m}\tilde{u}_2 \sin m\omega_2 + q'_1(\tilde{u}_1, \tilde{u}_2, \tilde{w}, \tilde{x}_1, \tilde{x}_2, \tilde{z}), \\
 u_2 &= -\gamma^{-m}\tilde{u}_1 \sin m\omega_2 + \gamma^{-m}\tilde{u}_2 \cos m\omega_2 + q'_2(\tilde{u}_1, \tilde{u}_2, \tilde{w}, \tilde{x}_1, \tilde{x}_2, \tilde{z}), \\
 \tilde{v} &= q'_3(\tilde{u}_1, \tilde{u}_2, \tilde{w}, \tilde{x}_1, \tilde{x}_2, \tilde{z}), \\
 w &= q'_4(\tilde{u}_1, \tilde{u}_2, \tilde{w}, \tilde{x}_1, \tilde{x}_2, \tilde{z}),
 \end{aligned}$$

where $\|q'_{1,2,4}\|_{C^1} = o(\gamma^{-m})$ and $\|q'_3\|_{C^1} = o(\hat{\lambda}^m)$.

Now, since $\cos m\omega_2$ is bounded away from zero, one can express \tilde{u}_1 as a function of $u_1, \tilde{u}_2, v, \tilde{w}$ from the u_1 -equation. Substituting the result into the equations for $u_2,$

\tilde{v} and w , we obtain

$$\begin{aligned}\gamma^{-m}\tilde{u}_1 \cos m\omega_2 &= u_1 - \gamma^{-m}\tilde{u}_2 \sin m\omega_2 + q_1''(u_1, \tilde{u}_2, \tilde{w}, \tilde{x}_1, \tilde{x}_2, \tilde{z}), \\ u_2 &= -\frac{u_1 \sin m\omega_2}{\cos m\omega_2} + \frac{\gamma^{-m}\tilde{u}_2}{\cos m\omega_2} + q_2''(u_1, \tilde{u}_2, \tilde{w}, \tilde{x}_1, \tilde{x}_2, \tilde{z}), \\ \tilde{v} &= q_3''(u_1, \tilde{u}_2, \tilde{w}, \tilde{x}_1, \tilde{x}_2, \tilde{z}), \\ w &= q_4''(u_1, \tilde{u}_2, \tilde{w}, \tilde{x}_1, \tilde{x}_2, \tilde{z}),\end{aligned}\tag{6.37}$$

where $\|q_{1,2,4}''\|_{C^1} = o(\gamma^{-m})$ and $\|q_3''\|_{C^1} = o(\hat{\lambda}^m)$.

Recall that by assumption $\cos m\omega_2 + a_{14} \sin m\omega_2$ is also bounded away from zero. Then, substituting the above u_2 -equation into the u_1 -equation in (6.32) yields

$$\begin{aligned}\left(1 + \frac{a_{14} \sin m\omega_2}{\cos m\omega_2}\right)u_1 &= a_{11}\tilde{x}_1 + a_{12}\tilde{x}_2 + a_{13}\tilde{z} + \frac{a_{14}\gamma^{-m}\tilde{u}_2}{\cos m\omega_2} + a_{15}w \\ &+ O(\tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{z}^2 + \tilde{u}_2^2 + w^2).\end{aligned}$$

Combining this with (6.37) and the \tilde{y} -equation in (6.32), yields that for a point $(\tilde{x}_1, \tilde{x}_2, \tilde{y}, \tilde{z})$ in a small neighborhood of M_1^- we have $(\tilde{u}_1, \tilde{u}_2, \tilde{v}, \tilde{w}) = F_2^m \circ F_{12}(\tilde{x}_1, \tilde{x}_2, \tilde{y}, \tilde{z})$ if and only if

$$\begin{aligned}\gamma^{-m}(\cos m\omega_2 + a_{14} \sin m\omega_2)\tilde{u}_1 &= \gamma^{-m}(a_{14} \cos m\omega_2 - \sin m\omega_2)\tilde{u}_2 \\ &+ a_{11}\tilde{x}_1 + a_{12}\tilde{x}_2 + a_{13}\tilde{z} \\ &+ O(\tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{z}^2) \\ &+ \hat{h}_1(\tilde{x}_1, \tilde{x}_2, \tilde{z}, \tilde{u}_2, \tilde{w}), \\ \tilde{y} - y^- &= O(|\tilde{x}_1| + |\tilde{x}_2| + \|\tilde{z}\|) \\ &+ \hat{h}_2(\tilde{x}_1, \tilde{x}_2, \tilde{z}, \tilde{u}_2, \tilde{w}), \\ \tilde{v} &= \hat{h}_3(\tilde{x}_1, \tilde{x}_2, \tilde{z}, \tilde{u}_2, \tilde{w}),\end{aligned}\tag{6.38}$$

where $\|\hat{h}_1\|_{C^1} = o(\gamma^{-m})$, $\|\hat{h}_2\|_{C^1} = O(\gamma^{-m})$, $\|\hat{h}_3\|_{C^1} = o(\hat{\lambda}^m)$.

Replacing $\tilde{u}_1, \tilde{u}_2, \tilde{v}, \tilde{w}$ by $\tilde{u}_1, \tilde{u}_2, \tilde{v}, \tilde{w}$ in the above formula, and combining it with (6.36) yields (6.34). \square

In formula (6.34), write

$$C_k = C \sin(k\omega_1 + \eta_1) \quad \text{and} \quad D_k = D \sin(k\omega_1 + \eta_2)\tag{6.39}$$

and

$$\begin{aligned}\cos m\omega_2 + a_{14} \sin m\omega_2 &= \sqrt{1 + a_{14}^2} \cos(m\omega_2 + \eta_3), \\ a_{14} \cos m\omega_2 - \sin m\omega_2 &= -\sqrt{1 + a_{14}^2} \sin(m\omega_2 + \eta_3).\end{aligned}\tag{6.40}$$

Since $a_{11}^2 + a_{12}^2 \neq 0$, $b_{21}^2 + b_{12}^2 \neq 0$, and $x^+ \neq 0$ by conditions C1–C3, we have

$$C \neq 0 \quad \text{and} \quad D \neq 0.$$

Moreover, by condition C4.2 and (6.35),

$$\tan \eta_1 \neq \tan \eta_2. \tag{6.41}$$

Introduce the coordinate transformation

$$\begin{aligned} U_1 &= \tilde{u}_1 - u_1^-, & \delta^{\frac{1}{2}} U_2 &= \tilde{u}_2 - u_2^- - b_{41}(\tilde{u}_1 - u_1^-) - b_{42}\tilde{v}, \\ \delta^{\frac{1}{2}} V &= \tilde{v}, & W &= \tilde{w} - w^- - b_{51}(\tilde{u}_1 - u_1^-) - b_{52}\tilde{v}, \end{aligned} \tag{6.42}$$

and consider the restriction of $F_{k,m}$ to

$$\Pi = [-\delta, \delta] \times [-\delta, \delta] \times [-\delta, \delta]^{d-d_1-1} \times [-\delta, \delta]^{d_1-1} \tag{6.43}$$

in the new coordinates. Then, formula (6.34), along with (6.39) and (6.40), implies that for a point $(U_1, U_2, V, W) \in \Pi$ we have $(\bar{U}_1, \bar{U}_2, \bar{V}, \bar{W}) = F_{k,m}(U_1, U_2, V, W)$ if and only if

$$\begin{aligned} &(1 + b_{41} \tan(m\omega_2 + \eta_3))\bar{U}_1 \\ &= \frac{\lambda^k \gamma^m C \sin(k\omega_1 + \eta_1)U_1 + \lambda^k \gamma^m D \sin(k\omega_1 + \eta_2)}{\sqrt{1 + a_{14}^2 \cos(m\omega_2 + \eta_3)}} - u_1^- \\ &\quad - \tan(m\omega_2 + \eta_3)u_2^- + \phi_1(U_1, \bar{U}_2, V, \bar{W}), \end{aligned} \tag{6.44}$$

$$U_2 = \phi_2(U_1, \bar{U}_2, V, \bar{W}), \quad \bar{V} = \phi_3(U_1, \bar{U}_2, V, \bar{W}),$$

$$W = \phi_4(U_1, \bar{U}_2, V, \bar{W}),$$

where

$$\begin{aligned} \phi_1 &= \lambda^k \gamma^m (O(\delta^{\frac{3}{2}}) + o(1)_{k,m \rightarrow \infty}) + O(\delta^{\frac{3}{2}}) + o(1)_{k,m \rightarrow \infty}, \\ \frac{\partial \phi_1}{\partial (U_1, \bar{U}_2, V, \bar{W})} &= \lambda^k \gamma^m (O(\delta^{\frac{1}{2}}) + o(1)_{k,m \rightarrow \infty}) + O(\delta^{\frac{1}{2}}) + o(1)_{k,m \rightarrow \infty}, \end{aligned} \tag{6.45}$$

$$\phi_{2,4} = O(\delta^{\frac{3}{2}}) + o(\hat{\gamma}^{-k}), \quad \frac{\partial \phi_{2,4}}{\partial (U_1, \bar{U}_2, V, \bar{W})} = O(\delta^{\frac{1}{2}}) + o(\hat{\gamma}^{-k}),$$

$$\|\phi_3\|_{C^1} = o(\hat{\lambda}^m).$$

6.4.4 Mutually activating blenders

For the proof of the next proposition, we will consider only such m that $\sin(m\omega_2 + \eta_3)$ is close to zero. In this case, the condition of Lemma 6.6 that $\cos m\omega_2$ and $(\cos m\omega_2 + a_{14} \sin m\omega_2)$ are both bounded away from zero are automatically satisfied, as follows from (6.40).

Proposition 6.7 *Let the system f have a heterodimensional cycle Γ of double-focus type, and $\theta, \frac{\omega_1}{2\pi}, \theta \frac{\omega_2}{2\pi}$ and 1 are rationally independent. There exist, arbitrarily close to Γ , a cs-blender with an activating pair (Π', C^{ss}) and a cu-blender with an activating pair (Π', C^{uu}) where*

$$\Pi' = [-q\delta, q\delta] \times [-\delta, \delta] \times [-\delta, \delta]^{d-d_1-1} \times [-\delta, \delta]^{d_1-1} \tag{6.46}$$

for some fixed $q \in (0, 1)$, and

$$C^{ss} = \{(\Delta U_1, \Delta U_2, \Delta V, \Delta W) : \max\{|\Delta U_1|, \|(\Delta U_2, \Delta W)\|\} \leq K \|\Delta V\|\},$$

$$C^{uu} = \{(\Delta U_1, \Delta U_2, \Delta V, \Delta W) : \max\{|\Delta U_1|, \|\Delta V\|\} \leq K \|(\Delta U_2, \Delta W)\|\}$$

for some sufficiently small K . The two blenders mutually activate each other (in the sense of Proposition 6.2).

Proof Denote

$$\begin{aligned} A_{k,m} &= \lambda^k \gamma^m \frac{C}{\sqrt{1+a_{14}^2}} \sin(k\omega_1 + \eta_1), \\ B_{k,m} &= \lambda^k \gamma^m \frac{D}{\sqrt{1+a_{14}^2}} \sin(k\omega_1 + \eta_2) - u_1^-. \end{aligned} \tag{6.47}$$

We see that if we replace

$$U_1 \rightarrow X_1, \quad (U_2, W) \rightarrow Y, \quad V \rightarrow (X_2, Z), \tag{6.48}$$

then the first-return map (6.44) takes the same form as the first-return map (6.18) in the saddle-focus case, provided we consider the values of (k, m) such that $\sin(m\omega_2 + \eta_3) \rightarrow 0$ and $\lambda^k \gamma^m$ stays bounded. Moreover, formulas (6.47) coincide with formulas (6.14) for the saddle-focus case if we rename the constants $A = \frac{C}{\sqrt{1+a_{14}^2}}, B = \frac{D}{\sqrt{1+a_{14}^2}}$,

and replace u_1^- by $b_{11}u^-$, and ω_1 by ω .

Thus, we obtain Proposition 6.7 in the same way as Proposition 6.2, if we show that given any $(s, t) \in \mathbb{R}^2$, one can find a sequence $\{(k_n, m_n, p_n)\}$ with $k_n, m_n \rightarrow \infty$ such that conditions (6.25) are satisfied, and

$$\sin(m_n\omega_2 + \eta_3) \rightarrow 0$$

(we do not need the evenness of m_n here, as $\gamma > 0$ in the double-focus case).

These requirements are equivalent to the existence of a sequence of integers $\{(k_n, m_n, p_n, l_n)\}$ such that

$$k_n\theta - m_n \rightarrow -t, \quad k_n \frac{\omega_1}{2\pi} - p_n \rightarrow s, \quad k_n \frac{1}{\pi} \theta \omega_2 - l_n \rightarrow -\frac{t\omega_2 + \eta_3}{\pi}.$$

This is guaranteed by our assumption that $\theta, \omega_1/2\pi, \theta\omega_2/2\pi$, and 1 are rationally independent. □

6.4.5 Local stabilization of heterodimensional cycles in the double-focus case

To complete the proof of Theorem 7 we need to show that the two periodic points O_1 and O_2 are homoclinically related to the cs-blender and, respectively, the cu-blender obtained in the above proposition.

Lemma 6.8 *Let Π' and C^{uu} be given by Proposition 6.7. There exists a sequence $\{m_j\}$ of positive integers with $m_j \rightarrow \infty$ such that the image $S_j^u := F_2^{m_j} \circ F_{12}(W_{loc}^u(O_1)) \cap \Pi$ is a disc of the form $(U_1, V) = s(U_2, W)$ for some smooth function s . The disc S_j^u crosses Π' properly with respect to the cone field C^{uu} .*

Proof By (6.38) and (6.42), the image $F_2^m \circ F_{12}(W_{loc}^u(O_1))$ satisfies

$$\begin{aligned} &(\cos m\omega_2 + a_{14} \sin m\omega_2)(U_1 + u_1^-) \\ &= (a_{14} \cos m\omega_2 - \sin m\omega_2)(\delta^{\frac{1}{2}}U_2 + u_2^- + b_{41}U_1 - b_{42}\delta^{\frac{1}{2}}V) + o(1)_{m \rightarrow \infty}, \\ &V = \delta^{-\frac{1}{2}}o(\hat{\lambda}^m), \end{aligned}$$

which by (6.40) reduces to

$$\begin{aligned} &(1 + b_{41} \tan(m\omega_2 + \eta_3))U_1 \\ &= -u_1^- - \tan(m\omega_2 + \eta_3)u_2^- - \tan(m\omega_2 + \eta_3)\delta^{\frac{1}{2}}U_2 + o(1)_{m \rightarrow \infty}, \\ &V = \delta^{-\frac{1}{2}}o(\hat{\lambda}^m). \end{aligned}$$

Now consider a sequence $\{m_j\} \rightarrow \infty$ such that $\tan(m_j\omega_2 + \eta_3) \rightarrow -u_1^-/u_2^-$. Since $1/b_{41} \neq u_1^-/u_2^-$ by condition C4.2 and (6.33), it follows that the above equations can be rewritten as

$$\begin{aligned} U_1 &= -\frac{\tan(m_j\omega_2 + \eta_3)}{1 + b_{41} \tan(m_j\omega_2 + \eta_3)}\delta^{\frac{1}{2}}U_2 + o(1)_{j \rightarrow \infty}, \\ V &= \delta^{-\frac{1}{2}}o(\hat{\lambda}^{m_j}). \end{aligned}$$

This immediately shows that for all sufficiently small δ and all sufficiently large j , and for all $(U_2, W) \in [-\delta, \delta] \times [-\delta, \delta]^{d_1-2}$ we have

$$|U_1| < \frac{1}{2}\delta, \quad \|V\| < \frac{1}{2}\delta, \quad \frac{\partial(U_1, V)}{\partial(U_2, W)} = O(\delta^{\frac{1}{2}}) + o(1)_{j \rightarrow \infty}.$$

The proper crossing of S_j^u with respect to C^{uu} is thus guaranteed. □

Lemma 6.9 *Let Π' and C^{ss} be given by Proposition 6.7. There exists a sequence $\{k_j\}$ of positive integers with $k_j \rightarrow \infty$ such that the preimage $S_j^s := F_{21}^{-1} \circ F_1^{-k_j} \circ F_{12}^{-1}(W_{loc}^s(O_2)) \cap \Pi$ is a disc of the form $(U_1, U_2, W) = s(V)$ for some smooth function s . The disc S_j^s crosses Π' properly with respect to the cone field C^{ss} .*

Proof By (6.32), the preimage $F_{12}^{-1}(W_{loc}^s(O_2))$ satisfies

$$0 = a_{11}\tilde{x}_1 + a_{12}\tilde{x}_2 + a_{13}\tilde{z} + O(\tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{z}^2),$$

$$\tilde{y} - y^- = a_{31}\tilde{x}_1 + a_{32}\tilde{x}_2 + a_{33}\tilde{z} + O(\tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{z}^2).$$

Substitute the equations for $\tilde{x}_1, \tilde{x}_2, \tilde{z}$ from formula (6.36) for the map $F_1^k \circ F_{21}$ into the second equation above. This yields $\tilde{y} = y^- + O(\lambda^k)$ as a function of (\tilde{u}_1, \tilde{v}) . Combining the first equation above with (6.36) and using the new expression for \tilde{y} , we obtain the following equation for $F_{21}^{-1} \circ F_1^{-k} \circ F_{12}^{-1}(W_{loc}^s(O_2))$ (see (6.35) and (6.39)):

$$\tilde{u}_1 - u_1^- = -\frac{D \sin(k\omega_1 + \eta_2)}{C \sin(k\omega_1 + \eta_1)} + O((\tilde{u}_1 - u_1^-)^2 + \|\tilde{v}\|) + o(1)_{k \rightarrow \infty},$$

$$\tilde{u}_2 - u_2^- = b_{41}(\tilde{u}_1 - u_1^-) + b_{42}\tilde{v} + O((\tilde{u}_1 - u_1^-)^2 + \tilde{v}^2) + o(\hat{\gamma}^{-k}),$$

$$\tilde{w} - w_2^- = b_{51}(\tilde{u}_1 - u_1^-) + b_{52}\tilde{v} + O((\tilde{u}_1 - u_1^-)^2 + \tilde{v}^2) + o(\hat{\gamma}^{-k}),$$

where the small terms are functions of \tilde{u}_1 and \tilde{v} . Since ω_1 is irrational, we can choose the sequence of values of $k \rightarrow \infty$ such that $\sin(k\omega_1 + \eta_1)$ stays bounded away from zero.

Now, after solving \tilde{u}_1 from the first equation, we apply the coordinate transformation (6.42) and obtain the intersection $F_{21}^{-1} \circ F_1^{-k} \circ F_{12}^{-1}(W_{loc}^s(O_2)) \cap \Pi$ as

$$U_1 = -\frac{D \sin(k\omega_1 + \eta_2)}{C \sin(k\omega_1 + \eta_1)} + O(\delta^{\frac{1}{2}}V) + o(1)_{k \rightarrow \infty},$$

$$U_2 = O(\delta^{-\frac{1}{2}}U_1^2 + \delta^{\frac{1}{2}}|V|) + o(1)_{k \rightarrow \infty}, \quad W = O(U_1^2 + \delta|V|) + o(1)_{k \rightarrow \infty}.$$

After comparing this with (6.30) using the correspondence (6.48), one just follows the proof of Lemma 6.5. □

It can be seen from (6.42) that $W_{loc}^u(O_2) = \{V = 0\}$, and, after additionally taking $k \rightarrow \infty$ in (6.36), that the intersection $F_{21}^{-1}(W_{loc}^s(O_1)) \cap \Pi$ is the graph of some smooth function $(U_2, W) = s(U_1, V)$ satisfying $s = O(U_1^2 + \delta V^2)$ and defined on $[-\delta, \delta] \times [-\delta, \delta]^{d-d_1-1}$.

Therefore, since the above two lemmas are completely analogous to Lemmas 6.4 and 6.5, one obtains the homoclinic relations between O_1 and the cs-blender, and between O_2 and the cu-blender in the same way as it is shown in the end of Sect. 6.3.

Appendix: Standard blenders

Here we give a detailed construction of standard blenders. Essentially, this is a summary (with a necessary generalization to the case of an arbitrary Markov partition) of the blender horseshoe construction [16]. We also show in Proposition A.4 that the blenders found in this paper are standard. The main purpose is to make our blender-related results more accessible for further use. As an example, see the application

in [44] where we show the C^1 -robustness of homoclinic tangencies near homoclinic tangencies of effective dimension larger than one.

Recall that a compact invariant set Λ of a diffeomorphism g is hyperbolic if it possesses a continuous Dg -invariant splitting of its tangent bundle: $\mathcal{T}_\Lambda \mathcal{M} = E^s \oplus E^u$, such that Dg is expanding in E^u and contracting in E^s . This implies the existence of invariant continuous cone fields C^s and C^u in a small neighborhood U of Λ : for any point $M \in U$, and any vectors $v \in cl(C_M^s) \subset \mathcal{T}_M U$ and $w \in cl(C_M^u) \subset \mathcal{T}_M U$, one has $Dg^{-1}(v) \in int(C_{g^{-1}(M)}^s)$ if $g^{-1}(M) \in U$, and $Dg(v) \in int(C_{g(M)}^u)$ if $g(M) \in U$; moreover, Dg is uniformly expanding in C^u and uniformly contracting in C^s .

The hyperbolic set Λ is cs-partially hyperbolic if the invariant subbundle E^s admits a further invariant splitting $E^s = E^{ss} \oplus E^{cs}$ with $E^{cs} \neq 0$, where the contraction in E^{ss} is uniformly stronger than in E^{cs} . Similarly, it is cu-partially hyperbolic if the invariant subbundle E^u admits an invariant splitting $E^u = E^{uu} \oplus E^{cu}$ with $E^{cu} \neq 0$, where the expansion in E^{uu} is uniformly stronger than in E^{cu} . The partial hyperbolicity implies the existence of a forward-invariant cone field $C^{uu} \subset C^u$ that contains E^{uu} or a backward-invariant cone field $C^{ss} \subset C^s$ that contains E^{ss} .

The compact hyperbolic set Λ is called *basic* if it is transitive and locally-maximal, i.e., there is an orbit which is dense in Λ and, when the neighborhood U of Λ is sufficiently small, any orbit which lies entirely in U lies in Λ . Note that any point whose forward orbit never leaves U belongs to the local stable manifold of some point in Λ , and any point whose backward orbit never leaves U belongs to the local unstable manifold of some point in Λ .

We specifically consider *zero-dimensional* basic sets. Such sets admit a description in terms of finite Markov chains. Namely, one can find a finite unidirectional graph G with k vertices such that the set of orbits in Λ is in 1-to-1 correspondence with the set of all infinite paths along the edges of G . The correspondence is established as follows. One finds a finite collection of disjoint open subsets U_1, \dots, U_n of U (a Markov partition) such that $\Lambda \subset U_1 \cup \dots \cup U_n$, and $g(cl(U_s)) \cap cl(U_j) \neq \emptyset$ if and only if there is an edge in the graph G that goes from the vertex s to the vertex j . Any orbit in Λ defines a coding sequence $\{j_i\}_{i \in \mathbb{Z}} \in \{1, \dots, n\}^{\mathbb{Z}}$ such that the i -th point of the orbit lies in U_{j_i} , for all i . Moreover, the partition $\{U_1, \dots, U_n\}$ can be chosen such that these sequences encode orbits in Λ uniquely (i.e., different orbits produce different sequences $\{j_i\}$) and a sequence $\{j_i\}_{i \in \mathbb{Z}}$ corresponds to some orbit of Λ if and only if the graph G has an edge from the vertex j_i to the vertex j_{i+1} for all i .

One can choose the Markov partition such that $cl(U_j)$ is, for each j , a diffeomorphic image of a Cartesian product $\mathbb{X}_j \times \mathbb{Y}_j \subset \mathbb{R}^{d^s} \times \mathbb{R}^{d^u}$ of two compact convex sets, where $d^s = \dim(E^s)$ and $d^u = \dim(E^u)$. Hence, we can use $x \in \mathbb{R}^{d^s}$ and $y \in \mathbb{R}^{d^u}$ as coordinates on $cl(U_j)$. Let us denote by dx and dy the vectors in the tangent spaces. The invariant cone fields in $U_1 \cup \dots \cup U_n$ are then given by $C^u = \{\|dx\| < K_j^u \|dy\|\}$ and $C^s = \{\|dy\| < K_j^s \|dx\|\}$ (where the numbers $K_j^{s,u}$ may depend on the point in $\mathbb{X}_j \times \mathbb{Y}_j$). Moreover, given any *complete u-disc* $S \subset cl(U_j)$, i.e., a graph of a smooth function $x = h(y)$ defined for all $y \in \mathbb{Y}_j$, taking values $x \in int(\mathbb{X}_j)$ for some j , and satisfying the Lipschitz property $(Dh(dy), dy) \in C^s$, if there is an edge of the graph G going from j to s , then the image $g(S \cap g^{-1}(cl(U_s)))$ is a complete u-disc in $cl(U_s)$, i.e., it is a graph of a smooth function satisfying the same Lipschitz property and defined for all $y \in \mathbb{Y}_s$. Similarly, given any *complete s-disc* $S \subset cl(U_s)$, i.e., a graph of a

smooth function $y = h(x)$ defined for all $x \in \mathbb{X}_s$, taking values $y \in \text{int}(\mathbb{Y}_s)$, and satisfying the Lipschitz property $(dx, Dh(dx)) \in \mathcal{C}^u$, the preimage $g^{-1}(S \cap g(\text{cl}(U_j)))$ is a complete s -disc in $\text{cl}(U_j)$.

Since the tangents to the s -discs lie in \mathcal{C}^s and the tangents to the u -discs lie in \mathcal{C}^u , the map g is uniformly expanding on the u -discs and uniformly contracting on the s -discs. The stable lamination $W_{loc}^s(\Lambda)$ is comprised by complete s -discs and the unstable lamination $W_{loc}^u(\Lambda)$ by complete u -discs. Moreover, $W_{loc}^s(\Lambda) = \bigcup_J W_{loc,J}^s$ where the union is taken over all forward paths $J = \{j_i\}_{i \geq 0}$ in the graph G and $W_{loc,J}^s$ is the uniquely defined complete s -disc that consists of all points $M \in \text{cl}(U_{j_0})$ such that $g^i(M) \in \text{cl}(U_{j_i})$ for all $i \geq 0$. Similarly, $W_{loc}^u(\Lambda) = \bigcup_J W_{loc,J}^u$ where the union is taken over all backward paths $J = \{j_i\}_{i \leq 0}$ in the graph G and $W_{loc,J}^u$ is the uniquely defined completely u -disc that consists of all points $M \in \text{cl}(U_{j_0})$ such that $g^i(M) \in \text{cl}(U_{j_i})$ for all $i \leq 0$.

If the zero-dimensional basic set is also cs -partially hyperbolic, we assume that the Markov partition is chosen such that $\mathbb{X}_j = \mathbb{X}_j^{ss} \times \mathbb{X}_j^{cs}$, where \mathbb{X}_j^{ss} and \mathbb{X}_j^{cs} are convex compact sets of dimension $\dim(E^{ss})$ and $\dim(E^{cs})$, respectively. Thus, a point in $\text{cl}(U_j)$ has coordinates (x^{ss}, x^{cs}, y) with $x^{ss} \in \mathbb{X}_j^{ss}$, $x^{cs} \in \mathbb{X}_j^{cs}$, $y \in \mathbb{Y}_j$. A complete ss -disc of U_j is the graph $\ell := (x^{cs}, y) = p(x^{ss})$ of a smooth function p defined for all $x^{ss} \in \mathbb{X}_j^{ss}$ and such that the tangents to ℓ lie in \mathcal{C}^{ss} at every point of ℓ .

We say that the covering property is satisfied if for any complete ss -disc ℓ there exists s such that the preimage $g^{-1}(\ell \cap g(\text{cl}(U_s)))$ is a complete ss -disc again.

Definition A.1 (Standard blenders) A basic zero-dimensional cs -partially hyperbolic set Λ is called a standard cs -blender of coindex $d = \dim(E^{cs})$ if it satisfies the covering property.

By the covering property, given any complete ss -disc, we have an infinite sequence of its preimages by g which lie in a small neighborhood U of a cs -blender Λ . So, every such disc has at least one point whose backward orbit never leaves U . Such points must belong to $W_{loc}^u(\Lambda)$, which means that every complete ss -disc has a non-empty intersection with the local unstable lamination of the standard cs -blender.

Since the sum of the dimensions of any leaf $W_{loc,J}^u$ of $W_{loc}^u(\Lambda)$ and any ss -disc is strictly less than $\dim(U)$, these intersections are non-transverse, so they get destroyed by arbitrarily small perturbations. However, one can easily show that the covering property is C^1 -open: the basic set Λ persists for all C^1 -small perturbations of g , i.e., it depends continuously on g , the Markov partition and the cone families $\mathcal{C}^{ss} \subset \mathcal{C}^s$ and \mathcal{C}^u remain unchanged, and the covering property holds for the continuation of Λ . It follows that, given any complete ss -disc, it has an intersection with some leaf of $W_{loc}^s(\Lambda)$ for every map which is C^1 -close to g . Thus, a standard cs -blender is a cs -blender in the usual terminology [13, 17, 19], and, particularly, satisfies Definition 1.9.

The set Λ is a standard cu -blender if it is a standard cs -blender of the system obtained from time inversion. More specifically, when Λ is cu -partially hyperbolic, one chooses the Markov partition such that $\mathbb{Y}_j = \mathbb{Y}_j^{uu} \times \mathbb{Y}_j^{cu}$, and defines complete uu -discs as graphs of functions of the form $(x, y^{cu}) = p(y^{uu})$. The set Λ is called a standard cu -blender of coindex $d = \dim(E^{cu})$ if it satisfies the following covering

property: for any complete uu-disc ℓ there exists s such that the preimage $g(\ell \cap g^{-1}(cl(U_s)))$ is a complete uu-disc again. Then every complete uu-disc activates (i.e., has a non-empty intersection with the local stable lamination) the standard blender Λ and this property persists for every C^1 -close map.

Remark A.2 Note that the same construction can be carried out in the case of continuous-time dynamical systems. Given a compact, transitive, locally-maximal, one-dimensional uniformly hyperbolic invariant set Λ of a smooth flow, we can take a cross-section U such that the flow near Λ defines, on U , the return map g – a diffeomorphism acting from a small neighborhood of $\Lambda \cap U$ into U such that $\Lambda \cap U$ is a zero-dimensional basic set of g . Then, Λ is a standard blender of the flow if and only if $\Lambda \cap U$ is a standard blender of g .

Remark A.3 Similarly to the case of a flow, given a diffeomorphism f and a zero-dimensional basic set Λ of f , take an open set U such that $\partial U \cap \Lambda = \emptyset$. Then, $\Lambda \cap U$ is a basic set of an induced map g defined on the union of finitely many disjoint open subsets U_j of U , by the restriction of certain iterations of f to U_j . Iterating by f the Markov partition $\{U_j\}$, one extends it to a small neighborhood of Λ , and proves that if $\Lambda \cap U$ is a standard blender for g , then Λ is a standard blender of f .

Finally, let us show that the blenders obtained in this paper are standard. By the symmetry of the problem, it suffices to consider only cs-blenders. Recall that the finite set $\mathcal{J}_{\delta'}$ of pairs in Lemma 4.10 corresponds to a hyperbolic basic set $\Lambda := \Lambda_{\mathcal{J}_{\delta'}}$ by Lemma 4.7, and it is just the cs-blender in Proposition 4.4.

Proposition A.4 *The cs-blender Λ is standard.*

Proof Let Π' be the cube of size δ' , as in Lemma 4.10. For each pair $(k_j, m_j) \in \mathcal{J}_{\delta'}$, the intersection $T_{k_j, m_j}^{-1}(\Pi') \cap \Pi'$ is a “horizontal strip” of the form

$$U_j = \{(X, Y, Z) \mid X \in [-\delta', \delta'], Y \in \phi_{m_j, k_j}(X, [-\delta', \delta']^{d_1}, Z), Z \in [-\delta', \delta']^{d-d_1-1}\},$$

where ϕ_{m_j, k_j} is the function $\phi_2 = o(\hat{\gamma}^{-k})$ from (3.26) (take $(k, m) = (k_j, m_j)$). These strips are pairwise disjoint, and their union contains $\Lambda \cap \Pi'$. If we define a map g on $\bigcup U_j$ by

$$g(M) = T_{k_j, m_j}(M) \quad \text{if} \quad M \in \text{int}(\sigma_{k_j, m_j}),$$

then the set $\Lambda \cap \Pi'$ is a partially hyperbolic basic set of g . Moreover, it is cs-partially hyperbolic, with the cone fields \mathcal{C}^{cs} , \mathcal{C}^{ss} and \mathcal{C}^{uu} described in Lemma 3.1.

By Remarks A.2 and A.3, it suffices to prove that $\Lambda \cap \Pi'$ is a standard cs-blender of g . Obviously, the sets U_j form a Markov partition for $\Lambda \cap \Pi'$ with the property that for any s and j we have $g(cl(U_s)) \cap cl(U_j)$ non-empty. Next, observe that every $cl(U_j)$ is the diffeomorphic image of

$$Q = \{(x^{cs}, x^{ss}, y) \mid x^{cs} \in [-\delta', \delta'], x^{ss} \in [-\delta', \delta']^{d-d_1-1}, Y \in [-\delta', \delta']^{d_1}\},$$

where the diffeomorphism is defined by

$$X = x^{cs}, \quad Y = \phi_{m_j, k_j}(x^{cs}, y, x^{ss}), \quad Z = x^{ss}.$$

The last thing to check is the covering property, but this is exactly Lemma 4.10. So $\Lambda \cap \Pi'$ is indeed a standard cs-blender of g . \square

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