Inventiones mathematicae (2024) 236:483–547
https://doi.org/10.1007/s00222-024-01237-5

Demazure crystals and the Schur positivity of Catalan functions

Jonah Blasiak[1] · Jennifer Morse[2] · Anna Pun[3]

Received: 14 February 2022 / Accepted: 12 January 2024 / Published online: 8 March 2024

Abstract
Catalan functions, the graded Euler characteristics of certain vector bundles on the flag variety, are a rich class of symmetric functions which include k-Schur functions and parabolic Hall-Littlewood polynomials. We prove that Catalan functions indexed by partition weight are the characters of $U_q(\widehat{\mathfrak{sl}}_\ell)$-generalized Demazure crystals as studied by Lakshmibai-Littelmann-Magyar and Naoi. We obtain Schur positive formulas for these functions, settling conjectures of Chen-Haiman and Shimozono-Weyman. Our approach more generally gives key positive formulas for graded Euler characteristics of certain vector bundles on Schubert varieties by matching them to characters of generalized Demazure crystals.

1 Introduction

The *Kostka-Foulkes polynomials* $K_{\lambda\mu}(q)$ originated in the character theory of $GL_\ell(\mathbb{F}_q)$ and their study has since flourished. They express the modified Hall-Littlewood polynomials in the Schur basis of the ring of symmetric functions, $H_\mu(\mathbf{x};q) = \sum_\lambda K_{\lambda\mu}(q)\, s_\lambda(\mathbf{x})$, and are q-weight multiplicities defined via a q-analog

✉ A. Pun
annapunying@gmail.com

J. Blasiak
jblasiak@gmail.com

J. Morse
morsej@virginia.edu

1 Department of Mathematics, Drexel University, Philadelphia, PA 19104, USA

2 Department of Mathematics, University of Virginia, Charlottesville, VA 22904, USA

3 Department of Mathematics, Baruch College (CUNY), New York, NY 10010, USA

of Kostant's partition function $\mathcal{P}$:

$$K_{\lambda\mu}(q) = \sum_{w\in\mathcal{S}_\ell} \operatorname{sgn}(w)\mathcal{P}_q(w(\lambda+\rho)-(\mu+\rho))$$

$$\text{for} \quad \prod_{\alpha\in\Delta^+} \frac{1}{(1-q\mathbf{x}^\alpha)} = \sum_{\gamma\in\mathbb{Z}^\ell} \mathcal{P}_q(\gamma)\mathbf{x}^\gamma .$$

The positivity property, $K_{\lambda\mu}(q) \in \mathbb{Z}_{\geq 0}[q]$, has deep geometric and combinatorial significance: $K_{\lambda\mu}(q)$ are affine Kazhdan-Lusztig polynomials [65, 66], give characters of cohomology rings of Springer fibers [30, 84], record the Brylinski filtration of weight spaces [14], and are sums over tableaux weighted by the Lascoux-Schützenberger charge statistic [56].

A broader framework has emerged over the last decades. Broer [13] and Shimozono-Weyman [82], in their study of nilpotent conjugacy class closures, replaced the set of all positive roots Δ^+ by a *parabolic* subset—the roots $\Delta(\eta) \subset \Delta^+$ above a block diagonal matrix. Panyushev [74] and Chen-Haiman [16] went further, taking any one of Catalan many upper order ideals $\Psi \subset \Delta^+$. The associated symmetric *Catalan functions*, $H(\Psi;\mu)(\mathbf{x};q) = \sum_\lambda K^\Psi_{\lambda\mu}(q)\, s_\lambda(\mathbf{x})$, indexed by Ψ and partition μ, are graded Euler characteristics of vector bundles on the flag variety.

The broader scope deepened ties to Kazhdan-Lusztig theory, advanced by the discovery of LLT polynomials [24, 39, 59, 60], and inspired a generalization of Jing's Hall-Littlewood vertex operators [83]. Catalan functions were connected to spaces of coinvariants of fusion products in the WZW theory [20, 21], k-Schur functions and Gromov-Witten invariants [10, 11, 49, 51], and affine crystals [61, 70, 78, 80]. Positivity remained a central theme; extending earlier work of Broer, Chen-Haiman [16] posed

Conjecture 1.1 *The Catalan functions $H(\Psi;\mu)$ are Schur positive*: $K^\Psi_{\lambda\mu}(q) \in \mathbb{Z}_{\geq 0}[q]$.

The picture in the *dominant rectangle* case, when $\Psi = \Delta(\eta)$ *and* μ is constant on parabolic blocks, is beautifully complete. These Catalan functions were equated with characters of $U_q(\widehat{\mathfrak{sl}}_\ell)$-Demazure crystals [80], and Schur positive formulas were established using Kirillov-Reshetikhin (KR) crystals [78, 79] and rigged configurations [40]. The view of Catalan functions as Euler characteristics ties their positivity to a conjecture on higher cohomology vanishing, first posed by Broer in the parabolic case and later extended by Chen-Haiman [16] to arbitrary Ψ and partition μ; it was settled by Broer [13] in the dominant rectangle case.

The cohomology of vector bundles associated to Catalan functions, particularly for $\Psi = \Delta^+$, has been extensively studied [12–14, 26, 28, 43, 69, 74]. Hague [26] extended Broer's cohomology vanishing result to some other classes of weights in the parabolic case, using Grauert-Riemenschneider vanishing and Frobenius splitting results of Mehta and van der Kallen [69]. Panyushev [74] established higher cohomology vanishing for a large subclass of Catalan functions; it includes the case μ is strictly decreasing and Ψ arbitrary. Nonetheless, for arbitrary partitions μ, the vanishing conjecture remains open even for parabolic Ψ.

The gold standard is to settle Conjecture 1.1 with a manifestly positive formula. Many attempts to extend the Lascoux-Schützenberger charge formula for Kostka-Foulkes polynomials were made. Shimozono-Weyman [82] conjectured such a formula for the parabolic Catalan functions $H(\Delta(\eta); \mu)$, hinging on an intricate tableau procedure called *katabolism*. Soon after, katabolism led to the origin of k-Schur functions [52], and more recently, Chen-Haiman [16] proposed a variant of katabolism to solve Conjecture 1.1 completely. However, katabolism offered no traction for proofs.

We are now able to paint the picture in its entirety by moving to a larger framework of *tame nonsymmetric Catalan functions* $H(\Psi; \mu; w)$, depending on an additional input $w \in \mathcal{S}_\ell$; they are Euler characteristics of vector bundles on Schubert varieties and specialize to Catalan functions when $w = w_0$. Our findings include

(1) Tame nonsymmetric Catalan functions are characters of $U_q(\widehat{\mathfrak{sl}}_\ell)$-generalized Demazure crystals, certain subsets of tensor products of $U_q(\widehat{\mathfrak{sl}}_\ell)$-highest weight crystals. Lakshmibai-Littelmann-Magyar [46] introduced these crystals in their study of Bott-Samelson varieties.
(2) Tame nonsymmetric Catalan functions are key positive, implying and generalizing Conjecture 1.1. By the powerful theory of Demazure crystals [32, 35, 46, 64], $U_q(\widehat{\mathfrak{sl}}_\ell)$-generalized Demazure crystals restrict to disjoint unions of $U_q(\mathfrak{sl}_\ell)$-Demazure crystals, implying that their characters are key positive.
(3) Positive combinatorial formulas for the key coefficients of (2). We draw on techniques of Naoi [71] to match generalized Demazure crystals with a family of *DARK crystals*, Demazure-like subsets of tensor products of KR crystals. Explicit katabolism combinatorics arises naturally from this point of view.
(4) A katabolism tableau formula for Catalan functions. In the parabolic case, it agrees with and settles the Shimozono-Weyman conjecture.
(5) A conjectural module-theoretic strengthening of (2), generalizing the earlier higher cohomology vanishing conjectures of Broer and Chen-Haiman.
(6) The $t = 0$ nonsymmetric Macdonald polynomials $E_\alpha(\mathbf{x}; q, 0)$ are tame nonsymmetric Catalan functions. Dating back to Sanderson [77], the $E_\alpha(\mathbf{x}; q, 0)$ are characters of certain $U_q(\widehat{\mathfrak{sl}}_\ell)$-Demazure crystals. This topic has recently regained popularity [1, 2, 4–6, 62, 63, 73], and in particular Assaf-Gonzalez [5, 6] gave a key positive formula for $E_\alpha(\mathbf{x}; q, 0)$. Our results yield a different key positive formula, which generalizes Lascoux's tableau formula for cocharge Kostka-Foulkes polynomials [53].

2 Main results

The basic approach of [11] is to open the door to powerful inductive techniques by realizing k-Schur functions as a subclass of (symmetric) Catalan functions. In a similar spirit, our inductive approach here depends crucially on viewing the Catalan functions as a subclass of a larger family of nonsymmetric Catalan functions.

Nonsymmetric Catalan functions are Euler characteristics of vector bundles on Schubert varieties and can be defined by a Demazure operator formula. Fix $\ell \in \mathbb{Z}_{\geq 0}$. The symmetric group $\mathcal{S}_\ell$ acts on $\mathbb{Z}[q][\mathbf{x}] = \mathbb{Z}[q][x_1, \ldots, x_\ell]$ by permuting the x_i; let

$s_i \in \mathcal{S}_\ell$ denote the simple transposition which swaps i and $i+1$. Let $\mathcal{H}_\ell$ denote the 0-Hecke monoid of $\mathcal{S}_\ell$ with generators $\mathsf{s}_1, \dots, \mathsf{s}_{\ell-1}$. It is obtained from $\mathcal{S}_\ell$ by replacing the relations $s_i^2 = id$ with $\mathsf{s}_i^2 = \mathsf{s}_i$. For $i \in [\ell-1] := \{1, 2, \dots, \ell-1\}$, the *Demazure operator* π_i is the linear operator on $\mathbb{Z}[q][\mathbf{x}]$ defined by

$$\pi_i(f) = \frac{x_i f - x_{i+1} s_i(f)}{x_i - x_{i+1}}. \tag{2.1}$$

More generally, for any $w \in \mathcal{H}_\ell$, let $w = \mathsf{s}_{i_1}\mathsf{s}_{i_2}\cdots\mathsf{s}_{i_m}$ and define the associated *Demazure operator* by $\pi_w := \pi_{i_1}\pi_{i_2}\cdots\pi_{i_m}$; this is well defined as the π_i satisfy the 0-Hecke relations.

A *root ideal* is an upper order ideal of the poset $\Delta^+ = \Delta^+_\ell := \{(i, j) \mid 1 \le i < j \le \ell\}$ with partial order given by $(a, b) \le (c, d)$ when $a \ge c$ and $b \le d$. A *labeled root ideal of length* ℓ is a triple (Ψ, γ, w) consisting of a root ideal $\Psi \subset \Delta^+_\ell$, a weight $\gamma \in \mathbb{Z}^\ell$, and $w \in \mathcal{H}_\ell$.

Definition 2.1 The *nonsymmetric Catalan function* associated to the labeled root ideal (Ψ, γ, w) of length ℓ is

$$H(\Psi; \gamma; w)(\mathbf{x}; q) := \pi_w\Big(\mathrm{poly}\Big(\prod_{(i,j)\in\Psi}\big(1 - q x_i/x_j\big)^{-1}\mathbf{x}^\gamma\Big)\Big) \in \mathbb{Z}[q][\mathbf{x}], \tag{2.2}$$

where poly denotes the polynomial truncation operator, defined by its action on key polynomials: $\mathrm{poly}(\kappa_\alpha) = \kappa_\alpha$ for $\alpha \in \mathbb{Z}^\ell_{\ge 0}$ and $\mathrm{poly}(\kappa_\alpha) = 0$ for $\alpha \in \mathbb{Z}^\ell \setminus \mathbb{Z}^\ell_{\ge 0}$ (see §5).

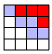
In the case $w = \mathsf{w}_0$, the longest element in $\mathcal{H}_\ell$, we recover the (symmetric) Catalan functions studied in [10, 11, 16, 74]. See §5 for more on the nonsymmetric Catalan functions, Fig. 1 (the second and third to last rows) for some examples, and [11, Example 4.5] for an example of a symmetric Catalan function explicitly expanded out as a sum of Schur functions starting from (2.2).

2.1 The rotation theorem

For a root ideal $\Psi \subset \Delta^+_\ell$, define the tuple $\mathbf{n}(\Psi) = (\mathbf{n}(\Psi)_1, \dots, \mathbf{n}(\Psi)_{\ell-1}) \in [\ell]^{\ell-1}$ by

$$\mathbf{n}(\Psi)_i := \Big|\big\{j \in \{i, i+1, \dots, \ell\} : (i, j) \notin \Psi\big\}\Big|. \tag{2.3}$$

For example, for the root ideal $\Psi = \{(1, 2), (1, 3), (1, 4), (2, 4)\} \subset \Delta^+_4$ illustrated below in red ■ (with matrix-style coordinates), $\mathbf{n}(\Psi) = (1, 2, 2)$ counts the number of light blue boxes □ in each row.

$\Psi =$ , $\qquad \mathbf{n}(\Psi) = (1, 2, 2)$. (2.4)

Definition 2.2 A labeled root ideal (Ψ, γ, w) of length ℓ is *tame* if the right descent set $\{i \in [\ell-1] \mid w\mathsf{s}_i = w\}$ of w contains $\mathbf{n}(\Psi)_1 + 1, \mathbf{n}(\Psi)_1 + 2, \dots, \ell - 1$; note that $\mathbf{n}(\Psi)_1 + 1, \mathbf{n}(\Psi)_1 + 2, \dots, \ell$ are the column indices j for which $(1, j)$ lies in Ψ, and being tame is equivalent to w having a reduced expression which ends in the long word for this interval. *We also say that the associated nonsymmetric Catalan function is tame.*

For example, for Ψ as in (2.4) and any $\gamma \in \mathbb{Z}^\ell$, exactly 4 of the 24 elements $w \in \mathcal{H}_4$ make (Ψ, γ, w) tame, namely, $\mathsf{s}_3\mathsf{s}_2\mathsf{s}_3$, $\mathsf{s}_1\mathsf{s}_3\mathsf{s}_2\mathsf{s}_3$, $\mathsf{s}_2\mathsf{s}_1\mathsf{s}_3\mathsf{s}_2\mathsf{s}_3$, and $\mathsf{s}_3\mathsf{s}_2\mathsf{s}_1\mathsf{s}_3\mathsf{s}_2\mathsf{s}_3$.

Define the $\mathbb{Z}[q]$-algebra homomorphism Φ of $\mathbb{Z}[q][\mathbf{x}]$ by

$$\Phi(x_i) = x_{i+1} \text{ for } i \in [\ell-1], \quad \Phi(x_\ell) = q x_1. \tag{2.5}$$

A crucial finding of this paper is the following operator formula for tame nonsymmetric Catalan functions.

Theorem 2.3 *For any tame labeled root ideal* (Ψ, γ, w) *with* $\gamma \in \mathbb{Z}^\ell_{\geq 0}$,

$$H(\Psi;\gamma;w) = \pi_w x_1^{\gamma_1} \Phi \pi_{\mathsf{s}(n_1)} x_1^{\gamma_2} \Phi \pi_{\mathsf{s}(n_2)} x_1^{\gamma_3} \cdots \Phi \pi_{\mathsf{s}(n_{\ell-1})} x_1^{\gamma_\ell}, \tag{2.6}$$

where $(n_1, \ldots, n_{\ell-1}) = \mathbf{n}(\Psi)$ *and* $\mathsf{s}(d) := \mathsf{s}_{\ell-1}\mathsf{s}_{\ell-2}\cdots\mathsf{s}_d \in \mathcal{H}_\ell$ *for* $d \in [\ell]$.

Its proof requires an in-depth understanding of polynomial truncation and is given in Sect. 5. The operator Φ arises in a recurrence for nonsymmetric Macdonald polynomials, and we will see in Sect. 8 that its appearance here is no coincidence.

2.2 Affine generalized Demazure crystals and key positivity

Theorem 2.3 allows us to connect tame nonsymmetric Catalan functions with affine Demazure crystals. We describe this connection here, but defer a thorough treatment of crystals to Sect. 4.

Let $U_q(\mathfrak{g})$ be the quantized enveloping algebra of a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$ (as in [37]). Among the data specifying a $U_q(\mathfrak{g})$-crystal B are maps $\tilde{f}_i : B \sqcup \{0\} \to B \sqcup \{0\}$ for i ranging over the Dynkin node set I. For a subset S of B and $i \in I$, define

$$\mathcal{F}_i\, S := \{\tilde{f}_i^m b \mid b \in S, m \geq 0\} \setminus \{0\} \subset B.$$

For a dominant integral weight $\Lambda \in P^+$, let $B(\Lambda)$ denote the highest weight $U_q(\mathfrak{g})$-crystal of highest weight Λ and u_Λ its highest weight element.

Definition 2.4 A $U_q(\mathfrak{g})$-*Demazure crystal* is a subset of a highest weight $U_q(\mathfrak{g})$-crystal $B(\Lambda)$ of the form $\mathcal{F}_{i_1} \cdots \mathcal{F}_{i_k}\{u_\Lambda\}$.

Now specialize to $\mathfrak{g} = \widehat{\mathfrak{sl}}_\ell$, our focus here. The associated data includes Dynkin nodes $I = \mathbb{Z}/\ell\mathbb{Z} = \{0, 1, \ldots, \ell-1\}$, fundamental weights $\{\Lambda_i \mid i \in I\}$, weight lattice $P = \sum_{i\in I} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\frac{\delta}{2\ell}$, and dominant weights $P^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\Lambda_i + \mathbb{Z}\frac{\delta}{2\ell} \subset P$. Let τ denote the Dynkin diagram automorphism $I \to I$, $i \mapsto i+1$. Let $\widetilde{S}_\ell$ denote the extended affine symmetric group and $\widetilde{\mathcal{H}}_\ell$ its 0-Hecke monoid. The generators of $\widetilde{\mathcal{H}}_\ell$ are denoted τ and s_i $(i \in I)$, and relations include $\tau \mathsf{s}_i \tau^{-1} = \mathsf{s}_{\tau(i)} = \mathsf{s}_{i+1}$, braid relations, and $\mathsf{s}_i^2 = \mathsf{s}_i$.

Definition 2.5 Let $\mathcal{D}(\widehat{\mathfrak{sl}}_\ell)$ be the set of all subsets $S \subset B$ such that B is a tensor product of highest weight $U_q(\widehat{\mathfrak{sl}}_\ell)$-crystals and the image of S under $B \xrightarrow{\cong} \bigsqcup_{\Lambda \in M} B(\Lambda)$ is a disjoint union of $U_q(\widehat{\mathfrak{sl}}_\ell)$-Demazure crystals. Here, M is a multiset of elements of P^+.

For $\Lambda \in P^+$, define the bijection of sets $\mathcal{F}_\tau \colon B(\Lambda) \to B(\tau(\Lambda))$ by $\tilde{f}_{j_1}^{d_1} \cdots \tilde{f}_{j_k}^{d_k}(u_\Lambda) \mapsto \tilde{f}_{\tau(j_1)}^{d_1} \cdots \tilde{f}_{\tau(j_k)}^{d_k}(u_{\tau(\Lambda)})$, for any $j_1, \dots, j_k \in I$ and $d_i \in \mathbb{Z}_{\geq 0}$; we also denote by $\mathcal{F}_\tau$ the bijection $B(\Lambda^1) \otimes \cdots \otimes B(\Lambda^p) \xrightarrow{\mathcal{F}_\tau \otimes \cdots \otimes \mathcal{F}_\tau} B(\tau(\Lambda^1)) \otimes \cdots \otimes B(\tau(\Lambda^p))$, for $\Lambda^1, \dots, \Lambda^p \in P^+$. We can regard $\mathcal{F}_i$ $(i \in I)$ and $\mathcal{F}_\tau$ as operators on $\mathcal{D}(\widehat{\mathfrak{sl}}_\ell)$ and as such they satisfy the relations of $\widetilde{\mathcal{H}}_\ell$ (by [35, 71]—see §4.7). This yields a well-defined operator $\mathcal{F}_w \colon \mathcal{D}(\widehat{\mathfrak{sl}}_\ell) \to \mathcal{D}(\widehat{\mathfrak{sl}}_\ell)$ for any $w \in \widetilde{\mathcal{H}}_\ell$. For $\Lambda \in P^+$ and $w \in \widetilde{\mathcal{H}}_\ell$, denote by $B_w(\Lambda) = \mathcal{F}_w\{u_\Lambda\}$ the associated $U_q(\widehat{\mathfrak{sl}}_\ell)$-Demazure crystal.

Theorem 2.6 (Combinatorial Excellent Filtration [32, 46]) *For any $\Lambda^1, \Lambda^2 \in P^+$ and $w \in \widetilde{\mathcal{H}}_\ell$, $B_w(\Lambda^2) \otimes u_{\Lambda^1}$ is isomorphic to a disjoint union of $U_q(\widehat{\mathfrak{sl}}_\ell)$-Demazure crystals.*

A $U_q(\widehat{\mathfrak{sl}}_\ell)$-*generalized Demazure crystal* is a subset of a tensor product of highest weight crystals of the form $\mathcal{F}_{w_1}\big(\mathcal{F}_{w_2}\big(\cdots \mathcal{F}_{w_{p-1}}\big(\mathcal{F}_{w_p}\{u_{\Lambda^p}\} \otimes u_{\Lambda^{p-1}}\big) \cdots \otimes u_{\Lambda^2}\big) \otimes u_{\Lambda^1}\big)$ for some $\Lambda^1, \dots, \Lambda^p \in P^+$ and $w_1, \dots, w_p \in \widetilde{\mathcal{H}}_\ell$. Theorem 2.6 and the well-definedness of $\mathcal{F}_w$ on $\mathcal{D}(\widehat{\mathfrak{sl}}_\ell)$ show that these are well-defined and yield the following corollary (this argument is essentially due to [46], with the extended affine setup treated carefully in [71]).

Corollary 2.7 *Any $U_q(\widehat{\mathfrak{sl}}_\ell)$-generalized Demazure crystal is isomorphic to a disjoint union of $U_q(\widehat{\mathfrak{sl}}_\ell)$-Demazure crystals.*

Our focus is on the following subclass of $U_q(\widehat{\mathfrak{sl}}_\ell)$-generalized Demazure crystals: for $\mathbf{w} = (w_1, w_2, \dots, w_p) \in (\mathcal{H}_\ell)^p$ and a partition $\mu = (\mu_1 \geq \cdots \geq \mu_p \geq 0)$, define the associated *affine generalized Demazure (AGD) crystal* by

$$\begin{aligned}&\mathrm{AGD}(\mu; \mathbf{w}) \\ &\quad := \mathcal{F}_{w_1}\big(\mathcal{F}_{\tau w_2}\big(\cdots \mathcal{F}_{\tau w_{p-1}}\big(\mathcal{F}_{\tau w_p}\{u_{\mu^p \Lambda_1}\} \otimes u_{\mu^{p-1}\Lambda_1}\big) \cdots \otimes u_{\mu^2\Lambda_1}\big) \otimes u_{\mu^1\Lambda_1}\big) \\ &\qquad \subset B(\mu^p \Lambda_p) \otimes \cdots \otimes B(\mu^1 \Lambda_1), \qquad (2.7)\end{aligned}$$

where $\mu^i = \mu_i - \mu_{i+1}$, with $\mu_{p+1} := 0$.

Let $\mathbb{Z}[P]$ denote the group ring of P with $\mathbb{Z}$-basis $\{e^\lambda\}_{\lambda \in P}$. The *character* of a $U_q(\widehat{\mathfrak{sl}}_\ell)$-crystal G is $\mathrm{char}(G) := \sum_{g \in G} e^{\mathrm{wt}(g)} \in \mathbb{Z}[P]$. Define the ring homomorphism ζ by

$$\zeta \colon \mathbb{Z}[q][\mathbf{x}] \to \mathbb{Z}[P], \quad x_i \mapsto e^{\Lambda_i - \Lambda_{i-1} + \frac{\ell+1-2i}{2\ell}\delta}, \; q \mapsto e^{-\delta}. \qquad (2.8)$$

Let $\mathbf{n}(\Psi)$ be as in (2.3) and $\mathsf{s}(d) = \mathsf{s}_{\ell-1}\mathsf{s}_{\ell-2} \cdots \mathsf{s}_d$. For a root ideal Ψ, set

$$\mathbf{s}(\Psi) := (\mathsf{s}(\mathbf{n}(\Psi)_1), \dots, \mathsf{s}(\mathbf{n}(\Psi)_{\ell-1})) \in (\mathcal{H}_\ell)^{\ell-1}. \qquad (2.9)$$

Theorem 2.8 *Tame nonsymmetric Catalan functions of partition weight are characters of AGD crystals: for any tame labeled root ideal (Ψ, μ, w) of length ℓ with* *partition* μ,

$$\zeta(H(\Psi;\mu;w)) = e^{-\mu_1\Lambda_0+n_\ell(\mu)\delta}\operatorname{char}\big(\operatorname{AGD}(\mu;(w,\mathbf{s}(\Psi)))\big), \tag{2.10}$$

where $n_\ell(\mu) = \frac{|\mu|(\ell-1)}{2\ell} - \frac{1}{\ell}\sum_{i=1}^{\ell}(i-1)\mu_i$.

Proof sketch From Corollary 2.7 and Kashiwara's results on Demazure crystals [35], one readily obtains a Demazure operator formula for the character of AGD(μ; $(w,\mathbf{s}(\Psi))$), which is not difficult to connect to the rotation Theorem 2.3. □

It can further be shown that the $U_q(\mathfrak{sl}_\ell)$-restriction of a $U_q(\widehat{\mathfrak{sl}}_\ell)$-Demazure crystal is isomorphic to a disjoint union of $U_q(\mathfrak{sl}_\ell)$-Demazure crystals (Theorem 4.1). Combining this with Corollary 2.7 and Theorem 2.8 proves that

Corollary 2.9 *The tame nonsymmetric Catalan functions are key positive.*

More detailed versions of Theorem 2.8 and Corollary 2.9—Theorem 7.5 and Corollary 7.15—are stated and proved in Sect. 7. They include explicit positive formulas for the key expansions.

2.3 DARK crystals

To extract key positive formulas from Theorem 2.8, we use a technique of Naoi [71] to match generalized Demazure crystals with subsets of tensor products of KR crystals, termed DARK crystals; the latter appears to have simpler combinatorics and, remarkably, exactly matches the katabolism combinatorics conjectured in [82].

Let $B^{1,s}$ denote the single row KR crystal; it is a seminormal crystal for the subalgebra $U'_q(\widehat{\mathfrak{sl}}_\ell) \subset U_q(\widehat{\mathfrak{sl}}_\ell)$ (see §4.4). Its elements are labeled by weakly increasing words of length s in the alphabet $[\ell]$. For $\mu = (\mu_1 \geq \cdots \geq \mu_p \geq 0)$, set $\mathcal{B}^\mu = B^{1,\mu_p} \otimes \cdots \otimes B^{1,\mu_1}$.

Definition 2.10 The *Kirillov-Reshetikhin affine Demazure (DARK) crystal* associated to $\mu = (\mu_1 \geq \cdots \geq \mu_p \geq 0)$ and $\mathbf{w} = (w_1,\dots,w_p) \in (\mathcal{H}_\ell)^p$, is the following subset of $\mathcal{B}^\mu$:

$$\mathcal{B}^{\mu;\mathbf{w}} := \mathcal{F}_{w_1}\big(\mathcal{F}_\tau\mathcal{F}_{w_2}\big(\cdots\mathcal{F}_\tau\mathcal{F}_{w_{p-1}}\big(\mathcal{F}_\tau\mathcal{F}_{w_p}\{\mathsf{b}_{\mu_p}\}\otimes\mathsf{b}_{\mu_{p-1}}\big)\cdots\otimes\mathsf{b}_{\mu_2}\big)\otimes\mathsf{b}_{\mu_1}\big), \tag{2.11}$$

where $\mathsf{b}_s \in B^{1,s}$ is the element labeled by the word 1^s, $\mathcal{F}_\tau\colon B^{1,\mu_p}\otimes\cdots\otimes B^{1,\mu_j} \to B^{1,\mu_p}\otimes\cdots\otimes B^{1,\mu_j}$ is given by adding 1 (mod ℓ) to each letter and then sorting each tensor factor to be weakly increasing (see Proposition 6.12), and $\mathcal{F}_{w_i} = \mathcal{F}_{j_1}\cdots\mathcal{F}_{j_k}$ for any chosen expression $w_i = \mathsf{s}_{j_1}\cdots\mathsf{s}_{j_k}$ $(i \in [p])$; the right side of (2.11) does not depend on these choices by [9, Theorem 3.7]. See §2.7 for examples.

The following modification of [71, Proposition 5.16] allows us to port results in crystal theory from AGD to DARK crystals.

Theorem 2.11 ([9, Corollary 3.11]) *Let* $\mathbf{w}$, μ, μ^i *be as in* (2.7). *There is a strict embedding of* $U_q'(\widehat{\mathfrak{sl}}_\ell)$*-seminormal crystals* (*see* §4.1)

$$\Theta_\mu : \mathcal{B}^\mu \otimes B(\mu_1\Lambda_0) \hookrightarrow B(\mu^p\Lambda_p) \otimes \cdots \otimes B(\mu^1\Lambda_1);$$

it is an isomorphism from the domain onto a disjoint union of connected components of the codomain. And under this map, $\Theta_\mu(\mathcal{B}^{\mu;\mathbf{w}} \otimes u_{\mu_1\Lambda_0}) = \mathrm{AGD}(\mu; \mathbf{w})$. *Here, the* $B(s\Lambda_i)$ *are regarded as* $U_q'(\widehat{\mathfrak{sl}}_\ell)$*-seminormal crystals by restriction—see* §4.4.

Remark 2.12 This article makes important use of the $U_q'(\widehat{\mathfrak{sl}}_\ell)$-crystal structures of $\mathcal{B}^\mu \otimes B(\mu_1\Lambda_0)$ and $B(\mu^p\Lambda_p) \otimes \cdots \otimes B(\mu^1\Lambda_1)$, but not of $\mathcal{B}^\mu$—it does not seem to be the right object for the combinatorics of interest here. However, the $U_q(\mathfrak{sl}_\ell)$-restriction of $\mathcal{B}^\mu$, being isomorphic to that of $\mathcal{B}^\mu \otimes B(\mu_1\Lambda_0)$, *is* of interest and will be frequently used.

2.4 Katabolism and Schur positive formulas

We establish the Schur positivity of Catalan functions in the strongest possible terms with a streamlined tableau formula. It arises naturally from DARK crystals by unraveling the $\mathcal{F}_{w_i}$, $\mathcal{F}_\tau$, and tensor operations in their construction (in the spirit of [45, 46]).

Given a weak composition $\alpha = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{Z}^\ell_{\geq 0}$, the *diagram* of α consists of a left justified array of boxes with α_i boxes in row i (rows are allowed to be empty). A *tabloid* T of shape α is a filling of the diagram of α with weakly increasing rows, drawn in English notation with rows labeled $1, 2, \ldots, \ell$ from the top down. Set $\mathrm{shape}(T) = \alpha$. The *content* of T is the vector $(c_1, \ldots, c_p)$, where c_i is the number of times letter i appears in T.

Let $\mathrm{Tabloids}_\ell$ denote the set of tabloids of any shape $\alpha \in \mathbb{Z}^\ell_{\geq 0}$, and $\mathrm{Tabloids}_\ell(\mu) \subset \mathrm{Tabloids}_\ell$ the subset with fixed content μ. Let $\mathrm{SSYT}_\ell(\mu)$ denote the subset of $\mathrm{Tabloids}_\ell(\mu)$ which are *tableaux*, tabloids with partition shape and where entries strictly increase down columns. Given a tabloid T, let T^i denote the i-th row of T and $T^{[i,j]}$ the subtabloid of T consisting of the rows in the interval $[i, j] := \{i, i+1, \ldots, j\}$; set $T^{[j]} = T^{[1,j]}$.

Definition 2.13 (Partial insertion) For $T \in \mathrm{Tabloids}_\ell$ such that $T^{[i,\ell-1]}$ is a tableau, define $P_{i,\ell}(T) \in \mathrm{Tabloids}_\ell$ to be the tabloid obtained by column inserting the ℓ-th row of T into $T^{[i,\ell-1]}$ and leaving rows 1 through $i-1$ of T fixed. (There is a way to extend this definition to any tabloid T but this simpler version is all we need for the results of this section—see Definition 6.8 and Remark 6.16.)

Example 2.14 Let $\ell = 5$. We compute $P_{2,\ell}(T)$ for the $T \in \mathrm{Tabloids}_\ell(4, 3, 3, 3, 2)$ below:

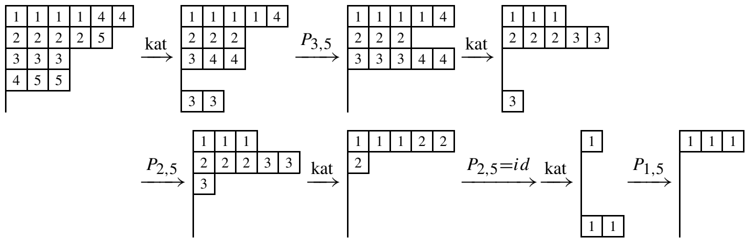
$$T = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 4 & 5 \\ \hline 2 & 3 \\ \cline{1-2} 3 & 4 \\ \cline{1-2} \\ \hline 2 & 2 & 3 & 4 & 5 \\ \cline{1-5} \end{array} \qquad P_{2,\ell}(T) = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 4 & 5 \\ \hline 2 & 2 & 2 & 3 & 3 \\ \cline{1-5} 3 & 4 \\ \cline{1-2} 4 & 5 \\ \cline{1-2} \\ \end{array}$$

$$\mathrm{shape}(T) = (6, 2, 2, 0, 5) \qquad \mathrm{shape}(P_{2,\ell}(T)) = (6, 5, 2, 2, 0)$$

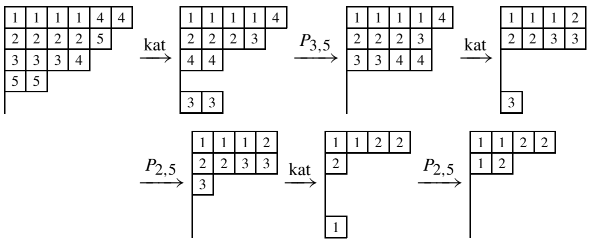
Definition 2.15 (Katabolism) For $T \in \text{Tabloids}_\ell$, define $\text{kat}(T) \in \text{Tabloids}_\ell$ as follows: remove all 1's from T and left justify rows, then remove the first (top) row and add it as the new ℓ-th row, and finally subtract 1 from all letters.

Let $\mathbf{n} = (n_1, \ldots, n_{p-1}) \in [\ell]^{p-1}$ and $\mu \in \mathbb{Z}^p_{\geq 0}$ for some $p \in \mathbb{Z}_{\geq 1}$. A tableau $T \in \text{SSYT}_\ell(\mu)$ is $\mathbf{n}$-*katabolizable* if, for all $i \in [p-1]$, the tabloid $P_{n_i,\ell} \circ \text{kat} \circ \cdots \circ P_{n_2,\ell} \circ \text{kat} \circ P_{n_1,\ell} \circ \text{kat}(T)$ has all its 1's on the first row.

Example 2.16 For $\ell = 5$ and $\mathbf{n} = (3, 2, 2, 1)$, the tableau below (left) is $\mathbf{n}$-katabolizable:



In contrast, the following tableau is not $\mathbf{n}$-katabolizable:



See also Example 6.18.

The elements of $\mathcal{B}^\mu$ are naturally labeled by biwords whose top word is $p^{\mu_p} \cdots 2^{\mu_2} 1^{\mu_1}$ and whose bottom word is weakly increasing on the intervals with constant top word. Define the bijection $\text{inv}\colon \mathcal{B}^\mu \xrightarrow{\cong} \text{Tabloids}_\ell(\mu)$ as follows: for all i, the i-th row of $\text{inv}(b)$ is obtained by sorting the letters above the i's in the bottom word of $b \in \mathcal{B}^\mu$. For example, with $\ell = 4$ and $\mu = (4, 3, 1)$,

$$\begin{pmatrix} 3\,222\,1111 \\ 2\,144\,1114 \end{pmatrix} \xrightarrow{\text{inv}} \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 3 & & & \\ \hline & & & \\ \hline 1 & 2 & 2 & \\ \hline \end{array} \; . \tag{2.12}$$

The bijection inv is essentially the well-known inverse map on biwords generalizing the inverse of a permutation. See §6.4 for more details.

Katabolism exactly characterizes the image of DARK crystals under inv.

Theorem 2.17 *For a partition* μ *and root ideal* Ψ, *the map* inv *gives a bijection*

$$\mathcal{B}^{\mu;(\mathrm{w}_0,\mathbf{s}(\Psi))} \xrightarrow{\text{inv}} \left\{ T \in \text{Tabloids}_\ell(\mu) \mid P(T) \text{ is } \mathbf{n}(\Psi)\text{-katabolizable} \right\}$$

which takes content to shape. Here, $P(T)$ denotes the insertion tableau of the row reading word $T^{\ell} \cdots T^{1}$ of T.

We settle Conjecture 1.1 with a manifestly positive formula.

Theorem 2.18 *For any root ideal $\Psi \subset \Delta_{\ell}^{+}$ and partition $\mu = (\mu_1 \geq \cdots \geq \mu_\ell \geq 0)$, the associated Catalan function has the following Schur positive expression*:

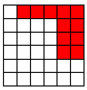
$$H(\Psi; \mu; \mathsf{w}_0)(\mathbf{x}; q) = \sum_{\substack{U \in \mathrm{SSYT}_\ell(\mu) \\ U \text{ is } \mathbf{n}(\Psi)\text{-katabolizable}}} q^{\mathrm{charge}(U)} s_{\mathrm{shape}(U)}(\mathbf{x}) . \tag{2.13}$$

Proof sketch (details in §7.3) Combine Theorems 2.17, 2.8, and 2.11 and select the tabloids which are inv of the $U_q(\mathfrak{sl}_\ell)$-highest weight elements of $\mathcal{B}^{\mu;(\mathsf{w}_0,\mathbf{s}(\Psi))}$. □

See Example 7.16. When $\Psi = \Delta^+$, every $U \in \mathrm{SSYT}_\ell(\mu)$ is $(\mathbf{n}(\Psi) =)$ $\mathbf{1}$-katabolizable and this is the Lascoux-Schützenberger [56] charge formula for the modified Hall-Littlewood polynomial $H_\mu(\mathbf{x}; q) = \sum_\lambda K_{\lambda\mu}(q) s_\lambda(\mathbf{x})$.

Theorem 2.18 resolves the Shimozono-Weyman conjecture [82, Conjecture 27] for the generalized Kostka polynomials $K_{\lambda\mu}^{\Delta(\eta)}(q)$. Indeed, Proposition 7.7 confirms that Shimozono-Weyman katabolizability agrees with $\mathbf{n}(\Psi)$-katabolizability for the *parabolic root ideal* $\Psi = \Delta(\eta)$, defined for $\eta \in \mathbb{Z}_{\geq 0}^{r}$ by

$$\Delta(\eta) := \left\{\alpha \in \Delta_{|\eta|}^{+} \text{ above the block diagonal with block sizes } \eta_1, \ldots, \eta_r\right\}. \tag{2.14}$$

For example, $\Delta(1, 3, 2) =$  .

This gives the first proof of positivity for the Catalan functions and generalized Kostka polynomials in the parabolic case.

Remark 2.19 Shimozono [79] and Schilling-Warnaar [78] give a positive formula for the dominant rectangle Catalan functions $H(\Delta(\eta); \mu; \mathsf{w}_0)$ (i.e., $\mu = a_1^{\eta_1} \cdots a_r^{\eta_r}$, $a_1 \geq \cdots \geq a_r$) using tensor products of arbitrary KR crystals in type A. Included in Theorem 2.18 is a different formula addressing this case, using *subsets* of tensor products of single row KR crystals. Conjecture 10 of [39] proposes a map to reconcile these two different formulas.

We further obtain a positive combinatorial formula for the key expansion of any tame nonsymmetric Catalan function of partition weight by similar methods (Corollary 7.15).

2.5 Consequences for $t = 0$ nonsymmetric Macdonald polynomials

A deep theory of nonsymmetric Macdonald polynomials has developed over the last 30 years, beginning with the work of Opdam-Heckman [72], Macdonald [68], and

Cherednik [17]. Our results apply to the type A nonsymmetric Macdonald polynomials at $t = 0$, $E_\alpha(\mathbf{x}; q, 0)$, a nonsymmetric generalization of the modified Hall-Littlewood polynomials. They were connected to affine Demazure characters by Sanderson [77] and the subject of recent results and conjectures on key positivity [1, 2, 4–6]. The $t = 0$ nonsymmetric Macdonald polynomials in other types have also received considerable attention [31, 62, 63, 73]. Our results yield the following.

Theorem 2.20 *For any* $\alpha \in \mathbb{Z}^\ell_{\geq 0}$, $E_\alpha(\mathbf{x}; q, 0)$ *is* (1) *the character of a* $U_q(\widehat{\mathfrak{sl}}_\ell)$-*Demazure crystal*, *and* (2) *key positive with key expansion*

$$E_\alpha(\mathbf{x}; q, 0) = \sum q^{\sum_i \binom{\alpha_i}{2} - \mathrm{charge}(P(T))} \kappa_{\mathrm{shape}(T)}(\mathbf{x}), \tag{2.15}$$

where the sum is over tabloids T satisfying the katabolizability conditions in Corollary 8.4/ *Definition* 7.12. *Further, up to a specialization* $x_{\ell+1} = \cdots = x_m = 0$ *when* $m = |\alpha| > \ell$, $E_\alpha(\mathbf{x}; q, 0)$ *is* (3) *a tame nonsymmetric Catalan function*, *and* (4) *the Euler characteristic of a vector bundle on a Schubert variety.*

Proof Statement (1) is due to Sanderson [77], and we also recover it as a special case of our character formula (7.8) for AGD crystals (see Theorem 8.3). Statement (2) is proved in Corollary 8.4, (3) in Theorem 8.11, and (4) follows from (3) and Theorem 3.2. □

The formula (2.15) generalizes Lascoux's formula for cocharge Kostka-Foulkes polynomials [53], answering a call put out in [2, Conjecture 15], [55, p. 267-268] for a description of the key coefficients of $E_\alpha(\mathbf{x}; q, 0)$ in this style. Assaf-Gonzalez [5, 6] studied the problem from a different point of view and realized the coefficients in terms of crystals on nonattacking fillings with no coinversion triples (objects defined in [25]). See also Remark 8.6.

2.6 Consequences for k-Schur functions

The k-Schur functions are a family of symmetric functions over $\mathbb{Q}(q)$ which arose in the study of Macdonald polynomials [52]; many conjecturally equivalent candidates for k-Schur functions have been proposed over the years. At $q = 1$, most of the different candidates were proven to be equal, and this case was connected to Gromov-Witten invariants and affine Schubert calculus [47, 50, 51].

For $\mu = (k \geq \mu_1 \geq \cdots \geq \mu_\ell \geq 0)$, define the *$k$-Schur Catalan function* by

$$\mathfrak{s}^{(k)}_\mu(\mathbf{x}; q) := H(\Delta^k(\mu); \mu; \mathsf{w}_0)(\mathbf{x}; q),$$

where the root ideal $\Delta^k(\mu)$ is determined by

$$\mathbf{n}(\Delta^k(\mu))_i = \min\{k - \mu_i + 1, \ell - i + 1\} \quad \text{for all } i \in [\ell].$$

It was established in [10, 11] that these Catalan functions agree with several of the (generic q) k-Schur candidates including one involving chains in Bruhat order on $\widetilde{S}_{k+1}$ [48] and one involving Jing operators [49]. It then follows from [47, 48, 50] that the $q = 1$ specializations $\{\mathfrak{s}_\mu^{(k)}(\mathbf{x};1)\}$ match two other candidates for k-Schur functions which have no q parameter; in particular, they represent Schubert classes in the homology of the affine Grassmannian $\mathrm{Gr}_{SL_{k+1}}$.

A combinatorial formula for the Schur expansion of $\mathfrak{s}_\mu^{(k)}(\mathbf{x};q)$ was given in [11] in terms of chains in Bruhat order on the affine symmetric group $\widehat{\mathcal{S}}_{k+1}$ and the spin statistic. Theorem 2.18 yields a very different formula:

Corollary 2.21 *The k-Schur function $\mathfrak{s}_\mu^{(k)}$ has the following Schur positive expansion*:

$$\mathfrak{s}_\mu^{(k)}(\mathbf{x};q) = \sum_{\substack{T \in \mathrm{SSYT}_\ell(\mu) \\ T \text{ is } \mathbf{n}(\Delta^k(\mu))\text{-katabolizable}}} q^{\mathrm{charge}(T)} s_{\mathrm{shape}(T)}(\mathbf{x}). \tag{2.16}$$

Namely, T occurs in the sum as follows: remove the μ_1 1's from the first row of T and column insert the remainder of row 1 into rows larger than $\min\{k-\mu_1, \ell-1\}$; remove μ_2 2's from first row and column insert its remainder into rows larger than $\min\{k-\mu_2, \ell-2\}$; continue until reaching an i such that there are not μ_i i's in the first row; T survives if no such i occurs.

Example 7.16 illustrates (2.16) for $\mathfrak{s}_{22211}^{(3)}$. This formula has the same spirit as the original definition of k-Schur functions [52], which expressed them in terms of sets of tableaux called super atoms $\mathbb{A}_\mu^{(k)}$, constructed using Shimozono-Weyman katabolism and crystal reflection operators.

Conjecture 2.22 *The set of tableaux appearing in (2.16) is equal to the super atom $\mathbb{A}_\mu^{(k)}$.*

2.7 Examples

We provide several examples for reference throughout the article.

Figure 1 (fifth column) depicts the DARK crystal $\mathcal{B}^{\mu;\mathbf{w}}$ for $\ell = 3$, $\mu = (2,1,1)$, $\mathbf{w} = (id, \mathsf{s}_2\mathsf{s}_1, \mathsf{s}_2\mathsf{s}_1)$; it can be constructed step by step using the $\mathcal{F}_i$, $\mathcal{F}_\tau$, and tensor operations as illustrated.

The first two lines in Figure 1 give two different names for each DARK crystal. The connected components of solid edges decompose them into $U_q(\mathfrak{sl}_\ell)$-Demazure crystals, each of which has character equal to a key polynomial; the key expansions of their charge weighted characters (see §7) are given in the third to last line, written so that reading left to right gives the components top to bottom, e.g., $\{3211\}$ has character $\kappa_{211} = x_1^2 x_2 x_3$. By Corollary 7.15, these characters are tame nonsymmetric Catalan functions (second to last line), though this requires rewriting the DARK crystals appropriately (last line), e.g., $\mathcal{F}_{\mathsf{s}_1}(\mathcal{F}_\tau\mathcal{F}_{\mathsf{s}_2\mathsf{s}_1}(\mathsf{b}_1)\otimes\mathsf{b}_1) = \mathcal{F}_{\mathsf{s}_1\mathsf{s}_2}(\mathcal{F}_\tau\mathcal{F}_{\mathsf{s}_2\mathsf{s}_1}(\mathcal{F}_\tau\mathcal{F}_{\mathsf{s}_2\mathsf{s}_1}(\mathsf{b}_0)\otimes\mathsf{b}_1)\otimes\mathsf{b}_1) = \mathcal{B}^{(1,1,0);(\mathsf{s}_1\mathsf{s}_2,\mathsf{s}_2\mathsf{s}_1,\mathsf{s}_2\mathsf{s}_1)}$. Here, b_s denotes the element of $B^{1,s}$ labeled by 1^s, with b_0 the empty word (see §6.1).

The dashed arrows are the $\tilde{f}_0$-edges of $\mathcal{B}^{\mu;\mathbf{w}} \otimes u_{2\Lambda_0}$ (technically this is just a subset of the $U_q'(\widehat{\mathfrak{sl}}_\ell)$-seminormal crystal $\mathcal{B}^\mu \otimes B(2\Lambda_0)$ but we often think of it as coming with the edges $\tilde{f}_i, \tilde{e}_i$ $(i \in I)$ which have both ends in the subset). By Theorem 2.11,

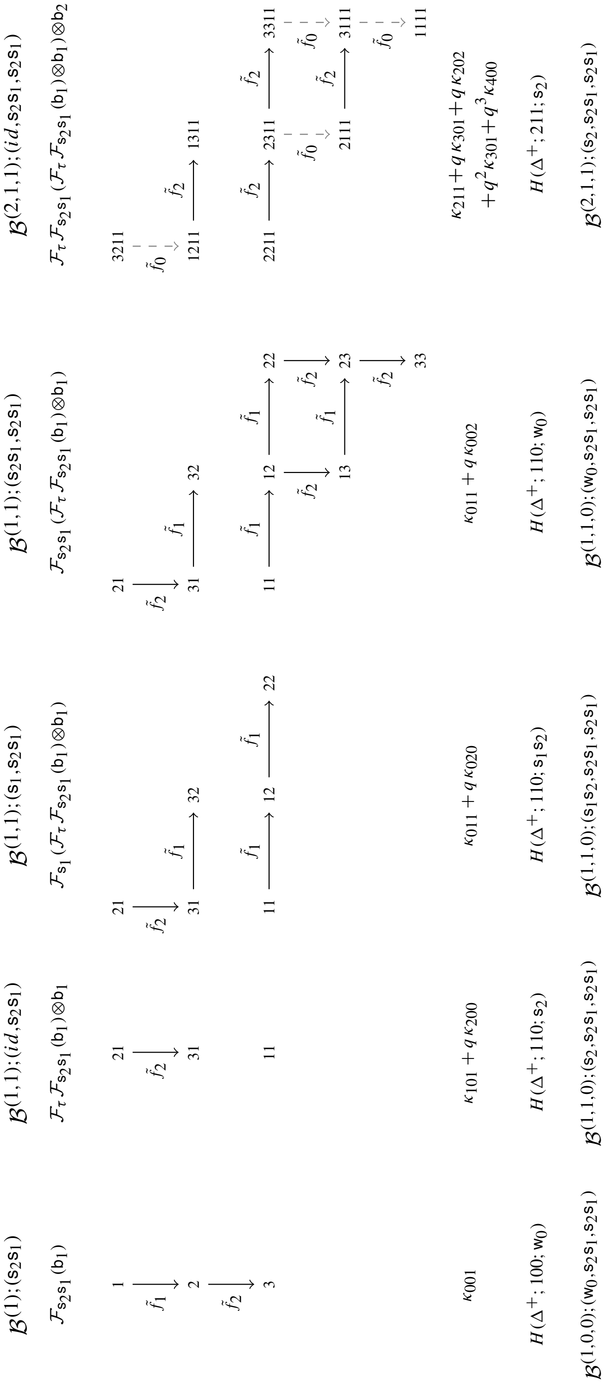


Fig. 1 Five DARK crystals and associated data

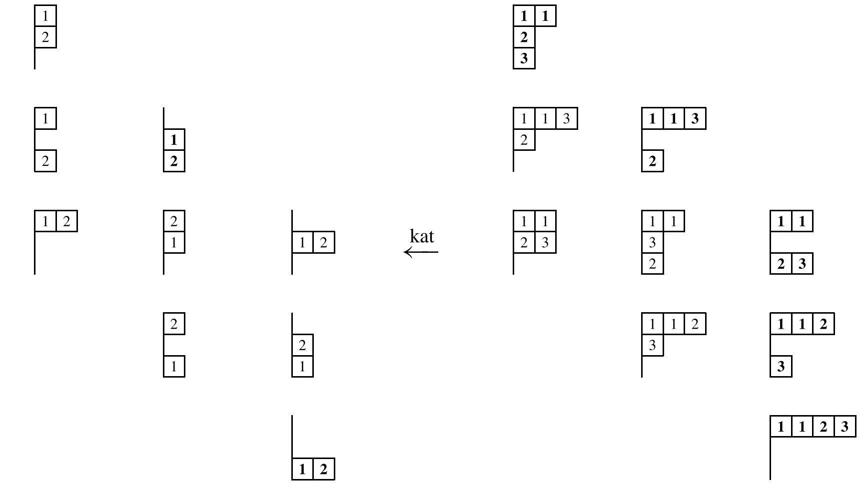


Fig. 2 The image under inv of the crystals in the fourth and fifth columns of Fig. 1

$\mathrm{AGD}(\mu;\mathbf{w}) = \Theta_\mu(B^{\mu;\mathbf{w}} \otimes u_{2\Lambda_0})$, which is isomorphic to a disjoint union of $U_q(\widehat{\mathfrak{sl}}_\ell)$-Demazure crystals; the corresponding decomposition of $B^{\mu;\mathbf{w}}$ is given by the components of dashed and solid edges (in the crystal in the fifth column). Here there are two such components, so $\mathrm{AGD}(\mu;\mathbf{w})$ is not a single $U_q(\widehat{\mathfrak{sl}}_\ell)$-Demazure crystal; this demonstrates a fundamental difference between this work and earlier work [44, 77, 80] relating generalizations of Kostka-Foulkes polynomials to Demazure crystals, where only single $U_q(\widehat{\mathfrak{sl}}_\ell)$-Demazure crystals were used.

Figure 2 depicts the tabloids obtained by applying inv to the two DARK crystals in fourth and fifth columns of Fig. 1. By Theorem 6.20 (the full version of Theorem 2.17), the tabloids on the right are also the $T \in \mathrm{Tabloids}_\ell(211)$ which are $\mathbf{w}$-katabolizable in the sense of Definition 6.14; the ones on the left are the $T \in \mathrm{Tabloids}_\ell(11)$ which are $(\mathsf{s}_2\mathsf{s}_1, \mathsf{s}_2\mathsf{s}_1)$-katabolizable. The bold tabloids, by reading off their shapes and charges, give the key expansions in the fourth and fifth columns of Fig. 1; this will be explained in Corollary 7.15.

Figure 3 depicts the tensor product of KR crystals $\mathcal{B}^\mu = B^{1,1} \otimes B^{1,1} \otimes B^{1,1}$, which is also the DARK crystal $\mathcal{B}^{\mu;\mathbf{w}}$ for $\mu = (1, 1, 1)$ and $\mathbf{w} = (\mathsf{s}_2\mathsf{s}_1, \mathsf{s}_2\mathsf{s}_1, \mathsf{s}_2\mathsf{s}_1)$. Its charge weighted character is the modified Hall-Littlewood polynomial $H_{111}(\mathbf{x}; q) = s_{111} + (q + q^2)s_{21} + q^3 s_3 = \kappa_{111} + (q + q^2)\kappa_{012} + q^3\kappa_{003}$. Horizontal and vertical arrows give the $\tilde{f}_1$, $\tilde{f}_2$ edges, respectively. The dashed arrows are the $\tilde{f}_0$-edges of $\mathcal{B}^\mu \otimes B(\Lambda_0)$ which have both ends in $\mathcal{B}^{\mu;\mathbf{w}} \otimes u_{\Lambda_0} \subset \mathcal{B}^\mu \otimes B(\Lambda_0)$. Since μ is constant, the corresponding generalized Demazure crystal $\mathrm{AGD}(\mu;\mathbf{w}) = \Theta_\mu(B^{\mu;\mathbf{w}} \otimes u_{\Lambda_0})$ (via Theorem 2.11) is an actual affine Demazure crystal, namely $\mathcal{F}_{\mathsf{s}_2\mathsf{s}_1\tau\mathsf{s}_2\mathsf{s}_1\tau\mathsf{s}_2\mathsf{s}_1}\{u_{\Lambda_1}\} \subset B(\Lambda_0)$. Shown bold is the DARK crystal $\mathcal{B}^{\mu;\mathbf{v}} = \big(\mathcal{F}_\tau\mathcal{F}_{\mathsf{s}_2\mathsf{s}_1}(\mathcal{F}_\tau\mathcal{F}_{\mathsf{s}_2\mathsf{s}_1}(\mathsf{b}_1) \otimes \mathsf{b}_1)\big) \otimes \mathsf{b}_1$, for $\mu = (1, 1, 1)$, $\mathbf{v} = (id, \mathsf{s}_2\mathsf{s}_1, \mathsf{s}_2\mathsf{s}_1)$, which has charge weighted character $\kappa_{111} + q\kappa_{102} + q^2\kappa_{201} + q^3\kappa_{300}$, also equal to the $t = 0$ nonsymmetric Macdonald polynomial $\tilde{E}_{300}(\mathbf{x}; q)$ and the nonsymmetric Catalan function $H(\Delta^+; \mu; \mathsf{s}_2)$. Again, the corresponding generalized Demazure crystal $\mathrm{AGD}(\mu;\mathbf{v}) = \Theta_\mu(B^{\mu;\mathbf{v}} \otimes u_{\Lambda_0}) = \mathcal{F}_{\tau\mathsf{s}_2\mathsf{s}_1\tau\mathsf{s}_2\mathsf{s}_1}\{u_{\Lambda_1}\} \subset B(\Lambda_0)$ is an affine Demazure crystal.

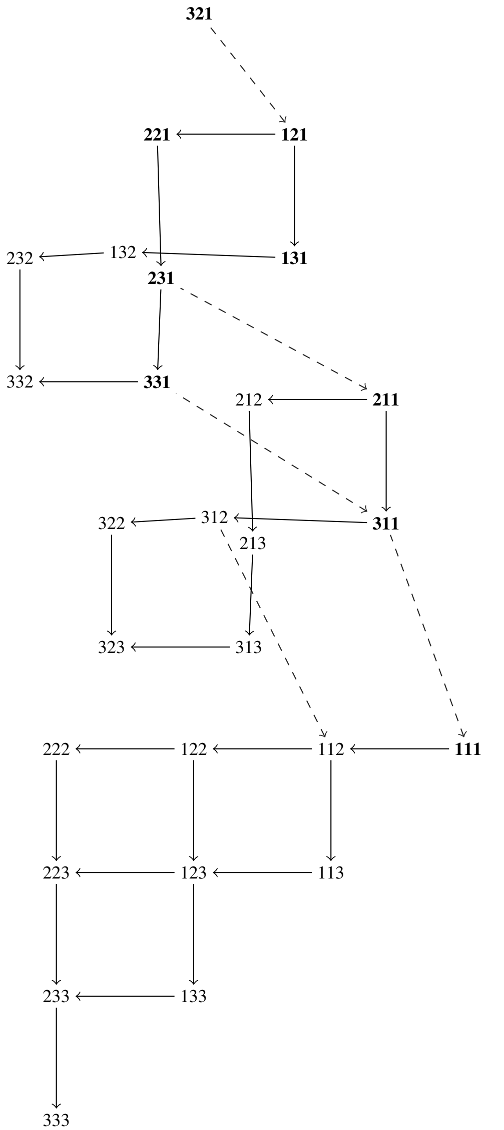


Fig. 3 The tensor product of KR crystals $B^{1,1} \otimes B^{1,1} \otimes B^{1,1}$ and, in bold, the DARK crystal $\mathcal{B}^{\mu;\mathbf{v}}$ for $\mu = (1,1,1)$, $\mathbf{v} = (id, \mathsf{s}_2\mathsf{s}_1, \mathsf{s}_2\mathsf{s}_1)$

3 Higher cohomology vanishing and nonsymmetric Catalan functions

This section uses notation in §1, (2.1)–(2.2), and Definition 5.1, but is otherwise notationally independent from the remainder of the paper.

Let $G = GL_\ell(\mathbb{C})$ and $B \subset G$ the standard upper triangular Borel subgroup. For $w \in \mathcal{S}_\ell$, let $X_w = \overline{B \cdot wB} \subset G/B$ denote the Schubert variety. Given a B-module N, let $G \times_B N$ denote the homogeneous G-vector bundle on G/B with fiber N above $B \in G/B$, and let $\mathscr{L}(N)$ denote the locally free $\mathcal{O}_{G/B}$-module of its sections. We also denote by $\mathscr{L}(N) = \mathscr{L}(N)|_{X_w}$ the restriction of $\mathscr{L}(N)$ to X_w.

Consider the adjoint action of B on the Lie algebra $\mathfrak{u}$ of strictly upper triangular matrices. The B-stable (or "ad-nilpotent") ideals of $\mathfrak{u}$ are in bijection with root ideals via the map sending the root ideal Ψ to the B-submodule, call it $\mathfrak{u}_\Psi$, of $\mathfrak{u}$ with weights $\{\epsilon_i - \epsilon_j \mid (i,j) \in \Psi\}$.

The *character* of a B-module N is $\operatorname{char}(N) = \sum_{\alpha \in \mathbb{Z}^\ell} \dim(N_\alpha)\mathbf{x}^\alpha$, where $N_\alpha = \{v \in N \mid \operatorname{diag}(x_1,\ldots,x_\ell)v = \mathbf{x}^\alpha v\}$ is the α-*weight space* of N and $\mathbf{x}^\alpha := x_1^{\alpha_1}\cdots x_\ell^{\alpha_\ell}$. Let d be the $\mathbb{Z}$-linear operator on $\mathbb{Z}[x_1^{\pm1},\ldots,x_\ell^{\pm1}]$ satisfying $\mathsf{d}(\mathbf{x}^\alpha) = \mathbf{x}^{-\alpha}$, so that $\operatorname{char}(N^*) = \mathsf{d}(\operatorname{char}(N))$; extend it to an operator on $\mathbb{Z}[x_1^{\pm1},\ldots,x_\ell^{\pm1}][[q]]$ by $\mathsf{d}(\sum_{d\ge0} f_d q^d) = \sum_{d\ge0}\mathsf{d}(f_d)q^d$.

For $\gamma \in \mathbb{Z}^\ell$, let $\mathbb{C}_\gamma$ denote the one-dimensional B-module of weight γ.

We need the following result of Demazure [18, §5.5] (this assumes G is semisimple; see also [15, II.14.18 (a)] where reductive G are allowed).

Theorem 3.1 *For any weight $\gamma \in \mathbb{Z}^\ell$ and $w \in \mathcal{S}_\ell$,*

$$\mathsf{d} \circ \pi_w \circ \mathsf{d}(\mathbf{x}^\gamma) = \sum_{i\ge0}(-1)^i \operatorname{char} H^i\big(X_w, \mathscr{L}(\mathbb{C}_\gamma)\big).$$

Nonsymmetric Catalan functions appear naturally as graded Euler characteristics, extending a description of the Catalan functions in [16, 74]:

Theorem 3.2 *For any labeled root ideal (Ψ, γ, w),*

$$H(\Psi;\gamma;w) = \operatorname{poly}\circ\mathsf{d}\Big(\sum_{i,j\ge0}(-1)^i q^j \operatorname{char} H^i\big(X_w, \mathscr{L}(S^j\mathfrak{u}_\Psi^* \otimes \mathbb{C}_\gamma^*)\big)\Big), \tag{3.1}$$

where $S^j\mathfrak{u}_\Psi^*$ denotes the j-th symmetric power of the B-module $\mathfrak{u}_\Psi^*$.

Proof The series $\mathsf{d}\big(\prod_{(i,j)\in\Psi}\big(1 - qx_i/x_j\big)^{-1}\mathbf{x}^\gamma\big)$ gives the character of $\bigoplus_j S^j\mathfrak{u}_\Psi^* \otimes \mathbb{C}_\gamma^*$ where q keeps track of the grading. Each homogeneous component $S^j\mathfrak{u}_\Psi^* \otimes \mathbb{C}_\gamma^*$ has a B-module filtration into one-dimensional weight spaces. Then by the additivity of the Euler characteristic and Theorem 3.1,

$$\sum_{i,j\ge0}(-1)^i q^j \operatorname{char} H^i\big(X_w, \mathscr{L}(S^j\mathfrak{u}_\Psi^* \otimes \mathbb{C}_\gamma^*)\big) = \mathsf{d}\circ\pi_w\circ\mathsf{d}\circ\mathsf{d}\Big(\prod_{(i,j)\in\Psi}\big(1 - qx_i/x_j\big)^{-1}\mathbf{x}^\gamma\Big).$$

Applying $\operatorname{poly}\circ\mathsf{d}$ to both sides, the right side becomes the nonsymmetric Catalan function $H(\Psi;\gamma;w)$ from Definition 2.1 after using $\operatorname{poly}\circ\pi_w = \pi_w\circ\operatorname{poly}$ (Proposition 5.5 (i)). □

Remark 3.3 A version of (3.1) holds for any B-module N, with the product over Ψ in the definition of $H(\Psi;\gamma;w)$ replaced by a product over the multiset of weights of N. However, restricting to the $\mathfrak{u}_\Psi$ is natural from the geometric perspective of [74], e.g., for $w = w_0$ and $\Psi = \Delta^+$, $H^i(G/B, \mathscr{L}(\bigoplus_j S^j\mathfrak{u}^* \otimes \mathbb{C}^*_\gamma)) \cong H^i(T^*(G/B), \theta^*\mathscr{L}(\mathbb{C}^*_\gamma))$, where $T^*(G/B)$ is the cotangent bundle of the flag variety and $\theta\colon T^*(G/B) \to G/B$ the projection.

For $\nu = (\nu_1 \geq \cdots \geq \nu_\ell) \in \mathbb{Z}^\ell$, let $V(\nu)$ be the irreducible G-module of highest weight ν. Let $\alpha \in \mathbb{Z}^\ell$ and α^+ be the weakly decreasing rearrangement of α. The *Demazure module* $D(\alpha) \subset V(\alpha^+)$ is the B-module $B\,u_\alpha$, where u_α is an element of the (one-dimensional) α-weight space of $V(\alpha^+)$. The *Demazure atom module* $\hat{D}(\alpha)$ is the quotient of $D(\alpha)$ by the sum of all Demazure modules properly contained in $D(\alpha)$. The characters $\kappa_\alpha(\mathbf{x}) = \mathrm{char}(D(\alpha))$ and $\hat{\kappa}_\alpha(\mathbf{x}) = \mathrm{char}(\hat{D}(\alpha))$ are the key polynomial and Demazure atom, respectively which will be discussed further in §4.8 and §5.2.

As in [85, §2.3], say a B-module N has an *excellent filtration* (resp. *relative Schubert filtration*) if its dual N^* has a B-module filtration whose subquotients are isomorphic to Demazure modules (resp. Demazure atom modules).

Conjecture 3.4 *Let* (Ψ, μ, w) *be a labeled root ideal with partition* μ *and* $j \geq 0$.

(i) *The nonsymmetric Catalan function* $H(\Psi;\mu;w)$ *is a positive sum of Demazure atoms, i.e.,* $H(\Psi;\mu;w)(\mathbf{x};q) = \sum_\alpha K^{\Psi,w}_{\alpha,\mu}(q)\,\hat{\kappa}_\alpha(\mathbf{x})$ *with* $K^{\Psi,w}_{\alpha,\mu}(q) \in \mathbb{Z}_{\geq 0}[q]$.
(ii) $H^i(X_w, \mathscr{L}(S^j\mathfrak{u}^*_\Psi \otimes \mathbb{C}^*_\mu)) = 0$ *for* $i > 0$.
(iii) $H^0(X_w, \mathscr{L}(S^j\mathfrak{u}^*_\Psi \otimes \mathbb{C}^*_\mu))$ *has a relative Schubert filtration.*
(iv) $H^0(X_w, \mathscr{L}(S^j\mathfrak{u}^*_\Psi \otimes \mathbb{C}^*_\mu))$ *has an excellent filtration when* (Ψ, μ, w) *is tame.*

For tame (Ψ, μ, w), Corollary 2.9 implies (i), while conjectures (ii) and (iv) constitute a module-theoretic strengthening of this corollary. Similarly, (ii) and (iii) give a module-theoretic strengthening of (i).

In this paragraph we discuss the $w = w_0$ ($X_w = G/B$) case of Conjecture 3.4. First note that the cohomology groups are G-modules, so (iii)–(iv) hold and (ii) implies (i). Conjecture (ii) was posed by Chen-Haiman [16, Conjecture 5.4.3]; this generalized a conjecture of Broer for parabolic Ψ, which he settled in the dominant rectangle case [13, Theorem 2.2]. Hague [26, Theorems 4.15 and 4.23] extended this result to some other classes of weights (still parabolic Ψ). Panyushev proved that (ii) holds when the weight $\mu - \rho + \sum_{(i,j)\in\Delta^+\setminus\Psi} \epsilon_i - \epsilon_j$ is weakly decreasing, where $\rho = (\ell-1, \ell-2, \ldots, 0)$. Frobenius splitting methods [43] give another proof of a subcase of Broer's result; this method has the advantage of applying to G over algebraically closed fields of prime characteristic.

When $\Psi = \varnothing$, $H^0(X_w, \mathscr{L}(\mathbb{C}^*_\mu))^* = D(w\mu)$ (implying (iii)-(iv)) and the cohomology vanishing (ii) are results of Demazure [18, 19]; a gap in [18] is bypassed by another proof method of Andersen [3, §4.3]; see also II.14.18 (b) and II.14.15 (e) of [15].

For non-tame (Ψ, μ, w), (i) does not seem amenable to the methods of this paper. The main barrier is that we do not know an analog of the rotation theorem (Theorem 2.3) in this setting. Also, since the $H(\Psi; \mu; w)$ are only conjecturally atom positive and not key positive in general, realizing them as characters of crystals is more difficult—for instance, they cannot be characters of generalized Demazure crystals.

4 Background on crystals

We begin by reviewing crystals for any symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$ and prove that restrictions of Demazure crystals are disjoint unions of Demazure crystals. We then fix notation and conventions for $\mathfrak{g} = \widehat{\mathfrak{sl}}_\ell$ following Naoi [71] and Kac [33]; note that the notation I, P, P^+, α_i, $\alpha_i^\vee$ is for general $\mathfrak{g}$ in §4.1–4.2 and for $\widehat{\mathfrak{sl}}_\ell$ from §4.3 through the remainder of the paper.

4.1 $U_q(\mathfrak{g})$-(seminormal) crystals

The quantized enveloping algebra $U_q(\mathfrak{g})$ is specified by a Dynkin node set I, coweight lattice P^*, weight lattice $P = \mathrm{Hom}_{\mathbb{Z}}(P^*, \mathbb{Z})$, simple coroots $\{\alpha_i^\vee\}_{i\in I} \subset P^*$, simple roots $\{\alpha_i\}_{i\in I} \subset P$, and a symmetric bilinear form $(\cdot,\cdot)\colon P \times P \to \mathbb{Q}$ subject to several conditions (see [37, §2.1]). This data given, a $U_q(\mathfrak{g})$-*seminormal crystal* is a set B equipped with a *weight function* $\mathrm{wt}\colon B \to P$ and *crystal operators* $\tilde{e}_i, \tilde{f}_i\colon B \sqcup \{0\} \to B \sqcup \{0\}$ $(i \in I)$ such that for all $i \in I$ and $b \in B$, there holds $\tilde{e}_i(0) = \tilde{f}_i(0) = 0$ and

$$\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i \text{ whenever } \tilde{e}_i b \neq 0, \quad \text{and}$$

$$\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i \text{ whenever } \tilde{f}_i b \neq 0;$$

$$\varepsilon_i(b) := \max\{k \geq 0 \mid \tilde{e}_i^k b \neq 0\} < \infty, \qquad \phi_i(b) := \max\{k \geq 0 \mid \tilde{f}_i^k b \neq 0\} < \infty;$$

$$\langle \alpha_i^\vee, \mathrm{wt}(b)\rangle = \phi_i(b) - \varepsilon_i(b);$$

$$\tilde{f}_i(\tilde{e}_i b) = b \text{ whenever } \tilde{e}_i b \neq 0, \quad \text{and } \tilde{e}_i(\tilde{f}_i b) = b \text{ whenever } \tilde{f}_i b \neq 0.$$

This agrees with the notion of a seminormal crystal in [37, §7], the notion of a crystal in [71], and the notion of a P-weighted I-crystal in [80].

A *strict embedding* of $U_q(\mathfrak{g})$-seminormal crystals B, B' is an injective map $\Psi\colon B \sqcup \{0\} \to B' \sqcup \{0\}$ such that $\Psi(0) = 0$ and Ψ commutes with wt, ε_i, ϕ_i, $\tilde{e}_i$, and $\tilde{f}_i$ for all $i \in I$. It is necessarily an isomorphism from B onto a disjoint union of connected components of B'.

For $U_q(\mathfrak{g})$-seminormal crystals B_1 and B_2, their tensor product $B_1 \otimes B_2 = \{b_1 \otimes b_2 \mid b_1 \in B_1, b_2 \in B_2\}$ is the $U_q(\mathfrak{g})$-seminormal crystal with weight function $\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$ and crystal operators (we use the convention opposite Kashiwara's)

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varepsilon_i(b_1) > \phi_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varepsilon_i(b_1) \leq \phi_i(b_2), \end{cases} \tag{4.1}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varepsilon_i(b_1) \geq \phi_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varepsilon_i(b_1) < \phi_i(b_2). \end{cases} \tag{4.2}$$

Assume for this paragraph that the roots and coroots are linearly independent. Let $\mathcal{O}_{\text{int}}(\mathfrak{g})$ denote the category whose objects are the $U_q(\mathfrak{g})$-modules isomorphic to a direct sum of integrable highest weight $U_q(\mathfrak{g})$-modules (see, e.g., [37, §2.4]). Any M in $\mathcal{O}_{\text{int}}(\mathfrak{g})$ has a unique local crystal basis $(\mathcal{L}, \mathcal{B})$ up to isomorphism [34], and extracting the associated combinatorial data yields a $U_q(\mathfrak{g})$-seminormal crystal (see [37, §4.2, §7.5]). We define a $U_q(\mathfrak{g})$-*crystal* to be a $U_q(\mathfrak{g})$-seminormal crystal arising in this way. For $\Lambda \in P^+ = \{\lambda \in P \mid \langle \alpha_i^\vee, \lambda \rangle \geq 0\}$, the *highest weight* $U_q(\mathfrak{g})$-*crystal* $B(\Lambda)$ is the $U_q(\mathfrak{g})$-crystal arising from the local crystal basis of the irreducible highest weight module $V(\Lambda)$ in $\mathcal{O}_{\text{int}}(\mathfrak{g})$. So with this notation, any $U_q(\mathfrak{g})$-crystal is a disjoint union of highest weight $U_q(\mathfrak{g})$-crystals by [34].

4.2 Restricting Demazure crystals

Let $U_q(\mathfrak{g})$, P^*, P, $\{\alpha_i^\vee\}_{i\in I}$, $\{\alpha_i\}_{i\in I}$ be as in §4.1. Let $J \subset I$ and $\hat{P}^* \subset P^*$ be such that $\{\alpha_i^\vee\}_{i\in J} \subset \hat{P}^*$. As $P = \text{Hom}_{\mathbb{Z}}(P^*, \mathbb{Z})$, restricting maps from P^* to $\hat{P}^*$ yields a projection $z\colon P \to \hat{P} := \text{Hom}_{\mathbb{Z}}(\hat{P}^*, \mathbb{Z})$. Assume that the sets $\{\alpha_i^\vee\}_{i\in I}$, $\{\alpha_i\}_{i\in I}$, $\{\alpha_i^\vee\}_{i\in J}$, $\{z(\alpha_i)\}_{i\in J}$ are linearly independent. The algebra $U_q(\mathfrak{g})$ has generators e_i, f_i, $i \in I$, and q^h, $h \in P^*$. Let $U_q(\mathfrak{g}_J) \subset U_q(\mathfrak{g})$ be the subalgebra generated by e_i, f_i, $i \in J$, and q^h, $h \in \hat{P}^*$; it is a quantized enveloping algebra and its defining data includes J, $\{\alpha_i^\vee\}_{i\in J} \subset \hat{P}^*$, $\{z(\alpha_i)\}_{i\in J} \subset \hat{P}$.

It is straightforward to verify that for any M in $\mathcal{O}_{\text{int}}(\mathfrak{g})$, the local crystal basis $(\mathcal{L}, \mathcal{B})$ of M is also a local crystal basis of the $U_q(\mathfrak{g}_J)$-restriction of M and so is isomorphic to the direct sum of local crystal bases of highest weight $U_q(\mathfrak{g}_J)$-modules by [34] (see [37, §4.6] for a similar result). Moreover, the associated $U_q(\mathfrak{g})$-crystal B of $(\mathcal{L}, \mathcal{B})$ and $U_q(\mathfrak{g}_J)$-crystal $\hat{B}$ of $\text{Res}_{U_q(\mathfrak{g}_J)}(\mathcal{L}, \mathcal{B})$ are related as follows: $\hat{B}$ is obtained from B by replacing its weight function with $z \circ \text{wt}\colon B \to \hat{P}$ and taking only the crystal operators $\tilde{e}_i$, $\tilde{f}_i$ for $i \in J$. We say $\hat{B}$ is the $U_q(\mathfrak{g}_J)$-*restriction* of B and denote it $\text{Res}_J B$ or similar—see §4.4.

The following crystal restriction theorem will be important for obtaining key positivity results. Its proof was communicated to us by Peter Littelmann, and we are also grateful to Wilberd van der Kallen who pointed us to his module-theoretic version [85, Theorem 6.3.1]. A more general module-theoretic version was recently given in [6, Appendix A].

Theorem 4.1 *Let $U_q(\mathfrak{g}_J) \subset U_q(\mathfrak{g})$ be as above. For any $U_q(\mathfrak{g})$-Demazure crystal S, $\text{Res}_J S$ is isomorphic to a disjoint union of $U_q(\mathfrak{g}_J)$-Demazure crystals.*

Here, $S = \mathcal{F}_{i_1} \cdots \mathcal{F}_{i_k}\{u_\Lambda\} \subset B(\Lambda)$ for some highest weight $U_q(\mathfrak{g})$-crystal $B(\Lambda)$ (as in Definition 2.4) and $\text{Res}_J S$ denotes the set S regarded as a subset of $\text{Res}_J B(\Lambda)$, which is isomorphic to a disjoint union of highest weight $U_q(\mathfrak{g}_J)$-crystals by the discussion above.

Proof As $\{\alpha_i^\vee\}_{i\in I}$ is linearly independent, we can choose $\{\Lambda_j\}_{j\in I} \subset P$ such that $\langle \alpha_i^\vee, \Lambda_j \rangle = m\delta_{ij}$ for $i, j \in I$ and $m \in \mathbb{Z}_{\geq 1}$. Set $\bar{J} = I \setminus J$ and $\rho_{\bar{J}} = \sum_{i\in \bar{J}} \Lambda_i$. Put $c = 1 + \max\{\varepsilon_i(b) \mid b \in S\}$. Consider the $U_q(\mathfrak{g})$-crystal $B(c\rho_{\bar{J}})$ with highest weight element $u_{c\rho_{\bar{J}}}$. Since $\phi_i(u_{c\rho_{\bar{J}}}) = \langle \alpha_i^\vee, c\rho_{\bar{J}} \rangle = cm$ for $i \in \bar{J}$, the tensor product rule (4.2) implies

$$\tilde{f}_i(b \otimes u_{c\rho_{\bar{J}}}) \notin S \otimes u_{c\rho_{\bar{J}}} \quad \text{for all } i \in \bar{J} \text{ and } b \in S. \tag{4.3}$$

By [32, §2.11], $S \otimes u_{c\rho_{\bar{J}}}$ is a disjoint union of $U_q(\mathfrak{g})$-Demazure crystals, each of the form $\mathcal{F}_{i_1} \cdots \mathcal{F}_{i_m}\{b \otimes u_{c\rho_{\bar{J}}}\}$ for some $b \in S$ and, by (4.3), we must have $i_j \in J$; moreover, $\mathcal{F}_{i_1} \cdots \mathcal{F}_{i_m}\{b \otimes u_{c\rho_{\bar{J}}}\} = (\mathcal{F}_{i_1} \cdots \mathcal{F}_{i_m}\{b\}) \otimes u_{c\rho_{\bar{J}}}$, which follows from (4.2) and $\phi_i(u_{c\rho_{\bar{J}}}) = \langle \alpha_i^\vee, c\rho_{\bar{J}} \rangle = 0$ for $i \in J$. Thus S is a disjoint union of sets of the form $\mathcal{F}_{i_1} \cdots \mathcal{F}_{i_m}\{b\}$, and these are $U_q(\mathfrak{g}_J)$-Demazure crystals as $\tilde{e}_i(b \otimes u_{c\rho_{\bar{J}}}) = 0$ implies $\tilde{e}_i(b) = 0$ for $i \in J$. □

Remark 4.2 Let $U_q(\mathfrak{g}_J) \subset U_q(\mathfrak{g})$ be as above and assume $J = I$. Then a subset S of a $U_q(\mathfrak{g})$-crystal B is isomorphic to a disjoint union of $U_q(\mathfrak{g})$-Demazure crystals if and only if $\mathrm{Res}_J\, S$ is isomorphic to a disjoint union of $U_q(\mathfrak{g}_J)$-Demazure crystals. This is immediate from the definitions since B and $\mathrm{Res}_J\, B$ have the same $\tilde{f}_i$-edges for all $i \in J = I$.

4.3 The affine Lie algebra $\widehat{\mathfrak{sl}}_\ell$

Let $\widehat{\mathfrak{sl}}_\ell$ be the complex affine Kac-Moody Lie algebra of type $A_{\ell-1}^{(1)}$, with associated Dynkin nodes $I = \mathbb{Z}/\ell\mathbb{Z} = \{0, 1, \ldots, \ell - 1\}$ and Cartan matrix $A = (a_{ij})_{i,j \in I}$. Let $\mathfrak{h} \subset \widehat{\mathfrak{sl}}_\ell$ be the Cartan subalgebra, which has a basis consisting of the simple coroots $\{\alpha_i^\vee \mid i \in I\} \subset \mathfrak{h}$ together with the scaling element $d \in \mathfrak{h}$. We have the simple roots $\{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$, with pairings $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$ and $\langle d, \alpha_i \rangle = \delta_{i0}$ $(i, j \in I)$. The fundamental weights $\{\Lambda_i \mid i \in I\} \subset \mathfrak{h}^*$ are defined by $\langle \alpha_i^\vee, \Lambda_j \rangle = \delta_{ij}$ for $i, j \in I$, $\langle d, \Lambda_0 \rangle = 0$, and

$$\langle d, \Lambda_i - \Lambda_{i-1} \rangle = \frac{2i - 1 - \ell}{2\ell} \quad \text{for } i \in [\ell]. \tag{4.4}$$

The $\{\Lambda_i \mid i \in I\}$ together with the null root $\delta = \sum_{i \in I} \alpha_i$ form a basis for $\mathfrak{h}^*$; note that $\langle \alpha_i^\vee, \delta \rangle = 0$ for $i \in I$ and $\langle d, \delta \rangle = 1$. The convention (4.4) is implicit in [71]; it ensures that the extended affine Weyl group acts on the α_i and Λ_i by permutating them cyclically (see §4.5), which has an important consequence for crystals (see §4.6).

Let $P = \bigoplus_{i \in I} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\frac{\delta}{2\ell} \subset \mathfrak{h}^*$ be the weight lattice and $P^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\Lambda_i + \mathbb{Z}\frac{\delta}{2\ell}$ the dominant weights. Let $\mathrm{cl}\colon \mathfrak{h}^* \to \mathfrak{h}^*/\mathbb{C}\delta$ be the canonical projection, and set $P_{\mathrm{cl}} = \mathrm{cl}(P) = \bigoplus_{i \in I} \mathbb{Z}\,\mathrm{cl}(\Lambda_i)$. Let $\mathrm{aff}\colon \mathfrak{h}^*/\mathbb{C}\delta \to \mathfrak{h}^*$ be the section of cl satisfying $\langle d, \mathrm{aff}(\lambda) \rangle = 0$ for all $\lambda \in h^*/\mathbb{C}\delta$. Set $\varpi_i = \mathrm{aff}(\mathrm{cl}(\Lambda_i - \Lambda_0))$ for $i \in I$ (hence $\varpi_0 = 0$).

Let $\mathfrak{sl}_\ell \subset \widehat{\mathfrak{sl}}_\ell$ be the simple Lie subalgebra with Dynkin nodes $I \setminus \{0\} = [\ell - 1]$, Cartan subalgebra $\mathring{\mathfrak{h}} = \bigoplus_{i \in [\ell-1]} \mathbb{C}\alpha_i^\vee \subset \mathfrak{h}$, and fundamental weights $\{\mathring{\varpi}_i \mid i \in [\ell - 1]\} \subset (\mathring{\mathfrak{h}})^*$. The associated weight lattice $\mathring{P} = \bigoplus_{i \in [\ell-1]} \mathbb{Z}\mathring{\varpi}_i$ is naturally viewed as the image of P under the projection $\mathfrak{h}^* \to \mathfrak{h}^*/(\mathbb{C}\delta \oplus \mathbb{C}\Lambda_0) = (\mathring{\mathfrak{h}})^*$; moreover, ϖ_i maps to $\mathring{\varpi}_i$ and $\bigoplus_{i \in [\ell-1]} \mathbb{Z}\varpi_i \subset \mathfrak{h}^*$ maps isomorphically onto $\mathring{P} \subset (\mathring{\mathfrak{h}})^*$.

4.4 Type A crystals

Let $U_q(\widehat{\mathfrak{sl}}_\ell)$ be the quantized enveloping algebra specified by the data $I, P^* = \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$, P, $\{\alpha_i^\vee\}_{i\in I}$, $\{\alpha_i\}_{i\in I}$ above and the symmetric bilinear form $(\cdot,\cdot)\colon P \times P \to \mathbb{Q}$ defined by $(\alpha_i, \alpha_j) = a_{ij}$, $(\alpha_i, \Lambda_0) = \delta_{i0}$, $(\Lambda_0, \Lambda_0) = 0$. The subalgebra $U_q(\mathfrak{sl}_\ell) \subset U_q(\widehat{\mathfrak{sl}}_\ell)$ fits the form in §4.2, with Dynkin node subset $[\ell-1] \subset I$, coweight lattice $\bigoplus_{i\in[\ell-1]} \mathbb{Z}\alpha_i^\vee$, and weight lattice $\mathring{P}$. Let $U_q(\mathfrak{gl}_\ell)$ be as in [37, §5]; data includes Dynkin nodes $[\ell-1]$, weight lattice $\mathbb{Z}^\ell$, and roots $\{\epsilon_i - \epsilon_{i+1}\}_{i\in[\ell-1]}$.

Let $U_q'(\widehat{\mathfrak{sl}}_\ell) \subset U_q(\widehat{\mathfrak{sl}}_\ell)$ be the subalgebra generated by e_i, f_i, $i \in I$, and q^h, $h \in P_{\mathrm{cl}}^* = \bigoplus_{i\in I} \mathbb{Z}\alpha_i^\vee$; it can be considered a quantized enveloping algebra with data I, $\{\alpha_i^\vee\}_{i\in I} \subset P_{\mathrm{cl}}^*$, $\{\mathrm{cl}(\alpha_i)\}_{i\in I} \subset P_{\mathrm{cl}}$ (it fits the form in [37, Definition 2.1]), but note that the roots are not linearly independent. *For $U_q'(\widehat{\mathfrak{sl}}_\ell)$, we work with $U_q'(\widehat{\mathfrak{sl}}_\ell)$-seminormal crystals so that we can work with both KR crystals and restrictions of $U_q(\widehat{\mathfrak{sl}}_\ell)$-crystals and treat them uniformly, while for $\mathfrak{g} = \mathfrak{sl}_\ell$, $\mathfrak{gl}_\ell$, or $\widehat{\mathfrak{sl}}_\ell$ we only need $U_q(\mathfrak{g})$-crystals.*

We fix some notation for restricting crystals and specify the projection z of weight lattices (as in (4.2)) for each case. For a $U_q(\mathfrak{gl}_\ell)$-crystal (resp. $U_q(\widehat{\mathfrak{sl}}_\ell)$-crystal) B, its $U_q(\mathfrak{sl}_\ell)$-restriction $\mathrm{Res}_{\mathfrak{sl}_\ell} B$ has edges $\tilde{e}_i$, $\tilde{f}_i$, $i \in [\ell-1]$, and z is the canonical projection $\mathbb{Z}^\ell \to \mathbb{Z}^\ell/\mathbb{Z}(1,\dots,1) \cong \mathring{P}$, $\epsilon_i \mapsto \mathring{\varpi}_i - \mathring{\varpi}_{i-1}$ (resp. $P \to \mathring{P}$). For a $U_q'(\widehat{\mathfrak{sl}}_\ell)$-seminormal crystal B, its $U_q(\mathfrak{sl}_\ell)$-restriction $\mathrm{Res}_{\mathfrak{sl}_\ell} B$ has edges $\tilde{e}_i$, $\tilde{f}_i$, $i \in [\ell-1]$, and z is the canonical projection $P_{\mathrm{cl}} \to \mathring{P}$ (this does not fit the form in §4.2 and it need not yield a $U_q(\mathfrak{sl}_\ell)$-crystal, but it does so for all $U_q'(\widehat{\mathfrak{sl}}_\ell)$-seminormal crystals considered in this paper). For a $U_q(\widehat{\mathfrak{sl}}_\ell)$-crystal B, its $U_q'(\widehat{\mathfrak{sl}}_\ell)$-restriction has the same edges as B and z is $\mathrm{cl}\colon P \to P_{\mathrm{cl}}$ (it is easily verified that this always yields a $U_q'(\widehat{\mathfrak{sl}}_\ell)$-seminormal crystal).

4.5 The affine symmetric group and 0-Hecke monoid

The *extended affine symmetric group* $\widetilde{\mathcal{S}}_\ell$ is the group generated by τ and s_i $(i \in I)$ with relations

$$s_i^2 = id, \tag{4.5}$$

$$s_i s_j = s_j s_i \qquad \text{if } a_{ij} = 0 \, (\text{equivalently}, i \notin \{j-1, j+1\}), \tag{4.6}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \tag{4.7}$$

$$\tau s_i = s_{i+1}\tau, \tag{4.8}$$

$$\tau^\ell = id. \tag{4.9}$$

Here, i, j denote arbitrary elements of $I = \mathbb{Z}/\ell\mathbb{Z}$. The *affine symmetric group* $\widehat{\mathcal{S}}_\ell$ is the subgroup of $\widetilde{\mathcal{S}}_\ell$ generated by the s_i for $i \in I$, and *the symmetric group* $\mathcal{S}_\ell$ is the subgroup generated by s_i for $i \in [\ell-1]$. We have $\widetilde{\mathcal{S}}_\ell = \Sigma \ltimes \widehat{\mathcal{S}}_\ell$, where $\Sigma = \{\tau^i \mid i \in [\ell]\} \cong \mathbb{Z}/\ell\mathbb{Z}$; as in §2.2, we also denote by τ the Dynkin diagram automorphism $I \to I$, $i \mapsto i+1$, so that $\tau s_i \tau^{-1} = s_{\tau(i)}$.

Following the conventions of [71], $\widetilde{\mathcal{S}}_\ell$ is also naturally realized as a subgroup of $GL(\mathfrak{h}^*)$: for $i \in I$, s_i acts by $s_i(\lambda) = \lambda - \langle\alpha_i^\vee, \lambda\rangle\alpha_i$ for $\lambda \in \mathfrak{h}^*$, and $\tau \in GL(\mathfrak{h}^*)$ is determined by $\tau(\Lambda_i) = \Lambda_{i+1}$ for $i \in I$ and $\tau(\delta) = \delta$. Another useful description of $\Sigma \subset \widetilde{\mathcal{S}}_\ell$ is as the subgroup of $\widetilde{\mathcal{S}}_\ell \subset GL(\mathfrak{h}^*)$ which takes the set $\{\alpha_i \mid i \in I\}$ to itself; moreover, $\sigma(\alpha_i) = \alpha_{\sigma(i)}$ for all $\sigma \in \Sigma$, $i \in I$.

The 0-*Hecke monoid* $\widetilde{\mathcal{H}}_\ell$ *of* $\widetilde{\mathcal{S}}_\ell$ is the monoid generated by τ and s_i $(i \in I)$ with relations (4.6)–(4.9) (with s_i's in place of s_i's) together with

$$\mathsf{s}_i^2 = \mathsf{s}_i \tag{4.10}$$

for $i \in I$. The 0-*Hecke monoid* $\mathcal{H}_\ell$ *of* $\mathcal{S}_\ell$ is the submonoid of $\widetilde{\mathcal{H}}_\ell$ generated by s_i for $i \in [\ell - 1]$.

The *length* of $w \in \widehat{\mathcal{S}}_\ell$, denoted $\text{length}(w)$, is the minimum m such that $w = s_{i_1} s_{i_2} \cdots s_{i_m}$ for some $i_j \in I$. For $w \in \widetilde{\mathcal{S}}_\ell$, we can write $w = \tau^i v$, $v \in \widehat{\mathcal{S}}_\ell$; define $\text{length}(w) = \text{length}(v)$. An expression for $w \in \widetilde{\mathcal{S}}_\ell$ as a product of τ's and s_i's is *reduced* if it uses $\text{length}(w)$ s_i's. Length and reduced expressions for elements of $\widetilde{\mathcal{H}}_\ell$ are defined similarly.

4.6 Dynkin diagram automorphisms and crystals

Any $\sigma \in \Sigma$, viewed as an element of $GL(\mathfrak{h}^*)$, satisfies $\sigma(P) = P$ and since $\sigma(\delta) = \delta$, it also yields an element of $GL(\mathfrak{h}^*/\mathbb{C}\delta)$ which satisfies $\sigma(P_{\text{cl}}) = P_{\text{cl}}$; hence σ yields automorphisms of P and P_{cl}.

For $\sigma \in \Sigma$ and $U_q(\widehat{\mathfrak{sl}}_\ell)$-crystals (resp. $U_q'(\widehat{\mathfrak{sl}}_\ell)$-seminormal crystals) B, B', a bijection of sets $\theta \colon B \to B'$ is a σ-*twist* if

$$\sigma(\text{wt}(b)) = \text{wt}(\theta(b)), \quad \text{and}$$
$$\theta(\tilde{e}_i b) = \tilde{e}_{\sigma(i)} \theta(b), \;\; \theta(\tilde{f}_i b) = \tilde{f}_{\sigma(i)} \theta(b) \;\; \text{for all } i \in I, \;\text{where } \theta(0) := 0.$$

For any $\Lambda \in P^+$, there is a unique σ-twist $\mathcal{F}_\sigma^\Lambda \colon B(\Lambda) \to B(\sigma(\Lambda))$, which follows from $\sigma(\alpha_i) = \alpha_{\sigma(i)}$ and the uniqueness of local crystal bases of highest weight modules [34].

It is easily verified that if $\theta_1 \colon B_1 \to B_1'$ and $\theta_2 \colon B_2 \to B_2'$ are σ-twists, then so is $\theta_1 \otimes \theta_2 \colon B_1 \otimes B_2 \to B_1' \otimes B_2'$. Thus the tensor product of maps

$$\mathcal{F}_\sigma^{\Lambda^1} \otimes \cdots \otimes \mathcal{F}_\sigma^{\Lambda^p} \colon B(\Lambda^1) \otimes \cdots \otimes B(\Lambda^p) \to B(\sigma(\Lambda^1)) \otimes \cdots \otimes B(\sigma(\Lambda^p))$$

is the natural choice of σ-twist from any tensor product $B(\Lambda^1) \otimes \cdots \otimes B(\Lambda^p)$ of highest weight $U_q(\widehat{\mathfrak{sl}}_\ell)$-crystals, $\Lambda^1, \ldots, \Lambda^p \in P^+$. We let $\mathcal{F}_\tau$ denote the operator on $\mathcal{D}(\widehat{\mathfrak{sl}}_\ell)$ (see Definition 2.5) which takes $S \subset B(\Lambda^1) \otimes \cdots \otimes B(\Lambda^p)$ to $\mathcal{F}_\tau^{\Lambda^1} \otimes \cdots \otimes \mathcal{F}_\tau^{\Lambda^p}(S)$. This agrees with and explains the definition of $\mathcal{F}_\tau$ in §2.2. Similarly, there is a unique τ-twist of $U_q'(\widehat{\mathfrak{sl}}_\ell)$-seminormal crystals $\mathcal{F}_\tau \colon \mathcal{B}^\mu \to \mathcal{B}^\mu$, explained in §6.6.

4.7 $U_q(\widehat{\mathfrak{sl}}_\ell)$-Demazure crystals

Recall that for a subset S of a seminormal crystal B and $i \in I$, $\mathcal{F}_i S = \{\tilde{f}_i^k b \mid b \in S, k \geq 0\} \setminus \{0\} \subset B$.

Proposition 4.3 *The operators $\mathcal{F}_i$ $(i \in I)$ and $\mathcal{F}_\tau$ take $U_q(\widehat{\mathfrak{sl}}_\ell)$-Demazure crystals to $U_q(\widehat{\mathfrak{sl}}_\ell)$-Demazure crystals. Hence they can be regarded as operators on $\mathcal{D}(\widehat{\mathfrak{sl}}_\ell)$ and as such they satisfy the* 0*-Hecke relations* (4.6)–(4.10) *of* $\widetilde{\mathcal{H}}_\ell$.

Proof This follows from [71, Lemma 4.3] and its proof (which is largely based on [35]). □

Thus for any $w \in \widetilde{\mathcal{H}}_\ell$, we can define $\mathcal{F}_w : \mathcal{D}(\widehat{\mathfrak{sl}}_\ell) \to \mathcal{D}(\widehat{\mathfrak{sl}}_\ell)$ by

$$\mathcal{F}_w = \mathcal{F}_{c_1}\mathcal{F}_{c_2}\cdots\mathcal{F}_{c_m},$$

where $w = c_1 \cdots c_m$ with each $c_j \in \{\mathsf{s}_i \mid i \in I\} \sqcup \{\tau\}$ and $\mathcal{F}_{\mathsf{s}_i} := \mathcal{F}_i$, and this is independent of the chosen expression for w. Recall that for $\Lambda \in P^+$ and $w \in \widetilde{\mathcal{H}}_\ell$, $B_w(\Lambda) := \mathcal{F}_w\{u_\Lambda\}$. We thus have $\mathcal{F}_{w'}B_w(\Lambda) = \mathcal{F}_{w'}\mathcal{F}_w\{u_\Lambda\} = \mathcal{F}_{w'w}\{u_\Lambda\} = B_{w'w}(\Lambda)$ for any $\Lambda \in P^+$ and $w, w' \in \widetilde{\mathcal{H}}_\ell$.

4.8 $U_q(\mathfrak{gl}_\ell)$-Demazure crystals and key polynomials

The symmetric group $\mathcal{S}_\ell$ acts on $\mathbb{Z}^\ell$ by permuting coordinates. It is also convenient to define an action of $\mathcal{H}_\ell$ on $\mathbb{Z}^\ell$ by

$$\mathsf{s}_i\,\alpha = \begin{cases} s_i\,\alpha & \text{if } \alpha_i \geq \alpha_{i+1}, \\ \alpha & \text{if } \alpha_i \leq \alpha_{i+1}. \end{cases} \tag{4.11}$$

Let $B^{\mathfrak{gl}}(\nu)$ denote the highest weight $U_q(\mathfrak{gl}_\ell)$-crystal and u_ν its highest weight element, parameterized by $\nu \in \{\lambda \in \mathbb{Z}^\ell \mid \lambda_1 \geq \cdots \geq \lambda_\ell\}$, the dominant integral weights for $U_q(\mathfrak{gl}_\ell)$. Definition 2.4 defines $U_q(\mathfrak{gl}_\ell)$-Demazure crystals but let us make this more explicit. They are indexed by elements of $\mathbb{Z}^\ell$. Let $\alpha \in \mathbb{Z}^\ell$. Denote by α^+ the weakly decreasing rearrangement of α and $\mathsf{p}(\alpha) \in \mathcal{H}_\ell$ the shortest element such that $\mathsf{p}(\alpha)\alpha^+ = \alpha$. Define the $U_q(\mathfrak{gl}_\ell)$-*Demazure crystal* indexed by α to be $BD(\alpha) = \mathcal{F}_{\mathsf{p}(\alpha)}\{u_{\alpha^+}\} \subset B^{\mathfrak{gl}}(\alpha^+)$.

Remark 4.4 Analogous results to §4.7 hold for $U_q(\mathfrak{gl}_\ell)$-Demazure crystals. In particular, the $\mathcal{F}_i$ $(i \in [\ell - 1])$ can be regarded as operators on the set of $U_q(\mathfrak{gl}_\ell)$-Demazure crystals and as such satisfy the 0-Hecke relations (4.6), (4.7), (4.10) of $\mathcal{H}_\ell$.

Consider the group ring of the $\mathfrak{gl}_\ell$-weight lattice $\mathbb{Z}[\mathbb{Z}^\ell] = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_\ell^{\pm 1}]$. It has the monomial basis $\mathbf{x}^\alpha := x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_\ell^{\alpha_\ell}$, as α ranges over $\mathbb{Z}^\ell$. Recall from (2.1) that the Demazure operators π_i are given by $\pi_i = \frac{x_i - x_{i+1}s_i}{x_i - x_{i+1}}$ for $i \in [\ell - 1]$. They were defined as operators on $\mathbb{Z}[q][\mathbf{x}]$, but we will also regard them as operators on $(\mathbb{Z}[x_1^{\pm 1}, \ldots, x_\ell^{\pm 1}])[[q]]$ or $\mathbb{A}[x_1^{\pm 1}, \ldots, x_\ell^{\pm 1}]$ for a ground ring $\mathbb{A}$, given by the same formula. They satisfy the 0-Hecke relations (4.6), (4.7), (4.10) of $\mathcal{H}_\ell$ (see e.g. [75]). Thus, just as we discussed for $\mathcal{F}_w$ in §4.7, π_w makes sense for any $w \in \mathcal{H}_\ell$ and $\pi_w\pi_{w'} = \pi_{ww'}$ for all $w, w' \in \mathcal{H}_\ell$.

Definition 4.5 For $\alpha \in \mathbb{Z}^\ell$, define the *key polynomial* or Demazure character by

$$\kappa_\alpha = \pi_{\mathsf{p}(\alpha)} \mathbf{x}^{\alpha^+}. \tag{4.12}$$

If $\alpha \in \mathbb{Z}^\ell$ is weakly decreasing, then κ_α is simply the monomial $\mathbf{x}^\alpha$, while if α is weakly increasing, then κ_α is the Schur function $s_{\alpha^+}(\mathbf{x}) = s_{\alpha^+}(x_1, x_2, \dots, x_\ell)$.

We record several facts about key polynomials for later use. First, it follows from $\pi_{\mathsf{s}_i}\pi_{w'} = \pi_{\mathsf{s}_i w'}$ for all $w' \in \mathcal{H}_\ell$, that

$$\pi_i \kappa_\alpha = \kappa_{\mathsf{s}_i \alpha}, \tag{4.13}$$

where $\mathsf{s}_i\alpha$ is as in (4.11).

Next, note that for $f \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_\ell^{\pm 1}]$, $s_i(f) = f$ if and only if $\pi_i(f) = f$ if and only if f is symmetric in x_i, x_{i+1}. Further, for $f, g \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_\ell^{\pm 1}]$ with $s_i(f) = f$,

$$\pi_i(fg) = f\pi_i(g). \tag{4.14}$$

It is immediate from Definition 4.5 and (4.14) that

$$(x_1 \cdots x_\ell)^d \kappa_\alpha = \kappa_{\alpha + (d, \dots, d)} \qquad \text{for all } d \in \mathbb{Z} \text{ and } \alpha \in \mathbb{Z}^\ell. \tag{4.15}$$

Proposition 4.6 *The key polynomials* $\{\kappa_\alpha \mid \alpha \in \mathbb{Z}_{\geq 0}^\ell\}$ *form a basis for* $\mathbb{Z}[x_1, \dots, x_\ell]$ *and* $\{\kappa_\alpha \mid \alpha \in \mathbb{Z}^\ell\}$ *form a basis for* $\mathbb{Z}[x_1^{\pm 1}, \dots, x_\ell^{\pm 1}]$.

Proof The first holds by [75, Corollary 7], and the second then follows from (4.15). □

Remark 4.7 We caution that though the key polynomials κ_α, $\alpha \in \mathbb{Z}_{\geq 0}^\ell$, have the $\mathfrak{gl}_\infty$-stability property $\kappa_\alpha = \kappa_{(\alpha,0)} = \kappa_{(\alpha,0,0)} = \cdots$, this is not so for the κ_β with $\beta \in \mathbb{Z}^\ell \setminus \mathbb{Z}_{\geq 0}^\ell$.

The *character* of a subset S of a $U_q(\mathfrak{gl}_\ell)$-crystal is $\operatorname{char}_{\mathfrak{gl}}(S) = \sum_{b \in S} \mathbf{x}^{\mathrm{wt}(b)} \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_\ell^{\pm 1}]$.

Proposition 4.8 *The characters of* $U_q(\mathfrak{gl}_\ell)$*-Demazure crystals are key polynomials: for any* $\alpha \in \mathbb{Z}^\ell$,

$$\operatorname{char}_{\mathfrak{gl}}(BD(\alpha)) = \sum_{b \in BD(\alpha)} \mathbf{x}^{\mathrm{wt}(b)} = \kappa_\alpha(x_1, \dots, x_\ell). \tag{4.16}$$

Proof This is a consequence of [35]. Note that the setup of [35] encompasses the $\mathfrak{gl}_\ell$ case with weight lattice $\mathbb{Z}^\ell$ (see [37, §5]), and the Demazure operators defined therein match the π_i in the definition of key polynomials. □

5 The rotation theorem for tame nonsymmetric Catalan functions

We give the proof of the rotation Theorem 2.3, which requires Demazure operator identities and an in-depth study of polynomial truncation. Interestingly, the expression it gives for tame nonsymmetric Catalan functions is automatically polynomially truncated, whereas we had to explicitly add the truncation in our definition of these functions.

Definition 5.1 The *polynomial truncation operator*, denoted poly, is the linear operator on $\mathbb{Z}[x_1^{\pm 1}, \dots, x_\ell^{\pm 1}]$ determined by its action on the basis $\{\kappa_\alpha \mid \alpha \in \mathbb{Z}^\ell\}$:

$$\operatorname{poly}(\kappa_\alpha) = \begin{cases} \kappa_\alpha & \text{if } \alpha \in \mathbb{Z}^\ell_{\geq 0}, \\ 0 & \text{otherwise.} \end{cases}$$

We extend this in the natural way to a linear operator on $\mathbb{Z}[x_1^{\pm 1}, \dots, x_\ell^{\pm 1}][[q]]$ by $\operatorname{poly}(\sum_{d \geq 0} f_d q^d) = \sum_{d \geq 0} \operatorname{poly}(f_d) q^d$ for any $f_d \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_\ell^{\pm 1}]$.

5.1 Root expansion

A straightforward yet surprisingly powerful recursion played an important role for the Catalan functions in [11]. This is easily generalized to the nonsymmetric setting. For a root ideal Ψ, we say $\alpha \in \Psi$ is a *removable root of* Ψ if $\Psi \setminus \alpha$ is a root ideal. For $\alpha = (i, j) \in \Delta_\ell^+$, write $\varepsilon_\alpha = \epsilon_i - \epsilon_j \in \mathbb{Z}^\ell$.

Proposition 5.2 *Let* (Ψ, γ, w) *be a labeled root ideal. For any removable root* α *of* Ψ,

$$H(\Psi; \gamma; w) = H(\Psi \setminus \alpha; \gamma; w) + q H(\Psi; \gamma + \varepsilon_\alpha; w). \tag{5.1}$$

Proof Apply the linear operator $\pi_w \circ \operatorname{poly}$ to the following identity of series:

$$\begin{aligned} \prod_{(i,j) \in \Psi} \big(1 - q x_i / x_j\big)^{-1} \mathbf{x}^\gamma &= \big(1 - q \mathbf{x}^{\varepsilon_\alpha}\big)^{-1} \prod_{(i,j) \in \Psi \setminus \alpha} \big(1 - q x_i / x_j\big)^{-1} \mathbf{x}^\gamma \\ &= \big(1 + q \mathbf{x}^{\varepsilon_\alpha} \big(1 - q \mathbf{x}^{\varepsilon_\alpha}\big)^{-1}\big) \prod_{(i,j) \in \Psi \setminus \alpha} \big(1 - q x_i / x_j\big)^{-1} \mathbf{x}^\gamma \\ &= \prod_{(i,j) \in \Psi \setminus \alpha} \big(1 - q x_i / x_j\big)^{-1} \mathbf{x}^\gamma \\ &\quad + q \prod_{(i,j) \in \Psi} \big(1 - q x_i / x_j\big)^{-1} \mathbf{x}^{\gamma + \varepsilon_\alpha}. \end{aligned}$$

□

5.2 Polynomial truncation

Polynomial truncation is better understood using the following symmetric bilinear form which comes from Macdonald theory and was given a self-contained treatment by Fu and Lascoux [22]. For $f, g \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_\ell^{\pm 1}]$, define

$$(f, g) = \mathrm{CT}\Big(f(x_1, \dots, x_\ell) g(x_\ell^{-1}, \dots, x_1^{-1}) \prod_{(i,j) \in \Delta_\ell^+} (1 - x_i/x_j)\Big),$$

where CT denotes taking the constant term.

For $\alpha \in \mathbb{Z}^\ell$, define the *Demazure atom* by $\hat{\kappa}_\alpha = \hat{\pi}_{\mathsf{p}(\alpha)} \mathbf{x}^{\alpha^+}$. Here, $\hat{\pi}_i := \pi_i - 1$ and $\hat{\pi}_w := \hat{\pi}_{i_1} \cdots \hat{\pi}_{i_m}$, where $w = \mathsf{s}_{i_1} \cdots \mathsf{s}_{i_m}$ is a reduced expression; this is well defined since the $\hat{\pi}_i$ satisfy the braid relations.

Theorem 5.3 ([22, Theorem 15]) *The key polynomials and Demazure atoms are dual bases with respect to* $(\cdot, \cdot)$: *for* $\alpha, \beta \in \mathbb{Z}^\ell$, $(\kappa_\alpha, \hat{\kappa}_{w_0 \beta}) = \delta_{\alpha,\beta}$, *where* δ *is the Kronecker delta and* w_0 *is the longest permutation in* $\mathcal{S}_\ell$.

Proof The statement in [22, Theorem 15] is for $\alpha, \beta \in \mathbb{Z}_{\geq 0}^\ell$, and this yields the statement for $\alpha, \beta \in \mathbb{Z}^\ell$ too since it implies that for d sufficiently large, $(\kappa_\alpha, \hat{\kappa}_{w_0\beta}) = ((x_1 \cdots x_\ell)^d \kappa_\alpha, (x_1 \cdots x_\ell)^d \hat{\kappa}_{w_0 \beta}) = (\kappa_{\alpha + (d,\dots,d)}, \hat{\kappa}_{w_0 \beta + (d, \dots, d)}) = \delta_{\alpha + (d,\dots,d), \beta + (d,\dots,d)} = \delta_{\alpha,\beta}$. □

Hence, letting $c_{\alpha,\beta} \in \mathbb{Z}_{\geq 0}$ denote the atom to monomial expansion coefficients, i.e., $\hat{\kappa}_\alpha = \sum_{\beta \in \mathbb{Z}^\ell} c_{\alpha,\beta} \mathbf{x}^\beta$, the coefficient of κ_α in the key expansion of any $f \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_\ell^{\pm 1}]$ is given by

$$\begin{aligned}
&\mathrm{CT}\Big(f \prod_{(i,j) \in \Delta^+} (1 - x_i/x_j) \hat{\kappa}_{w_0 \alpha}(x_\ell^{-1}, \dots, x_1^{-1})\Big) \\
&= \sum_{\beta \in \mathbb{Z}^\ell} c_{w_0\alpha,\beta} \, \mathrm{CT}\Big(f \prod_{(i,j) \in \Delta^+} (1 - x_i/x_j) \mathbf{x}^{-\mathrm{rev}(\beta)}\Big) \\
&= \sum_{\beta \in \mathbb{Z}^\ell} c_{w_0\alpha,\beta} \Big(\text{coef. of } \mathbf{x}^{\mathrm{rev}(\beta)} \text{ in the monomial expansion of } f \prod_{(i,j) \in \Delta^+} (1 - x_i/x_j)\Big),
\end{aligned}$$

where $\mathrm{rev}(\beta) := (\beta_\ell, \dots, \beta_1)$ denotes the reverse of any $\beta = (\beta_1, \dots, \beta_\ell) \in \mathbb{Z}^\ell$. We package this into the following corollary:

Corollary 5.4 *For* $f \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_\ell^{\pm 1}]$,

$$f = \sum_{\alpha, \beta \in \mathbb{Z}^\ell} c_{w_0\alpha,\beta} \Big(\textit{coefficient of } \mathbf{x}^{\mathrm{rev}(\beta)} \textit{ in } f \prod_{(i,j) \in \Delta^+} (1 - x_i/x_j)\Big) \kappa_\alpha, \tag{5.2}$$

$$\mathrm{poly}(f) = \sum_{\alpha, \beta \in \mathbb{Z}_{\geq 0}^\ell} c_{w_0\alpha,\beta} \Big(\textit{coefficient of } \mathbf{x}^{\mathrm{rev}(\beta)} \textit{ in } f \prod_{(i,j) \in \Delta^+} (1 - x_i/x_j)\Big) \kappa_\alpha. \tag{5.3}$$

Proposition 5.5 *Let $\gamma \in \mathbb{Z}^\ell$ and $w \in \mathcal{H}_\ell$ be arbitrary.*

(i) *For any $f \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_\ell^{\pm 1}][[q]]$,* $\mathrm{poly}(\pi_i(f)) = \pi_i(\mathrm{poly}(f))$.
(ii) *For any $\alpha \in \mathbb{Z}_{\geq 0}^\ell$,* $\mathrm{poly}(\mathbf{x}^\alpha) = \mathbf{x}^\alpha$.
(iii) *If $\sum_{a=k}^\ell \gamma_a < 0$ for some $k \in [\ell]$, then* $\mathrm{poly}(\mathbf{x}^\gamma) = 0$.
(iv) *If $\sum_{a=k}^\ell \gamma_a < 0$ for some $k \in [\ell]$, then $H(\Psi; \gamma; w) = 0$ for any root ideal $\Psi \subset \Delta_\ell^+$.*
(v) *If $\gamma_{m+1} = \cdots = \gamma_\ell = 0$, then $H(\Psi; \gamma; w) = H(\Psi'; \gamma; w)$ for any $\Psi, \Psi' \subset \Delta_\ell^+$ such that $\Psi \cap \Delta_m^+ = \Psi' \cap \Delta_m^+$.*

Proof Statement (i) is immediate from the definition of polynomial truncation and (4.13). Both $\{\mathbf{x}^\alpha \mid \alpha \in \mathbb{Z}_{\geq 0}^\ell\}$ and $\{\kappa_\alpha \mid \alpha \in \mathbb{Z}_{\geq 0}^\ell\}$ are $\mathbb{Z}$-bases for $\mathbb{Z}[x_1, \dots, x_\ell]$ (Proposition 4.6). Since poly acts as the identity on the latter basis by definition, (ii) follows.

To prove (iii), by (5.3), it suffices to show that for any term $c\mathbf{x}^\zeta$ in the monomial expansion of $\mathbf{x}^\gamma \prod_{(i,j)\in\Delta^+}(1 - x_i/x_j)$, we have $\zeta \notin \mathbb{Z}_{\geq 0}^\ell$. Indeed, for such a term we must have $\sum_{a=k}^\ell \zeta_a \leq \sum_{a=k}^\ell \gamma_a < 0$ as needed.

To prove (iv), recall $H(\Psi; \gamma; w) = \pi_w\big(\mathrm{poly}\big(\mathbf{x}^\gamma \prod_{(i,j)\in\Psi}\big(1 + qx_i/x_j + q^2(x_i/x_j)^2 + \cdots\big)\big)\big)$ from Definition 2.1. Any term $c\mathbf{x}^\zeta = \prod_{(i,j)\in\Psi} q^{d_{ij}}(x_i/x_j)^{d_{ij}}$ arising in the expansion of the product over Ψ satisfies $\sum_{a=k}^\ell \zeta_a \leq 0$, so $\sum_{a=k}^\ell (\gamma + \zeta)_a < 0$ and $\pi_w(\mathrm{poly}(c\,\mathbf{x}^{\gamma+\zeta})) = 0$ by (iii). Thus $H(\Psi; \gamma; w) = 0$. Statement (v) follows similarly from the observation that any term $c\mathbf{x}^\zeta = \prod_{(i,j)\in\Psi} q^{d_{ij}}(x_i/x_j)^{d_{ij}}$ with $d_{ij} > 0$ for some root (i, j) with $j > m$, satisfies $\sum_{a=j}^\ell (\gamma + \zeta)_a < 0$. □

Corollary 5.6 *The nonsymmetric Catalan functions lie in $(\mathbb{Z}[q])[x_1, \dots, x_\ell]$ rather than the larger $(\mathbb{Z}[x_1, \dots, x_\ell])[[q]]$, i.e., they are finite sums of key polynomials κ_α, $\alpha \in \mathbb{Z}_{\geq 0}^\ell$, with coefficients which are polynomials in q with integer coefficients.*

Proof Similar to the proof of (iv) above, one checks that in computing $H(\Psi; \gamma; w)$, any term $q^d\mathbf{x}^\zeta = \prod_{(i,j)\in\Psi} q^{d_{ij}}(x_i/x_j)^{d_{ij}}$ with $d > \ell|\gamma|$ satisfies $\sum_{a=k}^\ell (\gamma + \zeta)_a < 0$ for some $k \in [\ell]$. □

5.3 Identities for Demazure operators and polynomial truncation

Recall from (2.5) that Φ is the operator on $\mathbb{Z}[q][\mathbf{x}]$ given by $\Phi(f) = f(x_2, \dots, x_\ell, qx_1)$; here we will regard it as an operator on $\mathbb{Z}[q, q^{-1}][x_1^{\pm 1}, \dots, x_\ell^{\pm 1}]$.

Proposition 5.7 *For any $f \in \mathbb{Z}[q, q^{-1}][x_1^{\pm 1}, \dots, x_\ell^{\pm 1}]$,*

$$\pi_{i+1}\Phi(f) = \Phi\pi_i(f) \quad \textit{for } i = 1, \dots, \ell - 2.$$

Thus, recalling that $\tau \mathsf{s}_i \tau^{-1} = \mathsf{s}_{i+1}$, we have

$$\pi_{\tau v \tau^{-1}}\Phi(f) = \Phi\pi_v(f) \quad \textit{for any } v \in \mathcal{H}_{\ell-1} \times \mathcal{H}_1 \subset \mathcal{H}_\ell.$$

Proof This is a direct computation from the definition of the Demazure operator π_i:

$$\Phi(\pi_i(f)) = \Phi\Big(\frac{x_i f - x_{i+1}s_i(f)}{x_i - x_{i+1}}\Big) = \frac{x_{i+1}\Phi(f) - x_{i+2}s_{i+1}(\Phi(f))}{x_{i+1} - x_{i+2}} = \pi_{i+1}(\Phi(f)).$$
□

Lemma 5.8 *For any* $f \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_{\ell-1}^{\pm 1}]$ *and* $a \geq 0$, $\mathrm{poly}(x_1^a \Phi(f)) = x_1^a \Phi(\mathrm{poly}(f))$.

Proof Since poly and Φ are linear operators, it is enough to prove this for f ranging over the $\mathbb{Z}$-basis $\{\kappa_\zeta \mid \zeta \in \mathbb{Z}^{\ell-1}\}$ of $\mathbb{Z}[x_1^{\pm 1}, \dots, x_{\ell-1}^{\pm 1}]$. In light of Remark 4.7, computing $\mathrm{poly}(\kappa_\zeta)$ is nontrivial as we have defined polynomial truncation with respect to the basis $\{\kappa_\alpha \mid \alpha \in \mathbb{Z}^\ell\}$ of $\mathbb{Z}[x_1^{\pm 1}, \dots, x_\ell^{\pm 1}]$. However, we can use Demazure operators: write $\kappa_\zeta = \pi_v \mathbf{x}^{(\mu,0)}$ with $\mu = \zeta^+ \in \mathbb{Z}^{\ell-1}$ and $v = \mathsf{p}(\zeta) \in \mathcal{H}_{\ell-1}$ as in Definition 4.5 but for $\ell - 1$ in place of ℓ. Then

$$x_1^a \Phi(\mathrm{poly}(\pi_v \mathbf{x}^{(\mu,0)})) = \pi_{\tau v \tau^{-1}} x_1^a \Phi(\mathrm{poly}(\mathbf{x}^{(\mu,0)})) = \begin{cases} \pi_{\tau v \tau^{-1}} \mathbf{x}^{(a,\mu)} & \text{if } \mu \in \mathbb{Z}_{\geq 0}^{\ell-1}, \\ 0 & \text{otherwise}, \end{cases}$$

where the first equality is by Propositions 5.5 (i) and 5.7 and then (4.14); the second equality uses Proposition 5.5 (ii) for the top line and Proposition 5.5 (iii) for the bottom line (μ weakly decreasing implies $\mu_{\ell-1} < 0$ if $\mu \notin \mathbb{Z}_{\geq 0}^{\ell-1}$).

On the other hand, there holds

$$\begin{aligned} \mathrm{poly}(x_1^a \Phi(\pi_v \mathbf{x}^\mu)) &= \pi_{\tau v \tau^{-1}} \mathrm{poly}(x_1^a \Phi(\mathbf{x}^\mu)) = \pi_{\tau v \tau^{-1}} \mathrm{poly}(\mathbf{x}^{(a,\mu)}) \\ &= \begin{cases} \pi_{\tau v \tau^{-1}} \mathbf{x}^{(a,\mu)} & \text{if } \mu \in \mathbb{Z}_{\geq 0}^{\ell-1}, \\ 0 & \text{otherwise}. \end{cases} \end{aligned}$$

The justification is just as in the previous paragraph (the last equality uses $a \geq 0$). □

Lascoux [55, §4.1] gives a partial description of a Monk's rule for key polynomials, i.e. $x_i \kappa_\alpha$ expanded in key polynomials. The computations therein are similar to the next three lemmas, which we need for polynomial part computations. Recall that $\hat{\pi}_i = \pi_i - 1$.

Lemma 5.9 *For any* $f \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_\ell^{\pm 1}]$,

$$x_{i+1} \pi_i(f) = \hat{\pi}_i(x_i f) \qquad \textit{for } i \in [\ell - 1], \tag{5.4}$$

$$x_i^{-1} \pi_i(f) = \hat{\pi}_i(x_{i+1}^{-1} f) \qquad \textit{for } i \in [\ell - 1], \tag{5.5}$$

$$x_i^{-1} \pi_{i-1}(f) = \pi_{i-1}(x_{i-1}^{-1} f) + x_i^{-1} f \qquad \textit{for } 2 \leq i \leq \ell. \tag{5.6}$$

Proof The identity (5.4) is proved by direct computation:

$$\begin{aligned} \hat{\pi}_i(x_i f) &= \frac{x_i(x_i f) - x_{i+1} s_i(x_i f)}{x_i - x_{i+1}} - \frac{(x_i - x_{i+1}) x_i f}{x_i - x_{i+1}} \\ &= \frac{x_{i+1} x_i f - x_{i+1}^2 s_i(f)}{x_i - x_{i+1}} = x_{i+1} \pi_i(f). \end{aligned}$$

Multiplying both sides by $x_i^{-1} x_{i+1}^{-1}$ (which commutes with π_i) yields (5.5), and (5.6) is a rearrangement of (5.4). □

Lemma 5.10 *Let* $f \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_\ell^{\pm 1}]$ *such that* $s_i(f) = f$ *for* $a < i \le \ell - 1$. *Then*

$$x_\ell \pi_{\ell-1}\pi_{\ell-2}\cdots\pi_a(f) = \hat{\pi}_{\ell-1}\hat{\pi}_{\ell-2}\cdots\hat{\pi}_a(x_a f) = \hat{\pi}_{\ell-1}\pi_{\ell-2}\cdots\pi_a(x_a f). \quad (5.7)$$

Proof Applying (5.4) repeatedly yields the first equality of (5.7). For the second equality, we use that $\hat{\pi}_i(f) = 0$ for $i > a$ and $\hat{\pi}_i\pi_j = \pi_j\hat{\pi}_i$ for $i > j+1$, and compute as follows:

$$\begin{aligned}
&\hat{\pi}_{\ell-1}\cdots\hat{\pi}_a(x_a f) = \hat{\pi}_{\ell-1}\cdots\hat{\pi}_{a+1}\pi_a(x_a f) - \hat{\pi}_{\ell-1}\cdots\hat{\pi}_{a+1}(x_a f)\\
&= \hat{\pi}_{\ell-1}\cdots\hat{\pi}_{a+1}\pi_a(x_a f) = \hat{\pi}_{\ell-1}\cdots\hat{\pi}_{a+2}\pi_{a+1}\pi_a(x_a f) - \hat{\pi}_{\ell-1}\cdots\hat{\pi}_{a+2}\pi_a(x_a f)\\
&= \hat{\pi}_{\ell-1}\cdots\hat{\pi}_{a+2}\pi_{a+1}\pi_a(x_a f) = \cdots = \hat{\pi}_{\ell-1}\pi_{\ell-2}\cdots\pi_a(x_a f). \qquad \square
\end{aligned}$$

Lemma 5.11 *For* $i \in [\ell]$ *and* $\alpha \in \mathbb{Z}^\ell$,

$$x_i^{-1}\kappa_\alpha \in \mathbb{Z}\big\{\kappa_\beta \mid \beta^+ = \alpha^+ - \epsilon_j \text{ for some } j \in [\ell]\big\}.$$

Proof Write $\kappa_\alpha = \pi_v \mathbf{x}^\mu$ with $\mu = \alpha^+$ and $v = \mathsf{p}(\alpha)$ as in Definition 4.5. The proof is by induction on length(v). For the base case $v = id$, let z be the index such that $\mu_i = \mu_{i+1} = \cdots = \mu_z > \mu_{z+1}$ (interpret $\mu_{\ell+1} = -\infty$ so that $z = \ell$ if $\mu_i = \cdots = \mu_\ell$). Then $x_i^{-1}\mathbf{x}^\mu = \hat{\pi}_i\hat{\pi}_{i+1}\cdots\hat{\pi}_{z-1}x_z^{-1}\mathbf{x}^\mu$, which belongs to $\mathbb{Z}\{\kappa_\beta \mid \beta^+ = \mu - \epsilon_z\}$ by (4.13).

Now suppose $v \neq id$. Choose a length additive factorization $v = \mathsf{s}_j u$. Using (5.5) and (5.6) we obtain

$$x_i^{-1}\pi_v\mathbf{x}^\mu = x_i^{-1}\pi_j\pi_u\mathbf{x}^\mu = \begin{cases} \pi_j x_{i-1}^{-1}\pi_u\mathbf{x}^\mu + x_i^{-1}\pi_u\mathbf{x}^\mu & \text{if } j = i-1,\\ \hat{\pi}_j x_{i+1}^{-1}\pi_u\mathbf{x}^\mu & \text{if } j = i,\\ \pi_j x_i^{-1}\pi_u\mathbf{x}^\mu & \text{otherwise.}\end{cases}$$

By the inductive hypothesis, $x_{i-1}^{-1}\pi_u\mathbf{x}^\mu$, $x_i^{-1}\pi_u\mathbf{x}^\mu$, and $x_{i+1}^{-1}\pi_u\mathbf{x}^\mu$ belong to $\mathbb{Z}\{\kappa_\beta \mid \beta^+ = \mu - \epsilon_j$ for some $j \in [\ell]\}$. Hence the result follows from (4.13). $\square$

Lemma 5.12 *For any* $f \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_\ell^{\pm 1}]$, $x_\ell\,\mathrm{poly}(x_\ell^{-1}\hat{\pi}_{\ell-1}(f)) = \mathrm{poly}(\hat{\pi}_{\ell-1}(f))$.

Proof It is enough to prove this identity for f ranging over a $\mathbb{Z}$-basis of $\mathbb{Z}[x_1^{\pm 1}, \dots, x_\ell^{\pm 1}]$. We choose the basis $\{\mathbf{x}^\gamma \mid \gamma \in \mathbb{Z}_{\ge 0}^\ell\} \sqcup \{\kappa_\alpha \mid \alpha \in \mathbb{Z}^\ell \setminus \mathbb{Z}_{\ge 0}^\ell\}$. First consider $f = \mathbf{x}^\gamma$ with $\gamma \in \mathbb{Z}_{\ge 0}^\ell$; set $c = \gamma_{\ell-1}$, $d = \gamma_\ell$. Then

$$\hat{\pi}_{\ell-1}\mathbf{x}^\gamma = \begin{cases} x_1^{\gamma_1}\cdots x_{\ell-2}^{\gamma_{\ell-2}}(x_{\ell-1}^{c-1}x_\ell^{d+1} + x_{\ell-1}^{c-2}x_\ell^{d+2} + \cdots + x_{\ell-1}^{d}x_\ell^{c}) & \text{if } c > d,\\ 0 & \text{if } c = d,\\ -x_1^{\gamma_1}\cdots x_{\ell-2}^{\gamma_{\ell-2}}(x_{\ell-1}^{c}x_\ell^{d} + x_{\ell-1}^{c+1}x_\ell^{d-1} + \cdots + x_{\ell-1}^{d-1}x_\ell^{c+1}) & \text{if } c < d.\end{cases}$$

Since x_ℓ appears with a positive power in each summand, we have $x_\ell\,\mathrm{poly}(x_\ell^{-1}\hat{\pi}_{\ell-1}\mathbf{x}^\gamma) = \hat{\pi}_{\ell-1}\mathbf{x}^\gamma = \mathrm{poly}(\hat{\pi}_{\ell-1}\mathbf{x}^\gamma)$ by Proposition 5.5 (ii).

Now consider $\alpha \in \mathbb{Z}^\ell \setminus \mathbb{Z}_{\ge 0}^\ell$. Since $\hat{\pi}_{\ell-1}\kappa_\alpha$ is a sum of key polynomials indexed by rearrangements of α (by (4.13)), $\mathrm{poly}(\hat{\pi}_{\ell-1}\kappa_\alpha) = 0$. By Lemma 5.11, $x_\ell^{-1}\hat{\pi}_{\ell-1}\kappa_\alpha$ lies in $\mathbb{Z}\big\{\kappa_\beta \mid \beta^+ = \alpha^+ - \epsilon_j$ for some $j \in [\ell]\big\}$, so $\mathrm{poly}(x_\ell^{-1}\hat{\pi}_{\ell-1}\kappa_\alpha) = 0$ as well. $\square$

Let $\mathsf{w}_{[i,j)} \in \mathcal{H}_\ell$ be the longest element of the submonoid generated by $\mathsf{s}_i, \dots, \mathsf{s}_{j-1}$, i.e., the element of $\mathcal{H}_\ell$ corresponding to the permutation which reverses the interval $[i, j]$; we will also use the shorthand $\mathsf{w}_{\vec{a}} := \mathsf{w}_{[a,\ell)}$.

Corollary 5.13 *For any $g \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_\ell^{\pm 1}]$ and $a \in [\ell - 1]$,*

$$\mathrm{poly}(\pi_{\ell-2}\pi_{\ell-3}\cdots\pi_a\pi_{\mathsf{w}_{\vec{a+1}}}(x_a g)) + x_\ell\,\mathrm{poly}(\pi_{\mathsf{w}_{\vec{a}}}(g)) = \mathrm{poly}(\pi_{\mathsf{w}_{\vec{a}}}(x_a g)). \tag{5.8}$$

Proof Rewriting the right side of (5.8) using $\pi_{\mathsf{w}_{\vec{a}}} = \pi_{\ell-1}\cdots\pi_a\pi_{\mathsf{w}_{\vec{a+1}}}$, we obtain the equivalent statement

$$x_\ell\,\mathrm{poly}(\pi_{\mathsf{w}_{\vec{a}}}(g)) = \mathrm{poly}(\hat{\pi}_{\ell-1}\pi_{\ell-2}\cdots\pi_a\pi_{\mathsf{w}_{\vec{a+1}}}(x_a g)).$$

To prove this, we compute

$$\begin{aligned} x_\ell\,\mathrm{poly}(\pi_{\mathsf{w}_{\vec{a}}}(g)) &= x_\ell\,\mathrm{poly}\left(x_\ell^{-1}x_\ell\pi_{\ell-1}\pi_{\ell-2}\cdots\pi_a(\pi_{\mathsf{w}_{\vec{a+1}}}g)\right) \\ &= x_\ell\,\mathrm{poly}\left(x_\ell^{-1}\hat{\pi}_{\ell-1}\pi_{\ell-2}\cdots\pi_a(x_a\pi_{\mathsf{w}_{\vec{a+1}}}g)\right) && \text{by Lemma 5.10} \\ &= \mathrm{poly}\left(\hat{\pi}_{\ell-1}\pi_{\ell-2}\cdots\pi_a(x_a\pi_{\mathsf{w}_{\vec{a+1}}}g)\right) && \text{by Lemma 5.12} \\ &= \mathrm{poly}\left(\hat{\pi}_{\ell-1}\pi_{\ell-2}\cdots\pi_a\pi_{\mathsf{w}_{\vec{a+1}}}(x_a g)\right) && \text{by (4.14).} \end{aligned}$$

□

5.4 Proof of Theorem 2.3

The next theorem shows how to express a tame nonsymmetric Catalan function $H(\Psi; \gamma; \mathsf{w}_{\vec{a+1}})$ in terms of a smaller one $H(\mathsf{R}(\Psi); \mathsf{R}(\gamma); \mathsf{w}_{\vec{a}})$ by peeling off its first row, which we can then iterate to unravel it one row at a time and obtain the desired expression involving π_i's and Φ's.

Theorem 5.14 *Let $\gamma \in \mathbb{Z}^\ell$ and Ψ be a root ideal of length ℓ. Let $\mathsf{R}(\gamma) = (\gamma_2, \dots, \gamma_\ell, 0)$ and $\mathsf{R}(\Psi) \subset \Delta_\ell^+$ be $\mathsf{R}(\Psi) := \{(i-1, j-1) \mid (i, j) \in \Psi, i > 1\} \sqcup \{(i, \ell) \mid i \in [\ell - 1]\}$; this is the root ideal obtained from Ψ by removing its first row, shifting what remains up 1 and left 1, and adding a full column of roots on the right. Set $a = \mathbf{n}(\Psi)_1$. If $\gamma_1 \geq 0$, then*

$$H(\Psi; \gamma; \mathsf{w}_{\vec{a+1}}) = x_1^{\gamma_1}\Phi\big(H(\mathsf{R}(\Psi); \mathsf{R}(\gamma); \mathsf{w}_{\vec{a}})\big). \tag{5.9}$$

Remark 5.15 The last column of roots in $\mathsf{R}(\Psi)$ is just a place holder to make the right side a length ℓ nonsymmetric Catalan function: since $\mathsf{R}(\gamma)_\ell = 0$, by Proposition 5.5 (v), $H(\mathsf{R}(\Psi); \mathsf{R}(\gamma); \mathsf{w}_{\vec{a}}) = H(\Psi'; \mathsf{R}(\gamma); \mathsf{w}_{\vec{a}})$ for any $\Psi' \subset \Delta_\ell^+$ with $\Psi' \cap \Delta_{\ell-1}^+ = \mathsf{R}(\Psi) \cap \Delta_{\ell-1}^+$.

Example 5.16 Let us verify Theorem 5.14 for $\ell = 2$, $\gamma = (3, 2)$, $\Psi = \Delta^+$. Then $a = 1$, $\mathsf{w}_{\vec{a+1}} = id$, $\mathsf{w}_{\vec{a}} = \mathsf{s}_1$, $\mathsf{R}(\gamma) = (2, 0)$, $\mathsf{R}(\Delta^+) = \Delta^+$. We compute both sides of (5.9):

$$\begin{aligned} H(\Delta^+; \gamma; \mathsf{w}_{\vec{a+1}}) &= \mathrm{poly}\big(x_1^3x_2^2(1 - qx_1/x_2)^{-1}\big) \\ &= \mathrm{poly}\big(x_1^3x_2^2 + qx_1^4x_2 + q^2x_1^5 + q^3x_1^6x_2^{-1} + \cdots\big) = x_1^3x_2^2 + qx_1^4x_2 + q^2x_1^5. \end{aligned}$$

$$x_1^{\gamma_1}\Phi\big(H(\mathsf{R}(\Delta^+);\mathsf{R}(\gamma);\mathsf{w}_{\vec{a}})\big) = x_1^{\gamma_1}\Phi\pi_1\,\mathrm{poly}\big(x_1^2(1-qx_1/x_2)^{-1}\big) = x_1^{\gamma_1}\Phi\pi_1(x_1^2)$$
$$= x_1^{\gamma_1}\Phi\big(x_1^2+x_1x_2+x_2^2\big) = x_1^3(x_2^2+qx_1x_2+q^2x_1^2) = x_1^3x_2^2+qx_1^4x_2+q^2x_1^5.$$

Example 5.17 Let $\ell=3$, $\gamma=211$, and $\Psi=\Delta^+$. Then $a=1$ and Theorem 5.14 yields $H(\Delta^+;211;\mathsf{s}_2)=x_1^2\Phi(H(\Delta^+;110;\mathsf{w}_0))$. This can be viewed (via Corollary 7.15) as the identity of characters corresponding to going from the fourth to fifth crystal in Fig. 1.

Proof of Theorem 5.14 The proof is by induction on $\sum_{j=2}^{\ell}\sum_{i=j}^{\ell}\gamma_i$ and $|\Psi|$. The former quantity is not bounded below, so to make this induction valid we first handle the following "base case": suppose $\sum_{i=j}^{\ell}\gamma_i<0$ for some $2\le j\le\ell$. Thus $\sum_{i=j-1}^{\ell}\mathsf{R}(\gamma)_i<0$, and so by Proposition 5.5 (iv), $H(\Psi;\gamma;\mathsf{w}_{\vec{a+1}})=0=x_1^{\gamma_1}\Phi\big(H(\mathsf{R}(\Psi);\mathsf{R}(\gamma);\mathsf{w}_{\vec{a}})\big)$.

Next, the base case $|\Psi|=0$ holds by Lemma 5.8 (it is here we need $\gamma_1\ge 0$):

$$H(\varnothing;\gamma;id)=\mathrm{poly}(\mathbf{x}^{\gamma})=\mathrm{poly}(x_1^{\gamma_1}\Phi(\mathbf{x}^{\mathsf{R}(\gamma)}))=x_1^{\gamma_1}\Phi(\mathrm{poly}(\mathbf{x}^{\mathsf{R}(\gamma)}))$$
$$=x_1^{\gamma_1}\Phi\big(H(\mathsf{R}(\varnothing);\mathsf{R}(\gamma);id)\big);$$

we have also used Remark 5.15 for the last equality.

We may assume from now on that $|\Psi|>0$ and $\sum_{i=j}^{\ell}\gamma_i\ge 0$ for $j\ge 2$. If there is a removable root α of Ψ not in the first row, then

$$\begin{aligned}
&H(\Psi;\gamma;\mathsf{w}_{\vec{a+1}})\\
&\quad=H(\Psi\setminus\alpha;\gamma;\mathsf{w}_{\vec{a+1}})+qH(\Psi;\gamma+\varepsilon_\alpha;\mathsf{w}_{\vec{a+1}})\\
&\quad=x_1^{\gamma_1}\Phi\big(H(\mathsf{R}(\Psi\setminus\alpha);\mathsf{R}(\gamma);\mathsf{w}_{\vec{a}})\big)+qx_1^{\gamma_1}\Phi\big(H(\mathsf{R}(\Psi);\mathsf{R}(\gamma+\varepsilon_\alpha);\mathsf{w}_{\vec{a}})\big)\\
&\quad=x_1^{\gamma_1}\Phi\big(H(\mathsf{R}(\Psi);\mathsf{R}(\gamma);\mathsf{w}_{\vec{a}}\big),
\end{aligned}$$

where the first and third equalities are by Proposition 5.2 and the second is by the inductive hypothesis.

Now we may assume Ψ consists of a single nonempty row. Hence we can expand on the only removable root $(1,a+1)$ (Proposition 5.2) to obtain the first equality below:

$$\begin{aligned}
&H(\Psi;\gamma;\mathsf{w}_{\vec{a+1}})\\
&=H(\Psi\setminus(1,a+1);\gamma;\mathsf{w}_{\vec{a+1}})+qH(\Psi;\gamma+\epsilon_1-\epsilon_{a+1};\mathsf{w}_{\vec{a+1}})\\
&=\pi_{\ell-1}\pi_{\ell-2}\cdots\pi_{a+1}H(\Psi\setminus(1,a+1);\gamma;\mathsf{w}_{\vec{a+2}})+qH(\Psi;\gamma+\epsilon_1-\epsilon_{a+1};\mathsf{w}_{\vec{a+1}})\\
&=\pi_{\ell-1}\pi_{\ell-2}\cdots\pi_{a+1}x_1^{\gamma_1}\Phi\big(H(\varnothing;\mathsf{R}(\gamma);\mathsf{w}_{\vec{a+1}})\big)+qx_1^{\gamma_1+1}\Phi\big(H(\varnothing;\mathsf{R}(\gamma)-\epsilon_a;\mathsf{w}_{\vec{a}})\big)\\
&=\pi_{\ell-1}\pi_{\ell-2}\cdots\pi_{a+1}x_1^{\gamma_1}\Phi\pi_{\mathsf{w}_{\vec{a+1}}}\mathrm{poly}(\mathbf{x}^{\mathsf{R}(\gamma)})+qx_1^{\gamma_1+1}\Phi\pi_{\mathsf{w}_{\vec{a}}}\mathrm{poly}(\mathbf{x}^{\mathsf{R}(\gamma)-\epsilon_a})\\
&=x_1^{\gamma_1}\Phi\Big(\pi_{\ell-2}\pi_{\ell-3}\cdots\pi_a\pi_{\mathsf{w}_{\vec{a+1}}}\mathrm{poly}(\mathbf{x}^{\mathsf{R}(\gamma)})+x_\ell\pi_{\mathsf{w}_{\vec{a}}}\mathrm{poly}(\mathbf{x}^{\mathsf{R}(\gamma)-\epsilon_a})\Big)
\end{aligned}$$

$$= x_1^{\gamma_1} \Phi \pi_{\mathsf{w}_{\vec{a}}} \operatorname{poly}(\mathbf{x}^{\mathsf{R}(\gamma)})$$
$$= x_1^{\gamma_1} \Phi\big(H(\mathsf{R}(\varnothing); \mathsf{R}(\gamma); \mathsf{w}_{\vec{a}})\big).$$

The second equality is by $\pi_{\mathsf{w}_{\vec{a+1}}} = \pi_{\ell-1}\pi_{\ell-2}\dots\pi_{a+1}\pi_{\mathsf{w}_{\vec{a+2}}}$ and Definition 2.1, the third is by the inductive hypothesis and Remark 5.15 (note that we have the first part of $\gamma + \epsilon_1 - \epsilon_{a+1}$ is still ≥ 0), the fifth is by Proposition 5.7, (4.14), and $\Phi(x_\ell) = qx_1$, and the sixth is by Corollary 5.13 with $g = \mathbf{x}^{\mathsf{R}(\gamma)-\epsilon_a}$ and Proposition 5.5 (i). □

Proof of Theorem 2.3 Our goal is to prove (2.6), reproduced here for convenience:

$$H(\Psi; \gamma; w) = \pi_w x_1^{\gamma_1} \Phi \pi_{\mathsf{s}(n_1)} x_1^{\gamma_2} \Phi \pi_{\mathsf{s}(n_2)} x_1^{\gamma_3} \cdots \Phi \pi_{\mathsf{s}(n_{\ell-1})} x_1^{\gamma_\ell}.$$

We proceed by induction on m, the minimum index such that $\gamma_m = \gamma_{m+1} = \cdots = \gamma_\ell = 0$ (set $m = \ell + 1$ if $\gamma_\ell \neq 0$). The base case $m = 1$, $\gamma = \mathbf{0}$ holds since $H(\Psi; \gamma; w) = 1$ by Proposition 5.5 (v). Now assume $m > 1$. By the tameness assumption, w has a length additive factorization $w = v\,\mathsf{w}_{\vec{n_1+1}}$. Thus Theorem 5.14 gives

$$H(\Psi; \gamma; w) = \pi_v H(\Psi; \gamma; \mathsf{w}_{\vec{n_1+1}}) = \pi_v x_1^{\gamma_1} \Phi\big(H(\mathsf{R}(\Psi); \mathsf{R}(\gamma); \mathsf{w}_{\vec{n}_1})\big).$$

Applying the inductive hypothesis to $H(\mathsf{R}(\Psi); \mathsf{R}(\gamma); \mathsf{w}_{\vec{n}_1})$, we obtain

$$\pi_v x_1^{\gamma_1} \Phi\big(H(\mathsf{R}(\Psi); \mathsf{R}(\gamma); \mathsf{w}_{\vec{n}_1})\big)$$
$$= \pi_v x_1^{\gamma_1} \Phi \pi_{\mathsf{w}_{\vec{n}_1}} x_1^{\gamma_2} \Phi \pi_{\mathsf{s}(n_2)} x_1^{\gamma_3} \cdots \Phi \pi_{\mathsf{s}(n_{\ell-1})} x_1^{\gamma_\ell} \Phi \pi_{\mathsf{s}(1)} x_1^0$$
$$= \pi_v \pi_{\mathsf{w}_{\vec{n_1+1}}} x_1^{\gamma_1} \Phi \pi_{\mathsf{s}(n_1)} x_1^{\gamma_2} \Phi \pi_{\mathsf{s}(n_2)} x_1^{\gamma_3} \cdots \Phi \pi_{\mathsf{s}(n_{\ell-1})} x_1^{\gamma_\ell},$$

giving the desired (2.6); for the second equality, we have used the operator identity

$$x_1^{\gamma_1} \Phi \pi_{\mathsf{w}_{\vec{n}_1}} = x_1^{\gamma_1} \Phi \pi_{\mathsf{w}_{[n_1,\ell-1)}} \pi_{\mathsf{s}(n_1)} = \pi_{\mathsf{w}_{\vec{n_1+1}}} x_1^{\gamma_1} \Phi \pi_{\mathsf{s}(n_1)},$$

where the last equality is by Proposition 5.7 and (4.14). □

6 DARK crystals and katabolism

We show that for any DARK crystal $\mathcal{B}^{\mu;\mathbf{w}}$, katabolism is exactly the condition on $\text{Tabloids}_\ell(\mu)$ which detects membership in $\text{inv}(\mathcal{B}^{\mu;\mathbf{w}})$. A connection between KR crystals and Catalan functions in the dominant rectangle case has been well established (see Remark 2.19). One of our key insights is that to go beyond this case, DARK crystals are needed rather than full tensor products of KR crystals.

6.1 Single row Kirillov-Reshetikhin crystals

We will only need an explicit description of the KR crystals $B^{1,s}$ in type A. For any positive integer s, the $U_q'(\widehat{\mathfrak{sl}}_\ell)$-seminormal crystal $B^{1,s}$ consists of all weakly

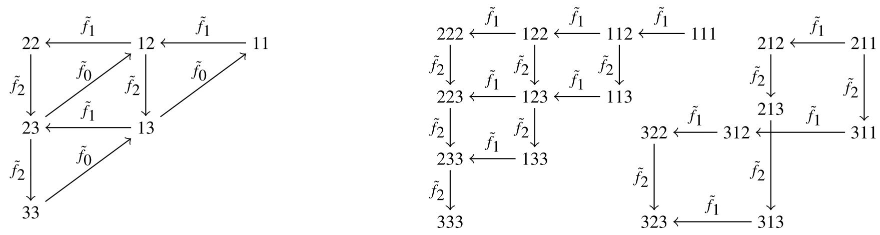


Fig. 4 For $\ell = 3$, the KR crystal $B^{1,2}$ (left) and $\mathrm{Res}_{\mathfrak{sl}_\ell} \mathcal{B}^{(2,1)} = \mathrm{Res}_{\mathfrak{sl}_\ell} B^{1,1} \otimes B^{1,2}$ (right)

increasing words of length s in the alphabet $[\ell]$, with weight function $\mathrm{wt}\colon B^{1,s} \to P_{\mathrm{cl}}$ given by

$$\mathrm{wt}(b) = \mathrm{cl}\big(c_1(\Lambda_1 - \Lambda_0) + c_2(\Lambda_2 - \Lambda_1) + \cdots + c_\ell(\Lambda_0 - \Lambda_{\ell-1})\big),$$
$$\text{for } b = 1^{c_1} 2^{c_2} \cdots \ell^{c_\ell} \tag{6.1}$$

(i.e., b is the weakly increasing word with content $(c_1, \ldots, c_\ell)$), and crystal operators defined as follows: for $i \in [\ell - 1]$ and $b \in B^{1,s}$, $\tilde{e}_i(b)$ is obtained from b by changing its leftmost $i + 1$ to an i, and $\tilde{f}_i(b)$ by changing its rightmost i to an $i + 1$; if there are no $i + 1$'s, $\tilde{e}_i(b) = 0$, and if there are no i's, $\tilde{f}_i(b) = 0$. The element $\tilde{e}_0(b)$ is obtained from b by removing a letter 1 from the beginning and adding a letter ℓ to the end, and $\tilde{f}_0(b)$ is obtained by removing a letter ℓ from the end and adding a letter 1 to the beginning; if there are no 1's, $\tilde{e}_0(b) = 0$, and if there are no ℓ's, $\tilde{f}_0(b) = 0$. See Fig. 4.

We also define $B^{1,0} = \{\mathsf{b}_0\}$ to be the trivial $U_q'(\widehat{\mathfrak{sl}}_\ell)$-seminormal crystal, i.e., $\mathrm{wt}(\mathsf{b}_0) = 0$ and $\tilde{e}_i(\mathsf{b}_0) = \tilde{f}_i(\mathsf{b}_0) = 0$ for all $i \in I$, and view b_0 as the empty word.

6.2 Products of KR crystals

We now describe in detail the crystals $\mathcal{B}^\mu$ which were briefly introduced in §2.3.

Definition 6.1 A *biword* is a pair of words $b = \begin{pmatrix} v_1 & v_2 & \cdots & v_m \\ w_1 & w_2 & \cdots & w_m \end{pmatrix}$ with $v_i, w_i \in \mathbb{Z}_{\geq 1}$, such that for $i < j$, $v_i > v_j$ or ($v_i = v_j$ and $w_i \leq w_j$). Define $\mathrm{top}(b) := v_1 \cdots v_m$, the *top word of* b, and $\mathrm{bottom}(b) := w_1 \cdots w_m$, the *bottom word of* b. The *i-th block* of b, denoted b^i, is the (contiguous) subword of $w_1 \cdots w_m$ below the letters i in $v_1 \cdots v_m$. Thus a pair of words b is a biword if and only if $\mathrm{top}(b)$ is weakly decreasing and its blocks are weakly increasing. The *content* of b, denoted $\mathrm{content}(b)$, is the vector $(c_1, c_2, \ldots, c_\ell)$, where c_i is the number of occurrences of the letter i in $\mathrm{bottom}(b)$.

Recall that for a partition $\mu = (\mu_1 \geq \cdots \geq \mu_p \geq 0)$, we let $\mathcal{B}^\mu = B^{1,\mu_p} \otimes \cdots \otimes B^{1,\mu_1}$, a $U_q'(\widehat{\mathfrak{sl}}_\ell)$-seminormal crystal. We identify its elements with the biwords whose bottom word has letters in $[\ell]$ and whose top word is $p^{\mu_p} \cdots 2^{\mu_2} 1^{\mu_1}$ (see Example 6.6); we use a biword b interchangeably with its bottom word when the crystal $\mathcal{B}^\mu$ it belongs to is clear.

Remark 6.2 We can also regard $\mathcal{B}^\mu$ as a $U_q(\mathfrak{gl}_\ell)$-crystal (temporarily denote it $\mathcal{B}^\mu_{\mathfrak{gl}}$) with weight function $\mathcal{B}^\mu_{\mathfrak{gl}} \to \mathbb{Z}^\ell$, $b \mapsto \mathrm{content}(b)$ and the same edges as $\mathrm{Res}_{\mathfrak{sl}_\ell} \mathcal{B}^\mu$ (the restriction from $U'_q(\widehat{\mathfrak{sl}}_\ell)$ to $U_q(\mathfrak{sl}_\ell)$); moreover, $\mathrm{Res}_{\mathfrak{sl}_\ell} \mathcal{B}^\mu_{\mathfrak{gl}} = \mathrm{Res}_{\mathfrak{sl}_\ell} \mathcal{B}^\mu$ by (6.1). From now on we write $\mathcal{B}^\mu$ for both the $U'_q(\widehat{\mathfrak{sl}}_\ell)$-seminormal and $U_q(\mathfrak{gl}_\ell)$-crystal, and will clarify when necessary.

The crystal operators $\tilde{e}_i$ and $\tilde{f}_i$ on $\mathcal{B}^\mu$ are determined by the above description of $\tilde{e}_i$ and $\tilde{f}_i$ on $B^{1,s}$ and the tensor product rule (4.1)–(4.2). For $i \in [\ell-1]$, they have the following streamlined description. Let $b \in \mathcal{B}^\mu$. Place a left parenthesis "(" below each letter $i+1$ in b and a right parenthesis ")" below each letter i. Match parentheses in the usual way. The unmatched parentheses correspond to a subword consisting of i's followed by $i+1$'s. Then $\tilde{e}_i(b)$ is obtained from b by changing the leftmost unmatched $i+1$ to an i, and $\tilde{f}_i(b)$ by changing the rightmost unmatched i to an $i+1$; if there are no unmatched $i+1$'s, $\tilde{e}_i(b) = 0$, and if there are no unmatched i's, $\tilde{f}_i(b) = 0$.

Example 6.3 We illustrate the parentheses matching rule for computing $\tilde{e}_2$ and $\tilde{f}_2$ of the element $b \in \mathcal{B}^{554}$ below, with the unmatched letters in bold.

$$
\begin{aligned}
b &= \mathbf{22}34\ 2\mathbf{23}33\ 11122 \\
  &\ ))(\ \ ))(((\ \ \ \ \ )) \\
\tilde{e}_2(b) &= \mathbf{22}34\ 2\mathbf{22}33\ 11122 \\
\tilde{f}_2(b) &= \mathbf{22}34\ 2\mathbf{33}33\ 11122
\end{aligned}
$$

6.3 RSK and crystals

We review the beautiful connection between $U_q(\mathfrak{gl}_\ell)$-crystals and classical tableau combinatorics, which may be attributed to Kashiwara-Nakashima [38], and Lascoux-Schützenberger [57] who anticipated much of the combinatorics before the development of crystals. Other good references include [81] and [29, Chap. 7].

The crystals $\mathcal{B}^\mu$ are compatible with the following variant of the Robinson-Schensted-Knuth correspondence described in [23, A.4.1, Proposition 2]. Let b be a biword. The *insertion tableau* $P(b)$ of b is the ordinary insertion tableau of the word bottom(b). It can be obtained by applying the Schensted row insertion algorithm to the letters of bottom(b) from left to right or by column inserting each letter from right to left. The *recording tableau* $Q(b)$ of b is obtained by column inserting the bottom word of b from right to left and recording each newly added box with the corresponding top letter. More precisely, $Q(b)$ is the tableau with the same shape as $P(b)$ such that the skew shape $\mathrm{shape}(P(b^i b^{i-1} \cdots b^1))/\mathrm{shape}(P(b^{i-1} \cdots b^1))$ is filled with i's for all i.

Recall from §2.4 that $\mathrm{SSYT}_\ell(\mu)$ denotes the subset of $\mathrm{Tabloids}_\ell(\mu)$ consisting of tabloids with partition shape whose columns strictly increase from top to bottom. (This is the set of semistandard Young tableaux of content μ with at most ℓ rows, but with the fine print that we regard them as having ℓ rows some of which may be empty.)

Theorem 6.4 (see [81, Theorem 3.6]) *The decomposition of the $U_q(\mathfrak{gl}_\ell)$-crystal $\mathcal{B}^\mu$ into highest weight $U_q(\mathfrak{gl}_\ell)$-crystals is given by*

$$\mathcal{B}^\mu = \bigsqcup_{U \in \mathrm{SSYT}_\ell(\mu)} \mathcal{C}_U, \quad \text{where } \mathcal{C}_U := \{b \in \mathcal{B}^\mu \mid Q(b) = U\} \cong B^{\mathfrak{gl}}(\mathrm{shape}(U)). \tag{6.2}$$

Here, $B^{\mathfrak{gl}}(\nu)$ denotes the highest weight $U_q(\mathfrak{gl}_\ell)$-crystal of highest weight ν.

6.4 The inv bijection and RSK

A biword can be thought of as a sequence of biletters $\binom{v_1}{w_1}\binom{v_2}{w_2}\cdots\binom{v_m}{w_m}$ which is weakly decreasing for the order $\binom{v}{w} \geq \binom{v'}{w'}$ if and only if $v > v'$ or ($v = v'$ and $w \leq w'$). Then, for a biword b, define $\mathrm{inv}(b)$ to be the result of exchanging the top and bottom words of b and then sorting biletters to be weakly decreasing.

It is natural to regard inv as an involution on the set of biwords. However, as discussed in Remark 6.7 below, we prefer to think of inv as a bijection between biwords and tabloids, which we can do since biwords and tabloids may be naturally identified by equating blocks with rows (see the right side of (6.3)). Since the contents of the top and bottom words are exchanged by inv, it restricts to a bijection $\mathrm{inv}\colon \mathcal{B}^\mu \xleftrightarrow{\cong} \mathrm{Tabloids}_\ell(\mu)$, which takes content to shape (we gave a direct description of the map $\mathcal{B}^\mu \to \mathrm{Tabloids}_\ell(\mu)$ in §2.3).

Proposition 6.5 ([23, A.4.1, Symmetry Theorem B]) *The insertion* (P) *and recording* (Q) *tableaux are exchanged by* inv. *In particular, for a biword $b \in \mathcal{B}^\mu$, $Q(b) = P(\mathrm{inv}(b))$ and for a tabloid $T \in \mathrm{Tabloids}_\ell(\mu)$, $P(T) = Q(\mathrm{inv}(T))$.*

Example 6.6 For the following biword $b \in \mathcal{B}^{554}$, we compute $\mathrm{inv}(b)$ and $Q(b)$:

$$b = \begin{pmatrix} 3333\,22222\,11111 \\ 2234\,13334\,11222 \end{pmatrix} \xrightarrow{\mathrm{inv}} \begin{pmatrix} 44\,3333\,22222\,111 \\ 23\,2223\,11133\,112 \end{pmatrix} = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 \\ \hline 1 & 1 & 1 & 3 & 3 \\ \hline 2 & 2 & 2 & 3 \\ \hline 2 & 3 \\ \hline \end{array} = T, \tag{6.3}$$

$$Q(b) = P(T) = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 2 & 2 & 3 & 3 \\ \hline 3 \\ \hline \end{array}.$$

Remark 6.7 Though it is possible to define a two-sided crystal structure on biwords in which crystal operators act on both a biword and its inverse, this is not the perspective we take here. Instead, we break the symmetry between the two sides by adopting the following conventions: crystal operators act only on the $\mathcal{B}^\mu$ side and not the $\mathrm{Tabloids}_\ell(\mu)$ side; we are mainly interested in $Q(b)$, not $P(b)$, for $b \in \mathcal{B}^\mu$, and $P(T)$, not $Q(T)$, for $T \in \mathrm{Tabloids}_\ell(\mu)$ as these are the ones which identify inv of the highest weight element of a $U_q(\mathfrak{gl}_\ell)$-component. Further, *elements of $\mathcal{B}^\mu$ will be written as biwords and never tabloids; their inverses will be written as tabloids, though occasionally thought of as biwords for the purposes of computing* inv.

6.5 Partial insertion and $\tilde{e}_i^{\max}$

In the remainder of Sect. 6, we match operations on the tabloids side with ones on the crystal side. The material in this subsection is similar to [82, §3.5], [54, §2] and perhaps can be considered folklore.

For an element b of a $U_q(\mathfrak{gl}_\ell)$-crystal, define

$$\tilde{e}_i^{\max}(b) = \tilde{e}_i^{\varepsilon(b)}(b), \tag{6.4}$$

i.e., the last element in the list $b, \tilde{e}_i(b), \tilde{e}_i^2(b), \dots$ which is not 0. For example, in the crystal $\mathcal{B}^{432}$, $\tilde{e}_1^{\max}(12\,122\,1222) = 12\,112\,1111$. More generally, for $w \in \mathcal{H}_\ell$, let $w = \mathsf{s}_{i_1} \cdots \mathsf{s}_{i_m}$ be any expression for w as a product of s_j's; define $\tilde{e}_w^{\max} = \tilde{e}_{i_1}^{\max} \cdots \tilde{e}_{i_m}^{\max}$; by Proposition 6.11 (ii) below, this is independent of the chosen expression for w.

Recall that T^i denotes the i-th row of a tabloid T.

Definition 6.8 (Partial insertion) Given a tabloid T, $P_i(T)$ is the tabloid obtained from T by replacing rows i and $i+1$ of T by the tableau $P(T^{i+1}T^i)$ (if $P(T^{i+1}T^i)$ has only one row, then the $i+1$-st row of $P_i(T)$ is empty). More generally, for $w = \mathsf{s}_{i_1} \cdots \mathsf{s}_{i_m} \in \mathcal{H}_\ell$, define $P_w = P_{i_1} \cdots P_{i_m}$; by Proposition 6.11 (iii) below, this is independent of the chosen expression for w. For how this is related to Definition 2.13, see Remark 6.16.

For example, $$P_2\left(\begin{array}{|l|} 1\;1\;2\;2 \\ 1\;2\;3\;3\;4 \\ 1\;1\;2\;2\;2\;3 \\ 2\;3\;4 \end{array}\right) = \begin{array}{|l|} 1\;1\;2\;2 \\ 1\;1\;1\;2\;2\;2\;3\;3\;4 \\ 2\;3 \\ 2\;3\;4 \end{array}.$$

The following commutative diagrams give a summary of §6.4–6.5 (the left holds by Proposition 6.9 and the right by Propositions 6.5, 6.9, and 6.11 (iv)).

$$\begin{array}{ccc} \mathcal{B}^\mu & \xleftrightarrow{\text{inv}} & \text{Tabloids}_\ell(\mu) \\ \tilde{e}_i^{\max}\downarrow & & \downarrow P_i \\ \mathcal{B}^\mu & \xleftrightarrow[\text{inv}]{} & \text{Tabloids}_\ell(\mu) \end{array} \qquad \begin{array}{ccc} \mathcal{B}^\mu & \xleftrightarrow{\text{inv}} & \text{Tabloids}_\ell(\mu) \\ \tilde{e}_{\mathsf{w}_0}^{\max}\downarrow & \searrow Q & \downarrow P = P_{\mathsf{w}_0} \\ \mathcal{B}^\mu & \xleftrightarrow[\text{inv}]{} & \text{Tabloids}_\ell(\mu) \end{array}$$

Proposition 6.9 *For any biword $b \in \mathcal{B}^\mu$ and $i \in [\ell-1]$, $\tilde{e}_i^{\max}(b) = \mathrm{inv}(P_i(\mathrm{inv}(b)))$.*

Proof Set $T = \mathrm{inv}(b)$. Recall from §6.2 that $\tilde{e}_i^{\max}(b)$ is obtained by viewing $i+1$'s and i's as left and right parentheses and then changing all unmatched $i+1$'s to i's. We claim that $\mathrm{inv}(P_i(\mathrm{inv}(b)))$, computed using the row bumping algorithm, is obtained by the same rule except with the following *greedy parentheses matching* in place of the ordinary one: read ")"s from right to left and match each with the rightmost unmatched "(". To see this, first note that the letters in top(b) above the i's (resp. $i+1$'s) in bottom(b) are the values of T^i (resp. T^{i+1}). The row bumping algorithm computes $P_i(T)$ by processing the letters of T^i from left to right; each letter x of T^i bumps the smallest entry of T^{i+1} greater than x not already bumped (if it exists).

Each bump corresponds to a greedy-matched pair in b and the unmatched $i+1$'s of b correspond to the entries of T^{i+1} not bumped, which are exactly the ones that move from T^{i+1} to $(P_i(T))^i$ in computing $P_i(T)$.

It remains to show that, given a string $w_1\cdots w_m$ in the letters "(" and ")", the ordinary and greedy matching rules produce the same unmatched "("s. We proceed by induction on m. Consider the subword $w_i\cdots w_m$ where w_i is the rightmost matched "("; it must look like $())\cdots)(\cdots($. Let (w_i, w_j) (resp. (w_i, w_{i+1})) be the greedy (resp. ordinary) matched pair in this subword. Though these pairs typically differ, deleting the greedy-matched pair yields the same string as deleting the ordinary matched pair. Since the position of the "(" in both pairs is the same, the result follows by the inductive hypothesis. □

Proposition 6.10 *Let $b\in\mathcal{B}^\mu$ and set $T=\mathrm{inv}(b)\in\mathrm{Tabloids}_\ell(\mu)$. Then b is a $U_q(\mathfrak{gl}_\ell)$-highest weight element if and only if any of the following equivalent conditions holds*:

(a) $\tilde{e}_i(b)=0$ *for all* $i\in[\ell-1]$,
(b) $P_i(T)=T$ *for all* $i\in[\ell-1]$,
(c) T *is a tableau, i.e.,* $T\in\mathrm{SSYT}_\ell(\mu)$.

Proof Condition (a) is the definition of b being a $U_q(\mathfrak{gl}_\ell)$-highest weight element. The equivalence (a) $\iff$ (b) is by Proposition 6.9, and (b) $\iff$ (c) is clear from computing $P(T^{i+1}T^i)$ by column insertion. □

Proposition 6.11 *Let $B^{\mathfrak{gl}}(\nu)$, $\nu=(\nu_1\geq\cdots\geq\nu_\ell)$, be a highest weight $U_q(\mathfrak{gl}_\ell)$-crystal and u_ν its highest weight element. Then*

(i) $\mathcal{F}_{\mathsf{w}_0}\{u_\nu\}=B^{\mathfrak{gl}}(\nu)$.
(ii) *The operators* $\tilde{e}_1^{\max},\dots,\tilde{e}_{\ell-1}^{\max}$ *on* $B^{\mathfrak{gl}}(\nu)$ *satisfy the* 0*-Hecke relations of* $\mathcal{H}_\ell$ ((4.6), (4.7), *and* (4.10)).
(iii) *The operators* $P_1,\dots,P_{\ell-1}$ *on* $\mathrm{Tabloids}_\ell$ *satisfy the* 0*-Hecke relations of* $\mathcal{H}_\ell$.
(iv) $\tilde{e}_{\mathsf{w}_0}^{\max}(b)=u_\nu$ *for any* $b\in B^{\mathfrak{gl}}(\nu)$ *and* $P_{\mathsf{w}_0}(T)=P(T)$ *for any* $T\in\mathrm{Tabloids}_\ell$.

Proof Statement (i) is well known; it can be deduced, for instance, from Remark 4.4 using that $B^{\mathfrak{gl}}(\nu)$ is finite. Statement (iii) holds by (ii) and Proposition 6.9. For (ii), the $\tilde{e}_i^{\max}$ clearly satisfy the relations (4.10); we now verify that they satisfy the braid relations (4.7) and omit the similar argument for (4.6). Let $b\in B^{\mathfrak{gl}}(\nu)$. Let $U_q(\mathfrak{g}_J)\subset U_q(\mathfrak{gl}_\ell)$ be the subalgebra isomorphic to $U_q(\mathfrak{gl}_3)$ associated to Dynkin node subset $J=\{i,i+1\}\subset[\ell-1]$. The component B' of $\mathrm{Res}_J\, B^{\mathfrak{gl}}(\nu)$ containing b is isomorphic to a highest weight $U_q(\mathfrak{gl}_3)$-crystal (see §4.2); let $u\in B'$ be its highest weight element. By (i) and Remark 4.4, $B'=\mathcal{F}_1\mathcal{F}_2\mathcal{F}_1\{u\}=\mathcal{F}_2\mathcal{F}_1\mathcal{F}_2\{u\}$; hence $\tilde{e}_i^{\max}\tilde{e}_{i+1}^{\max}\tilde{e}_i^{\max}(b)=u=\tilde{e}_{i+1}^{\max}\tilde{e}_i^{\max}\tilde{e}_{i+1}^{\max}(b)$.

For (iv), $\tilde{e}_{\mathsf{w}_0}^{\max}(b)=u_\nu$ holds by (i). Next, let $U=P(T)$ and $\mathcal{C}_U$ be the $U_q(\mathfrak{gl}_\ell)$-crystal component containing $\mathrm{inv}(T)$. By Proposition 6.10, the set $\mathrm{inv}(\mathcal{C}_U)=\{S\in\mathrm{Tabloids}_\ell(\mu)\mid P(S)=U\}$ has a unique element fixed by P_i for all $i\in[\ell-1]$. Both U and $P_{\mathsf{w}_0}(T)$ satisfy this property by Proposition 6.10 and (iii), hence $P(T)=P_{\mathsf{w}_0}(T)$. □

6.6 The kat and kat′ operators and the automorphism τ

Recall from §4.6 that for $\sigma \in \Sigma = \{\tau^j \mid j \in [\ell]\}$ and $U'_q(\widehat{\mathfrak{sl}}_\ell)$-seminormal crystals B, B', a σ-twist is a bijection $B \to B'$ taking i-edges to $\sigma(i)$-edges. For a word w, let $\mathrm{sort}(w)$ denote its weakly increasing rearrangement. For $m \in \mathbb{Z}$, let $\mathrm{mod}_\ell(m)$ be the unique $i \in \{1, 2, \dots, \ell\}$ such that $i \equiv m \bmod \ell$.

Proposition 6.12 ([71, Proposition 5.5]) *There is a unique σ-twist $\mathcal{F}_\sigma : \mathcal{B}^\mu \to \mathcal{B}^\mu$ for any $\sigma \in \Sigma$. The τ^{-1}-twist $\mathcal{F}_{\tau^{-1}}$ has the following explicit description: first, for $v = v_1 \cdots v_s \in B^{1,s}$, $\mathcal{F}_{\tau^{-1}}(v_1 \cdots v_s) = \mathrm{sort}\big(\mathrm{mod}_\ell(v_1 - 1) \cdots \mathrm{mod}_\ell(v_s - 1)\big)$. Then for a biword $b \in \mathcal{B}^\mu$ with blocks $b^p, \dots, b^1$, $\mathrm{bottom}(\mathcal{F}_{\tau^{-1}}(b)) = \mathcal{F}_{\tau^{-1}}(b^p) \cdots \mathcal{F}_{\tau^{-1}}(b^1)$ and $\mathrm{top}(\mathcal{F}_{\tau^{-1}}(b)) = \mathrm{top}(b)$.*

For example, with $\ell = 4$ and $b = 233\,1124\,12223 \in \mathcal{B}^{543}$, $\mathcal{F}_{\tau^{-1}}(b) = 122\,1344\,11124$.

For $b \in \mathcal{B}^\mu$, with blocks denoted $b^p, \dots, b^1$ as usual, define

$$\mathrm{kat}'(b) = \mathcal{F}_{\tau^{-1}}(b^p \cdots b^2) \in \mathcal{B}^{(\mu_2, \dots, \mu_p)}. \tag{6.5}$$

In other words, $\mathrm{kat}'(b)$ is obtained from the biword b as follows: remove the rightmost block of b, subtract 1 from all bottom letters, turn any 0's into ℓ's, sort each block, and finally subtract 1 from all top letters to obtain a biword in $\mathcal{B}^{(\mu_2, \dots, \mu_p)}$. For example, with $\ell = 4$ and $b = \begin{pmatrix} 444\,3333\,22222\,111111 \\ 233\,1124\,12223\,111111 \end{pmatrix} \in \mathcal{B}^{6543}$, $\mathrm{kat}'(b) = \begin{pmatrix} 333\,2222\,11111 \\ 122\,1344\,11124 \end{pmatrix} \in \mathcal{B}^{543}$.

Recall from Definition 2.15 that for $T \in \mathrm{Tabloids}_\ell$, $\mathrm{kat}(T)$ is defined as follows: remove all 1's from T and left justify rows, then shift rows up by one cycling the first row to become the ℓ-th row, and finally subtract 1 from all letters. The operators kat and kat′ are conjugate under inv:

Proposition 6.13 *For any $T \in \mathrm{Tabloids}_\ell$, $\mathrm{inv}(\mathrm{kat}(T)) = \mathrm{kat}'(\mathrm{inv}(T))$.*

Proof For $j > 1$, let $a_1 \cdots a_m$ be the row indices of the letters j in T, in weakly increasing order, which is also the block $\mathrm{inv}(T)^j$. Since kat rotates rows, these j's (which become $j-1$'s in $\mathrm{kat}(T)$) appear in rows $\mathrm{mod}_\ell(a_1 - 1) \cdots \mathrm{mod}_\ell(a_m - 1)$ of $\mathrm{kat}(T)$. Thus

$$(\mathrm{inv}(\mathrm{kat}(T)))^{j-1} = \mathrm{sort}\big(\mathrm{mod}_\ell(a_1 - 1) \cdots \mathrm{mod}_\ell(a_m - 1)\big) = (\mathrm{kat}'(\mathrm{inv}(T)))^{j-1},$$

where the second equality is by Proposition 6.12 ($j-1$ appears on the right and not j because of the final step in the computation of kat′). The result follows. □

6.7 Katabolism

Definition 6.14 Let $\mathbf{w} = (w_1, \dots, w_p) \in (\mathcal{H}_\ell)^p$. We say $T \in \mathrm{Tabloids}_\ell$ is $\mathbf{w}$-*katabolizable* if all the 1's of $P_{w_1^{-1}}(T)$ lie in its first row and $\mathrm{kat}(P_{w_1^{-1}}(T))$ is $(w_2, \dots, w_p)$-katabolizable. For $\mathbf{w}$ the empty sequence, the only $\mathbf{w}$-katabolizable tabloid is the empty one.

The streamlined version of katabolism from Definition 2.15 agrees with this one in the setting of Theorem 2.18, as we now verify.

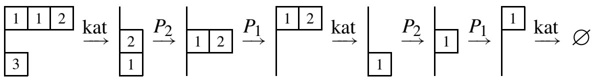
Proposition 6.15 *Let* $\mu \in \mathbb{Z}^p_{\geq 0}$ *and* $\mathbf{n} = (n_1, \ldots, n_{p-1}) \in [\ell]^{p-1}$ *satisfy* $n_{i+1} \geq n_i - 1$ *for all* $i \in [p-2]$. *A tableau* $U \in \mathrm{SSYT}_\ell(\mu)$ *is* $\mathbf{n}$*-katabolizable in the sense of Definition 2.15 if and only if it is* $(id, \mathsf{s}(n_1), \ldots, \mathsf{s}(n_{p-1}))$*-katabolizable.*

Proof We first verify the following claim: for any tabloid T such that its subtabloid $T^{[i,\ell-1]}$ is a tableau, $P_i \cdots P_{\ell-1}(T)$ can be obtained by column inserting T^ℓ into $T^{[i,\ell-1]}$, i.e., $P_i \cdots P_{\ell-1}(T) = P_{i,\ell}(T)$ in the notation of Definition 2.13. To ease notation, assume $i = 1$, as this easily implies the general case. We have $P_{1,\ell}(T) = P(T)$, the unique tableau with reading word Knuth equivalent to that of T. Then by Proposition 6.11 (iv), $P_{1,\ell}(T) = P(T) = P_{\mathsf{w}_0}(T) = P_1 \cdots P_{\ell-1} P_{\mathsf{w}_{[1,\ell-1)}}(T) = P_1 \cdots P_{\ell-1}(T)$, where the last equality uses that $T^{[\ell-1]}$ is a tableau.

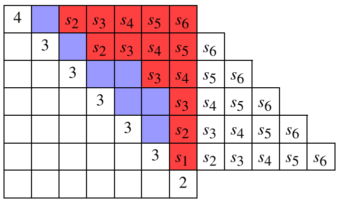
Let us now see that the tabloids produced in computing the two versions of katabolism are the same: set $\dot{U} = \mathrm{kat}(U)$. Since $\dot{U}^{[\ell-1]}$ is a tableau, $P_{n_1} \cdots P_{\ell-1}(\dot{U}) = P_{n_1,\ell}(\dot{U})$ by the claim. Since $(P_{n_1,\ell}(\dot{U}))^{[n_1,\ell]}$ is a tableau, so is $\ddot{U}^{[n_1-1,\ell-1]}$, for $\ddot{U} := \mathrm{kat}(P_{n_1,\ell}(\dot{U}))$. As $n_2 \geq n_1 - 1$, $\ddot{U}^{[n_2,\ell-1]}$ is also a tableau. Hence $P_{n_2} \cdots P_{\ell-1}(\ddot{U}) = P_{n_2,\ell}(\ddot{U})$ again by the claim, and so on. □

Remark 6.16 With the assumption of Proposition 6.15, $P_{n_i,\ell}(T) = P_{\mathsf{s}(n_i)^{-1}}(T) = P_{n_i} \cdots P_{\ell-1}(T)$ at every step of the katabolism algorithm, so in this sense the partial insertion of Definition 6.8 generalizes that of Definition 2.13. We caution however, that without this assumption, only Definition 6.14 should be used and not Definition 2.15.

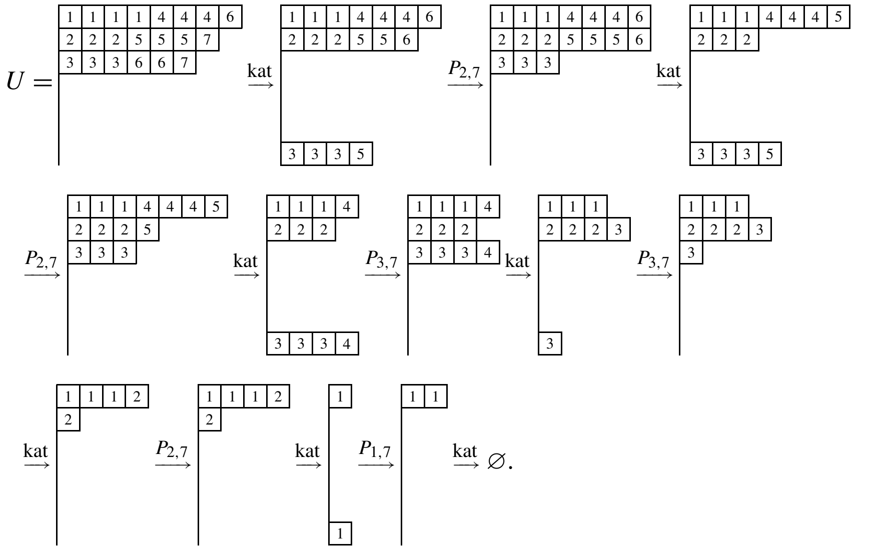
Example 6.17 The following tabloid from Fig. 2 (§2.7) is $(id, \mathsf{s}_2\mathsf{s}_1, \mathsf{s}_2\mathsf{s}_1)$-katabolizable:



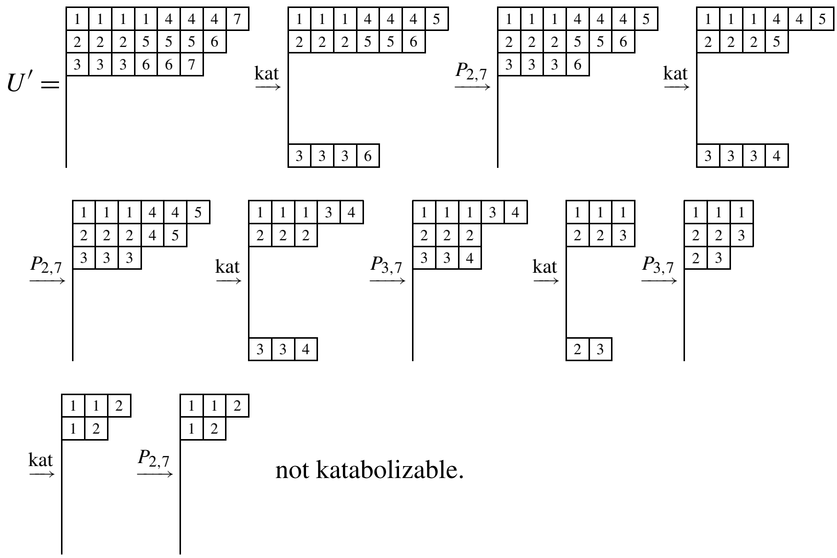
Example 6.18 Let $\ell = 7$, $\mu = 4333332$, and Ψ be the root ideal in red ■; $\Delta^+ \setminus \Psi$ is shown in blue ■. We have $\mathbf{n}(\Psi) = (2, 2, 3, 3, 2, 1)$. We can visualize $\mathbf{s}(\Psi) = (\mathsf{s}(\mathbf{n}(\Psi)_1), \ldots, \mathsf{s}(\mathbf{n}(\Psi)_{\ell-1}))$ partially overlaid on the root ideal, so that row i read right to left is $\mathbf{s}(\Psi)_i$.



The following computation shows the tableau U below to be $\mathbf{n}(\Psi)$-katabolizable or equivalently $(id, \mathbf{s}(\Psi))$-katabolizable (see Proposition 6.15). So this gives one term $q^{\mathrm{charge}(U)} s_{\mathrm{shape}(U)} = q^{14} s_{876}$ of the Schur expansion of $H(\Psi; \mu; \mathsf{w}_0)$ from Theorem 2.18.



In contrast, the tableau U' below is not $\mathbf{n}(\Psi)$-katabolizable since the katabolism algorithm produces a tabloid with a 1 outside its first row just after an application of $P_{n(\Psi)_i,\ell}$.



Remark 6.19 Let $\mu = (\mu_1 \geq \cdots \geq \mu_\ell \geq 0)$ and $\mathbf{w} = (w_1, \ldots, w_\ell) = (w_1, \mathbf{s}(\Psi)) \in (\mathcal{H}_\ell)^\ell$ for a root ideal $\Psi \subset \Delta_\ell^+$ which is empty in rows $\geq r$. Then $T \in \mathrm{Tabloids}_\ell(\mu)$ is $\mathbf{w}$-katabolizable $\iff$ T is $(w_1, \ldots, w_r, id, \ldots, id)$-katabolizable $\iff$ for $i \in [r-1]$, the tabloid U_i has all its 1's on the first row, and U_r is the superstandard

tableau of shape and content $(\mu_r, \dots, \mu_\ell)$, where $U_i := P_{w_i^{-1}} \circ \mathrm{kat} \circ \cdots \circ \mathrm{kat} \circ P_{w_2^{-1}} \circ \mathrm{kat} \circ P_{w_1^{-1}}(T)$.

Theorem 6.20 *For $\mu = (\mu_1 \geq \cdots \geq \mu_p \geq 0)$ and $\mathbf{w} = (w_1, \dots, w_p) \in (\mathcal{H}_\ell)^p$, inv gives a bijection*

$$\left\{T \in \mathrm{Tabloids}_\ell(\mu) \mid T \textit{ is } \mathbf{w}\textit{-katabolizable}\right\} \xrightarrow{\mathrm{inv}} \mathcal{B}^{\mu;\mathbf{w}} \tag{6.6}$$

which takes shape to content.

Proof We must show that for any $T \in \mathrm{Tabloids}_\ell(\mu)$, T is $\mathbf{w}$-katabolizable if and only if $\mathrm{inv}(T) \in \mathcal{B}^{\mu;\mathbf{w}}$. We prove this by induction on $p + \sum_i \mathrm{length}(w_i)$. The base case $p = 1$, $w_1 = id$ is clear. Now suppose $w_1 \neq id$. We can write $\mathcal{B}^{\mu;\mathbf{w}} = \mathcal{F}_{w_1}(\mathcal{B}^{\mu;(id,w_2,\dots,w_p)})$ and then

$$\begin{aligned}
&\mathrm{inv}(T) \in \mathcal{B}^{\mu;\mathbf{w}} \\
\iff &\tilde{e}^{\max}_{w_1^{-1}}(\mathrm{inv}(T)) \in \mathcal{B}^{\mu;(id,w_2,\dots,w_p)} \\
\iff &\mathrm{inv}\left(\tilde{e}^{\max}_{w_1^{-1}}\big(\mathrm{inv}(T)\big)\right) = P_{w_1^{-1}}(T) \text{ is } (id, w_2, \dots, w_p)\text{-katabolizable} \\
\iff &T \text{ is } (w_1, w_2, \dots, w_p)\text{-katabolizable,}
\end{aligned}$$

where the second equivalence uses Proposition 6.9 and the inductive hypothesis.

Next suppose $p > 1$ and $w_1 = id$. Note that $\mathcal{B}^{\mu;\mathbf{w}} = (\mathcal{F}_\tau\, \mathcal{B}^{(\mu_2,\dots,\mu_p);(w_2,\dots,w_p)}) \otimes \mathsf{b}_{\mu_1}$. Then $\mathrm{inv}(T) \in \mathcal{B}^{\mu;\mathbf{w}}$ if and only if (1) the rightmost block of $\mathrm{inv}(T)$ is $\mathsf{b}_{\mu_1} = 1^{\mu_1}$ and (2) $b' \in \mathcal{F}_\tau\, \mathcal{B}^{(\mu_2,\dots,\mu_p);(w_2,\dots,w_p)}$, where b' is the biword obtained from $\mathrm{inv}(T)$ by removing this block of 1's and subtracting 1 from its top letters. By (6.5), condition (2) is equivalent to $\mathrm{kat}'(\mathrm{inv}(T)) = \mathcal{F}_{\tau^{-1}}(b') \in \mathcal{B}^{(\mu_2,\dots,\mu_p);(w_2,\dots,w_p)}$; by the inductive hypothesis and Proposition 6.13, this is equivalent to $\mathrm{kat}(T) = \mathrm{inv}(\mathrm{kat}'(\mathrm{inv}(T)))$ being $(w_2, \dots, w_p)$-katabolizable. Hence, noting that (1) is equivalent to T having no 1's outside its first row, we conclude that (1) and (2) are equivalent to T being $\mathbf{w}$-katabolizable. □

Theorem 6.21 *Let μ and $\mathbf{w} = (w_1, \dots, w_p)$ be as in Theorem 6.20, but now with the additional assumption $w_1 = \mathsf{w}_0$. Then the DARK crystal $\mathcal{B}^{\mu;\mathbf{w}}$ (regarded as a subset of the $U_q(\mathfrak{gl}_\ell)$-crystal $\mathcal{B}^\mu$) is a disjoint union of highest weight $U_q(\mathfrak{gl}_\ell)$-crystals, with decomposition given by*

$$\mathcal{B}^{\mu;\mathbf{w}} = \bigsqcup_{\substack{U \in \mathrm{SSYT}_\ell(\mu) \\ U \textit{ is } (id, w_2, \dots, w_p)\textit{-katabolizable}}} \mathcal{C}_U, \quad \textit{where } \mathcal{C}_U = \{b \in \mathcal{B}^\mu \mid Q(b) = U\}. \tag{6.7}$$

Proof This follows from Theorems 6.4 and 6.20, using that, when $w_1 = \mathsf{w}_0$, $T \in \mathrm{Tabloids}_\ell(\mu)$ is $\mathbf{w}$-katabolizable if and only if $P(T) = P_{\mathsf{w}_0^{-1}}(T)$ is $(id, w_2, \dots, w_p)$-katabolizable. □

Let us now also prove Theorem 2.17: apply Theorem 6.20 with $\mathbf{w} = (\mathsf{w}_0, \mathbf{s}(\Psi))$, then the "if and only if" statement in the proof of Theorem 6.21, and then Proposition 6.15.

7 Schur and key positivity

We connect charge to $\widehat{\mathfrak{sl}}_\ell$-weights and then combine the results of Sect. 6 with Corollary 2.7 and Theorem 2.11 to give several character formulas for DARK and AGD crystals; this yields our katabolism formula Theorem 2.18 upon combining with the rotation theorem. Stronger key positivity results are then obtained via the restriction Theorem 4.1.

7.1 Characters

Let $\mathbb{Z}[P]$ denote the group ring of P with $\mathbb{Z}$-basis $\{e^\lambda\}_{\lambda \in P}$. The $\widehat{\mathfrak{sl}}_\ell$-*Demazure operators* are linear operators D_i on $\mathbb{Z}[P]$ defined for each $i \in I$ by

$$D_i(f) = \frac{f - e^{-\alpha_i} \cdot s_i(f)}{1 - e^{-\alpha_i}},$$

where s_i acts on $\mathbb{Z}[P]$ by $s_i(e^\lambda) = e^{s_i(\lambda)}$ (see §4.5). The action of $\Sigma = \{\tau^i \mid i \in [\ell]\}$ on P yields an action on $\mathbb{Z}[P]$ given by $\sigma(e^\lambda) = e^{\sigma(\lambda)}$ for $\sigma \in \Sigma$. Then τ and the D_i $(i \in I)$ satisfy the 0-Hecke relations (4.6)–(4.10) of $\widetilde{\mathcal{H}}_\ell$ (it is well known that they satisfy (4.6), (4.7) [42, Corollary 8.2.10] and the others are easily checked). Thus, just as we discussed for $\mathcal{F}_w$ in §4.7, D_w makes sense for any $w \in \widetilde{\mathcal{H}}_\ell$ and $D_w D_{w'} = D_{ww'}$ for all $w, w' \in \widetilde{\mathcal{H}}_\ell$.

The *character* of a subset S of a $U_q(\widehat{\mathfrak{sl}}_\ell)$-crystal is $\operatorname{char}(S) := \sum_{b \in S} e^{\operatorname{wt}(b)} \in \mathbb{Z}[P]$. Kashiwara [35] gave a Demazure operator formula for the character of any Demazure crystal, and Naoi extended this to encompass the action of Σ, as follows:

Corollary 7.1 ([71, Corollary 4.6]) *For any $w \in \widetilde{\mathcal{H}}_\ell$ and $S \in \mathcal{D}(\widehat{\mathfrak{sl}}_\ell)$ (see Definition 2.5)*,

$$\operatorname{char}(\mathcal{F}_w(S)) = D_w(\operatorname{char}(S)).$$

Set $\mathbb{A} = \mathbb{Z}[q^{1/2\ell}, q^{-1/2\ell}]$. Define the ring homomorphism ζ by

$$\zeta : \mathbb{A}[x_1^{\pm 1}, \ldots, x_\ell^{\pm 1}] \to \mathbb{Z}[P], \quad x_i \mapsto e^{\varpi_i - \varpi_{i-1}}, \ q^{1/2\ell} \mapsto e^{-\delta/2\ell}. \tag{7.1}$$

It is $\mathcal{S}_\ell$-equivariant (s_i acts on $\mathbb{A}[x_1^{\pm 1}, \ldots, x_\ell^{\pm 1}]$ by permuting the variables) and has kernel $(x_1 \cdots x_\ell - 1)$. It is an extension of the map ζ from (2.8) to a larger domain.

We wish to recover an element of $\mathbb{A}[x_1^{\pm 1}, \ldots, x_\ell^{\pm 1}]$ given its image under ζ, and this is possible if we know it to be homogenous of a given degree. Accordingly, let $\mathbf{X}_m \subset \mathbb{A}[x_1^{\pm 1}, \ldots, x_\ell^{\pm 1}]$ denote the homogeneous component of $\mathbf{x}$-degree m. The restricted map $\zeta : \mathbf{X}_m \to \mathbb{Z}[P]$ is injective; let $\zeta(\mathbf{X}_m)$ denote the image and $Z_m : \zeta(\mathbf{X}_m) \xrightarrow{\cong} \mathbf{X}_m$ the inverse of this restriction of ζ (which is only a $\mathbb{Z}$-linear map).

Let μ be a partition and set $m = |\mu|$. Suppose G is a $U_q(\widehat{\mathfrak{sl}}_\ell)$-crystal such that $e^{-\mu_1\Lambda_0}\,\mathrm{char}(G) \in \zeta(\mathbf{X}_m)$ (by the proof of Theorem 7.5 below, this holds for $G = \mathrm{AGD}(\mu; \mathbf{w})$, our main case of interest). We define the $\mathbf{x}$-*character* of G by

$$\mathrm{char}_{\mathbf{x};\mu}(G) = \sum_{g\in G} Z_m(e^{\mathrm{wt}(g)-\mu_1\Lambda_0}) \in \mathbf{X}_m\,. \tag{7.2}$$

In other words, if we find $f \in \mathbf{X}_m$ such that $\zeta(f) = \mathrm{char}(G)e^{-\mu_1\Lambda_0}$, then $f = \mathrm{char}_{\mathbf{x};\mu}(G)$.

We will need two facts which relate π_i and D_i, and Φ and τ via ζ. First, it is straightforward to show from the $\mathcal{S}_\ell$-equivariance of ζ that

$$\zeta(\pi_i(f)) = D_i(\zeta(f)) \quad \text{for } i \in [\ell-1]. \tag{7.3}$$

Second, we claim that for any $f \in \mathbf{X}_m$,

$$\zeta(\Phi(f)) = e^{-m\delta/\ell}\tau(\zeta(f)). \tag{7.4}$$

Since ζ, τ, and Φ are ring homomorphisms, it is enough to prove $\zeta(\Phi(x_i)) = e^{-\delta/\ell}\tau(\zeta(x_i))$. This is readily verified from the computation

$$\begin{aligned}\tau(\zeta(x_i)) &= \tau(e^{\varpi_i-\varpi_{i-1}}) = e^{\varpi_{i+1}-\varpi_i+\delta(m_{i+1}-m_i-(m_i-m_{i-1}))}\\ &= \begin{cases} e^{\varpi_{i+1}-\varpi_i+\delta/\ell} & \text{if } i \in [\ell-1],\\ e^{\varpi_{i+1}-\varpi_i+\delta/\ell-\delta} & \text{if } i = \ell,\end{cases}\end{aligned}$$

where $m_i := \langle d, \Lambda_i\rangle$ and the last equality is by (4.4).

7.2 Charge and $\widehat{\mathfrak{sl}}_\ell$-weights

The pairing $\langle d, \cdot\rangle$ on $\widehat{\mathfrak{sl}}_\ell$-weights gives a statistic on $U_q(\widehat{\mathfrak{sl}}_\ell)$-crystal elements, which is not available for $U_q'(\widehat{\mathfrak{sl}}_\ell)$-seminormal crystals. Naoi [71] showed that the strict embedding Θ_μ matches this statistic to energy, thereby effectively allowing the full information of $\widehat{\mathfrak{sl}}_\ell$-weights to be seen on the DARK side. Since energy on $\mathcal{B}^\mu$ matches charge on $\mathrm{Tabloids}_\ell(\mu) = \mathrm{inv}(\mathcal{B}^\mu)$ [70], the charge and $\langle d, \cdot\rangle$ statistics agree.

Remark 7.2 It is actually more natural to connect charge and $\langle d, \cdot\rangle$ directly as they both essentially measure the number of $\tilde{f}_0$-edges required to construct the crystal element, whereas energy is a more complicated statistic. In the interest of space, we just give the idea: for $b = \tilde{f}_{i_1}^{a_1}\cdots\tilde{f}_{i_k}^{a_k}u_\Lambda \in \mathcal{F}_{i_1}\cdots\mathcal{F}_{i_k}\{u_\Lambda\}$ with $i_j \in I$, $\langle -d, \mathrm{wt}(b) - \mathrm{wt}(u_\Lambda)\rangle$ is the number of $\tilde{f}_0$'s appearing in $\tilde{f}_{i_1}^{a_1}\cdots\tilde{f}_{i_k}^{a_k}$. A similar statement can be made for AGD crystals. Charge also has a similar flavor since $\tilde{f}_0$-edges are related to property (C3) below by the inv map—see [80, §4.2].

Charge is a statistic on words of partition content which is commonly defined by a circular-reading procedure (see, e.g., [82, §3.6]). We prefer to take the following theorem of Lascoux and Schützenberger as its definition.

Theorem 7.3 ([57], see [82, Theorem 24]) *Charge is the unique function from words of partition content to* $\mathbb{Z}_{\geq 0}$ *satisfying*

(C1) *Charge of the empty word is* 0.
(C2) *For a word of partition content* λ *and of the form* $u = v1^{\lambda_1}$, $\mathrm{charge}(u) = \mathrm{charge}(v^-)$, *where* v^- *is obtained from* v *by subtracting* 1 *from all its letters.*
(C3) *For a word of partition content and of the form* $u = vx$ *with* $x \neq 1$ *a letter,* $\mathrm{charge}(vx) = \mathrm{charge}(xv) + 1$.
(C4) *Charge is constant on Knuth equivalence classes.*

We will view charge as a statistic on tabloids by setting $\mathrm{charge}(T) = \mathrm{charge}(T^\ell \cdots T^2 T^1)$ for any $T \in \mathrm{Tabloids}_\ell$, where the concatenation $T^\ell \cdots T^2 T^1$ is the row reading word of T.

Corollary 7.4 *Let* μ *be a partition and* $\Theta_\mu \colon \mathcal{B}^\mu \otimes B(\mu_1\Lambda_0) \hookrightarrow B(\mu^p\Lambda_p) \otimes \cdots \otimes B(\mu^1\Lambda_1)$ *the strict embedding of* $U'_q(\widehat{\mathfrak{sl}}_\ell)$*-seminormal crystals from Theorem 2.11. For any* $b \in \mathcal{B}^\mu$,

$$\mathrm{wt}\left(\Theta_\mu(b \otimes u_{\mu_1\Lambda_0})\right) = \mu_1\Lambda_0 + \mathrm{aff}(\mathrm{wt}(b)) - \delta\big(\mathrm{charge}(\mathrm{inv}(b)) + n_\ell(\mu)\big), \quad (7.5)$$

$$\zeta\left(q^{\mathrm{charge}(\mathrm{inv}(b)) + n_\ell(\mu)}\mathbf{x}^{\mathrm{content}(b)}\right) = e^{-\mu_1\Lambda_0 + \mathrm{wt}(\Theta_\mu(b \otimes u_{\mu_1\Lambda_0}))}, \quad (7.6)$$

where $n_\ell(\mu) := \frac{|\mu|(\ell-1) - 2n(\mu)}{2\ell}$, *a variant of the well-known statistic* $n(\mu) := \sum_{i=1}^p (i-1)\mu_i$.

Proof As $\mathrm{wt}(b) \in P_{\mathrm{cl}}$ is given by (6.1), $\zeta(\mathbf{x}^{\mathrm{content}(b)}) = e^{\mathrm{aff}\,\mathrm{wt}(b)}$; hence (7.5) implies (7.6).

We now prove (7.5). Set $m_i = \langle d, \Lambda_i \rangle$ for $i \in I$. Since Θ_μ commutes with the P_{cl}-valued weight functions, $\mathrm{cl}(\mathrm{wt}\,\Theta_\mu(b \otimes u_{\mu_1\Lambda_0})) = \mathrm{cl}(\mu_1\Lambda_0) + \mathrm{wt}(b)$. Thus (7.5) is equivalent to

$$\mu_1 m_0 + \langle -d, \mathrm{wt}\,\Theta_\mu(b \otimes u_{\mu_1\Lambda_0})\rangle - \mathrm{charge}(\mathrm{inv}(b)) = n_\ell(\mu). \quad (7.7)$$

By [71, Theorem 7.1], $\langle -d, \mathrm{wt}\,\Theta_\mu(b \otimes u_{\mu_1\Lambda_0})\rangle = D(b) + C$, where $D(b)$ is the energy of b and C is a constant that depends on μ and ℓ but not b. Further, $D(b) = \mathrm{charge}(\mathrm{inv}(b))$ by [70] (see also [81, Proposition 4.25]). Hence, to pin down the constant, we need only verify (7.7) for a single $b \in \mathcal{B}^\mu$. We choose $b_{hw} := \mathcal{F}_\tau\big(\cdots \mathcal{F}_\tau(\mathcal{F}_\tau \mathsf{b}_{\mu_p} \otimes \mathsf{b}_{\mu_{p-1}}) \cdots \otimes \mathsf{b}_{\mu_2}\big) \otimes \mathsf{b}_{\mu_1}$, the element satisfying $\Theta_\mu(u_{hw}) = b_{hw} \otimes u_{\mu_1\Lambda_0}$ for $u_{hw} := u_{\mu^p\Lambda_p} \otimes \cdots \otimes u_{\mu^1\Lambda_1}$, as can be seen from the proof of [9, Theorem 3.7]. We compute

$$\begin{aligned}
&\mu_1 m_0 + \langle -d, \mathrm{wt}(u_{hw})\rangle - \mathrm{charge}(\mathrm{inv}(b_{hw})) \\
&= \mu_1 m_0 - \sum_{i=1}^p \mu^i m_i - \mathrm{charge}(\mathrm{inv}(b_{hw})) \\
&= -\sum_{i=1}^p \mu_i(m_i - m_{i-1}) - \mathrm{charge}(\mathrm{inv}(b_{hw}))
\end{aligned}$$

$$= -\sum_{i=1}^{p} \mu_i \tfrac{2\,\mathrm{mod}_\ell(i)-1-\ell}{2\ell} - \textstyle\sum_{i=1}^{p} \lfloor \tfrac{i-1}{\ell} \rfloor \mu_i$$

$$= \tfrac{|\mu|\,(\ell-1)}{2\ell} - \tfrac{1}{\ell} \textstyle\sum_{i=1}^{p} (i-1)\mu_i = n_\ell(\mu),$$

where $\{\mathrm{mod}_\ell(i)\} = (i + \ell\mathbb{Z}) \cap [\ell]$. The third equality is by (4.4) and a direct computation of the charge of $\mathrm{inv}(b_{hw})$, the tabloid of content μ with all letters i in row $\mathrm{mod}_\ell(i)$. □

7.3 A Schur positive formula for Catalan functions: proof of Theorem 2.18

Theorem 7.5 *Let* $\mathbf{w} = (w_1, w_2, \ldots, w_p) \in (\mathcal{H}_\ell)^p$ *and* $\mu = (\mu_1 \geq \cdots \geq \mu_p \geq 0)$ *be a partition; set* $\mu^i = \mu_i - \mu_{i+1}$, *where* $\mu_{p+1} := 0$. *The* $\mathbf{x}$*-character of the crystal* $\mathrm{AGD}(\mu; \mathbf{w})$ *agrees with the charge weighted character of the DARK crystal* $\mathcal{B}^{\mu;\mathbf{w}}$ *and these have an explicit description in terms of* π_i *and* Φ:

$$\pi_{w_1} x_1^{\mu_1} \Phi \pi_{w_2} x_1^{\mu_2} \Phi \pi_{w_3} x_1^{\mu_3} \cdots \Phi \pi_{w_p} x_1^{\mu_p} = q^{-n_\ell(\mu)}\, \mathrm{char}_{\mathbf{x};\mu}(\mathrm{AGD}(\mu; \mathbf{w})) \tag{7.8}$$

$$= \sum_{b \in \mathcal{B}^{\mu;\mathbf{w}}} q^{\mathrm{charge}(\mathrm{inv}(b))} \mathbf{x}^{\mathrm{content}(b)} = \sum_{\substack{T \in \mathrm{Tabloids}_\ell(\mu) \\ T \text{ is } \mathbf{w}\text{-katabolizable}}} q^{\mathrm{charge}(T)} \mathbf{x}^{\mathrm{shape}(T)}, \tag{7.9}$$

where $n_\ell(\mu) = \frac{|\mu|\,(\ell-1)}{2\ell} - \frac{1}{\ell} \sum_{i=1}^{p} (i-1)\mu_i$ *as in Corollary 7.4.*

Proof The first equality of (7.9) follows from Theorem 2.11 and (7.6), and the second holds by Theorem 6.20. We will establish (7.8) by proving $e^{-\mu_1\Lambda_0}\,\mathrm{char}(\mathrm{AGD}(\mu; \mathbf{w})) = e^{-\delta n_\ell(\mu)} \zeta(\pi_{w_1} x_1^{\mu_1} \Phi \pi_{w_2} x_1^{\mu_2} \cdots \Phi \pi_{w_p} x_1^{\mu_p})$. By Corollaries 2.7 and 7.1,

$$\mathrm{char}(\mathrm{AGD}(\mu; \mathbf{w})) = D_{w_1}\big(e^{\mu^1\Lambda_1} \cdot \tau D_{w_2}\big(e^{\mu^2\Lambda_1} \cdot \tau D_{w_3} \cdots \tau D_{w_p}(e^{\mu^p\Lambda_1})\big)\big). \tag{7.10}$$

(A similar character formula is proved in [71, §7] with this argument.) Using the operator identities $e^{\Lambda_1}\tau = \tau e^{\Lambda_0}$ and $e^{\Lambda_0} D_i = D_i e^{\Lambda_0}$ for $i \in [\ell - 1]$, we compute

$$\begin{aligned}
&e^{-\mu_1\Lambda_0}\,\mathrm{char}(\mathrm{AGD}(\mu; \mathbf{w}))\\
&\quad = e^{-\mu_1\Lambda_0} D_{w_1}\big(e^{\mu^1\Lambda_1} \cdot \tau D_{w_2}\big(e^{\mu^2\Lambda_1} \cdots \tau D_{w_p}(e^{\mu^p\Lambda_1})\big)\big)\\
&\quad = D_{w_1}\big(e^{\mu_1(\Lambda_1-\Lambda_0)-\mu_2\Lambda_1} \cdot \tau D_{w_2}\big(e^{\mu^2\Lambda_1} \cdots \tau D_{w_p}(e^{\mu^p\Lambda_1})\big)\big)\\
&\quad = D_{w_1}\big(e^{\mu_1(\Lambda_1-\Lambda_0)} \cdot \tau D_{w_2}\big(e^{-\mu_2\Lambda_0} e^{\mu^2\Lambda_1} \cdots \tau D_{w_p}(e^{\mu^p\Lambda_1})\big)\big)\\
&\quad = D_{w_1}\big(e^{\mu_1(\Lambda_1-\Lambda_0)} \cdot \tau D_{w_2}\big(e^{\mu_2(\Lambda_1-\Lambda_0)-\mu_3\Lambda_1} \cdots \tau D_{w_p}(e^{\mu^p\Lambda_1})\big)\big)\\
&\quad = \cdots\\
&\quad = D_{w_1}\big(e^{\mu_1(\Lambda_1-\Lambda_0)} \cdot \tau D_{w_2}\big(e^{\mu_2(\Lambda_1-\Lambda_0)} \cdots \tau D_{w_p}(e^{\mu_p(\Lambda_1-\Lambda_0)})\big)\big)\\
&\quad = e^{-\delta n_\ell(\mu)} \zeta\Big(\pi_{w_1} x_1^{\mu_1} \Phi \pi_{w_2} x_1^{\mu_2} \cdots \Phi \pi_{w_p} x_1^{\mu_p}\Big).
\end{aligned}$$

The last equality follows from (7.3) and (7.4); in particular, the constant $n_\ell(\mu)$ appears since, as we pull ζ to the right through the operators, we pick up a factor $e^{-\frac{\delta}{\ell}\sum_{i=1}^{p}(i-1)\mu_i}$ for converting Φ's to τ's and a factor $e^{\delta|\mu|\frac{\ell-1}{2\ell}}$ for converting multiplication by x_1 to multiplication by $e^{\Lambda_1-\Lambda_0}$ since $\zeta(x_1)=e^{\varpi_1}=e^{\delta\frac{\ell-1}{2\ell}}e^{\Lambda_1-\Lambda_0}$ by (4.4). □

Corollary 7.6 *In the case $w_1 = \mathsf{w}_0$ (the longest element in $\mathcal{H}_\ell$), the characters in Theorem 7.5 have the following Schur positive expansion*:

$$\pi_{w_1}x_1^{\mu_1}\Phi\pi_{w_2}x_1^{\mu_2}\Phi\pi_{w_3}x_1^{\mu_3}\cdots\Phi\pi_{w_p}x_1^{\mu_p}=q^{-n_\ell(\mu)}\operatorname{char}_{\mathbf{x};\mu}(\mathrm{AGD}(\mu;\mathbf{w}))$$

$$=\sum_{b\in\mathcal{B}^{\mu;\mathbf{w}}}q^{\operatorname{charge}(\operatorname{inv}(b))}\mathbf{x}^{\operatorname{content}(b)}=\sum_{\substack{U\in\mathrm{SSYT}_\ell(\mu)\\ U\text{ is }(id,w_2,\dots,w_p)\text{-katabolizable}}}q^{\operatorname{charge}(U)}s_{\operatorname{shape}(U)}(\mathbf{x}).$$

Proof Combine Theorems 6.21 and 7.5, noting that each component $\mathcal{C}_U$ of the $U_q(\mathfrak{gl}_\ell)$-crystal $\mathcal{B}^{\mu;\mathbf{w}}$ contributes $q^{\operatorname{charge}(U)}$ times $\sum_{b\in\mathcal{C}_U}\mathbf{x}^{\operatorname{content}(b)}=\sum_{b\in B^{\mathfrak{gl}}(\operatorname{shape}(U))}\mathbf{x}^{\operatorname{wt}(b)}=s_{\operatorname{shape}(U)}(\mathbf{x})$ to the left side of (7.9); this last (well-known) equality follows from Proposition 4.8. □

Combining Corollary 7.6, Theorem 2.3, and Proposition 6.15 yields Theorem 2.18. This proves the katabolism conjecture of Shimozono-Weyman [82, Conjecture 27] upon verifying that our katabolism Definition 2.15 agrees with that of [82] in the parabolic case:

Proposition 7.7 *When Ψ is the parabolic root ideal $\Delta(\eta)$ for some composition η of ℓ (see (2.14)), a tableau T of partition content μ is $\mathbf{n}(\Psi)$-katabolizable if and only if it is $R(\eta,\mu)$-katabolizable in the sense of* [82, §3.7].

Proof Checking whether T is $\mathbf{n}(\Psi)$-katabolizable begins with the computation $U = P_{1,\ell}\circ\operatorname{kat}\cdots P_{\eta_1-1,\ell}\circ\operatorname{kat}\circ P_{\eta_1,\ell}\circ\operatorname{kat}(T)$. The key observation is that each row $T^1, T^2,\dots,T^{\eta_1}$ of T is never touched by the column insertions until it is rotated to become the new ℓ-th row. Hence the computation of U amounts to the following: check whether T^1 contains μ_1 1's, remove these 1's, then column insert the result into $T^{[\eta_1+1,\ell]}$ to obtain a new tableau V, then check whether T^2 contains μ_2 2's, remove these 2's, column insert the result into V, and so on. These checks are equivalent to checking whether T contains the superstandard tableau Z of shape $(\mu_1,\dots,\mu_{\eta_1})$. Thus, T is not rejected in this computation if and only if T contains Z, and if so, U is obtained by column inserting $T^{[\eta_1]}\setminus Z$ into $T^{[\eta_1+1,\ell]}$ one row at a time, which is the same as the rectification of the skew tableau formed by placing $T^{[\eta_1]}\setminus Z$ and $T^{[\eta_1+1,\ell]}$ catty-corner. This is exactly the first step in the katabolism algorithm of [82]. Continuing in this way with $\eta_2,\eta_3,\dots$ gives the result. □

7.4 Key positivity

We generalize the results above to key positive formulas for characters of AGD and DARK crystals and tame nonsymmetric Catalan functions. To do this, we address the algorithmic problem of obtaining explicit key expansions for characters of subsets which we know to be disjoint unions of $U_q(\mathfrak{gl}_\ell)$-Demazure crystals; some of this material, in particular Proposition 7.9, is similar in spirit to [5, §4].

Let B be a $U_q(\mathfrak{gl}_\ell)$-crystal. The weight function takes values in $\mathbb{Z}^\ell$ and we write $\mathrm{wt}(b) = (\mathrm{wt}_1(b), \dots, \mathrm{wt}_\ell(b))$ for the entries of $\mathrm{wt}(b)$. The *crystal reflection operators* $S_i : B \to B$, $i \in [\ell - 1]$, are given by

$$S_i(b) = \begin{cases} \tilde{f}_i^{\,\mathrm{wt}_i(b)-\mathrm{wt}_{i+1}(b)}(b) & \text{if } \mathrm{wt}_i(b) \geq \mathrm{wt}_{i+1}(b), \\ \tilde{e}_i^{\,-\mathrm{wt}_i(b)+\mathrm{wt}_{i+1}(b)}(b) & \text{if } \mathrm{wt}_i(b) \leq \mathrm{wt}_{i+1}(b). \end{cases}$$

Note that $s_i(\mathrm{wt}(b)) = \mathrm{wt}(S_i(b))$. The operators S_i were first studied by Lascoux and Schützenberger [57], and later generalized by Kashiwara [36]. They satisfy the braid relations and therefore generate an action of $\mathcal{S}_\ell$ on B. For $1 \leq i < j \leq \ell$, let $s_{ij} = s_i s_{i+1} \cdots s_{j-2} s_{j-1} s_{j-2} \cdots s_i \in \mathcal{S}_\ell$ denote the transposition swapping i and j, and $S_{ij} = S_i S_{i+1} \cdots S_{j-2} S_{j-1} S_{j-2} \cdots S_i$ the corresponding reflection operator.

We define *Bruhat order on* $\mathbb{Z}^\ell$ by $\alpha < \beta$ if and only if $\alpha^+ = \beta^+$ and $p(\alpha) < p(\beta)$ in Bruhat order on $\mathcal{S}_\ell$, where α^+ denotes the weakly decreasing rearrangement of α and $p(\alpha) \in \mathcal{S}_\ell$ the shortest element such that $p(\alpha)\alpha^+ = \alpha$.

Proposition 7.8 *The relation $\beta > \alpha$ is a covering relation in Bruhat order on $\mathbb{Z}^\ell$ if and only if there exist $1 \leq i < k \leq \ell$ such that $\alpha = s_{ik}\beta$ with $\alpha_i > \alpha_k$, and $\alpha_j \notin [\alpha_i, \alpha_k]$ for all $j \in [i+1, k-1]$.*

Proof For permutations α and β, this is a well-known combinatorial description of the Bruhat order covering relations of $\mathcal{S}_\ell$ (see, e.g., [7, Lemma 2.1.4]). The general case can be deduced from this one by a standardization argument and the fact that any covering relation in the Bruhat order poset restricted to minimal coset representatives is actually a covering relation in the full Bruhat poset (by, e.g., [7, Theorem 2.5.5]). □

For example, $\beta = 32812852 > 52812832 = \alpha$ is a covering relation and $\alpha = s_{17}\beta$.

The next proposition is motivated by the following algorithmic problem: suppose we have access to the elements of a $U_q(\mathfrak{gl}_\ell)$-Demazure crystal G and want to determine the $\gamma \in \mathbb{Z}^\ell$ for which $G = BD(\gamma)$ (see §4.8 for the definition of $BD(\gamma)$).

Proposition 7.9 *Let G be a $U_q(\mathfrak{gl}_\ell)$-Demazure crystal. There is a unique element $u_{lw} \in G$ such that, setting $\gamma = \mathrm{wt}(u_{lw})$, (1) γ^+ is the highest weight of G, and (2) $S_{ij}(u_{lw}) \notin G$ for all covering relations $\gamma < s_{ij}\gamma$ in Bruhat order on $\mathbb{Z}^\ell$. Moreover, $G = BD(\gamma)$.*

Proof Consider a highest weight $U_q(\mathfrak{gl}_\ell)$-crystal $B^{\mathfrak{gl}}(\nu)$. For each weight α in the orbit $\mathcal{S}_\ell \cdot \nu$, there is a unique element $u_\alpha \in B^{\mathfrak{gl}}(\nu)$ of weight α; it belongs to $BD(\beta)$, $\beta \in \mathcal{S}_\ell \cdot \nu$, if and only if $\alpha \leq \beta$ in Bruhat order on $\mathbb{Z}^\ell$. It follows that if $G = BD(\tilde{\gamma})$, then $u_{\tilde{\gamma}} \in B^{\mathfrak{gl}}(\tilde{\gamma}^+)$ is the unique element $u_{lw} \in G$ satisfying (1) and (2), and $G = BD(\mathrm{wt}(u_{lw}))$. □

Definition 7.10 A tabloid T is *row-frank* if $\mathrm{shape}(T)$ is a rearrangement of $\mathrm{shape}(P(T))$. Let $\mathrm{RowFrank}_\ell(\mu) = \{T \in \mathrm{Tabloids}_\ell(\mu) \mid T \text{ is row-frank}\}$. This is also the set of inverses of the extremal weight elements of the crystal $\mathcal{B}^\mu$.

Remark 7.11 Row-frank tabloids are essentially in bijection with row-frank words in the sense of [58]: a word w is defined to be row-frank in [58] if $\mathrm{shape}(P(w))$ is a rearrangement of $(|w^1|, \dots, |w^k|)$, where $w = w^1 \cdots w^k$ is the factorization of w into weakly increasing contiguous subwords of maximal length. The map from row-frank tabloids with ℓ rows to words given by $T \mapsto T^\ell \cdots T^1$ has image the row-frank words having a factorization $w = w^1 \cdots w^k$ as above with $k \le \ell$, and the fiber of such a w consists of all tabloids with rows $w^1, \dots, w^k$ appearing in that order from bottom to top and $\ell - k$ empty rows interspersed arbitrarily between them.

For a tabloid $T \in \mathrm{Tabloids}_\ell(\mu)$ and $i \in [\ell - 1]$, define $S'_i := \mathrm{inv} \circ S_i \circ \mathrm{inv}(T)$ and $S'_{ij} = \mathrm{inv} \circ S_{ij} \circ \mathrm{inv}(T)$. Since $\mathrm{shape}(S'_i(T)) = s_i(\mathrm{shape}(T))$ for any $T \in \mathrm{RowFrank}_\ell(\mu)$, the S'_i and S'_{ij} preserve the set $\mathrm{RowFrank}_\ell(\mu)$. This also gives a simple description of $S'_{ij}(T)$ for $T \in \mathrm{RowFrank}_\ell(\mu)$: $S'_{ij}(T)$ is the unique row-frank tabloid Knuth equivalent to T with shape obtained from $\mathrm{shape}(T)$ by exchanging the i-th and j-th parts.

Definition 7.12 A tabloid $T \in \mathrm{RowFrank}_\ell(\mu)$ is *extreme* $\mathbf{w}$*-katabolizable* if T is $\mathbf{w}$-katabolizable and $S'_{ij}(T)$ is not $\mathbf{w}$-katabolizable for all $i < j$ such that $\mathrm{shape}(T) < s_{ij}(\mathrm{shape}(T))$ is a covering relation in Bruhat order on $\mathbb{Z}^\ell$.

Theorem 7.13 *The DARK crystal $\mathcal{B}^{\mu;\mathbf{w}}$ is isomorphic to a disjoint union of $U_q(\mathfrak{gl}_\ell)$-Demazure crystals, with decomposition given by*

$$\mathcal{B}^{\mu;\mathbf{w}} = \bigsqcup_{\substack{T \in \mathrm{RowFrank}_\ell(\mu) \\ T \text{ is extreme } \mathbf{w}\text{-katabolizable}}} \tilde{\mathcal{C}}_T,$$

$$\text{where } \tilde{\mathcal{C}}_T = \{b \in \mathcal{B}^{\mu;\mathbf{w}} \mid Q(b) = P(T)\} \cong BD(\mathrm{shape}(T)).$$

Proof By Corollary 2.7 and Theorem 4.1, the $U_q(\mathfrak{sl}_\ell)$-restriction of $\mathrm{AGD}(\mu; \mathbf{w})$ is isomorphic to a disjoint union of $U_q(\mathfrak{sl}_\ell)$-Demazure crystals. So the same is true of $\mathcal{B}^{\mu;\mathbf{w}} \otimes u_{\mu_1 \Lambda_0}$ (by Theorem 2.11) and therefore $\mathcal{B}^{\mu;\mathbf{w}}$ as well. Hence by Remark 4.2, $\mathcal{B}^{\mu;\mathbf{w}}$ is isomorphic to a disjoint union of $U_q(\mathfrak{gl}_\ell)$-Demazure crystals; this decomposition can be written as $\mathcal{B}^{\mu;\mathbf{w}} = \bigsqcup \mathcal{C}_U \cap \mathcal{B}^{\mu;\mathbf{w}}$, where $\mathcal{C}_U$ ranges over the $U_q(\mathfrak{gl}_\ell)$-components of $\mathcal{F}_{\mathsf{w}_0}\mathcal{B}^{\mu;\mathbf{w}}$ (see Theorem 6.21). Then by Proposition 7.9 and Theorem 6.20, each set $\mathrm{inv}(\mathcal{C}_U \cap \mathcal{B}^{\mu;\mathbf{w}})$ contains a unique $T \in \mathrm{RowFrank}_\ell(\mu)$ which is extreme $\mathbf{w}$-katabolizable, and $\mathcal{C}_U \cap \mathcal{B}^{\mu;\mathbf{w}} = \{b \in \mathcal{B}^{\mu;\mathbf{w}} \mid Q(b) = U\} = \tilde{\mathcal{C}}_T \cong BD(\mathrm{shape}(T))$. □

Corollary 7.14 *The characters in Theorem 7.5 are key positive with key expansion*

$$\pi_{w_1} x_1^{\mu_1} \Phi \pi_{w_2} x_1^{\mu_2} \Phi \pi_{w_3} x_1^{\mu_3} \cdots \Phi \pi_{w_p} x_1^{\mu_p} = q^{-n_\ell(\mu)} \operatorname{char}_{\mathbf{x};\mu}(\mathrm{AGD}(\mu; \mathbf{w}))$$

$$= \sum_{b \in \mathcal{B}^{\mu;\mathbf{w}}} q^{\mathrm{charge}(\mathrm{inv}(b))} \mathbf{x}^{\mathrm{content}(b)} = \sum_{\substack{T \in \mathrm{RowFrank}_\ell(\mu) \\ T \text{ is extreme } \mathbf{w}\text{-katabolizable}}} q^{\mathrm{charge}(T)} \kappa_{\mathrm{shape}(T)}(\mathbf{x}).$$

Proof Combine Theorems 7.13 and 7.5; each $U_q(\mathfrak{gl}_\ell)$-Demazure crystal $\tilde{\mathcal{C}}_T$ contributes $\sum_{b \in \tilde{\mathcal{C}}_T} q^{\mathrm{charge}(T)} \mathbf{x}^{\mathrm{content}(b)} = \sum_{b \in BD(\mathrm{shape}(T))} q^{\mathrm{charge}(T)} \mathbf{x}^{\mathrm{wt}(b)} = q^{\mathrm{charge}(T)} \kappa_{\mathrm{shape}(T)}(\mathbf{x})$ to the left side of (7.9), where we have used Proposition 4.8 for the second equality. □

Combining Corollary 7.14 and Theorem 2.3 yields a positive combinatorial formula for the key expansions of tame nonsymmetric Catalan functions, generalizing Theorem 2.18:

Corollary 7.15 *Let* (Ψ, μ, w) *be a tame labeled root ideal of length* ℓ *with partition* μ. *Set* $\mathbf{w} = (w, \mathbf{s}(\Psi)) \in (\mathcal{H}_\ell)^\ell$ *with* $\mathbf{s}(\Psi)$ *as in* (2.9). *The associated nonsymmetric Catalan function has the key positive expansion*

$$H(\Psi; \mu; w)(\mathbf{x}; q) = \sum_{\substack{T \in \mathrm{RowFrank}_\ell(\mu) \\ T \text{ is extreme } \mathbf{w}\text{-katabolizable}}} q^{\mathrm{charge}(T)} \kappa_{\mathrm{shape}(T)}(\mathbf{x}) \tag{7.11}$$

and is the character of a AGD crystal and DARK crystal:

$$H(\Psi; \mu; w)(\mathbf{x}; q) = q^{-n_\ell(\mu)} \operatorname{char}_{\mathbf{x};\mu}(\mathrm{AGD}(\mu; \mathbf{w}))$$

$$= \sum_{b \in \mathcal{B}^{\mu;\mathbf{w}}} q^{\mathrm{charge}(\mathrm{inv}(b))} \mathbf{x}^{\mathrm{content}(b)}. \tag{7.12}$$

See the last three lines of Fig. 1 (§2.7). The bold tabloids in Fig. 2 are the extreme $\mathbf{v}$-katabolizable tabloids, for $\mathbf{v} = (\mathsf{s}_1\mathsf{s}_2\mathsf{s}_1, \mathsf{s}_2\mathsf{s}_1, \mathsf{s}_2\mathsf{s}_1)$ (left), $\mathbf{v} = (\mathsf{s}_2, \mathsf{s}_2\mathsf{s}_1, \mathsf{s}_2\mathsf{s}_1)$ (right); reading off their shapes and charges yields the key expansions in the fourth and fifth columns of Fig. 1.

Example 7.16 Let $\ell = 5$, $\mu = 22211$, and Ψ be the root ideal defined by $\mathbf{n}(\Psi) = (2, 2, 2, 2)$. Let $w = \mathsf{s}_3\mathsf{s}_4\mathsf{s}_3$. Then $\mathbf{w} = (w, \mathbf{s}(\Psi)) = (\mathsf{s}_3\mathsf{s}_4\mathsf{s}_3, \mathsf{s}_4\mathsf{s}_3\mathsf{s}_2, \mathsf{s}_4\mathsf{s}_3\mathsf{s}_2, \mathsf{s}_4\mathsf{s}_3\mathsf{s}_2, \mathsf{s}_4\mathsf{s}_3\mathsf{s}_2)$. Figure 5 (right) depicts the set of T in $\mathrm{RowFrank}_\ell(\mu)$ such that T is extreme $\mathbf{w}$-katabolizable. By (7.11), reading off their shapes and charges yields the key positive expansion

$$\begin{aligned} H(\Psi; \mu; w) = {} & \kappa_{22112} + q\kappa_{32111} + q\kappa_{22013} + q^2\kappa_{33011} + q^2\kappa_{32012} + q^2\kappa_{23003} \\ & + q^2\kappa_{42011} + q^3\kappa_{42011} + q^3\kappa_{43001} + q^3\kappa_{42002} + q^3\kappa_{33002} + q^4\kappa_{43001} \\ & + q^4\kappa_{52001} + q^5\kappa_{53000}. \end{aligned}$$

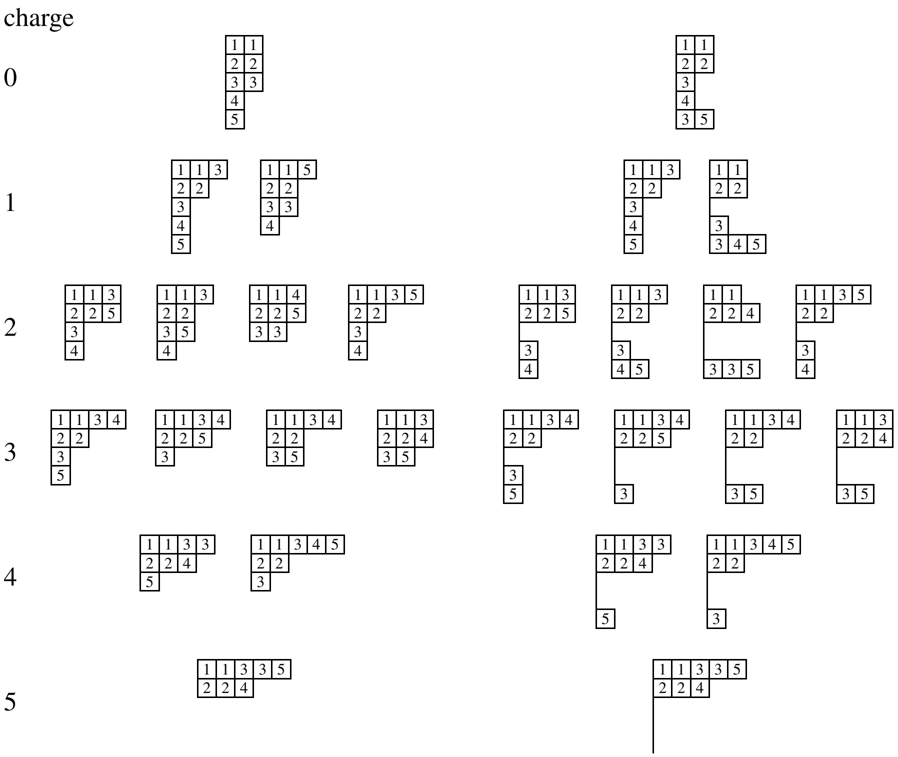
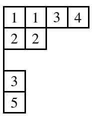
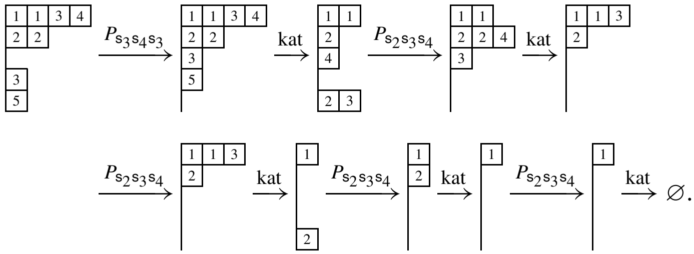


Fig. 5 The tabloids in $\mathrm{RowFrank}_5(22211)$ which are extreme $\mathbf{w}$-katabolizable (right) and their insertion tableaux (left), as explained in Example 7.16

On the left of Fig. 5 are the inverses of the $U_q(\mathfrak{gl}_\ell)$-highest weight elements of $\mathcal{B}^{\mu;\mathbf{w}}$ obtained by computing $P(T)$ of the tabloids on the right. This is also the set of $U \in \mathrm{SSYT}_\ell(\mu)$ which are $(id, \mathbf{s}(\Psi))$-katabolizable ($= \mathbf{n}(\Psi)$-katabolizable), providing an example of Theorem 2.18 and Corollary 7.6 as well: reading off their shapes and charges yields the following Schur positive expression for $H(\Psi;\mu;\mathsf{w}_0) = \sum_{b\in\mathcal{B}^{\mu;(\mathsf{w}_0,\mathbf{s}(\Psi))}} q^{\mathrm{charge}(\mathrm{inv}(b))}\mathbf{x}^{\mathrm{content}(b)}$.

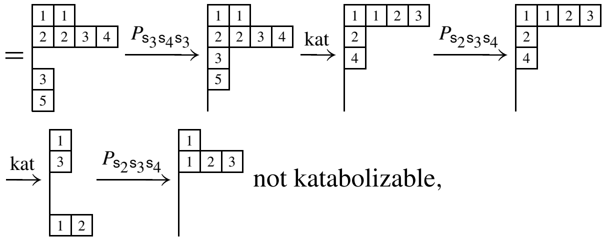
$$
\begin{aligned}
H(\Psi;\mu;\mathsf{w}_0) = s_{22211} &+ q s_{32111} + q s_{3122} + q^2 s_{3311} + q^2 s_{3221} + q^2 s_{332} \\
&+ q^2 s_{4211} + q^3 s_{4211} + q^3 s_{431} + q^3 s_{422} + q^3 s_{332} + q^4 s_{431} \\
&+ q^4 s_{521} + q^5 s_{53}.
\end{aligned}
$$

Let us check that the tabloid $T =$  is extreme $\mathbf{w}$-katabolizable. First, the following computation shows it is $\mathbf{w}$-katabolizable:



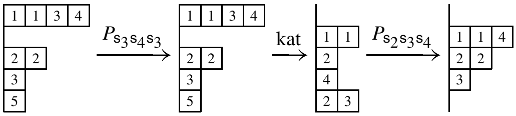
We must also show that $S'_{ij}(T)$ is not $\mathbf{w}$-katabolizable for all covering relations shape$(T) < s_{ij}$(shape(T)). We have shape$(T) = 42011$, and there are three covering relations corresponding to $(i, j) = (1, 2)$, $(2, 3)$, and $(2, 4)$.

$S'_{12}(T) =$



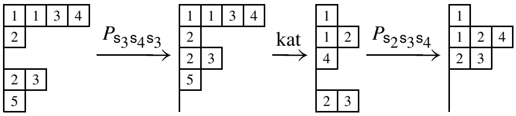
not katabolizable,

$S'_{23}(T) =$



not katabolizable,

$S'_{24}(T) =$



not katabolizable.

8 Consequences for $t = 0$ nonsymmetric Macdonald polynomials

We show that the $t = 0$ specialized nonsymmetric Macdonald polynomials are characters of AGD crystals and equal to certain nonsymmetric Catalan functions. We thus obtain a key positive formula for these polynomials as a special case of Corollary 7.14.

The Knop-Sahi recurrence [41, 76] determines the nonsymmetric Macdonald polynomials $E_\alpha(\mathbf{x}; q, t) = E_\alpha(x_1, \ldots, x_\ell; q, t)$ for all weak compositions $\alpha \in \mathbb{Z}^\ell_{\geq 0}$. At $t = 0$, the recurrence becomes

$$E_{(0,\ldots,0)}(\mathbf{x}; q, 0) = 1, \tag{8.1}$$

$$E_{\mathsf{s}_i\alpha}(\mathbf{x}; q, 0) = \pi_i(E_\alpha(\mathbf{x}; q, 0)), \tag{8.2}$$

$$E_{(\alpha_\ell+1,\alpha_1,\ldots,\alpha_{\ell-1})}(\mathbf{x}; q, 0) = q^{\alpha_\ell} x_1 E_\alpha(x_2, \ldots, x_\ell, x_1/q; q, 0), \tag{8.3}$$

which determines the specializations $E_\alpha(\mathbf{x}; q, 0)$. We have adopted the notation of [27, Equations (40)–(42)], except that in (8.2) we have used the action of $\mathcal{H}_\ell$ on $\mathbb{Z}^\ell$

from (4.11) to put what are often two equations into one. For this paper, it is more convenient to work with a renormalization of the $E_\alpha(\mathbf{x}; q, 0)$, denoted $\tilde{E}_\alpha = \tilde{E}_\alpha(\mathbf{x}; q)$ and defined by

$$\tilde{E}_{(0,\ldots,0)} = 1, \tag{8.4}$$

$$\tilde{E}_{\mathsf{s}_i\alpha} = \pi_i(\tilde{E}_\alpha), \tag{8.5}$$

$$\tilde{E}_{(\alpha_\ell+1,\alpha_1,\ldots,\alpha_{\ell-1})} = x_1\tilde{E}_\alpha(x_2,\ldots,x_\ell,qx_1) = x_1\Phi(\tilde{E}_\alpha). \tag{8.6}$$

The two versions are related by $E_\alpha(\mathbf{x}; q, 0) = q^{\sum_i \binom{\alpha_i}{2}}\tilde{E}_\alpha(\mathbf{x}; q^{-1})$; note that the exponent of q here is also $n(\eta) = \sum_i (i-1)\eta_i$ for $\eta = (\alpha^+)'$ the conjugate partition of α^+ (recall that α^+ denotes the weakly decreasing rearrangement of α). In addition, our notation $E_\alpha(\mathbf{x}; q, t)$ agrees with that of [5, 25, 27], while the version used by Sanderson [77], call it E^S_α, is related by $E^S_\alpha(x_1,\ldots,x_\ell; q, t) = E_{(\alpha_\ell,\ldots,\alpha_1)}(x_\ell,\ldots,x_1; q, t)$.

We suggest that on a first reading of this section, the reader focus on the case $|\alpha| = \ell$, as it captures the main ideas but with fewer technical details.

8.1 Sanderson's theorem and key positivity

Recall from §4.5 that $\widetilde{\mathcal{S}}_\ell \subset GL(\mathfrak{h}^*)$. Set $y_1 = \tau s_{\ell-1}\cdots s_1$ and $y_i = s_{i-1}\cdots s_1 y_1 s_1\cdots s_{i-1}$ for $i = 2,\ldots,\ell$. These elements commute pairwise and satisfy only one additional relation $y_1\cdots y_\ell = id$; hence they generate a subgroup of translations T, with $T \xrightarrow{\cong} \mathbb{Z}^\ell/\mathbb{Z}(1,\ldots,1) \xrightarrow{\cong} \bigoplus_{i\in I}\mathbb{Z}\varpi_i$ via $y_i \mapsto \epsilon_i \mapsto \varpi_i - \varpi_{i-1}$. Write $\mathbf{y}^\lambda \in T$ for the element mapping to $\lambda \in \bigoplus_{i\in I}\mathbb{Z}\varpi_i$, so that $\mathbf{y}^{\varpi_i} = y_1\cdots y_i$. These satisfy $\mathbf{y}^\lambda\mathbf{y}^\mu = \mathbf{y}^{\lambda+\mu}$ and $w\mathbf{y}^\lambda w^{-1} = \mathbf{y}^{w(\lambda)}$ for $\lambda,\mu \in \bigoplus_{i\in I}\mathbb{Z}\varpi_i$ and $w \in \mathcal{S}_\ell$. Hence $\widetilde{\mathcal{S}}_\ell = \mathcal{S}_\ell \ltimes T$. One can check that $\mathbf{y}^\lambda \in GL(\mathfrak{h}^*)$ is the same as the t_λ defined in [33, Equation 6.5.2], which is one way to verify the above facts about the y_i and T and matches our notation with [33, 71].

Lemma 8.1 *View a translation $\mathbf{y}^\lambda \in \widetilde{\mathcal{S}}_\ell$ as an element of the 0-Hecke monoid $\widetilde{\mathcal{H}}_\ell$ by taking any reduced expression for it. Let $d, d' \in [\ell]$. The following hold in $\widetilde{\mathcal{H}}_\ell$:*

(i) s_i *commutes with* $\tau\mathsf{s}(d)$ *for* $d < i \le \ell - 1$.
(ii) $(\tau\mathsf{s}(d))^d$ *is a reduced expression for* $\mathbf{y}^{\varpi_d}$ *in* $\widetilde{\mathcal{S}}_\ell$*, and thus* $\mathbf{y}^{\varpi_d} = (\tau\mathsf{s}(d))^d$ *in* $\widetilde{\mathcal{H}}_\ell$.
(iii) *For weights* $\lambda, \mu \in \sum_{i\in I}\mathbb{Z}_{\ge 0}\varpi_i$, $\mathbf{y}^\lambda\mathbf{y}^\mu = \mathbf{y}^{\lambda+\mu} = \mathbf{y}^\mu\mathbf{y}^\lambda$.
(iv) $(\tau\mathsf{s}(d))^d$ *commutes with* $(\tau\mathsf{s}(d'))^{d'}$.

Proof For (i), we compute using the relations (4.6)–(4.8):

$$\begin{aligned}\mathsf{s}_i(\tau\mathsf{s}_{\ell-1}\cdots\mathsf{s}_d) &= \tau\mathsf{s}_{i-1}\mathsf{s}_{\ell-1}\cdots\mathsf{s}_d = \tau\mathsf{s}_{\ell-1}\cdots\mathsf{s}_{i+1}\mathsf{s}_{i-1}\mathsf{s}_i\mathsf{s}_{i-1}\mathsf{s}_{i-2}\cdots\mathsf{s}_d\\ &= \tau\mathsf{s}_{\ell-1}\cdots\mathsf{s}_{i+1}\mathsf{s}_i\mathsf{s}_{i-1}\mathsf{s}_i\mathsf{s}_{i-2}\cdots\mathsf{s}_d = (\tau\mathsf{s}_{\ell-1}\cdots\mathsf{s}_d)\mathsf{s}_i.\end{aligned}$$

Statement (ii) can be proved using the description of $\widetilde{\mathcal{S}}_\ell$ as certain permutations of $\mathbb{Z}$; see, for instance, (15), (18), (19), and Proposition 4.1 of [8]. For $\lambda, \mu \in \sum_{i\in I}\mathbb{Z}_{\ge 0}\varpi_i$, $\text{length}(\mathbf{y}^\lambda) + \text{length}(\mathbf{y}^\mu) = \text{length}(\mathbf{y}^{\lambda+\mu})$ (see, e.g., [27, §4.1]), which gives (iii). Statement (iv) is immediate from (ii) and (iii). □

We will need the observation that affine generalized Demazure crystals $\mathrm{AGD}(\mu; \mathbf{w})$ for constant μ are just affine Demazure crystals.

Proposition 8.2 *Suppose* $\mu = (a^m, 0^{p-m})$ *for* $a \in \mathbb{Z}_{>0}$ *and* $\mathbf{w} = (w_1, w_2, \dots, w_p) \in (\mathcal{H}_\ell)^p$. *Then* $\mathrm{AGD}(\mu; \mathbf{w}) = \mathcal{F}_{w_1\tau w_2\cdots\tau w_m}\{u_{a\Lambda_1}\} = B_{w_1\tau w_2\cdots\tau w_m}(a\Lambda_1) \subset B(a\Lambda_m)$. *Further,* $B_{w_1\tau w_2\cdots\tau w_m}(a\Lambda_1) = B_{w_1\tau w_2\cdots\tau w_m\tau w}(a\Lambda_0)$ *for any* $w \in \mathcal{H}_\ell$.

Proof As $\mu^i = 0$ for $i \neq m$, the first statement is immediate from the definition of $\mathrm{AGD}(\mu; \mathbf{w})$ in (2.7). The second follows from the fact that $\tilde{f}_i u_{a\Lambda_0} = 0$ for $i \in [\ell\text{-}1]$. □

Recall from (7.2) the definition of the $\mathbf{x}$-character of a crystal. The next result is partially a restatement of Sanderson's theorem [77] (specifically, $\tilde{E}_\alpha = q^{\frac{p(p-\ell)}{2\ell}} \mathrm{char}_{\mathbf{x};\mu}(B_v(\Lambda_0))$). However, we now have the advantage of seeing it as part of the more general Theorem 7.5 and can make it combinatorially explicit in a way which encompasses earlier work of Lascoux [53] and Shimozono-Weyman [82] on cocharge Kostka-Foulkes polynomials.

Theorem 8.3 *The* $t = 0$ *nonsymmetric Macdonald polynomials are* $\mathbf{x}$*-characters of affine Demazure crystals: let* $\alpha \in \mathbb{Z}^\ell_{\geq 0}$ *and* $\eta = (\eta_1, \dots, \eta_k) = (\alpha^+)'$ *be the conjugate of* α^+. *Let* $\mathsf{z} \in \mathcal{H}_\ell$ *be any element satisfying* $\mathsf{z}\alpha^+ = \alpha$. *Set* $p = |\alpha|$ *and* $\mu = 1^p$. *Then*

$$\tilde{E}_\alpha(\mathbf{x}; q) = \pi_{\mathsf{z}}(x_1 \Phi \pi_{\mathsf{s}(\eta_k)})^{\eta_k} \cdots (x_1 \Phi \pi_{\mathsf{s}(\eta_1)})^{\eta_1} \cdot 1 \tag{8.7}$$

$$= q^{\frac{p(p-\ell)}{2\ell}} \mathrm{char}_{\mathbf{x};\mu}(B_v(\Lambda_0)) = \sum_{b \in \mathcal{B}^{\mu;\mathbf{w}}} q^{\mathrm{charge}(\mathrm{inv}(b))} \mathbf{x}^{\mathrm{content}(b)}, \tag{8.8}$$

where $v = \mathsf{z}(\tau\mathsf{s}(\eta_k))^{\eta_k}(\tau\mathsf{s}(\eta_{k-1}))^{\eta_{k-1}} \cdots (\tau\mathsf{s}(\eta_1))^{\eta_1} = \mathsf{z}\mathbf{y}^{\varpi_{\eta_1}+\cdots+\varpi_{\eta_k}} \in \widetilde{\mathcal{H}}_\ell$, *and*

$$\mathbf{w} = \big(\mathsf{z}, \underbrace{\mathsf{s}(\eta_k), \dots, \mathsf{s}(\eta_k)}_{\eta_k \text{ times}}, \dots, \underbrace{\mathsf{s}(\eta_2), \dots, \mathsf{s}(\eta_2)}_{\eta_2 \text{ times}}, \underbrace{\mathsf{s}(\eta_1), \dots, \mathsf{s}(\eta_1)}_{\eta_1 - 1 \text{ times}}\big).$$

Proof First, using (8.5), we obtain $\tilde{E}_\alpha = \pi_{\mathsf{z}} \tilde{E}_{\alpha^+}$. Let $\beta = \alpha^+ - \epsilon_{\eta_k}$ be the result of subtracting 1 from the rightmost occurrence of the largest part of α^+; by (8.5)–(8.6), $\tilde{E}_{\alpha^+} = x_1 \Phi \pi_{\mathsf{s}(d)}(\tilde{E}_\beta)$ for any d such that $\beta_d = (\alpha^+)_1 - 1$, which is equivalent to $\eta_k \leq d \leq \eta_{k-1}$. The same argument shows that $\tilde{E}_\beta = x_1 \Phi \pi_{\mathsf{s}(d')}(\tilde{E}_{\beta - \epsilon_{\eta_k - 1}})$ for any $\eta_k - 1 \leq d' \leq \eta_{k-1}$. Repeating this $\eta_k - 2$ more times, we obtain $\tilde{E}_{\alpha^+} = (x_1 \Phi \pi_{\mathsf{s}(\eta_k)})^{\eta_k}(\tilde{E}_{\alpha^+ - (\epsilon_1 + \cdots + \epsilon_{\eta_k})})$. Continuing in this way we obtain (8.7).

By Theorem 7.5, the right side of (8.7) is equal to $\sum_{b \in \mathcal{B}^{\mu;\mathbf{w}}} q^{\mathrm{charge}(\mathrm{inv}(b))} \times \mathbf{x}^{\mathrm{content}(b)} = q^{\frac{p(p-\ell)}{2\ell}} \mathrm{char}_{\mathbf{x};\mu}(\mathrm{AGD}(\mu; \mathbf{w}))$, and this is also equal to $q^{\frac{p(p-\ell)}{2\ell}} \mathrm{char}_{\mathbf{x};\mu}(B_v(\Lambda_0))$ by Proposition 8.2. Finally, the two descriptions of v are equal by Lemma 8.1 (ii)–(iii). □

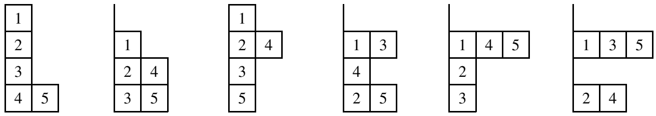
Combining (8.7) with Corollary 7.14 we obtain

Corollary 8.4 *Maintain the notation of Theorem 8.3. The $t=0$ nonsymmetric Macdonald polynomials are key positive with key expansion given by*

$$q^{\sum_i \binom{\alpha_i}{2}} E_\alpha(\mathbf{x}; q^{-1}, 0) = \tilde{E}_\alpha(\mathbf{x}; q)$$
$$= \sum_{\substack{T \in \mathrm{RowFrank}_\ell(\mu) \\ T \text{ is extreme } \mathbf{w}\text{-katabolizable}}} q^{\mathrm{charge}(T)} \kappa_{\mathrm{shape}(T)}(\mathbf{x}). \qquad (8.9)$$

Example 8.5 We illustrate Corollary 8.4 for $\ell = 4$, $\alpha = 0302$. We have $\alpha^+ = 3200$, $\eta = (\alpha^+)' = 221$, $\mu = 1^5$, and $\mathbf{w} = (\mathsf{s}_1\mathsf{s}_3\mathsf{s}_2, \mathsf{s}_3\mathsf{s}_2\mathsf{s}_1, \mathsf{s}_3\mathsf{s}_2, \mathsf{s}_3\mathsf{s}_2, \mathsf{s}_3\mathsf{s}_2)$.

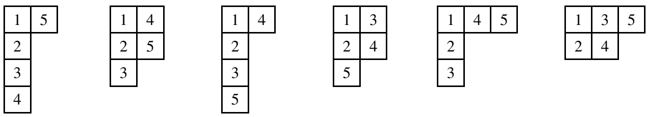
$T \in \mathrm{RowFrank}_\ell(\mu)$
T is extreme $\mathbf{w}$-katabolizable



$$\tilde{E}_{0302} = q\kappa_{1112} + q^2\kappa_{0122} + q^2\kappa_{1211} + q^3\kappa_{0212} + q^3\kappa_{0311} + q^4\kappa_{0302}$$

Here are the corresponding inverses of highest weight elements obtained by computing $P(T)$ of these tabloids. This is also the subset of tableaux in the Lascoux/Shimozono-Weyman formula for $\omega\tilde{H}_\eta$ with at most ℓ rows (see Theorems 8.7 and 8.15). Taking $\sum_U q^{\mathrm{charge}(U)} s_{\mathrm{shape}(U)}$ over these tableaux gives the symmetrization of $\tilde{E}_{0302}$.

$U \in \mathrm{SYT}_\ell^{|\alpha|}$
U is $\mathbf{n}(\Delta(\eta))$-katabolizable



$$\pi_{w_0}\tilde{E}_{0302} = \tilde{E}_{0023} = qs_{2111} + q^2 s_{221} + q^2 s_{2111} + q^3 s_{221} + q^3 s_{311} + q^4 s_{32}$$

Remark 8.6 Another key positive formula for the $t=0$ nonsymmetric Macdonald polynomials was given by Assaf and Gonzalez in [5, 6]. Their approach also uses crystals but their indexing combinatorial objects are rather different—compare Example 8.5 with [5, Fig. 31]. An interesting problem is to find an explicit bijection between the two objects.

Let us now explain how Corollary 8.4 is a nonsymmetric generalization of Lascoux's formula for the cocharge Kostka-Foulkes polynomials $\tilde{K}_{\lambda\mu}(q)$. For partition μ, let $\tilde{H}_\mu(x_1, x_2, \ldots; q) = \sum_\lambda \tilde{K}_{\lambda\mu}(q) s_\lambda$ be the *cocharge variant modified Hall-Littlewood polynomial*; it equals $q^{n(\mu)} Q'_\mu(x_1, x_2, \ldots; q^{-1})$ in the notation of [67, p. 234], specializes to the homogeneous symmetric function h_μ at $q=1$, and the coefficient of s_μ is $q^{n(\mu)}$. Lascoux [53] gave a formula for $\tilde{H}_\mu$ in terms of a function kattype from standard tableaux to partitions (see [82, §4.1]); this version of katabolism was shown [82, §4] to agree with a special case of the Shimozono-Weyman version despite its somewhat different looking definition. We assemble these results as follows:

Theorem 8.7 (Shimozono-Weyman [82, §4], Lascoux [53]) *Let η be a partition of ℓ. Recall that $\Delta(\eta)$ is the parabolic root ideal with blocks given by η (see (2.14)). Then*

$$\omega\tilde{H}_\eta = \sum_U q^{\mathrm{charge}(U)} s_{\mathrm{shape}(U)}(\mathbf{x}) = H(\Delta(\eta); 1^\ell; \mathsf{w}_0)(\mathbf{x}; q), \tag{8.10}$$

where the sum is over the set of standard tableaux U which are $R(\eta, 1^\ell)$-katabolizable in the sense of [82, *§*3.7], *and, moreover, this set equals* $\{U \mid \mathrm{kattype}(U^t) \trianglerighteq \eta\}$, *where U^t denotes the transpose of U, and $\trianglerighteq$ denotes dominance order on partitions. Here, $\mathbf{x} = (x_1, \ldots, x_\ell)$ and ω denotes the $\mathbb{Z}[q]$-algebra homomorphism from symmetric functions in $x_1, x_2, \ldots$ to $\mathbb{Z}[q][\mathbf{x}]^{\mathcal{S}_\ell}$ which satisfies $\omega(s_\lambda(x_1, x_2, \ldots)) = s_{\lambda'}(\mathbf{x})$.*

We have not defined either notion of katabolism appearing here, but by Proposition 7.7, $R(\eta, 1^\ell)$-katabolizability agrees with $\mathbf{n}(\Delta(\eta))$-katabolizability of standard tableaux.

For $\ell \geq |\eta|$, define $\Delta_\ell(\eta) \subset \Delta_\ell^+$ to be the root ideal $\Delta(\eta) \sqcup \{(i, j) \in \Delta_\ell^+ \mid j > |\eta|\}$. This is just a convenient way to extend $\Delta(\eta)$ to length ℓ—see Proposition 5.5 (v). Our key positive formula (8.9) "symmetrizes" to the Lascoux/Shimozono-Weyman formula (8.10) in the following sense:

Theorem 8.8 *Maintain the notation of Theorem* 8.3; *also set $m = p = |\alpha|$ and assume $m \leq \ell$. Let $\mathrm{SYT}^m = \mathrm{SSYT}_\ell(\mu)$, the standard Young tableaux with m boxes. Then*

$$\pi_{\mathsf{w}_0} \tilde{E}_\alpha(\mathbf{x}; q) = H(\Delta_\ell(\eta); \mu; \mathsf{w}_0)(\mathbf{x}; q) = \sum_{\substack{U \in \mathrm{SYT}^m \\ U \textit{ is } \mathbf{n}(\Delta_\ell(\eta))\textit{-katabolizable}}} q^{\mathrm{charge}(U)} s_{\mathrm{shape}(U)}(\mathbf{x}). \tag{8.11}$$

Moreover, $\mathcal{F}_{\mathsf{w}_0} \mathcal{B}^{\mu;\mathbf{w}} = \mathcal{B}^{\mu;(\mathsf{w}_0, \mathbf{s}(\Delta_\ell(\eta)))}$ as $U_q(\mathfrak{gl}_\ell)$-crystals, and so

$$\begin{aligned} &\{P(T) \mid T \in \mathrm{RowFrank}_\ell(\mu) \textit{ is extreme } \mathbf{w}\textit{-katabolizable}\} = \\ &\{U \in \mathrm{SYT}^m \mid U \textit{ is } \mathbf{n}(\Delta_\ell(\eta))\textit{-katabolizable}\} = \{U \in \mathrm{SYT}^m \mid \mathrm{kattype}(U^t) \trianglerighteq \eta\}. \end{aligned} \tag{8.12}$$

The proof is given in §8.3, along with a similar result for $m > \ell$.

8.2 Connection to nonsymmetric Catalan functions

For $\alpha \in \mathbb{Z}_{\geq 0}^\ell$, define $\tilde{\mathsf{p}}(\alpha) \in \mathcal{H}_\ell$ to be the longest element such that $\tilde{\mathsf{p}}(\alpha)\alpha^+ = \alpha$. This choice of z in Theorem 8.3 will be important below.

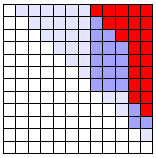
We now show that the $\tilde{E}_\alpha$ can be realized as certain tame nonsymmetric Catalan functions. Note that this is not immediate from Theorem 8.3 as typically $\eta_k + 1$ does not lie in the right descent set of $\tilde{\mathsf{p}}(\alpha)$ and so (8.7) does not match a tame nonsymmetric Catalan function via Theorem 2.3. However, there is a way to rewrite (8.7) which does the job. This adds to the list of interesting functions which are encompassed in the nonsymmetric Catalan functions, proves that the $\tilde{E}_\alpha$ are Euler characteristics of vector bundles on Schubert varieties (see Theorem 2.20), and is important in the proofs of Theorems 8.8 and 8.15.

Definition 8.9 For a partition $\eta = (\eta_1, \dots, \eta_k)$ of m, let $\Delta'(\eta) \subset \Delta_m^+$ be the root ideal determined by

$$\mathbf{n}(\Delta'(\eta)) = \big((\eta_1)^{\eta_2}, \eta_1, \eta_1 - 1, \eta_1 - 2, \dots, \eta_2 + 1, (\eta_2)^{\eta_3}, \eta_2, \eta_2 - 1, \eta_2 - 2, \dots,$$
$$\eta_3 + 1, \dots, (\eta_{k-1})^{\eta_k}, \eta_{k-1}, \eta_{k-1} - 1, \dots, \eta_k + 1, \eta_k, \eta_k - 1, \dots, 2\big),$$

where $(\eta_1)^{\eta_2}$ indicates that η_1 appears in the list η_2 times, and similarly for $(\eta_2)^{\eta_3}$, etc. Informally, $\Delta'(\eta)$ is obtained from the parabolic root ideal $\Delta(\eta)$ by removing trapezoids between consecutive blocks. For $\ell \geq m = |\eta|$, we also define $\Delta'_\ell(\eta) \subset \Delta_\ell^+$ to be the root ideal $\Delta'(\eta) \sqcup \{(i, j) \in \Delta_\ell^+ \mid j > |\eta|\}$.

Example 8.10 For $\eta = 732$, we have depicted $\Delta'(\eta)$ in red ■, $\Delta^+ \setminus \Delta(\eta)$ in light blue □, and $\Delta(\eta) \setminus \Delta'(\eta)$ in blue ■ (the two trapezoidal regions).

$\Delta'(\eta) =$ 

Theorem 8.11 *Let $\alpha \in \mathbb{Z}^\ell_{\geq 0}$ and set $\eta = (\alpha^+)'$. Set $m = |\alpha|$ and put $\mu = 1^m 0^{\ell - m}$ if $m \leq \ell$ and $\mu = 1^m$ otherwise. The $t = 0$ nonsymmetric Macdonald polynomials agree with certain nonsymmetric Catalan functions*:

$$\tilde{E}_\alpha(x_1, \dots, x_\ell; q)$$
$$= \begin{cases} H(\Delta'_\ell(\eta); \mu; \tilde{\mathsf{p}}(\alpha))(x_1, \dots, x_\ell; q) & \text{if } m \leq \ell, \\ H(\Delta'(\eta); \mu; \tilde{\mathsf{p}}(\alpha, 0^{m-\ell}))(x_1, \dots, x_m; q)|_{x_{\ell+1} = \cdots = x_m = 0} & \text{if } m > \ell. \end{cases} \tag{8.13}$$

Proof First assume $m \leq \ell$. The labeled root ideal $(\Delta'_\ell(\eta), \mu, \tilde{\mathsf{p}}(\alpha))$ is tame because the $\ell - \eta_1$ 0's in α^+ ensure that $\eta_1 + 1, \dots, \ell - 1$ is contained in the right descent set of $\tilde{\mathsf{p}}(\alpha)$. Hence by (7.12) and then Proposition 8.2,

$$H(\Delta'_\ell(\eta); \mu; \tilde{\mathsf{p}}(\alpha)) = q^{-n_\ell(\mu)} \operatorname{char}_{\mathbf{x};\mu}(\mathrm{AGD}(\mu; \mathbf{w})) = q^{-n_\ell(\mu)} \operatorname{char}_{\mathbf{x};\mu}(B_w(\Lambda_0)),$$

where $\mathbf{w} = (\tilde{\mathsf{p}}(\alpha), \mathsf{s}(\Delta'_\ell(\eta)))$, $w = \tilde{\mathsf{p}}(\alpha)\tau\mathsf{s}(n_1)\tau\mathsf{s}(n_2)\cdots\tau\mathsf{s}(n_{m-1})\tau\mathsf{s}(n_m) \in \widetilde{\mathcal{H}}_\ell$, $(n_1, \dots, n_{\ell-1}) = \mathbf{n}(\Delta'_\ell(\eta))$, and $n_m := 1$ (if $m < \ell$ this was already true otherwise we define it); the $\mathsf{s}(n_m)$ here (allowed by Proposition 8.2) makes w end in $\tau\mathsf{s}(\eta_k)\tau\mathsf{s}(\eta_k - 1)\cdots\tau\mathsf{s}(1)$, enabling the parts of η to be handled uniformly below. Thus by Theorem 8.3, to prove the top case of (8.13) it suffices to show $B_w(\Lambda_0) = B_v(\Lambda_0)$, where $v = \tilde{\mathsf{p}}(\alpha)(\tau\mathsf{s}(\eta_k))^{\eta_k}(\tau\mathsf{s}(\eta_{k-1}))^{\eta_{k-1}}\cdots(\tau\mathsf{s}(\eta_1))^{\eta_1}$ as in Theorem 8.3 with $\mathsf{z} = \tilde{\mathsf{p}}(\alpha)$. Since $\mathcal{F}_{\mathsf{w}_0}\{u_{\Lambda_0}\} = \{u_{\Lambda_0}\}$, it is enough to show $w\mathsf{w}_0 = v\mathsf{w}_0$ in the 0-Hecke monoid $\widetilde{\mathcal{H}}_\ell$.

For an interval $[i, j] \subset [m]$, set $w^{[i,j]} = \tau\mathsf{s}(n_i)\tau\mathsf{s}(n_{i+1})\cdots\tau\mathsf{s}(n_j)$, so that $w = \tilde{\mathsf{p}}(\alpha)w^{[m]}$. By definition of $\Delta'_\ell(\eta)$, $w^{[\eta_1]} = (\tau\mathsf{s}(\eta_1))^{\eta_2}\tau\mathsf{s}(\eta_1)\tau\mathsf{s}(\eta_1 - 1)\tau\mathsf{s}(\eta_1 -$

$2)\cdots\tau\mathsf{s}(\eta_2+1)$. By Lemma 8.1 (i), for $j\in[m]$ and $n_j<i\leq\ell-1$, $\mathsf{s}_i w^{[j,m]}\mathsf{w}_0 = w^{[j,m]}\mathsf{s}_i\mathsf{w}_0 = w^{[j,m]}\mathsf{w}_0$. Hence

$$\begin{aligned} w^{[m]}\mathsf{w}_0 &= (\tau\mathsf{s}(\eta_1))^{\eta_2+1}\tau\mathsf{s}(\eta_1-1)\tau\mathsf{s}(\eta_1-2)\cdots\tau\mathsf{s}(\eta_2+1)w^{[\eta_1+1,m]}\mathsf{w}_0 \\ &= (\tau\mathsf{s}(\eta_1))^{\eta_2+2}\tau\mathsf{s}(\eta_1-2)\cdots\tau\mathsf{s}(\eta_2+1)w^{[\eta_1+1,m]}\mathsf{w}_0 \\ &= \cdots = (\tau\mathsf{s}(\eta_1))^{\eta_1}w^{[\eta_1+1,m]}\mathsf{w}_0. \end{aligned}$$

In Example 8.12, this amounts to removing the triangle  as the first step in going from the left to middle diagram. Repeating this for the other parts of η we obtain

$$\begin{aligned} w\mathsf{w}_0 &= \tilde{\mathsf{p}}(\alpha)(\tau\mathsf{s}(\eta_1))^{\eta_1}\cdots(\tau\mathsf{s}(\eta_k))^{\eta_k}\mathsf{w}_0 = \tilde{\mathsf{p}}(\alpha)(\tau\mathsf{s}(\eta_k))^{\eta_k}\cdots(\tau\mathsf{s}(\eta_1))^{\eta_1}\mathsf{w}_0 \\ &= v\mathsf{w}_0, \end{aligned} \tag{8.14}$$

as desired. Here, we have used Lemma 8.1 (iv) for the second equality.

Now to handle the case $m>\ell$, we use the following stability property of the $t=0$ nonsymmetric Macdonald polynomials which is straightforward to verify from the Haglund-Haiman-Loehr formula [25, Theorem 3.5.1]: for any $\beta\in\mathbb{Z}^{\ell}_{\geq 0}$, $E_{(\beta,0)}(x_1,\ldots,x_{\ell+1};q,0)|_{x_{\ell+1}=0} = E_\beta(x_1,\ldots,x_\ell;q,0)$. Thus we also have $\tilde{E}_{(\beta,0)}(x_1,\ldots,x_{\ell+1};q)|_{x_{\ell+1}=0} = \tilde{E}_\beta(x_1,\ldots,x_\ell;q)$. Applying this to $\tilde{E}_{(\alpha,0^{m-\ell})}(x_1,\ldots,x_m;q) = H(\Delta'(\eta);1^m;\tilde{\mathsf{p}}(\alpha,0^{m-\ell}))(x_1,\ldots,x_m;q)$, which holds by the top case of (8.13), yields the bottom case of (8.13). □

Example 8.12 We assemble several expressions for $\tilde{E}_\alpha$ for $\alpha=3221110000$ ($\ell=10$). We have $\eta=(\alpha^+)'=631$ and $\Delta'(\eta)$ is indicated by the red squares ■ in the left diagram below. Let $\mu=1^\ell$ and note that $\tilde{\mathsf{p}}(\alpha)=\mathsf{s}_2\mathsf{s}_4\mathsf{s}_5\mathsf{s}_4\mathsf{s}_7\mathsf{s}_8\mathsf{s}_9\mathsf{s}_8\mathsf{s}_7\mathsf{s}_8$.

$$\tilde{E}_\alpha = x_1\Phi\pi_{\mathsf{s}(1)}(x_1\Phi\pi_{\mathsf{s}(3)})^3(x_1\Phi\pi_{\mathsf{s}(6)})^6\cdot 1 \tag{8.15}$$

$$= \mathrm{char}_{\mathbf{x};\mu}\big(B_{\tau\mathsf{s}(1)(\tau\mathsf{s}(3))^3(\tau\mathsf{s}(6))^6}(\Lambda_0)\big) \tag{8.16}$$

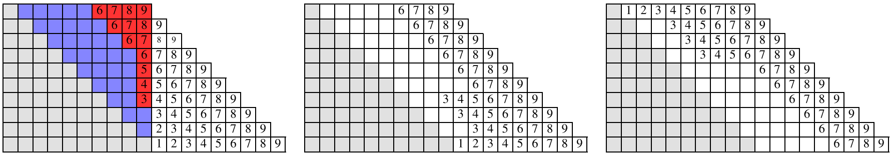
$$= \mathrm{char}_{\mathbf{x};\mu}\big(B_{(\tau\mathsf{s}(6))^6(\tau\mathsf{s}(3))^3(\tau\mathsf{s}(1))^1}(\Lambda_0)\big) \tag{8.17}$$

$$= \mathrm{char}_{\mathbf{x};\mu}\big(B_{(\tau\mathsf{s}(6))^4\tau\mathsf{s}(5)\tau\mathsf{s}(4)\tau\mathsf{s}(3)\tau\mathsf{s}(3)\tau\mathsf{s}(2)\tau\mathsf{s}(1)}(\Lambda_0)\big) \tag{8.18}$$

$$= H(\Delta'(\eta);\mu;\tilde{\mathsf{p}}(\alpha)) \tag{8.19}$$

$$= \pi_{\tilde{\mathsf{p}}(\alpha)}(x_1\Phi\pi_{\mathsf{s}(6)})^4 x_1\Phi\pi_{\mathsf{s}(5)}x_1\Phi\pi_{\mathsf{s}(4)}x_1\Phi\pi_{\mathsf{s}(3)}x_1\Phi\pi_{\mathsf{s}(3)}x_1\Phi\pi_{\mathsf{s}(2)}x_1\Phi\pi_{\mathsf{s}(1)}\cdot 1. \tag{8.20}$$

The formulas (8.15)–(8.19) come from (8.7), (8.8), (8.14), (8.14), and (8.13), respectively; the last equality holds by Theorem 2.3. The left diagram below gives a way of visualizing (8.18)–(8.20) (in the style of Example 6.18), the middle diagram corresponds to (8.17), and the right to (8.15)–(8.16).



8.3 Symmetrization to the Lascoux/Shimozono-Weyman formula

Proof of Theorem 8.8 By Theorem 8.11 and Definition 2.1, $\pi_{\mathsf{w}_0}\tilde{E}_\alpha = \pi_{\mathsf{w}_0}H(\Delta'_\ell(\eta); \mu; \tilde{\mathsf{p}}(\alpha)) = H(\Delta'_\ell(\eta); \mu; \mathsf{w}_0)$. Hence the first equality of (8.11) will follow from

$$H(\Delta'_\ell(\eta); \mu; \mathsf{w}_0) = H(\Delta_\ell(\eta); \mu; \mathsf{w}_0). \tag{8.21}$$

This identity can be seen by starting with $\Delta'_\ell(\eta)$ and filling in the trapezoidal regions of $\Delta_\ell(\eta) \setminus \Delta'_\ell(\eta)$ one root at a time, using [10, Lemma 8.9] to show that the corresponding Catalan functions remain the same (this is essentially the same argument used to prove [10, Lemma 10.1]). The second equality of (8.11) holds by Theorem 2.18.

Now to prove $\mathcal{F}_{\mathsf{w}_0}\mathcal{B}^{\mu;\mathbf{w}} = \mathcal{B}^{\mu;(\mathsf{w}_0,\mathbf{s}(\Delta_\ell(\eta)))}$, we first apply Proposition 8.2 to obtain $\mathrm{AGD}\big(\mu; (\mathsf{w}_0, \mathbf{s}(\Delta'_\ell(\eta)))\big) = B_w(\Lambda_0)$ and $\mathrm{AGD}\big(\mu; (\mathsf{w}_0, \mathbf{s}(\Delta_\ell(\eta)))\big) = B_y(\Lambda_0)$, where

$$w := \mathsf{w}_0\tau\, \mathbf{s}(\Delta'_\ell(\eta))_1 \cdots \tau\, \mathbf{s}(\Delta'_\ell(\eta))_{m-1}\tau,$$

$$y := \mathsf{w}_0\tau\, \mathbf{s}(\Delta_\ell(\eta))_1\tau\, \mathbf{s}(\Delta_\ell(\eta))_2 \cdots \tau\, \mathbf{s}(\Delta_\ell(\eta))_{m-1}\tau.$$

Considering the given expressions for w and y as words in the s_i's and τ, we see the expression for w is a subword of that of y, and hence $B_w(\Lambda_0) = \mathcal{F}_w\{u_{\Lambda_0}\} \subset \mathcal{F}_y\{u_{\Lambda_0}\} = B_y(\Lambda_0)$. But we also have $\mathrm{char}_{\mathbf{x};\mu}(B_w(\Lambda_0)) = \mathrm{char}_{\mathbf{x};\mu}(B_y(\Lambda_0))$ by Corollary 7.15 and (8.21). So equality $B_w(\Lambda_0) = B_y(\Lambda_0)$ holds.

Let v and $\mathbf{w}$ be as in Theorem 8.3 with $\mathsf{z} = \tilde{\mathsf{p}}(\alpha)$. Then $\mathsf{w}_0 v \mathsf{w}_0 = w\mathsf{w}_0$ by (8.14), and hence $\mathcal{F}_{\mathsf{w}_0}B_v(\Lambda_0) = B_w(\Lambda_0) = B_y(\Lambda_0)$ (by Proposition 4.3). By Theorem 2.11, the equality of AGD crystals $\mathcal{F}_{\mathsf{w}_0}B_v(\Lambda_0) = B_y(\Lambda_0)$ implies that the corresponding DARK crystals are equal (as subsets of $\mathcal{B}^\mu$): $\mathcal{F}_{\mathsf{w}_0}\mathcal{B}^{\mu;\mathbf{w}} = \mathcal{B}^{\mu;(\mathsf{w}_0,\mathbf{s}(\Delta_\ell(\eta)))}$. Equating the $U_q(\mathfrak{gl}_\ell)$-highest weights of these crystals then gives the first equality of (8.12) (with the help of Proposition 6.15); the connection to kattype follows from Proposition 7.7 and Theorem 8.7. $\square$

The companion result to Theorem 8.8 for $m > \ell$ is more technical and requires a crystal version of setting $x_{\ell+1} = \cdots = x_m = 0$, which we now describe.

Let B be a $U_q(\mathfrak{gl}_m)$-crystal which is a isomorphic to a disjoint union of highest weight crystals $B^{\mathfrak{gl}}(\nu)$ for $\nu = (\nu_1 \geq \cdots \geq \nu_m \geq 0)$. The weight function takes values in $\mathbb{Z}^m_{\geq 0}$ and we write $\mathrm{wt}(b) = (\mathrm{wt}_1(b), \dots, \mathrm{wt}_m(b))$ for the entries of $\mathrm{wt}(b)$. Let $S \subset B$

be isomorphic to a disjoint union of $U_q(\mathfrak{gl}_m)$-Demazure crystals. Let $\mathrm{Res}_J\, B$ denote the $U_q(\mathfrak{gl}_\ell)$-restriction of B corresponding to Dynkin node subset $J = [\ell-1] \subset [m-1]$ (see §4.2). By Theorem 4.1 $\mathrm{Res}_J\, S$ is isomorphic to a disjoint union of $U_q(\mathfrak{gl}_\ell)$-Demazure crystals. Define

$$\mathrm{R}_\ell^m S = \left\{b \in S \mid \mathrm{wt}_i(b) = 0 \text{ for all } i \in [\ell+1, m]\right\} \subset \mathrm{Res}_J\, S. \tag{8.22}$$

Since $\tilde{f}_j$, $j \in [\ell-1]$, fixes wt_i for $i > \ell$, $\mathrm{R}_\ell^m S$ is also a disjoint union of $U_q(\mathfrak{gl}_\ell)$-Demazure crystals and its character is obtained from that of S by setting $x_{\ell+1} = \cdots = x_m = 0$.

Below we work with $\mathcal{H}_m$ and its submonoid $\mathcal{H}_\ell$ generated by $\mathsf{s}_1, \ldots, \mathsf{s}_{\ell-1}$ ($\ell \leq m$); denote by $\iota\colon \mathcal{H}_\ell \hookrightarrow \mathcal{H}_m$ the inclusion and $\mathsf{w}_{[1,m)}$ and $\mathsf{w}_{[1,\ell)}$ their longest elements.

Lemma 8.13 *Let $m \geq \ell$ and $S \subset B$ as above. Then*

$$\mathrm{R}_\ell^m \mathcal{F}_{\mathsf{w}_{[1,m)}} S = \mathcal{F}_{\mathsf{w}_{[1,\ell)}} \mathrm{R}_\ell^m S. \tag{8.23}$$

Proof Let $\mathsf{s}_{i_1} \cdots \mathsf{s}_{i_k}$ be a reduced word for $\mathsf{w}_{[1,m)}$. For any $b \in B$, $\sum_{i=\ell+1}^m \mathrm{wt}_i(\tilde{f}_\ell(b)) > \sum_{i=\ell+1}^m \mathrm{wt}_i(b)$ and for $j \neq \ell$, $\sum_{i=\ell+1}^m \mathrm{wt}_i(\tilde{f}_j(b)) = \sum_{i=\ell+1}^m \mathrm{wt}_i(b)$; also, $\sum_{i=\ell+1}^m \mathrm{wt}_i(b) = 0$ implies $\tilde{f}_j(b) = 0$ for $j > \ell$. It follows that an arbitrary element $\tilde{f}_{i_1}^{a_{i_1}} \cdots \tilde{f}_{i_k}^{a_{i_k}}(b)$ of $\mathcal{F}_{\mathsf{w}_{[1,m)}} S$ lies in $\mathrm{R}_\ell^m \mathcal{F}_{\mathsf{w}_{[1,m)}} S$ if and only if $b \in \mathrm{R}_\ell^m S$ and $a_{i_j} = 0$ whenever $i_j \geq \ell$. Hence $\mathrm{R}_\ell^m \mathcal{F}_{\mathsf{w}_{[1,m)}} S = \mathcal{F}_v \mathrm{R}_\ell^m S$ where v is the product of the s_{i_j} with $i_j < \ell$. Since $\mathsf{s}_{i_1} \cdots \mathsf{s}_{i_k}$ contains a reduced word for $\mathsf{w}_{[1,\ell)}$, it follows from Remark 4.4 that $\mathcal{F}_v \mathrm{R}_\ell^m S = \mathcal{F}_{\mathsf{w}_{[1,\ell)}} \mathrm{R}_\ell^m S$. ☐

Lemma 8.14 *Given $\mathbf{n} = (n_1, \ldots, n_{p-1}) \in [\ell]^{p-1}$ and $\mathsf{z} \in \mathcal{H}_\ell$, define the tuples $\mathbf{w} = (\mathsf{z}, \mathsf{s}_{\ell-1} \cdots \mathsf{s}_{n_1}, \ldots, \mathsf{s}_{\ell-1} \cdots \mathsf{s}_{n_{p-1}}) \in (\mathcal{H}_\ell)^p$ and $\widetilde{\mathbf{w}} = (\iota(\mathsf{z}), \mathsf{s}_{m-1} \cdots \mathsf{s}_{n_1}, \ldots, \mathsf{s}_{m-1} \cdots \mathsf{s}_{n_{p-1}}) \in (\mathcal{H}_m)^p$. Then for any partition $\mu = (\mu_1, \ldots, \mu_p)$, $\mathrm{R}_\ell^m \mathcal{B}^{\mu;\widetilde{\mathbf{w}}} = \mathcal{B}^{\mu;\mathbf{w}}$.*

Proof By Theorem 6.20, this is equivalent to showing that $T \in \mathrm{Tabloids}_\ell$ is $\mathbf{w}$-katabolizable if and only if $\widetilde{T}$ is $\widetilde{\mathbf{w}}$-katabolizable, where $\widetilde{T}$ is the same as T but regarded as an element of $\mathrm{Tabloids}_m$. One checks easily by induction that these two katabolism computations are essentially identical, the only difference being that whenever kat is applied in the $\mathbf{w}$-katabolism algorithm, it matches the application of $P_{\mathsf{s}_\ell \cdots \mathsf{s}_{m-1}} \circ \mathrm{kat}$ in the $\widetilde{\mathbf{w}}$-katabolism algorithm; this holds because at every step (in either algorithm) just before kat is applied, the input tabloid is empty in rows $\ell+1, \ldots, m$. ☐

Theorem 8.15 *Maintain the notation of Theorem 8.3; also set $m = p = |\alpha|$ and assume $m > \ell$. Let $\mathrm{SYT}_\ell^m = \mathrm{SSYT}_\ell(\mu)$, the SYT with m boxes and at most ℓ rows. Then*

$$\begin{aligned} \pi_{\mathsf{w}_{[1,\ell)}} \tilde{E}_\alpha &= H(\Delta(\eta); \mu; \mathsf{w}_{[1,m)})|_{x_{\ell+1} = \cdots = x_m = 0} \\ &= \sum_{\substack{U \in \mathrm{SYT}_\ell^m \\ U \text{ is } \mathbf{n}(\Delta(\eta))\text{-katabolizable}}} q^{\mathrm{charge}(U)} s_{\mathrm{shape}(U)}(x_1, \ldots, x_\ell). \end{aligned} \tag{8.24}$$

Moreover, $\mathcal{F}_{\mathsf{w}_{[1,\ell)}}\mathcal{B}^{\mu;\mathbf{w}} = \mathrm{R}^m_\ell \mathcal{B}^{\mu;(\mathsf{w}_{[1,m)},\mathbf{s}(\Delta(\eta)))}$ *as* $U_q(\mathfrak{gl}_\ell)$*-crystals, and so*

$$\{P(T) \mid T \in \mathrm{RowFrank}_\ell(\mu) \text{ is extreme } \mathbf{w}\text{-katabolizable}\} = \tag{8.25}$$
$$\{U \in \mathrm{SYT}^m_\ell \mid U \text{ is } \mathbf{n}(\Delta(\eta))\text{-katabolizable}\} = \{U \in \mathrm{SYT}^m_\ell \mid \mathrm{kattype}(U^t) \trianglerighteq \eta\}.$$

Proof By Theorem 8.8, applied with m in place of ℓ and $(\alpha, 0^{m-\ell})$ in place of α, $\mathcal{F}_{\mathsf{w}_{[1,m)}}\mathcal{B}^{\mu;\widetilde{\mathbf{w}}} = \mathcal{B}^{\mu;(\mathsf{w}_{[1,m)},\mathbf{s}(\Delta(\eta)))}$ as $U_q(\mathfrak{gl}_m)$-crystals, where $\widetilde{\mathbf{w}} = (\iota(\mathsf{z}), \mathsf{s}_{m-1}\cdots\mathsf{s}_{n_1}, \ldots, \mathsf{s}_{m-1}\cdots\mathsf{s}_{n_{p-1}}) \in (\mathcal{H}_m)^p$ with $\mathbf{n} = \eta_k^{\eta_k}\cdots\eta_2^{\eta_2}\eta_1^{\eta_1-1}$. (Here z is one of the inputs to Theorem 8.15, which can be any element of $\mathcal{H}_\ell$ satisfying $\mathsf{z}\alpha^+ = \alpha$. In this application of Theorem 8.8 we must choose a $\tilde{\mathsf{z}} \in \mathcal{H}_m$ such that $\tilde{\mathsf{z}}(\alpha, 0^{m-\ell})^+ = (\alpha, 0^{m-\ell})$; we choose $\tilde{\mathsf{z}} = \iota(\mathsf{z})$.) Applying R^m_ℓ to both sides and then using Lemmas 8.13 and 8.14 yields the "moreover" statement:

$$\mathrm{R}^m_\ell \mathcal{B}^{\mu;(\mathsf{w}_{[1,m)},\mathbf{s}(\Delta(\eta)))} = \mathrm{R}^m_\ell \mathcal{F}_{\mathsf{w}_{[1,m)}}\mathcal{B}^{\mu;\widetilde{\mathbf{w}}} = \mathcal{F}_{\mathsf{w}_{[1,\ell)}}\mathrm{R}^m_\ell \mathcal{B}^{\mu;\widetilde{\mathbf{w}}} = \mathcal{F}_{\mathsf{w}_{[1,\ell)}}\mathcal{B}^{\mu;\mathbf{w}},$$

and the consequence (8.25) follows much like the proof of (8.12). Next, it follows from Theorem 8.3 that the charge weighted character of $\mathcal{F}_{\mathsf{w}_{[1,\ell)}}\mathcal{B}^{\mu;\mathbf{w}}$ is $\pi_{\mathsf{w}_{[1,\ell)}}\tilde{E}_\alpha$ and that of $\mathrm{R}^m_\ell\mathcal{F}_{\mathsf{w}_{[1,m)}}\mathcal{B}^{\mu;\widetilde{\mathbf{w}}}$ is $(\pi_{\mathsf{w}_{[1,m)}}\tilde{E}_{(\alpha,0^{m-\ell})})|_{x_{\ell+1}=\cdots=x_m=0}$. Hence $\big(\pi_{\mathsf{w}_{[1,m)}}\tilde{E}_{(\alpha,0^{m-\ell})}\big)|_{x_{\ell+1}=\cdots=x_m=0} = \pi_{\mathsf{w}_{[1,\ell)}}\tilde{E}_\alpha$. This fact given, (8.24) is obtained by applying Theorem 8.8 (specifically (8.11)), with $(\alpha, 0^{m-\ell})$ in place of α and then setting $x_{\ell+1} = \cdots = x_m = 0$. □

Index of notation

Root systems and Weyl groups

$\mathcal{H}_\ell$	0-Hecke monoid of $\mathcal{S}_\ell$ with generators s_i for $i \in [\ell-1]$	§2, §4.5
$\widetilde{\mathcal{H}}_\ell$	0-Hecke monoid of $\widetilde{\mathcal{S}}_\ell$, generators τ and s_i for $i \in \mathbb{Z}/\ell\mathbb{Z}$	§2.2, §4.5
w_0, w_0	longest element of $\mathcal{S}_\ell$ and $\mathcal{H}_\ell$, respectively	§2
$\mathsf{w}_{[i,j)}$	$\mathsf{w}_{[i,j)} \in \mathcal{H}_\ell$ corresp. to permutation reversing $\{i, \ldots, j\}$	§5.3
$\mathsf{w}_{\vec{a}}$	$\mathsf{w}_{[a,\ell)} \in \mathcal{H}_\ell$	§5.3
I, P	Dynkin nodes, weight lattice for general $\mathfrak{g}$	§4.1
I, P	Dynkin nodes $I = \mathbb{Z}/\ell\mathbb{Z}$, weight lattice for $\widehat{\mathfrak{sl}}_\ell$	§2.2, §4.3
$\mathfrak{h}$	Cartan subalgebra $\mathfrak{h} \subset \widehat{\mathfrak{sl}}_\ell$	§4.3
d	scaling element $d \in \mathfrak{h}$	§4.3
Λ_i	fundamental weights $\Lambda_i \in \mathfrak{h}^*$ $(i \in I)$ for $\widehat{\mathfrak{sl}}_\ell$	§4.3
δ	null root $\delta = \sum_{i\in I}\alpha_i \in \mathfrak{h}^*$	§4.3
cl	projection from $\mathfrak{h}^*$ to $\mathfrak{h}^*/\mathbb{C}\delta$	§4.3
aff	section of cl satisfying $\langle d, \mathrm{aff}(\lambda)\rangle = 0$	§4.3
ϖ_i	$\mathrm{aff}(\mathrm{cl}(\Lambda_i - \Lambda_0))$	§4.3
τ	Dynkin diagram automorphism, element of $\widetilde{\mathcal{H}}_\ell$ and $\widetilde{\mathcal{S}}_\ell$	§4.5

Crystals

$\mathcal{F}_i S$	$\{\tilde{f}_i^k b \mid b \in S, k \geq 0\} \setminus \{0\} \subset B$ for a subset S of a crystal B	§2.2, §4.7
$\mathcal{F}_\tau$	bijection $B(\Lambda) \to B(\tau(\Lambda))$, or $\mathcal{B}^\mu \to \mathcal{B}^\mu$	§2.2, §4.6, §6.6
$\mathrm{AGD}(\mu; \mathbf{w})$	affine generalized Demazure crystal	(2.7)
$B^{1,s}$	single row KR crystal	§6.1
$\mathcal{B}^\mu$	$B^{1,\mu_p} \otimes \cdots \otimes B^{1,\mu_1}$	§2.3, §6.2
$\mathcal{B}^{\mu;\mathbf{w}}$	DARK crystal	Def 2.10
$B^{\mathfrak{gl}}(\nu)$	highest weight $U_q(\mathfrak{gl}_\ell)$-crystal	§4.8
$BD(\alpha)$	$U_q(\mathfrak{gl}_\ell)$-Demazure crystal	§4.8
u_Λ	highest weight element of the crystal with highest weight Λ	§2.2
b_s	element of $B^{1,s}$ labeled by 1^s, with b_0 the empty word	§2.3
wt	weight function $B \to P$ of a $U_q(\mathfrak{g})$-seminormal crystal B	§4.1, (6.1)
$\tilde{e}_i^{\max}(b)$	the last element in the crystal string $b, \tilde{e}_i(b), \tilde{e}_i^2(b), \ldots$	§6.5
$\mathrm{char}_{\mathbf{x};\mu}(G)$	variant of the character of a crystal G	(7.2)

Words and tableaux

$\mathrm{Tabloids}_\ell(\mu)$	tabloids with ℓ rows and content μ	§2.4
T^i	i-th row of a tabloid T	§2.4
$T^{[i,j]}$	subtabloid of T consisting of rows $\{i, i+1, \ldots, j\}$	§2.4
biword	biwords with top word $p^{\mu_p} \cdots 2^{\mu_2} 1^{\mu_1}$ are labels of $\mathcal{B}^\mu$	§2.4, Def 6.1
inv	bijection from $\mathcal{B}^\mu$ to $\mathrm{Tabloids}_\ell(\mu)$	(2.12), §6.4
$P_{i,\ell}$, P_i	partial insertion, acting on tabloids	Defs 2.13, 6.8
$P(T)$	insertion tableau of $T^\ell \cdots T^1$, for a tabloid T	Thm 2.17
$Q(b)$	recording tableau of a biword b	§6.3
kat	operation on tabloids used in the definition of katabolism	Def 2.15, §6.6
katabolism	$\mathbf{n}$-katabolizable for $\mathbf{n} = (n_1, \ldots, n_{p-1}) \in [\ell]^{p-1}$	Def 2.15
	$\mathbf{w}$-katabolizable for $\mathbf{w} = (w_1, \ldots, w_p) \in (\mathcal{H}_\ell)^p$	Def 6.14
	extreme $\mathbf{w}$-katabolizable	Def 7.12
α^+	the weakly decreasing rearrangement of α	§3, §4.8
$\mathsf{p}(\alpha)$, $\tilde{\mathsf{p}}(\alpha)$	shortest and longest element $\mathsf{z} \in \mathcal{H}_\ell$ such that $\mathsf{z}\alpha^+ = \alpha$	§4.8, §8.2
charge	integer statistic on words and tabloids	Def 7.3
row-frank	tabloid T such that shape(T) rearranges to shape$(P(T))$	Def 7.10

Polynomials

$H(\Psi;\gamma;w)$	nonsymmetric Catalan function	Def 2.1
Φ	operator given by $\Phi(x_i)=x_{i+1}$ for $i\in[\ell-1]$, $\Phi(x_\ell)=qx_1$	(2.5), §5.3
π_i	Demazure operator $\pi_i(f)=\frac{x_i f-x_{i+1}s_i(f)}{x_i-x_{i+1}}$	(2.1), §4.8
π_w	$\pi_{i_1}\pi_{i_2}\cdots\pi_{i_m}$ for $w=\mathsf{s}_{i_1}\cdots\mathsf{s}_{i_m}\in\mathcal{H}_\ell$	§2, §4.8
$\hat{\pi}_i$	π_i-1	§5.2
κ_α	key polynomial or Demazure character, $\kappa_\alpha=\pi_{\mathsf{p}(\alpha)}\mathbf{x}^{\alpha^+}$	§3, Def 4.5
$\hat{\kappa}_\alpha$	Demazure atom, $\hat{\kappa}_\alpha=\hat{\pi}_{\mathsf{p}(\alpha)}\mathbf{x}^{\alpha^+}$	§3, §5.2
poly	polynomial truncation	Def 5.1

Root ideals

root ideal	upper order ideal of the poset $\Delta_\ell^+=\{(i,j)\mid 1\le i<j\le\ell\}$	§2
(Ψ,γ,w)	labeled root ideal, with $\Psi\subset\Delta_\ell^+$, $\gamma\in\mathbb{Z}^\ell$, $w\in\mathcal{H}_\ell$	§2
$\mathbf{n}(\Psi)$	$\mathbf{n}(\Psi)\in\mathbb{Z}^{\ell-1}$ with $\mathbf{n}(\Psi)_i:=\big|\{j\in\{i,\dots,\ell\}\mid(i,j)\notin\Psi\}\big|$	(2.3)
$\mathsf{s}(d)$	$\mathsf{s}_{\ell-1}\mathsf{s}_{\ell-2}\cdots\mathsf{s}_d\in\mathcal{H}_\ell$	Thm 2.3
$\mathbf{s}(\Psi)$	$(\mathsf{s}(\mathbf{n}(\Psi)_1),\dots,\mathsf{s}(\mathbf{n}(\Psi)_{\ell-1}))\in(\mathcal{H}_\ell)^{\ell-1}$	(2.9)
$\Delta(\eta)$	parabolic root ideal with block sizes η	(2.14)

Acknowledgements We are grateful to Peter Littelmann and Wilberd van der Kallen for the proof of Theorem 4.1. We thank Mark Haiman, George Seelinger, Mark Shimozono, and Weiqiang Wang for helpful discussions and Elaine So for help typing and typesetting figures.

Funding Authors were supported by NSF Grants DMS-1855784 (J. B.) and DMS-1855804 (J. M.).

References

1. Alexandersson, P.: Non-symmetric Macdonald polynomials and Demazure-Lusztig operators (2016). arXiv:1602.05153
2. Alexandersson, P., Sawhney, M.: A major-index preserving map on fillings. Electron. J. Comb. **24**(4), Paper 4.3, 30 (2017)
3. Andersen, H.H.: Schubert varieties and Demazure's character formula. Invent. Math. **79**(3), 611–618 (1985)
4. Assaf, S.: Nonsymmetric Macdonald polynomials and a refinement of Kostka-Foulkes polynomials. Trans. Am. Math. Soc. **370**(12), 8777–8796 (2018)

5. Assaf, S., Gonzalez, N.: Demazure crystals for specialized nonsymmetric Macdonald polynomials (2019). arXiv:1901.07520
6. Assaf, S., Gonzalez, N.: Affine Demazure crystals for specialized nonsymmetric Macdonald polynomials. With an appendix by Shrawan Kumar (2020). arXiv:2002.04141
7. Björner, A., Brenti, F.: Combinatorics of Coxeter Groups. Graduate Texts in Mathematics, vol. 231. Springer, New York (2005)
8. Blasiak, J.: Cyclage, catabolism, and the affine Hecke algebra. Adv. Math. **228**(4), 2292–2351 (2011)
9. Blasiak, J.: The DARK side of generalized Demazure crystals (2020). arXiv:2007.04888
10. Blasiak, J., Morse, J., Pun, A., Summers, D.: k-Schur expansions of Catalan functions. Adv. Math. (2018, to appear). arXiv:1811.02490
11. Blasiak, J., Morse, J., Pun, A., Summers, D.: Catalan functions and k-Schur positivity. J. Am. Math. Soc. **32**(4), 921–963 (2019)
12. Broer, B.: Line bundles on the cotangent bundle of the flag variety. Invent. Math. **113**(1), 1–20 (1993)
13. Broer, B.: Normality of some nilpotent varieties and cohomology of line bundles on the cotangent bundle of the flag variety. In: Lie Theory and Geometry. Progr. Math., vol. 123, pp. 1–19. Birkhäuser, Boston (1994)
14. Brylinski, R.K.: Limits of weight spaces, Lusztig's q-analogs, and fiberings of adjoint orbits. J. Am. Math. Soc. **2**(3), 517–533 (1989)
15. Carsten Jantzen, J.: Representations of Algebraic Groups, 2nd edn. Mathematical Surveys and Monographs, vol. 107. Am. Math. Soc., Providence (2003)
16. Chen, L.-C.: Skew-linked partitions and a representation-theoretic model for k-Schur functions. PhD thesis, UC Berkeley (2010)
17. Cherednik, I.: Nonsymmetric Macdonald polynomials. Int. Math. Res. Not. **10**, 483–515 (1995)
18. Demazure, M.: Désingularisation des variétés de Schubert généralisées. Ann. Sci. Éc. Norm. Supér. (4) **7**, 53–88 (1974)
19. Demazure, M.: Une nouvelle formule des caractères. Bull. Sci. Math. (2) **98**(3), 163–172 (1974)
20. Feigin, B., Loktev, S.: On generalized Kostka polynomials and the quantum Verlinde rule. In: Differential Topology, Infinite-Dimensional Lie Algebras, and Applications. Amer. Math. Soc. Transl. Ser. 2, vol. 194, pp. 61–79. Am. Math. Soc., Providence (1999)
21. Feigin, B., Jimbo, M., Kedem, R., Loktev, S., Miwa, T.: Spaces of coinvariants and fusion product. I. From equivalence theorem to Kostka polynomials. Duke Math. J. **125**(3), 549–588 (2004)
22. Fu, A.M., Lascoux, A.: Non-symmetric Cauchy kernels for the classical groups. J. Comb. Theory, Ser. A **116**(4), 903–917 (2009)
23. Fulton, W.: Young Tableaux. London Mathematical Society Student Texts, vol. 35. Cambridge University Press, Cambridge (1997). With applications to representation theory and geometry
24. Grojnowski, I., Haiman, M.: Affine Hecke algebras and positivity of LLT and Macdonald polynomials. Preprint (2007)
25. Haglund, J., Haiman, M., Loehr, N.: A combinatorial formula for nonsymmetric Macdonald polynomials. Am. J. Math. **130**(2), 359–383 (2008)
26. Hague, C.: Cohomology of flag varieties and the Brylinski-Kostant filtration. J. Algebra **321**(12), 3790–3815 (2009)
27. Haiman, M.: Cherednik algebras, Macdonald polynomials and combinatorics. In: International Congress of Mathematicians, vol. III, pp. 843–872. Eur. Math. Soc., Zürich (2006)
28. Hesselink, W.H.: Cohomology and the resolution of the nilpotent variety. Math. Ann. **223**(3), 249–252 (1976)
29. Hong, J., Kang, S.-J.: Introduction to Quantum Groups and Crystal Bases. Graduate Studies in Mathematics, vol. 42. Am. Math. Soc., Providence (2002)
30. Hotta, R., Springer, T.A.: A specialization theorem for certain Weyl group representations and an application to the Green polynomials of unitary groups. Invent. Math. **41**(2), 113–127 (1977)
31. Ion, B.: Nonsymmetric Macdonald polynomials and Demazure characters. Duke Math. J. **116**(2), 299–318 (2003)
32. Joseph, A.: A decomposition theorem for Demazure crystals. J. Algebra **265**(2), 562–578 (2003)
33. Kac, V.G.: Infinite-Dimensional Lie Algebras, 3rd edn. Cambridge University Press, Cambridge (1990)
34. Kashiwara, M.: On crystal bases of the Q-analogue of universal enveloping algebras. Duke Math. J. **63**(2), 465–516 (1991)
35. Kashiwara, M.: The crystal base and Littelmann's refined Demazure character formula. Duke Math. J. **71**(3), 839–858 (1993)

36. Kashiwara, M.: Crystal bases of modified quantized enveloping algebra. Duke Math. J. **73**(2), 383–413 (1994)
37. Kashiwara, M.: On crystal bases. In: Representations of Groups (Banff, AB, 1994). CMS Conf. Proc., vol. 16, pp. 155–197. Am. Math. Soc., Providence (1995)
38. Kashiwara, M., Nakashima, T.: Crystal graphs for representations of the q-analogue of classical Lie algebras. J. Algebra **165**(2), 295–345 (1994)
39. Kirillov, A.N., Shimozono, M.: A generalization of the Kostka-Foulkes polynomials. J. Algebraic Comb. **15**(1), 27–69 (2002)
40. Kirillov, A.N., Schilling, A., Shimozono, M.: A bijection between Littlewood-Richardson tableaux and rigged configurations. Sel. Math. New Ser. **8**(1), 67–135 (2002)
41. Knop, F.: Integrality of two variable Kostka functions. J. Reine Angew. Math. **482**, 177–189 (1997)
42. Kumar, S.: Kac-Moody Groups, Their Flag Varieties and Representation Theory. Progress in Mathematics, vol. 204. Birkhäuser Boston, Boston (2002)
43. Kumar, S., Lauritzen, N., Thomsen, J.F.: Frobenius splitting of cotangent bundles of flag varieties. Invent. Math. **136**(3), 603–621 (1999)
44. Kuniba, A., Misra, K.C., Okado, M., Takagi, T., Uchiyama Paths, J.: Demazure crystals, and symmetric functions. J. Math. Phys. **41**(9), 6477–6486 (2000)
45. Lakshmibai, V., Magyar, P.: Standard monomial theory for Bott-Samelson varieties of $\mathrm{GL}(n)$. Publ. Res. Inst. Math. Sci. **34**(3), 229–248 (1998)
46. Lakshmibai, V., Littelmann, P., Magyar, P.: Standard monomial theory for Bott-Samelson varieties. Compos. Math. **130**(3), 293–318 (2002)
47. Lam, T.: Schubert polynomials for the affine Grassmannian. J. Am. Math. Soc. **21**(1), 259–281 (2008)
48. Lam, T., Lapointe, L., Morse, J., Shimozono, M.: Affine insertion and Pieri rules for the affine Grassmannian. Mem. Am. Math. Soc. **208**(977), xii+82 (2010)
49. Lapointe, L., Morse, J.: Schur function analogs for a filtration of the symmetric function space. J. Comb. Theory, Ser. A **101**(2), 191–224 (2003)
50. Lapointe, L., Morse, J.: A k-tableau characterization of k-Schur functions. Adv. Math. **213**(1), 183–204 (2007)
51. Lapointe, L., Morse, J.: Quantum cohomology and the k-Schur basis. Trans. Am. Math. Soc. **360**(4), 2021–2040 (2008)
52. Lapointe, L., Lascoux, A., Morse, J.: Tableau atoms and a new Macdonald positivity conjecture. Duke Math. J. **116**(1), 103–146 (2003)
53. Lascoux, A.: Cyclic permutations on words, tableaux and harmonic polynomials. In: Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989), pp. 323–347. Manoj Prakashan, Madras (1991)
54. Lascoux, A.: Double crystal graphs. In: Studies in Memory of Issai Schur (Chevaleret/Rehovot, 2000). Progr. Math., vol. 210, pp. 95–114. Birkhäuser Boston, Boston (2003)
55. Lascoux, A.: Polynomials (2013). https://web.archive.org/web/20170312164301/http://phalanstere.univ-mlv.fr/~al//ARTICLES/CoursYGKM.pdf
56. Lascoux, A., Schützenberger, M.-P.: Sur une conjecture de H.O. Foulkes. C. R. Acad. Sci. Paris Sér. A–B **286**(7), A323–A324 (1978)
57. Lascoux, A., Schützenberger, M.-P.: Le monoïde plaxique. In: Noncommutative Structures in Algebra and Geometric Combinatorics (Naples, 1978). Quad. "Ricerca Sci.", vol. 109, pp. 129–156. CNR, Rome (1981)
58. Lascoux, A., Schützenberger, M.-P.: Keys & standard bases. In: Invariant Theory and Tableaux (Minneapolis, MN, 1988). IMA Vol. Math. Appl., vol. 19, pp. 125–144. Springer, New York (1990)
59. Lascoux, A., Leclerc, B., Thibon, J.-Y.: Ribbon tableaux, Hall-Littlewood functions, quantum affine algebras, and unipotent varieties. J. Math. Phys. **38**(2), 1041–1068 (1997)
60. Leclerc, B., Thibon, J.-Y.: Littlewood-Richardson coefficients and Kazhdan-Lusztig polynomials. In: Combinatorial Methods in Representation Theory (Kyoto, 1998). Adv. Stud. Pure Math., vol. 28, pp. 155–220. Kinokuniya, Tokyo (2000)
61. Lecouvey, C., Okado, M., Shimozono, M.: Affine crystals, one-dimensional sums and parabolic Lusztig q-analogues. Math. Z. **271**(3–4), 819–865 (2012)
62. Lenart, C., Naito, S., Sagaki, D., Schilling, A., Shimozono, M.: A uniform model for Kirillov-Reshetikhin crystals. In: 25th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2013). Discrete Math. Theor. Comput. Sci. Proc., AS, pp. 25–36. Assoc. Discrete Math. Theor. Comput. Sci., Nancy (2013)

63. Lenart, C., Naito, S., Sagaki, D., Schilling, A., Shimozono, M.: A uniform model for Kirillov-Reshetikhin crystals III: nonsymmetric Macdonald polynomials at $t = 0$ and Demazure characters. Transform. Groups **22**(4), 1041–1079 (2017)
64. Littelmann, P.: Crystal graphs and Young tableaux. J. Algebra **175**(1), 65–87 (1995)
65. Lusztig, G.: Green polynomials and singularities of unipotent classes. Adv. Math. **42**(2), 169–178 (1981)
66. Lusztig, G.: Singularities, character formulas, and a q-analog of weight multiplicities. In: Analysis and Topology on Singular Spaces, II, III (Luminy, 1981). Astérisque, vol. 101, pp. 208–229. Soc. Math. France, Paris (1983)
67. Macdonald, I.G.: Symmetric Functions and Hall Polynomials, 2nd edn. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York (1995)
68. Macdonald, I.G.: Affine Hecke algebras and orthogonal polynomials. Astérisque **237**, Exp. No. 797, 4, 189–207 (1996). Séminaire Bourbaki, vol. 1994/95
69. Mehta, V.B., van der Kallen, W.: A simultaneous Frobenius splitting for closures of conjugacy classes of nilpotent matrices. Compos. Math. **84**(2), 211–221 (1992)
70. Nakayashiki, A., Yamada, Y.: Kostka polynomials and energy functions in solvable lattice models. Sel. Math. New Ser. **3**(4), 547–599 (1997)
71. Naoi, K.: Demazure crystals and tensor products of perfect Kirillov-Reshetikhin crystals with various levels. J. Algebra **374**, 1–26 (2013)
72. Opdam, E.M.: Harmonic analysis for certain representations of graded Hecke algebras. Acta Math. **175**(1), 75–121 (1995)
73. Orr, D., Shimozono, M.: Specializations of nonsymmetric Macdonald-Koornwinder polynomials. J. Algebraic Comb. **47**(1), 91–127 (2018)
74. Panyushev, D.I.: Generalised Kostka-Foulkes polynomials and cohomology of line bundles on homogeneous vector bundles. Sel. Math. New Ser. **16**(2), 315–342 (2010)
75. Reiner, V., Shimozono, M.: Key polynomials and a flagged Littlewood-Richardson rule. J. Comb. Theory, Ser. A **70**(1), 107–143 (1995)
76. Sahi, S.: Interpolation, integrality, and a generalization of Macdonald's polynomials. Int. Math. Res. Not. **10**, 457–471 (1996)
77. Sanderson, Y.B.: On the connection between Macdonald polynomials and Demazure characters. J. Algebraic Comb. **11**(3), 269–275 (2000)
78. Schilling, A., Warnaar, S.O.: Inhomogeneous lattice paths, generalized Kostka polynomials and A_{n-1} supernomials. Commun. Math. Phys. **202**(2), 359–401 (1999)
79. Shimozono, M.: A cyclage poset structure for Littlewood-Richardson tableaux. Eur. J. Comb. **22**(3), 365–393 (2001)
80. Shimozono, M.: Affine type A crystal structure on tensor products of rectangles, Demazure characters, and nilpotent varieties. J. Algebraic Comb. **15**(2), 151–187 (2002)
81. Shimozono, M.: Crystals for dummies. Notes (2005)
82. Shimozono, M., Weyman, J.: Graded characters of modules supported in the closure of a nilpotent conjugacy class. Eur. J. Comb. **21**(2), 257–288 (2000)
83. Shimozono, M., Zabrocki, M.: Hall-Littlewood vertex operators and generalized Kostka polynomials. Adv. Math. **158**(1), 66–85 (2001)
84. Springer, T.A.: A construction of representations of Weyl groups. Invent. Math. **44**(3), 279–293 (1978)
85. van der Kallen, W.: Lectures on Frobenius Splittings and B-Modules. Published for the Tata Institute of Fundamental Research, Bombay. Springer, Berlin (1993). Notes by S.P. Inamdar