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# Nonuniformly elliptic Schauder theory

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*To Arrigo Cellina, with admiration for his pioneering work in the Calculus of Variations*

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**Abstract**

Local Schauder theory holds in the nonuniformly elliptic setting. Specifically, first derivatives of solutions to nonuniformly elliptic problems are locally Hölder continuous if so are their coefficients.

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## Contents



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## 1 Introduction

In this paper we give the first general solution to two different, but yet connected, longstanding and classical open problems in the regularity theory of variational integrals and elliptic equations. To begin with, we prove the first results concerning local gradient Hölder regularity of minimizers of nonuniformly elliptic integrals, that are not necessarily equipped with a Euler-Lagrange equation. In fact, in the cases we are going to consider here, such equations might not exist. Such type of results are classical in the uniformly elliptic case since the work of Giaquinta & Giusti [28–30], whilst nothing is known in the general nonuniformly elliptic one. Second, and most importantly, we prove Schauder estimates for nonuniformly elliptic problems. Both for variational problems and for elliptic equations, gradients of solutions are locally Hölder continuous provided coefficients are locally Hölder continuous. Again, while this is classical in the uniformly elliptic case – see again [29, 30], Manfredi's [61, 62] and Lieberman's [59] papers for full generality – no analog is recorded in the nonuniformly elliptic case. The crucial point in this setting is to obtain $L^\infty$-gradient bounds, after which, more classical perturbation methods can be combined with certain specific forms of the a priori estimates obtained, to prove gradient Hölder continuity. See for instance the comments in Lieberman's review of Giaquinta & Giusti's paper [30].[1] The central role of gradient bounds is also remarked by Ivanov [46, page 7][2] and was exploited by Ladyzhenskaya & Ural'tseva [55], as described in [46, page 15] too. To achieve our results, we employ a novel hybrid perturbation approach, suited for nonuniformly elliptic problems. This is aimed at replacing the classical ones used in the uniformly elliptic setting, that are ultimately based on plain freezing arguments. We believe that this approach has potential for applications in several other places. In fact, in this paper we present the main bulk of the technique and apply it in a certain number of different settings. Others are still possible. In particular, the boundary case, as well as the evolutionary one, will be treated in forthcoming papers.

So-called Schauder estimates for linear elliptic equations are actually a classic achievement of Hopf [43], Caccioppoli [12] and Schauder [74, 75] (see also [33]). Modern proofs are in [13, 32, 80, 87]. The nonlinear story goes back to the classical papers by Frehse [27], Giaquinta & Giusti [28–31], Ivert [47, 48] and Manfredi [61, 62]. There the first Schauder type results for nonlinear equations and nondifferentiable integral functionals, asserting local Hölder continuity of the gradient for some exponent, were proved. For the sake of simplicity, let us consider the following classical model example [14, 15, 29, 42, 51, 72, 81, 84]:

$$w \mapsto \int_\Omega [F(Dw) + \mathtt{h}(x, w)]\,\mathrm{d}x\,. \tag{1.1}$$

Here $F(\cdot) \geq 0$ is a sufficiently regular, uniformly elliptic integrand with $p$-growth - take for instance $F(Dw) \equiv |Dw|^p$, $p > 1$. Instead, $\mathtt{h}\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a bounded, merely Hölder continuous function; $\Omega \subset \mathbb{R}^n$ denotes a bounded open subset, $n \geq 2$. By uniform ellipticity of $F(\cdot)$ here we mean that the ellipticity ratio $\mathcal{R}_F(z)$ remains

[1] Indeed, in the MR review of [30], Lieberman states: "A comment needs to be made concerning their [i.e., of Giaquinta & Giusti's methods] brief application to equations when their growth properties fail. As they point out, such equations fall under their considerations provided a global gradient bound has been established; however, this gradient bound has only been proved when $A$ [i.e., the operator or functional considered in [30]] is differentiable with respect to all its arguments, and in many cases more smoothness of the coefficients is needed. The results of this paper are thus much more striking when applied to uniformly elliptic equations than to nonuniformly elliptic ones" [60]. The missing growth properties forcing smoothness of coefficients Lieberman is pointing at actually correspond to nonuniform ellipticity. This can be therefore treated, when coefficients are Hölder continuous, only upon assuming that solutions are a priori Lipschitz (and under certain additional assumptions, like non-degeneracy). In this paper we overcome these basic points.

[2] Ivanov remarks: "In view of the results of Ladyzhenskaya and Ural'tseva, the problem of solvability of boundary value problems for a nonuniformly elliptic or parabolic equation reduces to the question of constructing a priori estimates of the maximum moduli of the gradients of solutions for a suitable one-parameter family of similar equations". This means finding uniform a priori gradient estimates for regularized problems. This is shown to be possible in this paper without the unnatural assumptions considered before, i.e., without differentiability and smoothness of coefficients, in turn ruling out the Schauder setting. See the proof of Theorem 6.

bounded for $|z|$ large [46, 55, 78, 85], i.e.,

$$\sup_{|z|\geq 1} \mathcal{R}_F(z) < \infty, \quad \mathcal{R}_F(z) := \frac{\text{highest eigenvalue of } \partial_{zz}F(z)}{\text{lowest eigenvalue of } \partial_{zz}F(z)} . \tag{1.2}$$

This happens for instance in the $p$-Laplacean case $F(z) = |z|^p$, i.e., when

$$\partial_{zz}F(z) \approx |z|^{p-2}\mathbb{I}_{\mathrm{d}} \tag{1.3}$$

holds for $|z|$ large. As $\mathtt{h}(\cdot)$ is not assumed to be differentiable, the Euler-Lagrange equation

$$-\operatorname{div} \partial_z F(Du) + \partial_u \mathtt{h}(x, u) = 0 \tag{1.4}$$

of the functional in (1.1) just cannot be derived. Yet, in [29] a method is devised to get local gradient Hölder continuity of minima only using minimality, and without passing through (1.4). This goes roughly as follows. Given a minimizer $u$ of the functional in (1.1), one defines its lifting $v$ on the ball $B \Subset \Omega$ by solving

$$v \to \min_{w \in u + W^{1,p}_0(B)} \int_B F(Dw)\, \mathrm{d}x . \tag{1.5}$$

Solutions to corresponding Euler-Lagrange equation $\operatorname{div} \partial_z F(Dv) = 0$, are $C^{1,\alpha}$-regular and enjoy good *homogeneous* decay estimates. This is a direct consequence of the uniform ellipticity (1.2). Such estimates can then be matched with comparison ones between $u$ and $v$ on $B$ as they enjoy *the same degree of homogeneity* of the reference ones and this is another consequence of (1.2). Thanks to this common homogeneity, combining the two ingredients finally leads to transfer, at all scales, the $C^{1,\alpha}$-estimates available for solutions to (1.5), to the original minimizer $u$. This comparison scheme, based on minimality, is in spirit close to the one used to derive classical Schauder estimates and relying on Korn's argument.

Such classical perturbation schemes uniformly fail in the nonuniformly elliptic setting. In this paper we show a different route aimed at bypassing the lack of homogeneous estimates typical of nonuniformly elliptic problems. For this, we shall consider a general class of integrands for which (1.2) is not necessarily satisfied, and for which $\mathcal{R}_F(z)$ grows at most polynomially

$$\mathcal{R}_F(z) \lesssim |z|^{\delta} + 1 , \qquad \delta > 0 . \tag{1.6}$$

In fact, our main ellipticity assumption, replacing (1.3), will be of the type

$$|z|^{p-2}\mathbb{I}_{\mathrm{d}} \lesssim \partial_{zz}F(z) \lesssim |z|^{q-2}\mathbb{I}_{\mathrm{d}} \tag{1.7}$$

for $|z| \geq 1$, so that it is $\delta = q - p$ in (1.6); conditions (1.7) are the most general to describe (1.6). Polynomial nonuniform ellipticity as in (1.6) is a standard topic since the classical works of Ladyzhenskaya & Ural'tseva [55, 56], Hartman & Stampacchia [39], Trudinger [85, 86], Ivočkina & Oskolkov [49], Oskolkov [71], Serrin [77], Ivanov [44–46], Leon Simon [78, 79], Ural'tseva & Urdaletova [89], Lieberman [58], just to mention a few. In the variational setting, conditions (1.7) were systematically studied by Marcellini in a series of pioneering papers [65–67], who introduced functionals with so-called $(p, q)$-growth conditions, referring to the formulation in (1.7). Today a huge literature is devoted to such problems. With the current techniques, differentiability of coefficients is unescapable; Hölder regularity is forbidden.

To describe the methods employed here, we recall that a traditional, *non-perturbative and direct* way to get that minima are Lipschitz when equation (1.4) exists, is to first differentiate (1.4), and then invoking De Giorgi-Nash-Moser theory. Sticking to De Giorgi's method, this means to get first a Caccioppoli inequality involving derivatives of $Du$, and then to run a geometric iteration leading to gradient boundedness. Here we use, in a sense, both the direct and the perturbative approach, and that's where the word hybrid stems from. We also take advantage of various regularity tools and viewpoints developed over the last years in the Calculus of Variations [3, 21, 51] and in Nonlinear Potential Theory [50, 69]. The approach proposed here goes along the following points:

- We still prove $Du \in L^{\infty}_{\mathrm{loc}}$ via a direct De Giorgi type geometric iteration involving, up to minor corrections, truncations of a certain convex function of $|Du|$ (Bernstein method). The iteration is based on a Caccioppoli type inequality, that this time does not involve full derivatives of $Du$, as in the classical case. This is because the functionals/equations we are dealing with involve nondifferentiable coefficients, and therefore cannot be differentiated. Instead, notwithstanding the problem is local, the Caccioppoli inequality we use involves fractional derivatives of $Du$; see Sect. 5.
- In order to iterate the Caccioppoli inequality of the previous point, we use a renormalization that makes it homogeneous, as in uniformly elliptic problems. The price we pay is a controlled increase of the involved multiplicative constants. They now incorporate an additional, direct dependence on $\|Du\|^{\sigma}_{L^{\infty}}$, with $\sigma \equiv \sigma(q/p)$. Keeping track of such constants is a crucial part of the proof. For this, we impose a moderate polynomial growth rate on $\mathcal{R}_F(z)$ as in (1.6), that implies that $\delta \equiv q - p$ must be small enough, in turn making $\sigma$ small too. This kind of assumption is not technical and it is necessary (see Remark 1).
- The perturbative part, making the approach hybrid. This is in the proof of the Caccioppoli inequality. For this we exploit a delicate atomic like decomposition in Nikolski spaces. The argument still employs comparisons with liftings $v$ as in (1.5), used in a sense as atoms (see Sect. 5.3.4 below). At this stage, crucial use is made of precise a priori Lipschitz estimates for autonomous problems (see Lemma 5.3).
- We rely on the use of nonlinear potentials of the type originally introduced by Havin & Maz'ya [68]; see Sect. 4. Indeed, certain quantities apparently uncontrollable in the nonuniformly elliptic setting, are now treated by means of an optimized splitting between the $L^{\infty}$-norm of $Du$, and integral remainder terms building up nonlinear potentials along iterations (see Propositions 5.3-5.4). As an additional benefit, this nonlinear potential theoretic approach allows to treat cases involving unbounded data. In particular, it leads to discover new borderline conditions for regularity in Lorentz spaces. These extend known ones from standard settings and connect to a very large literature on borderline cases; see Remark 2.

This approach allows to settle the open problem of proving Schauder estimates in the nonuniformly elliptic setting. To fix the ideas, consider the model functional

$$\begin{cases} w \mapsto \mathcal{S}_{\mathrm{x}}(w,\Omega) := \displaystyle\int_\Omega \mathfrak{c}(x)F(Dw)\,\mathrm{d}x, \quad 0<\nu\le \mathfrak{c}(\cdot)\le L \\ |\mathfrak{c}(x_1)-\mathfrak{c}(x_2)|\le L|x_1-x_2|^\alpha, \quad \alpha\in(0,1], \end{cases} \tag{1.8}$$

for every choice of $x_1, x_2 \in \Omega$, where $F(\cdot)$ satisfies (1.7). When $p = q$, gradient Hölder regularity of minima can be found in [28, 29, 59, 61, 62]. When $p \neq q$, the first result is in Theorem 2 below, that in fact applies to functionals as in (1.8).

Further developing our approach, we treat also nondifferentiable functionals of the type

$$\begin{cases} w \mapsto \mathcal{S}(w,\Omega) := \displaystyle\int_\Omega [\mathfrak{c}(x,w)F(Dw)+\mathtt{h}(x,w)]\,\mathrm{d}x, \\ \quad 0<\nu\le \mathfrak{c}(\cdot)\le L, \\ |\mathfrak{c}(x_1,y_1)-\mathfrak{c}(x_2,y_2)|\le L|x_1-x_2|^\alpha + L|y_1-y_2|^\alpha, \quad \alpha\in(0,1], \end{cases} \tag{1.9}$$

for every choice of $x_1, x_2 \in \Omega$, $y_1, y_2 \in \mathbb{R}$, and thereby falling outside the realm of traditional Schauder estimates. The conditions we impose on $\mathtt{h}(\cdot)$ are detailed in (2.6) below; see also (2.23). Essentially, $\mathtt{h}(x, y)$ is assumed to be Hölder continuous with respect to $y$ and merely measurable with respect to $x$. Examples are obviously given by $\mathtt{h}(x, y) = \mathtt{f}(x)h(y)$, where $h(\cdot)$ is any Hölder continuous function and $\mathtt{f} \in L^{\mathfrak{q}}$ with $\mathfrak{q} > n/\alpha$. For functionals as in (1.9) we assume the additional lower bound $p > n$, that we suspect to be necessary; see Theorem 3 and subsequent Remark 3. We note that Theorem 3 is new already when $y \mapsto \mathfrak{c}(\cdot, y)$ is smooth, and, due to the peculiar growth conditions assumed on $\mathtt{h}(\cdot)$ with respect to $x$, even when $p = q$ (uniformly elliptic case; see Remark 2 below).

Our techniques apply to general nonuniformly elliptic equations in divergence forms of the type considered in [30, 55, 59, 61, 62] in the uniformly elliptic case. In order to present the main ideas and to keep presentation at a reasonable length, we confine ourselves to equations of the type

$$-\operatorname{div} A(x, Du) = 0\,, \tag{1.10}$$

but not necessarily stemming from variational integrals; see Sect. 2.6. Again, more general cases, for instance involving non-zero right-hand sides, can be treated by our means.

## 2 Results

### 2.1 Basic notation

We deal with integral functionals of the form

$$W^{1,1}_{\mathrm{loc}}(\Omega) \ni w \mapsto \mathcal{F}(w,\Omega) := \int_\Omega [\mathbb{F}(x,w,Dw)+\mathtt{h}(x,w)]\,\mathrm{d}x\,. \tag{2.1}$$

Here, as in the rest of the paper, $\Omega \subset \mathbb{R}^n$, $n \geq 2$, denotes a fixed open and bounded subset, $\mathbb{F}\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, \infty)$ and $\mathtt{h}\colon \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory regular functions, and $\mathtt{h}(\cdot)$ is such that $\mathtt{h}(\cdot, w) \in L^1_{\mathrm{loc}}(\Omega)$, whenever $w \in W^{1,1}_{\mathrm{loc}}(\Omega)$. Under such premises, we adopt the following

**Definition 1** A function $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a *(local) minimizer* of the functional $\mathcal{F}$ in (2.1) if, for every ball $B \Subset \Omega$, we have $\mathbb{F}(\cdot, u, Du) \in L^1(B)$ and $\mathcal{F}(u, B) \leq \mathcal{F}(w, B)$ holds for every competitor $w \in u + W^{1,1}_0(B)$.

For the rest of the paper, we denote by $c$, $\chi_1$, $\chi_2$ general constants such that $c, \chi_1, \chi_2 \geq 1$. Different occurrences from line to line will be still denoted using the same letters. Special occurrences of $c$ will be denoted by $c_*$, $\tilde{c}$ or likewise. Relevant dependencies on parameters will be as usual emphasized by putting them in parentheses; for instance $c \equiv c(n, p, q)$ means that $c$ depends on $n, p, q$. Next, we fix a set of real parameters denoted by $\mathtt{data} \equiv (n, p, q, \alpha, \nu, L)$, where $n \geq 2$ is an integer, $1 < p \leq q$, $0 < \nu \leq L$, and $\alpha \in (0, 1]$, and also set $\mathtt{data}_{\mathrm{e}} \equiv (n, p, q, \alpha)$. With $\nu$, $L$ being fixed, we denote by $\tilde{\nu} \equiv \tilde{\nu}(n, p, \nu)$ and $\tilde{L} \equiv \tilde{L}(n, q, L)$ two quantities such that $0 < \tilde{\nu} \leq \nu \leq L \leq \tilde{L}$. While $\nu$, $L$ are fixed here, the exact value of the quantities $\tilde{\nu}$, $\tilde{L}$ might vary in different occurrences, but still keeping the dependence on the constants specified above. For this reason, dependence on $\tilde{\nu}$, $\tilde{L}$ will be often incorporated in the dependence on $\mathtt{data}$ or on $\nu$, $L$. Unless otherwise specified, $\mu$ denotes a fixed constant such that $\mu \in [0, 1]$; we also denote

$$H_s(z) := |z|^2 + s^2 \ \text{ for } z \in \mathbb{R}^n, \quad \mu_s := \mu + s \ \text{ and } s \geq 0. \tag{2.2}$$

Further notation can be found in Sect. 3.1.

### 2.2 Functionals without the Euler-Lagrange equation

Here we concentrate on nondifferentiable functionals of the form

$$w \mapsto \mathcal{G}(w, \Omega) := \int_\Omega [F(Dw) + \mathtt{g}(x, w, Dw) + \mathtt{h}(x, w)] \, \mathrm{d}x \, . \tag{2.3}$$

The integrand $F : \mathbb{R}^n \to [0, \infty)$ satisfies the growth and (nonuniform) ellipticity conditions

$$\begin{cases} F(\cdot) \in C^1(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \{0_{\mathbb{R}^n}\}) \\ \nu[H_\mu(z)]^{p/2} \leq F(z) \leq L[H_\mu(z)]^{q/2} + L[H_\mu(z)]^{p/2} \\ \nu[H_\mu(z)]^{(p-2)/2}|\xi|^2 \leq \partial_{zz} F(z)\xi \cdot \xi \\ |\partial_{zz} F(z)| \leq L[H_\mu(z)]^{(q-2)/2} + L[H_\mu(z)]^{(p-2)/2}, \end{cases} \tag{2.4}$$

for all $z, \xi \in \mathbb{R}^n$, $|z| \neq 0$. The function $\mathrm{g}\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, \infty)$ satisfies

$$\begin{cases} z \mapsto \mathrm{g}(x, y, z) \text{ is convex and of class } C^1(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \{0_{\mathbb{R}^n}\}) \\ \mathrm{g}(x, y, z) + H_\mu(z)|\partial_{zz}\mathrm{g}(x, y, z)| \leq L[H_\mu(z)]^{p/2} \\ |\mathrm{g}(x_1, y_1, z) - \mathrm{g}(x_2, y_2, z)| \\ \quad \leq L\,(|x_1 - x_2|^\alpha + |y_1 - y_2|^\alpha)\,(|z|^2 + 1)^{\gamma/2} \\ \alpha + \gamma < p, \quad \gamma \geq 0 \end{cases} \tag{2.5}$$

for all $x, x_1, x_2 \in \Omega$, $y, y_1, y_2 \in \mathbb{R}$, $z \in \mathbb{R}^n$, $|z| \neq 0$ .[3] The function $\mathrm{g}(\cdot)$ is therefore only Hölder continuous with respect to $(x, y)$; a typical example can be $\mathrm{g}(x, y, z) = \mathfrak{c}(x, y)[H_\mu(z)]^{p/2}[H_1(z)]^{(\gamma-p)/2}$, where $\mathfrak{c}(\cdot)$ is as in (1.9). The function $\mathtt{h}\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is assumed to be Carathéodory regular and to satisfy conditions involving Lorentz spaces (see Sect. 4)

$$\begin{cases} |\mathtt{h}(x, y_1) - \mathtt{h}(x, y_2)| \leq f(x)|y_1 - y_2|^\alpha \\ f \in L(n/\alpha, 1/2)(\Omega) \\ \mathtt{h}(\cdot, 0) \in L^1(\Omega) \end{cases} \tag{2.6}$$

for a.e. $x \in \Omega$ and every $y_1, y_2 \in \mathbb{R}$. This time $\mathtt{h}(\cdot)$ is only measurable with respect to the $x$-variable. An example can be $\mathtt{h}(x, y) = \mathtt{f}(x)h(y)$, where $h(\cdot)$ is Hölder continuous and $\mathtt{f} \in L(n/\alpha, 1/2)(\Omega)$.

**Theorem 1** *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a minimizer* [4] *of the functional $\mathcal{G}$ in (2.3), under assumptions (2.4)-(2.6) and*

$$\frac{q}{p} \leq 1 + \frac{1}{5}\left(1 - \frac{\alpha + \gamma}{p}\right)\frac{\alpha}{n}\,. \tag{2.7}$$

*Then $Du \in L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. Moreover,*

$$\begin{aligned} &\|Du\|_{L^\infty(B_t)} \\ &\quad \leq \frac{c}{(r-t)^{\chi_1}}\left[\mathcal{G}(u, B_r) + \|\mathtt{h}(\cdot, u)\|_{L^1(B_r)} + \|f\|_{n/\alpha, 1/2; B_r} + 1\right]^{\chi_2} \end{aligned} \tag{2.8}$$

*holds whenever $B_t \Subset B_r \Subset \Omega$ are concentric balls with $r \leq 1$, where $c \equiv c(\mathtt{data}, \gamma)$ and $\chi_1, \chi_2 \equiv \chi_1, \chi_2(\mathtt{data}_{\mathrm{e}}, \gamma)$. If $f \in L^{\mathfrak{q}}_{\mathrm{loc}}(\Omega)$ for some $\mathfrak{q} > n/\alpha$, then $Du$ is locally Hölder continuous in $\Omega$.*

Note that Theorem 1 covers the model functional in (1.1) by simply taking $\mathrm{g}(\cdot) \equiv 0$. In this case the bound on $q/p$ becomes (2.7) with $\gamma = 0$ and condition $\alpha + \gamma < p$ becomes immaterial; see also Remark 8.

[3] The function $\mathrm{g}(\cdot)$ is continuous. This follows from (2.5) and Lemma 3.4.

[4] Assumptions (2.6) imply that $\mathtt{h}(\cdot, w) \in L^1_{\mathrm{loc}}(\Omega)$, whenever $w \in W^{1,1}_{\mathrm{loc}}(\Omega)$. This follows using Sobolev embedding theorem and (2.6) via (6.37) below. This is in accordance to Definition 1. This will always be the case for the rest of this paper, also when dealing with different functionals. As a consequence, requiring that $F(Dw), \mathrm{g}(\cdot, u, Dw) \in L^1_{\mathrm{loc}}(\Omega)$ implies that $\mathcal{G}(w, B)$ is finite for every ball $B \Subset \Omega$. It follows from $(2.4)_1$ that any minimizer automatically belongs to $W^{1,p}_{\mathrm{loc}}(\Omega)$.

***Remark 1*** (Gap bounds) Gap bounds of the form $q/p < 1 + \mathrm{o}(n)$ for $\mathrm{o}(n) \approx 1/n$, as in (2.7), are known to be necessary already for boundedness of minimizers, as shown in [65, 66], and in the case $w \mapsto \int F(Dw)\,\mathrm{d}x$. Already in such a plain situation, the optimal bound on $q/p$ implying $Du \in L^{\infty}_{\mathrm{loc}}$ remains unknown. See [4–6, 8, 19, 42, 73] for recent work in this direction. Counterexamples in [24], involving nonhomogeneous integrands show that the condition

$$\frac{q}{p} \leq 1 + \frac{\alpha}{n} \tag{2.9}$$

is necessary already for continuity of minima. This is crucial in our setting. Specifically, it implies that, when departing from standard ellipticity conditions, Schauder estimates do not hold in general. A subtle balance between regularity of $x \mapsto F(x, \cdot)$ and ellipticity of $z \mapsto F(\cdot, z)$ is needed. The bound in (2.7) reflects this fact and exhibits the same sharp asymptotic with respect to $\alpha/n$ of (2.9). A more delicate interaction occurs with the regularity of $y \mapsto \mathrm{g}(\cdot, y, \cdot)$, and reflects in conditions $\alpha + \gamma < p$ and (2.7). Indeed, assumptions as $\alpha + \gamma < p$ are bound to quantify the direct interaction between more irregular coefficients, as those yield by the presence of $v$ in the integrand, and gradient terms. Such interactions have already been considered in the literature, see for instance [29, 51, 52] and related references. See Remark 3 for a similar interaction and Remark 7 for the technical role assumption $\alpha + \gamma < p$ plays in the proof of Theorem 1. The bound in (2.7) can be further (slightly) improved by introducing a correction function $\kappa_1(\cdot)$. For this see Proposition 6.1 and subsequent Remark 8. Anyway this does not change the asymptotic of (2.7) with respect to $\alpha/n$ and $p - (\alpha + \gamma)$, that is what we are mainly interested in at this stage.

***Remark 2*** (New borderline conditions) When $p = q$ (uniformly elliptic case), the assumptions of Theorem 1 are standard [28–31, 51, 61, 62], but the one on $f$, which is usually taken in $L^{\infty}$. When $p \neq q$ assuming $f \in L^{\infty}$ would not improve the gap bound in (2.7) according to our techniques. Passing from $L^{\infty}$ to Lebesgue spaces, and eventually to Lorentz, is an automatic side benefit of our approach. By scaling arguments, we do not expect Hölder continuity of $Du$ without assuming that $f \in L^{\mathfrak{q}}(\Omega)$ for $\mathfrak{q} > n/\alpha$, and do not expect that $Du \in L^{\infty}$ when $f \in L^{n/\alpha}$. The new Lorentz condition (2.6) connects Theorem 1 to a large literature devoted to find optimal conditions on data $f$ implying regularity. For differentiable, uniformly elliptic functionals like

$$w \mapsto \int_{\Omega} [F(Dw) + fw]\,\mathrm{d}x, \tag{2.10}$$

and equations as $\operatorname{div} a(Du) = f$, the $L(n, 1)$-regularity of $f$ implies that minima and solutions are locally Lipschitz; $L^n$ is not sufficient. This can be considered as a nonlinear version of a classical result of Stein [82] on solutions to $\Delta u = f$. See [54] for local estimates, and [16, 17] for global ones. The same holds when assuming that $F(\cdot)$ is nonuniformly elliptic in the sense of (1.7) and (2.4), as in [3, 19, 21]. The remarkable fact is that the $L(n, 1)$-condition is independent of the integrand/operator considered, exactly as the condition $f \in L(n/\alpha, 1/2)$ considered here for (2.3). Anyway, this can be still improved in certain situations, see Theorem 5 below. As far as we know, due to the presence of the Lorentz condition, Theorem 1 is new already when $p = q$.

### 2.3 Nonuniformly elliptic Schauder and more nondifferentiable functionals

**Theorem 2** *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a minimizer of the functional $\mathcal{S}_{\mathrm{x}}$ in (1.8), under assumptions (2.4); in particular, $\mathfrak{c}(\cdot) \in C^{0,\alpha}(\Omega)$. If*

$$\frac{q}{p} \le 1 + \frac{\alpha^2}{5n^2}\,, \tag{2.11}$$

*then $Du$ is locally Hölder continuous in $\Omega$. Moreover*

$$\|Du\|_{L^\infty(B_t)} \le \frac{c}{(r-t)^{\chi_1}}\,[\mathcal{S}_{\mathrm{x}}(u, B_r) + 1]^{\chi_2} \tag{2.12}$$

*holds whenever $B_t \Subset B_r \Subset \Omega$ are concentric balls with $r \le 1$, where $c \equiv c(\mathtt{data})$ and $\chi_1, \chi_2 \equiv \chi_1, \chi_2(\mathtt{data}_{\mathrm{e}})$.*

**Corollary 1** (Hopf-Schauder-Caccioppoli, reloaded - I) *In the setting of Theorem 2, assume also that $p \ge 2$, $\mu > 0$, and that $\partial_{zz} F(\cdot)$ is continuous. Then $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$.*

In the linear case $F(Dw) \equiv |Dw|^2$, Corollary 1 is nothing but the content of classical (interior) Schauder estimates, where local $C^{0,\alpha}$-regularity of coefficients sharply reflects in local $C^{1,\alpha}$-regularity of solutions. Assuming the non-degeneracy condition $\mu > 0$ is necessary, as otherwise shown by counterexamples occurring in the $p$-Laplacean setting [53, 57, 88]. In the uniformly elliptic case $p = q = 2$, Corollary 1 is a classical result of Giaquinta & Giusti [29, 30]. In the case $p < 2$ we can still get results; for this see Remark 12 at the very end of this paper.

**Theorem 3** *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a minimizer of the functional $\mathcal{S}$ in (1.9), under assumptions (2.4), with $\mathtt{h}(\cdot)$ as in (2.6) and $p > n$. If*

$$\frac{q}{p} \le 1 + \frac{1}{5}\left(1 - \frac{n}{p}\right)\frac{\alpha^2}{n^2}, \tag{2.13}$$

*then $Du \in L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. Moreover,*

$$\begin{aligned} &\|Du\|_{L^\infty(B_t)} \\ &\quad \le \frac{c}{(r-t)^{\chi_1}}\left[\mathcal{S}(u, B_r) + \|\mathtt{h}(\cdot, u)\|_{L^1(B_r)} + \|f\|_{n/\alpha,1/2;B_r} + 1\right]^{\chi_2} \end{aligned} \tag{2.14}$$

*holds with the same notation relative to (2.12). If $f \in L^{\mathfrak{q}}_{\mathrm{loc}}(\Omega)$ for some $\mathfrak{q} > n/\alpha$, then $Du$ is locally Hölder continuous in $\Omega$.*

***Remark 3*** In Theorem 3, assuming $p > n$ ensures, via Sobolev embedding, some degree of Hölder continuity of $x \mapsto \mathfrak{c}(x, u(x))$, and rebalances the otherwise measurable interaction between coefficients and gradient terms. When $p = q$, this is not necessary since De Giorgi-Nash-Moser theory ensures that minimizers are a priori Hölder continuous, also when $x \mapsto \mathfrak{c}(x, \cdot)$ is measurable. This is in general false when $p \neq q$. The situation shares similarities with the uniformly elliptic vectorial theory, where De Giorgi type results are not available, and indeed singularities occur for minimizers no matter dependence on $u$-coefficients is smooth. In this respect, we note that some counterexamples of irregular minimizers in the scalar, nonuniformly elliptic case [24] resemble those occurring in the uniformly elliptic vectorial one [35]. By such a potential analogy, we would tend to believe that the assuming $p > n$ is unavoidable. For the same reasons (2.13) naturally connects to (2.7) as discussed in Remark 1.

**Corollary 2** *In the setting of Theorem 3, assume also that* $p \geq 2$, $\mu > 0$, *and that* $\partial_{zz} F(\cdot)$ *is continuous; finally, assume that* $f \in L^{\infty}_{\mathrm{loc}}(\Omega)$. *Then* $u \in C^{1,\alpha/2}_{\mathrm{loc}}(\Omega)$.

In comparison to Corollary 1, Corollary 2 exhibits a loss in the Hölder exponent of $Du$. This is typical when dealing with nondifferentiable functionals. It is not a technical fact, as $C^{1,\alpha}$-regularity cannot be reached in general, as shown in [72]. In the uniformly elliptic case $p = q = 2$, Corollary 2 is another classical result of Giaquinta & Giusti [29–31].

### 2.4 General functionals, relaxation and the Lavrentiev phenomenon

In order to deal with cases more general than (1.8), that is with integrals of the type

$$w \mapsto \mathcal{F}_{\mathrm{x}}(w, \Omega) := \int_{\Omega} F(x, Dw)\,\mathrm{d}x\,, \tag{2.15}$$

we need to recast a few basic facts concerning relaxed functionals and Lavrentiev phenomenon. Here, we assume that the integrand $F\colon \Omega \times \mathbb{R}^n \to [0,\infty)$ is Carathéodory regular and such that

$$\begin{cases} z \mapsto F(x,z) \text{ satisfies (2.4) uniformly with respect to } x \in \Omega \\ |\partial_z F(x_1,z) - \partial_z F(x_2,z)| \\ \quad \leq L|x_1-x_2|^{\alpha}([H_{\mu}(z)]^{(q-1)/2} + [H_{\mu}(z)]^{(p-1)/2}) \end{cases} \tag{2.16}$$

for every $x_1, x_2 \in \Omega$ and $z \in \mathbb{R}^n$ .[5] In this case, a natural obstruction to regularity of minimizers is the possible occurrence of the Lavrentiev phenomenon. For instance, it might happen that

$$\inf_{w \in u_0 + W^{1,p}_0(B)} \mathcal{F}_{\mathrm{x}}(w, B) < \inf_{w \in u_0 + W^{1,p}_0(B) \cap W^{1,q}(B)} \mathcal{F}_{\mathrm{x}}(w, B) \tag{2.17}$$

[5] $\partial_z F(\cdot)$ is continuous. This follows from $(2.16)_2$ and the upper bound on $\partial_{zz} F(\cdot)$ in $(2.4)_4$ via $(2.16)_1$. In fact $\partial_z F(\cdot)$ is uniformly continuous on $\Omega \times \mathcal{B}_M$, for every $M < \infty$. In fact, there is no loss of generality is assuming that $F(\cdot)$ is continuous. For this, it is sufficient to observe that we can assume $F(x, 0_{\mathbb{R}^n}) = 0$ (eventually passing to the new integrand $(x,z) \mapsto F(x,z) - F(x, 0_{\mathbb{R}^n})$). Then we can use (2.16) in combination with Lemma 3.4.

even when $u_0$ is a Lipschitz regular function and $B \Subset \Omega$ is a ball [24]. Examples of Lavrentiev phenomenon related to our setting, were given by Zhikov [90–92]; see also [24]. In the case energy gaps as (2.17) occur, one is led to consider the so-called relaxed functional [7, 25, 26, 63–65, 76][6]

$$\overline{\mathcal{F}_{\mathrm{x}}}(w,U) := \inf_{\{w_k\}\subset W^{1,q}(U)} \left\{ \liminf_k \mathcal{F}_{\mathrm{x}}(w_k,U) : \ w_k \rightharpoonup w \text{ in } W^{1,p}(U) \right\} \tag{2.18}$$

for every $w \in W^{1,1}(U)$ and every open subset $U \subseteq \Omega$. Accordingly, the Lavrentiev gap functional is defined by

$$\mathcal{L}_{\mathcal{F}_{\mathrm{x}}}(w,U) := \overline{\mathcal{F}_{\mathrm{x}}}(w,U) - \mathcal{F}_{\mathrm{x}}(w,U) \tag{2.19}$$

for every $w \in W^{1,1}(U)$ such that $\mathcal{F}_{\mathrm{x}}(w,U) < \infty$; we set $\mathcal{L}_{\mathcal{F}_{\mathrm{x}}}(w,U) = 0$ otherwise. The functional $\mathcal{L}_{\mathcal{F}_{\mathrm{x}}}(\cdot,U)$ provides a possible way to quantify phenomena like (2.17). Note that by $W^{1,p}$-weak lower semicontinuity of $\mathcal{F}_{\mathrm{x}}(\cdot,U)$,[7] we have $\mathcal{F}_{\mathrm{x}}(\cdot,U) \leq \overline{\mathcal{F}_{\mathrm{x}}}(\cdot,U)$ so that $\mathcal{L}_{\mathcal{F}_{\mathrm{x}}}(\cdot,U) \geq 0$. It trivially follows that $w \in W^{1,p}(U)$ whenever $\overline{\mathcal{F}_{\mathrm{x}}}(w,U)$ is finite.

**Definition 2** A minimizer $u \in W^{1,p}(\Omega)$ of $\overline{\mathcal{F}_{\mathrm{x}}}(\cdot,\Omega)$ is a function such that $\overline{\mathcal{F}_{\mathrm{x}}}(u,\Omega)$ is finite and $\overline{\mathcal{F}_{\mathrm{x}}}(u,\Omega) \leq \overline{\mathcal{F}_{\mathrm{x}}}(w,\Omega)$ holds whenever $w \in u + W^{1,1}_0(\Omega)$.

**Theorem 4** *Let $u \in W^{1,p}(\Omega)$ be a minimizer of the functional $\overline{\mathcal{F}_{\mathrm{x}}}(\cdot,\Omega)$, where $\Omega$ is a Lipschitz regular domain and $\mathcal{F}_{\mathrm{x}}$ is defined in (2.15). Assume (2.11) and (2.16). Then $Du$ is locally Hölder continuous in $\Omega$. Moreover*

$$\|Du\|_{L^\infty(\Omega_0)} \leq \frac{c}{[\operatorname{dist}(\Omega_0,\partial\Omega)]^{\chi_1}} \left[\overline{\mathcal{F}_{\mathrm{x}}}(u,\Omega)+1\right]^{\chi_2} \tag{2.20}$$

*holds whenever $\Omega_0 \Subset \Omega$ is an open subset, where $c$, $\chi_1$, $\chi_2$ are as in Theorem 2.*

If $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a minimizer of the original functional $\mathcal{F}_{\mathrm{x}}$ in (2.15) such that $\mathcal{L}_{\mathcal{F}_{\mathrm{x}}}(u,B) \equiv 0$ for every ball $B \Subset \Omega$, then

$$\overline{\mathcal{F}_{\mathrm{x}}}(u,B) = \mathcal{F}_{\mathrm{x}}(u,B) \leq \mathcal{F}_{\mathrm{x}}(w,B) \leq \overline{\mathcal{F}_{\mathrm{x}}}(w,B) \tag{2.21}$$

holds whenever $w \in u + W^{1,1}_0(B)$. Therefore $u$ is also a minimizer of $w \mapsto \overline{\mathcal{F}_{\mathrm{x}}}(w,B)$, for every ball $B \Subset \Omega$. Then from Theorem 4 it follows

**Corollary 3** *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a minimizer of the functional $\mathcal{F}_{\mathrm{x}}$ in (2.15), under assumptions (2.11) and (2.16). Assume that $\mathcal{L}_{\mathcal{F}_{\mathrm{x}}}(u,B) = 0$ for every ball $B$. Then*

[6]The idea of considering this type of lower semicontinuous envelope goes back to Lebesgue, Caccioppoli, Serrin and De Giorgi. In the nonuniformly elliptic setting, it appears for the first time in the work of Marcellini [63, 64].

[7]Lower semicontinuity, when $z \mapsto F(\cdot,z)$ is convex, follows by results of De Giorgi and Ioffe, see [34, Theorem 4.5].

*Du is locally Hölder continuous in* $\Omega$. *Moreover,*

$$\|Du\|_{L^\infty(B_t)} \le \frac{c}{(r-t)^{\chi_1}}\,[\mathcal{F}_{\mathrm{x}}(u,B_r)+1]^{\chi_2} \tag{2.22}$$

*holds with the same notation relative to* (2.12).

The Lavrentiev gap (2.19) vanishes in several common situations. For instance, if there exists a convex function $G\colon \mathbb{R}^n \to [0,\infty)$ such that $G(z) \lesssim F(x,z) \lesssim G(z)+1$, then $\mathcal{L}_{\mathcal{F}_{\mathrm{x}}}(\cdot,B) \equiv 0$ holds for every ball $B \Subset \Omega$. This is the case of Theorem 2, that in fact follows from Corollary 3. See [24, 90, 92] for cases where the Lavrentiev gap is zero.

### 2.5 More Lorentz conditions

In some cases, the Lorentz condition on data $(2.6)_2$ can still be improved in

$$f \in L(n/\alpha, \mathtt{l})(\Omega), \quad \text{where } \ \mathtt{l} := \min\left\{\frac{p}{2(p-\alpha)}, \frac{1}{2-\alpha}\right\}. \tag{2.23}$$

**Theorem 5** *Let* $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ *be a minimizer of the functional* $\mathcal{G}$ *in* (2.3), *under assumptions* (2.4)-(2.6), *and replace* $(2.6)_2$ *by the weaker* (2.23). *If*

$$\frac{q}{p} \le 1 + \frac{1}{5}\left(1 - \frac{\alpha+\gamma}{p}\right)\min\left\{\frac{\alpha}{n}, \frac{2(p-\alpha)}{p(2-\alpha)}\right\}, \tag{2.24}$$

*then* $Du \in L^\infty_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$ *and moreover*

$$\begin{aligned}\|Du\|_{L^\infty(B_t)} \\ \le \frac{c}{(r-t)^{\chi_1}}\left[\mathcal{G}(u,B_r) + \|\mathrm{h}(\cdot,u)\|_{L^1(B_r)} + \|f\|_{n/\alpha,\mathtt{l};B_r} + 1\right]^{\chi_2}\end{aligned} \tag{2.25}$$

*holds whenever* $B_t \Subset B_r \Subset \Omega$ *are concentric balls with* $r \le 1$, *where* $c \equiv c(\mathtt{data},\gamma)$ *and* $\chi_1, \chi_2 \equiv \chi_1, \chi_2(\mathtt{data_e}, \gamma)$.

Just note that, when $p \ge 2n\alpha/[2n - 2\alpha + \alpha^2]$, the upper bounds in (2.7) and (2.24) coincide; in particular, this happens when $p \ge 2$. As a consequence of Theorem 5 with $\alpha = 1$, we get the following result, that completes the ones in [3], where the $L(n,1)$-condition on data was proved to imply local Lipschitz regularity of minima for $n \ge 3$. Similar technical restrictions also occur elsewhere [16, 17].

**Corollary 4** (Nonlinear Stein Theorem in two dimensions) *Let* $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ *be a minimizer of the functional in* (2.10) *under assumptions* (2.4) *and* (2.24), *with* $n = 2$, $1 < p \le 2$, $\alpha = 1$, $\gamma = 0$. *If* $f \in L(2,1)$, *then u is locally Lipschitz continuous.*

## 2.6 Equations

Here we deal with equations of the type (1.10). The vector field $A\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is of class $C^1(\mathbb{R}^n \setminus \{0_{\mathbb{R}^n}\})$ with respect to gradient variable, and satisfies

$$\begin{cases} |A(x,z)| + [H_\mu(z)]^{1/2}|\partial_z A(x,z)| \\ \quad \leq L[H_\mu(z)]^{(q-1)/2} + L[H_\mu(z)]^{(p-1)/2} \\ \nu[H_\mu(z)]^{(p-2)/2}|\xi|^2 \leq \partial_z A(x,z)\xi\cdot\xi \\ |A(x_1,z) - A(x_2,z)| \\ \quad \leq L|x_1-x_2|^\alpha([H_\mu(z)]^{(q-1)/2} + [H_\mu(z)]^{(p-1)/2}) \end{cases} \tag{2.26}$$

whenever $x_1, x_2 \in \Omega$ and $z, \xi \in \mathbb{R}^n$, $|z| \neq 0$ .[8] In the nonuniformly elliptic setting, passing from minimizers of functionals to weak solutions of general equations raises additional issues. There are essentially two approaches available in the literature. The former prescribes to get a priori estimates for more regular, i.e. $W^{1,q}$-solutions [55, 66, 80]. The latter is to prove, simultaneously, the existence of regular solutions for assigned boundary value problems, say for instance Dirichlet problems [3, 46, 66]. These alternatives are ultimately linked to growth conditions $(2.26)_1$, that imply that the distributional form of (1.10) can be tested only by $W^{1,q}$-regular functions. Therefore an ambiguity arises concerning the space where to initially pick solutions from, and on the very same concept of energy solution. See Remark 4 below. Such an ambiguity does not exist when $p = q$. Of the two approaches mentioned above, we follow the second, and consider the Dirichlet problem

$$\begin{cases} -\operatorname{div} A(x, Du) = 0 & \text{in } \Omega \\ \qquad u \equiv u_0 & \text{on } \partial\Omega\,, \end{cases} \qquad u_0 \in W^{1,\frac{p(q-1)}{p-1}}(\Omega)\,, \tag{2.27}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded and Lipschitz domain.

**Theorem 6** *Assume that the vector field $A(\cdot)$ satisfies (2.26). If*

$$\frac{q}{p} \leq 1 + \frac{p-1}{10p}\frac{\alpha^2}{n^2}\,, \tag{2.28}$$

*then there exists a solution $u \in W^{1,p}(\Omega)$ to the Dirichlet problem (2.27), such that $Du$ is locally Hölder continuous in $\Omega$. Moreover, the estimate*

$$\|Du\|_{L^\infty(\Omega_0)} \leq \frac{c}{[\operatorname{dist}(\Omega_0, \partial\Omega)]^{\chi_1}}\left(\int_\Omega (|Du_0|+1)^{\frac{p(q-1)}{p-1}}\,\mathrm{d}x + 1\right)^{\chi_2} \tag{2.29}$$

*holds whenever $\Omega_0 \Subset \Omega$ is an open subset, where $c \equiv c(\texttt{data})$ and $\chi_1, \chi_2 \equiv \chi_1, \chi_2(\texttt{data}_{\mathrm{e}})$.*

[8] Assumptions (2.26) imply that $A(\cdot)$ is uniformly continuous on $\Omega \times \mathcal{B}_M$, for every $M < \infty$.

**Corollary 5** (Hopf-Schauder-Caccioppoli, reloaded - II) *In the setting of Theorem 6, assume also that* $p \geq 2$, $\mu > 0$, *and that* $\partial_z A(\cdot)$ *is continuous on* $\Omega \times \mathbb{R}^n$. *Then* $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$.

When $p = q = 2$, Corollary 1 is classical in the uniformly elliptic theory [12, 29, 30, 43, 74].

***Remark 4*** Already in the case of the classical $p$-Laplace equation $\operatorname{div}(|Du|^{p-2}Du) = 0$, starting from distributional solutions that only belong to $W^{1,s}$ for $s < p$, prevents higher regularity, and even $W^{1,p}$-regularity. This is a recent, striking achievement of Colombo & Tione [18].

### 2.7 Recent, related cases available in the literature

Schauder type estimates in the nonuniformly elliptic case attracted a lot of attention over decades, and especially in the last years. For functionals with nonstandard polynomial growth of the type in (2.15), and connected equations, recent related results hold under certain special structure assumptions [1, 2, 9–11, 38, 40, 41]. These include for instance the variable exponent case $F(x, Dw) \equiv |Dw|^{p(x)}$, $p(x) > 1$ as in [1], and the double phase case $F(x, Dw) \equiv |Dw|^p + a(x)|Dw|^q$, $a(x) \geq 0$, as in [2]. The common point of all these papers is that the frozen integrand $z \mapsto F(x_0, z)$ is still uniformly elliptic, for every fixed point $x_0 \in \Omega$. This means that (1.2) is satisfied upon taking $F(z) \equiv F(x_0, z)$. On the contrary, in this paper we deal with real, pointwise nonuniform ellipticity, allowing that

$$\sup_{|z|\geq 1} \frac{\text{highest eigenvalue of } \partial_{zz}F(x_0, z)}{\text{lowest eigenvalue of } \partial_{zz}F(x_0, z)} = \infty\,.$$

In (1.8), note that $z \mapsto \mathfrak{c}(x_0)F(z)$ is still nonuniformly elliptic in the sense of (1.6) if so is $F(\cdot)$. We mention that, when applying the techniques of this paper in the known settings mentioned above, we come up with the same sharp results available in the literature [22]. Another approach was described in [23], where the authors considered integrands depending on $|z|$ and with Sobolev-regular coefficients. This means that $x \mapsto F(x, \cdot)$ belongs to $W^{1,d}$, with $d \equiv d(p, q) > n$. In this case Hölder continuity of coefficients follows by Sobolev-Morrey embedding, but, again, differentiability of coefficients must be assumed; see also [21].

## 3 Preliminary facts and notation

### 3.1 Further notation

We denote $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and by $B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$ the open ball with center $x_0$ and radius $r > 0$; we omit denoting the center when it is not necessary, i.e., $B \equiv B_r \equiv B_r(x_0)$; this especially happens when various balls in the same context will share the same center. We also denote

$$\mathcal{B}_r \equiv B_r(0) := \{x \in \mathbb{R}^n : |x| < r\} \tag{3.1}$$

and often abbreviate $0 \equiv 0_{\mathbb{R}^n}$. Finally, with $B$ being a given ball with radius $r$ and $\gamma$ being a positive number, we denote by $\gamma B$ the concentric ball with radius $\gamma r$ and by $B/\gamma \equiv (1/\gamma)B$. Given a number $s \geq 1$, its Sobolev conjugate exponent is denoted by

$$s^* := \begin{cases} \frac{ns}{n-s} \text{ if } s < n \\ \infty \text{ if } s \geq n\,. \end{cases} \tag{3.2}$$

In denoting several function spaces like $L^s(\Omega)$, $W^{1,s}(\Omega)$, we shall denote the vector valued version by $L^s(\Omega, \mathbb{R}^k)$, $W^{1,p}(\Omega, \mathbb{R}^k)$ in the case the maps considered take values in $\mathbb{R}^k$, $k \in \mathbb{N}$. When clear from the contest, we shall also abbreviate $L^s(\Omega, \mathbb{R}^k)$, $W^{1,s}(\Omega, \mathbb{R}^k) \equiv L^s(\Omega)$, $W^{1,s}(\Omega)$ and so on. With $\mathcal{U} \subset \mathbb{R}^n$ being a measurable subset with bounded positive measure $0 < |\mathcal{U}| < \infty$, and with $g : \mathcal{U} \to \mathbb{R}^k$, $k \geq 1$, being an integrable map, we denote

$$(g)_{\mathcal{U}} \equiv \fint_{\mathcal{U}} g(x)\,\mathrm{d}x := \frac{1}{|\mathcal{U}|}\int_{\mathcal{U}} g(x)\,\mathrm{d}x\,,$$

and also $\|g\|^{\gamma}_{L^{\gamma}(\mathcal{U})} = \int_{\mathcal{U}} |g|^{\gamma}\,\mathrm{d}x$, for every $\gamma \geq 0$. Given a function $h : A \to \mathbb{R}$ we define its (essential) oscillation on $A$

$$\operatorname{osc}(h, A) := \operatorname*{ess\,sup}_{A} h - \operatorname*{ess\,inf}_{A} h\,. \tag{3.3}$$

With $\beta \in (0, 1]$ and $A \subset \mathbb{R}^n$, we use the standard notation

$$[w]_{0,\beta;A} := \sup_{x_1, x_2 \in A, x_1 \neq x_2} \frac{|w(x_1) - w(x_2)|}{|x_1 - x_2|^{\beta}}\,.$$

Given a ball $B \subset \mathbb{R}^n$, we denote by $Q_{\text{inn}} \equiv Q_{\text{inn}}(B)$ and $Q_{\text{out}} \equiv Q_{\text{out}}(B)$ the inner and outer (open) hypercubes of $B$. These are defined as the largest and the smallest hypercubes, with sides parallel to the coordinate axes and concentric to $B$, that are contained in and containing $B$, respectively:

$$Q_{\text{inn}}(B) \subset B \subset Q_{\text{out}}(B)\,. \tag{3.4}$$

If $B$ has radius $r$, then the sidelength of $Q_{\text{inn}}(B)$ is $2r/\sqrt{n}$ while that of $Q_{\text{out}}(B)$ is $2r$.

### 3.2 Fractional Sobolev spaces

Classical fractional Sobolev-Slobodeckij spaces are defined via Gagliardo norms as follows:

**Definition 3** Let $\beta \in (0, 1)$, $s \in [1, \infty)$, $k \in \mathbb{N}$, $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be an open subset. The space $W^{\beta,s}(\Omega, \mathbb{R}^k)$ consists of maps $w : \Omega \to \mathbb{R}^k$ such that

$$\|w\|_{W^{\beta,s}(\Omega)} := \|w\|_{L^s(\Omega)} + \left(\int_{\Omega}\int_{\Omega} \frac{|w(x) - w(y)|^s}{|x - y|^{n+\beta s}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/s}$$

$$=: \|w\|_{L^s(\Omega)} + [w]_{\beta,s;\Omega} < \infty\,. \tag{3.5}$$

The local variant $W^{\beta,s}_{\mathrm{loc}}(\Omega, \mathbb{R}^k)$ is defined by requiring that $w \in W^{\beta,s}_{\mathrm{loc}}(\Omega, \mathbb{R}^k)$ iff $w \in W^{\beta,s}(\tilde{\Omega}, \mathbb{R}^k)$ for every open subset $\tilde{\Omega} \Subset \Omega$.

Given $w\colon \Omega \to \mathbb{R}^k$, $k \geq 1$, an open subset $\Omega \subset \mathbb{R}^n$, and a vector $h \in \mathbb{R}^n$, we denote by $\tau_h\colon L^1(\Omega, \mathbb{R}^k) \to L^1(\Omega_{|h|}, \mathbb{R}^k)$ the standard finite difference operator

$$\tau_h w(x) := w(x+h) - w(x)\,, \tag{3.6}$$

for $x \in \Omega_{|h|}$, where $\Omega_{|h|} := \{x \in \Omega \,:\, \mathrm{dist}(x, \partial\Omega) > |h|\}$. We shall several times use the following elementary properties (with $B_r(x_0) \subset \mathbb{R}^n$ being a fixed ball):

$$\begin{cases} \|\tau_h w\|_{L^s(B_r(x_0))} \leq c(s)\|w\|_{L^s(B_{r+|h|}(x_0))} \\ \qquad \forall\, w \in L^s(B_{r+|h|}(x_0)),\ s \geq 1 \\ \|\tau_h w\|_{L^s(B_r(x_0))} \leq c(n,s)|h|\|Dw\|_{L^s(B_{r+|h|}(x_0))} \\ \qquad \forall\, w \in W^{1,s}(B_{r+|h|}(x_0)),\ s \geq 1\,. \end{cases} \tag{3.7}$$

Finite difference operators can be used to detect maps from fractional Sobolev spaces, as described in the following lemma, that in fact quantifies, locally, the imbedding properties of Nikolski spaces into Sobolev-Slobodeckij spaces $W^{\beta,s}$ (see [20]):

**Lemma 3.1** *Let $B_\varrho \Subset B_r \subset \mathbb{R}^n$ be concentric balls with $r \leq 1$, $w \in L^s(B_r, \mathbb{R}^k)$, $s \geq 1$ and assume that, for $\alpha_* \in (0,1]$, $\mathcal{H} \geq 1$, there holds*

$$\|\tau_h w\|_{L^s(B_\varrho)} \leq \mathcal{H}|h|^{\alpha_*}\,, \tag{3.8}$$

*for every $h \in \mathbb{R}^n$ with $0 < |h| \leq (r-\varrho)/K$, where $K \geq 1$. Then, for $c \equiv c(n,s)$, it holds that*

$$\|w\|_{W^{\beta,s}(B_\varrho)} \leq \frac{c}{(\alpha_*-\beta)^{1/s}}\left(\frac{r-\varrho}{K}\right)^{\alpha_*-\beta}\mathcal{H} + c\left(\frac{K}{r-\varrho}\right)^{n/s+\beta}\|w\|_{L^s(B_\varrho)} \tag{3.9}$$

*for every $\beta < \alpha_*$.*

In the case the domain considered is the ball $\mathcal{B}_{1/2}$ (which is the only one needed here) the fractional Sobolev embedding reads as

$$\|w\|_{L^{\frac{ns}{n-s\beta}}(\mathcal{B}_{1/2})} \leq c\|w\|_{W^{\beta,s}(\mathcal{B}_{1/2})} \tag{3.10}$$

and holds provided $s \geq 1$, $\beta \in (0,1)$ and $s\beta < n$, where $c \equiv c(n,s,\beta)$.

### 3.3 Additional preliminary material

We shall use the vector field $V_\mu : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$V_\mu(z) := (|z|^2 + \mu^2)^{(p-2)/4} z \tag{3.11}$$

where $\mu \in [0, 2]$ and $p > 1$ is defined in Sect. 2.1. Whenever $z_1, z_2 \in \mathbb{R}^n$, there holds that

$$|V_\mu(z_1) - V_\mu(z_2)|^2 \approx_p (|z_1|^2 + |z_2|^2 + \mu^2)^{(p-2)/2} |z_1 - z_2|^2, \tag{3.12}$$

see [37, Lemma 2.1]. As a consequence we find

$$|z_1 - z_2|^p \lesssim_p |V_\mu(z_1) - V_\mu(z_2)|^2 + \mathbb{1}_p |V_\mu(z_1) - V_\mu(z_2)|^p (|z_1| + \mu)^{p(2-p)/2}, \tag{3.13}$$

where

$$\mathbb{1}_p := \begin{cases} 0 \text{ if } p \geq 2 \\ 1 \text{ if } p < 2 \,. \end{cases} \tag{3.14}$$

When $p \geq 2$, (3.13) follows directly from (3.12). Instead, for $1 < p < 2$ inequality in (3.13) follows mimicking the proof of [54, Lemma 2], which is given in the case $\mu = 0$. We also record the following inequality, which is a direct consequence of the Mean Value Theorem:

$$|[H_\mu(z_2)]^{p/2} - [H_\mu(z_1)]^{p/2}| \lesssim_p (|z_1|^2 + |z_2|^2 + \mu^2)^{(p-1)/2} |z_2 - z_1| \,. \tag{3.15}$$

See (2.2) for the definition of $H_\mu(\cdot)$. Combining this last inequality with (3.12) yields

$$|[H_\mu(z_2)]^{p/2} - [H_\mu(z_1)]^{p/2}| \lesssim_p (|z_1|^2 + |z_2|^2 + \mu^2)^{p/4} |V_\mu(z_1) - V_\mu(z_2)| \,. \tag{3.16}$$

The last algebraic result of elementary nature we include is the following:

$$\int_0^1 [H_\mu(z_1 + t(z_2 - z_1))]^{s/2} \,\mathrm{d}t \approx_s (|z_1|^2 + |z_2|^2 + \mu^2)^{s/2} \tag{3.17}$$

that holds whenever $s > -1$, $\mu \in [0, 2]$ and $z_1, z_2 \in \mathbb{R}^n$; see for instance [37]. The above inequalities are useful to prove a few convexity and monotonicity properties concerning functionals and vector fields. These are scattered in the literature, but for the sake of the reader we briefly recall the proofs of some of them. Let us first consider a $C^1$-regular vector field $A_0 : \mathbb{R}^n \to \mathbb{R}^n$ satisfying (2.26) (recast for no $x$-dependence). We have that

$$\frac{1}{c} |V_\mu(z_2) - V_\mu(z_1)|^2 \leq (A_0(z_2) - A_0(z_1)) \cdot (z_2 - z_1) \tag{3.18}$$

holds whenever $z_1, z_2 \in \mathbb{R}^n$, where $c \equiv c(n, p, \nu) \geq 1$. This is a standard monotonicity property that follows by $(2.26)_2$ via the use of (3.12) and (3.17); see for instance [37] or [22, Remark 2]. Next we consider a $C^2$-regular integrand $F_0 : \mathbb{R}^n \mapsto \mathbb{R}$ satisfying (2.4) with $\mu \in (0, 2]$ (this is exactly the setting we are interested in, but, in fact, using mollifiers, everything works under the full assumptions (2.4)). The strict convexity inequality

$$\frac{1}{c}|V_\mu(z_2)-V_\mu(z_1)|^2 \le F_0(z_2)-F_0(z_1)-\partial_z F_0(z_1)\cdot(z_2-z_1) \tag{3.19}$$

holds whenever $z_1, z_2 \in \mathbb{R}^n$, for $c \equiv c(n,p,\nu) \ge 1$. For completeness we give a rapid proof of (3.19). Note that the vector field $\partial_z F_0(\cdot)$ is of the type $A_0(\cdot)$ considered a few lines above and therefore satisfies (3.18). Using this last fact, with $z_t := z_1 + t(z_2 - z_1)$, $0 \le t \le 1$, we have

$$\begin{aligned}
&F_0(z_2)-F_0(z_1)-\partial_z F_0(z_1)\cdot(z_2-z_1)\\
&\quad=\int_0^1 [\partial_z F_0(z_t)-\partial_z F_0(z_1)]\,\mathrm{d}t\cdot(z_2-z_1)\\
&\quad\ge \frac{1}{c}\int_{1/2}^1 t(|z_t|^2+|z_1|^2+\mu^2)^{\frac{p-2}{2}}\,\mathrm{d}t\,|z_2-z_1|^2\,.
\end{aligned}$$

Since $|z_1|+|z_t| \le 2|z_1|+2|z_2|$, in the case $1<p\le 2$ inequality (3.19) immediately follows from the above display and (3.12). In the case $p>2$ just observe that triangle inequality implies $t|z_2| \le (1-t)|z_1|+|z_t|$ so that $|z_1|+|z_t| \approx |z_1|+|z_2|$ for $1/2 \le t \le 1$ and (3.19) follows again using (3.12).

Next, two classical iteration lemmas from Campanato, and Giaquinta & Giusti.

**Lemma 3.2** [28, Lemma 1.1] *Let $h\colon [t,s] \to \mathbb{R}$ be a non-negative and bounded function, and let $a, b, \gamma$ be non-negative numbers. Assume that the inequality*

$$h(\tau_1) \le \frac{h(\tau_2)}{2} + \frac{a}{(\tau_2-\tau_1)^\gamma} + b$$

*holds whenever $t \le \tau_1 < \tau_2 \le s$. Then*

$$h(t) \le c(\gamma)\left[\frac{a}{(s-t)^\gamma}+b\right]$$

*holds too.*

**Lemma 3.3** [29, Lemma 2.2] *Let $h\colon [0,r_0] \to \mathbb{R}$ be a non-negative and non-decreasing function such that the inequality*

$$h(\varrho) \le a\left[\left(\frac{\varrho}{r}\right)^{\beta_*}+\varepsilon\right]h(r)+br^\beta$$

*holds whenever $0 \le \varrho \le r \le r_0$, where $a, b$ are positive constants, and $0<\beta<\beta_*$. There exists $\varepsilon_0 \equiv \varepsilon_0(a,\beta_*,\beta)$ such that, if $\varepsilon \le \varepsilon_0$, then*

$$h(\varrho) \le c\left(\frac{\varrho}{r}\right)^\beta [h(r)+br^\beta]$$

*holds whenever $0 \le \varrho \le r \le r_0$, where $c \equiv c(a,\beta_*,\beta)$.*

Finally, a convexity lemma of Marcellini, that follows as in [65, Lemma 2.1].

**Lemma 3.4** *Let $\mathbb{F}\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0,\infty)$ be a Carathéory function such that $z \mapsto \mathbb{F}(x,y,z)$ is convex for every $(x,y) \in \Omega \times \mathbb{R}$ and that $\mathbb{F}(x,y,z) \le c_*[H_\mu(z)]^{q/2} + c_*[H_\mu(z)]^{p/2}$, for some $c_* \ge 1$. Then*

$$|\partial_z \mathbb{F}(x,y,z)| \le c[H_\mu(z)]^{(q-1)/2} + c[H_\mu(z)]^{(p-1)/2}$$

*holds for every $(x,y,z) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, where $c \equiv c(n,p,q,c_*)$.*

## 4 Nonlinear potentials, Lorentz spaces, and iterations

Let us fix $t,\delta > 0$, $m,\theta \ge 0$, $f \in L^1(B_r(x_0))$ such that $|f|^m \in L^1(B_r(x_0))$ with $B_r(x_0) \subset \mathbb{R}^n$. The following quantity will play a crucial role in this paper:

$$\mathbf{P}^{m,\theta}_{t,\delta}(f;x_0,r) := \int_0^r \varrho^\delta \left( \fint_{B_\varrho(x_0)} |f|^m \,\mathrm{d}x \right)^{\theta/t} \frac{\mathrm{d}\varrho}{\varrho} \,. \tag{4.1}$$

This is a nonlinear potential of Havin-Maz'ya-Wolff type [68], used for instance in [3, 19, 21, 54]. $\mathbf{P}^{m,\theta}_{t,\delta}(f,\cdot)$ can be seen as a nonlinear generalization of the classical Riesz potential $\mathbf{I}_1(f,\cdot)$, that in its so-called truncated form is in turn defined as

$$\begin{aligned}\int_{B_{r/2}(x_0)} \frac{|f(x)|}{|x-x_0|^{n-1}} \,\mathrm{d}x &\lesssim \mathbf{I}_1(f;x_0,r) \\ &:= \int_0^r \frac{1}{\varrho^{n-1}} \int_{B_\varrho(x_0)} |f| \,\mathrm{d}x \,\frac{\mathrm{d}\varrho}{\varrho} \approx \mathbf{P}^{1,1}_{1,1}(f;x_0,r)\,.\end{aligned}$$

The variety of parameters considered in (4.1) makes $\mathbf{P}^{m,\theta}_{t,\delta}$ useful in settings where $\mathbf{I}_1$ cannot be directly employed, as we shall see via its use in this paper. We refer to [3, Sect. 2] for more details and references. Note that

$$\mathbf{P}^{m,\theta}_{t,\delta}(f;x_0,r) = r^\delta/\delta\,, \text{ when } m=0 \text{ or when } |f| \equiv 1 \text{ or when } \theta = 0\,. \tag{4.2}$$

The usual definition of Lorentz space $L(s,\gamma)(\Omega)$, with $s,\gamma \in (0,\infty)$, prescribes that the quantity

$$\begin{aligned}\|f\|_{s,\gamma;\Omega} &= \left( \int_0^\infty \left( f^*(\varrho)\varrho^{1/s} \right)^\gamma \frac{\mathrm{d}\varrho}{\varrho} \right)^{1/\gamma} \\ &= \left( s \int_0^\infty (\lambda^s |\{x \in \Omega : |f(x)| > \lambda\}|)^{\gamma/s} \frac{\mathrm{d}\lambda}{\lambda} \right)^{1/\gamma}\end{aligned}$$

is finite in order to say that the function $f\colon \Omega \to \mathbb{R}$ belongs to $L(s,\gamma)(\Omega)$, see [36, 1.4.6, 1.4.9]; other basic references in this setting are [70, 83]. Here $f^*\colon [0,\infty] \to \mathbb{R}$ denotes the non-increasing rearrangement of $f$, i.e.,

$$f^*(\tau) := \inf\{v > 0 : |\{x \in \Omega : |f(x)| > v\}| \le \tau\}\,.$$

Note that a simple change of variables yields that

$$\||f|^m\|_{s,\gamma;\Omega} = \|f\|^m_{ms,m\gamma;\Omega} \tag{4.3}$$

holds for every choice of $s, \gamma, m > 0$; see [36, 1.4.7]. In Lorentz spaces, the latter index tunes the former in the sense that, when $\Omega \subset \mathbb{R}^n$ has finite measure, it then holds that

$$\begin{cases} L(s_1,\gamma_1)(\Omega) \subset L(s_2,\gamma_2)(\Omega) \text{ for all } 0 < s_2 < s_1 < \infty,\ \gamma_1,\gamma_2 \in (0,\infty] \\ L(s,\gamma_1)(\Omega) \subset L(s,\gamma_2)(\Omega) \text{ for all } s \in (0,\infty),\ 0 < \gamma_1 \le \gamma_2 \le \infty \\ L(s,s)(\Omega) = L^s(\Omega) \text{ for all } s > 0, \end{cases} \tag{4.4}$$

with continuous inclusions. When $s > 1$, it is possible to define another quantity, which is equivalent to $\|f\|_{s,\gamma;\Omega}$, by considering the maximal operator of $f^*$, that is

$$f^{**}(\varrho) := \frac{1}{\varrho}\int_0^\varrho f^*(\tau)\,\mathrm{d}\tau \quad \text{for } \varrho > 0\,. \tag{4.5}$$

It turns out that

$$\|f\|^\gamma_{s,\gamma;\Omega} \le \int_0^\infty \left(f^{**}(\varrho)\varrho^{1/s}\right)^\gamma \,\frac{\mathrm{d}\varrho}{\varrho} \le c\|f\|^\gamma_{s,\gamma;\Omega} \tag{4.6}$$

holds provided $s > 1$ and $\gamma > 0$, where $c \equiv c(\gamma, s)$; see [70, (6.8)]. Nonlinear potentials and Lorentz spaces naturally connect, as for instance shown in the next lemma; see [3, 21, 54] for similar results.

**Lemma 4.1** *Let $n \ge 2$, $t, \delta, \theta > 0$ be numbers such that*

$$\frac{n\theta}{t\delta} > 1\,. \tag{4.7}$$

*Let $B_{\tau_1} \Subset B_{\tau_1+r_0} \subset \mathbb{R}^n$ be two concentric balls with $\tau_1, r_0 \le 1$, and let $f \in L^1(B_{\tau_1+r_0})$ be such that $|f|^m \in L^1(B_{\tau_1+r_0})$, where $m > 0$. Then*

$$\begin{aligned}\|\mathbf{P}^{m,\theta}_{t,\delta}(f;\cdot,r_0)\|_{L^\infty(B_{\tau_1})} &\le \tilde{c}\|f\|^{\frac{m\theta}{t}}_{\frac{mn\theta}{t\delta},\frac{m\theta}{t};B_{\tau_1+r_0}} \\ &\le c(\varepsilon)\tilde{c}\|f\|^{\frac{m\theta}{t}}_{L^{\frac{(1+\varepsilon)mn\theta}{t\delta}}(B_{\tau_1+r_0})}\end{aligned} \tag{4.8}$$

*holds for every $\varepsilon > 0$, with $\tilde{c} \equiv \tilde{c}(n,t,\delta,\theta)$.*

***Proof*** Basic properties of rearrangements give

$$\begin{aligned}\varrho^{t\delta/\theta} \fint_{B_\varrho(x_0)} |f|^m\,\mathrm{d}x \;&\le\; \frac{\varrho^{t\delta/\theta}}{|\mathcal{B}_1|\varrho^n}\int_0^{|\mathcal{B}_1|\varrho^n} (|f|^m)^*(\tau)\,\mathrm{d}\tau \\ &\overset{(4.5)}{\le} \varrho^{t\delta/\theta}(|f|^m)^{**}(|\mathcal{B}_1|\varrho^n)\end{aligned} \tag{4.9}$$

whenever $x_0 \in B_{\tau_1}$ and $\varrho \le r_0$. We further estimate

$$\begin{aligned}
\mathbf{P}^{m,\theta}_{t,\delta}(f;x_0,r_0) &\overset{(4.9)}{\le} \int_0^{r_0} [\varrho^{t\delta/\theta}(|f|^m)^{**}(|\mathcal{B}_1|\varrho^n)]^{\theta/t}\ \frac{\mathrm{d}\varrho}{\varrho}\\
&\le\ c\int_0^{\infty} [\varrho^{\frac{t\delta}{n\theta}}(|f|^m)^{**}(\varrho)]^{\theta/t}\ \frac{\mathrm{d}\varrho}{\varrho}\\
&\overset{(4.6)}{\le} c\||f|^m\|^{\theta/t}_{\frac{n\theta}{t\delta},\frac{\theta}{t};B_{\tau_1+r_0}(x_0)}\\
&\overset{(4.3)}{=} c(n,t,\delta,\theta)\|f\|^{\frac{m\theta}{t}}_{\frac{mn\theta}{t\delta},\frac{m\theta}{t};B_{\tau_1+r_0}(x_0)}
\end{aligned}$$

so that the first inequality in (4.8) follows (recall (4.4) and see [83] for the second). □

Next lemma extends [3, Lemma 3.1]. It features a pointwise version of classical De Giorgi's iteration that finds its origins in the work in Nonlinear Potential Theory of Kilpeläinen & Malý [50]. We report the full proof as the crucial point here is the explicit dependence on the constants.

**Lemma 4.2** *Let $B_{r_0}(x_0) \subset \mathbb{R}^n$ be a ball, $n \ge 2$, and consider functions $f_i$, $|f_i|^{m_i} \in L^1(B_{2r_0}(x_0))$, and constants $\chi > 1$, $t \ge 1$, $\delta_i, m_i, \theta_i > 0$ and $c_*, M_0 > 0$, $\kappa_0, M_i \ge 0$, for $i \in \{1,\dots,h\}$, $h \in \mathbb{N}$. Assume that $v \in L^t(B_{r_0}(x_0))$ is such that for all $\kappa \ge \kappa_0$, and for every ball $B_\varrho(x_0) \subset B_{r_0}(x_0)$, the inequality*

$$\begin{aligned}
\left(\fint_{B_{\varrho/2}(x_0)} (v-\kappa)_+^{t\chi}\,\mathrm{d}x\right)^{1/\chi} &\le c_* M_0^t \fint_{B_\varrho(x_0)} (v-\kappa)_+^t\,\mathrm{d}x\\
&\quad + c_* \sum_{i=1}^h M_i^t \varrho^{t\delta_i}\left(\fint_{B_\varrho(x_0)} |f_i|^{m_i}\,\mathrm{d}x\right)^{\theta_i} \qquad (4.10)
\end{aligned}$$

*holds, where we denote, as usual, $(v-\kappa)_+ := \max\{v-\kappa, 0\}$. If $x_0$ is a Lebesgue point of $v$ in the sense that*

$$\lim_{\varrho\to 0}(v)_{B_\varrho(x_0)} = v(x_0)\,, \qquad (4.11)$$

*then*

$$\begin{aligned}
v(x_0) &\le \kappa_0 + cM_0^{\frac{\chi}{\chi-1}}\left(\fint_{B_{r_0}(x_0)} (v-\kappa_0)_+^t\,\mathrm{d}x\right)^{1/t}\\
&\quad + cM_0^{\frac{1}{\chi-1}}\sum_{i=1}^h M_i \mathbf{P}^{m_i,\theta_i}_{t,\delta_i}(f_i;x_0,2r_0) \qquad (4.12)
\end{aligned}$$

*holds with $c \equiv c(n,\chi,\delta_i,\theta_i,c_*)$.*

***Proof*** We can assume that the right-hand side in (4.12) is finite, otherwise, there is nothing to prove. In the following all the balls will be centred at $x_0$. We define radii $\{\varrho_j\}_{j\in\mathbb{N}_0}$, where $\varrho_j := r_0/2^j$ integer $j \geq 0$ (so that $\varrho_0 = r_0$), and, for every $i \leq h$, numbers $\{W_{i,j}\}_{j\in\mathbb{N}_0}$, via

$$W_{i,j} := \varrho_j^{\delta_i}\left(\fint_{B_{\varrho_j}} |f_i|^{m_i}\,\mathrm{d}x\right)^{\theta_i/t} . \tag{4.13}$$

The next two sequences of numbers $\{\kappa_j\}_{j\in\mathbb{N}_0}$ and $\{V_j\}_{j\in\mathbb{N}_0}$ are defined inductively, with $\kappa_0$ given by the statement. With $\kappa_j$ having been defined, we set $V_j$ and then $\kappa_{j+1}$ as follows:

$$V_j := \left(\fint_{B_{\varrho_j}} (v-\kappa_j)_+^t\,\mathrm{d}x\right)^{1/t}, \qquad \kappa_{j+1} := \kappa_j + V_j/\tau, \tag{4.14}$$

where $\tau > 0$ is going to determined in due course of the proof as a function of $n$, $c_*$, $\chi$, $t$; see (4.19) below. It follows that $\{\kappa_j\}_j$ is non-decreasing, and $V_{j+1} \leq 2^{n/t}V_j$; therefore

$$\kappa_{j+2} - \kappa_{j+1} \leq 2^{n/t}(\kappa_{j+1} - \kappa_j) \tag{4.15}$$

holds for every $j \geq 0$. Using (4.10) and the definitions in (4.13), yields, for every $j \geq 0$

$$\left(\fint_{B_{\varrho_{j+1}}} (v-\kappa_j)_+^{t\chi}\,\mathrm{d}x\right)^{1/\chi} \leq c_* M_0^t V_j^t + c_* \sum_{i=1}^{h} M_i^t W_{i,j}^t .$$

By $\kappa_{j+1} \geq \kappa_j$ for every $j$ and (4.14), we estimate

$$\begin{aligned}
&(\kappa_{j+1}-\kappa_j)^{(\chi-1)/\chi} V_{j+1}^{1/\chi} \\
&\quad = (\kappa_{j+1}-\kappa_j)^{(\chi-1)/\chi}\left(\fint_{B_{\varrho_{j+1}}} (v-\kappa_{j+1})_+^t\,\mathrm{d}x\right)^{\frac{1}{t\chi}} \\
&\quad \leq \left(\fint_{B_{\varrho_{j+1}}} (v-\kappa_j)_+^{t\chi-t}(v-\kappa_{j+1})_+^t\,\mathrm{d}x\right)^{\frac{1}{t\chi}} \\
&\quad \leq \left(\fint_{B_{\varrho_{j+1}}} (v-\kappa_j)_+^{t\chi}\,\mathrm{d}x\right)^{\frac{1}{t\chi}} .
\end{aligned}$$

Recalling the definition in (4.13), the last two displays combine in

$$(\kappa_{j+1}-\kappa_j)^{(\chi-1)/\chi} V_{j+1}^{1/\chi} \leq c_*^{1/t} M_0 V_j + c_*^{1/t} \sum_{i=1}^{h} M_i W_{i,j} . \tag{4.16}$$

Now, in the case it is

$$\kappa_{j+2}-\kappa_{j+1}\geq \frac{1}{2}(\kappa_{j+1}-\kappa_j) \tag{4.17}$$

we deduce

$$\begin{aligned}
2^{-1/\chi}\tau^{1/\chi}(\kappa_{j+1}-\kappa_j) &\overset{(4.17)}{\leq} \tau^{1/\chi}(\kappa_{j+1}-\kappa_j)^{(\chi-1)/\chi}(\kappa_{j+2}-\kappa_{j+1})^{1/\chi}\\
&\overset{(4.14)}{=} (\kappa_{j+1}-\kappa_j)^{(\chi-1)/\chi}V_{j+1}^{1/\chi}\\
&\overset{(4.16)}{\leq} c_*^{1/t}M_0V_j + c_*^{1/t}\sum_{i=1}^{h} M_iW_{i,j}\\
&\overset{(4.14)}{=} c_*^{1/t}M_0\tau(\kappa_{j+1}-\kappa_j) + c_*^{1/t}\sum_{i=1}^{h} M_iW_{i,j}\,. 
\end{aligned} \tag{4.18}$$

We now take

$$\tau = 2^{-\frac{1+\chi}{\chi-1}}M_0^{-\frac{\chi}{\chi-1}}c_*^{-\frac{\chi}{t(\chi-1)}} \Longrightarrow 2^{1/\chi}c_*^{1/t}M_0\tau^{\frac{\chi-1}{\chi}} = \frac{1}{2} \tag{4.19}$$

so that, reabsorbing the first term in the right-hand side in the left-hand side of (4.18), and recalling (4.15), we get

$$\kappa_{j+2}-\kappa_{j+1}\leq c^*M_0^{\frac{1}{\chi-1}}\sum_{i=1}^{h} M_iW_{i,j}, \quad c^*:=2^{\frac{n}{t}+\frac{\chi+1}{\chi-1}}c_*^{\frac{\chi}{t(\chi-1)}}\,.$$

As this last inequality holds under condition (4.17), we can work in any case with

$$\kappa_{j+2}-\kappa_{j+1}\leq \frac{\kappa_{j+1}-\kappa_j}{2} + c^*M_0^{\frac{1}{\chi-1}}\sum_{i=1}^{h} M_iW_{i,j}$$

for all integers $j\geq 0$. Summing up such inequalities for $0\leq j\leq N$, re-absorbing terms, and then letting $N\to\infty$, gives

$$\sum_{j=0}^{\infty}(\kappa_{j+2}-\kappa_{j+1})\leq \kappa_1-\kappa_0+2c^*M_0^{\frac{1}{\chi-1}}\sum_{i=1}^{h}\sum_{j=0}^{\infty} M_iW_{i,j}\,,$$

so that, recalling also the definitions of $V_0$ and $\tau$, in (4.14) and (4.19), respectively,

$$\begin{aligned}
\lim_{j\to\infty}\kappa_j &= \sum_{j=0}^{\infty}(\kappa_{j+2}-\kappa_{j+1})+\kappa_1\\
&\leq \kappa_0+2(\kappa_1-\kappa_0)+2c^*M_0^{\frac{1}{\chi-1}}\sum_{i=1}^{h}\sum_{j=0}^{\infty} M_iW_{i,j}
\end{aligned}$$

$$
\begin{aligned}
&\le \kappa_0 + \frac{2V_0}{\tau} + 2c^* M_0^{\frac{1}{\chi-1}} \sum_{i=1}^{h}\sum_{j=0}^{\infty} M_i W_{i,j} \\
&\le \kappa_0 + 2c^* M_0^{\frac{\chi}{\chi-1}} V_0 + 2c^* M_0^{\frac{1}{\chi-1}} \sum_{i=1}^{h}\sum_{j=0}^{\infty} M_i W_{i,j}\,. \qquad (4.20)
\end{aligned}
$$

Setting $\varrho_{-1} := 2r_0$, we have, for every $i \le h$,

$$
\begin{aligned}
\sum_{j=0}^{\infty} W_{i,j} &= \sum_{j=0}^{\infty} \varrho_j^{\delta_i} \left( \fint_{B_{\varrho_j}} |f_i|^{m_i}\,\mathrm{d}x \right)^{\theta_i/t} \\
&= \frac{\delta_i}{2^{\delta_i}-1} \sum_{j=0}^{\infty} \int_{\varrho_j}^{\varrho_{j-1}} \varrho^{\delta_i} \frac{\mathrm{d}\varrho}{\varrho} \left( \fint_{B_{\varrho_j}} |f_i|^{m_i}\,\mathrm{d}x \right)^{\theta_i/t} \\
&\le \frac{2^{n\theta_i/t}\delta_i}{2^{\delta_i}-1} \sum_{j=0}^{\infty} \int_{\varrho_j}^{\varrho_{j-1}} \varrho^{\delta_i} \left( \fint_{B_{\varrho}(x_0)} |f_i|^{m_i}\,\mathrm{d}x \right)^{\theta_i/t} \frac{\mathrm{d}\varrho}{\varrho} \\
&\le \frac{2^{n\theta_i/t}\delta_i}{2^{\delta_i}-1} \int_{0}^{2r_0} \varrho^{\delta_i} \left( \fint_{B_{\varrho}(x_0)} |f_i|^{m_i}\,\mathrm{d}x \right)^{\theta_i/t} \frac{\mathrm{d}\varrho}{\varrho} \\
&= \frac{2^{n\theta_i/t}\delta_i}{2^{\delta_i}-1} \mathbf{P}_{t,\delta_i}^{m_i,\theta_i}(f_i; x_0, 2r_0)\,. \qquad (4.21)
\end{aligned}
$$

From the (4.20)-(4.21) we gain

$$
\lim_{j\to\infty} \kappa_j \le \kappa_0 + cM_0^{\frac{\chi}{\chi-1}} V_0 + cM_0^{\frac{1}{\chi-1}} \sum_{i=1}^{h} M_i \mathbf{P}_{t,\delta_i}^{m_i,\theta_i}(f_i; x_0, 2r_0) \qquad (4.22)
$$

with $c \equiv c(n, \chi, \delta_i, \theta_i, c_*)$. In particular, $\{\kappa_j\}_j$ converges to a finite limit and therefore (4.14) implies

$$
\lim_{j\to\infty} V_j = 0\,. \qquad (4.23)
$$

Using (4.11) now we have

$$
\begin{aligned}
v(x_0) &= \lim_{j\to\infty} (v)_{B_{\varrho_j}} \\
&\le \limsup_{j\to\infty} \fint_{B_{\varrho_j}} (v-\kappa_j)_+\,\mathrm{d}x + \limsup_{j\to\infty} \kappa_j \\
&\le \limsup_{j\to\infty} \left( \fint_{B_{\varrho_j}} (v-\kappa_j)_+^t\,\mathrm{d}x \right)^{1/t} + \lim_{j\to\infty} \kappa_j \\
&= \lim_{j\to\infty} V_j + \lim_{j\to\infty} \kappa_j \overset{(4.23)}{=} \lim_{j\to\infty} \kappa_j
\end{aligned}
$$

and (4.12) follows using (4.22). $\square$

# 5 Hybrid fractional Caccioppoli inequalities

In this section we provide various Caccioppoli type inequalities for minima of variational integrals and solutions to nonlinear equations. These are basic tools in order to prove Lipschitz estimates. Although the underlying principle is common to all cases, the specific shape of these inequalities varies according to the setting considered. The basic prototype is provided in Propositions 5.1-5.2. These contain fractional Caccioppoli type estimates of hybrid type. The word hybrid accounts this time for the fact that on the right-hand sides of (5.10) and (5.11), there still appears the $L^\infty$-norm of $Du$, (implicit in the presence of $M$). Inequalities (5.10) and (5.11) will be proved in the form of a priori estimates for minima of more regular, uniformly elliptic integrals with standard polynomial growth. They will be then incorporated in a suitable approximation scheme in order to cover the case of nonuniformly elliptic integrals. Propositions 5.1-5.2 refer to functionals of the type in (2.3). Later on, in Propositions 5.3-5.5, we shall present additional Caccioppoli inequalities, valid also for functionals of the type in (1.9) and (2.15), and for general elliptic equations of the type in (1.10). Although the basic scheme of proofs is the same, the estimation of the various terms must be different in each case, as every particular type of structure considered needs a specific treatment, eventually leading to different bounds on $q/p$. In the rest of Sect. 5, $B_r \Subset \Omega$ will always denote a ball such that $r \leq 1$. Moreover, when dealing with minimizers, we shall always assume that $p$ and $q$ satisfy at least the bound

$$\frac{q}{p} < 1 + \frac{2}{n} \,. \tag{5.1}$$

Accordingly, we consider the number

$$\mathfrak{s} := \frac{2q}{(n+2)p - nq} \geq 1 \,, \tag{5.2}$$

unless $n = 2, q > p$, when $\mathfrak{s}$ is any larger quantity. This is in fact well-defined by (5.1). Note that $\mathfrak{s} = 1$ if and only if $p = q$.

## 5.1 Model Caccioppoli estimates

In this section we consider the functional $\mathcal{G}(\cdot, B_r)$ in (2.3). The integrand $F\colon \mathbb{R}^n \to \mathbb{R}$ is assumed to satisfy stronger conditions that those considered in (2.4), namely

$$\begin{cases} F(\cdot) \in C^2(\mathbb{R}^n), \quad 0 < \mu \leq 2\,, \\ \nu_0[H_\mu(z)]^{q/2} + \tilde{\nu}[H_\mu(z)]^{p/2} \\ \quad \leq F(z) \leq \tilde{L}[H_\mu(z)]^{q/2} + \tilde{L}[H_\mu(z)]^{p/2} \\ \nu_0[H_\mu(z)]^{(q-2)/2}|\xi|^2 + \tilde{\nu}[H_\mu(z)]^{(p-2)/2}|\xi|^2 \\ \quad \leq \partial_{zz}F(z)\xi \cdot \xi \\ |\partial_{zz}F(z)| \leq \tilde{L}[H_\mu(z)]^{(q-2)/2} + \tilde{L}[H_\mu(z)]^{(p-2)/2} \end{cases} \tag{5.3}$$

for all $z, \xi \in \mathbb{R}^n$, where $\nu_0 > 0$ and the numbers $0 < \tilde{\nu} \leq \tilde{L}$ are as in Sect. 2.1. The function $\mathtt{g}\colon B_r \times \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}$ satisfies the following reinforcement of (2.5):

$$\begin{cases} \mathtt{g}(\cdot) \text{ satisfies (2.5) with } \mu \text{ as in } (5.3)_1 \text{ and } L \text{ replaced by } \tilde{L} \text{ in (5.3)} \\ z \mapsto \mathtt{g}(x, y, z) \in C^2(\mathbb{R}^n) \text{ for every } (x, y) \in \Omega \times \mathbb{R}. \end{cases} \tag{5.4}$$

We shall denote by $c_{\mathtt{g}}$ a generic constant, depending on parameters that will be specified, but such that $c_{\mathtt{g}} = 0$ when $\mathtt{g}$ is identically zero. Finally, the function $\mathtt{h}\colon B_r \times \mathbb{R} \mapsto \mathbb{R}$ is such that

$$\begin{cases} |\mathtt{h}(x, y_1) - \mathtt{h}(x, y_2)| \leq f(x)|y_1 - y_2|^\alpha, \quad \alpha \in (0, 1] \\ f(x) \leq L_0, \quad |\mathtt{h}(x, y)| \leq L_0(|y| + 1) \end{cases} \tag{5.5}$$

hold for all $x \in B_r$, $y, y_1, y_2 \in \mathbb{R}$, where $L_0 \geq 1$ is a fixed constant. Denoting $\tilde{F}(x, y, z) := F(z) + \mathtt{g}(x, y, z)$, using also that $\mu > 0$, this new integrand is seen to satisfy

$$\begin{cases} \frac{1}{c_*}[H_1(z)]^{q/2} \leq \tilde{F}(x, y, z) \leq c_*[H_1(z)]^{q/2} \\ \frac{1}{c_*}[H_1(z)]^{(q-2)/2}|\xi|^2 \leq \partial_{zz}\tilde{F}(x, y, z)\xi \cdot \xi \\ |\partial_{zz}\tilde{F}(x, y, z)| \leq c_*[H_1(z)]^{(q-2)/2} \\ |\tilde{F}(x_1, y_1, z) - \tilde{F}(x_2, y_2, z)| \\ \quad \leq c_*\left(|x_1 - x_2|^\alpha + |y_1 - y_2|^\alpha\right)[H_1(z)]^{\gamma/2} \end{cases} \tag{5.6}$$

with the same meaning of (5.3)-(5.5), but this time we have $c_* \equiv c_*(n, q, L, \mu, \nu_0)$. In (5.6) the constant $c_*$ is such that $c_* \to \infty$ when either $\mu \to 0$ or $\nu_0 \to 0$. Therefore we have that $\tilde{F}(\cdot)$ is a regular and non-degenerate integrand with $q$-growth. It follows that if $u \in W^{1,q}(B_r)$ is a minimizer of the functional $\mathcal{G}(\cdot, B_r)$ in (2.3) and (5.3)-(5.5) are in force, then standard regularity arguments give

$$u \in C^{1,\alpha_1}_{\mathrm{loc}}(B_r) \quad \text{for some } \alpha_1 \equiv \alpha_1(n, q, c_*, \alpha, L_0, \mu) \in (0, 1)\,. \tag{5.7}$$

This is shortly detailed in Sect. 5.7. We shall also use a couple of parameters $(\beta, \chi)$ such that

$$0 < \beta < \frac{\alpha_{\mathrm{m}}}{2 + \alpha_{\mathrm{m}}} \qquad \text{and} \qquad \chi \equiv \chi(\beta) := \frac{n}{n - 2\beta} > 1\,, \tag{5.8}$$

where $\alpha_{\mathrm{m}} \leq 1$ is a positive number that will be specified later. By denoting

$$E_\mu(z) := \frac{1}{p}\left[(|z|^2 + \mu^2)^{p/2} - \mu^p\right] = \frac{1}{p}\left[[H_\mu(z)]^{p/2} - \mu^p\right], \quad z \in \mathbb{R}^n \tag{5.9}$$

we now have

**Proposition 5.1** (Hybrid Fractional Caccioppoli) *Let $u \in W^{1,q}(B_r)$ be a minimizer of the functional $\mathcal{G}(\cdot, B_r)$ in (2.3), under assumptions (5.1) and (5.3)-(5.5). Let*

*$B_{\varrho}(x_0) \Subset B_r$ and let $M \geq 1$ be a constant such that $\|Du\|_{L^\infty(B_{\varrho}(x_0))} \leq M$. Then, for every number $\kappa \geq 0$*

$$
\begin{aligned}
&\varrho^{2\beta-n}[(E_\mu(Du)-\kappa)_+]^2_{\beta,2;B_{\varrho/2}(x_0)} + \left(\fint_{B_{\varrho/2}(x_0)} (E_\mu(Du)-\kappa)_+^{2\chi}\,\mathrm{d}x\right)^{1/\chi}\\
&\quad \leq cM^{\mathfrak{s}(q-p)} \fint_{B_{\varrho}(x_0)} (E_\mu(Du)-\kappa)_+^2\,\mathrm{d}x\\
&\qquad + c_{\mathrm{g}} M^{\mathfrak{s}q+\alpha+\gamma q/p}\varrho^{\alpha} + cM^{\mathfrak{s}q+\alpha}\varrho^{\alpha}\fint_{B_{\varrho}(x_0)} f\,\mathrm{d}x
\end{aligned}
\tag{5.10}
$$

*holds whenever $(\beta, \chi)$ are as in (5.8) with $\alpha_{\mathrm{m}} := \alpha$. The constants $c$, $c_{\mathrm{g}}$ in (5.10) depend on* `data`*, $\gamma$ and $\beta$, but are otherwise independent of $\nu_0$ and $L_0$; dependence on $\gamma$ only occurs when $\mathrm{g}(\cdot) \not\equiv 0$. The number $\mathfrak{s}$ is defined in (5.2).*

An alternative, and more flexible version of (5.10), is in the next

**Proposition 5.2** *Under the same assumptions of Proposition 5.1, for every number $\kappa \geq 0$*

$$
\begin{aligned}
&\varrho^{2\beta-n}[(E_\mu(Du)-\kappa)_+]^2_{\beta,2;B_{\varrho/2}(x_0)} + \left(\fint_{B_{\varrho/2}(x_0)} (E_\mu(Du)-\kappa)_+^{2\chi}\,\mathrm{d}x\right)^{1/\chi}\\
&\quad \leq cM^{\mathfrak{s}(q-p)} \fint_{B_{\varrho}(x_0)} (E_\mu(Du)-\kappa)_+^2\,\mathrm{d}x + c_{\mathrm{g}} M^{\mathfrak{s}q+\alpha+\gamma q/p}\varrho^{\alpha}\\
&\qquad + cM^{\mathfrak{s}q}\varrho^{\frac{p\alpha}{p-\alpha}}\left(\fint_{B_{\varrho}(x_0)} f^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}\,\mathrm{d}x\right)^{\theta(\mathfrak{p})}\\
&\qquad + c\mathbb{1}_p M^{\mathfrak{s}q+\frac{\alpha(2-p)}{2-\alpha}}\varrho^{\frac{2\alpha}{2-\alpha}}\left(\fint_{B_{\varrho}(x_0)} f^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}\,\mathrm{d}x\right)^{\sigma(\mathfrak{p})}
\end{aligned}
\tag{5.11}
$$

*holds whenever $\mathfrak{p} \in [p, p^*)$, where*

$$
\theta(\mathfrak{p}) := \frac{\mathfrak{p}-\alpha}{\mathfrak{p}}\frac{p}{p-\alpha} \geq 1 \qquad \text{and} \qquad \sigma(\mathfrak{p}) := \frac{\mathfrak{p}-\alpha}{\mathfrak{p}}\frac{2}{2-\alpha}\,, \tag{5.12}
$$

*and whenever $(\beta, \chi)$ are as in (5.8), with this time*

$$
\alpha_{\mathrm{m}} \equiv \alpha_{\mathrm{m}}(\mathfrak{p}) := \alpha \min\left\{1, \frac{p\mathfrak{a}(\mathfrak{p})}{p-\alpha}, \frac{2\mathfrak{a}(\mathfrak{p})}{2-\alpha}\right\} \quad \text{and} \quad \mathfrak{a}(\mathfrak{p}) := \frac{n}{\mathfrak{p}} - \frac{n}{p} + 1\,. \tag{5.13}
$$

*The constants $c$, $c_{\mathrm{g}}$, $\mathfrak{s}$ are as in Proposition 5.1 and $\mathbb{1}_p$ is defined in (3.14).*

***Remark 5*** From the proof of Proposition 5.2 it follows that in the case $\mathfrak{g}(\cdot) \equiv 0$, so that $c_{\mathfrak{g}} = 0$ in (5.11), the definition of $\alpha_{\mathrm{m}}$ in (5.13) can be changed into

$$\alpha_{\mathrm{m}} = \alpha \min\left\{\frac{p\mathfrak{a}(\mathfrak{p})}{p-\alpha}, \frac{2\mathfrak{a}(\mathfrak{p})}{2-\alpha}\right\}. \tag{5.14}$$

We also note that in both the proofs of Proposition 5.1 and 5.2, when deriving quantitative a priori estimates, we shall only use $(5.5)_1$. Instead we use $(5.5)_2$ only to derive the qualitative information in (5.7) and to justify the computations. This is explained in Sect. 5.7 below.

### 5.2 Preliminaries on equations

For the proof of Propositions 5.1-5.2, we need a few preliminary results concerning *non-degenerate* elliptic equations of the type

$$-\operatorname{div} A_0(Dv) = 0 \qquad \text{in } B \subset \mathbb{R}^n, \tag{5.15}$$

where $B$ is a ball. Here the vector field $A_0\colon \mathbb{R}^n \mapsto \mathbb{R}^n$ satisfies

$$\begin{cases} A_0(\cdot) \text{ is } C^1\text{-regular}, \quad 0 < \mu \le 2 \\ |A_0(z)| + [H_\mu(z)]^{1/2}|\partial_z A_0(z)| \\ \quad \le \tilde{L}[H_\mu(z)]^{(q-1)/2} + \tilde{L}[H_\mu(z)]^{(p-1)/2} \\ \nu_0[H_\mu(z)]^{(q-2)/2}|\xi|^2 + \tilde{\nu}[H_\mu(z)]^{(p-2)/2}|\xi|^2 \\ \quad \le \partial_z A_0(z)\xi\cdot\xi, \end{cases} \tag{5.16}$$

whenever $z, \xi \in \mathbb{R}^n$, where $\nu_0 > 0$ and the numbers $0 < \tilde{\nu} \le \tilde{L}$ are as in Sect. 2.1. The assumption $\mu > 0$ implies that $A_0(\cdot)$ is non-degenerate elliptic.

**Lemma 5.1** *Let $v \in W^{1,q}(B)$ be a weak solution to (5.15) under assumptions (5.16), then $[H_\mu(Dv)]^{p/2}, E_\mu(Dv) \in W^{1,2}_{\mathrm{loc}}(B)$, and $Dv$ is locally Hölder continuous in $B$. Moreover, assume also that there exists $\mathfrak{M} \ge 1$ such that $\|Dv\|_{L^\infty(B)} \le \mathfrak{M}$.*

- *Then*

$$\int_{B/2} |D(E_\mu(Dv)-\kappa)_+|^2\,\mathrm{d}x \le \frac{c\mathfrak{M}^{2(q-p)}}{|B|^{2/n}}\int_B (E_\mu(Dv)-\kappa)_+^2\,\mathrm{d}x \tag{5.17}$$

  *holds for every $\kappa \ge 0$, where $c \equiv c(n,p,q,\nu,L)$ is independent of $\nu_0$.*
- *If, in addition, $\partial A_0(\cdot)$ is symmetric, then*

$$\int_{B/2} |D(E_\mu(Dv)-\kappa)_+|^2\,\mathrm{d}x \le \frac{c\mathfrak{M}^{q-p}}{|B|^{2/n}}\int_B (E_\mu(Dv)-\kappa)_+^2\,\mathrm{d}x \tag{5.18}$$

  *holds with $c, \kappa$ as in (5.17).*

***Proof*** The standard regularity theory gives that $Dv$ is locally Hölder continuous in $\Omega$, as well as the differentiability of $H_\mu(Dv)$, and therefore of $E_\mu(Dv)$. For the first result see for instance [62] and Lemma 6.3. As for the second, let us outline the argument for completeness. Standard difference quotients techniques give that $V_\mu(Du) \in W^{1,2}_{\rm loc}(B, \mathbb{R}^n)$; see for instance [34, Chap. 8], and recall that $\mu > 0$ to adapt from there (still observe that by approximation $\mu > 0$ is not really needed at this stage). By (3.12) we have

$$|\tau_h Dv|^2 \lesssim (|Dv(\cdot + h)|^2 + |Dv|^2 + \mu^2)^{(2-p)/2} |\tau_h V_\mu(Dv)|^2$$

in $\{x \in B : \operatorname{dist}(x, \partial B) > |h|\}$, so that $\mu > 0$ and $Dv \in L^\infty_{\rm loc}(B, \mathbb{R}^n)$ imply $Dv \in W^{1,2}_{\rm loc}(B, \mathbb{R}^n)$ via difference quotients (note that, in fact, for this last result we can allow $\mu = 0$ when $1 < p \leq 2$). In turn, (3.15) implies

$$|\tau_h [H_\mu(Dv)]^{p/2}|^2 \lesssim (|Dv(\cdot + h)|^2 + |Dv|^2 + 1)^{p-1} |\tau_h Dv|^2 ,$$

from which $[H_\mu(Dv)]^{p/2} \in W^{1,2}_{\rm loc}(B)$ again follows via difference quotients method. We go for the proof of (5.17)-(5.18) and we first consider (5.18). By (5.16) we are in the setting of [3, Lemma 4.5]; in particular, by $\mu > 0$, $A_0(\cdot)$ satisfies [3, (4.26)] with $\bar{T} = 0$ and

$$\begin{cases} g_{2,\varepsilon}(t) \equiv g_2(t) \equiv \tilde{L}(t^2 + \mu^2)^{(q-2)/2} + \tilde{L}(t^2 + \mu^2)^{(p-2)/2} \\ g_1(t) \equiv \tilde{\nu}(t^2 + \mu^2)^{(p-2)/2} . \end{cases} \tag{5.19}$$

Note that, with respect to [3], we have that $g_{2,\varepsilon}(\cdot)$ is actually independent of $\varepsilon$ as indicated in (5.19) (that in fact appears only in the context of [3], where a family of vector fields is considered). Moreover, thanks to (5.16), we can in fact take $\bar{T} = 0$ (alternatively, take any $\bar{T} \in (0, \mathfrak{M})$ in [3] and then let $\bar{T} \to 0$ in [3]). Again by (5.16), $v$ enjoys the regularity in (5.7) and we can use [3, (4.29)] with $f \equiv 0$, that gives, with the notation of [3]

$$\int_{B/2} |D(G_{\bar{T}}(|Dv|) - k)_+|^2 \, dx \leq \frac{c(n)}{|B|^{2/n}} \frac{g_2(\mathfrak{M})}{g_1(\mathfrak{M})} \int_B (G_{\bar{T}}(|Dv|) - k)_+^2 \, dx . \tag{5.20}$$

In this setting (recall $\bar{T} = 0$) it is

$$G_{\bar{T}}(|z|) := \int_0^{|z|} g_1(s) s \, ds = \frac{\tilde{\nu}}{p} \left[ (|z|^2 + \mu^2)^{p/2} - \mu^p \right] = \tilde{\nu} E_\mu(|z|)$$

for $z \in \mathbb{R}^n$. Recalling (5.19) and that $\mathfrak{M} \geq 1$, we estimate

$$\frac{g_2(\mathfrak{M})}{g_1(\mathfrak{M})} \leq c(\mathfrak{M}^{q-p} + 1) \leq c \mathfrak{M}^{q-p} , \tag{5.21}$$

with $c \equiv c(n, p, q, \tilde{\nu}, \tilde{L})$, so that (5.18) follows from (5.20). For (5.17), we use [3, (4.37)], that is

$$\int_{B/2} |D(G_{\bar{T}}(|Dv|) - k)_+|^2 \, dx \leq \frac{c}{|B|^{2/n}} \left[ \frac{g_2(\mathfrak{M})}{g_1(\mathfrak{M})} \right]^2 \int_B (G_{\bar{T}}(|Dv|) - k)_+^2 \, dx ,$$

and (5.17) follows via (5.21). For the constant dependence, recall that $\tilde{\nu}$, $\tilde{L}$ depend on $n, p, q, \nu, L$. □

**Lemma 5.2** *Let $v \in u + W^{1,q}_0(B)$ be a weak solution to (5.15) under assumptions (5.16), with $u \in W^{1,q}(B) \cap L^\infty(B)$. Then $\operatorname{osc}(v, B) \le \operatorname{osc}(u, B)$ and $\|u - v\|_{L^\infty(B)} \le \operatorname{osc}(u, B)$.*

***Proof*** This is a small variant of the classical maximum principle, see for instance [62, Lemma 3.1], and we report the proof for the sake of completeness. By (5.16) it follows that

$$(|z_1|^2 + |z_2|^2 + \mu^2)^{(p-2)/2}|z_1 - z_2|^2 \lesssim (A_0(z_2) - A_0(z_1)) \cdot (z_2 - z_1)\,. \tag{5.22}$$

This is in fact a consequence of (3.12) and (3.18). Note that, up to replacing $A_0(z)$ by $A_0(z) - A_0(0_{\mathbb{R}^n})$, that does not change the validity of (5.22), we can assume $A_0(0_{\mathbb{R}^n}) = 0_{\mathbb{R}^n}$, so that (5.22) now gives that

$$(|z|^2 + \mu^2)^{(p-2)/2}|z|^2 \lesssim A_0(z) \cdot z \tag{5.23}$$

holds for every $z \in \mathbb{R}^n$. Let now $M$, $m$ be constants such that $m \le u \le M$ a.e. in $B$, and use $\varphi := (v - M)_+ \in W^{1,q}_0(B)$ as test function in $\int_B A_0(Dv) \cdot D\varphi \,\mathrm{d}x = 0$, which is in fact valid whenever $\varphi \in W^{1,q}_0(B)$ by $v \in W^{1,q}(B)$ and a standard density argument. By (5.23) it follows that $D(v - M)_+ = 0$ which implies, via Poincaré inequality, that $(v - M)_+ = 0$, that is, $v \le M$ a.e. Similarly, testing with $\varphi := (v - m)_- := \max\{m - v, 0\}$, we get $v \ge m$ a.e. Alternatively, note that $\tilde{v} := -v$ solves $-\operatorname{div} \tilde{A}_0(D\tilde{v}) = 0$ with $\tilde{A}_0(z) := -A_0(-z)$ and $\tilde{v} \equiv -u$ on $\partial B$, and apply the above argument to $\tilde{v}$ with $-m$ in place of $M$. This proves that $\operatorname{osc}(v, B) \le \operatorname{osc}(u, B)$. Moreover, this also implies that $m - M \le u(x) - v(x) \le M - m$ a.e., that is, $\|u - v\|_{L^\infty(B)} \le \operatorname{osc}(u, B)$. The proof is complete. Notice that the lemma still works assuming only $|A_0(z)| \lesssim |z|^{q-1} + 1$ and (5.23) in place of (5.16). □

We proceed with a few a priori estimates that will play a central role in our analysis. These are suitable modifications of estimates that can be found in the literature since [65, 66]. To get the full statements we need, we shall appeal to [3].

**Lemma 5.3** *Let $v \in W^{1,q}(B)$ be a weak solution to (5.15), under assumptions (5.16).*

- *If $q/p < 1 + 1/n$, then*

$$\|Dv\|_{L^\infty(B/2)} \le c \left( \fint_B (|Dv| + 1)^p \,\mathrm{d}x \right)^{\frac{1}{(n+1)p - nq}} \tag{5.24}$$

  *holds unless $n = 2, q > p$, when it holds for any larger exponent, whenever $B \Subset \Omega \subset \mathbb{R}^n$ is a ball, where $c \equiv c(n, p, q, \nu, L)$ is independent of $\nu_0$.*
- *If $q/p < 1 + 2/n$, that is (5.1) holds, and in addition $\partial A_0(\cdot)$ is symmetric, then*

$$\|Dv\|_{L^\infty(B/2)} \le c \left( \fint_B (|Dv| + 1)^p \,\mathrm{d}x \right)^{\frac{2}{(n+2)p - nq}} \tag{5.25}$$

  *holds in place of (5.24), unless $n = 2, q > p$, when it holds for any larger exponent.*

***Proof*** We start proving (5.24)-(5.25). The information concerning the local boundedness of $Dv$ can be found for instance in [3], but it is also a direct consequence of the arguments developed to prove (5.24)-(5.25) in the following lines, when using arbitrary balls $\tilde{B} \Subset B$ rather than $B/2$. So, we concentrate on the proof of (5.24)-(5.25). First, note that we can reduce to the case $B \equiv \mathcal{B}_1$ by a standard scaling argument (which is for instance detailed in Sect. 5.3.1). The proof of (5.24) can be obtained following the estimates in [3, Lemma 7.2], where we must take $f \equiv 0$ and $\Omega \equiv \mathcal{B}_1$, and the choices are those in (5.19); as in [3], we first treat the case $n > 2$. Specifically, using [3, (4.38)] with $\vartheta = 1$, we arrive at

$$\|Dv\|^p_{L^\infty(\mathcal{B}_{\tau_1})} \le \frac{c}{(\tau_2-\tau_1)^{n/2}}\left[\|Dv\|^{\frac{(q-p)n+p}{2}}_{L^\infty(\mathcal{B}_{\tau_2})}+1\right]\left[\|Dv\|^{p/2}_{L^p(\mathcal{B}_1)}+1\right], \tag{5.26}$$

which holds whenever $1/2 \le \tau_1 < \tau_2 \le 1$. This is nothing but the estimate in the second display at [3, Page 1026], where $\mathcal{D}$ is replaced by $\|Dv\|_{L^p(\mathcal{B}_1)}+1$. Using $q/p < 1+1/n$ - implying $[(q-p)n+p]/2 < p$ - allows to apply Young's inequality in (5.26) and

$$\|Dv\|_{L^\infty(\mathcal{B}_{\tau_1})} \le \frac{1}{2}\|Dv\|_{L^\infty(\mathcal{B}_{\tau_2})} + \frac{c}{(\tau_2-\tau_1)^{\frac{n}{(n+1)p-nq}}}\left[\|Dv\|_{L^p(\mathcal{B}_1)}+1\right]^{\frac{p}{(n+1)p-nq}}$$

from which (5.24) follows - modulo rescaling back to $B$ - applying Lemma 3.2 with the choice $h(\tau) \equiv \|Du\|_{L^\infty(B_\tau)}$, $1/2 \le \tau \le 1$. For the case $n = 2$, again from [3, (4.38)] we find that (5.26) still holds replacing the exponent $[(q-p)n+p]/2 = q-p/2$ with any larger number still smaller than $p$ (no need when $q = p$), and the conclusion follows as in the case $n > 2$. For (5.25) the argument is similar (and we outline it when $n > 2$). We again go back to [3, Lemma 7.2] and apply [3, (4.38)] with $\vartheta = 0$, that this time gives the following analog of (5.26):

$$\|Dv\|^p_{L^\infty(\mathcal{B}_{\tau_1})} \le \frac{c}{(\tau_2-\tau_1)^{n/2}}\left[\|Dv\|^{\frac{(q-p)n+2p}{4}}_{L^\infty(\mathcal{B}_{\tau_2})}+1\right]\left[\|Dv\|^{p/2}_{L^p(\mathcal{B}_1)}+1\right].$$

Applying Young's inequality as in (5.26), thanks to (5.1), we arrive at (5.25). The case $n = 2$ can again be dealt with as for (5.24). □

### 5.3 Proof of Propositions 5.1-5.2

The proof goes in five different steps; in the first four, given in Sects. 5.3.1-5.3.4, we complete the proof of (5.11). The last one, in Sect. 5.3.5, is instead devoted to the proof of (5.10).

#### 5.3.1 Blow-up

In order to prove Proposition 5.1, we can reduce to the case $B_\varrho(x_0) \equiv B_1(0) \equiv \mathcal{B}_1$ (recall the notation in (3.1)). Indeed, take $u_\varrho \in W^{1,q}(\mathcal{B}_1)$ defined as

$$u_\varrho(x) := \frac{u(x_0+\varrho x)}{\varrho}, \qquad x \in \mathcal{B}_1. \tag{5.27}$$

It follows that

$$\|Du_\varrho\|_{L^\infty(\mathcal{B}_1)} \le M \tag{5.28}$$

and that $u_\varrho$ is a minimizer of the functional (defined on $W^{1,q}(\mathcal{B}_1)$)

$$w \mapsto \mathcal{G}_\varrho(w, \mathcal{B}_1) := \int_{\mathcal{B}_1} [F(Dw) + \mathrm{g}_\varrho(x, w, Dw) + \mathrm{h}_\varrho(x, w)]\,\mathrm{d}x\,, \tag{5.29}$$

where $\mathrm{g}_\varrho(x, y, z) := \mathrm{g}(x_0 + \varrho x, \varrho y, z)$ and $\mathrm{h}_\varrho(x, y) := \mathrm{h}(x_0 + \varrho x, \varrho y)$. By (5.4) we have

$$\begin{cases} z \mapsto \mathrm{g}_\varrho(x, y, z) \text{ is convex, non-negative and } z \mapsto \mathrm{g}_\varrho(x, y, z) \in C^2(\mathbb{R}^n) \\ \mathrm{g}_\varrho(x, y, z) + H_\mu(z)|\partial_{zz}\mathrm{g}_\varrho(x, y, z)| \le \tilde{L}[H_\mu(z)]^{p/2} \\ |\mathrm{g}_\varrho(x_1, y_1, z) - \mathrm{g}_\varrho(x_2, y_2, z)| \le \tilde{L}\varrho^\alpha\,(|x_1 - x_2|^\alpha + |y_1 - y_2|^\alpha)\,[H_1(z)]^{\gamma/2} \end{cases} \tag{5.30}$$

for all $x, x_1, x_2 \in \mathcal{B}_1$, $y, y_1, y_2 \in \mathbb{R}$, $z \in \mathbb{R}^n$. As for $\mathrm{h}_\varrho(\cdot)$, by (5.5) we have that

$$|\mathrm{h}_\varrho(x, y_1) - \mathrm{h}_\varrho(x, y_2)| \le f_\varrho(x)|y_1 - y_2|^\alpha, \quad f_\varrho(x) := \varrho^\alpha f(x_0 + \varrho x) \tag{5.31}$$

hold for all $x \in \mathcal{B}_1$, $y_1, y_2 \in \mathbb{R}$. As $B_\varrho(x_0) \Subset B_r$, (5.7) implies $u_\varrho \in C^{1,\alpha_1}(\overline{\mathcal{B}_1})$. Therefore we only need to prove that

$$\begin{aligned} &[(E_\mu(Du_\varrho) - \kappa)_+]_{\beta,2;\mathcal{B}_{1/2}} + \|(E_\mu(Du_\varrho) - \kappa)_+\|_{L^{2\chi}(\mathcal{B}_{1/2})} \\ &\quad \le cM^{\mathfrak{s}(q-p)/2}\|E_\mu(Du_\varrho) - \kappa)_+\|_{L^2(\mathcal{B}_1)} + c_{\mathrm{g}} M^{(\mathfrak{s}q+\alpha+\gamma q/p)/2}\varrho^{\alpha/2} \\ &\qquad + cM^{\frac{\mathfrak{s}q}{2}}\|f_\varrho\|_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}(\mathcal{B}_1)}^{\frac{\mathfrak{p}\theta(\mathfrak{p})}{2(\mathfrak{p}-\alpha)}} + c\mathbb{1}_p M^{\frac{\mathfrak{s}q}{2}+\frac{\alpha(2-p)}{2(2-\alpha)}}\|f_\varrho\|_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}(\mathcal{B}_1)}^{\frac{\mathfrak{p}\sigma(\mathfrak{p})}{2(\mathfrak{p}-\alpha)}} \end{aligned} \tag{5.32}$$

holds for $c, c_{\mathrm{g}}$ as in Proposition 5.1, and then (5.11) follows scaling back (5.32) from $u_\varrho$ to $u$.

### 5.3.2 Estimates on balls

Consider a ball $B \subset \mathcal{B}_1$, centred at $x_{\mathrm{c}}$, and define the functional

$$w \mapsto \int_B F_0(Dw)\,\mathrm{d}x, \quad \text{where } F_0(z) := F(z) + \mathrm{g}_\varrho(x_{\mathrm{c}}, (u_\varrho)_B, z)\,. \tag{5.33}$$

As a consequence of (5.3),(5.30) and Lemma 3.4, and eventually choosing new $\tilde{\nu} \le \tilde{L}$ still as in Sect. 2.1, the integrand $F_0(\cdot)$ satisfies itself conditions (5.3). Therefore (3.19) applies to $F_0(\cdot)$, with $c \equiv c(n, p, \tilde{\nu}) \ge 1$. From now on, and up to Sect. 6, we shall often abbreviate $E_\mu(\cdot) \equiv E(\cdot)$, $V_\mu(\cdot) \equiv V(\cdot)$ and $H_\mu(\cdot) \equiv H(\cdot)$. To proceed with the proof (5.32), we fix $\beta_0 \in (0, 1)$, and $h \in \mathbb{R}^n$, such that

$$0 < |h| \le \frac{1}{2^{8/\beta_0}}\,. \tag{5.34}$$

We take

$$x_{\mathrm{c}} \in \mathcal{B}_{1/2+2|h|^{\beta_0}} \tag{5.35}$$

and fix a ball centred at $x_{\mathrm{c}}$ with radius $|h|^{\beta_0}$, denoted by $B_h := B_{|h|^{\beta_0}}(x_{\mathrm{c}})$. By (5.34) we have $8B_h \Subset \mathcal{B}_1$. We then define $v \equiv v_{B_h} \in u_{\varrho} + W_0^{1,q}(8B_h)$ as the solution to

$$v \mapsto \min_{w \in u_{\varrho}+W_0^{1,q}(8B_h)} \int_{8B_h} F_0(Dw)\,\mathrm{d}x \tag{5.36}$$

where the integrand $F_0(\cdot)$ has been fixed in (5.33), with $B \equiv B_h$, and verifies (5.6). It follows that $v$ solves the Euler-Lagrange equation

$$\int_{8B_h} \partial_z F_0(Dv) \cdot D\varphi\,\mathrm{d}x = 0 \qquad \text{for every } \varphi \in W_0^{1,q}(8B_h). \tag{5.37}$$

Note that $v$ is in fact Hölder continuous (for some exponent) up to the boundary $\partial(8B_h)$, see [34, Theorem 7.8], [50]. By (5.3) and (5.4), the vector field $\partial_z F_0(\cdot) \equiv A_0(\cdot)$ satisfies (5.16) for suitable constants $0 < \tilde{\nu} \leq \tilde{L}$ depending as described in Sect. 2.1, and, needless to say, $\partial_z A_0(\cdot) \equiv \partial_{zz} F_0(\cdot)$ is symmetric. Therefore equation (5.37) is of the type considered in (5.15) and, via (5.1), Lemmas 5.1-5.3 apply. In particular, Lemma 5.2 implies

$$\begin{cases} \operatorname{osc}(v, 8B_h) \leq \operatorname{osc}(u_{\varrho}, 8B_h) \\ \|u_{\varrho} - v\|_{L^\infty(8B_h)} \leq \operatorname{osc}(u_{\varrho}, 8B_h)\,. \end{cases} \tag{5.38}$$

As a consequence of (5.28) and (5.38), we find

$$\begin{cases} |v - (v)_{8B_h}| \leq 16M|h|^{\beta_0} \\ \|u_{\varrho} - v\|_{L^\infty(8B_h)} \leq 16M|h|^{\beta_0}\,. \end{cases} \tag{5.39}$$

Next, applying Lemma 5.3, estimate (5.25) yields

$$\|Dv\|_{L^\infty(4B_h)} \leq c \left( \fint_{8B_h} (|Dv|+1)^p\,\mathrm{d}x \right)^{\mathfrak{s}/q}. \tag{5.40}$$

On the other hand, by minimality of $v$, $(5.3)_2$ and (5.4), we have

$$\begin{aligned} \fint_{8B_h} |Dv|^p\,\mathrm{d}x &\leq c \fint_{8B_h} F_0(Dv)\,\mathrm{d}x \\ &\leq c \fint_{8B_h} F_0(Du_{\varrho})\,\mathrm{d}x \overset{(5.28)}{\leq} c(M^q+1) \leq cM^q\,. \end{aligned} \tag{5.41}$$

Matching the content of the last two inequalities, yields

$$\|Dv\|_{L^\infty(4B_h)} \leq \tilde{c} M^{\mathfrak{s}}\,, \quad \tilde{c} \equiv \tilde{c}(n,p,q,\nu,L)\,. \tag{5.42}$$

Finally, by (5.42) we apply Lemma 5.1 with $\mathfrak{M} \equiv \tilde{c} M^{\mathfrak{s}}$ as follows:

$$
\begin{aligned}
\int_{B_h} |\tau_h (E(Dv)-\kappa)_+|^2 \,\mathrm{d}x &\overset{(3.7)_2}{\leq} c|h|^2 \int_{2B_h} |D(E(Dv)-\kappa)_+|^2 \,\mathrm{d}x \\
&\overset{(5.18)}{\leq} c|h|^{2(1-\beta_0)} M^{\mathfrak{s}(q-p)} \int_{4B_h} (E(Dv)-\kappa)_+^2 \,\mathrm{d}x\,, \qquad (5.43)
\end{aligned}
$$

for $c \equiv c(n, p, q, \nu, L)$; recall that $\tau_h$ has been defined in (3.6). Note that in the first line we have used (5.34), that ensures

$$
B_h \equiv B_{|h|^{\beta_0}}(x_c) \subset B_{|h|^{\beta_0}+|h|}(x_c) \subset B_{2|h|^{\beta_0}}(x_c) = 2B_h\,. \qquad (5.44)
$$

Let us now quantify the $L^2$-distance between $V(Du_\varrho)$ and $V(Dv)$. Recalling that $u_\varrho$ minimizes $\mathcal{G}_\varrho(\cdot, \mathcal{B}_1)$ defined in (5.29), we have

$$
\begin{aligned}
&\frac{1}{\tilde{c}} \int_{8B_h} |V(Du_\varrho)-V(Dv)|^2 \,\mathrm{d}x \\
&\overset{(3.19)}{\leq} \int_{8B_h} \left[F_0(Du_\varrho) - F_0(Dv) - \partial_z F_0(Dv)\cdot(Du_\varrho - Dv)\right] \mathrm{d}x \\
&\overset{(5.37)}{=} \int_{8B_h} \left[F_0(Du_\varrho) - F_0(Dv)\right] \mathrm{d}x \\
&= \mathcal{G}_\varrho(u_\varrho, 8B_h) - \mathcal{G}_\varrho(v, 8B_h) \\
&\quad + \int_{8B_h} [\mathrm{g}_\varrho(x_c, (u_\varrho)_{8B_h}, Du_\varrho) - \mathrm{g}_\varrho(x, u_\varrho, Du_\varrho)]\,\mathrm{d}x \\
&\quad + \int_{8B_h} [\mathrm{g}_\varrho(x, v, Dv) - \mathrm{g}_\varrho(x_c, (u_\varrho)_{8B_h}, Dv)]\,\mathrm{d}x \\
&\quad + \int_{8B_h} [\mathrm{h}_\varrho(x, v) - \mathrm{h}_\varrho(x, u_\varrho)]\,\mathrm{d}x \\
&=: \overbrace{\mathcal{G}_\varrho(u_\varrho, 8B_h) - \mathcal{G}_\varrho(v, 8B_h)}^{\leq 0 \text{ by minimality of } u_\varrho} + (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) \\
&\leq (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III})\,. \qquad (5.45)
\end{aligned}
$$

We proceed estimating (I), using $(5.30)_3$ as follows:

$$
\begin{aligned}
(\mathrm{I}) &\leq c_{\mathrm{g}} \varrho^\alpha \int_{8B_h} \left(|x-x_c|^\alpha + |u_\varrho - (u_\varrho)_{8B_h}|^\alpha\right)(|Du_\varrho|+1)^\gamma \,\mathrm{d}x \\
&\leq c_{\mathrm{g}} |h|^{\beta_0\alpha} M^\gamma \varrho^\alpha |B_h| + c_{\mathrm{g}} |h|^{\beta_0\alpha} M^{\alpha+\gamma} \varrho^\alpha |B_h| \\
&\leq c_{\mathrm{g}} |h|^{\beta_0\alpha} M^{\alpha+\gamma} \varrho^\alpha |B_h| \\
&\leq c_{\mathrm{g}} |h|^{\beta_0\alpha} M^{\alpha+\gamma q/p} \varrho^\alpha |B_h|\,.
\end{aligned}
$$

For (II) we have

$$
\begin{aligned}
\text{(II)} &\overset{(5.30)_3}{\le} c_{\mathrm{g}}\varrho^{\alpha}\int_{8B_h}\left(|x-x_{\mathrm{c}}|^{\alpha}+|v-(v)_{8B_h}|^{\alpha}+|(v)_{8B_h}-(u_{\varrho})_{8B_h}|^{\alpha}\right)\\
&\qquad\qquad\cdot(|Dv|+1)^{\gamma}\,\mathrm{d}x\\
&\overset{(5.39)}{\le} c_{\mathrm{g}}|h|^{\alpha\beta_0}M^{\alpha}\varrho^{\alpha}\fint_{8B_h}(|Dv|+1)^{\gamma}\,\mathrm{d}x\,|B_h|\\
&\le c_{\mathrm{g}}|h|^{\alpha\beta_0}M^{\alpha}\varrho^{\alpha}\left(\fint_{8B_h}(|Dv|+1)^{p}\,\mathrm{d}x\right)^{\gamma/p}|B_h|\\
&\overset{(5.41)}{\le} c_{\mathrm{g}}|h|^{\beta_0\alpha}M^{\alpha+\gamma q/p}\varrho^{\alpha}|B_h|\,.
\end{aligned}
$$

For (III) we use Sobolev and Morrey embeddings in the form

$$
\|u_{\varrho}-v\|_{L^{\mathfrak{p}}(8B_h)}\le c|h|^{\beta_0\mathfrak{a}(\mathfrak{p})}\|Du_{\varrho}-Dv\|_{L^{p}(8B_h)}\,, \tag{5.46}
$$

where $\mathfrak{a}(\mathfrak{p})$ has been defined in (5.13). Note that, according to the definition (3.2), inequality (5.46) holds for every $\mathfrak{p}\ge p$ when $p\ge n$. Using also (3.13), (5.31) and (5.46), we find

$$
\begin{aligned}
\text{(III)} &\le c\int_{8B_h} f_{\varrho}(x)|u_{\varrho}-v|^{\alpha}\,\mathrm{d}x\\
&\le c\|f_{\varrho}\|_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}(8B_h)}\|u_{\varrho}-v\|^{\alpha}_{L^{\mathfrak{p}}(8B_h)}\\
&\le c|h|^{\beta_0\alpha\mathfrak{a}(\mathfrak{p})}\|f_{\varrho}\|_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}(8B_h)}\|Du_{\varrho}-Dv\|^{\alpha}_{L^{p}(8B_h)}\\
&\le c|h|^{\beta_0\alpha\mathfrak{a}(\mathfrak{p})}\|f_{\varrho}\|_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}(8B_h)}\left(\int_{8B_h}|V(Du_{\varrho})-V(Dv)|^{2}\,\mathrm{d}x\right)^{\alpha/p}\\
&\quad+c\mathbb{1}_{p}|h|^{\beta_0\alpha\mathfrak{a}(\mathfrak{p})}\|f_{\varrho}\|_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}(8B_h)}\\
&\qquad\cdot\left(\int_{8B_h}|V(Du_{\varrho})-V(Dv)|^{p}(|Du_{\varrho}|+\mu)^{p(2-p)/2}\,\mathrm{d}x\right)^{\alpha/p}\\
&\le c|h|^{\beta_0\alpha\mathfrak{a}(\mathfrak{p})}\|f_{\varrho}\|_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}(8B_h)}\left(\int_{8B_h}|V(Du_{\varrho})-V(Dv)|^{2}\,\mathrm{d}x\right)^{\alpha/p}\\
&\quad+c\mathbb{1}_{p}|h|^{\beta_0\alpha\mathfrak{a}(\mathfrak{p})}M^{\frac{\alpha(2-p)}{2}}|h|^{\frac{\beta_0 n}{p}\frac{\alpha(2-p)}{2}}\|f_{\varrho}\|_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}(8B_h)}\\
&\qquad\cdot\left(\int_{8B_h}|V(Du_{\varrho})-V(Dv)|^{2}\,\mathrm{d}x\right)^{\alpha/2}\\
&\le\frac{1}{2\tilde{c}}\int_{8B_h}|V(Du_{\varrho})-V(Dv)|^{2}\,\mathrm{d}x+c|h|^{\frac{\beta_0\alpha p\mathfrak{a}(\mathfrak{p})}{p-\alpha}}\|f_{\varrho}\|^{\frac{\mathfrak{p}\theta(\mathfrak{p})}{\mathfrak{p}-\alpha}}_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}(8B_h)}
\end{aligned}
$$

$$+ c\mathbb{1}_p |h|^{\frac{\beta_0 \alpha 2\mathfrak{a}(\mathfrak{p})}{2-\alpha}} M^{\frac{\alpha(2-p)}{2-\alpha}} |h|^{\frac{\beta_0 n}{p}\frac{\alpha(2-p)}{2-\alpha}} \|f_\varrho\|^{\frac{\mathfrak{p}\sigma(\mathfrak{p})}{\mathfrak{p}-\alpha}}_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}(8B_h)}, \tag{5.47}$$

where $\theta(\mathfrak{p})$, $\sigma(\mathfrak{p})$ are defined in (5.12); in the last line we have used Young's inequality twice. Using the estimates found for (I),(II),(III) in (5.45), and reabsorbing terms, we come up with

$$\begin{aligned}\int_{8B_h} |V(Du_\varrho) - V(Dv)|^2 \,\mathrm{d}x &\le c_{\mathfrak{g}} |h|^{\beta_0 \alpha} M^{\alpha+\gamma q/p} \varrho^\alpha |B_h| \\ &\quad + c|h|^{\frac{\beta_0 \alpha p \mathfrak{a}(\mathfrak{p})}{p-\alpha}} \|f_\varrho\|^{\frac{\mathfrak{p}\theta(\mathfrak{p})}{\mathfrak{p}-\alpha}}_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}(8B_h)} \\ &\quad + c\mathbb{1}_p |h|^{\frac{\beta_0 \alpha 2\mathfrak{a}(\mathfrak{p})}{2-\alpha}} M^{\frac{\alpha(2-p)}{2-\alpha}} |h|^{\frac{\beta_0 n}{p}\frac{\alpha(2-p)}{2-\alpha}} \|f_\varrho\|^{\frac{\mathfrak{p}\sigma(\mathfrak{p})}{\mathfrak{p}-\alpha}}_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}(8B_h)}\end{aligned} \tag{5.48}$$

where $c, c_{\mathfrak{g}} \equiv c, c_{\mathfrak{g}}(n, p, q, \nu, L)$. We now use this last inequality to bound $E(Du_\varrho) - E(Dv)$ in $L^2$. Recalling (2.2) and (5.9), and that $t \mapsto (t-k)_+$ is 1-Lipschitz regular, we find

$$\begin{aligned}&|(E(Du_\varrho) - \kappa)_+ - (E(Dv) - \kappa)_+|^2 \\ &\quad \le |E(Du_\varrho) - E(Dv)|^2 \\ &\quad \le |[H(Du_\varrho)]^{p/2} - [H(Dv)]^{p/2}|^2 \\ &\quad \overset{(3.16)}{\lesssim_p} (|Du_\varrho|^2 + |Dv|^2 + \mu^2)^{p/2} |V(Du_\varrho) - V(Dv)|^2, \quad \forall\, x \in 8B_h .\end{aligned}$$

By finally using (5.42), we conclude with

$$\begin{aligned}&|(E(Du_\varrho) - \kappa)_+ - (E(Dv) - \kappa)_+|^2 \\ &\quad \le c M^{\mathfrak{s}p} |V(Du_\varrho) - V(Dv)|^2, \quad \text{for every } x \in 4B_h\end{aligned} \tag{5.49}$$

for $c \equiv c(n, p, q, \nu, L)$. We then estimate using, in order, $(3.7)_1$, (5.49) (twice) and (5.43)

$$\begin{aligned}\int_{B_h} |\tau_h (E(Du_\varrho) - \kappa)_+|^2 \,\mathrm{d}x &\le 2\int_{B_h} |\tau_h (E(Dv) - \kappa)_+|^2 \,\mathrm{d}x \\ &\quad + 2\int_{B_h} |\tau_h \big((E(Du_\varrho) - \kappa)_+ - (E(Dv) - \kappa)_+\big)|^2 \,\mathrm{d}x \\ &\le 2\int_{B_h} |\tau_h (E(Dv) - \kappa)_+|^2 \,\mathrm{d}x \\ &\quad + 2\int_{2B_h} |(E(Du_\varrho) - \kappa)_+ - (E(Dv) - \kappa)_+|^2 \,\mathrm{d}x \\ &\le 2\int_{B_h} |\tau_h (E(Dv) - \kappa)_+|^2 \,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&+ cM^{\mathfrak{s}p} \int_{2B_h} |V(Du_\varrho) - V(Dv)|^2 \,\mathrm{d}x \\
&\leq c|h|^{2(1-\beta_0)} M^{\mathfrak{s}(q-p)} \int_{4B_h} (E(Dv) - \kappa)_+^2 \,\mathrm{d}x \\
&\quad + cM^{\mathfrak{s}p} \int_{2B_h} |V(Du_\varrho) - V(Dv)|^2 \,\mathrm{d}x \\
&\leq c|h|^{2(1-\beta_0)} M^{\mathfrak{s}(q-p)} \int_{4B_h} (E(Du_\varrho) - \kappa)_+^2 \,\mathrm{d}x \\
&\quad + cM^{\mathfrak{s}q} \int_{4B_h} |V(Du_\varrho) - V(Dv)|^2 \,\mathrm{d}x \,. \qquad (5.50)
\end{aligned}
$$

We have again used (5.44). Matching this last inequality with (5.48) we get

$$
\begin{aligned}
\int_{B_h} |\tau_h (E(Du_\varrho) - \kappa)_+|^2 \,\mathrm{d}x &\leq c|h|^{2(1-\beta_0)} M^{\mathfrak{s}(q-p)} \int_{8B_h} (E(Du_\varrho) - \kappa)_+^2 \,\mathrm{d}x \\
&\quad + c_{\mathfrak{g}} |h|^{\beta_0 \alpha} M^{\mathfrak{s}q + \alpha + \gamma q/p} \varrho^\alpha |B_h| + c|h|^{\frac{\beta_0 \alpha p \mathfrak{a}(\mathfrak{p})}{p-\alpha}} M^{\mathfrak{s}q} \|f_\varrho\|_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}(8B_h)}^{\frac{\mathfrak{p}\theta(\mathfrak{p})}{\mathfrak{p}-\alpha}} \\
&\quad + c\mathbb{1}_p |h|^{\frac{\beta_0 \alpha 2 \mathfrak{a}(\mathfrak{p})}{2-\alpha}} M^{\mathfrak{s}q + \frac{\alpha(2-p)}{2-\alpha}} |h|^{\frac{\beta_0 n}{p} \frac{\alpha(2-p)}{2-\alpha}} \|f_\varrho\|_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}(8B_h)}^{\frac{\mathfrak{p}\sigma(\mathfrak{p})}{\mathfrak{p}-\alpha}} \qquad (5.51)
\end{aligned}
$$

with $c, c_{\mathfrak{g}} \equiv c, c_{\mathfrak{g}}(n, p, q, \nu, L)$. Recalling that $|h| \leq 1$, we first estimate

$$
|h|^{\beta_0 \alpha} + |h|^{\frac{\beta_0 \alpha p \mathfrak{a}(\mathfrak{p})}{p-\alpha}} + |h|^{\frac{\beta_0 \alpha 2 \mathfrak{a}(\mathfrak{p})}{2-\alpha}} \leq 3|h|^{\beta_0 \alpha_{\mathrm{m}}} \qquad (5.52)
$$

in (5.51), with $\alpha_{\mathrm{m}}$ which has been defined in (5.13), and then equalize the resulting exponents by taking

$$
\beta_0 = \frac{2}{2+\alpha_{\mathrm{m}}} \Longleftrightarrow \beta_0 \alpha_{\mathrm{m}} = 2(1-\beta_0)\,. \qquad (5.53)
$$

We conclude with

$$
\begin{aligned}
&\int_{B_h} |\tau_h (E(Du_\varrho) - \kappa)_+|^2 \,\mathrm{d}x \\
&\leq c|h|^{\frac{2\alpha_{\mathrm{m}}}{2+\alpha_{\mathrm{m}}}} M^{\mathfrak{s}(q-p)} \int_{8B_h} (E(Du_\varrho) - \kappa)_+^2 \,\mathrm{d}x \\
&\quad + c_{\mathfrak{g}} |h|^{\frac{2\alpha_{\mathrm{m}}}{2+\alpha_{\mathrm{m}}}} M^{\mathfrak{s}q + \alpha + \gamma q/p} \varrho^\alpha |B_h| + c|h|^{\frac{2\alpha_{\mathrm{m}}}{2+\alpha_{\mathrm{m}}}} M^{\mathfrak{s}q} \|f_\varrho\|_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}(8B_h)}^{\frac{\mathfrak{p}\theta(\mathfrak{p})}{\mathfrak{p}-\alpha}} \\
&\quad + c\mathbb{1}_p |h|^{\frac{2\alpha_{\mathrm{m}}}{2+\alpha_{\mathrm{m}}}} M^{\mathfrak{s}q + \frac{\alpha(2-p)}{2-\alpha}} |h|^{\frac{\beta_0 n}{p} \frac{\alpha(2-p)}{2-\alpha}} \|f_\varrho\|_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}(8B_h)}^{\frac{\mathfrak{p}\sigma(\mathfrak{p})}{\mathfrak{p}-\alpha}} \,. \qquad (5.54)
\end{aligned}
$$

### 5.3.3 Covering

We consider a fixed, standard lattice $\mathcal{L}_{|h|}$ of hypercubes of $\mathbb{R}^n$, with sidelength equal to $2|h|^{\beta_0}/\sqrt{n}$ (these are mutually disjoint open hypercubes with sides parallel to the coordinate axes, and whose union of closures covers $\mathbb{R}^n$). We also consider $\tilde{\mathcal{L}}_{|h|} := \{Q \in \mathcal{L}_{|h|} : \ Q \cap \mathcal{B}_{1/2} \neq \emptyset\}$. With $\mathfrak{n}$ denoting the cardinality of $\tilde{\mathcal{L}}_{|h|}$, we write $\tilde{\mathcal{L}}_{|h|} = \{Q_k \equiv Q_k(x_k)\}_{k\le \mathfrak{n}}$, with $x_k$ being the center of $Q_k$. Obviously, it is

$$|\mathcal{B}_{1/2} \setminus \bigcup_{k\le \mathfrak{n}} Q_k| = 0, \qquad Q_i \cap Q_j = \emptyset \ \Leftrightarrow \ i \neq j\,. \tag{5.55}$$

This family of hypercubes corresponds to a family of balls $\mathcal{L}_{|h|} := \{B_k \equiv B_k(x_k)\}_{k\le \mathfrak{n}}$ in the sense of (3.4), that is, such cubes can be identified as inner cubes of balls, i.e.,

$$Q_k \equiv Q_k(x_k) \equiv Q_{\mathrm{inn}}(B_k) \subset B_k := B_{|h|^{\beta_0}}(x_k)\,. \tag{5.56}$$

Note that the centers $x_k \equiv x_{\mathrm{c}}$ satisfy (5.35) as the diameter of such cubes is equal to $2|h|^{\beta_0}$. Summarizing, we find

$$\begin{cases} 8B_k \Subset \mathcal{B}_1\,, \ \ x_k \in \mathcal{B}_{1/2+2|h|^{\beta_0}} \ \text{ for all } \ k \le \mathfrak{n} \\ \mathfrak{n} \lesssim n^{n/2} 2^{-n} |h|^{-\beta_0 n} \equiv c(n)|h|^{-\beta_0 n}\,. \end{cases} \tag{5.57}$$

Each of the dilated balls $8B_k$ intersects the similar ones $8B_i$ (including itself) less than a finite number $c_{\mathrm{t}}(n)$, depending only on $n$ (uniform finite intersection property). This can be easily seen by observing that the outer cubes of the dilated balls $Q_{\mathrm{out}}(8B_k)$ in the sense of (3.4), whose sidelength is $16|h|^{\beta_0}$, touch similar ones $c_{\mathrm{t}}(n)$ times, as they are obtained by dilating of a factor $8\sqrt{n}$ the original ones $Q_k \equiv Q_{\mathrm{inn}}(B_k)$, which are mutually disjoint and have sides parallel to the coordinate axes. By considering the family of enlarged balls $8\mathcal{L}_{|h|} := \{8B_k : \ B_k \in \mathcal{L}_{|h|}\}$, we can therefore write

$$\begin{cases} 8\mathcal{L}_{|h|} = \displaystyle\bigcup_{i \le \tilde{c}_{\mathrm{t}}(n)} 8\mathcal{L}^i_{|h|}\,, \ \ \text{with } \tilde{c}_{\mathrm{t}}(n) \le c_{\mathrm{t}}(n) \text{ and } 8\mathcal{L}^i_{|h|} \cap 8\mathcal{L}^j_{|h|} \neq \emptyset \Longrightarrow i = j \\ 8\mathcal{L}^i_{|h|} \text{ is made of mutually disjoint balls, for every } i \le c_{\mathrm{t}}(n)\,. \end{cases}$$

As a consequence, given a Radon measure $\lambda$ defined on $\mathcal{B}_1$, recalling (5.57), we find

$$\sum_{k\le \mathfrak{n}} \lambda(8B_k) = \sum_{i=1}^{\tilde{c}_{\mathrm{t}}(n)} \sum_{8B \in 8\mathcal{L}^i_{|h|}} \lambda(8B) \le \sum_{i=1}^{\tilde{c}_{\mathrm{t}}(n)} \lambda(\mathcal{B}_1) = \tilde{c}_{\mathrm{t}}(n)\lambda(\mathcal{B}_1)\,. \tag{5.58}$$

Another inequality we shall often use is, for $t \in (0, 1]$

$$\sum_{k\le \mathfrak{n}} a_k^t \le \mathfrak{n}^{1-t} \Big(\sum_{k\le \mathfrak{n}} a_k\Big)^t \overset{(5.57)}{\le} c(n)|h|^{-\beta_0 n(1-t)} \Big(\sum_{k\le \mathfrak{n}} a_k\Big)^t\,, \tag{5.59}$$

that holds whenever $\{a_k\}_{k\le \mathfrak{n}}$ are non-negative numbers. This is a simple consequence of the discrete Hölder's inequality (see [36, 1.1.4, page 12]).

### 5.3.4 Proof of (5.11)

By (5.57) we can now consider the minimization problems in (5.36) for each one of the enlarged balls $8B_k$, i.e., we take $B_h \equiv B_k$, thereby getting (5.54). Inequalities (5.54) can be summed over $k \leq \mathfrak{n}$, and this yields

$$
\begin{aligned}
\int_{\mathcal{B}_{1/2}} |\tau_h(E(Du_\varrho)-\kappa)_+|^2\,\mathrm{d}x &\stackrel{(5.55)}{\leq} \sum_{k\leq\mathfrak{n}} \int_{Q_k} |\tau_h(E(Du_\varrho)-\kappa)_+)|^2\,\mathrm{d}x \\
&\stackrel{(5.56)}{\leq} \sum_{k\leq\mathfrak{n}} \int_{B_k} |\tau_h(E(Du_\varrho)-\kappa)_+)|^2\,\mathrm{d}x \\
&\stackrel{(5.54)}{\leq} c|h|^{\frac{2\alpha\mathrm{m}}{2+\alpha\mathrm{m}}} M^{\mathfrak{s}(q-p)} \sum_{k\leq\mathfrak{n}} \int_{8B_k} (E(Du_\varrho)-\kappa)_+^2\,\mathrm{d}x \\
&\qquad + c_{\mathrm{g}}|h|^{\frac{2\alpha\mathrm{m}}{2+\alpha\mathrm{m}}} M^{\mathfrak{s}q+\alpha+\gamma q/p}\varrho^{\alpha} \sum_{k\leq\mathfrak{n}} |B_k| \\
&\qquad + c|h|^{\frac{2\alpha\mathrm{m}}{2+\alpha\mathrm{m}}} M^{\mathfrak{s}q} \sum_{k\leq\mathfrak{n}} \|f_\varrho\|_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}(8B_k)}^{\frac{\mathfrak{p}\theta(\mathfrak{p})}{\mathfrak{p}-\alpha}} \\
&\qquad + c\mathbb{1}_p|h|^{\frac{2\alpha\mathrm{m}}{2+\alpha\mathrm{m}}} M^{\mathfrak{s}q+\frac{\alpha(2-p)}{2-\alpha}} |h|^{\frac{\beta_0 n}{p}\frac{\alpha(2-p)}{2-\alpha}} \sum_{k\leq\mathfrak{n}} \|f_\varrho\|_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}(8B_k)}^{\frac{\mathfrak{p}\sigma(\mathfrak{p})}{\mathfrak{p}-\alpha}}\,. 
\end{aligned} \tag{5.60}
$$

To estimate the first two sums appearing in the right-hand side of (5.60) we can use (5.58) in an obvious way

$$
\begin{cases}
\displaystyle\sum_{k\leq\mathfrak{n}} \int_{8B_k} (E(Du_\varrho)-\kappa)_+^2\,\mathrm{d}x \leq \tilde{c}_{\mathrm{t}}(n) \int_{\mathcal{B}_1} (E(Du_\varrho)-\kappa)_+^2\,\mathrm{d}x \\
\displaystyle\sum_{k\leq\mathfrak{n}} |B_k| \leq \tilde{c}_{\mathrm{t}}(n)|\mathcal{B}_1|\,.
\end{cases} \tag{5.61}
$$

For the third we proceed similarly, using that $\theta(\mathfrak{p}) \geq 1$

$$
\sum_{k\leq\mathfrak{n}} \|f_\varrho\|_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}(8B_k)}^{\frac{\mathfrak{p}\theta(\mathfrak{p})}{\mathfrak{p}-\alpha}} \leq \|f_\varrho\|_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}(\mathcal{B}_1)}^{\frac{\mathfrak{p}[\theta(\mathfrak{p})-1]}{\mathfrak{p}-\alpha}} \sum_{k\leq\mathfrak{n}} \int_{8B_k} f_\varrho^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}\,\mathrm{d}x \tag{5.62}
$$

$$
\stackrel{(5.58)}{\leq} \tilde{c}_{\mathrm{t}}(n)\|f_\varrho\|_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}(\mathcal{B}_1)}^{\frac{\mathfrak{p}\theta(\mathfrak{p})}{\mathfrak{p}-\alpha}}\,. \tag{5.63}
$$

As for the last sum in (5.60), this appears only when $p<2$ and here we distinguish two cases. The first is when $\sigma(\mathfrak{p}) \geq 1$. In this case we again argue as in (5.63), using (5.58), and we get (recall $|h| \leq 1$ by (5.34))

$$
|h|^{\frac{\beta_0 n}{p}\frac{\alpha(2-p)}{2-\alpha}} \sum_{k\leq\mathfrak{n}} \|f_\varrho\|_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}(8B_k)}^{\frac{\mathfrak{p}\sigma(\mathfrak{p})}{\mathfrak{p}-\alpha}} \leq \tilde{c}_{\mathrm{t}}(n)\|f_\varrho\|_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}(\mathcal{B}_1)}^{\frac{\mathfrak{p}\sigma(\mathfrak{p})}{\mathfrak{p}-\alpha}}\,. \tag{5.64}
$$

The second case is when, instead, $\sigma(\mathfrak{p}) < 1$ and we are led to use (5.59), with $t \equiv \sigma(\mathfrak{p})$ and $a_k \equiv \|f_\varrho\|^{\mathfrak{p}/(\mathfrak{p}-\alpha)}_{L^{\mathfrak{p}/(\mathfrak{p}-\alpha)}(8B_k)}$, thereby getting

$$|h|^{\frac{\beta_0 n}{p}\frac{\alpha(2-p)}{2-\alpha}}\sum_{k\leq \mathfrak{n}}\|f_\varrho\|^{\frac{\mathfrak{p}\sigma(\mathfrak{p})}{\mathfrak{p}-\alpha}}_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}(8B_k)} \leq c|h|^{\frac{\beta_0 n}{p}\frac{\alpha(2-p)}{2-\alpha}-\beta_0 n(1-\sigma(\mathfrak{p}))}\|f_\varrho\|^{\frac{\mathfrak{p}\sigma(\mathfrak{p})}{\mathfrak{p}-\alpha}}_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}(\mathcal{B}_1)}$$
$$\leq c\|f_\varrho\|^{\frac{\mathfrak{p}\sigma(\mathfrak{p})}{\mathfrak{p}-\alpha}}_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}(\mathcal{B}_1)}. \tag{5.65}$$

We have used that $|h| \leq 1$ and the identity

$$\frac{\beta_0 n}{p}\frac{\alpha(2-p)}{2-\alpha} - \beta_0 n(1-\sigma(\mathfrak{p})) = \frac{2\beta_0 n\alpha}{p\mathfrak{p}(2-\alpha)}(\mathfrak{p}-p) \overset{\mathfrak{p}\geq p}{\geq} 0.$$

Connecting the content of (5.61)-(5.65) to (5.60), we finally arrive at

$$\int_{\mathcal{B}_{1/2}}|\tau_h(E(Du_\varrho)-\kappa)_+|^2\,\mathrm{d}x$$
$$\leq c|h|^{\frac{2\alpha_{\mathrm{m}}}{2+\alpha_{\mathrm{m}}}}M^{\mathfrak{s}(q-p)}\int_{\mathcal{B}_1}(E(Du_\varrho)-\kappa)_+^2\,\mathrm{d}x + c_{\mathrm{g}}|h|^{\frac{2\alpha_{\mathrm{m}}}{2+\alpha_{\mathrm{m}}}}M^{\mathfrak{s}q+\alpha+\gamma q/p}\varrho^\alpha$$
$$+ c|h|^{\frac{2\alpha_{\mathrm{m}}}{2+\alpha_{\mathrm{m}}}}M^{\mathfrak{s}q}\|f_\varrho\|^{\frac{\mathfrak{p}\theta(\mathfrak{p})}{\mathfrak{p}-\alpha}}_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}(\mathcal{B}_1)} + c\mathbb{1}_p|h|^{\frac{2\alpha_{\mathrm{m}}}{2+\alpha_{\mathrm{m}}}}M^{\mathfrak{s}q+\frac{\alpha(2-p)}{2-\alpha}}\|f_\varrho\|^{\frac{\mathfrak{p}\sigma(\mathfrak{p})}{\mathfrak{p}-\alpha}}_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}(\mathcal{B}_1)}$$

for $c \equiv c(n, p, q, \nu, L)$. From this last inequality and Lemma 3.1, we deduce that $(E(Du_\varrho)-\kappa)_+ \in W^{\beta,2}(\mathcal{B}_{1/2})$ for all $\beta \in (0, \alpha_{\mathrm{m}}/(2+\alpha_{\mathrm{m}}))$ and that the inequality

$$\|(E(Du_\varrho)-\kappa)_+\|_{W^{\beta,2}(\mathcal{B}_{1/2})} \leq cM^{\frac{\mathfrak{s}(q-p)}{2}}\|(E(Du_\varrho)-\kappa)_+\|_{L^2(\mathcal{B}_1)}$$
$$+ c_{\mathrm{g}}M^{\frac{\mathfrak{s}q+\alpha+\gamma q/p}{2}}\varrho^{\frac{\alpha}{2}} + cM^{\frac{\mathfrak{s}q}{2}}\|f_\varrho\|^{\frac{\mathfrak{p}\theta(\mathfrak{p})}{2(\mathfrak{p}-\alpha)}}_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}(\mathcal{B}_1)}$$
$$+ c\mathbb{1}_p M^{\frac{\mathfrak{s}q}{2}+\frac{\alpha(2-p)}{2(2-\alpha)}}\|f_\varrho\|^{\frac{\mathfrak{p}\sigma(\mathfrak{p})}{2(\mathfrak{p}-\alpha)}}_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha}}(\mathcal{B}_1)}$$

holds with $c \equiv c(\texttt{data}, \beta)$. From this the full form of (5.32) follows via (3.10). This brings to a conclusion the proof of Proposition 5.1.

### 5.3.5 Proof of (5.10)

The only real modification with respect to the proof of (5.11) occurs in the estimate of the term (III) in (5.47), that we can now replace as follows:

$$\text{(III)} \leq c\int_{8B_h} f_\varrho(x)|u_\varrho - v|^\alpha\,\mathrm{d}x \leq \|u_\varrho - v\|^\alpha_{L^\infty(8B_h)}\|f_\varrho\|_{L^1(8B_h)}$$
$$\overset{(5.39)}{\leq} c|h|^{\beta_0\alpha}M^\alpha\|f_\varrho\|_{L^1(8B_h)}. \tag{5.66}$$

Using this estimate in place of (5.47), we can replace (5.48) by

$$\int_{8B_h} |V(Du_\varrho) - V(Dv)|^2 \, dx \\ \leq c_{\mathrm{g}} |h|^{\beta_0 \alpha} M^{\alpha+\gamma q/p} \varrho^\alpha |B_h| + c|h|^{\beta_0 \alpha} M^\alpha \|f_\varrho\|_{L^1(8B_h)} \tag{5.67}$$

with $c, c_{\mathrm{g}} \equiv c, c_{\mathrm{g}}(n, p, q, \nu, L)$. Proceeding as for the proof of (5.11), choosing $\beta_0$ as in (5.53) with $\alpha_{\mathrm{m}} = \alpha$, we arrive at (5.10) and Proposition 5.1 is proved.

### 5.4 Functionals of the type in (1.9)

We consider functionals $\mathcal{S}(\cdot, B_r)$ as in (1.9), assuming that

$$F(\cdot) \text{ satisfies (5.3), } \mathfrak{c}(\cdot) \text{ is as in (1.9), and } \mathtt{h}(\cdot) \text{ satisfies (5.5)}. \tag{5.68}$$

**Proposition 5.3** *Let $u \in W^{1,q}(B_r)$ be a minimizer of the functional $\mathcal{S}(\cdot, B_r)$ in (1.9), under assumptions (5.1) and (5.68). Let $B_\varrho(x_0) \Subset B_r$ and let $M \geq 1$ be a constant such that $\|Du\|_{L^\infty(B_\varrho(x_0))} \leq M$. Then, for every number $\kappa \geq 0$*

$$\left( \fint_{B_{\varrho/2}(x_0)} (E_\mu(Du) - \kappa)_+^{2\chi} \, dx \right)^{1/\chi} \leq c M^{\mathfrak{s}(q-p)} \fint_{B_\varrho(x_0)} (E_\mu(Du) - \kappa)_+^2 \, dx \\ + c M^{\mathfrak{s}q+p+\alpha-\mathrm{b}} \varrho^\alpha \fint_{B_\varrho(x_0)} (|Du|+1)^{q-p+\mathrm{b}} \, dx \\ + c M^{\mathfrak{s}q+\alpha} \varrho^\alpha \fint_{B_\varrho(x_0)} f \, dx \tag{5.69}$$

*holds for every $\mathrm{b} \in [0, p]$, and $(\beta, \chi)$ as in (5.8) with $\alpha_{\mathrm{m}} := \alpha$, where $c \equiv c(\mathtt{data}, \beta)$.*

***Proof*** We keep the notation introduced in Proposition 5.1. First we rescale $u$ as in (5.27), thereby passing to $u_\varrho \in W^{1,q}(\mathcal{B}_1)$. This is a minimizer of the functional

$$w \mapsto \mathcal{S}_\varrho(w, \mathcal{B}_1) := \int_{\mathcal{B}_1} [\mathfrak{c}_\varrho(x, w) F(Dw) + \mathtt{h}_\varrho(x, w)] \, dx, \tag{5.70}$$

which is defined on $W^{1,q}(\mathcal{B}_1)$, where $\mathfrak{c}_\varrho(x, y) := \mathfrak{c}(x_0 + \varrho x, \varrho y)$ and $\mathtt{h}_\varrho(x, y) := \mathtt{h}(x_0 + \varrho x, \varrho y)$. As a consequence of (5.68), and therefore of (1.9),

$$|\mathfrak{c}_\varrho(x_1, y_1) - \mathfrak{c}_\varrho(x_2, y_2)| \leq L\varrho^\alpha \left( |x_1 - x_2|^\alpha + |y_1 - y_2|^\alpha \right), \quad \nu \leq \mathfrak{c}_\varrho(\cdot) \leq L \tag{5.71}$$

holds for all $x_1, x_2 \in \mathcal{B}_1$, $y_1, y_2 \in \mathbb{R}$, and $\mathtt{h}_\varrho(\cdot)$ satisfies (5.31). It is sufficient to prove that

$$\|(E(Du_\varrho) - \kappa)_+\|^2_{L^{2\chi}(\mathcal{B}_{1/2})} \leq c M^{\mathfrak{s}(q-p)} \|(E(Du_\varrho) - \kappa)_+\|^2_{L^2(\mathcal{B}_1)}$$

$$
\begin{aligned}
&+ cM^{\mathfrak{s}q+p+\alpha-\mathrm{b}}\varrho^{\alpha}\int_{\mathcal{B}_1}(|Du_{\varrho}|+1)^{q-p+\mathrm{b}}\,\mathrm{d}x\\
&+ cM^{\mathfrak{s}q+\alpha}\|f_{\varrho}\|_{L^1(\mathcal{B}_1)}
\end{aligned} \tag{5.72}
$$

holds for $c \equiv c(\mathtt{data}, \beta)$, so that (5.69) follows scaling back to $u$. We next consider $v \in u_{\varrho} + W^{1,q}_0(8B_h)$ as the solution to (5.36), where this time it is $F_0(z) := \mathfrak{c}_{\varrho}(x_{\mathrm{c}}, (u_{\varrho})_{8B_h})F(z)$, and $x_{\mathrm{c}}$ is the center of $B_h$. Note that, thanks to (5.68), inequalities (5.38)-(5.42) apply to $v$. Recalling (5.70), we have, similarly to (5.45)

$$
\begin{aligned}
\frac{1}{\tilde{c}}\int_{8B_h}&|V(Du_{\varrho})-V(Dv)|^2\,\mathrm{d}x\\
&\le \int_{8B_h}\big[F_0(Du_{\varrho})-F_0(Dv)\big]\,\mathrm{d}x\\
&= \mathcal{S}_{\varrho}(u_{\varrho},8B_h)-\mathcal{S}_{\varrho}(v,8B_h)\\
&\quad+\int_{8B_h}[\mathfrak{c}_{\varrho}(x_{\mathrm{c}},(u_{\varrho})_{8B_h})F(Du_{\varrho})-\mathfrak{c}_{\varrho}(x,u_{\varrho})F(Du_{\varrho})]\,\mathrm{d}x\\
&\quad+\int_{8B_h}[\mathfrak{c}_{\varrho}(x,v)F(Dv)-\mathfrak{c}_{\varrho}(x_{\mathrm{c}},(u_{\varrho})_{8B_h})F(Dv)]\,\mathrm{d}x\\
&\quad+\int_{8B_h}[\mathrm{h}_{\varrho}(x,v)-\mathrm{h}_{\varrho}(x,u_{\varrho})]\,\mathrm{d}x\\
&=: \overbrace{\mathcal{S}_{\varrho}(u_{\varrho},8B_h)-\mathcal{S}_{\varrho}(v,8B_h)}^{\le 0 \text{ by minimality of } u_{\varrho}}+(\mathrm{I})+(\mathrm{II})+(\mathrm{III})\\
&\le (\mathrm{I})+(\mathrm{II})+(\mathrm{III})\,.
\end{aligned} \tag{5.73}
$$

Using (5.71) yields

$$
\begin{aligned}
(\mathrm{I}) &\le c\varrho^{\alpha}\int_{8B_h}\big(|x-x_{\mathrm{c}}|^{\alpha}+|u_{\varrho}-(u_{\varrho})_{8B_h}|^{\alpha}\big)(|Du_{\varrho}|+1)^q\,\mathrm{d}x\\
&\le c|h|^{\beta_0\alpha}(M^{\alpha}+1)\varrho^{\alpha}\int_{8B_h}(|Du_{\varrho}|+1)^q\,\mathrm{d}x\\
&\le c|h|^{\beta_0\alpha}M^{p+\alpha-\mathrm{b}}\varrho^{\alpha}\int_{8B_h}(|Du_{\varrho}|+1)^{q-p+\mathrm{b}}\,\mathrm{d}x
\end{aligned} \tag{5.74}
$$

For (II), note that the minimality of $v$ and $\nu \le \mathfrak{c}_{\varrho}(\cdot) \le L$ give

$$
\begin{aligned}
\int_{8B_h}F(Dv)\,\mathrm{d}x &\le \frac{1}{\nu}\int_{8B_h}\mathfrak{c}_{\varrho}(x_{\mathrm{c}},(u_{\varrho})_{8B_h})F(Dv)\,\mathrm{d}x\\
&\le \frac{1}{\nu}\int_{8B_h}\mathfrak{c}_{\varrho}(x_{\mathrm{c}},(u_{\varrho})_{8B_h})F(Du_{\varrho})\,\mathrm{d}x\\
&\le c(\nu,L)\int_{8B_h}(|Du_{\varrho}|+1)^q\,\mathrm{d}x\,.
\end{aligned} \tag{5.75}
$$

Using the content of the last display, (5.71) and then (5.39), we get

$$
\begin{aligned}
\text{(II)} &\le c\varrho^{\alpha} \int_{8B_h} \left(|x-x_{\rm c}|^{\alpha} + |v-(v)_{8B_h}|^{\alpha} + |(v)_{8B_h} - (u_{\varrho})_{8B_h}|^{\alpha}\right) F(Dv)\,{\rm d}x \\
&\le c|h|^{\beta_0\alpha}(M^{\alpha}+1)\varrho^{\alpha} \int_{8B_h} F(Dv)\,{\rm d}x \\
&\le c|h|^{\beta_0\alpha} M^{\alpha}\varrho^{\alpha} \int_{8B_h} (|Du_{\varrho}|+1)^{q}\,{\rm d}x \\
&\le c|h|^{\beta_0\alpha} M^{p+\alpha-{\rm b}}\varrho^{\alpha} \int_{8B_h} (|Du_{\varrho}|+1)^{q-p+{\rm b}}\,{\rm d}x\,. 
\end{aligned} \tag{5.76}
$$

The term (III) can be estimated exactly as in (5.66). Using this together with (5.73)-(5.76) yields

$$
\begin{aligned}
\int_{8B_h} |V(Du_{\varrho})-V(Dv)|^2\,{\rm d}x &\le c|h|^{\beta_0\alpha} M^{p+\alpha-{\rm b}}\varrho^{\alpha} \int_{8B_h} (|Du_{\varrho}|+1)^{q-p+{\rm b}}\,{\rm d}x \\
&\quad + c|h|^{\beta_0\alpha} M^{\alpha} \|f_{\varrho}\|_{L^1(8B_h)}\,.
\end{aligned} \tag{5.77}
$$

Using (5.49)-(5.50), employing (5.77) in (5.50), and choosing $\beta_0 := 2/(2+\alpha)$, we arrive at

$$
\begin{aligned}
\int_{B_h} |\tau_h(E(Du_{\varrho})-\kappa)_+|^2\,{\rm d}x &\le c|h|^{\frac{2\alpha}{2+\alpha}} M^{\mathfrak{s}(q-p)} \int_{8B_h} (E(Du_{\varrho})-\kappa)_+^2\,{\rm d}x \\
&\quad + c|h|^{\frac{2\alpha}{2+\alpha}} M^{\mathfrak{s}q+p+\alpha-{\rm b}}\varrho^{\alpha} \int_{8B_h} (|Du_{\varrho}|+1)^{q-p+{\rm b}}\,{\rm d}x \\
&\quad + c|h|^{\frac{2\alpha}{2+\alpha}} M^{\mathfrak{s}q+\alpha} \|f_{\varrho}\|_{L^1(8B_h)}
\end{aligned}
$$

for $c \equiv c(\texttt{data})$. This estimate can be used as a replacement of (5.54) and proceeding as in the proofs of Proposition 5.1 and 5.2 we conclude with

$$
\begin{aligned}
\int_{\mathcal{B}_{1/2}} |\tau_h(E(Du_{\varrho})-\kappa)_+|^2\,{\rm d}x &\le c|h|^{\frac{2\alpha}{2+\alpha}} M^{\mathfrak{s}(q-p)} \int_{\mathcal{B}_1} (E(Du_{\varrho})-\kappa)_+^2\,{\rm d}x \\
&\quad + c|h|^{\frac{2\alpha}{2+\alpha}} M^{\mathfrak{s}q+p+\alpha-{\rm b}}\varrho^{\alpha} \int_{\mathcal{B}_1} (|Du_{\varrho}|+1)^{q-p+{\rm b}}\,{\rm d}x \\
&\quad + c|h|^{\frac{2\alpha}{2+\alpha}} M^{\mathfrak{s}q+\alpha} \|f_{\varrho}\|_{L^1(\mathcal{B}_1)}\,.
\end{aligned}
$$

As done for Proposition 5.1, from this last inequality, Lemma 3.1 and (3.10), we deduce that $(E(Du_{\varrho})-\kappa)_+ \in W^{\beta,2}(\mathcal{B}_{1/2})$ for all $\beta < \alpha/(2+\alpha)$, with (5.72) that follows accordingly. $\square$

### 5.5 Functionals of the type in (2.15)

The assumptions on the integrand $F\colon B_r \times \mathbb{R}^n \to [0,\infty)$ we consider here are

$$\begin{cases} z \mapsto F(x,z) \text{ satisfies (5.3) uniformly with respect to } x \in B_r \\ |\partial_z F(x_1,z) - \partial_z F(x_2,z)| \\ \quad \le \tilde{L}|x_1-x_2|^{\alpha}([H_\mu(z)]^{(q-1)/2} + [H_\mu(z)]^{(p-1)/2}) \end{cases} \tag{5.78}$$

whenever $x_1, x_2 \in B_r$, $z \in \mathbb{R}^n$, and where $0 < \mu \le 2$. In the following, we denote $A(\cdot) := \partial_z F(\cdot)$, so that the Euler-Lagrange equation of $\mathcal{F}_{\mathrm{x}}$ reads as $\operatorname{div} A(x, Du) = 0$ and any $W^{1,q}$-regular minimizer is an energy solution. Moreover, this time $(\beta,\chi)$ might also be such that

$$\beta < \frac{\alpha}{1+\alpha} \qquad \text{and} \qquad \chi := \frac{n}{n-2\beta}\,. \tag{5.79}$$

**Proposition 5.4** *Let $u \in W^{1,q}(B_r)$ be a minimizer of the functional $\mathcal{F}_{\mathrm{x}}(\cdot, B_r)$ in (2.15), under assumptions (5.1) and (5.78). Let $B_\varrho(x_0) \Subset B_r$ and let $M \ge 1$ be a constant such that $\|Du\|_{L^\infty(B_\varrho(x_0))} \le M$. Let $\kappa \ge 0$ be a number.*

- *If $p \ge 2$, then*

$$\left(\fint_{B_{\varrho/2}(x_0)} (E_\mu(Du)-\kappa)_+^{2\chi}\,\mathrm{d}x\right)^{1/\chi} \le cM^{\mathfrak{s}(q-p)} \fint_{B_\varrho(x_0)} (E_\mu(Du)-\kappa)_+^2\,\mathrm{d}x$$
$$+ cM^{\mathfrak{s}q+p-\mathrm{b}}\varrho^{2\alpha} \fint_{B_\varrho(x_0)} (|Du|+1)^{2q-2p+\mathrm{b}}\,\mathrm{d}x \tag{5.80}$$

  *holds for every* $\mathrm{b} \in [0,p]$ *and $(\beta,\chi)$ as in (5.79), where $c \equiv c(\mathtt{data}, \beta)$.*
- *If $1 < p < 2$, then*

$$\left(\fint_{B_{\varrho/2}(x_0)} (E_\mu(Du)-\kappa)_+^{2\chi}\,\mathrm{d}x\right)^{1/\chi} \le cM^{\mathfrak{s}(q-p)} \fint_{B_\varrho(x_0)} (E_\mu(Du)-\kappa)_+^2\,\mathrm{d}x$$
$$+ cM^{(\mathfrak{s}+1)q-\mathrm{b}/p}\varrho^{\alpha} \left(\fint_{B_\varrho(x_0)} (|Du|+1)^{q-p+\mathrm{b}}\,\mathrm{d}x\right)^{1/p} \tag{5.81}$$

  *holds for every* $\mathrm{b} \in [0,p]$, *and $(\beta,\chi)$ as in (5.8) with $\alpha_{\mathrm{m}} := \alpha$, where $c \equiv c(\mathtt{data}, \beta)$.*

***Proof of Proposition 5.4*** We again build on the general arguments exposed in Proposition 5.1. The rescaled function $u_\varrho$ defined in (5.27) minimizes

$$w \mapsto \int_{\mathcal{B}_1} F_\varrho(x, Dw)\,\mathrm{d}x,$$

where $F_\varrho(x,z) := F(x_0+\varrho x, z)$, and therefore solves

$$-\operatorname{div} A_\varrho(x, Du_\varrho) = 0 \quad \text{in } \mathcal{B}_1, \text{ where } \; A_\varrho(x,z) := A(x_0+\varrho x, z)\,. \tag{5.82}$$

Note that

$$|A_\varrho(x_1,z)-A_\varrho(x_2,z)|\le \tilde L\varrho^\alpha|x_1-x_2|^\alpha([H_\mu(z)]^{(q-1)/2}+[H_\mu(z)]^{(p-1)/2}) \quad (5.83)$$

holds for every choice of $x_1, x_2 \in \mathcal{B}_1$ and $z\in\mathbb{R}^n$, as a consequence of (5.78)$_2$. We define $A_0(z) := A_\varrho(x_c, z)$, where $B_h$ is centred at $x_c$, which is strictly $p$-monotone in the sense that (3.18) holds (with $c\equiv c(n,p,\tilde\nu)$). We then define $v$ as the unique solution to the Dirichlet problem

$$\begin{cases}\displaystyle\int_{8B_h} A_0(Dv)\cdot D\varphi\,\mathrm{d}x = 0 \quad \text{for every } \varphi\in W^{1,q}_0(8B_h)\\ v\in u_\varrho + W^{1,q}_0(8B_h)\,,\end{cases} \quad (5.84)$$

that coincides with (5.37), where $F_0(z)=F_\varrho(x_c,z)$, so that we can use (5.38)-(5.42). We have

$$\begin{aligned}
&\int_{8B_h}|V(Du_\varrho)-V(Dv)|^2\,\mathrm{d}x\\
&\stackrel{(3.18)}{\le} c\int_{8B_h}\big(A_0(Du_\varrho)-A_0(Dv)\big)\cdot\big(Du_\varrho-Dv\big)\,\mathrm{d}x\\
&\stackrel{(5.84)}{=} c\int_{8B_h}\big(A_0(Du_\varrho)-A_\varrho(x,Du_\varrho)\big)\cdot\big(Du_\varrho-Dv\big)\,\mathrm{d}x\\
&\stackrel{(5.83)}{\le} c\varrho^\alpha|h|^{\beta_0\alpha}\int_{8B_h}\Big([H(Du_\varrho)]^{(q-1)/2}+[H(Du_\varrho)]^{(p-1)/2}\Big)|Du_\varrho-Dv|\,\mathrm{d}x\\
&=: c(\mathrm{I})\,, \qquad (5.85)
\end{aligned}$$

where $c\equiv c(\texttt{data})$. We focus on (5.80) and therefore on the case $p\ge 2$. Young's inequality and (3.12) give

$$\begin{aligned}
c(\mathrm{I}) &\le \frac12\int_{8B_h}|V(Du_\varrho)-V(Dv)|^2\,\mathrm{d}x\\
&\quad + c\varrho^{2\alpha}|h|^{2\beta_0\alpha}\int_{8B_h}\Big([H(Du_\varrho)]^{q-1}+[H(Du_\varrho)]^{p-1}\Big)\\
&\qquad\qquad \cdot(|Du_\varrho|^2+|Dv|^2+\mu^2)^{(2-p)/2}\,\mathrm{d}x\\
&\le \frac12\int_{8B_h}|V(Du_\varrho)-V(Dv)|^2\,\mathrm{d}x\\
&\quad + c|h|^{2\beta_0\alpha}\varrho^{2\alpha}\int_{8B_h}\Big([H(Du_\varrho)]^{(2q-p)/2}+[H(Du_\varrho)]^{p/2}\Big)\,\mathrm{d}x\\
&\le \frac12\int_{8B_h}|V(Du_\varrho)-V(Dv)|^2\,\mathrm{d}x\\
&\quad + c|h|^{2\beta_0\alpha}M^{p-\mathrm{b}}\varrho^{2\alpha}\int_{8B_h}(|Du_\varrho|+1)^{2q-2p+\mathrm{b}}\,\mathrm{d}x\,.
\end{aligned}$$

Connecting this last inequality to (5.85) yields

$$\begin{aligned}\int_{8B_h} &|V(Du_\varrho)-V(Dv)|^2\,\mathrm{d}x\\ &\le c|h|^{2\beta_0\alpha}M^{p-\mathrm{b}}\varrho^{2\alpha}\int_{8B_h}(|Du_\varrho|+1)^{2q-2p+\mathrm{b}}\,\mathrm{d}x\,. \end{aligned}\tag{5.86}$$

Inserting this last estimate in (5.50) we can repeat the scheme of proof of Proposition 5.1, and this eventually leads to (5.80). The only difference is that this time, in order to equalize the exponents of $|h|$, we choose $\beta_0 := 1/(1+\alpha)$ instead of making the choice in (5.53). We now consider the range $1 < p < 2$, to which we specialize for the rest of the proof. We estimate the term $c(\mathrm{I})$ appearing in (5.85) in a different way, also using the minimality of $v$, as follows:

$$\begin{aligned}c(\mathrm{I}) &\le c|h|^{\beta_0\alpha}(M^{q-1}+M^{p-1}+1)\varrho^\alpha\int_{8B_h}|Du_\varrho-Dv|\,\mathrm{d}x\\
&\le c|h|^{\beta_0\alpha}M^{q-1}|B_h|^{(p-1)/p}\varrho^\alpha\left(\int_{8B_h}|Du_\varrho-Dv|^p\,\mathrm{d}x\right)^{1/p}\\
&\le c|h|^{\beta_0\alpha}M^{q-1}|B_h|^{(p-1)/p}\varrho^\alpha\left(\int_{8B_h}[F_0(Du_\varrho)+F_0(Dv)]\,\mathrm{d}x\right)^{1/p}\\
&\le c|h|^{\beta_0\alpha}M^{q-1}|B_h|^{(p-1)/p}\varrho^\alpha\left(\int_{8B_h}F_0(Du_\varrho)\,\mathrm{d}x\right)^{1/p}\\
&\le c|h|^{\beta_0\alpha}M^{q-1}|B_h|^{(p-1)/p}\varrho^\alpha\left(\int_{8B_h}(|Du_\varrho|+1)^q\,\mathrm{d}x\right)^{1/p}\\
&\le c|h|^{\beta_0\alpha}M^{q-\mathrm{b}/p}|B_h|^{(p-1)/p}\varrho^\alpha\left(\int_{8B_h}(|Du_\varrho|+1)^{q-p+\mathrm{b}}\,\mathrm{d}x\right)^{1/p}\,.\end{aligned}\tag{5.87}$$

Connecting the content of the last displays to (5.85) yields

$$\begin{aligned}\int_{8B_h}&|V(Du_\varrho)-V(Dv)|^2\,\mathrm{d}x\\ &\le c|h|^{\beta_0\alpha}M^{q-\mathrm{b}/p}\varrho^\alpha|h|^{\frac{\beta_0 n(p-1)}{p}}\||Du_\varrho|+1\|_{L^{q-p+\mathrm{b}}(8B_h)}^{\frac{q-p+\mathrm{b}}{p}}\end{aligned}\tag{5.88}$$

where $c\equiv c(\texttt{data})$. Using (5.88) in (5.50), and taking $\beta_0 = 2/(2+\alpha)$, we arrive at the following analog of (5.54):

$$\begin{aligned}\int_{B_h}|\tau_h(E(Du_\varrho)-\kappa)_+|^2\,\mathrm{d}x &\le c|h|^{\frac{2\alpha}{2+\alpha}}M^{\mathfrak{s}(q-p)}\int_{8B_h}(E(Du_\varrho)-\kappa)_+^2\,\mathrm{d}x\\
&\quad + c|h|^{\frac{2\alpha}{2+\alpha}}M^{(\mathfrak{s}+1)q-\mathrm{b}/p}\varrho^\alpha|h|^{\frac{\beta_0 n(p-1)}{p}}\||Du_\varrho|+1\|_{L^{q-p+\mathrm{b}}(8B_h)}^{\frac{q-p+\mathrm{b}}{p}}\,.\end{aligned}\tag{5.89}$$

The last step is to perform the covering argument in Sect. 5.3.3, summing up inequalities in (5.89) over balls $B_h \equiv B_k$ for $k \leq \mathfrak{n}$ as done in (5.60). The sum of the first terms in the right-hand sides can be dealt with as in (5.61). For the second ones, we use (5.59) with $t = 1/p$ and $a_k \equiv \||Du_\varrho| + 1\|^{q-p+\mathfrak{b}}_{L^{q-p+\mathfrak{b}}(8B_k)}$, arguing as in (5.64) we obtain

$$|h|^{\frac{\beta_0 n(p-1)}{p}} \sum_{k\leq \mathfrak{n}} \||Du_\varrho| + 1\|^{\frac{q-p+\mathfrak{b}}{p}}_{L^{q-p+\mathfrak{b}}(8B_k)} \stackrel{(5.59)}{\leq} c\left(\sum_{k\leq \mathfrak{n}} \||Du_\varrho| + 1\|^{q-p+\mathfrak{b}}_{L^{q-p+\mathfrak{b}}(8B_k)}\right)^{1/p}$$
$$\stackrel{(5.58)}{\leq} c\||Du_\varrho| + 1\|^{\frac{q-p+\mathfrak{b}}{p}}_{L^{q-p+\mathfrak{b}}(\mathcal{B}_1)}, \tag{5.90}$$

for $c \equiv c(n, p)$. From (5.89), after summation, we arrive at

$$\int_{\mathcal{B}_{1/2}} |\tau_h(E(Du_\varrho) - \kappa)_+|^2 \,\mathrm{d}x \leq c|h|^{\frac{2\alpha}{2+\alpha}} M^{\mathfrak{s}(q-p)} \int_{\mathcal{B}_1} (E(Du_\varrho) - \kappa)_+^2 \,\mathrm{d}x$$
$$+ c|h|^{\frac{2\alpha}{2+\alpha}} M^{(\mathfrak{s}+1)q-\mathfrak{b}/p} \varrho^\alpha \||Du_\varrho| + 1\|^{\frac{q-p+\mathfrak{b}}{p}}_{L^{q-p+\mathfrak{b}}(\mathcal{B}_1)}$$

and the rest of the proof of (5.81) can now be obtained as in Proposition 5.1. □

### 5.6 Equations

The proof of Proposition 5.4 makes little use of minimality of $u$. This leads to extend its content to solutions to general equations as

$$-\operatorname{div} A(x, Du) = 0 \qquad \text{in } B_r\,, \tag{5.91}$$

that are not necessarily arising as Euler-Lagrange equations of any functional. For this, we consider a general vector field $A\colon B_r \times \mathbb{R}^n \to \mathbb{R}^n$ satisfying (2.26), with $\nu$, $L$ replaced by general constants $\tilde{\nu}$, $\tilde{L}$ as in Sect. 2.1, with $0 < \mu \leq 2$ and such that

$$\nu_0[H_\mu(z)]^{(q-2)/2}|\xi|^2 + \tilde{\nu}[H_\mu(z)]^{(p-2)/2}|\xi|^2 \leq \partial_z A(x, z)\xi \cdot \xi \tag{5.92}$$

holds with the same notation of (2.26). Let us record the following coercivity inequality:

$$|z_2|^p \leq c[H_1(z_1)]^{p(q-1)/[2(p-1)]} + cA(x, z_2) \cdot (z_2 - z_1)\,, \tag{5.93}$$

which is valid under the assumptions satisfied by $A_0(\cdot)$ for every $z_1, z_2 \in \mathbb{R}^n$, $x \in B_r$, where $c \equiv c(\mathtt{data}) \geq 1$. The proof of (5.93) relies on (3.18) and is a minor variant of the one in [65, Lemma 4.4]. We shall argue under the permanent assumption

$$\frac{q}{p} < 1 + \frac{1}{n}\,. \tag{5.94}$$

Accordingly, in the present setting a relevant exponent is given by

$$\mathfrak{t} := \frac{p}{(n+1)p - nq}\frac{q-1}{p-1}\,, \tag{5.95}$$

unless $n=2, q>p$, when $\mathfrak{t}$ is any larger quantity. Note that $\mathfrak{t}\geq 1$ and $\mathfrak{t}=1$ when $p=q$; $\mathfrak{t}$ is well defined thanks to (5.94). This exponent plays for equations the same role that $\mathfrak{s}$ in (5.2) plays for functionals. Note that $\mathfrak{t}\geq\mathfrak{s}$ for $n>2$.

**Proposition 5.5** *Let $u\in W^{1,q}(B_r)$ be a weak solution to* (5.91), *under assumptions* (2.26) *with* $0<\mu\leq 2$ *and* $\nu$, $L$ *replaced by* $\tilde{\nu}$, $\tilde{L}$ *(as in Sect.* 2.1*); also assume* (5.92) *and* (5.94)*. Let $B_\varrho(x_0)\Subset B_r$ and let $M\geq 1$ be a constant such that $\|Du\|_{L^\infty(B_\varrho(x_0))}\leq M$. Let $\kappa\geq 0$ be a number.*

- *If $p\geq 2$, then*

$$\left(\fint_{B_{\varrho/2}(x_0)}(E_\mu(Du)-\kappa)_+^{2\chi}\,\mathrm{d}x\right)^{1/\chi}\leq cM^{2\mathfrak{t}(q-p)}\fint_{B_\varrho(x_0)}(E_\mu(Du)-\kappa)_+^2\,\mathrm{d}x$$
$$+cM^{\mathfrak{t}(2q-p)+p-\mathrm{b}}\varrho^{2\alpha}\fint_{B_\varrho(x_0)}(|Du|+1)^{2q-2p+\mathrm{b}}\,\mathrm{d}x \tag{5.96}$$

  *holds for every* $\mathrm{b}\in[0,p]$ *and* $(\beta,\chi)$ *as in* (5.79), *where* $c\equiv c(\mathtt{data},\beta)$.
- *If $1<p<2$, then*

$$\left(\fint_{B_{\varrho/2}(x_0)}(E_\mu(Du)-\kappa)_+^{2\chi}\,\mathrm{d}x\right)^{1/\chi}\leq cM^{2\mathfrak{t}(q-p)}\fint_{B_\varrho(x_0)}(E_\mu(Du)-\kappa)_+^2\,\mathrm{d}x$$
$$+cM^{(\mathfrak{t}+1)q+\mathfrak{t}(q-p)-\mathrm{b}/p}\varrho^{\alpha}\left(\fint_{B_\varrho(x_0)}(|Du|+1)^{\frac{p(q-1)}{p-1}-p+\mathrm{b}}\,\mathrm{d}x\right)^{1/p} \tag{5.97}$$

  *holds for every* $\mathrm{b}\in[0,p]$, *and* $(\beta,\chi)$ *as in* (5.8) *with* $\alpha_{\mathrm{m}}:=\alpha$, *where* $c\equiv c(\mathtt{data},\beta)$.

***Proof of Proposition 5.5*** We modify the proof of Proposition 5.4. Keeping the notation introduced there, we arrive up to (5.84), where again it is $A_0(z):=A_\varrho(x_{\mathrm{c}},z)$. Now, while we can still use (5.38)-(5.39), that hold for solutions to general equations, we have to find replacements for (5.40)-(5.42), that are linked to minimality. Thanks to (5.94), we can use (5.24), that yields

$$\|Dv\|_{L^\infty(4B_h)}\leq c\left(\fint_{8B_h}(|Dv|+1)^p\,\mathrm{d}x\right)^{\frac{\mathfrak{t}(p-1)}{p(q-1)}}.$$

Using (5.93) (applied to $A_0(\cdot)$) with $z_2\equiv Dv$, $z_1\equiv Du_\varrho$, and integrating over $8B_h$, gives

$$\fint_{8B_h}|Dv|^p\,\mathrm{d}x\leq c\fint_{8B_h}(|Du_\varrho|+1)^{\frac{p(q-1)}{p-1}}\,\mathrm{d}x\,. \tag{5.98}$$

Matching the inequalities in the last two displays we conclude with the following analog of (5.42):

$$\|Dv\|_{L^\infty(4B_h)}\leq cM^{\mathfrak{t}},\quad c\equiv c(n,p,q,\nu,L)\,. \tag{5.99}$$

This inequality allows to get analogs of (5.49)-(5.50), via this time the use of (5.17), that is

$$\int_{B_h}|\tau_h(E(Du_\varrho)-\kappa)_+|^2\,\mathrm{d}x \le c|h|^{2(1-\beta_0)}M^{2\mathfrak{t}(q-p)}\int_{4B_h}(E(Du_\varrho)-\kappa)_+^2\,\mathrm{d}x + cM^{\mathfrak{t}(2q-p)}\int_{4B_h}|V(Du_\varrho)-V(Dv)|^2\,\mathrm{d}x\,. \tag{5.100}$$

When $p \ge 2$, combining this last estimate with (5.86), that holds for general equations too, we finally arrive at (5.96) as in Proposition 5.4. It again remains to treat the case $p < 2$ as in Proposition 5.4 we have used minimality to deal with this case and here we need to take a different route. By looking at $c(\mathrm{I})$, defined as in (5.85), as in (5.87), we have

$$\begin{aligned}(\mathrm{I}) \ &\le \ c|h|^{\beta_0\alpha}M^{q-1}|B_h|^{(p-1)/p}\varrho^\alpha\left(\int_{8B_h}(|Du_\varrho|^p+|Dv|^p)\,\mathrm{d}x\right)^{1/p} \qquad (5.101)\\ &\overset{(5.98)}{\le} c|h|^{\beta_0\alpha}M^{q-\mathfrak{b}/p}|B_h|^{(p-1)/p}\varrho^\alpha\left(\int_{8B_h}(|Du_\varrho|+1)^{\frac{p(q-1)}{p-1}-p+\mathfrak{b}}\,\mathrm{d}x\right)^{1/p}.\end{aligned}$$

Connecting this to (5.85), and the resulting inequality to (5.100), yields

$$\begin{aligned}\int_{B_h}|\tau_h(E(Du_\varrho)-\kappa)_+|^2\,\mathrm{d}x &\le c|h|^{\frac{2\alpha}{2+\alpha}}M^{2\mathfrak{t}(q-p)}\int_{8B_h}(E(Du_\varrho)-\kappa)_+^2\,\mathrm{d}x\\ &+c|h|^{\frac{2\alpha}{2+\alpha}}cM^{(\mathfrak{t}+1)q+\mathfrak{t}(q-p)-\mathfrak{b}/p}\varrho^\alpha|h|^{\frac{\beta_0 n(p-1)}{p}}\left(\int_{8B_h}(|Du_\varrho|+1)^{\frac{p(q-1)}{p-1}-p+\mathfrak{b}}\,\mathrm{d}x\right)^{1/p}\end{aligned}$$

where $c \equiv c(\mathtt{data})$. After this, we can proceed as after (5.89), including the summation argument in (5.90), that works verbatim also with the new exponents, finally leading to (5.97). □

### 5.7 A priori Hölder

We briefly justify the validity of (5.7), which is in fact completely standard when $\mathtt{h}(\cdot) \equiv 0$ [29, 61, 62]. We report some details of the proof of the result, that, in the form stated here, does not seem to be explicitly mentioned in the literature, although it can be obtained by totally standard arguments. Here we consider a minimizer $u \in W^{1,q}(B_r)$ of the functional $\mathcal{F}(\cdot, B_r)$ in (2.1), and we assume (5.5) and that $\mathbb{F}(\cdot) \equiv \tilde{F}(\cdot)$ satisfies conditions (5.6). Therefore we are covering the functionals described in Sects. 5.1, 5.4 and 5.5. With $B_\tau(x_c) \Subset B_r$ being a ball, we define $v \in u + W^{1,q}_0(B_\tau)$ as the unique minimizer of $w \mapsto \int_{B_\tau}\mathbb{F}(x_c, (u)_{B_\tau}, Dw)\,\mathrm{d}x$ in the Dirichlet class $u + W^{1,q}_0(B_\tau)$. This satisfies the following a priori estimates

$$\begin{cases}\|Dv\|^q_{L^\infty(B_{\tau/2})} \le c\displaystyle\fint_{B_\tau}(|Dv|+1)^q\,\mathrm{d}x\\ \operatorname{osc}(Dv, B_\varrho) \le c\left(\dfrac{\varrho}{\tau}\right)^{\beta_1}\|Dv\|_{L^\infty(B_\tau)}\end{cases} \tag{5.102}$$

for every $\varrho \le \tau$, where both $c \ge 1$ and $\beta_1 \in (0,1)$ depend on $n, q, L, L_0, \mu, \nu_0$; see for instance [61, (2.4)-(2.5)]. Moreover, by Lemma 5.2 it is $\operatorname{osc}(v, B_\tau) \le \operatorname{osc}(u, B_\tau)$ and $\|u - v\|_{L^\infty(B_\tau)} \le \operatorname{osc}(u, B_\tau)$. Finally, by minimality it is $\|Dv\|_{L^q(B_\tau)} \lesssim \||Du| + 1\|_{L^q(B_\tau)}$. Next, note that also thanks to $(5.5)_2$, the integrand $\mathbb{F}(\cdot) + \mathtt{h}(\cdot)$ satisfies the growth conditions $|z|^q - |y| - 1 \lesssim \mathbb{F}(x, y, z) + \mathtt{h}(x, y) \lesssim |z|^q + |y| + 1$, with implied constants depending on $n, q, L, L_0, \mu, \nu_0$. Therefore, by De Giorgi-Nash-Moser theory it follows that $u \in C^{0,\beta_2}_{\mathrm{loc}}(\Omega)$ for some $\beta_2 \equiv \beta_2(n, q, L, L_0, \mu, \nu_0) \in (0,1)$; see for instance [34, Theorem 7.6]. Using these last facts, a modification of the comparison arguments in [28, 62], and especially [51, Lemma 4.9], taking into account the presence of $\mathtt{h}(\cdot)$, gives

$$\int_{B_\tau} (|Du|^2 + |Dv|^2 + 1)^{\frac{q-2}{2}} |Du - Dv|^2 \,\mathrm{d}x \le c\tau^{\alpha\beta_3} \int_{B_\tau} (|Du| + 1)^q \,\mathrm{d}x \qquad (5.103)$$

where $c \equiv c(n, q, L, L_0, \mu, \nu_0) \ge 1$ and $\beta_3 \equiv \beta_3(n, q, L, L_0, \mu, \nu_0) \in (0,1)$. See for instance [51, Lemma 4.9]. Estimates (5.102)-(5.103) can be combined in a by now standard way to prove (5.7), as for instance described in [61] or in [1].

## 6 Theorems 1 and 5

We first develop a priori estimates in Sect. 6.1. Those for Theorem 1 are in Proposition 6.1, while Proposition 6.2 contains those for Theorem 5. In both cases, the setting is that of Sect. 5.1. These estimates, obtained for minima of more regular functionals, are then embedded in an approximation argument which is contained in Sect. 6.2. At that stage we have proved that $Du$ is locally bounded in Theorems 1 and 5, thereby completing the proof of the latter. The proof that $Du$ is locally Hölder continuous when $f \in L^{\mathfrak{q}}$ for $\mathfrak{q} > n/\alpha$ is given in Sect. 6.4, and completes the derivation of Theorem 1. Some of the arguments developed in Sects. 6.1 and 6.2 will be employed also for the proofs of the remaining theorems of this paper.

***Remark 6*** (Tilting of $\mathfrak{s}$) The exponent $\mathfrak{s}$ in (5.2) reflects the structural properties of the functionals. Replacing it with a larger/smaller number leads to more/less restrictive bounds on $q/p$ and to cover more cases. Furthermore, since all the forthcomig algebraic inequalities concerning $\mathfrak{s}$ are strict and involve continuous functions, all the forthcoming results and proofs still hold for slightly larger values of $\mathfrak{s}$, giving room for micro-improvements along with those addressed in Remarks 5 and 8. Examples of possible tilting will occur in (6.8), (7.6) and (8.3) below. In particular, as it is standard in this setting [3, 67] and to ease and unify the presentation, in the following proofs we always formally use (5.2) when $n = 2$ too, and this is justified by slightly increasing $\mathfrak{s}$ when $n = 2$. The same applies to $\mathfrak{t}$ in (5.95) below. Notice, anyway, that all the results in Sect. 5 continue to hold when replacing $\mathfrak{s}$ in (5.2) by any larger number; the effective value in (5.2) plays a role from now on.

### 6.1 A priori $L^\infty$-bounds for Theorems 1 and 5

**Proposition 6.1** *Let $u \in W^{1,q}(B_r)$ be a minimizer of the functional $\mathcal{G}(\cdot, B_r)$ in* (2.3), *where $B_r \Subset \Omega$ and $r \le 1$, under assumptions* (5.3)-(5.5). *There exists an explicitly* *computable function* $\kappa_1(n,p,\alpha,\gamma)$, *with*

$$1/5<\kappa_1(\cdot)<1\,, \tag{6.1}$$

*such that, if*

$$\frac{q}{p}<1+\kappa_1(n,p,\alpha,\gamma)\left(1-\frac{\alpha+\gamma}{p}\right)\frac{\alpha}{n}\,, \tag{6.2}$$

*then*

$$\|Du\|_{L^\infty(B_t)}\le\frac{c}{(s-t)^{\chi_1}}\left[\|Du\|_{L^p(B_s)}+\|f\|_{n/\alpha,1/2;B_s}+1\right]^{\chi_2} \tag{6.3}$$

*holds whenever* $B_t\Subset B_s\Subset B_r$ *are concentric balls, where* $c\equiv c(\texttt{data},\gamma)\ge 1$ *and* $\chi_1,\chi_2\equiv\chi_1,\chi_2(\texttt{data}_{\rm e},\gamma)\ge 1$, *and such constants are independent of* $\nu_0$. *In particular, by* (6.1) *it follows that condition* (6.2) *implies* (5.1) *and that* (2.7) *implies* (6.2). *Ultimately, if* (2.7) *holds, then* (6.3) *holds as well.*

***Proof*** The function $\kappa_1(\cdot)$ will be computed in due course of the proof; it will be such that $p,q$ will verify the initial condition in (5.1), which is necessary in order to apply Propositions 5.1-5.2. Therefore we proceed assuming (5.1). By the assumptions considered we observe that $u$ satisfies (5.7); in particular, $Du$ is locally bounded in $B_r$, that is what we ultimately need at this stage. Let $B_t\Subset B_s\Subset B_r$ be balls as in the statement of Proposition 6.1, and note that we can assume that $\|Du\|_{L^\infty(B_t)}\ge 1$, otherwise (6.3) is trivial. Next we consider further concentric balls $B_t\Subset B_{\tau_1}\Subset B_{\tau_2}\Subset B_s$, and a generic point $x_0\in B_{\tau_1}$; note that every one of such points is a Lebesgue point for $E_\mu(Du)$ by virtue of (5.7). We let $r_0:=(\tau_2-\tau_1)/8$, so that $B_{2r_0}(x_0)\Subset B_{\tau_2}$. By (5.10) used with $M\equiv\|Du\|_{L^\infty(B_{\tau_2})}\ge 1$, we can apply Lemma 4.2 on $B_{r_0}(x_0)$ with $h\equiv 2$, $\kappa_0\equiv 0$, $v\equiv E_\mu(Du)$, $f_1\equiv 1$, $f_2\equiv f$, $M_0\equiv M^{\mathfrak{s}(q-p)/2}$, $M_1\equiv M^{(\mathfrak{s}q+\alpha+\gamma q/p)/2}$, $M_2\equiv M^{(\mathfrak{s}q+\alpha)/2}$, $t\equiv 2$, $\delta_1=\delta_2\equiv\alpha/2$, $m_1=m_2\equiv 1$, $\theta_1=\theta_2\equiv 1$; also using (4.2), this yields

$$\begin{aligned}E_\mu(Du(x_0))&\le c\|Du\|_{L^\infty(B_{\tau_2})}^{\frac{\mathfrak{s}(q-p)\chi}{2(\chi-1)}}\left(\fint_{B_{r_0}(x_0)}[E_\mu(Du)]^2\,{\rm d}x\right)^{1/2}\\&\quad+c_{\mathfrak{g}}\|Du\|_{L^\infty(B_{\tau_2})}^{\frac{\mathfrak{s}(q-p)\chi}{2(\chi-1)}+\frac{\mathfrak{s}p+\alpha+\gamma q/p}{2}}r_0^{\frac{\alpha}{2}}\\&\quad+c\|Du\|_{L^\infty(B_{\tau_2})}^{\frac{\mathfrak{s}(q-p)\chi}{2(\chi-1)}+\frac{\mathfrak{s}p+\alpha}{2}}\mathbf{P}^{1,1}_{2,\alpha/2}(f;x_0,2r_0)\end{aligned} \tag{6.4}$$

with $c\equiv c(\texttt{data},\beta)$ and $\chi\equiv\chi(\beta)$ is as in (5.8) with $\alpha_{\rm m}:=\alpha$. Since $x_0\in B_{\tau_1}$ is arbitrary we also gain, after a few elementary manipulations

$$\begin{aligned}\|Du\|_{L^\infty(B_{\tau_1})}&\le\frac{c}{(\tau_1-\tau_2)^{\frac{n}{2p}}}\|Du\|_{L^\infty(B_{\tau_2})}^{\left(\frac{q}{p}-1\right)\frac{\mathfrak{s}\chi}{2(\chi-1)}+\frac{1}{2}}\left(\int_{B_s}(|Du|+1)^p\,{\rm d}x\right)^{\frac{1}{2p}}\\&\quad+c_{\mathfrak{g}}\|Du\|_{L^\infty(B_{\tau_2})}^{\left(\frac{q}{p}-1\right)\frac{\mathfrak{s}\chi}{2(\chi-1)}+\frac{\mathfrak{s}}{2}+\frac{\alpha+\gamma q/p}{2p}}r_0^{\frac{\alpha}{2p}}\end{aligned}$$

$$+ c\|Du\|_{L^\infty(B_{\tau_2})}^{\left(\frac{q}{p}-1\right)\frac{\mathfrak{s}\chi}{2(\chi-1)}+\frac{\mathfrak{s}}{2}+\frac{\alpha}{2p}} \|\mathbf{P}^{1,1}_{2,\alpha/2}(f;\cdot,2r_0)\|^{1/p}_{L^\infty(B_{\tau_1})} + c\,. \tag{6.5}$$

We have used that $B_{r_0}(x_0) \subset B_s$. In the above display, the constants $c, c_{\mathrm{g}}$ depend on $\texttt{data}$ and $\beta$ in such a way that $c, c_{\mathrm{g}} \to \infty$ as $\beta$ approaches its upper limit in (5.8) (with $\alpha_{\mathrm{m}} = \alpha$; this ultimately comes from the application of Lemma 3.1 at the end of the proof of Proposition 5.1). Note also that, in order to derive (6.5), the second integral in (6.4) has been estimated as follows (we also use $\|Du\|_{L^\infty(B_{\tau_2})} \geq 1$):

$$\begin{aligned}\fint_{B_{r_0}(x_0)} [E_\mu(Du)]^2\,\mathrm{d}x &\leq \|E_\mu(Du)\|_{L^\infty(B_{r_0}(x_0))} \fint_{B_{r_0}(x_0)} (|Du|+1)^p\,\mathrm{d}x \\ &\leq c\|Du\|^p_{L^\infty(B_{\tau_2})} \fint_{B_{r_0}(x_0)} (|Du|+1)^p\,\mathrm{d}x\,. \end{aligned} \tag{6.6}$$

To proceed with the proof, by (4.8) and $B_{\tau_1+(\tau_2-\tau_1)/4} \subset B_s$, we have

$$\begin{aligned}\|\mathbf{P}^{1,1}_{2,\alpha/2}(f;\cdot,2r_0)\|_{L^\infty(B_{\tau_1})} &= \|\mathbf{P}^{1,1}_{2,\alpha/2}(f;\cdot,(\tau_2-\tau_1)/4)\|_{L^\infty(B_{\tau_1})} \\ &\leq c\|f\|^{1/2}_{n/\alpha,1/2;B_s}\,. \end{aligned} \tag{6.7}$$

Looking at (6.5), we are led to consider the term with the highest power of $\|Du\|_{L^\infty(B_{\tau_2})}$, which happens to be the second one appearing in the right-hand side. We are now interested in studying the range of parameters $n, p, q, \alpha, \beta, \gamma$ for which the following inequality holds true:

$$\left(\frac{q}{p}-1\right)\frac{\mathfrak{s}\chi(\beta)}{2[\chi(\beta)-1]} + \frac{\mathfrak{s}}{2} + \frac{\alpha+\gamma q/p}{2p} = \left(\frac{q}{p}-1\right)\frac{\mathfrak{s}n}{4\beta} + \frac{\mathfrak{s}}{2} + \frac{\alpha+\gamma q/p}{2p} < 1\,. \tag{6.8}$$

More precisely, as $n$, $p$, $q$, $\alpha$ and $\gamma$ are fixed, we want to determine the existence of a value $\beta$, within the range fixed in (5.8) here with $\alpha_{\mathrm{m}} = \alpha$, for which (6.8) holds. Recalling (5.2), (5.8) and Remark 6, reformulating (6.8) in terms of $q-p$ leads to

$$\frac{(q-p)}{2}\left(\frac{n[(q-p)+p]}{\beta[2p-n(q-p)]} + \frac{\gamma}{p}\right) + \frac{p[(q-p)+p]}{2p-n(q-p)} + \frac{\alpha+\gamma}{2} < p\,.$$

Via routine computations, we arrive at the following second order polynomial inequality in $q-p$:

$$\mathfrak{A}(\beta)(q-p)^2 + \mathfrak{B}(\beta)(q-p) + \mathfrak{C}(\beta) < 0\,, \tag{6.9}$$

where

$$\begin{cases} \mathfrak{A}(\beta) := n(p-\gamma\beta) > 0 \\ \mathfrak{B}(\beta) := p[np + 2\beta(p+\gamma) + n\beta(2p-\alpha-\gamma)] > 0 \\ \mathfrak{C}(\beta) := -2p^2\beta[p-(\alpha+\gamma)] < 0\,. \end{cases} \tag{6.10}$$

Computing the roots of (6.9), yields a first bound on $q-p$, i.e.,

$$
\begin{aligned}
q-p &< \frac{\mathfrak{B}(\beta)}{2\mathfrak{A}(\beta)}\left(\sqrt{1-\frac{4\mathfrak{A}(\beta)\mathfrak{C}(\beta)}{[\mathfrak{B}(\beta)]^2}}-1\right)\\
&= \frac{\mathfrak{B}(\beta)}{2\mathfrak{A}(\beta)}\left(\sqrt{1+\left(1-\frac{\alpha+\gamma}{p}\right)\frac{\beta}{n}\mathfrak{H}(\beta)}-1\right)
\end{aligned} \tag{6.11}
$$

where, by (6.10), it is

$$
\mathfrak{H}(\beta) := \frac{8p(p-\gamma\beta)}{[p+2\beta(p+\gamma)/n+\beta(2p-\alpha-\gamma)]^2} < \frac{8}{(1+\beta)^2}\,. \tag{6.12}
$$

For the last inequality drop the second term at the denominator and use $p>\alpha+\gamma$ again. From definition of $\mathfrak{H}(\beta)$ it then follows that

$$
\frac{\mathfrak{B}(\beta)\mathfrak{H}(\beta)}{2p\mathfrak{A}(\beta)} = \frac{4p}{p+2\beta(p+\gamma)/n+\beta(2p-\alpha-\gamma)} < 4\,. \tag{6.13}
$$

Introducing the function

$$
\kappa_{\alpha_{\mathrm{m}}}(n,p,\beta,\gamma) := \frac{1}{2+\alpha_{\mathrm{m}}}\frac{\mathfrak{B}(\beta)\mathfrak{H}(\beta)}{2p\mathfrak{A}(\beta)}\mathfrak{S}\left(\left(1-\frac{\alpha+\gamma}{p}\right)\frac{\beta}{n}\mathfrak{H}(\beta)\right), \tag{6.14}
$$

where

$$
\mathfrak{S}(t) := \frac{\sqrt{1+t}-1}{t}, \qquad t>0\,, \tag{6.15}
$$

we can rewrite (6.11) as

$$
\frac{q}{p} < 1+\kappa_{\alpha_{\mathrm{m}}}(n,p,\beta,\gamma)\frac{\beta(2+\alpha_{\mathrm{m}})}{\alpha_{\mathrm{m}}}\left(1-\frac{\alpha+\gamma}{p}\right)\frac{\alpha_{\mathrm{m}}}{n}\,. \tag{6.16}
$$

Note that, although here it is $\alpha_{\mathrm{m}}=\alpha$, we keep on using the notation $\alpha_{\mathrm{m}}$ to employ the same computations later on, in Proposition 6.2. We determine $\kappa_1(\cdot)$ in (6.2) as

$$
\kappa_1(n,p,\alpha,\gamma) := \kappa_{\alpha_{\mathrm{m}}}\left(n,p,\frac{\alpha_{\mathrm{m}}}{2+\alpha_{\mathrm{m}}},\gamma\right)
$$

so that (6.1) follows from Lemma 6.1 below. Recalling that $\kappa_{\alpha_{\mathrm{m}}}(\cdot)$ in (6.14) is a continuous function, we conclude that, if (6.2) holds, then we can find $\beta<\alpha_{\mathrm{m}}/(2+\alpha_{\mathrm{m}})$ such that (6.16), and therefore (6.8), holds. This allows to apply Young's inequality in (6.5); taking also into account (6.7) to estimate the terms containing the potentials, we come up with

$$
\begin{aligned}
&\|Du\|_{L^\infty(B_{\tau_1})}\\
&\quad\le \frac{1}{2}\|Du\|_{L^\infty(B_{\tau_2})}+\frac{c}{(\tau_2-\tau_1)^{\chi_1}}\left[\|Du\|_{L^p(B_s)}+\|f\|_{n/\alpha,1/2;B_s}+1\right]^{\chi_2}
\end{aligned}
$$

where $c$, $\chi_1$, $\chi_2$ depend as described in the statement. We recall that $Du$ is locally bounded in $B_r$, therefore, applying Lemma 3.2 to the last inequality with the choice $h(\tau) \equiv \|Du\|_{L^\infty(B_\tau)}$, $s \leq \tau \leq t$, yields (6.3). □

***Remark 7*** The crucial assumption $\alpha + \gamma < p$ enters the proof of Proposition 6.1 in (6.9) and makes the term $\mathfrak{C}(\beta)$ negative for every considered value of $\beta$. In turn, this ensures the existence of non-negative solutions to the quadratic inequality (6.9) for $q$ suitably close to $p$ and in a way which is indeed quantified via (2.7). Once this is secured, the reabsorption condition (6.8) holds and leads to the final a priori estimate. This is a typical mechanism in nonuniformly elliptic problems, which is several times displayed in this paper, i.e., a set of sophisticated integral estimates finally leads to verify certain simple but not always transparent algebraic conditions involving the gap $q/p$. Note that we have used $\alpha + \gamma < p$ also in deriving the upper bound on $\mathfrak{H}(\beta)$ in (6.12).

**Lemma 6.1** *The function $\kappa_{\alpha_{\mathrm{m}}}(\cdot)$ defined in (6.14) satisfies $1/5 < \kappa_{\alpha_{\mathrm{m}}}(n, p, t, \gamma) < 1$, for every $t$ such that $0 < t \leq 1/3$ and for every $\alpha_{\mathrm{m}} \in (0, 1]$.*

***Proof*** The function $\mathfrak{S}(\cdot)$ in (6.15) is decreasing, so that $\mathfrak{S}(\cdot) \leq 1/2$ and by (6.13) it follows that $\kappa_{\alpha_{\mathrm{m}}}(n, p, t, \gamma) < 1$ for every $t > 0$. Thanks to (6.12) and (6.13), elementary estimations give

$$\frac{12}{7} < \frac{\mathfrak{B}(t)\mathfrak{H}(t)}{2p\mathfrak{A}(t)} \quad \text{and} \quad \left(1 - \frac{\alpha+\gamma}{p}\right)\frac{t}{n}\mathfrak{H}(t) < \frac{8t}{n(1+t)^2} < \frac{3}{2n} \leq \frac{3}{4}\,.$$

Using again that $t \mapsto \mathfrak{S}(t)$ is decreasing, and the content of the previous display, we get

$$\frac{1}{5} < \frac{16}{21}\left(\sqrt{7/4} - 1\right) < \frac{1}{2+\alpha_{\mathrm{m}}}\frac{\mathfrak{B}(t)\mathfrak{H}(t)}{2p\mathfrak{A}(t)}\mathfrak{S}(3/4) < \kappa_{\alpha_{\mathrm{m}}}(n, p, t, \gamma)\,,$$

and the lemma is proved. □

**Proposition 6.2** *Let $u \in W^{1,q}(B_r)$ be a minimizer of the functional $\mathcal{G}(\cdot, B_r)$ in (2.3), where $B_r \Subset \Omega$ and $r \leq 1$, under assumptions (5.3)-(5.5). Assume that (6.1)-(6.2) hold together with*

$$p > \frac{2n\alpha}{2n - 2\alpha + \alpha^2}\,. \tag{6.17}$$

*Then*

$$\|Du\|_{L^\infty(B_t)} \leq \frac{c}{(s-t)^{\chi_1}}\left[\|Du\|_{L^p(B_s)} + \|f\|_{n/\alpha,1;B_s} + 1\right]^{\chi_2} \tag{6.18}$$

*holds whenever $B_t \Subset B_s \Subset B_r$ are concentric balls, where $\mathtt{1}$ has been defined in (2.23), $c \equiv c(\mathtt{data}, \gamma)$, and $\chi_1, \chi_2 \equiv \chi_1, \chi_2(\mathtt{data}_{\mathrm{e}})$. In the case (6.17) does not* *hold, there exists an explicitly computable function $\kappa_2(n, p, \alpha, \gamma)$, with $1/5 < \kappa_2(\cdot) < 1$, such that if*

$$\frac{q}{p} < 1 + \kappa_2(n, p, \alpha, \gamma)\left(1 - \frac{\alpha+\gamma}{p}\right)\frac{2(p-\alpha)}{p(2-\alpha)}, \tag{6.19}$$

*then (6.18) holds (with different constants $c, \chi_1, \chi_2$, but having the same dependence of the case (6.17)).*

***Proof*** We argue as for Proposition 6.1. This time we employ (5.11) with $M \equiv \|Du\|_{L^\infty(B_{\tau_2})} \geq 1$, so that, for any $x_0 \in B_{\tau_1}$, we apply Lemma 4.2 on $B_{r_0}(x_0)$, with $h \equiv 3$, $\kappa_0 \equiv 0$, $v \equiv E_\mu(Du)$, $f_1 \equiv 1$, $f_2 \equiv f_3 \equiv f$, $M_0 \equiv M^{\mathfrak{s}(q-p)/2}$, $M_1 \equiv M^{(\mathfrak{s}q+\alpha+\gamma q/p)/2}$, $M_2 \equiv M^{\mathfrak{s}q/2}$, $M_3 \equiv \mathbb{1}_p M^{\mathfrak{s}q/2+\alpha(2-p)/[2(2-\alpha)]}$, $t \equiv 2$, $\delta_1 \equiv \alpha/2$, $\delta_2 \equiv p\alpha/[2(p-\alpha)]$, $\delta_3 \equiv \alpha/(2-\alpha)$, $m_1 \equiv 1$, $m_2 = m_3 \equiv \mathfrak{p}/(\mathfrak{p}-\alpha)$, $\theta_1 \equiv 1$, $\theta_2 \equiv \theta(\mathfrak{p})$ and $\theta_3 \equiv \sigma(\mathfrak{p})$. Similarly to Proposition 6.1, we arrive at

$$\begin{aligned}
\|Du\|_{L^\infty(B_{\tau_1})} &\leq \frac{c}{(\tau_1-\tau_2)^{\frac{n}{2p}}}\|Du\|_{L^\infty(B_{\tau_2})}^{\left(\frac{q}{p}-1\right)\frac{\mathfrak{s}\chi}{2(\chi-1)}+\frac{1}{2}}\left(\int_{B_s}(|Du|+1)^p\,\mathrm{d}x\right)^{\frac{1}{2p}} \\
&\quad + c_{\mathfrak{g}}\|Du\|_{L^\infty(B_{\tau_2})}^{\left(\frac{q}{p}-1\right)\frac{\mathfrak{s}\chi}{2(\chi-1)}+\frac{\mathfrak{s}}{2}+\frac{\alpha+\gamma q/p}{2p}} r_0^{\frac{\alpha}{2p}} \\
&\quad + c\|Du\|_{L^\infty(B_{\tau_2})}^{\left(\frac{q}{p}-1\right)\frac{\mathfrak{s}\chi}{2(\chi-1)}+\frac{\mathfrak{s}}{2}}\left\|\mathbf{P}_{2,\frac{p\alpha}{2(p-\alpha)}}^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha},\theta(\mathfrak{p})}(f;\cdot,(\tau_2-\tau_1)/4)\right\|_{L^\infty(B_{\tau_1})}^{1/p} \\
&\quad + c\mathbb{1}_p\|Du\|_{L^\infty(B_{\tau_2})}^{\left(\frac{q}{p}-1\right)\frac{\mathfrak{s}\chi}{2(\chi-1)}+\frac{\mathfrak{s}}{2}+\left(\frac{2}{p}-1\right)\frac{\alpha}{2(2-\alpha)}} \\
&\qquad \cdot\left\|\mathbf{P}_{2,\frac{\alpha}{2-\alpha}}^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha},\sigma(\mathfrak{p})}(f;\cdot,(\tau_2-\tau_1)/4)\right\|_{L^\infty(B_{\tau_1})}^{1/p} + c\,.
\end{aligned} \tag{6.20}$$

In the following we shall always take $\mathfrak{p} \in [p, p^*)$ to be such that

$$\mathfrak{p} > \frac{n\alpha}{n-\alpha} =: \mathfrak{p}_{\mathrm{m}} \leq 2\,, \tag{6.21}$$

and proceed by checking that Lemma 4.1 can be applied to estimate the last two terms featuring potentials from (6.20); note that $\mathfrak{p}_{\mathrm{m}} < p^*$ therefore it is always possible to choose $\mathfrak{p} \in [p, p^*)$ satisfying (6.21), provided $\mathfrak{p}$ is close enough to $\mathfrak{p}_{\mathrm{m}}$. This ultimately boils down to check that (4.7) is satisfied with the current choice of parameters. Indeed we have

$$\frac{n\theta}{t\delta} \equiv \frac{n\theta(\mathfrak{p})}{2\delta_2} = \frac{n\sigma(\mathfrak{p})}{2\delta_3} = \frac{n(\mathfrak{p}-\alpha)}{\mathfrak{p}\alpha} \overset{(6.21)}{>} 1\,.$$

Therefore Lemma 4.1 applies, giving, in any case $p > 1$

$$\left\|\mathbf{P}_{2,\frac{p\alpha}{2(p-\alpha)}}^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha},\theta(\mathfrak{p})}(f;\cdot,(\tau_2-\tau_1)/4)\right\|_{L^\infty(B_{\tau_1})} \quad \leq \quad c\|f\|_{\frac{n}{\alpha},\frac{p}{2(p-\alpha)};B_s}^{\frac{p}{2(p-\alpha)}}$$

$$\stackrel{(2.23),(4.4)}{\leq} c\|f\|_{n/\alpha,1;B_s}^{\frac{p}{2(p-\alpha)}} \tag{6.22}$$

and (this one occurring only when $p < 2$)

$$\begin{aligned}\left\|\mathbf{P}_{2,\frac{\alpha}{2-\alpha}}^{\frac{\mathfrak{p}}{\mathfrak{p}-\alpha},\sigma(\mathfrak{p})}(f;\cdot,(\tau_2-\tau_1)/4)\right\|_{L^\infty(B_{\tau_1})} &\leq c\|f\|_{\frac{n}{\alpha},\frac{1}{2-\alpha};B_s}^{\frac{1}{2-\alpha}} \\ &\stackrel{(2.23)}{=} c\|f\|_{n/\alpha,1;B_s}^{1},\end{aligned} \tag{6.23}$$

where $c \equiv c(n, p, \alpha)$. Back to (6.20), and noting that

$$\begin{aligned}&\left(\frac{q}{p}-1\right)\frac{\mathfrak{s}\chi}{2(\chi-1)}+\frac{\mathfrak{s}}{2}+\mathbb{1}_p\left(\frac{2}{p}-1\right)\frac{\alpha}{2(2-\alpha)} \\ &\quad\leq \left(\frac{q}{p}-1\right)\frac{\mathfrak{s}\chi}{2(\chi-1)}+\frac{\mathfrak{s}}{2}+\frac{\alpha+\gamma q/p}{2p},\end{aligned} \tag{6.24}$$

we are again led to determine $\mathfrak{p} \in [p, p^*)$ and a positive $\beta < \alpha_{\mathrm{m}}(\mathfrak{p})/(2+\alpha_{\mathrm{m}}(\mathfrak{p}))$ such that (6.8) holds. For this, we distinguish three different cases, as now the identity $\alpha_{\mathrm{m}} = \alpha$ used in Proposition 6.1 is not always ensured.

- Case 1: $p > \mathfrak{p}_{\mathrm{m}}$. We take $\mathfrak{p} = p$, that gives $\mathfrak{a}(\mathfrak{p}) = \mathfrak{a}(p) = 1$. It follows that $\alpha_{\mathrm{m}} = \alpha$ in (5.13) and we conclude exactly as in Proposition 6.1, using the bound (6.16) with $\alpha_{\mathrm{m}} = \alpha$. Specifically, we take $\beta < \alpha_{\mathrm{m}}/(2+\alpha_{\mathrm{m}})$ in order to satisfy (6.8), then, using Young's inequality in (6.20), and taking also into account (6.22)-(6.23), we come up with

$$\begin{aligned}\|Du\|_{L^\infty(B_{\tau_1})} &\leq \frac{1}{2}\|Du\|_{L^\infty(B_{\tau_2})} \\ &\quad+\frac{c}{(\tau_2-\tau_1)^{\chi_1}}\left[\|Du\|_{L^p(B_s)}+\|f\|_{n/\alpha,1;B_s}+1\right]^{\chi_2}\end{aligned}$$

  where $c$, $\chi_1$, $\chi_2$ depend as described in the statement. Applying Lemma 3.2 yields (6.18) and the proof is complete.
- Case 2: $2n\alpha/(2n-2\alpha+\alpha^2) < p \leq \mathfrak{p}_{\mathrm{m}}$. This implies $p \leq 2$ and therefore

$$\alpha_{\mathrm{m}}(\mathfrak{p}) = \alpha \min\left\{1, \frac{2\mathfrak{a}(\mathfrak{p})}{2-\alpha}\right\}. \tag{6.25}$$

  On the other hand, the lower bound on $p$ implies $2\mathfrak{a}(\mathfrak{p}_{\mathrm{m}})/(2-\alpha) > 1$ so that $\alpha_{\mathrm{m}}(\mathfrak{p}_{\mathrm{m}}) = \alpha$. It follows we can take $\mathfrak{p} > \mathfrak{p}_{\mathrm{m}}$ close enough to $\mathfrak{p}_{\mathrm{m}}$ in order to have $2\mathfrak{a}(\mathfrak{p})/(2-\alpha) > 1$ so that it is $\alpha_{\mathrm{m}} = \alpha_{\mathrm{m}}(\mathfrak{p}) = \alpha$ in (6.25). We then argue as in Case 1.
- Case 3: $1 < p \leq 2n\alpha/(2n-2\alpha+\alpha^2)$. The upper bound on $p$ implies that $2\mathfrak{a}(\mathfrak{p}_{\mathrm{m}})/(2-\alpha) \leq 1$ so that $\alpha_{\mathrm{m}}(\mathfrak{p}_{\mathrm{m}}) = 2\alpha\mathfrak{a}(\mathfrak{p}_{\mathrm{m}})/(2-\alpha)$. Recalling that $t \mapsto \mathfrak{a}(t)$ is decreasing, we have that

$$\mathfrak{p} > \mathfrak{p}_{\mathrm{m}} \Longrightarrow \alpha_{\mathrm{m}}(\mathfrak{p}) = \frac{2\alpha\mathfrak{a}(\mathfrak{p})}{2-\alpha} < \frac{2\alpha\mathfrak{a}(\mathfrak{p}_{\mathrm{m}})}{2-\alpha} = \alpha_{\mathrm{m}}(\mathfrak{p}_{\mathrm{m}}) = \frac{2(p-\alpha)}{p(2-\alpha)}n \leq \alpha \leq 1.$$

We formally take the limiting value $\mathfrak{p} = \mathfrak{p}_{\mathrm{m}}$ and consider the corresponding version of (6.16) with $\alpha_{\mathrm{m}} \equiv \alpha_{\mathrm{m}}(\mathfrak{p}_{\mathrm{m}})$, $\beta = \alpha_{\mathrm{m}}(\mathfrak{p}_{\mathrm{m}})/(2+\alpha_{\mathrm{m}}(\mathfrak{p}_{\mathrm{m}}))$, and finally set

$$\kappa_2(n, p, \alpha, \gamma) := \kappa_{\alpha_{\mathrm{m}}(\mathfrak{p}_{\mathrm{m}})}\left(n, p, \frac{\alpha_{\mathrm{m}}(\mathfrak{p}_{\mathrm{m}})}{2+\alpha_{\mathrm{m}}(\mathfrak{p}_{\mathrm{m}})}, \gamma\right).$$

This leads to (6.19); note that $1/5 < \kappa_2(n, p, \alpha, \gamma) < 1$ follows as in Lemma 6.1. Summarizing, if (6.19) holds, then we can find $\mathfrak{p} > \mathfrak{p}_{\mathrm{m}}$ and $\beta < \alpha_{\mathrm{m}}(\mathfrak{p})/(2+\alpha_{\mathrm{m}}(\mathfrak{p}))$ such that (6.16) this time holds with $\alpha_{\mathrm{m}} \equiv \alpha_{\mathrm{m}}(\mathfrak{p})$, and therefore also (6.8) holds. We then conclude as for Case 1.

□

***Remark 8*** (Refinements) The lower bound $\kappa_1(\cdot) > 1/5$ in (6.1) can be improved by using Taylor expansion of the function $\mathfrak{S}(\cdot)$ in (6.15). This eventually leads to a slightly better bound than that in (2.7). Anyway, this does not lead to a different asymptotic in terms of the ratio $\alpha/n$ and we shall not pursue this path here. Further improvements come when considering more specific structures, as for instance in (1.1). Let us for simplicity consider the range $p \geq 2$. In this case, looking at (6.20) and recalling that $c_{\mathrm{g}} = 0$, we arrive at

$$\left(\frac{q}{p} - 1\right)\frac{\mathfrak{s}n}{4\beta} + \frac{\mathfrak{s}}{2} < 1, \tag{6.26}$$

that replaces (6.8) in this setting. Then, taking also into account the content of Remark 5, and performing the same reasoning of (6.8)-(6.16) and Lemma 6.1, we obtain a refinement of (2.7), but still preserving the same asymptotic with respect to $\alpha/n$.

### 6.2 Approximation

Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a (local) minimizer of $\mathcal{G}$ as in Theorem 5; in particular, $F(\cdot)$, $\mathrm{g}(\cdot)$, $\mathrm{h}(\cdot)$ satisfy (2.4)-(2.6). With $B_r \Subset \Omega$, $r \leq 1$, in the following by $\varepsilon, \delta \equiv \{\varepsilon\}, \{\delta\} \equiv \{\varepsilon_k\}, \{\delta_k\}$ we denote two decreasing sequences of positive numbers such that $\varepsilon, \delta \to 0$, $\varepsilon \leq \operatorname{dist}(B_r, \partial\Omega)/10$ and $\varepsilon, \delta \leq 1$; we shall several times extract subsequences and these will still be denoted by $\varepsilon, \delta$. We denote by $\mathrm{o}(\varepsilon, B)$ a quantity, also depending on a considered ball $B$ (or on an open subset) but independent of $\delta$, such that $\mathrm{o}(\varepsilon, B) \to 0$ as $\varepsilon \to 0$. Similarly, we denote by $\mathrm{o}_{\varepsilon}(\delta, B)$ a quantity, depending both on $\varepsilon$ and $\delta$, such that $\mathrm{o}_{\varepsilon}(\delta, B) \to 0$ as $\delta \to 0$ for each fixed $\varepsilon$. As usual, the exact value of such quantities might change on different occurences. In the following we shall denote, for $x \in B_r$ and $y, z \in \mathbb{R}^n$

$$\begin{cases} \mathbb{F}(x, y, z) := F(z) + \mathrm{g}(x, y, z) \\ \mathbb{G}_{\varepsilon}(u, B_r) := \mathcal{G}(u, B_r) + \|\mathrm{h}(\cdot, u)\|_{L^1(B_r)} + \|f\|^{p/(p-\alpha)}_{n/\alpha,1;B_{(1+\varepsilon)r}} + 1. \end{cases} \tag{6.27}$$

We fix a family of radially symmetric, non-negative mollifiers $\{\phi_s\}_{0<s\leq 1} \subset C^{\infty}(\mathbb{R}^n)$, defined in a standard way as $\phi_s(x) := \phi(x/s)/s^n$, where $0 \leq \phi \in C^{\infty}_c(\mathcal{B}_1)$, $\|\phi\|_{L^1(\mathbb{R}^n)} = 1$, $\mathcal{B}_{3/4} \subset \operatorname{supp}\phi$, with $\phi > 0$ when $|z| \leq 3/4$. We next define

$$\tilde{u}_{\varepsilon} = u * \phi_{\varepsilon}, \quad f_{\varepsilon} := f * \phi_{\varepsilon} + 1, \tag{6.28}$$

and

$$\begin{aligned}\mathbb{F}_{\varepsilon,\delta}(x,y,z) &:= (\mathbb{F}(x,y,\cdot)*\phi_\delta)(z)+\sigma_\varepsilon[H_{\mu_\delta}(z)]^{q/2}\\ &= \int_{\mathcal{B}_1}\mathbb{F}(x,y,z+\delta\lambda)\phi(\lambda)\,\mathrm{d}\lambda+\sigma_\varepsilon[H_{\mu_\delta}(z)]^{q/2}\\ &=: \mathbb{F}_\delta(x,y,z)+\sigma_\varepsilon[H_{\mu_\delta}(z)]^{q/2}\,, \end{aligned} \tag{6.29}$$

where

$$\begin{aligned}\sigma_\varepsilon &:= \left(1+\varepsilon^{-1}+\|D\tilde{u}_\varepsilon\|^{2q}_{L^q(B_r)}\right)^{-1}\\ &\Longrightarrow \sigma_\varepsilon\int_{B_r}[H_{\mu_\delta}(D\tilde{u}_\varepsilon)]^{q/2}\,\mathrm{d}x \stackrel{\varepsilon\to 0}{\to} 0\end{aligned}$$

uniformly with respect to $\delta$; we recall that $\mu_\delta=\mu+\delta\in(0,2]$. Specifically, by the Mean Value Theorem we can write

$$\sigma_\varepsilon\int_{B_r}[H_{\mu_\delta}(D\tilde{u}_\varepsilon)]^{q/2}\,\mathrm{d}x = \mathrm{o}(\varepsilon,B)+\mathrm{o}_\varepsilon(\delta,B) \tag{6.30}$$

with the second quantity in the right-hand side that converges to zero as $\delta\to 0$, uniformly with respect to $\varepsilon$. From (6.29) it obviously follows that, with $x\in B_r$, $y\in\mathbb{R}$, $z\in\mathbb{R}^n$

$$\begin{cases}\mathbb{F}_{\varepsilon,\delta}(x,y,z)=F_{\varepsilon,\delta}(z)+\mathtt{g}_\delta(x,y,z)\\ F_{\varepsilon,\delta}(z):=(F*\phi_\delta)(z)+\sigma_\varepsilon[H_{\mu_\delta}(z)]^{q/2}\\ \mathtt{g}_\delta(x,y,z):=(\mathtt{g}(x,y,\cdot)*\phi_\delta)(z)\,.\end{cases} \tag{6.31}$$

(Be careful here: $\mathtt{g}_\delta(\cdot)$ defined in $(6.31)_3$ and used in this section and in Sects. 6.3-6.4, has nothing to do with the rescaled function $\mathtt{g}_\varrho(\cdot)$ introduced in (5.29)). Finally, we define $\mathrm{h}_\varepsilon\colon B_r\times\mathbb{R}\to\mathbb{R}$ by

$$\begin{cases}\mathrm{h}_\varepsilon(x,y):=\tilde{\mathrm{h}}_\varepsilon(x,y)+|\tilde{u}_\varepsilon(x)-y|^\alpha\\ \tilde{\mathrm{h}}_\varepsilon(x,y):=(\mathrm{h}(\cdot,y)*\phi_\varepsilon)(x)=\displaystyle\int_{\mathcal{B}_1}\mathrm{h}(x+\varepsilon\lambda,y)\phi(\lambda)\,\mathrm{d}\lambda\\ \mathrm{h}_{0,\varepsilon}(x):=(|\mathrm{h}|(\cdot,0)*\phi_\varepsilon)(x)=\displaystyle\int_{\mathcal{B}_1}|\mathrm{h}(x+\varepsilon\lambda,0)|\phi(\lambda)\,\mathrm{d}\lambda\,.\end{cases} \tag{6.32}$$

From (6.29) and Lemma 3.4 it easily follows that

$$|\mathbb{F}_\delta(x,y,z)-\mathbb{F}(x,y,z)|\lesssim\delta(|z|+1)^{q-1} \tag{6.33}$$

(see also Lemma 9.1 below) so that, since $D\tilde{u}_\varepsilon$ is bounded for every $\varepsilon$, we have

$$\|\mathbb{F}_\delta(\cdot,\tilde{u}_\varepsilon,D\tilde{u}_\varepsilon)-\mathbb{F}(\cdot,\tilde{u}_\varepsilon,D\tilde{u}_\varepsilon)\|_{L^1(B_r)}=\mathrm{o}_\varepsilon(\delta,B_r)\,, \tag{6.34}$$

for every fixed $\varepsilon$. Up to not relabelled subsequences, we can also assume that

$$\begin{cases} \|D\tilde{u}_\varepsilon - Du\|_{L^p(B_r)} = \mathrm{o}(\varepsilon, B_r) \\ \tilde{u}_\varepsilon \to u \text{ in } L^\gamma(B_r), \text{ where } \gamma > n/(n-1) \text{ and a.e.} \end{cases} \tag{6.35}$$

By Remark 9 below, we have $F(D\tilde{u}_\varepsilon) \to F(Du)$ in $L^1(B_r)$. On the other hand by $(2.5)_2$, (6.35) and again Lebesgue domination, it follows $\mathrm{g}(\cdot, \tilde{u}_\varepsilon, D\tilde{u}_\varepsilon) \to \mathrm{g}(\cdot, u, Du)$ in $L^1(B_r)$. We conclude with

$$\|\mathbb{F}(\cdot, \tilde{u}_\varepsilon, D\tilde{u}_\varepsilon) - \mathbb{F}(\cdot, u, Du)\|_{L^1(B_r)} = \mathrm{o}(\varepsilon, B_r) \tag{6.36}$$

up to not relabelled subsequences. Next, observe that (2.6) gives

$$|\mathrm{h}(x, y)| \le |\mathrm{h}(x, 0)| + Lf(x)|y|^\alpha \le |\mathrm{h}(x, 0)| + L[f(x)]^{n/\alpha} + |y|^{n\alpha/(n-\alpha)} \tag{6.37}$$

for every $x \in \Omega$, $y \in \mathbb{R}$. This, and (6.32), easily imply the (nonuniform in $\varepsilon$) estimate

$$|\tilde{\mathrm{h}}_\varepsilon(x, y)| + |\mathrm{h}_\varepsilon(x, y)| \le c_\varepsilon(|y|^\alpha + 1) \le c_\varepsilon(|y| + 1)\,. \tag{6.38}$$

Again by (6.28) and (6.32), given any $x \in B_r$ and $y_1, y_2 \in \mathbb{R}$, we have

$$\begin{aligned} &|\mathrm{h}_\varepsilon(x, y_1) - \mathrm{h}_\varepsilon(x, y_2)| + |\tilde{\mathrm{h}}_\varepsilon(x, y_1) - \tilde{\mathrm{h}}_\varepsilon(x, y_2)| \\ &\qquad \le 2f_\varepsilon(x)|y_1 - y_2|^\alpha \le 2\|f_\varepsilon\|_{L^\infty(B_r)}|y_1 - y_2|^\alpha\,. \end{aligned} \tag{6.39}$$

By $(6.35)_2$, we now have that

$$\|\mathrm{h}_\varepsilon(\cdot, \tilde{u}_\varepsilon) - \mathrm{h}(\cdot, u)\|_{L^1(B_r)} = \|\tilde{\mathrm{h}}_\varepsilon(\cdot, \tilde{u}_\varepsilon) - \mathrm{h}(\cdot, u)\|_{L^1(B_r)} = \mathrm{o}(\varepsilon, B_r)\,. \tag{6.40}$$

The proof of this fact involves a commutator type estimate on mollifiers, and it is postponed to Lemma 6.2 below. We next consider the functional

$$W^{1,q}(B_r) \ni w \mapsto \mathcal{G}_{\varepsilon,\delta}(w, B_r) := \int_{B_r} [\mathbb{F}_{\varepsilon,\delta}(x, w, Dw) + \mathrm{h}_\varepsilon(x, w)]\,\mathrm{d}x \tag{6.41}$$

and note that, using (6.30),(6.34),(6.36) and (6.40), we have

$$\mathcal{G}_{\varepsilon,\delta}(\tilde{u}_\varepsilon, B_r) = \mathcal{G}(u, B_r) + \mathrm{o}(\varepsilon, B_r) + \mathrm{o}_\varepsilon(\delta, B_r)\,. \tag{6.42}$$

It is at this stage worth remarking that the original assumptions (2.4)-(2.5) from Theorem 1 imply that $F_{\varepsilon,\delta}(\cdot)$ satisfies the following version of (5.3):

$$\begin{cases} \sigma_\varepsilon[H_{\mu_\delta}(z)]^{q/2} + \tilde{\nu}[H_{\mu_\delta}(z)]^{p/2} \\ \quad \le F_{\varepsilon,\delta}(z) \le \tilde{L}[H_{\mu_\delta}(z)]^{q/2} + \tilde{L}[H_{\mu_\delta}(z)]^{p/2} \\ \nu_0(\varepsilon)[H_{\mu_\delta}(z)]^{(q-2)/2}|\xi|^2 + \tilde{\nu}[H_{\mu_\delta}(z)]^{(p-2)/2}|\xi|^2 \\ \quad \le \partial_{zz}F_{\varepsilon,\delta}(z)\xi\cdot\xi \\ |\partial_{zz}F_{\varepsilon,\delta}(z)| \le \tilde{L}[H_{\mu_\delta}(z)]^{(q-2)/2} + \tilde{L}[H_{\mu_\delta}(z)]^{(p-2)/2} \end{cases} \tag{6.43}$$

for new constants $0 < \tilde{\nu} \leq \tilde{L}$ as in Sect. 2.1, depending on $\texttt{data}$, that are independent of $\varepsilon$, $\delta$; note that $\mu_\delta = \mu + \delta > 0$. On the contrary, $\nu_0(\varepsilon) = q\min\{q-1, 1\}\sigma(\varepsilon)$ depends on $\varepsilon$. These properties can be checked using (6.29) and (6.31) in conjunction with assumptions (2.4); see for instance [20, Sect. 4.5] for more details. Similarly, by (2.5) the function $\mathrm{g}_\delta(\cdot)$ satisfies (5.4) with $\mu \equiv \mu_\delta \in (0, 2]$, and a suitable $\tilde{L}$, as it happens in (6.43) (eventually we enlarge $\tilde{L}$ to fit both (5.3) and (5.4)). Finally, $\mathrm{h}_\varepsilon(\cdot)$ satisfies a version of (5.5) thanks to (6.38)-(6.39), with $f(\cdot)$ replaced by $2f_\varepsilon(\cdot)$. We conclude that the functional in (6.41) is of the type considered in Sect. 5.1 and Propositions 5.1-5.2, as assumptions (5.3)-(5.5) are satisfied for a suitable choice of the parameters. In particular, the integrand $z \mapsto \mathbb{F}_{\varepsilon,\delta}(\cdot, z)$ is convex with $q$-polynomial growth; keeping also in mind (6.38), we conclude we can apply Direct Methods (lower semicontinuity $\oplus$ coercivity) to define $u_{\varepsilon,\delta} \in \tilde{u}_\varepsilon + W^{1,q}_0(B_r)$ as a solution to

$$u_{\varepsilon,\delta} \mapsto \min_{w \in \tilde{u}_\varepsilon + W^{1,q}_0(B_r)} \mathcal{G}_{\varepsilon,\delta}(w, B_r)\,. \tag{6.44}$$

Indeed, standard coercivity estimates - see for instance [3, Pages 986-987] or directly (6.45) below - give that

$$\|Dw\|^q_{L^q(B_r)} \lesssim \mathcal{G}_{\varepsilon,\delta}(w, B_r) + \|f_\varepsilon\|^{q/(q-\alpha)}_{L^{n/\alpha}(B_r)} + \|D\tilde{u}_\varepsilon\|^q_{L^q(B_r)} + \|\mathrm{h}_\varepsilon(\cdot, \tilde{u}_\varepsilon)\|_{L^1(B_r)}$$

holds for every $w \in \tilde{u}_\varepsilon + W^{1,q}_0(B_r)$, where the involved constants also depend on $\varepsilon$. On the other hand, keeping (6.38) in mind, $W^{1,q}$-weak lower semicontinuity of $\mathcal{G}_{\varepsilon,\delta}(\cdot)$ follows for instance as in [34, Theorem 4.5] and [34, Remark 4.1]. Alternatively, see [34, Theorem 4.6] and the subsequent discussion on Dirichlet data for the applications of the Direct Methods in the present situation. To proceed, using the minimality of $u_{\varepsilon,\delta}$, Sobolev and Young's inequalities, we find

$$\begin{aligned}
&\tilde{\nu}\|Du_{\varepsilon,\delta}\|^p_{L^p(B_r)} + \sigma_\varepsilon\|Du_{\varepsilon,\delta}\|^q_{L^q(B_r)} \\
&\quad \stackrel{(6.43)_1}{\leq} \int_{B_r} \mathbb{F}_{\varepsilon,\delta}(x, u_{\varepsilon,\delta}, Du_{\varepsilon,\delta})\,\mathrm{d}x \\
&\quad \leq \mathcal{G}_{\varepsilon,\delta}(u_{\varepsilon,\delta}, B_r) + \|\mathrm{h}_\varepsilon(\cdot, u_{\varepsilon,\delta})\|_{L^1(B_r)} \\
&\quad \leq \mathcal{G}_{\varepsilon,\delta}(\tilde{u}_\varepsilon, B_r) + \|\mathrm{h}(\cdot, u)\|_{L^1(B_r)} \\
&\qquad + \|\mathrm{h}_\varepsilon(\cdot, \tilde{u}_\varepsilon) - \mathrm{h}(\cdot, u)\|_{L^1(B_r)} + \|\mathrm{h}_\varepsilon(\cdot, \tilde{u}_\varepsilon) - \mathrm{h}_\varepsilon(\cdot, u_{\varepsilon,\delta})\|_{L^1(B_r)} \\
&\quad \stackrel{(6.39),(6.40),(6.42)}{\leq} \mathcal{G}(u, B_r) + \mathrm{o}(\varepsilon, B_r) + \mathrm{o}_\varepsilon(\delta, B_r) + \|\mathrm{h}(\cdot, u)\|_{L^1(B_r)} \\
&\qquad\qquad + c\|f_\varepsilon\|_{L^{n/\alpha}(B_r)}\left[\|D\tilde{u}_\varepsilon\|^\alpha_{L^p(B_r)} + \|Du_{\varepsilon,\delta}\|^\alpha_{L^p(B_r)}\right] \\
&\quad \stackrel{(2.4)_2,(6.35)_1}{\leq} \frac{\tilde{\nu}}{2}\|Du_{\varepsilon,\delta}\|^p_{L^p(B_r)} + c\,\mathcal{G}(u, B_r) + c\,\|\mathrm{h}(\cdot, u)\|_{L^1(B_r)} \\
&\qquad\qquad + c\,\|f+1\|^{p/(p-\alpha)}_{L^{n/\alpha}(B_{(1+\varepsilon)r})} + \mathrm{o}(\varepsilon, B_r) + \mathrm{o}_\varepsilon(\delta, B_r)\,.
\end{aligned} \tag{6.45}$$

Recalling the notation fixed in (6.27), we therefore conclude with

$$
\begin{aligned}
&\|Du_{\varepsilon,\delta}\|_{L^p(B_r)}^p + \sigma_\varepsilon \|Du_{\varepsilon,\delta}\|_{L^q(B_r)}^q \\
&\qquad\qquad \leq c\,\mathbb{G}_\varepsilon(u, B_r) + \mathrm{o}(\varepsilon, B_r) + \mathrm{o}_\varepsilon(\delta, B_r)
\end{aligned} \tag{6.46}
$$

where $c \equiv c(\texttt{data})$. In (6.46) we have used the embedding $\|f\|_{L^{n/\alpha}(B_{(1+\varepsilon)r})} \lesssim \|f\|_{n/\alpha,1;B_{(1+\varepsilon)r}}$, since $1 < n/\alpha$ (recall $(4.4)_{2,3}$ and (2.23)).

***Remark 9*** In the proof of (6.36) we have used a standard argument that it is better to recall. By Jensen's inequality we find $F(D\tilde{u}_\varepsilon) \leq F(Du) * \phi_\varepsilon$ on $B_r$, therefore, as $F(Du) \in L^1(B_r)$, a well-known variant of Lebesgue dominated convergence gives that $F(D\tilde{u}_\varepsilon) \to F(Du)$ in $L^1(B_r)$. In view of the proof of Theorem 3 it is then useful to observe that the same happens when considering the integrand from (1.9). In this case the very same argument gives that $\mathfrak{c}(\cdot, \tilde{u}_\varepsilon)F(D\tilde{u}_\varepsilon) \to \mathfrak{c}(\cdot, u)F(Du)$ and this follows from the upper bound on $\mathfrak{c}(\cdot)$ and its continuity.

***Remark 10*** By (6.43) and the subsequent discussion, and in particular the part concerning $\mathrm{g}_\delta(\cdot)$, it follows that the integrand $z \mapsto \mathbb{F}_{\varepsilon,\delta}(x, y, z)$ satisfies (6.43) uniformly with respect to $(x, y)$.

### 6.3 Proof of (2.8) and (2.25), and proof of Theorem 5 concluded

We start by (2.25); from now on, and until the end of Sect. 6.3, $\tau$ denotes a free parameter such that $0 < t \leq \tau < r$, while $t$ is the one fixed in (2.25). By (6.43) and the subsequent discussion, we are in the setting of Sects. 5.1 and 6.1. Applying estimate (6.18) to $u_{\varepsilon,\delta}$, and using it in conjunction to (6.46), and yet letting $s \to r$, we find, for any $\tau$ as above

$$
\|Du_{\varepsilon,\delta}\|_{L^\infty(B_\tau)} \leq \frac{c}{(r-\tau)^{\chi_1}}\,[\mathbb{G}_\varepsilon(u, B_r)]^{\chi_2} + \frac{\mathrm{o}(\varepsilon, B_r) + \mathrm{o}_\varepsilon(\delta, B_r)}{(r-\tau)^{\chi_1}} \tag{6.47}
$$

for a new constant $c \equiv c(\texttt{data})$, new exponents $\chi_1, \chi_2 \equiv \chi_1, \chi_2(\texttt{data}_{\mathrm{e}})$, and new quantities $\mathrm{o}(\varepsilon, B_r)$, $\mathrm{o}_\varepsilon(\delta, B_r)$. All in all, we have proved that for every $\tau < r$, and for every $\varepsilon$, there exists a constant $M(\tau, \varepsilon)$ such that

$$
\sup_\delta \|Du_{\varepsilon,\delta}\|_{L^\infty(B_\tau)} \leq M(\tau, \varepsilon)\,. \tag{6.48}
$$

By (6.33) we conclude that

$$
\|\mathbb{F}_\delta(\cdot, u_{\varepsilon,\delta}, Du_{\varepsilon,\delta}) - \mathbb{F}(\cdot, u_{\varepsilon,\delta}, Du_{\varepsilon,\delta})\|_{L^1(B_\tau)} = \mathrm{o}_\varepsilon(\delta, B_\tau) \tag{6.49}
$$

holds for every $\varepsilon$. By (6.46) and (6.48), up to passing to not relabelled subsequences, we find that for every $\varepsilon$ there exists $u_\varepsilon \in \tilde{u}_\varepsilon + W^{1,q}_0(B_r) \cap W^{1,\infty}(B_t)$ such that, as $\delta \to 0$, it holds that

$$
\begin{cases}
u_{\varepsilon,\delta} \rightharpoonup u_\varepsilon \text{ in } W^{1,q}(B_r) \\
u_{\varepsilon,\delta} \rightharpoonup^* u_\varepsilon \text{ in } W^{1,\infty}(B_t) \\
u_{\varepsilon,\delta} \to u_\varepsilon \text{ strongly in } L^{n/(n-1)}(B_r) \text{ and a.e.} \\
\mathrm{h}_\varepsilon(\cdot, u_{\varepsilon,\delta}) \to \mathrm{h}_\varepsilon(\cdot, u_\varepsilon) \text{ strongly in } L^1(B_r)\,.
\end{cases} \tag{6.50}
$$

Note that $(6.50)_4$ follows from $(6.50)_3$, a well-known variant of Lebesgue dominated convergence and (6.38). By convexity and non-negativity of $z \mapsto \mathbb{F}(\cdot, z)$, we can use standard weak lower semicontinuity theorems (see [34, Theorem 4.5]) to deduce that

$$\int_{B_\tau} \mathbb{F}(x, u_\varepsilon, Du_\varepsilon)\,\mathrm{d}x \le \liminf_{\delta} \int_{B_\tau} \mathbb{F}(x, u_{\varepsilon,\delta}, Du_{\varepsilon,\delta})\,\mathrm{d}x \tag{6.51}$$

holds whenever $0 < t \le \tau < r$ (actually this holds in the limiting case $\tau = r$ too). Next, letting $\delta \to 0$ in (6.46) and (6.47) allows to get uniform bounds (with respect to $\varepsilon$) on $\|Du_\varepsilon\|_{L^p(B_r)}$ and on $\|Du_\varepsilon\|_{L^\infty(B_t)}$, respectively. Again up to not relabelled subsequences we can assume that there exists $v \in u + W^{1,p}_0(B_r) \cap W^{1,\infty}(B_t)$ such that

$$\begin{cases} u_\varepsilon \rightharpoonup v \text{ in } W^{1,p}(B_r) \\ u_\varepsilon \rightharpoonup^* v \text{ in } W^{1,\infty}(B_t) \\ u_\varepsilon \to v \text{ strongly in } L^\gamma(B_r) \text{ for some } \gamma > n/(n-1), \text{ and a.e.} \\ \mathtt{h}(\cdot, u_\varepsilon) \to \mathtt{h}(\cdot, v) \text{ strongly in } L^1(B_r). \end{cases} \tag{6.52}$$

Note that $(6.52)_4$ follows from (6.37) and $(6.52)_3$ via Lebesgue domination. Moreover, we have

$$\|\tilde{\mathtt{h}}_\varepsilon(\cdot, u_\varepsilon) - \mathtt{h}(\cdot, u_\varepsilon)\|_{L^1(B_r)} = \mathrm{o}(\varepsilon, B_r)\,, \tag{6.53}$$

the proof of which is again postponed to Lemma 6.2 below. Then observe that

$$\begin{aligned}
&\liminf_{\delta} \mathcal{G}_{\varepsilon,\delta}(u_{\varepsilon,\delta}, B_r) \\
&\quad \ge \liminf_{\delta} \int_{B_r} \mathbb{F}_\delta(x, u_{\varepsilon,\delta}, Du_{\varepsilon,\delta})\,\mathrm{d}x + \lim_{\delta} \int_{B_r} \mathtt{h}_\varepsilon(x, u_{\varepsilon,\delta})\,\mathrm{d}x \\
&\quad \overset{(6.50)_4}{\ge} \liminf_{\delta} \int_{B_\tau} \mathbb{F}_\delta(x, u_{\varepsilon,\delta}, Du_{\varepsilon,\delta})\,\mathrm{d}x + \int_{B_r} \mathtt{h}_\varepsilon(x, u_\varepsilon)\,\mathrm{d}x \\
&\quad \overset{(6.53)}{=} \liminf_{\delta} \int_{B_\tau} \mathbb{F}_\delta(x, u_{\varepsilon,\delta}, Du_{\varepsilon,\delta})\,\mathrm{d}x \\
&\qquad\quad + \int_{B_r} [\mathtt{h}(x, u_\varepsilon) + |\tilde{u}_\varepsilon - u_\varepsilon|^\alpha]\,\mathrm{d}x + \mathrm{o}(\varepsilon, B_r) \\
&\quad \overset{(6.49),(6.51)}{\ge} \int_{B_\tau} \mathbb{F}(x, u_\varepsilon, Du_\varepsilon)\,\mathrm{d}x \\
&\qquad\quad + \int_{B_r} [\mathtt{h}(x, u_\varepsilon) + |\tilde{u}_\varepsilon - u_\varepsilon|^\alpha]\,\mathrm{d}x + \mathrm{o}(\varepsilon, B_r) \qquad (6.54)\\
&\quad = \mathcal{G}(u_\varepsilon, B_\tau) + \int_{B_r \setminus B_\tau} \mathtt{h}(x, u_\varepsilon)\,\mathrm{d}x + \int_{B_r} |\tilde{u}_\varepsilon - u_\varepsilon|^\alpha\,\mathrm{d}x + \mathrm{o}(\varepsilon, B_r)\,.
\end{aligned}$$

Letting $\tau \to r$ in the above display gives

$$\mathcal{G}(u_\varepsilon, B_r) + \int_{B_r} |\tilde{u}_\varepsilon - u_\varepsilon|^\alpha \, dx \le \liminf_{\delta} \mathcal{G}_{\varepsilon,\delta}(u_{\varepsilon,\delta}, B_r) + o(\varepsilon, B_r)\,. \tag{6.55}$$

Using, in order: (6.52) and lower semicontinuity, (6.55), the minimality of $u_{\varepsilon,\delta}$, (6.42) and the minimality of $u$, we obtain

$$\begin{aligned}
\mathcal{G}(v, B_r) + \int_{B_r} |u - v|^\alpha \, dx &\le \liminf_{\varepsilon} \mathcal{G}(u_\varepsilon, B_r) + \lim_{\varepsilon} \int_{B_r} |\tilde{u}_\varepsilon - u_\varepsilon|^\alpha \, dx \\
&\le \liminf_{\varepsilon} \liminf_{\delta} \mathcal{G}_{\varepsilon,\delta}(u_{\varepsilon,\delta}, B_r) \\
&\le \lim_{\varepsilon} \lim_{\delta} \mathcal{G}_{\varepsilon,\delta}(\tilde{u}_\varepsilon, B_r) \\
&= \mathcal{G}(u, B_r) \le \mathcal{G}(v, B_r)\,.
\end{aligned} \tag{6.56}$$

We deduce $u = v$. Letting first $\delta \to 0$, and then $\varepsilon \to 0$ in (6.47) used with $\tau = t$, leads to (2.25), that implies, via a standard covering argument, the local Lipschitz continuity of $u$. This concludes the proof of Theorem 5. Finally, we observe that the same arguments developed here, and in the preceding Sect. 6.2, apply verbatim to the setting of Theorem 1, and in this case one replaces $\|f\|_{n/\alpha,1;B_r}$ by $\|f\|_{n/\alpha,1/2;B_r}$ in (2.25). The outcome is estimate (2.14), and the local Lipschitz continuity assertion of Theorem 1. The proof of Theorem 1 will be indeed completed in Sect. 6.4, devoted to gradient local Hölder continuity under reinforced assumptions of $f$.

**Lemma 6.2** *Let $v \in L^\gamma(B_r)$ and let $\{v_\varepsilon\} \subset L^\gamma(B_r)$ be a sequence such that $v_\varepsilon \to v \in L^\gamma(B_r)$, where $\gamma > n/(n-1)$. Then, as $\varepsilon \to 0$*

$$\begin{cases} \|\tilde{\mathrm{h}}_\varepsilon(\cdot, v_\varepsilon) - \mathrm{h}(\cdot, v_\varepsilon)\|_{L^1(B_r)} \to 0 \\ \|\tilde{\mathrm{h}}_\varepsilon(\cdot, v_\varepsilon) - \mathrm{h}(\cdot, v)\|_{L^1(B_r)} \to 0\,. \end{cases} \tag{6.57}$$

***Proof*** Using (6.32) and (6.37), it follows that

$$|\tilde{\mathrm{h}}_\varepsilon(x, y)| \lesssim \mathrm{h}_{0,\varepsilon}(x) + [f_\varepsilon(x)]^{n/\alpha} + |y|^{n\alpha/(n-\alpha)}$$

holds for every $x \in B_r$ and $y \in \mathbb{R}$. By this, (6.37) and the equintegrability of the sequences $\{|v_\varepsilon|^{n/(n-1)}\}$, $\{\mathrm{h}_{0,\varepsilon}(x)\}$ and $\{[f_\varepsilon(x)]^{n/\alpha}\}$, it is sufficient to prove that (6.57) holds with $B_r$ replaced by any concentric ball $B_\varrho \Subset B_r$ and considering the range $\varepsilon < (r - \varrho)/10$. Let us first show that

$$\lim_{\varepsilon} \|\tilde{\mathrm{h}}_\varepsilon(\cdot, v) - \mathrm{h}(\cdot, v)\|_{L^1(B_\varrho)} = 0\,. \tag{6.58}$$

For this we estimate

$$\begin{aligned}
&\|\tilde{\mathrm{h}}_\varepsilon(\cdot, v) - \mathrm{h}(\cdot, v)\|_{L^1(B_\varrho)} \\
&\quad \le \|\tilde{\mathrm{h}}_\varepsilon(\cdot, v) - [\mathrm{h}(\cdot, v)] * \phi_\varepsilon\|_{L^1(B_\varrho)} + \|[\mathrm{h}(\cdot, v)] * \phi_\varepsilon - \mathrm{h}(\cdot, v)\|_{L^1(B_\varrho)}\,.
\end{aligned} \tag{6.59}$$

The last term in (6.59) goes to zero, as $\varepsilon \to 0$, by basic properties of convolutions (note that (6.37) guarantees that $\mathtt{h}(\cdot, v) \in L^1(B_r)$). For the first one, for every $k \in \mathbb{N}$ and $x \in B_\varrho$ we have

$$
\begin{aligned}
|\tilde{\mathtt{h}}_\varepsilon(x, v(x)) - [[\mathtt{h}(\cdot, v)] * \phi_\varepsilon](x)| &\le \int_{\mathcal{B}_1} f(x+\varepsilon\lambda)|v(x+\varepsilon\lambda) - v(x)|^\alpha \phi(\lambda)\,\mathrm{d}\lambda \\
&\le \frac{1}{k}\int_{\mathcal{B}_1} [f(x+\varepsilon\lambda)]^{n/\alpha}\phi(\lambda)\,\mathrm{d}\lambda + c(k)\int_{\mathcal{B}_1} |v(x+\varepsilon\lambda) - v(x)|^{\frac{n\alpha}{n-\alpha}}\phi(\lambda)\,\mathrm{d}\lambda\,.
\end{aligned}
$$

Integrating the previous inequality over $B_\varrho$ and using Fubini, we obtain

$$
\|\tilde{\mathtt{h}}_\varepsilon(\cdot, v) - [\mathtt{h}(\cdot, v)] * \phi_\varepsilon\|_{L^1(B_\varrho)} \le \frac{c}{k}\|f\|^{n/\alpha}_{L^{n/\alpha}(B_r)} + c(k)\mathrm{o}(\varepsilon, B_\varrho)\,.
$$

Using this in (6.59), letting first $\varepsilon \to 0$ and then $k \to \infty$ yields (6.58). For $(6.57)_1$ we estimate

$$
\begin{aligned}
\|\tilde{\mathtt{h}}_\varepsilon(\cdot, v_\varepsilon) - \mathtt{h}(\cdot, v_\varepsilon)\|_{L^1(B_\varrho)} \le\; & \|\tilde{\mathtt{h}}_\varepsilon(\cdot, v_\varepsilon) - \tilde{\mathtt{h}}_\varepsilon(\cdot, v)\|_{L^1(B_\varrho)} \\
& + \|\tilde{\mathtt{h}}_\varepsilon(\cdot, v) - \mathtt{h}(\cdot, v)\|_{L^1(B_\varrho)} \\
& + \|\mathtt{h}(\cdot, v) - \mathtt{h}(\cdot, v_\varepsilon)\|_{L^1(B_\varrho)}\,.
\end{aligned} \tag{6.60}
$$

Then, using (6.39), Fubini and finally Hölder's inequality (recall that $\gamma > n\alpha/(n-\alpha)$), we get

$$
\|\tilde{\mathtt{h}}_\varepsilon(\cdot, v_\varepsilon) - \tilde{\mathtt{h}}_\varepsilon(\cdot, v)\|_{L^1(B_\varrho)} \le c\|f+1\|_{L^{n/\alpha}(B_r)}\|v_\varepsilon - v\|^\alpha_{L^\gamma(B_r)} \to 0\,,
$$

as $\varepsilon \to 0$, and (2.6) gives

$$
\|\mathtt{h}(\cdot, v) - \mathtt{h}(\cdot, v_\varepsilon)\|_{L^1(B_\varrho)} \le c\|f\|_{L^{n/\alpha}(B_r)}\|v_\varepsilon - v\|^\alpha_{L^\gamma(B_r)} \to 0\,. \tag{6.61}
$$

Using the content of the last two displays and (6.58) in (6.60) yields $(6.57)_1$. As for $(6.57)_2$, this follows via $(6.57)_1$ and (6.61) via triangle inequality. $\square$

### 6.4 Gradient Hölder continuity and proof of Theorem 1 concluded

Here we assume that $f \in L^{\mathfrak{q}}$ and $q > n/\alpha$ and prove that $Du$ is locally Hölder continuous. This completes the proof of Theorem 1. Once we know that the gradient of minimizers is locally bounded, we can use more standard perturbation methods to prove its local Hölder continuity. Still we have to be careful at several points, since the boundedness of the gradient cannot be directly used, but must be transferred with an explicit control on the $L^\infty$-norm. For this, we reconsider some of the arguments explained in Sects. 5.3.2-5.3.4 and use the same approximation described in Sect. 6.2. First of all, as we are going to prove a local result, there is no loss of generality in assuming that $\mathcal{G}(u, \Omega) < \infty$, $h(\cdot, u) \in L^1(\Omega)$ and $f \in L^{\mathfrak{q}}(\Omega)$. Fix an open subset $\Omega_0 \Subset \Omega$, set $r := \min\{\mathrm{dist}(\Omega_0, \partial\Omega)/4, 1\}$; choose $B_r \equiv B_r(x_c)$ with $x_c \in \Omega_0$ (that implies that $B_r \Subset \Omega$) and use (6.47) to obtain, for any concentric ball $B_\tau \subset B_r$ with $\tau \le r/2$

$$
\begin{aligned}
\|Du_{\varepsilon,\delta}\|_{L^\infty(B_\tau)} &\le cr^{-\chi_1}\left[\mathcal{G}(u,\Omega)+\|\mathrm{h}(\cdot,u)\|_{L^1(\Omega)}+\|f\|_{L^{\mathfrak{q}}(\Omega)}+1\right]^{\chi_2}\\
&\qquad + cr^{-\chi_1}[\mathrm{o}(\varepsilon,B_r)+\mathrm{o}_\varepsilon(\delta,B_r)]\\
&=: M + cr^{-\chi_1}[\mathrm{o}(\varepsilon,B_r)+\mathrm{o}_\varepsilon(\delta,B_r)]\\
&=: M_{B_r}(\varepsilon,\delta) \ge 1\,,
\end{aligned}
\tag{6.62}
$$

where $c \equiv c(\mathtt{data}, |\Omega|)$. Note that the constant $M$ does not depend on the ball chosen $B_r$, but only on the radius $r$, that is, on $\mathrm{dist}(\Omega_0, \partial\Omega)$. Recalling the meaning of $\mathrm{o}(\varepsilon, B_r)$ and $\mathrm{o}_\varepsilon(\delta, B_r)$ introduced at the beginning of Sect. 6.2, letting first $\delta \to 0$ (keeping $\varepsilon$ fixed), and then $\varepsilon \to 0$, we get that $M_{B_r}(\varepsilon,\delta) \to M$ in the sense that

$$
M_{B_r}(\varepsilon,\delta) \stackrel{\delta\to 0}{\to} M + cr^{-\chi_1}\mathrm{o}(\varepsilon,B_r) \stackrel{\varepsilon\to 0}{\to} M\,. \tag{6.63}
$$

We define $v \in u_{\varepsilon,\delta} + W^{1,q}_0(B_\tau)$ as the solution to

$$
v \mapsto \min_{w \in u_{\varepsilon,\delta}+W^{1,q}_0(B_\tau)} \int_{B_\tau} F_0(Dw)\,\mathrm{d}x\,, \qquad F_0(z) := \mathbb{F}_{\varepsilon,\delta}(x_{\mathrm{c}}, (u_{\varepsilon,\delta})_{B_\tau}, z) \tag{6.64}
$$

so that, by (6.29) and (6.31) it turns out that $\mathbb{F}_{\varepsilon,\delta}(x_{\mathrm{c}}, (u_{\varepsilon,\delta})_{B_\tau}, z) = F_{\varepsilon,\delta}(z) + \mathfrak{g}_\delta(x_{\mathrm{c}}, (u_{\varepsilon,\delta})_{B_\tau}, z)$. Recalling Remark 10, observe that $F_{\varepsilon,\delta}(\cdot)$ satisfies (6.43), and that $\mathfrak{g}_\delta(\cdot)$ satisfies (5.4) with $\mu \equiv \mu_\delta > 0$ (see the discussion after (6.43)). Then, three implied features of (6.64) are:

- The integrand $F_0(\cdot)$ is exactly of the form considered in (5.33) (without scaling, therefore $\varrho = 1$) and we can use some of the estimates developed in Sect. 5.3.2.
- Applying Lemma 5.3, estimate (5.25), and then the minimality of $v$ exactly as in (5.40)-(5.41), and finally recalling (6.62), we gain an analog of (5.42), i.e.,

$$
\|Dv\|_{L^\infty(B_{\tau/2})} \le \tilde{c}[M_{B_r}(\varepsilon,\delta)]^{\mathfrak{s}}\,, \qquad \tilde{c} \equiv \tilde{c}(n,p,q,\nu,L)\,. \tag{6.65}
$$

- Let us fix a constant $\mathfrak{M} \ge 1$. The integrand $F_0(\cdot)$ is $C^2$-regular and satisfies

$$
\begin{cases}
\tilde{\nu}[H_{\mu_\delta}(z)]^{p/2} \le F_0(z) \le c\mathfrak{M}^{q-p}[H_{\mu_\delta}(z)]^{p/2}\\
\tilde{\nu}[H_{\mu_\delta}(z)]^{(p-2)/2}|\xi|^2 \le \partial_{zz}F_0(z)\xi\cdot\xi\\
|\partial_{zz}F_0(z)| \le c\mathfrak{M}^{q-p}[H_{\mu_\delta}(z)]^{(p-2)/2}
\end{cases}
\tag{6.66}
$$

  whenever $z, \xi \in \mathbb{R}^n$ with $|z| \le \mathfrak{M}$, where $c \equiv c(n,p,q,\tilde{L}) \equiv c(n,p,q,L)$.

We are now in position to apply Lemma 6.3 below to $v$ in the ball $B_{\tau/2}$, with $\mathfrak{M} \equiv \tilde{c}[M_{B_r}(\varepsilon,\delta)]^{\mathfrak{s}}$, as appearing in (6.65). We conclude with

$$
⨍_{B_\varrho} |Dv-(Dv)_{B_\varrho}|^p\,\mathrm{d}x \le c_{\varepsilon,\delta}\left(\frac{\varrho}{\tau}\right)^{p\beta_{\varepsilon,\delta}} \tag{6.67}
$$

that holds whenever $B_\varrho \subset B_{\tau/2}$ are concentric balls. Here

$$\begin{cases} c_{\varepsilon,\delta} \equiv c_{\varepsilon,\delta}(n,p,q,\nu,L,M_{B_r}(\varepsilon,\delta)) := c_{\mathrm{h}}(\mathfrak{M}) \geq 1 \\ \beta_{\varepsilon,\delta} \equiv \beta_{\varepsilon,\delta}(n,p,q,\nu,L,M_{B_r}(\varepsilon,\delta)) := \beta_{\mathrm{h}}(\mathfrak{M}) \in (0,1) \end{cases}$$

are non-decreasing and non-increasing functions of their last argument, respectively. This follows by the monotonicity properties of the functions $c_{\mathrm{h}}(\cdot)$ and $\beta_{\mathrm{h}}(\cdot)$ asserted in Lemma 6.3. By (6.63), we get, up to not relabelled subsequences

$$\begin{cases} c_{\varepsilon,\delta} \overset{\delta\to 0}{\to} c_\varepsilon \overset{\varepsilon\to 0}{\to} c_{\mathrm{l}} \equiv c_{\mathrm{l}}(\texttt{data}, M) < \infty \\ \beta_{\varepsilon,\delta} \overset{\delta\to 0}{\to} \beta_\varepsilon \overset{\varepsilon\to 0}{\to} \beta_{\mathrm{l}} \equiv \beta_{\mathrm{l}}(\texttt{data}, M) > 0 \end{cases} \tag{6.68}$$

in the same sense of (6.63). From now on we continue to denote by $c_{\varepsilon,\delta}$ a double sequence of constants, depending in the most general case on $\texttt{data}$, $\mathfrak{q}$ and $M_{B_r}(\varepsilon,\delta)$, such that (6.68) takes place. The exact value of $c_{\varepsilon,\delta}$ might change on different occurrences, but still keeping the property in (6.68), as it will be clear by the way they will be determined, starting from (6.63) and (6.68). To proceed, we argue as for the proof of (5.67); there replace $8|h|^{\beta_0} \equiv \tau$ and $f_\varrho$ by $f_\varepsilon$ and $\varrho$ by 1, in order to adapt it to the present setting. In particular the original (5.4)-(5.5) must be used instead of the rescaled ones (5.30)-(5.31). This said, we find

$$\begin{aligned} \fint_{B_\tau} |V_{\mu_\delta}(Du_{\varepsilon,\delta}) - V_{\mu_\delta}(Dv)|^2\,\mathrm{d}x &\leq c\tau^\alpha [M_{B_r}(\varepsilon,\delta)]^{\alpha+\gamma q/p} \\ &\quad + c\tau^\alpha [M_{B_r}(\varepsilon,\delta)]^\alpha \fint_{B_\tau} f_\varepsilon\,\mathrm{d}x \\ &\leq c_{\varepsilon,\delta}\tau^\alpha + c_{\varepsilon,\delta}\tau^{\alpha-n/\mathfrak{q}}\|f+1\|_{L^{\mathfrak{q}}(B_r)} \\ &\leq c_{\varepsilon,\delta}\tau^{\alpha-n/\mathfrak{q}}\,. \end{aligned} \tag{6.69}$$

Setting $\sigma := \min\{1, p/2\}(\alpha - n/\mathfrak{q}) > 0$, by (3.13) and Hölder's inequality, we further obtain

$$\begin{aligned} \fint_{B_\tau} |Du_{\varepsilon,\delta} - Dv|^p\,\mathrm{d}x &\leq c_{\varepsilon,\delta}\tau^{\alpha-n/\mathfrak{q}} + c_{\varepsilon,\delta}\mathbb{1}_p [M_{B_r}(\varepsilon,\delta)]^{p(2-p)/2}\tau^{p(\alpha-n/\mathfrak{q})/2} \\ &\leq c_{\varepsilon,\delta}\tau^\sigma\,. \end{aligned} \tag{6.70}$$

Estimates (6.67) and (6.70) are the two basic ingredients needed to apply a variant of the comparison argument implying local Hölder continuity. For this see for instance [62, pp. 43-45] and [1, Proof of Theorem 2.2]. Specifically, we first find

$$\fint_{B_\varrho} |Du_{\varepsilon,\delta} - (Du_{\varepsilon,\delta})_{B_\varrho}|^p\,\mathrm{d}x \leq c_{\varepsilon,\delta}\left(\frac{\varrho}{\tau}\right)^{p\beta_{\varepsilon,\delta}} + c_{\varepsilon,\delta}\left(\frac{\tau}{\varrho}\right)^n \tau^\sigma\,,$$

and this holds whenever $\varrho \leq \tau/2$. By taking $\varrho = (\tau/2)^{1+\sigma/(n+p\beta_{\varepsilon,\delta})}$ we arrive at

$$\fint_{B_\varrho} |Du_{\varepsilon,\delta} - (Du_{\varepsilon,\delta})_{B_\varrho}|^p\,\mathrm{d}x \leq c_{\varepsilon,\delta}\varrho^{\frac{p\sigma\beta_{\varepsilon,\delta}}{n+\sigma+p\beta_{\varepsilon,\delta}}}\,, \quad \forall\, \varrho \leq (r/4)^{1+\sigma/(n+p\beta_{\varepsilon,\delta})}\,.$$

Letting first $\delta \to 0$ and then $\varepsilon \to 0$ in the above display, and recalling (6.50),(6.52) and (6.68), we finally conclude with

$$\fint_{B_\varrho} |Du - (Du)_{B_\varrho}|^p \,dx \le c\varrho^{\frac{p\sigma\beta_1}{n+\sigma+p\beta_1}} , \quad \forall\, \varrho \le (r/4)^{1+\sigma/(n+p\beta_1)} , \tag{6.71}$$

where $c \equiv c(\texttt{data}, M)$ and the exponent $\beta_1 \equiv \beta_1(\texttt{data}, M) \in (0, 1)$ is defined in (6.68). Summarizing, we have proved that (6.71) holds whenever $B_\varrho$ is centred in $\Omega_0$, and where $r = \min\{\operatorname{dist}(\Omega_0, \partial\Omega)/4, 1\}$. Since $\Omega_0$ is arbitrary, the Campanato-Meyers integral characterization of Hölder continuity yields that for every open subset $\Omega_0 \Subset \Omega$ it holds that

$$[Du]_{0,\alpha_*;\Omega_0} \le c , \quad \alpha_* := \frac{\sigma\beta_1}{n + \sigma + p\beta_1} , \tag{6.72}$$

where $c \equiv c(\texttt{data}, \mathcal{G}(u, \Omega), \|\mathrm{h}(\cdot, u)\|_{L^1(\Omega)}, \|f\|_{L^q(\Omega)}, \operatorname{dist}(\Omega_0, \partial\Omega))$. The proof of Theorem 1 is finally complete.

**Lemma 6.3** (Theorem 2 from [62], revisited) *Let $v \in W^{1,q}(B)$, for some ball $B \subset \mathbb{R}^n$, be a weak solution to $\operatorname{div} \partial_z F_0(Dv) = 0$ in $B$, such that $\|Dv\|_{L^\infty(B)} \le \mathfrak{M}$, where $\mathfrak{M} \ge 1$ is a fixed constant and $F_0(\cdot)$ is defined in (6.64). There exist two constants $c_{\mathrm{h}}(\mathfrak{M}) \ge 1$ and $\beta_{\mathrm{h}}(\mathfrak{M}) \in (0, 1]$, depending on $n$, $p$, $q$, $\nu$, $L$ and $\mathfrak{M}$, but otherwise independent of $\mu$, $\varepsilon$, $\delta$, $\nu_0$ and the integrand considered $F_0(\cdot)$, such that*

$$\operatorname{osc}(Dv, sB) \le c_{\mathrm{h}}(\mathfrak{M}) s^{\beta_{\mathrm{h}}(\mathfrak{M})} \tag{6.73}$$

*holds whenever $s \in (0, 1]$. Moreover, the functions $\mathfrak{M} \mapsto c_{\mathrm{h}}(\mathfrak{M})$ and $\mathfrak{M} \mapsto \beta_{\mathrm{h}}(\mathfrak{M})$ are non-decreasing and non-increasing, respectively.*

***Proof*** There are essentially two crucial remarks here. The first is that, when applying the methods for [62, Theorem 2] to $\operatorname{div} \partial_z F_0(Dv) = 0$, we only see $z \equiv Dv$ as arguments of $\partial_z F_0(z)$ and $\partial_{zz} F_0(z)$. See also the methods in [22, Sect. 5.10], generating essentially the same outcomes. Therefore we can argue as conditions (6.66) hold for every $z \in \mathbb{R}^n$ when dealing with weak solutions as in [62]. At this point (6.73) follows, with the dependence of $c_{\mathrm{h}}(\mathfrak{M})$, $\beta_{\mathrm{h}}(\mathfrak{M})$ described in the statement, by tracking the constants in [62]. Specifically, the ratio $\gamma_1/\gamma_0$ appearing in [62] is in this setting replaced by $c\mathfrak{M}^{q-p}$, where $c \equiv c(n, p, q, \nu, L)$. The second remark is that here we cannot easily use an approximation procedure, as in [62], since we have to keep the condition $\|Dv\|_{L^\infty(B)} \le \mathfrak{M}$. Such an approximation is used in [62] to deal with $C^2$-solutions; such higher regularity allows for certain computations. It is not difficult to see that essentially the only point where this enters is the derivation and the testing of the differentiated equation $\operatorname{div}(\partial_{zz} F_0(Dv) DD_s v) = 0$, $s \in \{1, \dots, n\}$. On the other hand, in order to carry out the computation of [62], it is sufficient to have $Dv \in W^{1,2}_{\mathrm{loc}}(B, \mathbb{R}^n) \cap L^\infty_{\mathrm{loc}}(B, \mathbb{R}^n)$. As for $Dv \in L^\infty$, this is assumed in Lemma 6.3. Finally, $Dv \in W^{1,2}_{\mathrm{loc}}$ comes as in Lemma 5.1, whose application is allowed as $\mu \equiv \mu_\delta > 0$ in the present situation. This allows to avoid the approximation in [62], that was in fact implemented to reduce to the case $\mu > 0$ (denoted by $\varepsilon$ in [62]). Note that, again, we come to the same conclusions if, instead of using the methods in [62], we use those in [22, Sect. 5.10]. $\square$

## 7 Theorems 2, 4 and Corollary 3

**Proposition 7.1** *Let $u \in W^{1,q}(B_r)$ be a minimizer of the functional $\mathcal{F}_{\mathtt{x}}(\cdot, B_r)$ in* (2.15), *where $B_r \Subset \Omega$ and $r \leq 1$, under assumptions* (5.78) *and*

$$\frac{q}{p} < 1 + \mathtt{k}\frac{\alpha^2}{n^2}\,, \qquad \textit{where } \mathtt{k} := \begin{cases} 4/9 & \textit{if } p \geq 2 \\ 8/33 & \textit{if } 1 < p < 2\,. \end{cases} \tag{7.1}$$

*Then*

$$\|Du\|_{L^\infty(B_t)} \leq \frac{c}{(s-t)^{\chi_1}}\left[\|Du\|_{L^p(B_s)} + 1\right]^{\chi_2} \tag{7.2}$$

*holds whenever $B_t \Subset B_s \Subset B_r$ are concentric balls, with $c \equiv c(\mathtt{data})$ and $\chi_1, \chi_2 \equiv \chi_1, \chi_2(\mathtt{data_e})$.*

The setting of Proposition 7.1 is the one of Section 5.5. We shall use the following fact (keep Remark 6 in mind):

$$\frac{q}{p} < 1 + \frac{\alpha^2}{(C+1/4)n^2} \Longrightarrow \mathfrak{s} < 1 + \frac{\alpha}{Cn}\,, \tag{7.3}$$

that holds whenever $C \geq 1$ is a fixed number, where $\mathfrak{s}$ is defined in (5.2); keep Remark 6 in mind.

***Remark 11*** The standard regularity theory for nonlinear elliptic equations [62] gives that, under the assumptions of Proposition 5.5 (except (5.94)), any $W^{1,q}$-solution $u$ to (5.91) satisfies (5.7). The same applies to the minimizers $u$ considered in Proposition 7.1, as they are energy solutions to the Euler-Lagrange equation of the functional $\mathcal{F}_{\mathtt{x}}$; see comments after (5.78). Such equations are of the type in (5.91) considered in Proposition 5.5, by assumptions (5.78). Therefore in proving Proposition 7.1 we can assume that $Du$ is locally bounded in $B_r$.

### 7.1 Proposition 7.1, case $p \geq 2$

We first discuss the case $\alpha < 1$; at the end we give the modifications for the case $\alpha = 1$. We take balls $B_t \Subset B_s$ as in the statement of Proposition 7.1; we can assume that $\|Du\|_{L^\infty(B_t)} \geq 1$. Next, we consider further concentric balls $B_t \Subset B_{\tau_1} \Subset B_{\tau_2} \Subset B_s$, and a generic point $x_0 \in B_{\tau_1}$. By (5.80), with $M \equiv \|Du\|_{L^\infty(B_{\tau_2})}$, we apply Lemma 4.2 on $B_{r_0}(x_0) \equiv B_{(\tau_1-\tau_1)/8}(x_0)$ with $h \equiv 1$, $\kappa_0 \equiv 0$, $v \equiv E_\mu(Du)$, $f_1 \equiv |Du| + 1$, $M_0 \equiv M^{\mathfrak{s}(q-p)/2}$, $M_1 \equiv M^{(\mathfrak{s}q+p-\mathrm{b})/2}$, $t \equiv 2$, $\delta_1 \equiv \alpha$, $m_1 \equiv 2q - 2p + \mathrm{b}$, $\theta_1 \equiv 1$. We obtain

$$\begin{aligned} E_\mu(Du(x_0)) &\leq c\|Du\|_{L^\infty(B_{\tau_2})}^{\frac{\mathfrak{s}(q-p)\chi}{2(\chi-1)}}\left(\fint_{B_{r_0}(x_0)} [E_\mu(Du)]^2\,\mathrm{d}x\right)^{1/2} \\ &\quad + c\|Du\|_{L^\infty(B_{\tau_2})}^{\frac{\mathfrak{s}(q-p)}{2(\chi-1)}+\frac{\mathfrak{s}q+p-\mathrm{b}}{2}} \mathbf{P}_{2,\alpha}^{2q-2p+\mathrm{b},1}(|Du|+1; x_0, (\tau_2-\tau_1)/4)\,, \end{aligned} \tag{7.4}$$

where $\chi \equiv \chi(\beta) = n/(n-2\beta)$ for every $\beta < \alpha/(1+\alpha)$, $\mathrm{b} \in (0, p]$ and $c \equiv c(\mathtt{data}, \beta)$. Using (6.6), after a few manipulations we conclude with

$$\begin{aligned}\|Du\|_{L^\infty(B_{\tau_1})} &\le \frac{c}{(\tau_1-\tau_2)^{\frac{n}{2p}}}\|Du\|_{L^\infty(B_{\tau_2})}^{\left(\frac{q}{p}-1\right)\frac{\mathfrak{s}\chi}{2(\chi-1)}+\frac{1}{2}}\left(\int_{B_s}(|Du|+1)^p\,\mathrm{d}x\right)^{\frac{1}{2p}}\\ &\quad + c\|Du\|_{L^\infty(B_{\tau_2})}^{\left(\frac{q}{p}-1\right)\frac{\mathfrak{s}\chi}{2(\chi-1)}+\frac{\mathfrak{s}+1}{2}-\frac{\mathrm{b}}{2p}}\\ &\quad \cdot \|\mathbf{P}_{2,\alpha}^{2q-2p+\mathrm{b},1}(|Du|+1;\cdot,(\tau_2-\tau_1)/4)\|_{L^\infty(B_{\tau_1})}^{1/p}+c\,. \end{aligned} \tag{7.5}$$

The highest power of $\|Du\|_{L^\infty(B_{\tau_2})}$ appears in the second line of the above display; this follows by $\mathrm{b} \le p$ and $\mathfrak{s} \ge 1$. We now want to show that there exist $\beta < \alpha/(1+\alpha)$ and $\mathrm{b} \equiv \mathrm{b}(\beta) \in (0, p]$ such that

$$\left(\frac{q}{p}-1\right)\frac{\mathfrak{s}\chi}{2(\chi-1)}+\frac{\mathfrak{s}+1}{2}-\frac{\mathrm{b}}{2p} = \left(\frac{q}{p}-1\right)\frac{\mathfrak{s}n}{4\beta}+\frac{\mathfrak{s}+1}{2}-\frac{\mathrm{b}}{2p} < 1 \tag{7.6}$$

and the $L^\infty$-norm of $\mathbf{P}_{2,\alpha}^{2q-2p+\mathrm{b},1}$ appearing in (7.5) can be estimated by the $L^p$-norm of $Du$. In terms of $q/p$, condition (7.6) translates into

$$\frac{q}{p} < 1 + \left(1-\mathfrak{s}+\frac{\mathrm{b}}{p}\right)\frac{2\beta}{\mathfrak{s}n}\,. \tag{7.7}$$

For the $\mathbf{P}_{2,\alpha}^{2q-2p+\mathrm{b},1}$-term, we start checking that condition (4.7) is satisfied, that is

$$\frac{n\theta}{t\delta} \equiv \frac{n}{2\alpha} > 1\,. \tag{7.8}$$

This is the point where we use that $\alpha < 1$, as the above quantity turns out to be equal to one when $\alpha = 1$ and $n = 2$. As mentioned above, the case $\alpha = 1$ will be treated later. Now we get an $L^\infty$-bound for $\mathbf{P}^{2q-2p+\mathrm{b},1}$. First, thanks to (7.8), we apply (4.8) to get

$$\|\mathbf{P}_{2,\alpha}^{2q-2p+\mathrm{b},1}(|Du|+1;\cdot,(\tau_2-\tau_1)/4)\|_{L^\infty(B_{\tau_1})} \le c\||Du|+1\|_{L^p(B_s)}^{q-p+\mathrm{b}/2} \tag{7.9}$$

where $c \equiv c(n, p, q, \alpha, \mathrm{b})$, and provided

$$\frac{mn\theta}{t\delta} \equiv \frac{n(2q-2p+\mathrm{b})}{2\alpha} < p \Longleftrightarrow \frac{q}{p} < 1 + \frac{\alpha}{n} - \frac{\mathrm{b}}{2p}\,. \tag{7.10}$$

This provides a second condition on $(p, q)$, the one in (7.7) being the first. As specified in the statement of Proposition 5.4, (7.5) and (7.9) remain valid for any choice of $\mathrm{b} \in (0, p]$ satisfying (7.10). We optimize b matching both (7.7) and (7.10); this leads to equalize the two right-hand sides in such inequalities and therefore to the choice

$$\mathrm{b} \equiv \mathrm{b}(\beta) = \frac{2p[\mathfrak{s}\alpha + 2\beta(\mathfrak{s}-1)]}{\mathfrak{s}n+4\beta} > 0\,. \tag{7.11}$$

Notice that

$$\mathrm{b} < \frac{2p\alpha}{n} \Longleftrightarrow \mathfrak{s} < 1 + \frac{2\alpha}{n}\,. \tag{7.12}$$

By (7.3), that we use with $C = 2$, we find

$$(7.1) \Longrightarrow \frac{q}{p} < 1 + \frac{4\alpha^2}{9n^2} \Longrightarrow \mathfrak{s} < 1 + \frac{\alpha}{2n} \leq \frac{5}{4} \tag{7.13}$$

so that (7.12) is satisfied, in turn implying $\mathrm{b} \leq p$; the above conditions are independent of $\beta$ although b is not (indeed $\beta \mapsto \mathrm{b}(\beta)$ is a decreasing function provided the last inequality in (7.12) holds). Therefore b in (7.11) is an admissible value in (5.80) and (7.5), for any $\beta < \alpha/(1+\alpha)$, and the (equal) right-hand sides of (7.7) and (7.10) are larger than one. Plugging b from (7.11) in (7.10) yields

$$\frac{q}{p} < 1 + \frac{2\beta}{n}\left[\frac{2\alpha - n(\mathfrak{s}-1)}{\mathfrak{s}n + 4\beta}\right] \tag{7.14}$$

and the right-hand side is an increasing function of $\beta$ by (7.13). Formally taking the limiting value $\beta = \alpha/(1+\alpha)$ in (7.14), we come up with

$$\frac{q}{p} < 1 + \frac{2\alpha}{(1+\alpha)n}\left[\frac{2\alpha - n(\mathfrak{s}-1)}{\mathfrak{s}n + 4\alpha/(1+\alpha)}\right] =: 1 + \mathcal{R}_1(n,p,q,\alpha)\frac{\alpha^2}{n^2}\,. \tag{7.15}$$

By (7.13) it turns out that $\mathcal{R}_1(n,p,q,\alpha) > 2/3 > 4/9 = \mathtt{k}$, so that (7.15) is again implied by (7.1). In conclusion, assuming (7.1) leads to find $\beta < \alpha/(1+\alpha)$, close enough to $\alpha/(1+\alpha)$, such that (7.14) and therefore both (7.6) and (7.10) hold with the specific choice of b made in (7.11). As a consequence, (7.9) holds too and inserting this one in (7.5), and applying Young's inequality thanks to (7.6), we arrive at

$$\|Du\|_{L^\infty(B_{\tau_1})} \leq \frac{1}{2}\|Du\|_{L^\infty(B_{\tau_2})} + \frac{c}{(\tau_2-\tau_1)^{\chi_1}}\left[\|Du\|_{L^p(B_s)} + 1\right]^{\chi_2}\,,$$

with $c \equiv c(\mathtt{data})$ and $\chi_1, \chi_2 \equiv \chi_1, \chi_2(n,p,q,\alpha)$. Using Lemma 3.2 with $h(\tau) \equiv \|Du\|_{L^\infty(B_\tau)}$ (for this recall that $Du$ is locally bounded in $B_r$ by Remark 11) leads to (7.2), which is now proved when $\alpha < 1$. It remains to fix the case $\alpha = 1$, which in view of (7.8) is not covered only when $n = 2$ (otherwise the same proof above works). Observe that (5.80) still holds replacing $\varrho^{2\alpha}$, by $\varrho^{2\tilde{\alpha}}$ for any $\tilde{\alpha} < 1$. In particular, we take $\tilde{\alpha} < 1$ such that $q/p < 1 + \mathtt{k}\tilde{\alpha}^2/n^2$. We can now argue as in the case $\alpha < 1$ and the proof of Proposition 7.1 is complete when $p \geq 2$.

### 7.2 Proposition 7.1, case $p < 2$

The proof is similar to that for the case $p \geq 2$, but we are going to use (5.81) instead of (5.80). With $M \equiv \|Du\|_{L^\infty(B_{\tau_2})}$, we apply Lemma 4.2 on $B_{r_0}(x_0) \equiv B_{(\tau_1-\tau_1)/8}(x_0)$ with $h \equiv 1$, $\kappa_0 \equiv 0$, $v \equiv E_\mu(Du)$, $f_1 \equiv |Du| + 1$, $M_0 \equiv M^{\mathfrak{s}(q-p)/2}$,

$M_1 \equiv M^{[(\mathfrak{s}+1)q-\mathrm{b}/p]/2}$, $t \equiv 2$, $\delta_1 \equiv \alpha/2$, $m_1 \equiv q-p+\mathrm{b}$ and $\theta_1 \equiv 1/p$. As in the case $p \geq 2$, we obtain

$$\begin{aligned}\|Du\|_{L^\infty(B_{\tau_1})} &\leq \frac{c}{(\tau_1-\tau_2)^{\frac{n}{2p}}}\|Du\|_{L^\infty(B_{\tau_2})}^{\left(\frac{q}{p}-1\right)\frac{\mathfrak{s}\chi}{2(\chi-1)}+\frac{1}{2}}\left(\int_{B_s}(|Du|+1)^p\,\mathrm{d}x\right)^{\frac{1}{2p}}\\ &\quad + c\|Du\|_{L^\infty(B_{\tau_2})}^{\left(\frac{q}{p}-1\right)\frac{\mathfrak{s}\chi}{2(\chi-1)}+\frac{\mathfrak{s}}{2}+\frac{q-\mathrm{b}/p}{2p}}\\ &\quad\quad \cdot \|\mathbf{P}_{2,\alpha/2}^{q-p+\mathrm{b},1/p}(|Du|+1;\cdot,(\tau_2-\tau_1)/4)\|_{L^\infty(B_{\tau_1})}^{1/p}+c. \end{aligned} \tag{7.16}$$

Reabsorbing the $\|Du\|_{L^\infty}$-terms and estimating the $\mathbf{P}_{2,\alpha/2}^{q-p+\mathrm{b},1/p}$-ones in (7.16) (via (4.8)), leads to impose the conditions

$$\left(\frac{q}{p}-1\right)\frac{\mathfrak{s}n}{4\beta}+\frac{\mathfrak{s}}{2}+\frac{q-\mathrm{b}/p}{2p}<1 \qquad \text{and} \qquad \frac{n(q-p+\mathrm{b})}{p\alpha}<p \tag{7.17}$$

for some $\mathrm{b} \in (0,p]$ to be chosen. These parallel (7.7) and (7.10), respectively, from the case $p \geq 2$. The potential $\mathbf{P}_{2,\alpha/2}^{q-p+\mathrm{b},1/p}$ can be used here as we have $n\theta/(t\delta) = n/(p\alpha) > 1$ as now $p<2$, so that Lemma 4.1 applies and yields

$$\|\mathbf{P}_{2,\alpha/2}^{q-p+\mathrm{b},1/p}(|Du|+1;\cdot,(\tau_2-\tau_1)/4)\|_{L^\infty(B_{\tau_1})} \leq c\|Du\|_{L^p(B_s)}^{\frac{q-p+\mathrm{b}}{2p}}+c\,. \tag{7.18}$$

Arguing as for the case $p \geq 2$ in Sect. 7.1, relations in (7.17) translate into

$$\frac{q}{p}<1+\left(1-\mathfrak{s}+\frac{\mathrm{b}}{p^2}\right)\frac{2\beta}{\mathfrak{s}n+2\beta} \qquad \text{and} \qquad \frac{q}{p}<1+\frac{\alpha p}{n}-\frac{\mathrm{b}}{p} \tag{7.19}$$

and this leads to consider

$$\mathrm{b} \equiv \mathrm{b}(\beta) = \frac{p^2}{n}\left[\frac{p\alpha(\mathfrak{s}n+2\beta)+2\beta n(\mathfrak{s}-1)}{p(\mathfrak{s}n+2\beta)+2\beta}\right]>0\,. \tag{7.20}$$

By (7.3), now used with $C=3$, we have

$$(7.1) \Longrightarrow \frac{q}{p}<1+\frac{4\alpha^2}{13n^2} \Longrightarrow \mathfrak{s}<1+\frac{\alpha}{3n}\leq\frac{7}{6}\,. \tag{7.21}$$

This time the last inequality implies $\mathrm{b} \leq \alpha p^2/n \leq p$ (as $p \leq 2$), which is in fact equivalent to $\mathfrak{s}<1+\alpha/n$, so that the choice in (7.20) is admissible. This works whenever $\beta<\alpha/(2+\alpha)$. Using b from (7.20) in (7.19), and formally taking $\beta = \alpha/(2+\alpha)$, we find

$$\frac{q}{p}<1+\frac{2\alpha p}{n(2+\alpha)}\left[\frac{\alpha-n(\mathfrak{s}-1)}{\mathfrak{s}np+2\alpha(p+1)/(2+\alpha)}\right]=:1+\mathcal{R}_2(n,p,q,\alpha)\frac{\alpha^2}{n^2}\,.$$

Using (7.21) yields $\mathcal{R}_2(n,p,q,\alpha)>8/33=\mathtt{k}$, and we can conclude as in the case $p \geq 2$.

### 7.3 Proof of Theorem 4

Once (7.2) is secured, we can modify the arguments of Sects. 6.2-6.3 to get (2.20). In Sect. 6.2, replace the ball $B_r$ by $\Omega$; by the definition in (2.18) and the $q$-growth of the integrand $F(\cdot)$, we find a sequence $\{\tilde{u}_\varepsilon\} \subset W^{1,\infty}(\Omega)$ such that $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega)$ and $\mathcal{F}_{\mathrm{x}}(\tilde{u}_\varepsilon,\Omega) = \overline{\mathcal{F}_{\mathrm{x}}}(u,\Omega) + \mathrm{o}(\varepsilon,\Omega)$. The sequence $\{\tilde{u}_\varepsilon\}$ will play the role of the similarly denoted one defined in (6.28). We can proceed as in Sect. 6.2, where now it is $\mathbb{F}(x,y,z) \equiv F(x,z)$. All the terms involving $f(\cdot)$ and $\mathtt{h}(\cdot)$, including $\mathtt{h}_\varepsilon(\cdot)$ defined in (6.32), are absent. This time we define

$$\begin{cases} \mathbb{F}_{\varepsilon,\delta}(x,z) := (\mathbb{F}(x,\cdot) * \phi_\delta)(z) + \sigma_\varepsilon [H_{\mu_\delta}(z)]^{q/2} \\ \sigma_\varepsilon := (1 + \varepsilon^{-1} + \|D\tilde{u}_\varepsilon\|^{2q}_{L^q(\Omega)})^{-1} \\ \mathcal{F}_{\mathrm{x},\varepsilon,\delta}(w,\Omega) := \displaystyle\int_\Omega \mathbb{F}_{\varepsilon,\delta}(x,Dw)\,\mathrm{d}x \end{cases}$$

and $u_{\varepsilon,\delta}$ as the unique minimizer of $w \mapsto \mathcal{F}_{\mathrm{x},\varepsilon,\delta}(w,\Omega)$ in the Dirichlet class $\tilde{u}_\varepsilon + W^{1,q}_0(\Omega)$. We note that the integrand $z \mapsto \mathbb{F}_{\varepsilon,\delta}(x,z) (\equiv F_{\varepsilon,\delta}(z))$ satisfies (6.43), uniformly with respect to $x \in \Omega$, for a suitable choice of the parameters $\tilde{\nu}$, $\tilde{L}$ as in Sect. 2.1. We can now repeat the arguments of Sects. 6.2-6.3 with the following replacements. Instead of (6.42) we have $\mathcal{F}_{\mathrm{x},\varepsilon,\delta}(\tilde{u}_\varepsilon,\Omega) = \overline{\mathcal{F}_{\mathrm{x}}}(u,\Omega) + \mathrm{o}(\varepsilon,\Omega) + \mathrm{o}_\varepsilon(\delta,\Omega)$, and (6.46) becomes

$$\|Du_{\varepsilon,\delta}\|^p_{L^p(\Omega)} + \sigma_\varepsilon \|Du_{\varepsilon,\delta}\|^q_{L^q(\Omega)} \le c\overline{\mathcal{F}_{\mathrm{x}}}(u,\Omega) + \mathrm{o}(\varepsilon,\Omega) + \mathrm{o}_\varepsilon(\delta,\Omega)\,. \tag{7.22}$$

With $\Omega_0 \Subset \Omega$ being an open subset as in (2.20), we fix an increasing family of invading open subset $\{\Omega_\tau\}_{\tau>0}$, such that $\Omega_\tau \to \Omega$ as $\tau \to \infty$ and $\Omega_0 \subset \Omega_\tau \Subset \Omega$ for every $\tau$. Using (7.22) in combination with (7.2), and a covering argument, we find the analog of (6.47)

$$\begin{aligned} &\|Du_{\varepsilon,\delta}\|_{L^\infty(\Omega_\tau)} \\ &\quad \le \frac{c(\mathtt{data})}{[\operatorname{dist}(\Omega_\tau,\partial\Omega)]^{\chi_1}} \left[\overline{\mathcal{F}_{\mathrm{x}}}(u,\Omega) + 1\right]^{\chi_2} + \frac{\mathrm{o}(\varepsilon,\Omega) + \mathrm{o}_\varepsilon(\delta,\Omega)}{[\operatorname{dist}(\Omega_\tau,\partial\Omega)]^{\chi_1}}\,, \end{aligned} \tag{7.23}$$

that holds for every $\Omega_\tau \Subset \Omega$. In comparison to (6.47), here we use $\Omega$ in place of $B_r$ and with $\Omega_\tau$ in place of $B_\tau$. We proceed as in Sect. 6.3, with $u_{\varepsilon,\delta} \rightharpoonup u_\varepsilon \in \tilde{u}_\varepsilon + W^{1,q}_0(\Omega)$ weakly in $W^{1,q}(\Omega)$ and weakly* in $W^{1,\infty}(\Omega_0)$; then $u_\varepsilon \rightharpoonup v \in u + W^{1,p}_0(\Omega)$ weakly in $W^{1,p}(\Omega)$ and weakly* in $W^{1,\infty}(\Omega_0)$. As in (6.54) we get $\mathcal{F}_{\mathrm{x}}(u_\varepsilon,\Omega_\tau) \le \liminf_\delta \mathcal{F}_{\mathrm{x},\varepsilon,\delta}(u_{\varepsilon,\delta},\Omega)$. Letting $\tau \to \infty$ in this last inequality yields $\mathcal{F}_{\mathrm{x}}(u_\varepsilon,\Omega) \le \liminf_\delta \mathcal{F}_{\mathrm{x},\varepsilon,\delta}(u_{\varepsilon,\delta},\Omega) \le \liminf_\delta \mathcal{F}_{\mathrm{x},\varepsilon,\delta}(\tilde{u}_\varepsilon,\Omega) = \overline{\mathcal{F}_{\mathrm{x}}}(u,\Omega) + \mathrm{o}(\varepsilon,\Omega)$. Then by the definition in (2.18) we have $\overline{\mathcal{F}_{\mathrm{x}}}(v,\Omega) \le \liminf_\varepsilon \mathcal{F}_{\mathrm{x}}(u_\varepsilon,\Omega) \le \overline{\mathcal{F}_{\mathrm{x}}}(u,\Omega)$ and therefore $\overline{\mathcal{F}_{\mathrm{x}}}(u,\Omega) = \overline{\mathcal{F}_{\mathrm{x}}}(v,\Omega)$ by minimality of $u$. The equality $u \equiv v$ is then implied by the strict convexity of $w \mapsto \overline{\mathcal{F}_{\mathrm{x}}}(w,\Omega)$, see [24, pp. 47-50]. This settles the local boundedness of $Du$ as (2.20) follows taking $\tau = 0$, and letting first $\delta \to 0$ and then $\varepsilon \to 0$ in (7.23). For its local Hölder continuity, we revisit the arguments of Sect. 6.4. Using estimate (7.2) instead of (6.18), we derive a suitable analog of (6.62), where the constant $M$ in (6.62) is now of the form $M = cr^{-\chi_1}[\overline{\mathcal{F}_{\mathrm{x}}}(u,\Omega) + 1]^{\chi_2}$, where $r := \min\{\operatorname{dist}(\Omega_0, \partial\Omega)/4, 1\}$. The integrand $F_0(\cdot)$ and $v$ are again defined via (6.64) with the current definition of $\mathbb{F}_{\varepsilon,\delta}(\cdot)$, so that the properties in (6.66)-(6.67) are still in force. Finally, proceeding as in the proof of (5.86) and (5.88), we find

$$\begin{aligned}&\fint_{B_\tau} |V_{\mu_\delta}(Du_{\varepsilon,\delta}) - V_{\mu_\delta}(Dv)|^2 \,\mathrm{d}x\\ &\qquad \leq c\tau^{2\alpha}[M_{B_r}(\varepsilon,\delta)]^{2q-p} + c\mathbb{1}_p\tau^{\alpha}[M_{B_r}(\varepsilon,\delta)]^{q+q/p-1} \leq c_{\varepsilon,\delta}\tau^{\alpha}\,,\end{aligned}$$

that in fact replaces (6.69) in the present setting. After this, the rest of the proof proceeds as in Sect. 6.3 almost unchanged and leads to (6.72) for a different Hölder exponent $\alpha_*$.

### 7.4 Proof of Corollary 3

By the discussion before Corollary 3 and (2.21), $u$ is a minimizer of the functional $w \mapsto \overline{\mathcal{F}_{\mathrm{x}}}(w, B_r)$ for every ball $B_r \Subset \Omega$. Therefore Corollary 3 follows from Theorem 4 applied with $\Omega \equiv B_r$ and then a standard covering argument. In particular, estimate (2.22) follows directly from (2.20) by taking $B_t \Subset B_r$ as $\Omega' \Subset \Omega$ in Theorem 4, and recalling that $\mathcal{L}_{\mathcal{F}_{\mathrm{x}}}(u, B_r) = 0$ means that $\overline{\mathcal{F}_{\mathrm{x}}}(u, B_r) = \mathcal{F}_{\mathrm{x}}(u, B_r)$.

### 7.5 Proof of Theorem 2

The integrand $\mathfrak{c}(x)F(z)$ is of the type considered in Theorem 4. Indeed, by Lemma 3.4, the convexity of $F(\cdot)$ and $(2.4)_2$ imply

$$|\partial_z F(z)| \leq c(n, L)[H_\mu(z)]^{(q-1)/2} + c(n, L)[H_\mu(z)]^{(p-1)/2}\,,$$

so that assumption $(2.16)_2$ is verified. At this stage, it is sufficient to prove that $\mathcal{L}_{\mathcal{S}_{\mathrm{x}}}(u, B) = 0$ holds whenever $B \Subset \Omega$ is a ball, and Theorem 2 would follow from Corollary 3. For this, observe that the sequence $\{\tilde{u}_\varepsilon\}$ considered in (6.28), is such that $\mathfrak{c}(\cdot)F(D\tilde{u}_\varepsilon) \to \mathfrak{c}(\cdot)F(Du)$ in $L^1(B)$ by the convolution argument explained in Remark 9. This implies that $\overline{\mathcal{S}_{\mathrm{x}}}(u, B) \leq \mathcal{S}_{\mathrm{x}}(u, B)$ so that $\mathcal{L}_{\mathcal{S}_{\mathrm{x}}}(u, B) = 0$ and the proof is complete.

## 8 Theorem 3

**Proposition 8.1** *Let $u \in W^{1,q}(B_r)$ be a minimizer of the functional $\mathcal{S}(\cdot, B_r)$ in (1.9), where $B_r \Subset \Omega$ and $r \leq 1$, under assumptions (2.13), (5.68) and $p > n$. Then*

$$\|Du\|_{L^\infty(B_t)} \leq \frac{c}{(s-t)^{\chi_1}}\left[\|Du\|_{L^p(B_s)} + \|f\|_{n/\alpha,1/2;B_s} + 1\right]^{\chi_2} \tag{8.1}$$

*holds whenever $B_t \Subset B_s \Subset B_r$ are concentric balls, where $c \equiv c(\mathtt{data})$, $\chi_1, \chi_2 \equiv \chi_1, \chi_2(\mathtt{data}_{\mathrm{e}})$.*

***Proof*** Here the setting is the one of Sect. 5.4. By the discussion after (5.6), this time with $\tilde{F}(x,y,z)\equiv \mathfrak{c}(x,y)F(z)$, and the assumptions considered in Proposition 8.1, $u$ satisfies (5.7). This allows us to use that $Du$ is bounded in $B_s$. As in Proposition 7.1, we can assume that $\|Du\|_{L^\infty(B_t)}\geq 1$ and consider further concentric balls $B_t\Subset B_{\tau_1}\Subset B_{\tau_2}\Subset B_s$, $x_0\in B_{\tau_1}$. By (5.69) with $M\equiv \|Du\|_{L^\infty(B_{\tau_2})}$, we apply Lemma 4.2 on $B_{r_0}(x_0)\equiv B_{(\tau_1-\tau_1)/8}(x_0)$ with $h\equiv 2$, $\kappa_0\equiv 0$, $v\equiv E_\mu(Du)$, $f_1\equiv |Du|+1$, $f_2\equiv f$, $M_0\equiv M^{\mathfrak{s}(q-p)/2}$, $M_1\equiv M^{(\mathfrak{s}q+p+\alpha-\mathrm{b})/2}$, $M_2\equiv M^{(\mathfrak{s}q+\alpha)/2}$, $t\equiv 2$, $\delta_1=\delta_2\equiv\alpha/2$, $m_1\equiv q-p+\mathrm{b}$, $m_2\equiv 1$, $\theta_1=\theta_2\equiv 1$. Proceeding as for (7.4)-(7.5) leads to

$$
\begin{aligned}
\|Du\|_{L^\infty(B_{\tau_1})} &\leq \frac{c}{(\tau_1-\tau_2)^{\frac{n}{2p}}}\|Du\|_{L^\infty(B_{\tau_2})}^{\left(\frac{q}{p}-1\right)\frac{\mathfrak{s}\chi}{2(\chi-1)}+\frac{1}{2}}\left(\int_{B_s}(|Du|+1)^p\,\mathrm{d}x\right)^{\frac{1}{2p}}\\
&\quad + c\|Du\|_{L^\infty(B_{\tau_2})}^{\left(\frac{q}{p}-1\right)\frac{\mathfrak{s}\chi}{2(\chi-1)}+\frac{\mathfrak{s}+1}{2}+\frac{\alpha-\mathrm{b}}{2p}}\|\mathbf{P}^{q-p+\mathrm{b},1}_{2,\alpha/2}(|Du|+1;\cdot,(\tau_2-\tau_1)/4)\|^{1/p}_{L^\infty(B_{\tau_1})}\\
&\quad + c\|Du\|_{L^\infty(B_{\tau_2})}^{\left(\frac{q}{p}-1\right)\frac{\mathfrak{s}\chi}{2(\chi-1)}+\frac{\mathfrak{s}}{2}+\frac{\alpha}{2p}}\|\mathbf{P}^{1,1}_{2,\alpha/2}(f;\cdot,(\tau_2-\tau_1)/4)\|^{1/p}_{L^\infty(B_{\tau_1})}+c\,,
\end{aligned}
\tag{8.2}
$$

where $c\equiv c(\texttt{data},\beta)$. Note that the highest power of $\|Du\|_{L^\infty(B_{\tau_2})}$ appears in the second line of the above display since it is $\mathrm{b}\leq p$ and $\mathfrak{s}\geq 1$. As in Proposition 7.1, recalling also condition (4.7) to apply Lemma 4.1 and estimate the terms involving the potentials in (8.2) (observe that (4.7) is automatically satisfied since $n\geq 2$), this time we impose

$$
\left(\frac{q}{p}-1\right)\frac{\mathfrak{s}n}{4\beta}+\frac{\mathfrak{s}+1}{2}+\frac{\alpha-\mathrm{b}}{2p}<1 \quad\text{and}\quad \frac{n(q-p+\mathrm{b})}{\alpha}<p\,, \tag{8.3}
$$

that is

$$
\frac{q}{p}<1+\left(1-\mathfrak{s}+\frac{\mathrm{b}-\alpha}{p}\right)\frac{2\beta}{\mathfrak{s}n} \quad\text{and}\quad \frac{q}{p}<1+\frac{\alpha}{n}-\frac{\mathrm{b}}{p}\,, \tag{8.4}
$$

respectively. Equalizing the right-hand sides leads to consider

$$
\mathrm{b}\equiv\mathrm{b}(\beta)=\frac{p(2\beta+\alpha)(\mathfrak{s}-1)+\alpha(2\beta+p)}{\mathfrak{s}n+2\beta}\,. \tag{8.5}
$$

Now, note that

$$
\frac{\alpha}{n}-\frac{\mathrm{b}}{p}=\frac{\alpha}{n}-\frac{2\beta+\alpha}{\mathfrak{s}n+2\beta}(\mathfrak{s}-1)-\frac{\alpha(2\beta+p)}{p(\mathfrak{s}n+2\beta)}=\frac{2\alpha\beta(1-n/p)}{n(\mathfrak{s}n+2\beta)}-\frac{2\beta(\mathfrak{s}-1)}{\mathfrak{s}n+2\beta}\,. \tag{8.6}
$$

On the other hand, observe that

$$
\begin{aligned}
\frac{2\beta(\mathfrak{s}-1)}{\mathfrak{s}n+2\beta}<\frac{\alpha\beta(1-n/p)}{n(\mathfrak{s}n+2\beta)} &\Longleftrightarrow \mathfrak{s}-1<\frac{\alpha(1-n/p)}{2n}\\
&\overset{(5.2)}{\Longleftrightarrow} \frac{q}{p}<1+\frac{2\alpha(1-n/p)}{n[2(n+2)+\alpha(1-n/p)]}\\
&\Longleftarrow (2.13)\,.
\end{aligned}
\tag{8.7}
$$

Using the first inequality in (8.7) in (8.6) yields

$$\frac{\alpha}{n} - \frac{\mathrm{b}}{p} > \frac{\alpha\beta(1-n/p)}{n(\mathfrak{s}n+2\beta)} > 0 \tag{8.8}$$

that is $\mathrm{b} < p\alpha/n < p$, so that b in (8.5) is admissible in (5.69) and in (8.2), independently of the value of $\beta < \alpha/(2+\alpha)$. Taking (8.4) and (8.8) into account, we conclude that, in order to check that (8.4) holds for some $\beta < \alpha/(2+\alpha)$, it suffices to verify

$$\frac{q}{p} < 1 + \frac{\alpha\beta(1-n/p)}{n(\mathfrak{s}n+2\beta)}\,. \tag{8.9}$$

Note that the right-hand side of the above inequality is an increasing function of $\beta$. By formally taking the limiting value $\beta = \alpha/(2+\alpha)$ here, we come to

$$\frac{q}{p} < 1 + \frac{1-n/p}{[\mathfrak{s}+2\alpha/(n(2+\alpha))](2+\alpha)}\frac{\alpha^2}{n^2} =: 1 + \mathcal{R}_3(n,p,q,\alpha)\left(1-\frac{n}{p}\right)\frac{\alpha^2}{n^2}. \tag{8.10}$$

Noting that it is $\mathfrak{s} < 5/4$ by the second inequality in display (8.7), we infer the lower bound $\mathcal{R}_3(n,p,q,\alpha) > 4/19 > 1/5$, so that (8.10) is again implied by (2.13). We deduce that we can find $\beta < \alpha/(2+\alpha)$ such that (8.9) and therefore (8.4) and (8.3) are satisfied. Finally, thanks to the second inequality in (8.3) we can use (4.8), that yields

$$\|\mathbf{P}^{q-p+\mathrm{b},1}_{2,\alpha/2}(|Du|+1;\cdot,(\tau_2-\tau_1)/4)\|_{L^\infty(B_{\tau_1})} \le c\||Du|+1\|^{\frac{q-p+\mathrm{b}}{2}}_{L^p(B_s)}\,.$$

Using this last inequality and (6.7) in (8.2), we can now conclude as for Proposition 7.1. □

Once the a priori estimate of Proposition 8.1 is available, we can proceed as for the proof of the Theorems 1 and 5. Specifically, the proof of (2.14) is totally analogous to the one of (2.8), via the approximation arguments of Sects. 6.2-6.3 applied with the new definition

$$\mathbb{F}_{\varepsilon,\delta}(x,y,z) := \mathfrak{c}(x,y)\left[(F*\phi_\delta)(z) + \sigma_\varepsilon[H_{\mu_\delta}(z)]^{q/2}\right] \stackrel{(6.31)_2}{=} \mathfrak{c}(x,y)F_{\varepsilon,\delta}(z)\,.$$

This guarantees that the approximating integrands still preserve the product structure used in Proposition 5.3 and fit the assumptions considered there. In particular, $F_{\varepsilon,\delta}(\cdot)$ satisfies (6.43) and therefore conditions (5.68) are satisfied too. The convergence in (6.42) still takes place by Remark 9. As for the local Hölder continuity of $Du$, we proceed exactly as in Sect. 6.4, with (6.64) used with the current definition of $\mathbb{F}_{\varepsilon,\delta}(\cdot)$. In this case the analogs of (6.69)-(6.70) can be obtained estimating as in (5.77).

## 9 Theorem 6

**Proposition 9.1** *Let $u \in W^{1,q}(B_r)$ be a weak solution to* (5.91), *under assumptions* (2.26) *with $0 < \mu \le 2$ and $\nu$, $L$ replaced by $\tilde{\nu}$, $\tilde{L}$ (as in Sect.* 2.1), *and assume also*

(5.92). *If* (2.28) *is in force, then* (7.2) *holds whenever* $B_t \Subset B_s \Subset B_r$ *are concentric balls, where* $c \equiv c(\texttt{data})$, $\chi_1, \chi_2 \equiv \chi_1, \chi_2(\texttt{data}_{\rm e})$.

***Proof*** The setting of Proposition 9.1 is the one of Sect. 5.6 and by Remark 11 we have that $Du$ is locally bounded in $B_r$; in particular, the number $\mathfrak{t}$ has been defined in (5.95). Note that (2.28) implies (5.94) and therefore Proposition 5.5 can be used. The proof closely follows the one of Proposition 7.1. We therefore confine ourselves to give a sketch of it. First, we note that

$$\frac{q}{p} \leq 1 + \frac{p-1}{p}\frac{\alpha}{2Cn} \Longleftrightarrow \frac{q-1}{p-1} \leq 1 + \frac{\alpha}{2Cn}$$

whenever $C \geq 1$. Using this, and the definition of $\mathfrak{t}$ in (5.95), it is not difficult to see that (keep Remark 6 in mind)

$$\frac{q}{p} \leq 1 + \frac{p-1}{p}\frac{\alpha^2}{4Cn^2} \Longrightarrow \frac{q}{p} < 1 + \frac{\alpha^2}{2Cn(n+1)} \Longrightarrow \mathfrak{t} < 1 + \frac{\alpha}{Cn}\,. \tag{9.1}$$

*Case* $p \geq 2$. We take $C = 1$ when $p \geq 2$, so that the right-hand side inequality in (9.1) is implied by (2.28). Proceeding as in Sect. 7.1 for the case $p \geq 2$, but using (5.96) instead of (5.80), we arrive at the following analog of (7.5):

$$\begin{aligned}\|Du\|_{L^\infty(B_{\tau_1})} &\leq \frac{c}{(\tau_1-\tau_2)^{\frac{n}{2p}}}\|Du\|_{L^\infty(B_{\tau_2})}^{\left(\frac{q}{p}-1\right)\frac{\mathfrak{t}\chi}{\chi-1}+\frac{1}{2}}\left(\int_{B_s}(|Du|+1)^p\,{\rm d}x\right)^{\frac{1}{2p}}\\ &\quad + c\|Du\|_{L^\infty(B_{\tau_2})}^{\left(\frac{q}{p}-1\right)\frac{\mathfrak{t}\chi}{\chi-1}+\frac{\mathfrak{t}+1}{2}-\frac{\mathrm{b}}{2p}}\\ &\qquad \cdot \|\mathbf{P}^{2q-2p+\mathrm{b},1}_{2,\alpha}(|Du|+1;\cdot,(\tau_2-\tau_1)/4)\|_{L^\infty(B_{\tau_1})}^{1/p} + c\,,\end{aligned}$$

valid for every $\mathrm{b} \in (0, p]$, and this leads to consider the conditions

$$\frac{q}{p} < 1 + \left(1 - \mathfrak{t} + \frac{\mathrm{b}}{p}\right)\frac{\beta}{\mathfrak{t}n} \quad \text{and} \quad \frac{q}{p} < 1 + \frac{\alpha}{n} - \frac{\mathrm{b}}{2p}\,. \tag{9.2}$$

As done in Proposition 7.1, we can restrict to the case $\alpha < 1$. We choose

$$\mathrm{b} \equiv \mathrm{b}(\beta) = \frac{2p[\mathfrak{t}\alpha + \beta(\mathfrak{t}-1)]}{\mathfrak{t}n + 2\beta}\,,$$

which is admissible by (9.1) (with $C = 1$) for every $\beta < \alpha/(1+\alpha)$ (note that $\mathrm{b} < 2p\alpha/n$ iff $\mathfrak{t} < 1 + 2\alpha/n$, which is in turn implied by (9.1)). Using such $\mathrm{b}$ in (9.2), and formally taking $\beta = \alpha/(1+\alpha)$, we get

$$\frac{q}{p} < 1 + \frac{\alpha}{(1+\alpha)n}\left[\frac{2\alpha - n(\mathfrak{t}-1)}{\mathfrak{t}n + 2\alpha/(1+\alpha)}\right] =: 1 + \mathcal{R}_4(n,p,q,\alpha)\frac{\alpha^2}{n^2}\,. \tag{9.3}$$

By (9.1) it is $\mathcal{R}_4(n,p,q,\alpha) > 1/4$, so that (9.3) is implied by (2.28). We can now conclude as in Proposition 7.1, after (7.15).

*Case* $1 < p < 2$. We use (9.1) with $C = 2$, and (5.97) gives this time

$$\|Du\|_{L^\infty(B_{\tau_1})} \le \frac{c}{(\tau_1-\tau_2)^{\frac{n}{2p}}}\|Du\|_{L^\infty(B_{\tau_2})}^{\left(\frac{q}{p}-1\right)\frac{\mathfrak{t}\chi}{\chi-1}+\frac{1}{2}}\left(\int_{B_s}(|Du|+1)^p\,\mathrm{d}x\right)^{\frac{1}{2p}}$$
$$+ c\|Du\|_{L^\infty(B_{\tau_2})}^{\left(\frac{q}{p}-1\right)\frac{\mathfrak{t}\chi}{\chi-1}+\frac{\mathfrak{t}}{2}+\frac{q-\mathrm{b}/p}{2p}}$$
$$\cdot\|\mathbf{P}_{2,\alpha/2}^{p(q-p)/(p-1)+\mathrm{b},1/p}(|Du|+1;\cdot,(\tau_2-\tau_1)/4)\|_{L^\infty(B_{\tau_1})}^{1/p} + c\,,$$

as an analog of (7.16). Therefore we consider the conditions

$$\left(\frac{q}{p}-1\right)\frac{\mathfrak{t}n}{2\beta}+\frac{\mathfrak{t}}{2}+\frac{q-\mathrm{b}/p}{2p}<1 \quad \text{and} \quad \frac{n}{p\alpha}\left[\frac{p(q-1)}{p-1}-p+\mathrm{b}\right]<p\,,$$

that are equivalent to

$$\begin{cases} \dfrac{q}{p} < 1+\left(1-\mathfrak{t}+\dfrac{\mathrm{b}}{p^2}\right)\dfrac{\beta}{\mathfrak{t}n+\beta} \\ \dfrac{q-1}{p-1} < 1+\dfrac{\alpha p}{n}-\dfrac{\mathrm{b}}{p} \Longleftrightarrow \dfrac{q}{p}<1+\left(\dfrac{p\alpha}{n}-\dfrac{\mathrm{b}}{p}\right)\dfrac{p-1}{p}\,, \end{cases} \tag{9.4}$$

respectively. Note that we can apply Lemma 4.1 as $n\theta/(t\delta) = n/(p\alpha) > 1$ as now it is $p < 2$. We take $\mathrm{b} = 14p^2\alpha/(15n) \le p$; using this value in (9.4), and (9.1) with $C = 2$, makes the two inequalities in (9.4) implied by (2.28) provided we take $\beta$ close enough to $\alpha/(2+\alpha)$, and we conclude again as in Proposition 7.1. □

**Proposition 9.2** *Let* $u \in W^{1,q}(B_r)$ *be a weak solution to* (5.91), *under assumptions* (2.26) *with* $0 < \mu \le 2$ *and* $\nu$, $L$ *replaced by* $\tilde{\nu}$, $\tilde{L}$ (*as in Sect.* 2.1), *and assume also* (5.92). *Then*

$$[Du]_{\tilde{\alpha},p;B_t} \le \frac{c}{(s-t)^{\chi_1}}\left[\|Du\|_{L^q(B_s)}+1\right]^{\chi_2} \tag{9.5}$$

*holds whenever* $B_t \Subset B_s \Subset B_r$ *are concentric balls and* $\tilde{\alpha} < \min\{1/p, 1/2\}\alpha$, *where* $c \equiv c(\texttt{data})$ *and* $\chi_1, \chi_2 \equiv \chi_1, \chi_2(\texttt{data}_{\mathrm{e}})$.

***Proof*** In the case of minimizers, (9.5) is hidden in [24, Proof of Theorem 4]. The arguments in [24] rely on the use of the Euler-Lagrange equation and they work in the case of the general equations considered here. Indeed, from [24, (51)] we have that

$$\int_{B_t}|\tau_h V_\mu(Du)|^2\,\mathrm{d}x \le \frac{c|h|^\alpha}{(s-t)^\theta}\int_{B_s}(|Du|+1)^q\,\mathrm{d}x \tag{9.6}$$

holds whenever $B_t \Subset B_s \Subset B_r$ are concentric balls and $h \in \mathbb{R}^n$ such that $|h| \le (s-t)/4$, with $c \equiv c(\texttt{data})$ and $\theta \equiv \theta(\texttt{data}_{\mathrm{e}})$. Using (9.6) with (3.13), we obtain

$$\int_{B_t}|\tau_h Du|^p\,\mathrm{d}x \le c\int_{B_t}|\tau_h V_\mu(Du)|^2\,\mathrm{d}x + c\mathbb{1}_p\int_{B_t}|\tau_h V_\mu(Du)|^p(|Du|+1)^{p(2-p)/2}\,\mathrm{d}x$$

$$
\begin{aligned}
&\leq c \int_{B_t} |\tau_h V_\mu(Du)|^2 \,\mathrm{d}x \\
&\quad + c\mathbb{1}_p \left( \int_{B_t} |\tau_h V_\mu(Du)|^2 \,\mathrm{d}x \right)^{\frac{p}{2}} \left( \int_{B_t} (|Du|+1)^p \,\mathrm{d}x \right)^{\frac{2-p}{2}} \\
&\leq \frac{c|h|^{\min\{1,p/2\}\alpha}}{(s-t)^\theta} \int_{B_s} (|Du|+1)^q \,\mathrm{d}x
\end{aligned}
$$

for new constants $c \equiv c(\mathtt{data})$, $\theta \equiv \theta(\mathtt{data_e})$. The information in the last display and Lemma 3.1 now imply (9.5). Note that (9.5) does not require any upper bound on $q/p$ as on the right-hand side there appears the $L^q$-norm of $Du$. $\square$

We now complete the proof of Theorem 6. With the same notation on sequences $\{\varepsilon\}$ and mollifiers $\{\phi_s\}_{0<s\leq 1}$ of Sect. 6.2 (defined after (6.27)), we set

$$
A_\varepsilon(x,z) := (A(x,\cdot) * \phi_\varepsilon)(z) + \varepsilon[H_{\mu_\varepsilon}(z)]^{(q-2)/2} z\,, \quad (x,z) \in \Omega \times \mathbb{R}^n\,, \tag{9.7}
$$

compare with (6.29). We then have

**Lemma 9.1** *Under assumptions* (2.26), *let* $M \geq 1$; *for any* $(x,z) \in \Omega \times \mathbb{R}^n$

$$
\begin{cases}
|A_\varepsilon(x,z) - A(x,z)| \leq c\varepsilon^{\min\{1,p-1\}} & \text{holds when } |z| \leq M \\
|A_\varepsilon(x,z)| \leq c[H_1(z)]^{(q-1)/2}\,,
\end{cases} \tag{9.8}
$$

*where c is again independent of* $\varepsilon$.

***Proof*** We have, using the very definition in (9.7) and $(2.26)_1$,

$$
\begin{aligned}
|A_\varepsilon(x,z) - A(x,z)| &\leq c\varepsilon \int_{\mathcal{B}_1}\int_0^1 |\partial_z A(x, z+t\varepsilon\lambda)| \,\mathrm{d}t\, \phi(\lambda)\,\mathrm{d}\lambda + c\varepsilon M^{q-1} \\
&\leq c\varepsilon \int_{\mathcal{B}_1}\int_0^1 [H_\mu(z+t\varepsilon\lambda)]^{\frac{q-2}{2}} \,\mathrm{d}t\, \phi(\lambda)\,\mathrm{d}\lambda \\
&\quad + c\varepsilon \int_{\mathcal{B}_1}\int_0^1 [H_\mu(z+t\varepsilon\lambda)]^{\frac{p-2}{2}} \,\mathrm{d}t\, \phi(\lambda)\,\mathrm{d}\lambda + c\varepsilon M^{q-1} \\
&\overset{(3.17)}{\leq} c\varepsilon \int_{\mathcal{B}_1} (|z+\varepsilon\lambda|^2 + |z|^2 + \mu^2)^{\frac{q-2}{2}} \phi(\lambda)\,\mathrm{d}\lambda \\
&\quad + c\varepsilon \int_{\mathcal{B}_1} (|z+\varepsilon\lambda|^2 + |z|^2 + \mu^2)^{\frac{p-2}{2}} \phi(\lambda)\,\mathrm{d}\lambda + c\varepsilon M^{q-1} \\
&=: c\varepsilon \mathbb{I}_q(z) + c\varepsilon \mathbb{I}_p(z) + c\varepsilon M^{q-1} \\
&\leq c\varepsilon^{\min\{1,p-1\}} c(M)\,.
\end{aligned}
$$

It remains to justify the estimate in the last line. We treat $\varepsilon\mathbb{I}_q(z)$, the estimate for $\varepsilon\mathbb{I}_p(z)$ being completely similar. When $q \geq 2$, we find $\varepsilon\mathbb{I}_q(z) \leq c\varepsilon[H_{\mu_\varepsilon}(z)]^{(q-2)/2} \leq$

$c\varepsilon M^{q-2} \le \varepsilon c(M)$. When $q < 2$, instead we further distinguish two cases. The first is when $|z| \ge \varepsilon$; in this case it is $\varepsilon \mathbb{I}_q(z) \le c\varepsilon |z|^{q-2} \le c\varepsilon^{q-1} \le c\varepsilon^{p-1}$. Finally, if $q < 2$ and $|z| \le \varepsilon$, then we have, by changing variables

$$\varepsilon \mathbb{I}_q(z) \le c\varepsilon \int_{\mathcal{B}_1} |z + \varepsilon\lambda|^{q-2}\, d\lambda \le c\varepsilon^{1-n} \int_{\mathcal{B}_{2\varepsilon}} |\lambda|^{q-2}\, d\lambda \le c\varepsilon^{q-1} \le c\varepsilon^{p-1}.$$

We have proved $(9.8)_1$, while $(9.8)_2$ trivially follows from $(2.26)_1$. □

To proceed with the proof of Theorem 6, we define $u_\varepsilon$ as the (unique) solution to $\operatorname{div} A_\varepsilon(x, Du_\varepsilon) = 0$ in $\Omega$ such that $u_\varepsilon \in u_0 + W^{1,q}_0(\Omega)$. For every $\varepsilon$, the vector field $A_\varepsilon(\cdot)$ satisfies the assumptions required on $A(\cdot)$ in Proposition 5.5. In particular, (2.26) are satisfied with $\mu$ replaced by $\mu_\varepsilon := \mu + \varepsilon > 0$, and for new constants $0 < \tilde{\nu} \le \tilde{L}$ as in Sect. 2.1, replacing $\nu$, $L$, and independent of $\varepsilon$. This can be easily proved using the arguments of [20, Sect. 4.5]. Testing $\operatorname{div} A_\varepsilon(x, Du_\varepsilon) = 0$ by $u_\varepsilon - u_0$, and using (5.93), yields the uniform bound

$$\|Du_\varepsilon\|_{L^p(\Omega)} \le c \||Du_0| + 1\|^{(q-1)/(p-1)}_{L^{p(q-1)/(p-1)}(\Omega)} \tag{9.9}$$

where $c \equiv c(\mathtt{data})$; note that $p(q-1)/(p-1) \ge q$ and this allows for testing. Combining this with (7.2), we get the local estimate

$$\|Du_\varepsilon\|_{L^\infty(B/2)} \le \frac{c}{|B|^{\chi_1}} \left( \int_\Omega (|Du_0| + 1)^{\frac{p(q-1)}{p-1}}\, \mathrm{d}x + 1 \right)^{\chi_2}. \tag{9.10}$$

This holds whenever $B \Subset \Omega$ is a ball, where $c \equiv c(\mathtt{data}) \ge 1$, $\chi_1, \chi_2 \equiv \chi_1, \chi_2(\mathtt{data_e}) \ge 1$ are suitable constants otherwise independent of $\varepsilon$ (as usual, these are not necessarily the same appearing in (7.2)). By (9.9) and (9.10), up to not relabelled subsequences and a covering argument we can assume that $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega)$, for some $u \in u_0 + W^{1,p}_0(\Omega)$, and that $\{Du_\varepsilon\}$ is bounded in $L^\infty_{\operatorname{loc}}(\Omega, \mathbb{R}^n)$. This and Proposition 9.2 yield a uniform bound on $\{Du_\varepsilon\}$ in $W^{\tilde{\alpha},p}_{\operatorname{loc}}(\Omega, \mathbb{R}^n)$. With $\Omega_0 \Subset \Omega$ being an arbitrary open subset, again up to not relabelled subsequences, we can use the compact embedding properties of $W^{\tilde{\alpha},p}$ to assume that $Du_\varepsilon \to Du$ in $L^\gamma(\Omega_0, \mathbb{R}^n)$ for some $\gamma < np/(n - p\tilde{\alpha})$, and a.e. Using this and the uniform bound of $\{Du_\varepsilon\}$ in $L^\infty_{\operatorname{loc}}$, by interpolation it then follows that $Du_\varepsilon \to Du$ in $L^\gamma(\Omega_0, \mathbb{R}^n)$ for every $\gamma < \infty$. This and the fact that $\Omega_0$ is arbitrary imply that $u$ is a distributional solution to $(2.27)_1$ by Lemma 9.1 and dominated convergence. Letting $\varepsilon \to 0$ in (9.10) leads to (2.29) via coverings. Finally, the local Hölder continuity of $Du$ follows by the methods employed in Sect. 6.4. Note that in defining the comparison functions $v$ in (6.64), now we take $v$ as the solution to $\operatorname{div} A_\varepsilon(x_c, Dv) = 0$ such that $v \equiv u_\varepsilon$ on $B_\tau$. We remark that the analog of (6.62) is obtained via (9.10), and the one of (6.65) follows as in (5.99). Finally, the analog of (6.69) follows estimating as for (5.86) and (5.101).

## 10 Corollaries 1, 2 and 5

### 10.1 Proof of Corollary 1

Under the assumptions of Corollary 1, once $Du$ is known to be locally Hölder continuous, its Hölder exponent can be upgraded up to the maximal one via a combination of a few classical regularity arguments and estimates for nonuniformly elliptic problems. Here we give the details. Since the result is local in nature, we can assume that

$$\|Du\|_{L^\infty(\Omega)} + [Du]_{0,\beta;\Omega} =: M < \infty \tag{10.1}$$

and that $M \geq 1$; here $\beta$ is the Hölder exponent of $Du$ provided by Theorem 2. We can also assume that $\beta < \alpha$, otherwise there is nothing to prove. In the following, the constants denoted by $c$ will depend on $\texttt{data}$; additional dependencies will be emphasized in parentheses. We take a ball $B_r \equiv B_r(x_c) \Subset \Omega$ such that $r \leq 1$, denote $A(x,z) = \mathfrak{c}(x)\partial_z F(z)$ and $A_r(z) := A(x_c, z) + r^\alpha [H_\mu(z)]^{(q-2)/2} z$. Since $A_r(\cdot)$ is $q$-monotone, we can take $v \in u + W^{1,q}_0(B_r)$ such that $\operatorname{div} A_r(Dv) = 0$ in $B_r$. Note that this is the same that requiring that $v$ minimizes the functional $w \mapsto \int_{B_r} F_r(x_c, Dw)\,dx$ in the Dirichlet class $u + W^{1,q}_0(B_r)$, where $F_r(x,z) := \mathfrak{c}(x)F(z) + r^\alpha [H_\mu(z)]^{q/2}/q$. As in the proof of (5.86) in Proposition 5.4, we find

$$\fint_{B_r} |V_\mu(Du) - V_\mu(Dv)|^2 \,\mathrm{d}x \leq c M^{2q-p} r^{2\alpha}\,. \tag{10.2}$$

Using (3.12) in the above inequality, and recalling that $p \geq 2$ and $\mu > 0$, yields

$$\fint_{B_r} |Du - Dv|^2 \,\mathrm{d}x \leq c\mu^{2-p} M^{2q-p} r^{2\alpha}\,. \tag{10.3}$$

This last inequality and (10.1) gives

$$\fint_{B_r} |Dv - (Du)_{B_r}|^2 \,\mathrm{d}x \leq c(\mu, M) r^{2\beta}\,. \tag{10.4}$$

By $\mu > 0$, standard regularity theory (see also Lemmas 5.1 and 5.3) give $Dv \in L^\infty_{\mathrm{loc}}(B_r) \cap W^{1,2}_{\mathrm{loc}}(B_r)$. Note that (10.1) and (10.2) imply $\|Dv\|^p_{L^p(B_r)} \leq c(M) r^n$. This and (5.24) give

$$\|Dv\|_{L^\infty(B_{r/2})} \leq c(M)\,. \tag{10.5}$$

Moreover, every component $\mathfrak{v} \equiv D_s v$, $s \in \{1, \ldots, n\}$ is an energy solution to the linear elliptic equation

$$\operatorname{div}(\mathbb{A}(x) D\mathfrak{v}) = 0, \quad [\mathbb{A}(x)]_{ij} := \partial_{z_j} A^i_r(x_c, Dv(x)) = \partial_{z_i z_j} F_r(x_c, Dv(x))\,.$$

Again by $\mu > 0$ and thanks to (10.5), we find $\lambda \equiv \lambda(\texttt{data}, \mu, M) > 0$, independent of $r$, such that $\lambda \mathbb{I}_{\mathrm{d}} \leq \mathbb{A}(x) \leq (1/\lambda)\mathbb{I}_{\mathrm{d}}$ holds for a.e. $x \in B_{r/2}$. This allows to apply

De Giorgi-Nash-Moser theory, that yields $\beta_0 \equiv \beta_0(\mathtt{data}, \mu, M) \in (0,1)$, and $c \equiv c(\mathtt{data}, \mu, M) \geq 1$, such that

$$r^{2-n}\|DD_s v\|^2_{L^2(B_{r/4})} + r^{2\beta_0}[D_s v]^2_{0,\beta_0;B_{r/4}} \leq c \fint_{B_{r/2}} |D_s v - a|^2 \,\mathrm{d}x \tag{10.6}$$

holds for every $a \in \mathbb{R}$ and $s \in \{1, \ldots, n\}$. Note that we have also incorporated in (10.6) the standard Caccioppoli inequality for linear elliptic equations with measurable coefficients [34, Theorem 6.5]. As there is no loss of generality in assuming that $\beta_0 \leq \beta$, choosing $a \equiv (D_s u)_{B_r}$ and using (10.4) in (10.6), yields $[Dv]_{0,\beta_0;B_{r/4}} \leq c_* \equiv c_*(\mathtt{data}, \mu, M)$. We now denote by $\omega(\cdot)$ the modulus of continuity of $\partial_z A_r(x_c, \cdot)$ on $\mathcal{B}_M$. This is independent of $r$ and of the point $x_c$. Recalling the definition of $\mathbb{A}(\cdot)$, we therefore have

$$|\mathbb{A}(x_1) - \mathbb{A}(x_2)| \leq \omega(c_*|x_1 - x_2|^{\beta_0}) =: \tilde{\omega}(|x_1 - x_2|)$$

whenever $x_1, x_2 \in B_{r/4}$, so that the entries $\mathbb{A}(\cdot)$ are continuous, with a modulus of continuity that depends in a quantitative way on $\mathtt{data}$, $\mu$ and $M$. The above arguments apply for every $s \in \{1, \ldots, n\}$, therefore Campanato's perturbation theory and (10.6), now imply

$$\int_{B_\varrho} |D^2 v|^2 \,\mathrm{d}x \leq \frac{c}{r^2}\left[\left(\frac{\varrho}{r}\right)^n + [\tilde{\omega}(r)]^2\right]\int_{B_r} |Dv - (Du)_{B_r}|^2 \,\mathrm{d}x$$

holds whenever $\varrho \leq r/4$, where $c \equiv c(\mathtt{data}, \mu, M)$; here we have also used [34, (10.42)]. Poincaré inequality now gives, this time whenever $\varrho \leq r$

$$\begin{aligned} &\int_{B_\varrho} |Dv - (Dv)_{B_\varrho}|^2 \,\mathrm{d}x \\ &\quad \leq c\left[\left(\frac{\varrho}{r}\right)^{n+2} + \left(\frac{\varrho}{r}\right)^2 [\tilde{\omega}(r)]^2\right]\int_{B_r} |Dv - (Du)_{B_r}|^2 \,\mathrm{d}x\,. \end{aligned} \tag{10.7}$$

Using (10.7) in combination with (10.3), and a standard comparison argument, we conclude with

$$\int_{B_\varrho} |Du - (Du)_{B_\varrho}|^2 \,\mathrm{d}x \leq c\left[\left(\frac{\varrho}{r}\right)^{n+2} + [\tilde{\omega}(r)]^2\right]\int_{B_r} |Du - (Du)_{B_r}|^2 \,\mathrm{d}x + cr^{n+2\alpha}$$

again for every $\varrho \leq r$, where $c \equiv c(\mathtt{data}, \mu, M)$. Since

$$h(\varrho) := \|Du - (Du)_{B_\varrho}\|^2_{L^2(B_\varrho)}$$

is non-decreasing, we are in position to use Lemma 3.3. It follows that there exists $r_0$ depending on $\mathtt{data}$, $\mu$, $M$ and on the local modulus of continuity of $\partial_{zz} F(\cdot)$, such that

$$\int_{B_\varrho} |Du - (Du)_{B_\varrho}|^2 \,\mathrm{d}x \leq c\varrho^{n+2\alpha}$$

holds provided $\varrho \le r_0$, where $c$ depends on $\texttt{data}$, $\mu$, $M$ and the modulus of continuity of $\partial_{zz}F(\cdot)$. At this stage, the local $C^{1,\alpha}$ continuity of $Du$ follows from Campanato-Meyers integral characterization of Hölder continuity. This completes the proof of Corollary 1.

### 10.2 Proof of Corollary 2

The proof is a modification of the one for Corollary 1, and we keep the notation used there. Without loss of generality we can assume that $\|f\|_{L^\infty(\Omega)} \le 1$ and $\beta < \alpha/2$. This time we define $v \in u + W^{1,q}_0(B_r)$ as the unique minimizer of $w \mapsto \int_{B_r} F_r(Dw)\,dx$ in its Dirichlet class, where $F_r(z) := \mathfrak{c}(x_c, u(x_c))F(z) + r^\alpha[H_\mu(z)]^{q/2}$. We proceed as for (5.73)-(5.77) (with $8|h|^{\beta_0} \equiv r$ and $\varrho \equiv 1$), getting

$$
\begin{aligned}
\int_{B_r} |V_\mu(Du) - V_\mu(Dv)|^2\,\mathrm{d}x &\le c\int_{B_r} [F_r(Du) - F_r(Dv)]\,\mathrm{d}x\\
&\le c\int_{B_r} \mathfrak{c}(x_c, u(x_c))[F(Du) - F(Dv)]\,\mathrm{d}x + cr^\alpha \int_{B_r} [H_\mu(Du)]^{q/2}\,\mathrm{d}x\\
&\le c\int_{B_r} \mathfrak{c}(x_c, u(x_c))[F(Du) - F(Dv)]\,\mathrm{d}x + cM^q r^{n+\alpha}\\
&\le cM^{q+\alpha} r^{n+\alpha} + cM^q r^{n+\alpha} \le c(M) r^{n+\alpha}\,.
\end{aligned}
$$

In turn, via (3.12) this implies

$$
\int_{B_r} |Du - Dv|^2\,\mathrm{d}x \le c(\mu, M) r^{n+\alpha}\,.
$$

These estimates are the counterparts of (10.2) and (10.3) and from this point on the proof develops as in Corollary 1, replacing $\alpha$ by $\alpha/2$ everywhere.

### 10.3 Proof of Corollary 5

Let $u$ be a distributional solution to $\operatorname{div} A(x, Du) = 0$, satisfying (10.1), and under the assumptions of Corollary 5. It is easy to see that the proof of Corollary 1 applies verbatim to this situation, as it does not use the minimality of $u$ beyond the fact that $u$ solves the Euler-Lagrange equation. Indeed, up to passing to inner domains of $\Omega$, we can assume that $\partial_z A(\cdot)$ is uniformly continuous on $\Omega \times \mathcal{B}_M$, so that we can find a modulus of continuity of $z \mapsto \partial_z A(x_c, z)$, which is independent of the chosen point $x_c$. We conclude that Corollary 5 follows from Theorem 6 (used to get (10.1) up to passing to smaller open subsets).

***Remark 12*** When $p < 2$ it is still possible to get a quantitative information on the gradient Hölder exponent of minima and solutions. For this, we shall confine ourselves to the case of Corollary 1, where, assuming this time that $p < 2$, we can still prove that $Du \in C^{0,\alpha/2}_{\mathrm{loc}}$. Quantitative results in the remaining cases can be obtained in a similar way. We give the necessary modifications to the proof of Corollary 1, from which we keep the notation. Proceeding this time as in the proof of (5.88), we have the analog of (10.2), i.e.,

$$\fint_{B_r} |V_\mu(Du) - V_\mu(Dv)|^2 \, dx \le cM^{q(1+1/p)-1} r^\alpha \,. \tag{10.8}$$

Using Lemma 5.3 and (5.25), exactly as in the derivation of (5.42), we find $\|Dv\|_{L^\infty(B_{r/2})} \le cM^{\mathfrak{s}}$ where $c \equiv \tilde{c}(n, p, q, \nu, L)$. Using (3.12), we get

$$\begin{aligned}\fint_{B_{r/2}} |Du - Dv|^2 \, dx &\le c \fint_{B_{r/2}} (|Du|^2 + |Dv|^2 + 1)^{(2-p)/2} |V_\mu(Du) - V_\mu(Dv)|^2 \, dx \\ &\le cM^{(2-p)\mathfrak{s}} \fint_{B_{r/2}} |V_\mu(Du) - V_\mu(Dv)|^2 \, dx \,.\end{aligned}$$

Connecting this last inequality with (10.8) finally yields the analog of (10.3), i.e.,

$$\fint_{B_{r/2}} |Du - Dv|^2 \, dx \le cM^{(2-p)\mathfrak{s}+q(1+1/p)-1} r^\alpha \equiv c(M) r^\alpha$$

where $c(M)$ also depends on $\texttt{data}$. The crucial difference with (10.3) is that $\alpha$ has been replaced by $\alpha/2$. With this last inequality we finally come also to the analog of (10.4), that is

$$\|Dv - (Du)_{B_{r/2}}\|^2_{L^2(B_{r/2})} \le cr^{n+\beta}$$

with $c \equiv c(\texttt{data}, M)$. From this point on one can proceed exactly as in the case $p \ge 2$.

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## Declarations

**Competing Interests** The authors declare no competing interests.

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