Inventiones mathematicae (2023) 235:1009–1018
https://doi.org/10.1007/s00222-023-01234-0

A phantom on a rational surface

Johannes Krah[1]

Received: 19 April 2023 / Accepted: 15 December 2023 / Published online: 21 December 2023

Abstract
We construct a non-full exceptional collection of maximal length consisting of line bundles on the blow-up of the projective plane in 10 general points. As a consequence, the orthogonal complement of this collection is a universal phantom category. This provides a counterexample to a conjecture of Kuznetsov and to a conjecture of Orlov.

Mathematics Subject Classification 14F08 · 14J26 · 14C20

1 Introduction

Let X be a smooth projective variety over the field of complex numbers and denote by $\mathsf{D}^b(X)$ the bounded derived category of coherent sheaves on X. A nontrivial admissible subcategory $\mathcal{A} \subseteq \mathsf{D}^b(X)$ is called a *phantom* if the Grothendieck group $\mathsf{K}_0(\mathcal{A})$ vanishes. The first examples of phantom categories were constructed by Gorchinskiy–Orlov [6] and Böhning–Graf von Bothmer–Katzarkov–Sosna [1]. It follows from a result of Efimov that a so-called *universal phantom* can be embedded into a proper dg-category admitting a full exceptional collection [5]; see Sect. 2 for the definition of a universal phantom. We provide a simple example of a variety which admits a full exceptional collection and a phantom subcategory.

Theorem 1.1 *Let X be the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in 10 general closed points $p_1, \ldots, p_{10} \in \mathbb{P}^2_{\mathbb{C}}$. Denote by H the divisor class obtained by pulling back the class of a hyperplane in $\mathbb{P}^2_{\mathbb{C}}$ and denote by E_i the class of the exceptional divisor over the point p_i, $1 \le i \le 10$. Then*

$$\langle \mathcal{O}_X, \mathcal{O}_X(D_1), \ldots, \mathcal{O}_X(D_{10}), \mathcal{O}_X(F), \mathcal{O}_X(2F) \rangle \subseteq \mathsf{D}^b(X), \tag{1.2}$$

$$\textit{where} \quad D_i := -6H + 2\sum_{j=1}^{10} E_j - E_i \quad \textit{and} \quad F := -19H + 6\sum_{i=1}^{10} E_i,$$

is an exceptional collection of maximal length which is not full.

J. Krah
jkrah@math.uni-bielefeld.de

1 Fakultät für Mathematik, Universität Bielefeld, D-33501 Bielefeld, Germany

It was previously shown in [12, Thm. 6.35] that a del Pezzo surface Y does not admit a phantom in $\mathsf{D}^b(Y)$. Moreover, we showed in earlier work [7] that on the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in 9 very general points every exceptional collection of maximal length consisting of line bundles is full. We discovered the exceptional collection (1.2) while trying to increase the number of blown up points in [7, Thm. 1.3].

Since any blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in a finite set of points admits a full exceptional collection, Theorem 1.1 disproves the following conjecture of Kuznetsov:

Conjecture 1.3 ([8, Conj. 1.10]) *Let $\mathcal{T} = \langle E_1, \dots, E_n \rangle$ be a triangulated category generated by an exceptional collection. Then any exceptional collection of length n in $\mathcal{T}$ is full.*

As a consequence of Theorem 1.1, the right- or left-orthogonal complement of the collection (1.2) is a phantom category. In general, if $\mathcal{A}$ is an admissible subcategory of $\mathsf{D}^b(X)$ and $\mathsf{D}^b(X)$ admits a full exceptional collection, then by [10, Cor. 3.4] $\mathcal{A}$ has a dg-enhancement quasi-equivalent to $\mathcal{P}erf-\mathcal{R}$, where $\mathcal{R}$ is a smooth finite-dimensional dg-algebra. Hence, Theorem 1.1 disproves the following conjecture of Orlov:

Conjecture 1.4 ([10, Conj. 3.7]) *There are no phantoms of the form $\mathcal{P}erf-\mathcal{R}$, where $\mathcal{R}$ is a smooth finite-dimensional dg-algebra and $\mathcal{P}erf-\mathcal{R}$ is the dg-category of perfect dg-modules over $\mathcal{R}$.*

Recently, Chang–Haiden–Schroll gave an example of a triangulated category admitting a full exceptional collection such that the braid group action by mutations does not act transitively on the set of full exceptional collections up to shifts [2]. Since mutations of exceptional collections do not change the generated subcategory, our example provides a surface where the braid group does not act transitively on the set of exceptional collections of maximal length.

Conventions We work over the complex numbers $\mathbb{C}$ since we rely on [3, Thm. 0.1]. Replacing the usage of [3, Thm. 0.1] by a computer aided computation, it is possible to deduce that the conclusion of Theorem 1.1 also holds over an algebraically closed field of characteristic zero.

The term "n general points in $\mathbb{P}^2_{\mathbb{C}}$" means that there exists a Zariski open subset $U \subseteq (\mathbb{P}^2_{\mathbb{C}})^n$ such that for any $(p_1, \dots, p_n) \in U$ [...] holds.

2 Exceptional collections

We recall the basic definitions and properties of exceptional collections and semiorthogonal decompositions. For a detailed reference we refer to [8] and the references therein.

Let X be a smooth projective variety over $\mathbb{C}$ and denote by $\mathsf{D}^b(X)$ the bounded derived category of coherent sheaves on X. A *semiorthogonal decomposition* of $\mathsf{D}^b(X)$ is an ordered collection $(\mathcal{A}_1, \dots, \mathcal{A}_n)$ of full triangulated subcategories such that

$$\mathrm{Hom}_{\mathsf{D}^b(X)}(A_i, A_j) = 0 \text{ for all } A_i \in \mathcal{A}_i, A_j \in \mathcal{A}_j, j < i$$

and the smallest triangulated subcategory of $\mathsf{D}^b(X)$ containing $\mathcal{A}_1, \dots, \mathcal{A}_n$ is $\mathsf{D}^b(X)$. We write

$$\mathsf{D}^b(X) = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$$

for such a semiorthogonal decomposition. A full triangulated subcategory $\mathcal{A} \subseteq \mathsf{D}^b(X)$ is called *admissible* if the inclusion functor $\mathcal{A} \hookrightarrow \mathsf{D}^b(X)$ admits both a right and a left adjoint. Such an admissible subcategory gives rise to the semiorthogonal decompositions $\mathsf{D}^b(X) = \langle \mathcal{A}^\perp, \mathcal{A} \rangle = \langle \mathcal{A}, {}^\perp\mathcal{A} \rangle$, where

$$^\perp\mathcal{A} := \{F \in \mathsf{D}^b(X) \mid \mathrm{Hom}_{\mathsf{D}^b(X)}(F, A) = 0 \text{ for all } A \in \mathcal{A}\}$$

$$\text{and } \mathcal{A}^\perp := \{F \in \mathsf{D}^b(X) \mid \mathrm{Hom}_{\mathsf{D}^b(X)}(A, F) = 0 \text{ for all } A \in \mathcal{A}\}$$

are the *left-* and *right-orthogonal complements* of $\mathcal{A}$. If $\mathcal{A}$ is admissible, so are ${}^\perp\mathcal{A}$ and $\mathcal{A}^\perp$. An object $E \in \mathsf{D}^b(X)$ is called *exceptional* if $\mathrm{Hom}_{\mathsf{D}^b(X)}(E, E) = \mathbb{C}$ and $\mathrm{Hom}_{\mathsf{D}^b(X)}(E, E[k]) = 0$ for all $k \neq 0$. A collection $(E_1, \dots, E_n)$ of exceptional objects is called an *exceptional collection* if

$$\mathrm{Hom}_{\mathsf{D}^b(X)}(E_i, E_j[k]) = 0 \quad \text{for all } j < i \text{ and all } k \in \mathbb{Z}.$$

The full triangulated subcategory $\langle E_1, \dots, E_n \rangle \subseteq \mathsf{D}^b(X)$ generated by an exceptional collection $(E_1, \dots, E_n)$ is always admissible; in particular, its left- and right-orthogonal complements are again admissible. An exceptional collection $(E_1, \dots, E_n)$ is *full* if it generates $\mathsf{D}^b(X)$, i.e. $\langle E_1, \dots, E_n \rangle = \mathsf{D}^b(X)$; equivalently $\langle E_1, \dots, E_n \rangle^\perp = 0 = {}^\perp\langle E_1, \dots, E_n \rangle$.

A semiorthogonal decomposition $\mathsf{D}^b(X) = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ yields a direct sum decomposition of the Grothendieck group of $\mathsf{D}^b(X)$:

$$\mathsf{K}_0(X) = \mathsf{K}_0(\mathcal{A}_1) \oplus \cdots \oplus \mathsf{K}_0(\mathcal{A}_n).$$

An exceptional collection $(E_1, \dots, E_n)$ is of *maximal length* if there exists no further exceptional object $F \in \mathsf{D}^b(X)$ such that $(E_1, \dots, E_n, F)$ is an exceptional collection. Because $\langle E_i \rangle \cong \mathsf{D}^b(\mathrm{Spec}\,\mathbb{C})$ for an exceptional object E_i, we have $\mathsf{K}_0(\langle E_i \rangle) = \mathbb{Z}[E_i]$. Thus, if $\mathsf{K}_0(X)$ is finitely generated as an abelian group and $n = \mathrm{rk}\,\mathsf{K}_0(X)$, then any exceptional collection of length n is of maximal length.

Assume that $\mathsf{K}_0(X)$ is finitely generated and $(E_1, \dots, E_n)$ is an exceptional collection of length $n = \mathrm{rk}\,\mathsf{K}_0(X)$. The additivity of K_0 among semiorthogonal decompositions implies that $\mathsf{K}_0(\mathcal{A}) = \mathrm{tors}(\mathsf{K}_0(X))$ is a finite group, where $\mathcal{A} = \langle E_1, \dots, E_n \rangle^\perp$. If $\mathcal{A} \subseteq \mathsf{D}^b(X)$ is a nonzero admissible subcategory with finite $\mathsf{K}_0(\mathcal{A})$, then by definition $\mathcal{A}$ is a *quasi phantom* and if additionally $\mathsf{K}_0(\mathcal{A}) = 0$, then $\mathcal{A}$ is called a *phantom*.

Let $\mathcal{A} \subseteq \mathsf{D}^b(X)$ and $\mathcal{B} \subseteq \mathsf{D}^b(Y)$ be full triangulated subcategories. Then $\mathcal{A} \boxtimes \mathcal{B} \subseteq \mathsf{D}^b(X \times Y)$ denotes the smallest full triangulated subcategory of $\mathsf{D}^b(X \times Y)$ which is closed under direct summands and contains all objects of the form $p_X^* A \otimes^{\mathsf{L}} p_Y^* B$ for $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Following [6, Def. 1.9] an admissible subcategory $\mathcal{A} \subseteq \mathsf{D}^b(X)$ is called a *universal phantom* if for all smooth projective varieties Y the category $\mathcal{A} \boxtimes \mathsf{D}^b(Y)$ is a phantom.

3 SHGH conjecture

Let X be the blow-up of the projective plane $\mathbb{P}^2_{\mathbb{C}}$ in a set of closed points $p_1, \dots, p_n \in \mathbb{P}^2_{\mathbb{C}}$. Denote by $E_i \subseteq X$ the (-1)-curve over the point p_i and recall that $\mathrm{Pic}(X) = \mathbb{Z}H \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_n$, where H is the pullback of a hyperplane in $\mathbb{P}^2_{\mathbb{C}}$. The class of a divisor D on X can be uniquely written as

$$D = dH - \sum_{i=1}^{n} m_i E_i$$

for some $d, m_i \in \mathbb{Z}$. Moreover, the intersection product satisfies $H^2 = 1$, $E_i^2 = -1$, $H \cdot E_i = 0$, and $E_i \cdot E_j = 0$ for all $i \neq j$. If $d > 0$ and $m_i \geq 0$, the space of global sections $H^0(X, \mathcal{O}_X(D))$ can be identified with the space of homogeneous polynomials $P \in \mathbb{C}[X, Y, Z]$ of degree d such that P vanishes to order $\geq m_i$ at p_i. If the points are chosen in general position, meaning that $h^0(D) := \dim H^0(X, \mathcal{O}_X(D))$ is minimal, then the following conjecture due to Segre–Harbourne–Gimigliano–Hirschowitz predicts the value of $h^0(D)$.

Conjecture 3.1 (SHGH) *Let $d > 0$ and $m_i \geq 0$, $1 \leq i \leq n$, be integers. For X the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in n general points, the divisor $D := dH - \sum_{i=1}^n m_i E_i$ satisfies*

$$\dim H^0(X, \mathcal{O}_X(D)) = \max(0, \chi(X, \mathcal{O}_X(D)))$$

or there exists a (-1)-curve $C \subseteq X$ such that $C \cdot D \leq -2$.

Note that the possible choices of blown up points depend on the divisor D, i.e. on the tuple $(d, m_1, \dots, m_n)$. If one requires $h^0(D)$ to be minimal for *all* tuples $(d, m_1, \dots, m_n)$, then the points have to be chosen very general.

A divisor $D = dH - \sum_{i=1}^n m_i E_i$ is said to be in *standard form* if $d > 0$, $m_i \geq 0$, $d \geq m_1 \geq \cdots \geq m_n$, and $d - m_1 - m_2 - m_3 \geq 0$. The following Lemma 3.2 is certainly well-known, see, e.g., [3, Prop. 1.4]. As it will be used in the proof of Theorem 1.1, we provide a proof here.

Lemma 3.2 *Let X be the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in n points. If $D = dH - \sum_{i=1}^n m_i E_i$ is in standard form and $C \subseteq X$ is a (-1)-curve, then $D \cdot C \geq 0$.*

Proof Let $C \subseteq X$ be a (-1)-curve. If $C = E_i$ for some i, then $D \cdot C = m_i \geq 0$. If $C \neq E_i$ for all i, then C is the strict transform of a curve in $\mathbb{P}^2_{\mathbb{C}}$, thus linearly equivalent to $eH - \sum_i f_i E_i$ with $e > 0$, and $f_i \geq 0$ for $1 \leq i \leq n$. Consider the divisors $G_1 := H - E_1$, $G_2 := 2H - E_1 - E_2$, and $G_j := 3H - \sum_{i=1}^j E_i$ for $3 \leq j \leq n$. By assumption, D is a linear combination of H and G_j, $1 \leq j \leq n$, with nonnegative coefficients. The divisors H, G_1, and G_2 are nef. Further, $G_j \cdot C \geq G_n \cdot C = -K_X \cdot C$ for $3 \leq j \leq n$. Since $-K_X \cdot C = 1$, the lemma follows. $\square$

The SHGH Conjecture is known to be true in various cases of low multiplicity. Alternatively, for a single explicit divisor D it is possible to compute the actual value of $h^0(D)$ using a computer. We will use the following known cases to show that the collection in Theorem 1.1 is exceptional:

Theorem 3.3 ([4, Thm. 34], [3, Thm. 0.1]) *Let X be the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in n general points and let $D = dH - \sum_{i=1}^n m_i E_i$ be a divisor with $d > 0$ and $m_i \geq 0$.*

(i) *If either all $m_i \leq 11$, or*
(ii) *if $n = 10$, $m_1 = m_2 = \cdots = m_{10}$, and $d/m_1 \geq 174/55$,*

then the SHGH Conjecture holds for D, i.e.

$$\dim H^0(X, \mathcal{O}_X(D)) = \max(0, \chi(X, \mathcal{O}_X(D))),$$

or there exists a (-1)-curve $C \subseteq X$ such that $C \cdot D \leq -2$.

4 Height and pseudoheight of exceptional collections

Kuznetsov introduced in [9] the so-called *height* of an exceptional collection $\langle E_1, \ldots, E_n \rangle \subseteq \mathsf{D}^b(X)$: If $\mathcal{D}$ is a smooth and proper dg-category and $\mathcal{B} \subseteq \mathcal{D}$ a dg-subcategory, Kuznetsov defines the *normal Hochschild cohomology* $\mathrm{NHH}^\bullet(\mathcal{B}, \mathcal{D})$ of $\mathcal{B}$ in $\mathcal{D}$ as a certain dg-module [9, Def. 3.2]. The height of an exceptional collection $(E_1, \ldots, E_n)$ is then defined as

$$\mathrm{h}(E_1, \ldots, E_n) := \min\{k \in \mathbb{Z} \mid \mathrm{NHH}^k(\mathcal{E}, \mathcal{D}) \neq 0\}$$

where $\mathcal{D}$ is a dg-enhancement of $\mathsf{D}^b(X)$ and $\mathcal{E}$ the dg-subcategory of $\mathcal{D}$ generated by the exceptional objects $(E_1, \ldots, E_n)$. In general, the normal Hochschild cohomology $\mathrm{NHH}^\bullet(\mathcal{E}, \mathcal{D})$ can be computed using a spectral sequence [9, Prop. 3.7]. For our purpose it will be sufficient to consider a coarser invariant of an exceptional collection, the so-called *pseudoheight*.

Definition 4.1 ([9, Def. 4.4, Def. 4.9]) For any two objects $F, F' \in \mathsf{D}^b(X)$ define the *relative height* as

$$\mathrm{e}(F, F') := \inf\{k \in \mathbb{Z} \mid \mathrm{Ext}^k(F, F') \neq 0\}.$$

For an exceptional collection $(E_1, \ldots, E_n)$ the *pseudoheight* is

$$\begin{aligned} &\mathrm{ph}(E_1, \ldots, E_n) \\ &\quad := \min_{1 \leq a_0 < \cdots < a_p \leq n} \Big(\mathrm{e}(E_{a_0}, E_{a_1}) + \cdots + \mathrm{e}(E_{a_{p-1}}, E_{a_p}) + \mathrm{e}(E_{a_p}, S^{-1}(E_{a_0})) - p \Big), \end{aligned}$$

where $S = - \otimes \omega_X[\dim X]$ is the Serre functor of $\mathsf{D}^b(X)$. The *anticanonical pseudoheight* is

$$\begin{aligned} &\mathrm{ph}_{\mathrm{ac}}(E_1, \ldots, E_n) \\ &\quad := \min_{1 \leq a_0 < \cdots < a_p \leq n} \Big(\mathrm{e}(E_{a_0}, E_{a_1}) + \cdots + \mathrm{e}(E_{a_{p-1}}, E_{a_p}) + \mathrm{e}(E_{a_p}, E_{a_0} \otimes \omega_X^{-1}) - p \Big). \end{aligned}$$

Clearly, $\mathrm{ph}_{\mathrm{ac}} = \mathrm{ph} - \dim X$.

Lemma 4.2 ([9, Lem. 4.5]) *For an exceptional collection* $(E_1, \dots, E_n)$ *in* $\mathsf{D}^b(X)$ *we have* $\mathrm{h}(E_1, \dots, E_n) \geq \mathrm{ph}(E_1, \dots, E_n)$.

We will use the following criterion to show that the exceptional collection in Theorem 1.1 is not full.

Proposition 4.3 ([9, Prop. 6.1]) *Let* X *be a smooth projective variety and* $(E_1, \dots, E_n)$ *an exceptional collection in* $\mathsf{D}^b(X)$. *If* $\mathrm{h}(E_1, \dots, E_n) > 0$, *then* $(E_1, \dots, E_n)$ *is not full.*

In particular, if $\mathrm{ph}_{\mathrm{ac}}(E_1, \dots, E_n) > -\dim X$, then the collection is not full.

5 Proof of Theorem 1.1

Let X be the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in 10 general points. Using the Beilinson collection $\mathsf{D}^b(\mathbb{P}^2_{\mathbb{C}}) = \langle \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}, \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(H), \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(2H) \rangle$, applying Orlov's blow-up formula [11], and applying right mutations to the torsion sheaves, one obtains the full exceptional collection

$$\mathsf{D}^b(X) = \langle \mathcal{O}_X, \mathcal{O}_X(E_1), \dots, \mathcal{O}_X(E_{10}), \mathcal{O}_X(H), \mathcal{O}_X(2H) \rangle$$

consisting of line bundles. In particular, we obtain

$$\begin{aligned} \mathsf{K}_0(X) &= \mathbb{Z}[\mathcal{O}_X] \oplus \mathbb{Z}[\mathcal{O}_X(E_1)] \oplus \cdots \oplus \mathbb{Z}[\mathcal{O}_X(E_{10})] \oplus \mathbb{Z}[\mathcal{O}_X(H)] \oplus \mathbb{Z}[\mathcal{O}_X(2H)] \\ &\cong \mathbb{Z}^{13}. \end{aligned}$$

Since the canonical class $K_X = -3H + \sum_{i=1}^{10} E_i$ satisfies $K_X^2 = -1$, the Picard lattice admits an orthogonal decomposition $\mathrm{Pic}(X) = K_X^{\perp} \oplus \mathbb{Z}K_X$ and one can compute that a basis of $K_X^{\perp}$ is given by $H - E_1 - E_2 - E_3, E_1 - E_2, \dots, E_9 - E_{10}$. Consider the orthogonal transformation $\iota\colon \mathrm{Pic}(X) \to \mathrm{Pic}(X)$ which multiplies an element of $K_X^{\perp}$ by -1 and is the identity on $\mathbb{Z}K_X$. We compute

$$D_i := \iota(E_i) = -6H + 2\sum_{j=1}^{10} E_j - E_i \quad \text{and} \quad F := \iota(H) = -19H + 6\sum_{i=1}^{10} E_i.$$

Since ι fixes the canonical class, one can deduce from the Riemann–Roch formula that

$$(\mathcal{O}_X, \mathcal{O}_X(D_1), \dots, \mathcal{O}_X(D_{10}), \mathcal{O}_X(F), \mathcal{O}_X(2F)) \tag{5.1}$$

is a *numerically exceptional collection*, i.e. it is semiorthogonal with respect to the Euler pairing

$$\chi(F, G) := \sum_{i \in \mathbb{Z}} (-1)^i \dim \mathrm{Hom}_{\mathsf{D}^b(X)}(F, G[i])$$

and each object F in the collection satisfies $\chi(F, F) = 1$. Moreover, it is clear that the image of (5.1) is a basis of the Grothendieck group $\mathsf{K}_0(X) \cong \mathbb{Z}^{13}$, thus the collection is of maximal length.

Proof of the Theorem 1.1 We first verify that the collection (5.1) is exceptional. Since the collection is numerically exceptional and consists of sheaves, it suffices to check the vanishing of Hom- and Ext^2-spaces. Via Serre duality, the computation of an Ext^2-space can be done by computing global sections of a divisor. Thus, abbreviating $\hom(-,-) = \dim \mathrm{Hom}(-,-)$ and $\mathrm{ext}^k(-,-) = \dim \mathrm{Ext}^k(-,-)$, we have to show that the following dimensions are zero:

$$\begin{aligned}
\hom(\mathcal{O}_X(2F), \mathcal{O}_X(F)) &= h^0(-F),\\
\mathrm{ext}^2(\mathcal{O}_X(2F), \mathcal{O}_X(F)) &= h^2(-F) = h^0(K_X + F),\\
\hom(\mathcal{O}_X(2F), \mathcal{O}_X(D_i)) &= h^0(D_i - 2F),\\
\mathrm{ext}^2(\mathcal{O}_X(2F), \mathcal{O}_X(D_i)) &= h^2(D_i - 2F) = h^0(K_X - D_i + 2F),\\
\hom(\mathcal{O}_X(2F), \mathcal{O}_X) &= h^0(-2F),\\
\mathrm{ext}^2(\mathcal{O}_X(2F), \mathcal{O}_X) &= h^2(-2F) = h^0(K_X + 2F),\\
\hom(\mathcal{O}_X(F), \mathcal{O}_X(D_i)) &= h^0(D_i - F),\\
\mathrm{ext}^2(\mathcal{O}_X(F), \mathcal{O}_X(D_i)) &= h^2(D_i - F) = h^0(K_X - D_i + F),\\
\hom(\mathcal{O}_X(F), \mathcal{O}_X) &= h^0(-F),\\
\mathrm{ext}^2(\mathcal{O}_X(F), \mathcal{O}_X) &= h^2(-F) = h^0(K_X + F),\\
\hom(\mathcal{O}_X(D_i), \mathcal{O}_X(D_j)) &= h^0(D_j - D_i),\\
\mathrm{ext}^2(\mathcal{O}_X(D_i), \mathcal{O}_X(D_j)) &= h^2(D_j - D_i) = h^0(K_X - D_j + D_i),\\
\hom(\mathcal{O}_X(D_i), \mathcal{O}_X) &= h^0(-D_i),\\
\mathrm{ext}^2(\mathcal{O}_X(D_i), \mathcal{O}_X) &= h^2(-D_i) = h^0(K_X + D_i),
\end{aligned}$$

where $1 \le i, j \le 10$, $i \ne j$. The vanishing holds trivially if the divisor has negative intersection with H or is of the form $D_j - D_i = E_i - E_j$. The remaining cases are

$$-F = 19H - 6\sum_{j=1}^{10} E_j, \quad -2F = 38H - 12\sum_{j=1}^{10} E_j, \tag{5.2}$$

$$-D_i = 6H - 2\sum_{j=1}^{10} E_j + E_i, \quad D_i - F = 13H - 4\sum_{j=1}^{10} E_j - E_i, \quad \text{and}$$

$$D_i - 2F = 32H - 10\sum_{j=1}^{10} E_j - E_i.$$

Up to permutation of the points, these divisors are in standard form. Thus by Lemma 3.2, if D is one of the divisors in (5.2), then $C \cdot D \geq 0$ holds for any (-1)-curve $C \subseteq X$. If $D \neq -2F$, then the multiplicities of D are bounded by 11, thus $h^0(D) = \chi(D) = 0$ by Theorem 3.3 (i). If $D = -2F$, then we compute $38/12 \geq 174/55$. Hence, $h^0(-2F) = \chi(-2F) = 0$ by Theorem 3.3 (ii). Therefore, (5.1) is exceptional.

To show that (5.1) is not full, by Proposition 4.3 and Lemma 4.2 it suffices to show that the anticanonical pseudoheight $\mathrm{ph}_{\mathrm{ac}}$ of (5.1) is at least -1. In the following, we show that $\mathrm{ph}_{\mathrm{ac}} \geq 0$. Recall that

$$\begin{aligned}\mathrm{ph}_{\mathrm{ac}} = \min_{1\leq a_0<\cdots<a_p\leq 13} \Big(& \mathrm{e}(E_{a_0}, E_{a_1}) + \cdots + \mathrm{e}(E_{a_{p-1}}, E_{a_p}) \\ & + \mathrm{e}(E_{a_p}, E_{a_0} \otimes \omega_X^{-1}) - p\Big),\end{aligned} \tag{5.3}$$

where the E_{a_i} are the exceptional objects in (5.1). Since (5.1) consists of sheaves, $\mathrm{e}(E_{a_i}, E_{a_{i+1}})$ and $\mathrm{e}(E_{a_p}, E_{a_0} \otimes \omega_X^{-1})$ take values in $\{0, 1, 2, \infty\}$. First, if $p = 0$, then the expression under the minimum in (5.3) is $\mathrm{e}(E_{a_0}, E_{a_0} \otimes \omega_X^{-1}) \geq 0$. Next, if $p \geq 1$ and if we know that $\mathrm{e}(E_{a_i}, E_{a_{i+1}}) \geq 1$ for all $0 \leq i \leq p-1$, then the expression under the minimum in (5.3) is greater or equal than $\mathrm{e}(E_{a_p}, E_{a_0} \otimes \omega_X^{-1}) \geq 0$. Hence, it is enough to show that the following dimensions vanish:

$$\hom(\mathcal{O}_X, \mathcal{O}_X(D_i)) = h^0(D_i),$$

$$\hom(\mathcal{O}_X, \mathcal{O}_X(F)) = h^0(F),$$

$$\hom(\mathcal{O}_X, \mathcal{O}_X(2F)) = h^0(2F),$$

$$\hom(\mathcal{O}_X(D_i), \mathcal{O}_X(D_j)) = h^0(D_j - D_i),$$

$$\hom(\mathcal{O}_X(D_i), \mathcal{O}_X(F)) = h^0(F - D_i),$$

$$\hom(\mathcal{O}_X(D_i), \mathcal{O}_X(2F)) = h^0(2F - D_i),$$

$$\hom(\mathcal{O}_X(F), \mathcal{O}_X(2F)) = h^0(F),$$

where $1 \leq i, j \leq 10$ and $i \neq j$. All these divisors have either negative intersection with H or are of the form $D_j - D_i = E_i - E_j$, thus the vanishing holds for trivial reasons. Hence, $\mathrm{ph}_{\mathrm{ac}} \geq 0$ and we conclude that (5.1) is not full. □

Corollary 5.4 *The admissible subcategory*

$$\mathcal{A} = \langle \mathcal{O}_X, \mathcal{O}_X(D_1), \ldots, \mathcal{O}_X(D_{10}), \mathcal{O}_X(F), \mathcal{O}_X(2F)\rangle^{\perp}$$

is a universal phantom subcategory of $\mathsf{D}^b(X)$.

Proof The Chow motive of X with integer coefficients is of Lefschetz type and $\mathsf{K}_0(\mathcal{A}) = 0$. By [6, Cor. 4.3] the K-motive of $\mathcal{A}$ with integer coefficients vanishes and [6, Prop. 4.4] shows that $\mathcal{A}$ is a universal phantom. □

Remark 5.5 Using the Hochschild–Kostant–Rosenberg isomorphism, one can compute that the Hochschild cohomology of X satisfies:

$$\dim \mathrm{HH}^0(X) = 1, \quad \dim \mathrm{HH}^1(X) = 0, \quad \dim \mathrm{HH}^2(X) = 12, \quad \text{and}$$
$$\dim \mathrm{HH}^i(X) = 0 \quad \text{for } i \geq 3.$$

Applying the techniques from [9] it is further possible to compute that the height of (5.1) is 4 and the Hochschild cohomology of $\mathcal{A}$ has the following dimensions:

$$\begin{aligned} &\dim \mathrm{HH}^0(\mathcal{A}) = 1, && \dim \mathrm{HH}^1(\mathcal{A}) = 0, && \dim \mathrm{HH}^2(\mathcal{A}) = 12, \\ &\dim \mathrm{HH}^3(\mathcal{A}) = 446, && \dim \mathrm{HH}^4(\mathcal{A}) = 853, && \dim \mathrm{HH}^5(\mathcal{A}) = 420, \quad \text{and} \\ &\dim \mathrm{HH}^i(\mathcal{A}) = 0 \text{ for } i \geq 6. \end{aligned}$$

In particular, the restriction morphism $\mathrm{HH}^i(X) \to \mathrm{HH}^i(\mathcal{A})$ is an isomorphism for $0 \leq i \leq 2$ and a monomorphism for $i = 3$. As explained in [9, Prop. 4.8], this implies that the formal deformation spaces of $\mathsf{D}^b(X)$ and $\mathcal{A}$ are isomorphic.

Acknowledgements This work is part of the author's dissertation, supervised by Charles Vial whom we wish to thank for helpful discussions and explanations. We discovered the existence of the exceptional collection (1.2) in the context of our previous work [7], where we study the transitivity of the braid group action on (numerically) exceptional collections on surfaces using a classification obtained by Vial in [13]. Further, we thank the anonymous referees for carefully reading our manuscript.

Funding Open Access funding enabled and organized by Projekt DEAL. The research was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – SFB-TRR 358/1 2023 – 491392403.

References

1. Böhning, C., Graf von Bothmer, H.-C., Katzarkov, L., Sosna, P.: Determinantal Barlow surfaces and phantom categories. J. Eur. Math. Soc. **17**(7), 1569–1592 (2015)
2. Chang, W., Haiden, F., Schroll, S.: Braid group actions on branched coverings and full exceptional sequences (2023). arXiv:2301.04398v2 [math.RT]
3. Ciliberto, C., Miranda, R.: Homogeneous interpolation on ten points. J. Algebraic Geom. **20**(4), 685–726 (2011)
4. Dumnicki, M., Jarnicki, W.: New effective bounds on the dimension of a linear system in $\mathbb{P}^2$. J. Symb. Comput. **42**(6), 621–635 (2007)
5. Efimov, A.: Private communication. Bielefeld (2023)
6. Gorchinskiy, S., Orlov, D.: Geometric phantom categories. Publ. Math. Inst. Hautes Études Sci. **117**, 329–349 (2013)

7. Krah, J.: Mutations of Numerically Exceptional Collections on Surfaces (2022). arXiv:2211.07724v2 [math.AG]
8. Kuznetsov, A.: Semiorthogonal decompositions in algebraic geometry. In: Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II, pp. 635–660. Kyung Moon Sa, Seoul (2014)
9. Kuznetsov, A.: Height of exceptional collections and Hochschild cohomology of quasiphantom categories. J. Reine Angew. Math. **708**, 213–243 (2015)
10. Orlov, D.: Finite-dimensional differential graded algebras and their geometric realizations. Adv. Math. **366**, 107096 (2020)
11. Orlov, D.: Projective bundles, monoidal transformations, and derived categories of coherent sheaves. Izv. Ross. Akad. Nauk, Ser. Mat. **56**(4), 852–862 (1992)
12. Pirozhkov, D.: Admissible subcategories of del Pezzo surfaces (2020). arXiv:2006.07643v1 [math.AG]
13. Vial, C.: Exceptional collections, and the Néron-Severi lattice for surfaces. Adv. Math. **305**, 895–934 (2017)