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Endomorphisms of the projective plane and the image of the Suslin–Hurewicz map

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Abstract The endomorphism ring of the projective plane over a field F of characteristic neither two nor three is slightly more complicated in the Morel–Voevodsky motivic stable homotopy category than in Voevodsky's derived category of motives. In particular, it is not commutative precisely if there exists a square in F which does not admit a sixth root. A byproduct of these computations is a proof of Suslin's conjecture on the Suslin–Hurewicz homomorphism from Quillen to Milnor K-theory in degree four, based on work of Asok et al. (Invent Math 219:39-73, 2020).

1 Introduction

Automorphisms of geometric objects describe their symmetries, and hence important geometric information. It depends on the context which type of morphisms are considered useful. In the case of a projective space $\mathbf{P}^n$ over the complex numbers, one may consider its linear automorphisms (a group denoted $\mathrm{PGL}_{n+1}(\mathbb{C})$), birational automorphisms (the Cremona group $\mathrm{Cr}_n(\mathbb{C})$), diffeomorphisms, homeomorphisms, and self-homotopy-equivalences, just to name a few. The Morel–Voevodsky $\mathbf{A}^1$-homotopy theory provides an interesting way to consider self-homotopy-equivalences of varieties [21]. Although this setup is conceptionally very satisfying, concrete determinations of endomorphisms in the $\mathbf{A}^1$-homotopy category are hard to come by. For example, the endomor-

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phism ring of $\mathbf{P}^1$ over a perfect field F in the pointed $\mathbf{A}^1$-homotopy category is given by the Grothendieck ring of isomorphism classes of symmetric inner product spaces over F with a chosen basis, where the isomorphisms preserve the inner product and have determinant 1 with respect to the chosen bases [17, Remark 7.37].

Stabilization with respect to smashing with a projective line $\mathbf{P}^1 \wedge -$ provides a simpler categorical setting, the motivic stable homotopy category $\mathbf{SH}(F)$, which is still richer than the corresponding derived category of motives [27]. Part of the gain from leaving the unstable realm is an additive (in fact triangulated) structure, whence the set of endomorphisms of any object is always a ring. For example, it is a deep theorem of Morel's that the endomorphism ring of the projective line in the motivic stable homotopy category over a field F is the Grothendieck–Witt ring of symmetric bilinear forms with coefficients in F [15]. The addition in the Grothendieck-Witt ring, whose elements are formal differences of symmetric bilinear forms, is induced by orthogonal sum, and the multiplication by tensor product of forms. By construction, it coincides with the endomorphism ring of the unit for the symmetric monoidal structure given by the smash product. Hence it has to be commutative. This is already different for the projective plane.

Theorem *Let F be a field of characteristic neither 2 nor 3, with group of units $F^\times$. The endomorphism ring $[\mathbf{P}^2, \mathbf{P}^2]_{\mathbf{SH}(F)}$ in the motivic stable homotopy category of F has an underlying additive group isomorphic to $\mathbb{Z}\oplus\mathbb{Z}\oplus F^\times/(F^\times)^6$. The multiplication corresponds to the multiplication given by*

$$(x_1, x_2, x_3) \circ (y_1, y_2, y_3) = (x_1y_1, x_1y_2 + x_2y_1 \\ +2x_2y_2, x_1y_3 + x_3y_1 + 2x_3y_2).$$

In particular, the ring $[\mathbf{P}^2, \mathbf{P}^2]_{\mathbf{SH}(F)}$ is non-commutative if and only if there exists a square in F which does not admit a sixth root, or, equivalently, if the cube map $u \mapsto u^3$ is not surjective on F. Its group of units (which could be called the group of $\mathbf{P}^1$-stable self-$\mathbf{A}^1$-homotopy-equivalences of $\mathbf{P}^2$) consists of all triples $(x_1, x_2, x_3) \in \mathbb{Z}\oplus\mathbb{Z}\oplus F^\times/(F^\times)^6$ where either $x_1 = \pm 1$ and $x_2 = 0$, or $x_1 = \pm 1$ and $x_2 = -x_1$. It is as non-commutative as the endomorphism ring it belongs to. Along the way, the homotopy modules $\pi_1\mathbf{P}^2$ and $\pi_2\mathbf{P}^2$ will be determined, based on computations in [23] and [22]. These computations provide an ingredient to complete the program Aravind Asok, Jean Fasel and Ben Williams developed in [3] to prove Suslin's conjecture on the Suslin-Hurewicz homomorphism from Quillen to Milnor K-theory in degree four.

Theorem *Let F be an infinite field of characteristic different from* 2 *and* 3*, and A an essentially smooth local F-algebra. The image of the Suslin-Hurewicz homomorphism $K_4^{\mathrm{Quillen}}(A) \to K_4^{\mathrm{Milnor}}(A)$ coincides with $6K_4^{\mathrm{Milnor}}(A)$.*

This closes the gap between three and five in the set of degrees for which Suslin's conjecture was previously known. See the introduction of [3] for more details on its history, as well as Suslin's original paper [25].

2 Topology

Let $\mathbf{CP}^n$ denote complex projective space of complex dimension n. This section contains rather elementary calculations in the classical stable homotopy theory, which determine the endomorphism ring of $\mathbf{CP}^2$ in the stable homotopy category. These calculations are based on stable homotopy groups of spheres $\pi_m\mathbb{S}$ in degree $m < 6$ and the action of the topological Hopf map $\eta\colon S^3 \to \mathbf{CP}^1 \cong S^2$, whose cofiber is $\mathbf{CP}^2$, on them. The standard reference here is [26]. As is customary in stable homotopy theory, the notation for a map and its (de)suspensions coincide if the context allows it. The purpose of this section is not to present original results (there aren't any), but instead to document the similarities and differences to the situation in the motivic stable homotopy category.

Choose a basepoint for $\mathbf{CP}^1$, and hence $\mathbf{CP}^2$, which will not appear in the notation. The cofiber sequence

$$S^3 \xrightarrow{\eta} \mathbf{CP}^1 \xrightarrow{i} \mathbf{CP}^2 \xrightarrow{q} S^4 \tag{2.1}$$

induces a long exact sequence of stable homotopy groups

$$\begin{aligned}\cdots \xrightarrow{\eta} \pi_m\mathbf{CP}^1 &= \pi_{m-2}\mathbb{S} \xrightarrow{i_*} \pi_m\mathbf{CP}^2 \xrightarrow{q_*} \pi_m S^4 = \pi_{m-4}\mathbb{S} \xrightarrow{\eta} \pi_{m-1}\mathbf{CP}^1 \\ &= \pi_{m-3}\mathbb{S} \to \cdots\end{aligned}$$

terminating with $\pi_2\mathbf{CP}^1 = \pi_2\mathbf{CP}^2$. The induced short exact sequences

$$0 \to \pi_{m-2}\mathbb{S}/\eta\pi_{m-3}\mathbb{S} \to \pi_m\mathbf{CP}^2 \to {}_{\eta}\pi_{m-4}\mathbb{S} \to 0$$

express the stable homotopy group of the complex projective plane as an extension of two groups, the subgroup annihilated by η, and the cokernel of multiplication by η, on the respective stable homotopy group of spheres. Since $\eta\colon \pi_0\mathbb{S} \cong \mathbb{Z} \to \pi_1\mathbb{S} \cong \mathbb{Z}/2\mathbb{Z}$ is surjective, $\eta\colon \pi_1\mathbb{S} \to \pi_2\mathbb{S}$ is an isomorphism, $\eta\colon \pi_2\mathbb{S} \to \pi_3\mathbb{S} \cong \mathbb{Z}/24$ is injective, and $\pi_4\mathbb{S} \cong \pi_5\mathbb{S} \cong 0$, the following table results, without any extension problem to solve.

m	2	3	4	5	6	7
$\pi_m\mathbf{CP}^2$	$\mathbb{Z}$	0	$2\mathbb{Z}$	$\mathbb{Z}/12$	0	$\mathbb{Z}/24$

Here $2\mathbb{Z}$ denotes the abelian group of even integers under addition. The vanishing $\pi_3\mathbf{CP}^2 = 0$ implies that the cofiber sequence (2.1) induces a short exact sequence

$$0 \to \pi_4\mathbf{CP}^2 = [S^4, \mathbf{CP}^2] \xrightarrow{q^*} [\mathbf{CP}^2, \mathbf{CP}^2] \xrightarrow{i^*} [\mathbf{CP}^1, \mathbf{CP}^2] = \pi_2\mathbf{CP}^2 \to 0 \tag{2.2}$$

of stable homotopy groups. Hence the abelian group $[\mathbf{CP}^2, \mathbf{CP}^2]$ is an extension of two free abelian groups, each on one generator. Since $\pi_2\mathbf{CP}^2 \cong \mathbb{Z}$, the short exact sequence (2.2) splits. In order to describe the ring structure, it helps to be more specific. A generator for $\pi_2\mathbf{CP}^2$ is the inclusion $i\colon \mathbf{CP}^1 \hookrightarrow \mathbf{CP}^2$. It is the image of $\mathrm{id}_{\mathbf{CP}^2}$ under $[\mathbf{CP}^2, \mathbf{CP}^2] \xrightarrow{i^*} [\mathbf{CP}^1, \mathbf{CP}^2]$. To describe a generator for $\pi_4\mathbf{CP}^2$, observe that there exists a unique map $\omega\colon S^4 \to \mathbf{CP}^2$ such that $q \circ \omega = 2\mathrm{id}_{S^4}$. The short exact sequence (2.2) then implies that every element $x \in [\mathbf{CP}^2, \mathbf{CP}^2]$ can uniquely be expressed as a sum $x_1\mathrm{id}_{\mathbf{CP}^2} + x_2(\omega \circ q)$, where $x_1, x_2 \in \pi_0\mathbb{S} \cong \mathbb{Z}$. The ring structure is then given as

$$\begin{aligned} x \circ y &= \big(x_1\mathrm{id}_{\mathbf{CP}^2} + x_2(\omega \circ q)\big) \circ \big(y_1\mathrm{id}_{\mathbf{CP}^2} + y_2(\omega \circ q)\big) \\ &= x_1y_1\mathrm{id}_{\mathbf{CP}^2} + (x_1y_2 + x_2y_1)(\omega \circ q) + x_2y_2(\omega \circ q \circ \omega \circ q) \\ &= x_1y_1\mathrm{id}_{\mathbf{CP}^2} + (x_1y_2 + x_2y_1 + 2x_2y_2)(\omega \circ q) \end{aligned}$$

and is in particular commutative. The group of units consists of the following 4 elements: $\{\mathrm{id}_{\mathbf{CP}^2}, -\mathrm{id}_{\mathbf{CP}^2}, \mathrm{id}_{\mathbf{CP}^2} - \omega\circ q, -\mathrm{id}_{\mathbf{CP}^2} + \omega\circ q\}$ Of course knowledge does not stop at the dimension two. For example, [20] determines the groups $[\mathbf{CP}^n, \mathbf{CP}^n]$ for $n \leq 7$. The complex dimension 7 is the smallest dimension where this group contains torsion; in fact, $[\mathbf{CP}^7, \mathbf{CP}^7] \cong \mathbb{Z}^7 \oplus \mathbb{Z}/2\mathbb{Z}$. The ring structure is commutative in all these dimensions.

The table above allows to determine $[\Sigma\mathbf{CP}^2, \mathbf{CP}^2]$ as well, which is useful, because this group contains the interesting element $\eta\mathrm{id}_{\mathbf{CP}^2} = \eta \wedge \mathbf{CP}^2$. The suspension of the homotopy cofiber sequence (2.1) induces a long exact sequence on $[-, \mathbf{CP}^2]$. Since $\pi_3\mathbf{CP}^2$ is the zero group, there results an isomorphism $\pi_5\mathbf{CP}^2/\eta\pi_4\mathbf{CP}^2 \xrightarrow{\cong} [\Sigma\mathbf{CP}^2, \mathbf{CP}^2]$. As $\pi_5\mathbf{CP}^2$ is cyclic, generated by $i \circ \nu$, the abelian group $[\Sigma\mathbf{CP}^2, \mathbf{CP}^2]$ is generated by the map

$$\Sigma\mathbf{CP}^2 \xrightarrow{q} S^5 \xrightarrow{\nu} S^2 = \mathbf{CP}^1 \xrightarrow{i} \mathbf{CP}^2.$$

Its order can be determined by identifying $\omega \circ \eta \in \pi_5\mathbf{CP}^2$, which is $\pm 6(i \circ \nu)$, as the Toda bracket $\langle \eta, 2, \eta\rangle = \{6\nu, -6\nu\}$ shows, together with [26, Prop. 1.8] applied to $\alpha = \gamma = \eta$ and $\beta = 2$, using $\omega = \widetilde{\beta}$. (See Appendix B for a definition and some properties of Toda brackets, and in particular Proposition B.1 for a restatement of [26, Prop. 1.8].) Hence $[\Sigma \mathbf{CP}^2, \mathbf{CP}^2]$ is cyclic of order 6. The element $\eta \mathrm{id}_{\mathbf{CP}^2}$ turns out to be the unique nonzero element of order 2, as the Toda bracket $\langle \eta, 2 = q \circ \omega, \eta \rangle$ also implies. The properties of Toda brackets supply an inclusion

$$\begin{aligned}\langle \eta, q, \eta \wedge \mathbf{CP}^2 \rangle \circ \omega \subset \langle \eta, q, \eta \wedge \mathbf{CP}^2 \circ \omega = \omega \circ \eta \rangle \subset \langle \eta, q \circ \omega, \eta \rangle \\ = \{6\nu, -6\nu\}\end{aligned}$$

which shows that it does not contain the zero element. More precisely, since the composition

$$\pi_3 \mathbb{S} \xrightarrow{q^*} [\Sigma^2 \mathbf{CP}^2, S^3] \xrightarrow{\omega^*} \pi_3 \mathbb{S}$$

is multiplication with 2 on a cyclic group with 24 elements, and the homomorphism $q^* \colon \pi_3 \mathbb{S} \to [\Sigma^2 \mathbf{CP}^2, S^3]$ is the projection onto a cyclic group with 12 elements (as one deduces from the action of η on $\pi_n \mathbb{S}$ for $n \in \{1, 2\}$), the homomorphism $\omega^* \colon [\Sigma^2 \mathbf{CP}^2, S^3] \to \pi_3 \mathbb{S}$ is injective. One obtains $\langle \eta, q, \eta \wedge \mathbf{CP}^2 \rangle \subset \{3\nu \circ q, -3\nu \circ q\}$. Hence the identity $\mathrm{id}_{\mathbf{CP}^2} \in [\mathbf{CP}^2, \mathbf{CP}^2]$ satisfies

$$\eta \mathrm{id}_{\mathbf{CP}^2} = \eta \wedge \mathbf{CP}^2 = \pm 3(i \circ \nu \circ q). \tag{2.3}$$

For comparison purposes with the motivic situation, it is instructive to look at the real case as well. Let $\mathbf{RP}^n$ denote real projective space. The cofiber sequence[1]

$$S^1 \xrightarrow{2} \mathbf{RP}^1 \xrightarrow{i} \mathbf{RP}^2 \xrightarrow{q} S^2 \tag{2.4}$$

induces a long exact sequence of stable homotopy groups

$$\begin{aligned}\cdots \xrightarrow{2} \pi_m \mathbf{RP}^1 = \pi_{m-1} \mathbb{S} \xrightarrow{i_*} \pi_m \mathbf{RP}^2 \xrightarrow{q_*} \pi_m S^2 = \pi_{m-2} \mathbb{S} \xrightarrow{2} \pi_{m-1} \mathbf{RP}^1 \\ = \pi_{m-2} \mathbb{S} \to \cdots\end{aligned}$$

terminating with $\pi_1 \mathbf{RP}^1 \to \pi_1 \mathbf{RP}^2$. The induced short exact sequences

$$0 \to \pi_{m-1} \mathbb{S} / 2\pi_{m-1} \mathbb{S} \to \pi_m \mathbf{RP}^2 \to {}_2\pi_{m-2} \mathbb{S} \to 0$$

[1] It would be better to use -2 and not 2, since the real realization of the algebraic Hopf map is negative.

express the stable homotopy group of the real projective plane as an extension of two groups, the 2-torsion subgroup, and the cokernel of multiplication by 2, on the respective stable homotopy group of spheres. The groups $\pi_3\mathbf{RP}^2$ and $\pi_4\mathbf{RP}^2$ are both extensions of $\mathbb{Z}/2$ by $\mathbb{Z}/2$. The Toda bracket $\langle 2, \eta, 2\rangle = \{\eta^2\}$ implies that $\pi_3\mathbf{RP}^2$ is given by the nontrivial extension, the extension for $\pi_4\mathbf{RP}^2$ turns out to be trivial [32, Lemma 5.2]. The following table results.

m	1	2	3	4	5	6
$\pi_m\mathbf{RP}^2$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0

The portion for $m < 3$ of this table implies that the cofiber sequence (2.4) induces a short exact sequence

$$0 \to \pi_2\mathbf{RP}^2 = [S^2, \mathbf{RP}^2] \xrightarrow{q^*} [\mathbf{RP}^2, \mathbf{RP}^2] \xrightarrow{i^*} [\mathbf{RP}^1, \mathbf{RP}^2] = \pi_1\mathbf{RP}^2 \to 0 \tag{2.5}$$

and hence the abelian group $[\mathbf{RP}^2, \mathbf{RP}^2]$ is an extension of two groups of order two. As in the computation of $\pi_3\mathbf{RP}^2$, the extension is nontrivial, meaning that $\mathrm{id}_{\mathbf{RP}^2} \in [\mathbf{RP}^2, \mathbf{RP}^2]$ is an element of order 4, as already proven in [7]. There cannot be any doubt whatsoever regarding the ring structure of $[\mathbf{RP}^2, \mathbf{RP}^2]$.

3 Over a field

Let F be a field, and let $\mathbf{SH}(F)$ denote the motivic stable homotopy category of F [27]. For a motivic spectrum $\mathsf{E} \in \mathbf{SH}(F)$ and integers $s, w \in \mathbb{Z}$, let $\pi_{s,w}\mathsf{E}$ denote the abelian group $[\Sigma^{s,w}\mathbf{1}, \mathsf{E}]$, where E is a motivic spectrum and $\mathbf{1}_F = \mathbf{1}$ is the motivic sphere spectrum. The grading conventions are such that the suspension functor $\Sigma^{2,1} = \Sigma^{1+(1)}$ is suspension with $\mathbf{P}^1$, and $\Sigma^{1,0} = \Sigma^{1+(0)} = \Sigma^1 = \Sigma$ is suspension with the simplicial circle. Set $\pi_{s+(w)}\mathsf{E} := \pi_{s+w,w}\mathsf{E}$, and let

$$\pi_{s+(\star)}\mathsf{E} = \bigoplus_{w\in\mathbb{Z}} \pi_{s+w,w}\mathsf{E} = \bigoplus_{w\in\mathbb{Z}} \pi_{s+(w)}\mathsf{E}$$

denote the direct sum, considered as a $\mathbb{Z}$-graded module over the $\mathbb{Z}$-graded ring $\pi_{0+(\star)}\mathbf{1}$. The notation $\pi_{s-(\star)}\mathsf{E} := \pi_{s+(-\star)}\mathsf{E}$ will be used frequently. The strictly $\mathbf{A}^1$-invariant sheaf obtained as the associated Nisnevich sheaf of $U \mapsto [\Sigma^{s,w}U_+, \mathsf{E}]$ for $U \in \mathbf{Sm}_F$ is denoted $\underline{\pi}_{s,w}\mathsf{E}$, which gives rise to the homotopy module $\underline{\pi}_{s+(\star)}\mathsf{E}$. In the following, every occurrence of "π" can be replaced by "$\underline{\pi}$" without affecting the truth of the (suitably reinterpreted) statements. See [15] for the following fundamental result.

Theorem 3.1 (Morel) *Let F be a field. Then $\pi_{0-(\star)}\mathbf{1}$ is the Milnor-Witt K-theory of F.*

The Milnor-Witt K-theory of F is denoted $\mathbf{K}^{\mathsf{MW}}(F)$, or simply $\mathbf{K}^{\mathsf{MW}}$, following the convention that the base field or scheme may be ignored in the notation. The definition and some details regarding $\mathbf{K}^{\mathsf{MW}}$ and modules over it are contained in the Appendix A. Theorem 3.1 implies that for every motivic spectrum E and for every integer s, $\pi_{s+(\star)}\mathsf{E}$ has a canonical structure as a graded $\mathbf{K}^{\mathsf{MW}}$-module. The conventions dictate that this structure comes with a sign change in the sense that elements in $\mathbf{K}^{\mathsf{MW}}_d$ provide homomorphisms $\pi_{s+(w)}\mathsf{E} \to \pi_{s+(w-d)}\mathsf{E}$ for every $d, s, w \in \mathbb{Z}$.

Choose a basepoint for $\mathbf{P}^1$, and hence $\mathbf{P}^2$, which will not appear in the notation. Neither will the base field F most of the time. The main cofiber sequence over a field is

$$S^{1+(2)} \xrightarrow{\eta} \mathbf{P}^1 \xrightarrow{i} \mathbf{P}^2 \xrightarrow{q} S^{2+(2)}. \tag{3.1}$$

It induces a long exact sequence of $\mathbf{K}^{\mathsf{MW}}$-modules

$$\cdots \xrightarrow{\eta} \pi_{m+(\star)}\mathbf{P}^1 \xrightarrow{i_*} \pi_{m+(\star)}\mathbf{P}^2 \xrightarrow{q_*} \pi_{m+(\star)}S^{2+(2)} \xrightarrow{\eta} \pi_{m-1+(\star)}\mathbf{P}^1 \to \cdots$$

terminating with $\pi_{1+(\star)}\mathbf{P}^2$ by connectivity [16]. The induced short exact sequences

$$0 \to \pi_{m-1+(\star-1)}\mathbf{1}/\eta\pi_{m-1+(\star-2)}\mathbf{1} \to \pi_{m+(\star)}\mathbf{P}^2 \to {}_{\eta}\pi_{m-2+(\star-2)}\mathbf{1} \to 0 \tag{3.2}$$

express $\pi_{m+(\star)}\mathbf{P}^2$ as an extension of two $\mathbf{K}^{\mathsf{MW}}$-modules, the submodule of $\pi_{m-2+(\star-2)}\mathbf{1}$ annihilated by η, and the cokernel of multiplication by η on $\pi_{m-1+(\star-1)}\mathbf{1}$. This justifies the relevance of the following statement.

Theorem 3.2 (Gille–Scully–Zhong) *Let F be a field of characteristic not two. The submodule of $\mathbf{K}^{\mathsf{MW}}$ annihilated by η coincides with the image of multiplication by the hyperbolic plane on $\mathbf{K}^{\mathsf{MW}}$:*

$${}_{\eta}\mathbf{K}^{\mathsf{MW}} = \mathsf{h}\mathbf{K}^{\mathsf{MW}}$$

Proof This follows from the injectivity of the homomorphism $\epsilon^R_\star$ in [12, Theorem 5.4] for R a field. □

Theorem 3.2 provides the exactness of the sequence mentioned in [8, Remark 4.3]. It is quite special that the kernel of multiplication by η on $\mathbf{K}^{\mathsf{MW}}$ is generated by a single element, but then the element η is also quite special. As a consequence of Theorem 3.2, the short exact sequence

$$0 \to \pi_{1+(\star-1)}\mathbf{1}/\eta\pi_{1+(\star-2)}\mathbf{1} \to \pi_{2+(\star)}\mathbf{P}^2 \to {}_{\eta}\pi_{0+(\star-2)}\mathbf{1} \to 0$$

specializes to an isomorphism $\pi_{1+(w-1)}\mathbf{1}/\eta\pi_{1+(w-2)}\mathbf{1} \cong \pi_{2+(w)}\mathbf{P}^2$ for $w > 2$. Therefore knowing $\pi_{1+(\star-1)}\mathbf{1}/\eta\pi_{1+(\star-2)}\mathbf{1}$ is essential.

Theorem 3.3 *Let F be a field of characteristic not two or three. The unit map $\mathbf{1} \to \mathbf{kq}$ induces a surjection $\pi_{1+(\star)}\mathbf{1}/\eta\pi_{1+(\star-1)}\mathbf{1} \to \pi_{1+(\star)}\mathbf{kq}/\eta\pi_{1+(\star-1)}\mathbf{kq}$ whose kernel is $\mathbf{K}^{\mathsf{M}}_{2-\star}/12$ (generated in $\star = 2$) after inverting the exponential characteristic e of F. In particular, the vanishing $\pi_{1+(w)}\mathbf{kq}/\eta\pi_{1+(w-1)}\mathbf{kq}$ for $w > 1$ implies that the nontrivial group of highest weight is $\pi_{1+(2)}\mathbf{1}[e^{-1}]/\eta\pi_{1+(1)}\mathbf{1}[e^{-1}] \cong \mathbb{Z}/12$, generated by the image of the Hopf map ν.*

Proof This is a consequence of [23, Theorem 5.5] in the formulation given in [22, Theorem 2.5]; see also [24, Theorem 1.1]. First of all, the unit map $\pi_{1+(\star)}\mathbf{1} \to \pi_{1+(\star)}\mathbf{kq}$ is surjective, whence the same is true for the induced map on the quotients. Set e to be the exponential characteristic of F. Consider the following natural transformation

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{K}^{\mathsf{M}}_{2-\star}/24[\frac{1}{e}] & \longrightarrow & \pi_{1+(\star)}\mathbf{1}[\frac{1}{e}] & \longrightarrow & \pi_{1+(\star)}\mathbf{kq}[\frac{1}{e}] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & \pi_{1+(\star)}\mathbf{1}[\frac{1}{e}]/\eta & \longrightarrow & \pi_{1+(\star)}\mathbf{kq}[\frac{1}{e}]/\eta & \longrightarrow & 0
\end{array}$$

of short exact sequences. The snake lemma implies that the map $\mathbf{K}^{\mathsf{M}}/24[\frac{1}{e}] \to A$ is surjective, because the map $\eta\pi_{1+(\star)}\mathbf{1} \to \eta\pi_{1+(\star)}\mathbf{kq}$ on the kernels is surjective. The map $\eta\colon \Sigma^{(1)}\mathbf{1} \to \mathbf{1}$ factors by construction as $\eta\colon \Sigma^{(1)}\mathbf{1} \to \mathsf{f}_1\mathbf{1} \to \mathbf{1}$, where $\mathsf{f}_1\mathbf{1} \to \mathbf{1}$ denotes the first effective cover. The presentations given in [22, Lemma 2.3, Theorem 2.5] then imply that the kernel of the map $\eta\pi_{1+(\star)}\mathbf{1} \to \eta\pi_{1+(\star)}\mathbf{kq}$ is generated by $12\nu = \eta^2\eta_{\mathsf{top}}$, where $\eta_{\mathsf{top}} \in \pi_{1+(0)}\mathbf{1}$ denotes the topological Hopf map.[2] Hence the snake lemma also implies that $A \cong \mathbf{K}^{\mathsf{M}}/12[\frac{1}{e}]$. □

The same proof shows that Theorem 3.3 is valid in characteristic 3 as well in the sense that the kernel of

$$\pi_{1+(\star)}\mathbf{1}/\eta\pi_{1+(\star-1)}\mathbf{1} \to \pi_{1+(\star)}\mathbf{kq}/\eta\pi_{1+(\star-1)}\mathbf{kq}$$

[2] This element appeared in Sect. 2 without the subscript.

is isomorphic to $\mathbf{K}^{\mathsf{M}}/4[\frac{1}{3}]$ after inverting 3. Theorems 3.2 and 3.3 provide sufficient information about the outer terms in the short exact sequence

$$0 \to \pi_{1+(\star-1)}\mathbf{1}/\eta\pi_{1+(\star-2)}\mathbf{1} \to \pi_{2+(\star)}\mathbf{P}^2 \to {}_{\eta}\pi_{0+(\star-2)}\mathbf{1} \to 0 \quad (3.3)$$

of $\mathbf{K}^{\mathsf{MW}}$-modules. Actually the outer terms are $\mathbf{K}^{\mathsf{M}}$-modules in a natural way; η acts trivially on these. However, η acts nontrivially on the middle term. The reason is the Toda bracket $\langle \eta, \mathsf{h}, \eta\rangle = \{6\nu, -6\nu\}$ from [22, Proposition 4.1].

Lemma 3.4 *Let F be a field of characteristic neither 2 nor 3. The action of η on the $\mathbf{K}^{\mathsf{MW}}$-module $\pi_{2+(\star)}\mathbf{P}^2$ in the extension*

$$0 \to \pi_{1+(\star-1)}\mathbf{1}/\eta\pi_{1+(\star-2)}\mathbf{1} \to \pi_{2+(\star)}\mathbf{P}^2 \to {}_{\eta}\pi_{0+(\star-2)}\mathbf{1} \to 0$$

is determined by the fact that $\mathsf{h}' \circ \eta = 6(i \circ \nu)$, where $\mathsf{h}' \in \pi_{2+(2)}\mathbf{P}^2$ is any lift of $\mathsf{h} \in \pi_{0+(0)}\mathbf{1}$.

Proof The Toda bracket $\langle \eta, \mathsf{h}, \eta\rangle = \{6\nu, -6\nu\}$ from [22, Proposition 4.1] implies by Proposition B.1 that there exists an element $\mathsf{h}' \in \pi_{2+(2)}\mathbf{P}^2$ which on the one hand maps to $\mathsf{h} \in {}_{\eta}\pi_{0+(0)}\mathbf{1}$, and on the other hand is such that $\mathsf{h}'\circ\eta$ is the image of $6\nu \in \pi_{1+(2)}\mathbf{1}$. Inspecting the short exact sequence (3.3) in weight 3 gives an isomorphism $\pi_{1+(2)}\mathbf{1}/\eta\pi_{1+(1)}\mathbf{1} \cong \pi_{2+(3)}\mathbf{P}^2$ by Theorem 3.2. Hence $\pi_{2+(3)}\mathbf{P}^2$ is cyclic of order 12 by Theorem 3.3, with the image $i \circ \nu$ of ν as a generator. It follows that $\mathsf{h}' \circ \eta$ is the unique nonzero element of order two in this group, and this is true for any choice of h' lifting h. Inspecting the short exact sequence (3.3) in weight 2 provides

$$0 \to \pi_{1+(1)}\mathbf{1}/\eta\pi_{1+(0)}\mathbf{1} \to \pi_{2+(2)}\mathbf{P}^2 \to {}_{\eta}\pi_{0+(0)}\mathbf{1} \to 0.$$

that for any two lifts $\mathsf{h}', \mathsf{h}''$ of h, there exists $\{u\} \in \mathbf{K}^{\mathsf{M}}_1/12$ with $\mathsf{h}'-\mathsf{h}'' = i\{u\}\nu$.

In order to describe the extension group more precisely, set $A_\star := \pi_{1-(\star-1)}\mathbf{1}/\eta\pi_{1-(\star)}\mathbf{1}$. The extension (3.3) is given by an element in

$$\mathrm{Ext}^1_{\mathbf{K}^{\mathsf{MW}}}({}_{\eta}\pi_{0-(\star+2)}\mathbf{1}, A_{\star+2}) \cong \mathrm{Ext}^1_{\mathbf{K}^{\mathsf{MW}}}(2\mathbf{K}^{\mathsf{M}}_{\star+2}, A_{\star+2}) \cong \mathrm{Ext}^1_{\mathbf{K}^{\mathsf{MW}}}(2\mathbf{K}^{\mathsf{M}}, A)$$

by Theorem 3.2. The short exact sequence

$$0 \to {}_2\mathbf{K}^{\mathsf{M}} \to \mathbf{K}^{\mathsf{M}} \to 2\mathbf{K}^{\mathsf{M}} \to 0$$

of $\mathbf{K}^{\mathsf{MW}}$-modules induces a long exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}(2\mathbf{K}^{\mathsf{M}}, A) \to \mathrm{Hom}(\mathbf{K}^{\mathsf{M}}, A) \to \mathrm{Hom}({}_2\mathbf{K}^{\mathsf{M}}, A) \\ &\to \mathrm{Ext}^1(2\mathbf{K}^{\mathsf{M}}, A) \to \mathrm{Ext}^1(\mathbf{K}^{\mathsf{M}}, A) \to \cdots \end{aligned}$$

where the subscript "$\mathbf{K}^{\mathsf{MW}}$" is suppressed. Lemma A.3 applies to provide an isomorphism between the group $\mathrm{Ext}^1_{\mathbf{K}^{\mathsf{MW}}}(\mathbf{K}^{\mathsf{M}}, A = \pi_{1-(\star-1)}\mathbf{1}/\eta\pi_{1-(\star)}\mathbf{1})$ and

$$_{\mathsf{h}}(\pi_{1+(2)}\mathbf{1}/\eta\pi_{1+(1)}\mathbf{1}) \cong {}_2(\pi_{1+(2)}\mathbf{1}/\eta\pi_{1+(1)}\mathbf{1}) \cong {}_2\mathbb{Z}/12$$

where the last isomorphism follows from Theorem 3.3. Note that multiplication with η on the $\mathbf{K}^{\mathsf{MW}}$-module $\pi_{1+(\star)}\mathbf{1}/\eta\pi_{1+(\star)}\mathbf{1}$ is the zero homomorphism by construction, which simplifies the term appearing in Lemma A.3. Hence $\mathrm{Ext}^1_{\mathbf{K}^{\mathsf{MW}}}(\mathbf{K}^{\mathsf{M}}, \pi_{1-(\star-1)}\mathbf{1}/\eta\pi_{1-(\star)}\mathbf{1}) = \{0, 6\nu\}$ does not depend on the base field F. The homomorphism $\mathrm{Ext}^1(2\mathbf{K}^{\mathsf{M}}, A) \to \mathrm{Ext}^1(\mathbf{K}^{\mathsf{M}}, A)$ is surjective, because the extension in question corresponds to the unique nonzero element in $\mathbf{A}_{-1} = \pi_{1+(2)}\mathbf{1}/\eta\pi_{1+(1)}\mathbf{1}$ of order two by the Toda bracket $\langle \eta, \mathsf{h}, \eta\rangle = \{6\nu, -6\nu\}$ from [22, Proposition 4.1]. There results an exact sequence

$$\begin{aligned}\mathrm{Hom}_{\mathbf{K}^{\mathsf{MW}}}(\mathbf{K}^{\mathsf{M}}, A) &\to \mathrm{Hom}_{\mathbf{K}^{\mathsf{MW}}}({}_2\mathbf{K}^{\mathsf{M}}, A) \to \mathrm{Ext}_{\mathbf{K}^{\mathsf{MW}}}(2\mathbf{K}^{\mathsf{M}}, A)\\ &\to \mathrm{Ext}_{\mathbf{K}^{\mathsf{MW}}}(\mathbf{K}^{\mathsf{M}}, A) \to 0 \end{aligned} \tag{3.4}$$

where $\mathrm{Hom}({}_2\mathbf{K}^{\mathsf{M}}, A) \subset \mathrm{Hom}(\mathbf{K}^{\mathsf{M}}(-1), A) = A_1$ via the surjection $\mathbf{K}^{\mathsf{M}}(-1) \to {}_2\mathbf{K}^{\mathsf{M}}$ obtained by multiplying with $\{-1\} \in \mathbf{K}^{\mathsf{M}}_1$, see Theorem A.1. Since $\mathrm{Hom}_{\mathbf{K}^{\mathsf{MW}}}(\mathbf{K}^{\mathsf{M}}, A) = \mathrm{Hom}_{\mathbf{K}^{\mathsf{M}}}(\mathbf{K}^{\mathsf{M}}, A)$ and $\mathrm{Hom}_{\mathbf{K}^{\mathsf{MW}}}({}_2\mathbf{K}^{\mathsf{M}}, A) = \mathrm{Hom}_{\mathbf{K}^{\mathsf{M}}}({}_2\mathbf{K}^{\mathsf{M}}, A)$, the exact sequence (3.4) induces a short exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbf{K}^{\mathsf{M}}}(2\mathbf{K}^{\mathsf{M}}, A) \to \mathrm{Ext}^1_{\mathbf{K}^{\mathsf{MW}}}(2\mathbf{K}^{\mathsf{M}}, A) \to \mathrm{Ext}_{\mathbf{K}^{\mathsf{MW}}}(\mathbf{K}^{\mathsf{M}}, A) \to 0$$

in which $\mathrm{Ext}^1_{\mathbf{K}^{\mathsf{M}}}(2\mathbf{K}^{\mathsf{M}}, A) \cong \mathrm{Hom}_{\mathbf{K}^{\mathsf{M}}}({}_2\mathbf{K}^{\mathsf{M}}, A)/\rho A_0$. In particular, the sought-after element in $\mathrm{Ext}^1_{\mathbf{K}^{\mathsf{MW}}}(2\mathbf{K}^{\mathsf{M}}, A)$ classifying the extension in question is determined by the relation $\mathsf{h}' \circ \eta = 6(i \circ \nu)$ and an element in the group $\mathrm{Ext}^1_{\mathbf{K}^{\mathsf{M}}}(2\mathbf{K}^{\mathsf{M}}, A)$ depending solely on the $\mathbf{K}^{\mathsf{M}}$-module structures of $2\mathbf{K}^{\mathsf{M}}$ and A. □

Remark 3.5 Regarding the unstable situation, the $\mathbf{A}^1$-fiber sequence

$$\mathbf{A}^1 \smallsetminus \{0\} \to \mathbf{A}^3 \smallsetminus \{0\} \to \mathbf{P}^2$$

and the $\mathbf{A}^1$-discreteness of $\mathbf{A}^1 \smallsetminus \{0\}$ provide an identification $\underline{\pi}_2^{\mathbf{A}^1}\mathbf{P}^2 \cong \underline{\pi}_2^{\mathbf{A}^1}(\mathbf{A}^3 \smallsetminus \{0\}) \cong \underline{\mathbf{K}}_3^{\mathsf{MW}}$ of (unstable) $\mathbf{A}^1$-homotopy sheaves, by [17, Theorem 1.23]. Let $\underline{\pi}_{2+(3)}^{\mathbf{A}^1}\mathbf{P}^2$ denote the threefold contraction of $\underline{\pi}_2^{\mathbf{A}^1}\mathbf{P}^2$, which coincides with the Nisnevich sheaf associated with the presheaf $X \mapsto \mathrm{Hom}_{\mathbf{H}_\bullet^{\mathbf{A}^1}(X)}(\mathbf{A}^3_X \smallsetminus \{0\}, \mathbf{P}^2_X)$ [17, p. 72, Theorem 6.13]. The generator of $\underline{\pi}_{2+(3)}^{\mathbf{A}^1}\mathbf{P}^2 \cong \underline{\mathbf{K}}_0^{\mathsf{MW}}$ is thus the class of the canonical map $\mathbf{A}^3 \smallsetminus \{0\} \to \mathbf{P}^2$. Stabilization with respect to $\mathbf{P}^1$ provides a homomorphism

$$\underline{\pi}^{\mathbf{A}^1}_{2+(3)}\mathbf{P}^2 \to \underline{\pi}_{2+(3)}\mathbf{P}^2$$

to the stable homotopy sheaf computed in Lemma 3.4. It sends the generator to $m(i \circ \nu)$, where m is an integer unique up to multiples of 12, and $i \circ \nu$ is the generator of the target. A comparison with the classical topological situation via complex or étale realization, which is possible since the target does not depend on the base field, shows that $m = \pm 2$, because the order of the canonical map $S^5 \to \mathbf{CP}^2$ is 6 after one suspension [19, Theorem 1.2]. Note that étale realization sends $\mathbf{P}^n$ to the profinite completion of its complex realization $\mathbf{CP}^n$ by [5, Theorem 12.9]; see also [10, Theorem 8.4]. The same applies to the maps involved here.

The computations provided by $\pi_{1-(\star-1)}\mathbf{P}^2 \cong \mathbf{K}^{\mathsf{M}}$ and Lemma 3.4 suffice to conclude the following statement.

Theorem 3.6 *Let F be a field of characteristic not in $\{2, 3\}$. The cofiber sequence (3.1) induces a short exact sequence*

$$0 \to \pi_{2+(\star+2)}\mathbf{P}^2/\eta\pi_{2+(\star+1)}\mathbf{P}^2 \to [\Sigma^{(\star)}\mathbf{P}^2, \mathbf{P}^2] \to \pi_{1+(\star+1)}\mathbf{P}^2 \to 0 \tag{3.5}$$

of $\mathbf{K}^{\mathsf{MW}}$-modules. In particular, after inverting the exponential characteristic, there is an isomorphism

$$\mathbb{Z}/6 \cong \pi_{2+(3)}\mathbf{P}^2/\eta\pi_{2+(2)}\mathbf{P}^2 \cong [\Sigma^{(1)}\mathbf{P}^2, \mathbf{P}^2]$$

with $i \circ \nu \circ q$ as a generator, and an isomorphism

$$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbf{K}^{\mathsf{M}}_1/6(F) \cong [\mathbf{P}^2, \mathbf{P}^2]$$

of abelian groups, with $\mathrm{id}_{\mathbf{P}^2}$ and $\mathsf{h}' \circ q$ each generating one free summand. The equality $\eta \cdot \mathrm{id}_{\mathbf{P}^2} = \eta \wedge \mathrm{id}_{\mathbf{P}^2} = 3(i \circ \nu \circ q)$ determines the action of η on $[\Sigma^{(\star)}\mathbf{P}^2, \mathbf{P}^2]$.

Proof As before, the cofiber sequence (3.1) induces a short exact sequence

$$0 \to \pi_{2+(\star+2)}\mathbf{P}^2/\eta\pi_{2+(\star+1)}\mathbf{P}^2 \to [\Sigma^{(\star)}\mathbf{P}^2, \mathbf{P}^2] \to {}_{\eta}\pi_{1+(\star+1)}\mathbf{P}^2 \to 0.$$

The short exact sequence for $\pi_{1+(\star)}\mathbf{P}^2$ specializes to the identification $\pi_{1+(\star)}\mathbf{P}^2 \cong \mathbf{K}^{\mathsf{M}}$ mentioned already above. Since η acts as zero on this $\mathbf{K}^{\mathsf{MW}}$-module, the short exact sequence (3.5) of $\mathbf{K}^{\mathsf{MW}}$-modules follows. The identity $\mathrm{id}_{\mathbf{P}^2}$ hits the canonical generator $i \in \pi_{1+(1)}\mathbf{P}^2$. The action of η in the $\mathbf{K}^{\mathsf{MW}}$-module structure on $[\Sigma^{(\star)}\mathbf{P}^2, \mathbf{P}^2]$ is then determined by specifiying

$\eta \mathrm{id}_{\mathbf{P}^2} \in [\Sigma^{(1)}\mathbf{P}^2, \mathbf{P}^2]$. More precisely, as in the proof of Lemma 3.4 there results a short exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbf{K}^{\mathsf{M}}}(2\mathbf{K}^{\mathsf{M}}, A) \to \mathrm{Ext}^1_{\mathbf{K}^{\mathsf{MW}}}(2\mathbf{K}^{\mathsf{M}}, A) \to \mathrm{Ext}^1_{\mathbf{K}^{\mathsf{MW}}}(\mathbf{K}^{\mathsf{M}}, A) \to 0$$

of abelian groups, where $A_\star := \pi_{2-(\star-2)}\mathbf{P}^2/\eta\pi_{2-(\star-1)}\mathbf{P}^2$. Lemma A.3 identifies the last extension group as

$$\mathrm{Ext}^1_{\mathbf{K}^{\mathsf{MW}}}(\mathbf{K}^{\mathsf{M}}, \pi_{2-(\star-2)}\mathbf{P}^2/\eta\pi_{2-(\star-1)}\mathbf{P}^2) \cong {}_{\mathsf{h}}(\pi_{2+(3)}\mathbf{P}^2/\eta\pi_{2+(2)}\mathbf{P}^2) \cong {}_2\mathbb{Z}/6.$$

Here the description of $\pi_{2+(\star)}\mathbf{P}^2$ as a $\mathbf{K}^{\mathsf{MW}}$-module from Lemma 3.4 supplies the last isomorphism in this sequence, as well as the first isomorphism mentioned in the statement of the theorem. Hence $\eta \mathrm{id}_{\mathbf{P}^2} = m(i \circ \nu \circ q)$ for some $m \in \mathbb{Z}$ which is unique up to multiples of 6. The element $\eta \mathrm{id}_{\mathbf{P}^2}$ turns out to be the unique nonzero element of order 2, as the Toda bracket $\langle \eta, \mathsf{h} = q \circ \mathsf{h}', \eta\rangle$ implies. The properties of Toda brackets supply an inclusion

$$\begin{aligned}\langle \eta, q, \eta \wedge \mathbf{P}^2\rangle \circ \mathsf{h}' &\subset \langle \eta, q, \eta \wedge \mathbf{P}^2 \circ \mathsf{h}' = \mathsf{h}' \circ \eta\rangle \subset \langle \eta, q \circ \mathsf{h}', \eta\rangle \\ &= \{6\nu, -6\nu\}\end{aligned}$$

which shows that it does not contain zero. This already suffices to conclude. More precisely, since the composition

$$\pi_{1+(2)}\mathbf{1} \xrightarrow{q^*} [\Sigma^{1+(1)}\mathbf{P}^2, S^{2+(1)}] \xrightarrow{(\mathsf{h}')^*} \pi_{1+(2)}\mathbf{1}$$

is multiplication with h, one obtains $\langle \eta, q, \eta \wedge \mathbf{P}^2\rangle \subset \{3\nu \circ q, -3\nu \circ q\}$. Note that the homomorphism $[\Sigma^{1+(1)}\mathbf{P}^2, S^{2+(1)}] \xrightarrow{(\mathsf{h}')^*} \pi_{1+(2)}\mathbf{1}$ identifies with the inclusion $\mathbb{Z}/12 \hookrightarrow \mathbb{Z}/24$, as one may deduce from the exact sequence

$$\pi_{1+(2)}\mathbf{1} \xrightarrow{q^*} [\Sigma^{1+(1)}\mathbf{P}^2, S^{2+(1)}] \xrightarrow{i^*} \pi_{0+(1)}\mathbf{1} \xrightarrow{\eta^*} \pi_{0+(2)}\mathbf{1}.$$

Hence $\mathrm{id}_{\mathbf{P}^2} \in [\mathbf{P}^2, \mathbf{P}^2]$ satisfies $\eta \wedge \mathbf{P}^2 = 3(i \circ \nu \circ q)$. □

Theorem 3.6 implies that every element $x \in [\mathbf{P}^2, \mathbf{P}^2]$ can be expressed uniquely as a sum $x_1 \mathrm{id}_{\mathbf{P}^2} + x_2(\mathsf{h}' \circ q) + x_3(i \circ \nu \circ q)$, where $x_1, x_2 \in \mathbb{Z}$ and $x_3 \in K^{\mathsf{M}}_1/6$. It would probably be more honest to think of the integers x_1, x_2 as the ranks of virtual quadratic forms. In particular, the hyperbolic form “h” corresponds to the integer “2”. Using that the composition $q \circ i$ is the zero map, the ring structure is then given as

$$
\begin{aligned}
x \circ y &= \big(x_1 \mathrm{id}_{\mathbf{P}^2} + x_2(\mathsf{h}' \circ q) + x_3(i \circ \nu \circ q)\big) \\
&\quad \circ \big(y_1 \mathrm{id}_{\mathbf{P}^2} + y_2(\mathsf{h}' \circ q) + y_3(i \circ \nu \circ q)\big) \\
&= x_1 y_1 \mathrm{id}_{\mathbf{P}^2} + (x_1 y_2 + x_2 y_1 + 2x_2 y_2)(\mathsf{h}' \circ q) \\
&\quad + (x_1 y_3 + x_3 y_1 + 2x_3 y_2)(i \circ \nu \circ q)
\end{aligned}
$$

and in particular is not commutative if $2K_1^{\mathsf{M}}(F)/6$ is nonzero. The group of units in $[\mathbf{P}^2, \mathbf{P}^2]$ consists of the elements

$$\{\pm \mathrm{id}_{\mathbf{P}^2} + x_3(i \circ \nu \circ q), \pm \mathrm{id}_{\mathbf{P}^2} \mp \mathsf{h}' \circ q + x_3(i \circ \nu \circ q) \colon x_3 \in K_1^{\mathsf{M}}/6\}.$$

If $F \subset \mathbb{C}$, then complex realization $[\Sigma^{(1)}\mathbf{P}^2, \mathbf{P}^2] \to [\Sigma\mathbf{CP}^2, \mathbf{CP}^2]$ is an isomorphism, but $[\mathbf{P}^2, \mathbf{P}^2] \to [\mathbf{CP}^2, \mathbf{CP}^2]$ is possibly only surjective, not injective. Nevertheless, every map in $[\mathbf{P}^2, \mathbf{P}^2]$ such that its complex realization is a unit in $[\mathbf{CP}^2, \mathbf{CP}^2]$ is already a unit in $[\mathbf{P}^2, \mathbf{P}^2]$.

Remark 3.7 A different motivic type of endomorphisms of the projective plane occurs in the motivic stable homotopy category $\mathbf{SH}(\mathbf{P}^2_F)$ for $\mathbf{P}^2_F$, with unit $\mathbf{1}_{\mathbf{P}^2_F} = \varphi^*(\mathbf{1}_F)$. Here $\varphi \colon \mathbf{P}^2_F \to \mathrm{Spec}(F)$ is the structure morphism. The choice of a rational point provides a splitting $\mathbf{1}_F \to \varphi_\sharp(\mathbf{1}_{\mathbf{P}^2_F}) \to \mathbf{1}_F$ of the counit, whence $\varphi_\sharp(\mathbf{1}_{\mathbf{P}^2_F}) \simeq \mathbf{P}^2_F \vee \mathbf{1}_F$. Then $\varphi_\sharp$ induces a ring homomorphism

$$\pi_{0+(\star)}\mathbf{1}_{\mathbf{P}^2_F} \xrightarrow{\varphi_\sharp} [\Sigma^{(\star)}\mathbf{P}^2_+, \mathbf{P}^2_+] \cong [\Sigma^{(\star)}\mathbf{P}^2, \mathbf{P}^2] \oplus \pi_{0+(\star)}\mathbf{1}_F \oplus [\Sigma^{(\star)}\mathbf{P}^2, \mathbf{1}]$$

where the source

$$\pi_{0+(\star)}\mathbf{1}_{\mathbf{P}^2_F} \cong [\Sigma^{(\star)}\varphi_\sharp(\mathbf{1}_{\mathbf{P}^2_F}), \mathbf{1}_F] \cong [\Sigma^{(\star)}\mathbf{P}^2, \mathbf{1}_F] \oplus \pi_{0+(\star)}\mathbf{1}_F$$

is a commutative ring by definition. Using [22, Theorem 2.7], the short exact sequence

$$
\begin{aligned}
0 &\to \pi_{2+(2)}\mathbf{1}_F/\eta\pi_{2+(1)}\mathbf{1}_F \cong \mathbf{K}_2^{\mathsf{M}}(F)/2 \to [\mathbf{P}^2, \mathbf{1}] \\
&\to {}_\eta\pi_{1+(1)}\mathbf{1} \cong \mathbf{K}_1^{\mathsf{M}}(F)/24 \to 0
\end{aligned}
$$

implies that $\pi_{0+(0)}\mathbf{1}_{\mathbf{P}^2_F}$ is not isomorphic to the Grothendieck-Witt ring of $\mathbf{P}^2_F$, which is isomorphic to $\mathbf{K}_0^{\mathsf{MW}}(F) \oplus \mathbf{K}_0^{\mathsf{M}}(F)$ [29].

4 Suslin's conjecture

An application of some of the computations performed in Sect. 3 is a proof of Suslin's conjecture on the Hurewicz homomorphism from Quillen to Milnor

K-theory in degree four, exploiting the beautiful work [3]. Unstable homotopy sheaves will occur, as already in Remark 3.5. Let

$$Q_{2n-1} := \left\{ (a,b) \in \mathbf{A}^n \times \mathbf{A}^n \colon \sum_{j=1}^{n} a_j b_j = 1 \right\}$$

be the smooth affine quadric hypersurface which is weakly equivalent to $\mathbf{A}^n \smallsetminus \{0\}$ via projection to the first n coordinates [9, Example 2.12(3)]. The quotient scheme $Q_{2n-1}/\mathbf{G}_{\mathfrak{m}}$ with respect to the free action $\lambda \cdot (a,b) := (\lambda a, \lambda^{-1} b)$ is then weakly equivalent via projection to the first n coordinates to $\mathbf{P}^{n-1}$. Given $(a,b,u) \in Q_{2n-1} \times \mathbf{A}^1 \smallsetminus \{0\}$, let $[a,b,u] \in \mathbf{GL}_n$ denote the matrix whose entry at (j,k) is $\delta_{jk} + (u-1)a_j b_k$, where δ_{jk} is the Kronecker symbol. For a unit $u \in \mathbf{G}_{\mathfrak{m}}$, let $\Delta(u)$ denote the diagonal matrix whose entries are $(u,1,\ldots,1)$. The map

$$\psi_n \colon Q_{2n-1} \times (\mathbf{A}^1 \smallsetminus \{0\}) \to \mathbf{SL}_n,\ \psi_n(a,b,u) := \Delta(u^{-1}) \cdot [a,b,u] \tag{4.1}$$

is compatible with the given $\mathbf{G}_{\mathfrak{m}}$ action on the first factor (and trivial actions on the other factor and $\mathbf{SL}_n$) and sends $Q_{2n-1} \times \{1\} \cup \big\{\big((1,0,\ldots,0),(1,0,\ldots,0)\big)\big\} \times (\mathbf{A}^1 \smallsetminus \{0\})$ to the identity matrix, the canonical basepoint in $\mathbf{SL}_n$. Let $\psi_n \colon \Sigma^{(1)} Q_{2n-1}/\mathbf{G}_{\mathfrak{m}} \to \mathbf{SL}_n$ denote also the induced pointed map, a variant of the map (with the same notation) to $\mathbf{GL}_n$ constructed in [31, Section 5]. Its complex realization is denoted j_n in [19, p. 180], and $f_{\mathbf{SU}(n)}$ in the even more classical source [33, Section 4]. It is straightforward to check that the diagram

$$\begin{array}{ccc} \Sigma^{(1)} Q_{2n-1}/\mathbf{G}_{\mathfrak{m}} & \xrightarrow{\psi_n} & \mathbf{SL}_n \\ \downarrow & & \downarrow \\ \Sigma^{(1)} Q_{2n+1}/\mathbf{G}_{\mathfrak{m}} & \xrightarrow{\psi_{n+1}} & \mathbf{SL}_{n+1} \end{array} \tag{4.2}$$

with obvious inclusions as vertical maps commutes.

Lemma 4.1 *The map $\psi_2 \colon \Sigma^{(1)} Q_3/\mathbf{G}_{\mathfrak{m}} \to \mathbf{SL}_2$ is a weak equivalence over* $\operatorname{Spec}(\mathbb{Z})$.

Proof Let $\{b_1 = 0\} \hookrightarrow Q_3$ denote the smooth closed subscheme where $b_1 = 0$, and $\{b_1 \neq 0\} \hookrightarrow Q_3$ its open complement. The map

$$\mathbf{A}^1 \times (\mathbf{A}^1 \smallsetminus \{0\}) \to \{b_1 = 0\},\ (x,y) \mapsto (x,y,0,y^{-1})$$

is an isomorphism. Its image $\overline{\{b_1 = 0\}} \hookrightarrow Q_3/\mathbf{G}_{\mathfrak{m}}$ is a smooth closed subscheme isomorphic to $\mathbf{A}^1$, with trivial normal bundle. The map $\psi_2 \colon Q_3 \times$

$\mathbf{A}^1 \smallsetminus \{0\} \to \mathbf{SL}_2$ sends the product $\{b_1 = 0\} \times \mathbf{A}^1 \smallsetminus \{0\}$ to the smooth closed subscheme $\{c_{21} = 0\} \hookrightarrow \mathbf{SL}_2$, and the induced map $\overline{\{b_1 = 0\}} \times \mathbf{A}^1 \smallsetminus \{0\} \to \{c_{21} = 0\}$ is a weak equivalence. The latter follows from placing it at the top in the commutative diagram

$$\begin{tikzcd} \overline{\{b_1 = 0\}} \times \mathbf{A}^1 \smallsetminus \{0\} \arrow[r] \arrow[d, "\mathrm{pr}_2"] & \{c_{21} = 0\} \arrow[d, "c_{22}"] \\ \mathbf{A}^1 \smallsetminus \{0\} \arrow[r, "\mathrm{id}"] & \mathbf{A}^1 \smallsetminus \{0\} \end{tikzcd}$$

where the vertical projections are weak equivalences. The open complement $\{b_1 \neq 0\} \hookrightarrow Q_3$ contains the closed subscheme $\{b_2 = 0\} \hookrightarrow Q_3$ as a strong $\mathbf{A}^1$-deformation retract, as the map

$$\{b_1 \neq 0\} \times \mathbf{A}^1 \to \{b_1 \neq 0\},\ \big((a, b), t\big) \mapsto (ta_1 + (1-t)b_1^{-1}, a_2, b_1, tb_2)$$

shows. This strong $\mathbf{A}^1$-deformation retraction is $\mathbf{G}_{\mathrm{m}}$-equivariant, whence also the inclusion $\overline{\{b_2 = 0\}} \hookrightarrow \overline{\{b_1 \neq 0\}}$ is a strong $\mathbf{A}^1$-deformation retract. Note that $\overline{\{b_2 = 0\}}$ is isomorphic to $\mathbf{A}^1$. Homotopy purity [21, Theorem 3.2.23] supplies a homotopy cofiber sequence

$$\mathbf{A}^1 \cong \overline{\{b_2 = 0\}} \simeq \overline{\{b_1 \neq 0\}} \hookrightarrow Q_3/\mathbf{G}_{\mathrm{m}} \to \Sigma^{1+(1)}\{b_1 = 0\}_+$$

inducing a homotopy cofiber sequence after applying $\Sigma^{(1)}$. In particular, since $\overline{\{b_2 = 0\}}$ is $\mathbf{A}^1$-contractible, the map $\Sigma^{(1)}Q_3/\mathbf{G}_{\mathrm{m}} \to \Sigma^{(1)}\Sigma^{1+(1)}\{b_1 = 0\}_+$ is a weak equivalence.

Similarly, the smooth closed subscheme $\{c_{21} = 0\} \hookrightarrow \mathbf{SL}_2$ gives rise, via homotopy purity, to a homotopy cofiber sequence

$$\{c_{21} \neq 0\} \hookrightarrow \mathbf{SL}_2 \to \Sigma^{1+(1)}\{c_{21} = 0\}_+$$

which can be related to the homotopy cofiber sequence above as follows. While the map $\psi_2\colon \overline{\{b_1 = 0\}} \times \mathbf{A}^1 \smallsetminus \{0\} \to \{c_{21} = 0\}$ is a weak equivalence, $\psi_2(\{b_1 \neq 0\} \times \mathbf{A}^1 \smallsetminus \{0\})$ is not contained in $\{c_{21} \neq 0\}$ but instead coincides with the union $U := \{c_{21} \neq 0\} \cup \{c_{11} = c_{22} = 1 \text{ and } c_{21} = 0\}$. The strong $\mathbf{A}^1$-deformation retraction

$$\begin{aligned} &\{c_{21} \neq 0\} \times \mathbf{A}^1 \to \{c_{21} \neq 0\},\ (C, t) \\ &\quad \mapsto \begin{pmatrix} c_{11} + t(1 - c_{11}) & c_{12} + t(1 - c_{11})\frac{c_{22}}{c_{21}} \\ c_{21} & c_{22} \end{pmatrix} \end{aligned}$$

extends via the constant $\mathbf{A}^1$-homotopy to a strong $\mathbf{A}^1$-deformation retration of U to $V := \{C \in U : c_{11} = 1\}$, which in turn deforms via

$$V \times \mathbf{A}^1 \to V,\ (C,t) \mapsto \begin{pmatrix} c_{11}=1 & c_{12} \\ tc_{21} & tc_{22}+1-t \end{pmatrix}$$

to the affine line $\{c_{11} = c_{22} = 1 \text{ and } c_{21} = 0\} \hookrightarrow \mathbf{SL}_2$. The induced diagram of homotopy cofiber sequences

$$\begin{array}{ccccc} \{c_{21} \neq 0\} & \longrightarrow & \mathbf{SL}_2 & \longrightarrow & \Sigma^{1+(1)}\{c_{21}=0\}_+ \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{A}^1 \simeq U & \longrightarrow & \mathbf{SL}_2 & \xrightarrow{\simeq} & \mathbf{SL}_2/U \end{array}$$

identifies $\Sigma^{1+(1)}\{c_{21}=0\}_+/\Sigma^{1+(1)}\{1\}_+ \simeq \mathbf{SL}_2/U$ and induces a commutative diagram

$$\begin{array}{ccc} \Sigma^{(1)}Q_3/\mathbf{G}_\mathfrak{m} & \xrightarrow{\simeq} & \Sigma^{1+(2)}\overline{\{b_1=0\}}_+ \\ \downarrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \simeq} \\ \mathbf{SL}_2 & \xrightarrow{\simeq} & \Sigma^{1+(1)}\{c_{21}=0\}_+/\Sigma^{1+(1)}\{1\}_+ \end{array}$$

in which the vertical map on the right hand side is a weak equivalence, because it is induced by the isomorphism $\psi_2 \colon \overline{\{b_1=0\}} \times \mathbf{A}^1 \smallsetminus \{0\} \xrightarrow{\cong} \{c_{21}=0\}$. Hence ψ_2 is a weak equivalence as claimed. □

The proof of the following statement is essentially a modification of Jean Fasel's unpublished proof for the corresponding statement on symplectic groups; I thank him sincerely for the inspiration.

Proposition 4.2 *The inclusion $\mathbf{SL}_n \hookrightarrow \mathbf{SL}_{n+1}$ fits into a homotopy cofiber sequence*

$$\mathbf{SL}_n \hookrightarrow \mathbf{SL}_{n+1} \to \Sigma^{n+(n+1)}(\mathbf{SL}_n)_+$$

over $\mathrm{Spec}(\mathbb{Z})$.

Proof A matrix $C \in \mathbf{SL}_n$ has entries denoted $c_{1,1},\ldots,c_{1,n},c_{2,1},\ldots,c_{n,n}$. The homotopy purity theorem [21, Theorem 3.2.23], applied to the smooth closed subscheme $W := \{c_{n+1,1} = \ldots = c_{n+1,n} = 0\} \hookrightarrow \mathbf{SL}_{n+1}$ (which is a global complete intersection and thus has a trivial normal bundle), supplies a homotopy cofiber sequence:

$$\mathbf{SL}_{n+1} \smallsetminus W \hookrightarrow \mathbf{SL}_{n+1} \to \Sigma^{n+(n)}W_+$$

The inclusion $(\mathbf{SL}_{n+1} \smallsetminus W) \cap \{c_{n+1,n+1} = 1\} \hookrightarrow \mathbf{SL}_{n+1} \smallsetminus W$ is a weak equivalence. Let $q : \mathbf{SL}_{n+1} \smallsetminus W \to \mathbf{A}^n \smallsetminus \{0\}$ send C to $(c_{n+1,1}, \ldots, c_{n+1,n})$. Let X and Y be defined by taking pullbacks

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\simeq"] & Y \arrow[r] \arrow[d, "\simeq"] & Q_{2n-1} \arrow[d, "\mathrm{pr}_1"] \\ (\mathbf{SL}_{n+1} \smallsetminus W) \cap \{c_{n+1,n+1} = 1\} \arrow[r] & \mathbf{SL}_{n+1} \smallsetminus W \arrow[r, "q"] & \mathbf{A}^n \smallsetminus \{0\} \end{tikzcd} \tag{4.3}$$

where the vertical maps are Zariski locally trivial fibrations with $\mathbf{A}^{n-1}$ as fiber, and hence weak equivalences. An element in Y is a pair (E, d) with $E \in \mathbf{SL}_{n+1} \smallsetminus W$ and $d \in \mathbf{A}^n \smallsetminus \{0\}$ such that $\sum_{j=1}^n e_{n+1,j} \cdot d_j = 1$. The map

$$Y \times \mathbf{A}^1 \to Y,\ ((E, d), t) \mapsto E \cdot \left(\begin{pmatrix} 1_n & t(1 - e_{n+1,n+1})d \\ 0 & 1 \end{pmatrix}, d \right)$$

is a strong $\mathbf{A}^1$-deformation retraction onto X. Moreover,

$$(E, t) \mapsto \begin{pmatrix} 1_n & -tE_{j,n+1} \\ 0 & 1 \cdot E \end{pmatrix}$$

is an $\mathbf{A}^1$-deformation retraction of $(\mathbf{SL}_{n+1} \smallsetminus W) \cap \{c_{n+1,n+1} = 1\}$ onto the closed subscheme

$$\mathbf{SL}_n \times \big(\mathbf{A}^n \smallsetminus \{0\}\big) \hookrightarrow (\mathbf{SL}_{n+1} \smallsetminus W) \cap \{c_{n+1,n+1} = 1\},\ (E, d) \mapsto \begin{pmatrix} E & 0 \\ d & 1 \end{pmatrix}.$$

As in the proof of Lemma 4.1, the homotopy purity theorem provides a homotopy cofiber sequence

$$\mathbf{SL}_n \times \big(\mathbf{A}^n \smallsetminus \{0\}\big) \hookrightarrow \mathbf{SL}_{n+1} \to \Sigma^{n+(n)} W_+ \tag{4.4}$$

in which $W = \{c_{n+1,1} = \ldots = c_{n+1,n} = 0\} \cong \mathbf{SL}_n \times (\mathbf{A}^1 \smallsetminus \{0\}) \times \mathbf{A}^n \simeq \mathbf{SL}_n \times (\mathbf{A}^1 \smallsetminus \{0\})$. Enlarging the subscheme $\mathbf{SL}_n \times \big(\mathbf{A}^n \smallsetminus \{0\}\big)$ in (4.4) to $\mathbf{SL}_n \times \mathbf{A}^n$ then provides the desired homotopy cofiber sequence:

$$\mathbf{SL}_n \simeq \mathbf{SL}_n \times \mathbf{A}^n \hookrightarrow \mathbf{SL}_{n+1} \to \Sigma^{n+(n+1)}(\mathbf{SL}_n)_+$$

□

Corollary 4.3 *In the commutative diagram*

$$\begin{tikzcd}
\Sigma^{(1)}Q_{2n-1}/\mathbf{G}_{\mathfrak{m}} \arrow[r] \arrow[d, "\psi_n"] & \Sigma^{(1)}Q_{2n+1}/\mathbf{G}_{\mathfrak{m}} \arrow[r] \arrow[d, "\psi_{n+1}"] & S^{n+(n+1)} \arrow[d] \\
\mathbf{SL}_n \arrow[r] & \mathbf{SL}_{n+1} \arrow[r] & \mathbf{SL}_{n+1}/\mathbf{SL}_n
\end{tikzcd}$$

in which the top row is a homotopy cofiber sequence and the bottom row is a homotopy fiber sequence, the canonically induced map $S^{n+(n+1)} \to \mathbf{SL}_{n+1}/\mathbf{SL}_n$ *is a weak equivalence over* $\mathrm{Spec}(\mathbb{Z})$.

Proof Proposition 4.2 and the standard homotopy cofiber sequence for $Q_{2n-1}/\mathbf{G}_{\mathfrak{m}} \hookrightarrow Q_{2n+1}/\mathbf{G}_{\mathfrak{m}}$ give a diagram of homotopy cofiber sequences

$$\begin{tikzcd}
\Sigma^{(1)}Q_{2n-1}/\mathbf{G}_{\mathfrak{m}} \arrow[r] \arrow[d, "\psi_n"] & \Sigma^{(1)}Q_{2n+1}/\mathbf{G}_{\mathfrak{m}} \arrow[r] \arrow[d, "\psi_{n+1}"] & \Sigma^{(1)}\mathbf{P}^n/\mathbf{P}^{n-1} = \Sigma^{n+(n+1)}(\mathrm{Spec}(\mathbb{Z}))_+ \arrow[d] \\
\mathbf{SL}_n \arrow[r] & \mathbf{SL}_{n+1} \arrow[r] & \Sigma^{n+(n+1)}(\mathbf{SL}_n)_+
\end{tikzcd}$$

where the vertical map on the right hand side is induced by the inclusion of the identity matrix. The canonical map $\Sigma^{n+(n+1)}(\mathbf{SL}_n)_+ \to \mathbf{SL}_{n+1}/\mathbf{SL}_n$ is induced by the structure map $\mathbf{SL}_n \to \mathrm{Spec}(\mathbb{Z})$, because the pullback square

$$\begin{tikzcd}
W \arrow[r] \arrow[d, "\ell"] & \mathbf{SL}_{n+1} \arrow[d, "\text{last row}"] \\
\{0\} \times (\mathbf{A}^1 \smallsetminus \{0\}) \arrow[r] & \mathbf{A}^{n+1} \smallsetminus \{0\}
\end{tikzcd}$$

of smooth schemes induces a commutative diagram

$$\begin{tikzcd}
(\mathbf{A}^n/\mathbf{A}^n \smallsetminus \{0\}) \wedge W_+ \arrow[r, "\simeq"] \arrow[d, "(\mathbf{A}^n/\mathbf{A}^n \smallsetminus\{0\})\wedge \ell_+"] & \mathbf{SL}_{n+1}/(\mathbf{SL}_{n+1} \smallsetminus W) \arrow[d] \\
(\mathbf{A}^n/\mathbf{A}^n \smallsetminus \{0\}) \wedge (\mathbf{A}^1 \smallsetminus \{0\})_+ \arrow[r, "\simeq"] & \mathbf{A}^{n+1} \smallsetminus \{0\}/(\mathbf{A}^n \smallsetminus \{0\}) \times \mathbf{A}^1
\end{tikzcd}$$

of homotopy purity transformations, where the map ℓ sends a matrix in $W \simeq \mathbf{SL}_n \times (\mathbf{A}^1 \smallsetminus \{0\})$ to its last diagonal element and hence is induced by the structure map $\mathbf{SL}_n \to \mathrm{Spec}(\mathbb{Z})$. Proceeding through the zigzag relating $\mathbf{SL}_{n+1}/(\mathbf{SL}_{n+1} \smallsetminus W)$ with $\mathbf{SL}_{n+1}/\mathbf{SL}_n$ produced in the proof of Proposition 4.2 provides the statement. □

With the help of ψ_3, the cell structure of $\mathbf{SL}_3$ looks as follows.

Lemma 4.4 *The homotopy cofiber of* $\psi_3\colon \Sigma^{(1)}\mathbf{P}^2 \to \mathbf{SL}_3$ *over* $\mathrm{Spec}(\mathbb{Z})$ *is given by* $S^{3+(5)}$.

Proof Lemma 4.1, Proposition 4.2 and elementary properties of homotopy pushout diagrams imply that the diagram

$$\begin{tikzcd}
\Sigma^{(1)}Q_5/\mathbf{G}_{\mathfrak{m}} \arrow[r, "\psi_3"] \arrow[d] & \mathbf{SL}_3 \arrow[d] \\
S^{2+(3)} \arrow[r] & \Sigma^{2+(3)}(\mathbf{SL}_2)_+
\end{tikzcd}$$

in which the vertical maps are the canonical quotient maps and the bottom horizontal map is the canonical inclusion is a homotopy pushout diagram. The result follows. □

More generally, the total homotopy cofiber of diagram (4.2) can be determined as follows.

Lemma 4.5 *The homotopy cofiber of the map* $\Sigma^{(1)}Q_{2n+1}/\mathbf{G}_{\mathfrak{m}} \cup_{\Sigma^{(1)}Q_{2n-1}/\mathbf{G}_{\mathfrak{m}}} \mathbf{SL}_n \to \mathbf{SL}_{n+1}$ *induced by the commutative diagram*

$$\begin{array}{ccc} \Sigma^{(1)}Q_{2n-1}/\mathbf{G}_{\mathfrak{m}} & \longrightarrow & \Sigma^{(1)}Q_{2n+1}/\mathbf{G}_{\mathfrak{m}} \\ \downarrow{\scriptstyle \psi_n} & & \downarrow{\scriptstyle \psi_{n+1}} \\ \mathbf{SL}_n & \longrightarrow & \mathbf{SL}_{n+1} \end{array}$$

is equivalent to $\Sigma^{n+(n+1)}\mathbf{SL}_n$ *over* $\mathrm{Spec}(\mathbb{Z})$.

Proof This follows from Proposition 4.2, Corollary 4.3 and a straightforward manipulation of homotopy pushout squares. □

In the following, $Q_{2n-1}/\mathbf{G}_{\mathfrak{m}}$ will be identified via projection to the first n coordinates with $\mathbf{P}^{n-1}$, which gives rise to maps such as $\psi_n\colon \Sigma^{(1)}\mathbf{P}^{n-1} \simeq \Sigma^{(1)}Q_{2n-1}/\mathbf{G}_{\mathfrak{m}} \to \mathbf{SL}_n$ in the homotopy category. Recall from [6, Convention 2.3.5] that a map f of pointed motivic spaces is $\mathbf{A}^1$-n-connected if its homotopy fiber[3] $\mathrm{hofib}(f)$ is $\mathbf{A}^1$-$(n-1)$-connected, which is equivalent to the homomorphism $\underline{\pi}_j^{\mathbf{A}^1} f$ being an isomorphism for $j < n$ and an epimorphism for $j = n$.

Lemma 4.6 *Let F be a field. The inclusion* $\Sigma^{(1)}\mathbf{P}^1 \to \Sigma^{(1)}\mathbf{P}^2$ *induces the canonical projection*

$$\mathbf{K}_2^{\mathsf{MW}} \cong \underline{\pi}_1^{\mathbf{A}^1}\Sigma^{(1)}\mathbf{P}^1 \to \underline{\pi}_1^{\mathbf{A}^1}\Sigma^{(1)}\mathbf{P}^2 \cong \mathbf{K}_2^{\mathsf{M}}$$

on $\mathbf{A}^1$*-fundamental groups. The inclusion* $\Sigma^{(1)}\mathbf{P}^{n-1} \to \Sigma^{(1)}\mathbf{P}^n$ *is* $\mathbf{A}^1$-$(n-1)$-*connected for all* $n > 0$.

Proof The assumption that F is perfect may be imposed by pulling back from a perfect subfield, over which everything in sight is defined. As a suspension of an $\mathbf{A}^1$-0-connected variety, $\Sigma^{(1)}\mathbf{P}^n$ is $\mathbf{A}^1$-0-connected for all n [6, Lemma 3.3.1]. The determination of $\underline{\pi}_1^{\mathbf{A}^1}\mathbf{P}^n$ from [17, Theorem 7.13 and Theorem 7.29] implies that the map $\underline{\pi}_1^{\mathbf{A}^1}\mathbf{P}^1 \to \underline{\pi}_1^{\mathbf{A}^1}\mathbf{P}^n$ is surjective for all $n > 0$.

[3] One has to take homotopy fibers at all $\mathbf{A}^1$-path components in the case (which will not occur here) that the target motivic space is not $\mathbf{A}^1$-0-connected.

In other words, the inclusion $\mathbf{P}^1 \to \mathbf{P}^n$ is $\mathbf{A}^1$-1-connected for all $n > 1$. While smashing with $\mathbf{G}_{\mathfrak{m}}$ preserves the simplicial connectivity of a map, it is a priori not clear whether smashing with $\mathbf{G}_{\mathfrak{m}}$ preserves the $\mathbf{A}^1$-connectivity of a map. Let $\mathrm{Sing}^{\mathbf{A}^1}$ denote the endofunctor on (pointed) simplicial presheaves on $\mathbf{Sm}_F$ introduced in [21, p. 87]. It commutes with limits and colimits. Since the natural transformation $\mathbf{G}_{\mathfrak{m}} \to \mathrm{Sing}^{\mathbf{A}^1}(\mathbf{G}_{\mathfrak{m}})$ is an isomorphism, there results a natural isomorphism $\Sigma^{(1)}\mathrm{Sing}^{\mathbf{A}^1}(B) \xrightarrow{\cong} \mathrm{Sing}^{\mathbf{A}^1}(\Sigma^{(1)}B)$ for every pointed simplicial presheaf B. Choosing appropriate $\mathbf{A}^1$-naive models for projective spaces – which is possible by [4, Example 4.2.13] – provides a model for the canonical inclusion $\mathbf{P}^1 \to \mathbf{P}^n$ such that $\mathrm{Sing}^{\mathbf{A}^1}(\mathbf{P}^1) \to \mathrm{Sing}^{\mathbf{A}^1}(\mathbf{P}^n)$ is simplicially 1-connected, and hence so is

$$\Sigma^{(1)}\mathrm{Sing}^{\mathbf{A}^1}(\mathbf{P}^1) \cong \mathrm{Sing}^{\mathbf{A}^1}(\Sigma^{(1)}\mathbf{P}^1) \to \mathrm{Sing}^{\mathbf{A}^1}(\Sigma^{(1)}\mathbf{P}^n) \cong \Sigma^{(1)}\mathrm{Sing}^{\mathbf{A}^1}(\mathbf{P}^n). \tag{4.5}$$

The source is stalkwise equivalent to $\mathrm{Sing}^{\mathbf{A}^1}(\mathbf{SL}_2)$ by [4, Lemma 4.2.4]. Since $\mathbf{SL}_2$ is $\mathbf{A}^1$-naive by [4, Theorem 4.2.1] (applied to $Q_3 \cong \mathbf{SL}_2$), there results an isomorphism

$$\underline{\pi}_0^{\mathbf{A}^1}\Omega\Sigma^{(1)}\mathrm{Sing}^{\mathbf{A}^1}(\mathbf{P}^1) \cong \underline{\pi}_1^{\mathbf{A}^1}\Sigma^{(1)}\mathbf{P}^1 \cong \underline{\pi}_1^{\mathbf{A}^1}\mathbf{SL}_2 \cong \mathbf{K}_2^{\mathsf{MW}}$$

where the last isomorphism follows from [17, Theorem 1.27] (see also [18, Theorem 1]). Thus $\underline{\pi}_0^{\mathbf{A}^1}\Omega\Sigma^{(1)}\mathrm{Sing}^{\mathbf{A}^1}(\mathbf{P}^1)$ is a strictly $\mathbf{A}^1$-invariant sheaf. To prove the same for $\underline{\pi}_0^{\mathbf{A}^1}\Omega\Sigma^{(1)}\mathrm{Sing}^{\mathbf{A}^1}(\mathbf{P}^n)$, let G denote the (simplicial) homotopy fiber of the map (4.5). The aforementioned connectivity of the map (4.5) implies that G is simplicially 0-connected, and hence $\mathbf{A}^1$-0-connected by [21, Cor. 2.3.22]. Since $\Omega\Sigma^{(1)}\mathrm{Sing}^{\mathbf{A}^1}(\mathbf{P}^n)$ is the simplicial homotopy fiber of the canonical map $G \to \Sigma^{(1)}\mathrm{Sing}^{\mathbf{A}^1}(\mathbf{P}^1)$, [17, Theorem 6.56] (see also [6, Corollary 2.3.6]) provides an exact sequence

$$\underline{\pi}_1^{\mathbf{A}^1}G \to \underline{\pi}_1^{\mathbf{A}^1}\Sigma^{(1)}\mathrm{Sing}^{\mathbf{A}^1}(\mathbf{P}^1) \to \underline{\pi}_0^{\mathbf{A}^1}\Omega\Sigma^{(1)}\mathrm{Sing}^{\mathbf{A}^1}(\mathbf{P}^n) \to \underline{\pi}_0^{\mathbf{A}^1}G = 0$$

using that $\underline{\pi}_0^{\mathbf{A}^1}\Omega\Sigma^{(1)}\mathrm{Sing}^{\mathbf{A}^1}(\mathbf{P}^1)$ is a strictly $\mathbf{A}^1$-invariant sheaf. In this exact sequence $\underline{\pi}_1^{\mathbf{A}^1}G$ is strongly $\mathbf{A}^1$-invariant by [17, Theorem 6.1], its image in $\underline{\pi}_1^{\mathbf{A}^1}\Sigma^{(1)}\mathrm{Sing}^{\mathbf{A}^1}(\mathbf{P}^1)$ is strongly $\mathbf{A}^1$-invariant by [8, Theorem 1.6], and moreover abelian, and the cokernel $\underline{\pi}_0^{\mathbf{A}^1}\Omega\Sigma^{(1)}\mathrm{Sing}^{\mathbf{A}^1}(\mathbf{P}^n)$ is then a strictly $\mathbf{A}^1$-invariant sheaf by [17, Corollary 6.24]. As a consequence [17, Theorem 6.56] applies to show that $\Sigma^{(1)}\mathbf{P}^1 \to \Sigma^{(1)}\mathbf{P}^n$ is $\mathbf{A}^1$-1-connected.

To determine $\underline{\pi}_1^{\mathbf{A}^1}\Sigma^{(1)}\mathbf{P}^n$ for $n > 1$, Morel's $\mathbf{A}^1$-Hurewicz theorem [17, Theorem 6.35] implies that the Hurewicz transformation $\underline{\pi}_1^{\mathbf{A}^1}\Sigma^{(1)}\mathbf{P}^n \to \underline{H}_1^{\mathbf{A}^1}\Sigma^{(1)}\mathbf{P}^n$ is an isomorphism for all n. For $n = 2$, the latter can be determined via the cofiber sequence

$$S^{1+(3)} \xrightarrow{\eta} \Sigma^{(1)}\mathbf{P}^1 \to \Sigma^{(1)}\mathbf{P}^2$$

as $\underline{H}_1^{\mathbf{A}^1}\Sigma^{(1)}\mathbf{P}^2 \cong \mathbf{K}_2^{\mathsf{M}}$. Similarly, for $n > 2$ the inclusion $\mathbf{P}^{n-1} \hookrightarrow \mathbf{P}^n$ is $\mathbf{A}^1$-$(n-1)$-connected, as one may deduce from the homotopy fiber of the "covering" map $\mathbf{A}^n \smallsetminus \{0\} \hookrightarrow \mathbf{A}^{n+1} \smallsetminus \{0\}$ of universal $\mathbf{A}^1$-coverings. Again smashing with $\mathbf{G}_{\mathfrak{m}}$ preserves the simplicial connectivity of $\mathbf{P}^{n-1} \hookrightarrow \mathbf{P}^n$. Arguing with $\mathbf{A}^1$-naive models and the functor $\mathrm{Sing}^{\mathbf{A}^1}$ as before provides that $\Sigma^{(1)}\mathbf{P}^{n-1} \to \Sigma^{(1)}\mathbf{P}^n$ is $\mathbf{A}^1$-$(n-1)$-connected, again invoking [17, Theorem 6.56] or [6, Corollary 2.3.6] and the strict $\mathbf{A}^1$-invariance of $\underline{\pi}_0^{\mathbf{A}^1}\Omega\Sigma^{(1)}\mathrm{Sing}^{\mathbf{A}^1}(\mathbf{P}^n)$ already established. □

The pushout in diagram (4.2) gives rise to a map

$$\theta_{n+1}\colon \Sigma^{(1)}\mathbf{P}^n \cup_{\Sigma^{(1)}\mathbf{P}^{n-1}} \mathbf{SL}_n \to \mathbf{SL}_{n+1} \tag{4.6}$$

which factors $\psi_{n+1}\colon \Sigma^{(1)}\mathbf{P}^n \to \mathbf{SL}_{n+1}$ for every $n > 0$.

Proposition 4.7 *Let F be a field. The image of the homomorphism $\underline{\pi}_n^{\mathbf{A}^1}\mathbf{SL}_{n+1} \to \underline{\pi}_n^{\mathbf{A}^1}S^{n+(n+1)}$ induced by taking the last column of a matrix is isomorphic to the image of the homomorphism $\underline{\pi}_n^{\mathbf{A}^1}\Sigma^{(1)}\mathbf{P}^n \cup_{\Sigma^{(1)}\mathbf{P}^{n-1}} \mathbf{SL}_n \to \underline{\pi}_n^{\mathbf{A}^1}S^{n+(n+1)}$ induced by the canonical quotient map collapsing $\mathbf{SL}_n$ to the basepoint.*

Proof In case $n = 1$, both the last column map and the quotient map are equivalences, whence the induced homomorphisms are isomorphisms. Let $n > 1$. Diagram (4.2) and the map θ_{n+1} defined in (4.6) induce the following commutative diagram

$$\begin{tikzcd}
\underline{\pi}_n^{\mathbf{A}^1}\Sigma^{(1)}\mathbf{P}^{n-1} \arrow[r] \arrow[d, "\underline{\pi}_n^{\mathbf{A}^1}\psi_n"] & \underline{\pi}_n^{\mathbf{A}^1}\Sigma^{(1)}\mathbf{P}^n \arrow[r] \arrow[d] & \underline{\pi}_n^{\mathbf{A}^1}S^{n+(n+1)} \arrow[d, "\cong"] \\
\underline{\pi}_n^{\mathbf{A}^1}\mathbf{SL}_n \arrow[r] \arrow[d, "\mathrm{id}"] & \underline{\pi}_n^{\mathbf{A}^1}\Sigma^{(1)}\mathbf{P}^n \cup_{\Sigma^{(1)}\mathbf{P}^{n-1}} \mathbf{SL}_n \arrow[r] \arrow[d, "\underline{\pi}_n^{\mathbf{A}^1}\theta_{n+1}"] & \underline{\pi}_n^{\mathbf{A}^1}S^{n+(n+1)} \arrow[d, "\cong"] \\
\underline{\pi}_n^{\mathbf{A}^1}\mathbf{SL}_n \arrow[r] & \underline{\pi}_n^{\mathbf{A}^1}\mathbf{SL}_{n+1} \arrow[r] & \underline{\pi}_n^{\mathbf{A}^1}S^{n+(n+1)}
\end{tikzcd} \tag{4.7}$$

of sheaves of $\mathbf{A}^1$-homotopy groups, in which the vertical homomorphisms on the right hand side are isomorphisms by Corollary 4.3 and by construction. In particular, the image of the homomorphism $\underline{\pi}_n^{\mathbf{A}^1}\Sigma^{(1)}\mathbf{P}^n \cup_{\Sigma^{(1)}\mathbf{P}^{n-1}} \mathbf{SL}_n \to \underline{\pi}_n^{\mathbf{A}^1} S^{n+(n+1)}$ embeds in the image of the homomorphism $\underline{\pi}_n^{\mathbf{A}^1}\mathbf{SL}_{n+1} \to \underline{\pi}_n^{\mathbf{A}^1} S^{n+(n+1)}$. To prove that the image of $\underline{\pi}_n^{\mathbf{A}^1}\mathbf{SL}_{n+1} \to \underline{\pi}_n^{\mathbf{A}^1} S^{n+(n+1)}$ coincides with the image of $\underline{\pi}_n^{\mathbf{A}^1}\Sigma^{(1)}\mathbf{P}^n \cup_{\Sigma^{(1)}\mathbf{P}^{n-1}} \mathbf{SL}_n \to \underline{\pi}_n^{\mathbf{A}^1} S^{n+(n+1)}$ induced by collapsing $\mathbf{SL}_n$, it suffices to prove that the homomorphism $\underline{\pi}_n^{\mathbf{A}^1}\theta_{n+1}\colon \underline{\pi}_n^{\mathbf{A}^1}\Sigma^{(1)}\mathbf{P}^n \cup_{\Sigma^{(1)}\mathbf{P}^{n-1}} \mathbf{SL}_n \to \underline{\pi}_n^{\mathbf{A}^1}\mathbf{SL}_{n+1}$ is surjective. This in turn follows if the map θ_{n+1} is $\mathbf{A}^1$-n-connected. This is the connectivity of its homotopy cofiber $\mathrm{cone}(\theta_{n+1})$ by Lemma 4.5. Unfortunately this does not necessarily imply that its homotopy fiber $\mathrm{hofib}(\theta_{n+1})$ is $\mathbf{A}^1$-$(n-1)$-connected. To conclude this nevertheless, start with the case $n = 2$. Then the map θ_3 in question coincides with $\psi_3\colon \Sigma^{(1)}\mathbf{P}^2 \to \mathbf{SL}_3$ up to equivalence by Lemma 4.1. Lemma 4.6 implies together with the determination of $\underline{\pi}_1^{\mathbf{A}^1}\mathbf{SL}_n$ from [18, Theorem 1] that $\underline{\pi}_1^{\mathbf{A}^1}\psi_3$ is an isomorphism. There results an exact sequence

$$\underline{\pi}_2^{\mathbf{A}^1}\Sigma^{(1)}\mathbf{P}^2 \xrightarrow{\underline{\pi}_2^{\mathbf{A}^1}\psi_3} \underline{\pi}_2^{\mathbf{A}^1}\mathbf{SL}_3 \to \underline{\pi}_1^{\mathbf{A}^1}\mathrm{hofib}(\psi_3) \to 0$$

whence it remains to prove that $\underline{\pi}_2^{\mathbf{A}^1}\psi_3$ is an epimorphism. Consider the commutative diagram

$$\begin{array}{ccccccccccc}
\underline{\pi}_2^{\mathbf{A}^1}\Sigma^{(1)}\mathbf{P}^1 & \longrightarrow & \underline{\pi}_2^{\mathbf{A}^1}\Sigma^{(1)}\mathbf{P}^2 & \longrightarrow & \underline{\pi}_2^{\mathbf{A}^1}S^{2+(3)} & \longrightarrow & \underline{\pi}_1^{\mathbf{A}^1}\Sigma^{(1)}\mathbf{P}^1 & \longrightarrow & \underline{\pi}_1^{\mathbf{A}^1}\Sigma^{(1)}\mathbf{P}^2 & \longrightarrow & 0 \\
\downarrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \psi_3} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \psi_3} & & \\
\underline{\pi}_2^{\mathbf{A}^1}\mathbf{SL}_2 & \longrightarrow & \underline{\pi}_2^{\mathbf{A}^1}\mathbf{SL}_3 & \longrightarrow & \underline{\pi}_2^{\mathbf{A}^1}\mathbf{A}^3 \smallsetminus \{0\} & \longrightarrow & \underline{\pi}_1^{\mathbf{A}^1}\mathbf{SL}_2 & \longrightarrow & \underline{\pi}_1^{\mathbf{A}^1}\mathbf{SL}_3 & \longrightarrow & 0
\end{array} \tag{4.8}$$

induced by diagram (4.2) in which the bottom row is induced by the homotopy fiber sequence

$$\mathbf{SL}_2 \to \mathbf{SL}_3 \xrightarrow{\text{last column}} \mathbf{A}^3 \smallsetminus \{0\}$$

and in particular exact. The top row in diagram (4.8) is induced by the homotopy cofiber sequence

$$\Sigma^{(1)}\mathbf{P}^1 \to \Sigma^{(2)}\mathbf{P}^2 \to S^{2+(3)}$$

and the weak equivalence $\psi_2\colon \Sigma^{(1)}\mathbf{P}^1 \to \mathbf{SL}_2$ from Lemma 4.1, in the sense that the homomorphism $\underline{\pi}_2^{\mathbf{A}^1}S^{2+(3)} \to \underline{\pi}_1^{\mathbf{A}^1}\Sigma^{(1)}\mathbf{P}^1$ is the composition of the inverse of $\underline{\pi}_1^{\mathbf{A}^1}\psi_2$ and $\underline{\pi}_2^{\mathbf{A}^1}S^{2+(3)} \cong \underline{\pi}_2^{\mathbf{A}^1}\mathbf{A}^3 \smallsetminus \{0\} \to \underline{\pi}_1^{\mathbf{A}^1}\mathbf{SL}_2$. Here the first isomorphism follows from Corollary 4.3. It follows that the upper row in diagram (4.8) is exact at the spots involving $\underline{\pi}_1^{\mathbf{A}^1}$. To conclude exactness at

$\underline{\pi}_2^{\mathbf{A}^1} S^{2+(3)}$, it remains to prove that the image of $\underline{\pi}_2^{\mathbf{A}^1} \Sigma^{(1)}\mathbf{P}^2 \to \underline{\pi}_2^{\mathbf{A}^1} S^{2+(3)}$ contains the kernel of $\underline{\pi}_2^{\mathbf{A}^1} S^{2+(3)} \to \underline{\pi}_1^{\mathbf{A}^1} \Sigma^{(1)}\mathbf{P}^1$. By construction of the homotopy cofiber sequence, the latter coincides with the kernel of $\eta\colon \mathbf{K}_3^{\mathsf{MW}} \to \mathbf{K}_2^{\mathsf{MW}}$, which is $\mathsf{h}\mathbf{K}_3^{\mathsf{MW}}$ by Theorem 3.2. The map

$$\mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^2, \ \big((a_0 : a_1), (b_0 : b_1)\big) \mapsto (a_0b_0 : a_0b_1 + a_1b_0 : a_1b_1)$$

induces a commutative diagram

$$\begin{tikzcd}
\mathbf{P}^1 \times \mathbf{P}^1 \arrow[r] \arrow[d, "\text{can.}"] & \mathbf{P}^2 \arrow[d, "\text{can.}"] \\
\mathbf{P}^1 \wedge \mathbf{P}^1 \arrow[r, "\mathsf{h}"] & \mathbf{P}^2/\mathbf{P}^1
\end{tikzcd}$$

in which the identification of the lower horizontal map can be deduced from $\mathbf{P}^1$-stabilizing first, then observing that its $\mathbf{P}^1$-stabilization is in the kernel of multiplication with η on the Grothendieck-Witt ring (hence an integer multiple of h by Theorem 3.2), and finally a degree argument using motivic cohomology or realization, showing that the rank of the integer multiple of h is 2. Morel's Theorem [17, Theorem 1.23] then implies that the image of the homomorphism $\underline{\pi}_2^{\mathbf{A}^1} \Sigma^{(1)}\mathbf{P}^1 \times \mathbf{P}^1 \to \underline{\pi}_2^{\mathbf{A}^1} \Sigma^{(1)}\mathbf{P}^2 \to \underline{\pi}_2^{\mathbf{A}^1} S^{2+(3)}$ contains $\mathsf{h}\mathbf{K}_3^{\mathsf{MW}}$. In particular, the top row in diagram (4.8) is also exact at $\underline{\pi}_2^{\mathbf{A}^1} S^{2+(3)}$. Together with the isomorphism $\underline{\pi}_2^{\mathbf{A}^1}\psi_2$ from Lemma 4.1, a diagram chase then provides that $\underline{\pi}_2^{\mathbf{A}^1}\psi_3$ is an epimorphism.

Diagram (4.2) furthermore implies that ψ_{n+1} is at least $\mathbf{A}^1$-2-connected for all $n > 1$. Invoking the $\mathbf{A}^1$-van Kampen theorem [30, Theorem 3.10], [17, Theorem 7.12] provides with Lemma 4.6 that $\underline{\pi}_1^{\mathbf{A}^1}\theta_{n+1}$ is an isomorphism for all $n > 1$. Hence $\theta_{n+1}\colon \Sigma^{(1)}\mathbf{P}^n \cup_{\Sigma^{(1)}\mathbf{P}^{n-1}} \mathbf{SL}_n \to \mathbf{SL}_{n+1}$ is $\mathbf{A}^1$-2-connected as well. To reach further, let $n > 2$ and consider the transformation

$$\begin{tikzcd}
\Omega S^{n+(n+1)} \arrow[r] \arrow[d] & \mathbf{SL}_n \arrow[r] \arrow[d] & \mathbf{SL}_{n+1} \arrow[d, "\mathrm{id}"] \\
\mathrm{hofib}(\theta_{n+1}) \arrow[r] & \Sigma^{(1)}\mathbf{P}^n \cup_{\Sigma^{(1)}\mathbf{P}^{n-1}} \mathbf{SL}_n \arrow[r, "\theta_{n+1}"] & \mathbf{SL}_{n+1}
\end{tikzcd} \tag{4.9}$$

of homotopy fiber sequences, inducing a transformation of long exact sequences of homotopy groups. The vertical map in the middle of diagram (4.9) is $\mathbf{A}^1$-$(n-1)$-connected. In fact, as the cobase change of the map $\Sigma^{(1)}\mathbf{P}^{n-1} \to \Sigma^{(1)}\mathbf{P}^n$ which is $\mathbf{A}^1$-$(n-1)$-connected by Lemma 4.6, it is simplicially $(n-1)$-connected. To apply [17, Theorem 6.56] or [6, Corollary 2.3.6] and conclude the desired $\mathbf{A}^1$-connectivity, it remains to prove that

$\underline{\pi}_0^{\mathbf{A}^1}\Omega\Sigma^{(1)}\mathbf{P}^n\cup_{\Sigma^{(1)}\mathbf{P}^{n-1}}\mathbf{SL}_n$ is strongly $\mathbf{A}^1$-invariant. The simplicial van Kampen theorem [30, Corollary 3.5] provides that the Nisnevich sheaf associated with the presheaf of fundamental groups of $\mathrm{Sing}^{\mathbf{A}^1}(\Sigma^{(1)}\mathbf{P}^n)\cup_{\mathrm{Sing}^{\mathbf{A}^1}(\Sigma^{(1)}\mathbf{P}^{n-1})}\mathrm{Sing}^{\mathbf{A}^1}(\mathbf{SL}_n)$ (using $\mathbf{A}^1$-naiveness provided by [17, Theorem 8.1] for $\mathbf{GL}_n$ and [4, Example 4.2.13]) coincides with $\mathbf{K}_2^{\mathsf{M}}$. In particular, it is strictly $\mathbf{A}^1$-invariant.

The long exact sequence diagram (4.9) induces on $\mathbf{A}^1$-homotopy groups then provides epimorphisms $\underline{\pi}_{j+1}^{\mathbf{A}^1}S^{n+(n+1)}\cong\underline{\pi}_j^{\mathbf{A}^1}\Omega S^{n+(n+1)}\to\underline{\pi}_j^{\mathbf{A}^1}\operatorname{hofib}(\theta_{n+1})$ for all $j<n$. In particular, Morel's $\mathbf{A}^1$-connectivity for $S^{n+(n+1)}$ provides $\underline{\pi}_j^{\mathbf{A}^1}\operatorname{hofib}(\theta_{n+1})=0$ for $j<n-1$, so that θ_{n+1} is at least $\mathbf{A}^1$-$(n-1)$-connected. To conclude the vanishing of $\underline{\pi}_{n-1}^{\mathbf{A}^1}\operatorname{hofib}(\theta_{n+1})$, observe that it is a quotient of $\underline{\pi}_n^{\mathbf{A}^1}S^{n+(n+1)}\cong\mathbf{K}_{n+1}^{\mathsf{MW}}$. The $\mathbf{A}^1$-homotopy groups induced by diagram (4.9) are modules over $\underline{\pi}_1^{\mathbf{A}^1}\mathbf{SL}_n\cong\underline{\pi}_1^{\mathbf{A}^1}(\Sigma^{(1)}\mathbf{P}^n\cup_{\Sigma^{(1)}\mathbf{P}^{n-1}}\mathbf{SL}_n)\cong\mathbf{K}_2^{\mathsf{M}}$, and all homomorphisms involved are $\mathbf{K}_2^{\mathsf{M}}$-equivariant. However, the action of $\underline{\pi}_1^{\mathbf{A}^1}\mathbf{SL}_n$ on $\underline{\pi}_j^{\mathbf{A}^1}\Omega S^{n+(n+1)}$ is trivial because it factors through $\underline{\pi}_1^{\mathbf{A}^1}*$, as the commutative diagram

$$\begin{tikzcd}
\Omega S^{n+(n+1)} \arrow[r] \arrow[d, "\mathrm{id}"] & \mathbf{SL}_n \arrow[r] \arrow[d] & \mathbf{SL}_{n+1} \arrow[d] \\
\Omega S^{n+(n+1)} \arrow[r] & * \arrow[r] & S^{n+(n+1)}
\end{tikzcd}$$

of fiber sequences implies. Hence the action of $\underline{\pi}_1^{\mathbf{A}^1}(\Sigma^{(1)}\mathbf{P}^n\cup_{\Sigma^{(1)}\mathbf{P}^{n-1}}\mathbf{SL}_n)$ on the quotient $\underline{\pi}_j^{\mathbf{A}^1}\operatorname{hofib}(\theta_{n+1})$ of $\underline{\pi}_j^{\mathbf{A}^1}\Omega S^{n+(n+1)}$ is also trivial. It follows that the relative Hurewicz homomorphism $\underline{\pi}_{n-1}^{\mathbf{A}^1}\operatorname{hofib}(\theta_{n+1})\to\underline{H}_n^{\mathbf{A}^1}\operatorname{cone}(\theta_{n+1})$ introduced in [2, Section 4.2] is an isomorphism by [2, Theorem 4.2.1]. The latter group is trivial by Lemma 4.5. Hence θ_{n+1} is $\mathbf{A}^1$-n-connected. □

Theorem 4.8 *Let F be an infinite field of characteristic different from 2 and 3, and A an essentially smooth local F-algebra. The image of the Suslin-Hurewicz homomorphism $K_4^{\mathrm{Quillen}}(A)\to\mathbf{K}_4^{\mathsf{M}}(A)$ coincides with $6\mathbf{K}_4^{\mathsf{M}}(A)$.*

Proof By [3, Theorem 2.18] it suffices to prove that a certain canonical surjection $\mathbf{K}_4^{\mathsf{M}}/6\to\mathbf{S}_4$, obtained from Suslin's Hurewicz homomorphism, is an isomorphism. One description of $\mathbf{S}_4$ goes as follows. The $\mathbf{A}^1$-fiber sequence

$$S^{3+(4)}\to\mathrm{B}\mathbf{SL}_3\to\mathrm{B}\mathbf{SL}_4$$

of motivic spaces induces a long exact sequence of unstable homotopy sheaves terminating with

$$\cdots \to \underline{\pi}_3^{\mathbf{A}^1}\mathbf{SL}_4 \to \underline{\pi}_3^{\mathbf{A}^1} S^{3+(4)} \to \underline{\pi}_3^{\mathbf{A}^1}\mathrm{B}\mathbf{SL}_3 \to \underline{\pi}_3^{\mathbf{A}^1}\mathrm{B}\mathbf{SL}_4 \to 0. \quad (4.10)$$

The sheaf $\mathbf{S}_4$ is isomorphic to the kernel of $\underline{\pi}_3^{\mathbf{A}^1}\mathrm{B}\mathbf{SL}_3 \to \underline{\pi}_3^{\mathbf{A}^1}\mathrm{B}\mathbf{SL}_4$ by [1, Lemma 3.5]. Homotopical stability provides that $\underline{\pi}_3^{\mathbf{A}^1}\mathrm{B}\mathbf{SL}_4 \cong \underline{\pi}_3^{\mathbf{A}^1}\mathrm{B}\mathbf{SL}_\infty \cong \mathbf{K}_3^{\mathsf{Q}}$ is the third algebraic K-theory sheaf, as explained in [3]. Morel's unstable computations provide $\underline{\pi}_3^{\mathbf{A}^1} S^{3+(4)} \cong \mathbf{K}_4^{\mathsf{MW}}$ [17, Theorem 1.23]. Hence there exists a surjection $\mathbf{K}_4^{\mathsf{MW}} \cong \underline{\pi}_3^{\mathbf{A}^1} S^{3+(4)} \to \mathbf{S}_4$, which, as is also explained in [3], factors over $\mathbf{K}_4^{\mathsf{M}}/6$. In the case where F is a field of characteristic $p > 3$, the map $\mathbf{K}_4^{\mathsf{M}}/6 \to \mathbf{K}_4^{\mathsf{M}}[p^{-1}]/6$ is injective by [13]. Therefore it suffices to prove that the induced map $\mathbf{K}_4^{\mathsf{M}}[p^{-1}]/6 \to \mathbf{S}_4[p^{-1}]$ is injective. In the following, the characteristic p of the field F may be implicitly inverted if $p > 3$.

Using the exact sequence (4.10), the image of $\underline{\pi}_3^{\mathbf{A}^1}\mathbf{SL}_4 \to \underline{\pi}_3^{\mathbf{A}^1} S^{3+(4)}$ (induced by taking the last column of a matrix in $\mathbf{SL}_4$) contains at least the subsheaves $6\mathbf{K}_4^{\mathsf{MW}}$ and $\eta\mathbf{K}_5^{\mathsf{MW}}$, and hence their sum. Proposition 4.7 shows that this image coincides with the image of $\underline{\pi}_3^{\mathbf{A}^1}\Sigma^{(1)}\mathbf{P}^3 \cup_{\Sigma^{(1)}\mathbf{P}^2} \mathbf{SL}_3 \xrightarrow{\underline{\pi}_3^{\mathbf{A}^1}\theta_4} \underline{\pi}_3^{\mathbf{A}^1}\mathbf{SL}_4 \to \underline{\pi}_3^{\mathbf{A}^1} S^{3+(4)}$. Passage to motivic suspension spectra simplifies the situation via the following commutative diagram

$$\begin{array}{ccc}
\underline{\pi}_3^{\mathbf{A}^1}\Sigma^{(1)}\mathbf{P}^3 \cup_{\Sigma^{(1)}\mathbf{P}^2} \mathbf{SL}_3 & \longrightarrow & \underline{\pi}_3^{\mathbf{A}^1} S^{3+(4)} \\
\downarrow & & \downarrow \\
\underline{\pi}_{3+(0)}\Sigma^{(1)}\mathbf{P}^3 \cup_{\Sigma^{(1)}\mathbf{P}^2} \mathbf{SL}_3 & \longrightarrow & \underline{\pi}_{3+(0)} S^{3+(4)}
\end{array}$$

in which the vertical homomorphisms are induced by taking $\mathbf{P}^1$-suspension spectra. In particular, the vertical homomorphism on the right hand side is an isomorphism by Morel's theorem. Hence the image in question is contained in the image of the homomorphism $\underline{\pi}_{3+(0)}\Sigma^{(1)}\mathbf{P}^3 \cup_{\Sigma^{(1)}\mathbf{P}^2} \mathbf{SL}_3 \to \underline{\pi}_{3+(0)} S^{3+(4)}$ which coincides with the kernel of the homomorphism $\underline{\pi}_{3+(0)} S^{3+(4)} \to \underline{\pi}_{2+(0)}\mathbf{SL}_3$ because of the cofiber sequence

$$\mathbf{SL}_3 \to \Sigma^{(1)}\mathbf{P}^3 \cup_{\Sigma^{(1)}\mathbf{P}^2} \mathbf{SL}_3 \to S^{3+(4)}.$$

Using the map $\psi_3\colon \Sigma^{(1)}\mathbf{P}^2 \to \mathbf{SL}_3$, this kernel contains the kernel of $\underline{\pi}_{3+(0)} S^{3+(4)} \to \underline{\pi}_{2+(0)}\Sigma^{(1)}\mathbf{P}^2$. These kernels are equal, because the map $\underline{\pi}_{2+(0)}\psi_3\colon \underline{\pi}_{2+(0)}\Sigma^{(1)}\mathbf{P}^2 \to \underline{\pi}_{2+(0)}\mathbf{SL}_3$ is not only surjective (a straightforward consequence of Lemma 4.4), but injective as well (at least after inverting the exponential characteristic of F if it is odd). The latter can be seen from Lemma 4.4 which supplies a homotopy cofiber sequence

$$\Sigma^{(1)}\mathbf{P}^2 \xrightarrow{\psi_3} \mathbf{SL}_3 \to S^{3+(5)}.$$

It induces a map $S^{3+(5)} \to \Sigma^{1+(1)}\mathbf{P}^2$ whose composition with the canonical quotient map $\Sigma^{1+(1)}\mathbf{P}^2 \to S^{3+(3)}$ gives rise to an element in the Witt ring $\underline{\pi}_{3+(5)}S^{3+(3)} \cong \mathbf{K}^{\mathsf{MW}}_{-2}$ by Morel's theorem. This element is the zero element, because the image of $\underline{\pi}_{3+(5)}\Sigma^{1+(1)}\mathbf{P}^2 \to \underline{\pi}_{3+(5)}S^{3+(3)}$ is zero by the short exact sequence (3.3). Hence the map $S^{3+(5)} \to \Sigma^{1+(1)}\mathbf{P}^2$ factors over the inclusion $\Sigma^{1+(1)}\mathbf{P}^1 \to \Sigma^{1+(1)}\mathbf{P}^2$ and gives rise to an element in the group $\underline{\pi}_{3+(5)}S^{2+(2)} = \underline{\pi}_{1+(3)}\mathbf{1}_F$ which is trivial by [23, Theorem 5.5] after inverting the characteristic of F if it is odd.

To summarize, the image of the homomorphism $\underline{\pi}_3^{\mathbf{A}^1}\mathbf{SL}_4 \to \underline{\pi}_3^{\mathbf{A}^1}S^{3+(4)}$ of unstable $\mathbf{A}^1$-homotopy sheaves is contained in the kernel of the homomorphism of stable $\mathbf{A}^1$-homotopy sheaves $\underline{\pi}_{3+(0)}S^{3+(4)} \to \underline{\pi}_{2+(0)}\Sigma^{(1)}\mathbf{P}^2$. Using $\mathbf{P}^1$-stability, the latter homomorphism can be identified with the homomorphism $\underline{\pi}_{2-(1)}S^{2+(3)} \to \underline{\pi}_{2-(1)}\mathbf{P}^2$ induced by the attaching map $\gamma_2\colon \mathbf{A}^3 \smallsetminus \{0\} \to \mathbf{P}^2$ of smooth schemes because of the cofibration sequence

$$\mathbf{A}^3 \smallsetminus \{0\} \xrightarrow{\gamma_2} \mathbf{P}^2 \hookrightarrow \mathbf{P}^3.$$

The composition

$$S^{2+(3)} \xrightarrow{\gamma_2} \mathbf{P}^2 \xrightarrow{q} S^{2+(2)}$$

is nullhomotopic essentially because 2 is even. More precisely, by Morel's Theorem, it suffices to prove this after applying $\Sigma^{(1)}$. The composition of the attaching map $\Sigma^{(1)}\gamma_2$ with the canonical inclusion to $\Sigma^{(1)}\mathbf{P}^3$ is nullhomotopic by construction, hence also the composition with $\psi_4\colon \Sigma^{(1)}\mathbf{P}^3 \to \mathbf{SL}_4$. By the commutative diagram appearing in Corollary 4.3 and exactness, the composition $\psi_3 \circ \Sigma^{(1)}\gamma_2$ is in the image of the connecting map $\delta_3\colon \underline{\pi}^{\mathbf{A}^1}_{3+(4)}S^{3+(4)} \to \underline{\pi}^{\mathbf{A}^1}_{2+(4)}\mathbf{SL}_3$. The result follows from [1, Lemma 3.5] and Corollary 4.3. Hence the attaching map γ_2 corresponds $\mathbf{P}^1$-stably to an element in the image of $\pi_{2+(3)}\mathbf{P}^1 \to \pi_{2+(3)}\mathbf{P}^2$. This element is $\gamma_2 = \pm 2(i\nu) \in \pi_{2+(3)}\mathbf{P}^2 \cong \mathbb{Z}/12\{i\nu\}$ by Remark 3.5. It follows that the kernel of the induced homomorphism

$$\mathbf{K}^{\mathsf{MW}}_{\star} \cong \underline{\pi}_{2-(\star-3)}S^{2+(3)} \xrightarrow{\gamma_2 = 2i\circ\nu} (\mathbf{K}^{\mathsf{M}}_{\star}/12)\{i \circ \nu\} \hookrightarrow \underline{\pi}_{2-(\star-3)}\mathbf{P}^2$$

is generated by η and 6 as a $\mathbf{K}^{\mathsf{MW}}$-module. In particular, in the relevant degree, the image of $\underline{\pi}_3^{\mathbf{A}^1}\mathbf{SL}_4 \to \underline{\pi}_3^{\mathbf{A}^1}S^{3+(4)}$ is contained in the subsheaf $\eta\mathbf{K}^{\mathsf{MW}}_5 +$

$6\mathbf{K}_4^{\mathsf{MW}}$. Since it also contains this subsheaf, equality follows. This completes the proof. □

Corollary 4.9 *Let F be an infinite field with characteristic coprime to* 6*. The inclusion* $\mathrm{B}\mathbf{SL}_3 \hookrightarrow \mathrm{B}\mathbf{SL}_\infty$ *induces the following short exact sequence of sheaves:*

$$0 \to \mathbf{K}_4^{\mathsf{M}}/6 \to \underline{\pi}_3^{\mathbf{A}^1}\mathrm{B}\mathbf{SL}_3 \to \mathbf{K}_3^{\mathrm{Quillen}} \to 0$$

Proof This is a single case of [1, Theorem 1.1], having identified $\mathbf{S}_4$ with $\mathbf{K}_4^{\mathsf{M}}/6$ in the proof of Theorem 4.8. □

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Appendix A: modules over Milnor–Witt K-theory

Let F be a field, usually suppressed from the notation. The Milnor-Witt K-theory of F, as defined by Hopkins-Morel [15], is the unital graded associative algebra generated by $\eta \in \mathbf{K}_{-1}^{\mathsf{MW}} \cong \pi_{1,1}\mathbf{1}$ and the symbols $[u] \in \mathbf{K}_1^{\mathsf{MW}}(F) \cong \pi_{-1,-1}\mathbf{1}_F$ for every unit $u \in F^\times$, subject to the following relations.

Steinberg relation $[u][v] = 0$ whenever $u + v = 1$.
η-twisted logarithm $[uv] = [u] + [v] + \eta[u][v]$
Commutativity $[u]\eta = \eta[u]$
Hyperbolic plane $\eta + \eta^2[-1] = -\eta$

If $u_1, \ldots, u_m \in F$ are units, then the element $(1 + \eta[u_1]) + \cdots + (1 + \eta[u_m]) \in \mathbf{K}_0^{\mathsf{MW}}(F)$ corresponds to the quadratic form $\langle u_1, \ldots, u_m \rangle$ given by the appropriate diagonal matrix under the identification of $\mathbf{K}_0^{\mathsf{MW}}(F)$ with the Grothendieck-Witt ring $\mathbf{GW}(F)$ of F. Milnor K-theory [14] is expressed as the quotient $\mathbf{K}_\star^{\mathsf{M}} \cong \mathbf{K}_\star^{\mathsf{MW}}/\eta\mathbf{K}_\star^{\mathsf{MW}}$. Often the grading will be suppressed from the notation. Its reduction modulo 2 is denoted $\mathbf{k}^{\mathsf{M}}$ for brevity. If A is a (graded) $\mathbf{K}^{\mathsf{MW}}$-module, its degree d part is A_d. For any $x \in \mathbf{K}_d^{\mathsf{MW}}$, the kernel and cokernel of multiplication with x on A_ℓ are denoted ${}_xA_\ell$ and $A_\ell/xA_{\ell-d}$. As a warm-up, $\mathbf{K}^{\mathsf{M}}$-modules will be treated first.

Theorem A.1 *There is an equality* $\{-1\}\mathbf{K}^{\mathsf{M}} = {}_2\mathbf{K}^{\mathsf{M}}$ *of* $\mathbf{K}^{\mathsf{M}}$*-modules.*

Proof This follows as an equality of zero modules from [13] if F has characteristic 2. Suppose now that the characteristic of F is different from 2. Since $2\{-1\} = \{1\} = 0 \in \mathbf{K}_1^{\mathsf{M}}$, there is an inclusion $\{-1\}\mathbf{K}^{\mathsf{M}} \subset {}_2\mathbf{K}^{\mathsf{M}}$ of $\mathbf{K}^{\mathsf{M}}$-modules. The cofiber sequence

$$\mathbf{M}\mathbb{Z} \xrightarrow{2} \mathbf{M}\mathbb{Z} \xrightarrow{\mathrm{pr}_2^\infty} \mathbf{M}\mathbb{Z}/2 \xrightarrow{\partial_\infty^2} \Sigma\mathbf{M}\mathbb{Z}$$

induces a short exact sequence

$$0 \to \pi_{1+(\star)}\mathbf{M}\mathbb{Z}/2\pi_{1+(\star)}\mathbf{M}\mathbb{Z} \to \pi_{1+(\star)}\mathbf{M}\mathbb{Z}/2 \xrightarrow{\partial_\infty^2} {}_2\pi_{0+(\star)}\mathbf{M}\mathbb{Z} \to 0 \tag{A.1}$$

of $\mathbf{K}^{\mathsf{M}}$-modules. Let $\tau \in \pi_{1-(1)}\mathbf{M}\mathbb{Z}/2 = h^{0,1}$ denote the class of $-1 \in F$. Then $\partial_\infty^2(\tau) = \{-1\} \in \pi_{0-(1)}\mathbf{M}\mathbb{Z} = \mathbf{K}_1^{\mathsf{M}}$ by inspecting the short exact sequence (A.1) in the given weight, using that -1 is the unique nontrivial unit of order two in F. Voevodsky's solution of the Milnor conjecture on Galois cohomology [28] (or rather the Rost-Voevodsky solution of the Beilinson-Lichtenbaum conjecture relating motivic and étale cohomology with coefficients in $\mathbb{Z}/2$) provides that multiplication with τ is an isomorphism $\pi_{0+(\star)}\mathbf{M}\mathbb{Z}/2 \cong \pi_{1+(\star)}\mathbf{M}\mathbb{Z}/2$ of $\mathbf{K}^{\mathsf{M}}$-modules. It follows that multiplication with $\{-1\}$ on $\mathbf{K}^{\mathsf{M}}$ factors as

$$\begin{aligned}\mathbf{K}^{\mathsf{M}} &= \pi_{0+(\star)}\mathbf{M}\mathbb{Z} \xrightarrow{\mathrm{pr}_2^\infty} \pi_{0+(\star)}\mathbf{M}\mathbb{Z}/2 \xrightarrow{\tau\cdot} \pi_{1+(\star)}\mathbf{M}\mathbb{Z}/2 \xrightarrow{\partial_\infty^2} {}_2\pi_{0+(\star)}\mathbf{M}\mathbb{Z} \\ &= {}_2\mathbf{K}^{\mathsf{M}} \subset \mathbf{K}^{\mathsf{M}}\end{aligned}$$

in which the first three maps are surjective. The result follows. □

Lemma A.2 *Let* A *be a* $\mathbf{K}^{\mathsf{M}}$*-module. There is an isomorphism*

$$\mathrm{Ext}^1_{\mathbf{K}^{\mathsf{M}}}(\mathbf{k}^{\mathsf{M}}, A) \cong {}_{\{-1\}}A_0/2A_0$$

which is natural in A.

Proof The short exact sequence

$$0 \to 2\mathbf{K}^{\mathsf{M}} \to \mathbf{K}^{\mathsf{M}} \to \mathbf{k}^{\mathsf{M}} \to 0$$

induces an exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathbf{K}^{\mathsf{M}}}(\mathbf{k}^{\mathsf{M}}, A) &\to \mathrm{Hom}_{\mathbf{K}^{\mathsf{M}}}(\mathbf{K}^{\mathsf{M}}, A) \to \mathrm{Hom}_{\mathbf{K}^{\mathsf{M}}}(2\mathbf{K}^{\mathsf{M}}, A) \\ &\to \mathrm{Ext}^1_{\mathbf{K}^{\mathsf{M}}}(\mathbf{k}^{\mathsf{M}}, A) \to 0. \end{aligned}$$

The short exact sequence

$$0 \to {}_2\mathbf{K}^{\mathsf{M}} \to \mathbf{K}^{\mathsf{M}} \to 2\mathbf{K}^{\mathsf{M}} \to 0$$

induces an exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_{\mathbf{K}^{\mathsf{M}}}(2\mathbf{K}^{\mathsf{M}}, A) \to \mathrm{Hom}_{\mathbf{K}^{\mathsf{M}}}(\mathbf{K}^{\mathsf{M}}, A) \to \mathrm{Hom}_{\mathbf{K}^{\mathsf{M}}}({}_2\mathbf{K}^{\mathsf{M}}, A) \\ &\to \mathrm{Ext}^1_{\mathbf{K}^{\mathsf{M}}}(2\mathbf{K}^{\mathsf{M}}, A) \to 0. \end{aligned}$$

The composition $\mathbf{K}^{\mathsf{M}} \to 2\mathbf{K}^{\mathsf{M}} \to \mathbf{K}^{\mathsf{M}}$ coincides with multiplication by 2, whence naturally $\mathrm{Hom}_{\mathbf{K}^{\mathsf{M}}}(\mathbf{k}^{\mathsf{M}}, A) = {}_2A_0$. In order to identify $\mathrm{Hom}_{\mathbf{K}^{\mathsf{M}}}(2\mathbf{K}^{\mathsf{M}}, A)$ in a similar way, observe the equality

$${}_2\mathbf{K}^{\mathsf{M}} = \{-1\}\mathbf{K}^{\mathsf{M}}$$

of $\mathbf{K}^{\mathsf{M}}$-modules from Theorem A.1. The composition $\mathbf{K}^{\mathsf{M}} \to \{-1\}\mathbf{K}^{\mathsf{M}} = {}_2\mathbf{K}^{\mathsf{M}} \to \mathbf{K}^{\mathsf{M}}$ coincides with multiplication by $\{-1\}$, whence naturally $\mathrm{Hom}_{\mathbf{K}^{\mathsf{M}}}(2\mathbf{K}^{\mathsf{M}}, A) = {}_{\{-1\}}A_0$. The statement follows. □

Lemma A.3 *Let A be a* $\mathbf{K}^{\mathsf{MW}}$*-module. There is a natural isomorphism*

$$\mathrm{Ext}^1_{\mathbf{K}^{\mathsf{MW}}}(\mathbf{K}^{\mathsf{M}}, A) \cong {}_{\mathsf{h}}A_{-1}/\eta A_0.$$

Proof Abbreviate $\mathrm{Hom}(A, B) := \mathrm{Hom}_{\mathbf{K}^{\mathsf{MW}}}(A, B)$. The short exact sequence

$$0 \to \eta\mathbf{K}^{\mathsf{MW}} \to \mathbf{K}^{\mathsf{MW}} \to \mathbf{K}^{\mathsf{M}} \to 0$$

induces an exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}(\mathbf{K}^{\mathsf{M}}, A) \to \mathrm{Hom}(\mathbf{K}^{\mathsf{MW}}, A) \to \mathrm{Hom}(\eta\mathbf{K}^{\mathsf{MW}}, A) \\ &\to \mathrm{Ext}^1_{\mathbf{K}^{\mathsf{MW}}}(\mathbf{K}^{\mathsf{M}}, A) \to 0. \end{aligned}$$

The short exact sequence

$$0 \to {}_{\eta}\mathbf{K}^{\mathsf{MW}}_{\star+1} \to \mathbf{K}^{\mathsf{MW}}_{\star+1} \to \eta\mathbf{K}^{\mathsf{MW}} \to 0$$

induces an exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}(\eta\mathbf{K}^{\mathsf{MW}}, A) \to \mathrm{Hom}(\mathbf{K}^{\mathsf{MW}}_{\star+1}, A) \to \mathrm{Hom}({}_{\eta}\mathbf{K}^{\mathsf{MW}}_{\star+1}, A) \\ &\to \mathrm{Ext}^1_{\mathbf{K}^{\mathsf{MW}}}(\eta\mathbf{K}^{\mathsf{MW}}, A) \to 0. \end{aligned}$$

The composition $\mathbf{K}^{\mathsf{MW}}_{\star+1} \to \eta\mathbf{K}^{\mathsf{MW}} \to \mathbf{K}^{\mathsf{MW}}$ coincides with multiplication by η, whence naturally $\mathrm{Hom}(\mathbf{K}^{\mathsf{M}}, A) = {}_{\eta}A_0$. In order to identify $\mathrm{Hom}(\eta\mathbf{K}^{\mathsf{MW}}, A)$ in a similar way, observe the existence of a further short exact sequence

$$0 \to {}_{\mathsf{h}}\mathbf{K}^{\mathsf{MW}} \to \mathbf{K}^{\mathsf{MW}} \to \mathsf{h}\mathbf{K}^{\mathsf{MW}} = {}_{\eta}\mathbf{K}^{\mathsf{MW}} \to 0$$

of $\mathbf{K}^{\mathsf{MW}}$-modules, the surjection being induced by multiplication with h. Note that $\mathsf{h}\mathbf{K}^{\mathsf{MW}} = \mathsf{h}\mathbf{K}^{\mathsf{M}} = {}_{\eta}\mathbf{K}^{\mathsf{MW}}$ by Theorem 3.2. The composition $\mathbf{K}^{\mathsf{MW}}_{\star+1} \to {}_{\eta}\mathbf{K}^{\mathsf{MW}}_{\star+1} \to \mathbf{K}^{\mathsf{MW}}_{\star+1}$ coincides with multiplication by h, whence naturally $\mathrm{Hom}_{\mathbf{K}^{\mathsf{MW}}}(\eta\mathbf{K}^{\mathsf{MW}}, A) = {}_{\mathsf{h}}A_{-1}$. The statement follows. □

Lemma A.4 *Let A be a $\mathbf{K}^{\mathsf{MW}}$-module. There is a natural exact sequence*

$$0 \to {}_{\{-1\}}\big({}_{\eta}A_0\big)/2\big({}_{\eta}A_0\big) \to \mathrm{Ext}^1_{\mathbf{K}^{\mathsf{MW}}}(\mathbf{k}^{\mathsf{M}}, A) \to {}_{\mathsf{h}}A_{-1}/\eta A_0$$

Proof Abbreviate $\mathrm{Hom}(A, B) := \mathrm{Hom}_{\mathbf{K}^{\mathsf{MW}}}(A, B)$ and $\mathrm{Ext}(A, B) := \mathrm{Ext}_{\mathbf{K}^{\mathsf{MW}}}(A, B)$. The short exact sequence

$$0 \to 2\mathbf{K}^{\mathsf{M}} \to \mathbf{K}^{\mathsf{M}} \to \mathbf{k}^{\mathsf{M}} \to 0$$

of $\mathbf{K}^{\mathsf{MW}}$-modules induces an exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}(\mathbf{k}^{\mathsf{M}}, A) \to \mathrm{Hom}(\mathbf{K}^{\mathsf{M}}, A) \\ &\to \mathrm{Hom}(2\mathbf{K}^{\mathsf{M}}, A) \to \mathrm{Ext}^1(\mathbf{k}^{\mathsf{M}}, A) \to \mathrm{Ext}^1(\mathbf{K}^{\mathsf{M}}, A) \to \cdots . \end{aligned}$$

The natural map $\mathrm{Hom}_{\mathbf{K}^{\mathsf{MW}}}(B, A) \to \mathrm{Hom}_{\mathbf{K}^{\mathsf{M}}}(B, {}_{\eta}A)$ is an isomorphism whenever $\eta B = 0$. The result then follows from Lemma A.2 and Lemma A.3. □

Appendix B: Toda brackets

Consider three composable maps

$$\mathsf{D} \xrightarrow{\gamma} \mathsf{E} \xrightarrow{\beta} \mathsf{F} \xrightarrow{\alpha} \mathsf{G}$$

of pointed motivic spaces or motivic spectra (over any base scheme) such that the compositions $\beta\circ\gamma$ and $\alpha\circ\beta$ are nullhomotopic. Choosing nullhomotopies $\mathrm{cone}(\mathsf{D}) := \mathsf{D}\wedge(\Delta^1,1)\to\mathsf{F}$ and $\mathrm{cone}(\mathsf{E}) = \mathsf{E}\wedge(\Delta^1,1)\to\mathsf{G}$ defines extensions of these compositions displayed in the diagram

$$\begin{tikzcd} & \mathrm{cone}(\mathsf{D}) \arrow[drr] & & & \\ \mathsf{D} \arrow[ur] \arrow[r, "\gamma"] & \mathsf{E} \arrow[rr, "\beta"] \arrow[drr] & & \mathsf{F} \arrow[r, "\alpha"] & \mathsf{G} \\ & & & \mathrm{cone}(\mathsf{E}) \arrow[ur] & \end{tikzcd}$$

and hence a map $\Sigma\mathsf{D}\simeq\mathrm{cone}(\mathsf{D})\cup_{\mathsf{D}}\mathrm{cone}(\mathsf{E})\to\mathsf{G}$, where the identification of $\Sigma\mathsf{D}$ with the displayed pushout is canonical. The *Toda bracket*

$$\langle\alpha,\beta,\gamma\rangle\subset[\Sigma\mathsf{D},\mathsf{G}]$$

is the subset of (pointed) homotopy classes maps which can be obtained this way. It depends only on the homotopy classes of α,β,γ. This definition provides a secondary composition on the level of homotopy categories and is sufficiently general to apply in manifold cases, for example in any pointed simplicial model category. In case $[\Sigma\mathsf{D},\mathsf{G}]$ is abelian (which is always the case for motivic spectra), then $\langle\alpha,\beta,\gamma\rangle$ is a coset of the subgroup $\alpha\circ[\Sigma\mathsf{D},\mathsf{F}]+[\Sigma\mathsf{E},\mathsf{G}]\circ\Sigma\gamma\subset[\Sigma\mathsf{D},\mathsf{G}]$ [26, Lemma 1.1]. An equivalent description of $\langle\alpha,\beta,\gamma\rangle$ is obtained by taking homotopy classes of compositions

$$\Sigma\mathsf{D}\xrightarrow{\gamma'}\mathrm{cone}(\beta)=\mathrm{cone}(\mathsf{E})\cup_{\mathsf{E}}\mathsf{F}\xrightarrow{\widetilde{\alpha}}\mathsf{G}$$

where $\widetilde{\alpha}$ is an extension of α to $\mathrm{cone}(\beta)$ and γ' is a coextension of γ with respect to β [26, Prop. 1.7]. Any coextension of γ with respect to β is a lift of $\Sigma\gamma$ along $\mathrm{cone}(\beta)\to\Sigma\mathsf{E}$. In the stable case of motivic spectra, a coextension of γ with respect to β is the same as a lift of $\Sigma\gamma$ along $\mathrm{cone}(\beta)\to\Sigma\mathsf{E}$ [11, Prop. 5.2]. Two straightforward extensions of statements from [26] will be stated explicitly for reference purposes.

Proposition B.1 *Let three composable maps*

$$\mathsf{D}\xrightarrow{\gamma}\mathsf{E}\xrightarrow{\beta}\mathsf{F}\xrightarrow{\alpha}\mathsf{G}$$

of pointed motivic spaces or motivic spectra (over any base scheme) be given such that the compositions $\beta\circ\gamma$ and $\alpha\circ\beta$ are nullhomotopic. Let $\mathsf{G}\xrightarrow{i}$

$\mathrm{cone}(\alpha) \xrightarrow{q} \Sigma\mathsf{F}$ *denote the canonical maps. There is an equality*

$$-i \circ \langle \alpha, \beta, \gamma \rangle = \{\beta' \circ \Sigma\gamma : \beta' \text{ coextension of } \beta \text{ w.r.t. } \alpha\}$$

of subsets of $[\Sigma\mathsf{D}, \mathsf{G}]$.

Proof This follows by translating [26, Proposition 1.8] to the motivic setting. □

Proposition B.2 *Let three composable maps*

$$\mathsf{D} \xrightarrow{\gamma} \mathsf{E} \xrightarrow{\beta} \mathsf{F} \xrightarrow{\alpha} \mathsf{G}$$

of pointed motivic spaces or motivic spectra (over any base scheme) be given such that the compositions $\beta \circ \gamma$ *and* $\alpha \circ \beta$ *are nullhomotopic. Then for every* $\lambda \in \langle \alpha, \beta, \gamma \rangle$ *there exists an extension* $\widetilde{\beta} \colon \mathrm{cone}(\gamma) \to \mathsf{F}$ *of* β *such that the diagram*

$$\begin{tikzcd} \mathrm{cone}(\gamma) \arrow[r, "\mathrm{can}"] \arrow[d, "\widetilde{\beta}"] & \Sigma\mathsf{D} \arrow[d, "\lambda"] \\ \mathsf{F} \arrow[r, "\alpha"] & \mathsf{G} \end{tikzcd} \tag{B.1}$$

commutes up to homotopy. Conversely, for every extension $\widetilde{\beta} \colon \mathrm{cone}(\gamma) \to \mathsf{F}$ *of* β *there exists an element* $\lambda \in \langle \alpha, \beta, \gamma \rangle$ *such that diagram (B.1) commutes up to homotopy.*

Proof This follows by translating [26, Proposition 1.9] to the motivic setting. □

By adjointness, an equivalent description in terms of maps to loop spaces can be given. From the definition it follows quite immediately that any functor Φ induced by a pointed simplicial left Quillen functor on homotopy categories preserves Toda brackets in the sense of providing an inclusion

$$\Phi(\langle \alpha, \beta, \gamma \rangle) \subset \langle \Phi(\alpha), \Phi(\beta), \Phi(\gamma) \rangle$$

whenever the Toda bracket $\langle \alpha, \beta, \gamma \rangle$ is defined. In particular, if the target Toda bracket $\langle \Phi(\alpha), \Phi(\beta), \Phi(\gamma) \rangle$ does not contain the zero element, neither does the source Toda bracket $\langle \alpha, \beta, \gamma \rangle$. This applies in particular to base change functors, stabilization, and to complex and étale realization.

For example, the Toda bracket $\langle 2, \eta_{\mathsf{top}}, 2 \rangle \subset \pi_5 S^3$ in the pointed homotopy category of topological spaces defined by

$$S^4 \xrightarrow{2} S^4 \xrightarrow{\eta_{\mathsf{top}}} S^3 \xrightarrow{2} S^3$$

consists of the single element $\eta_{\mathsf{top}}^2 \colon S^5 \to S^3$ by [26, Lemma 5.2 and (5.2)]. With the help of the unstable considerations over Spec($\mathbb{Z}$) in [9] (see in particular [9, Remark A.10]), the maps

$$S^{2+(2)} \xrightarrow{\mathsf{h}} S^{2+(2)} \xrightarrow{\eta} S^{2+(1)} \xrightarrow{\mathsf{h}} S^{2+(1)}$$

define a Toda bracket $\langle \mathsf{h}, \eta, \mathsf{h} \rangle \subset [S^{3+(2)}, S^{2+(1)}]$. Comparison with the $\mathbf{P}^1$-stable situation and [23, Theorem 5.5] shows that any element in this Toda bracket maps to $\eta \circ \eta_{\mathsf{top}} \in \pi_{1+(1)}\mathbf{1}_F$ modulo the subgroup $2(\mathbf{K}_1^{\mathsf{M}}(F)/24)$ generated by ν over fields F of characteristic not two. Similarly, any element in the Toda bracket

$$\langle \eta, \mathsf{h}, \eta \rangle \subset [S^{3+(3)}, S^{2+(1)}]$$

maps to $\pm 6\nu \in \pi_{1+(2)}\mathbf{1}_F$ over fields F of characteristic not two or three, by [23, Theorem 5.5] and [26, (5.4), Prop. 5.6 and its proof]. See [22, Prop. 4.1] for these two and further examples.

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