Exponential mixing of geodesic flows for geometrically finite hyperbolic manifolds with cusps

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Abstract Let Γ be a geometrically finite discrete subgroup in $\mathrm{SO}(d+1,1)^\circ$ with parabolic elements. We establish exponential mixing of the geodesic flow on the unit tangent bundle $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^{d+1})$ with respect to the Bowen–Margulis–Sullivan measure, which is the unique probability measure on $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^{d+1})$ with maximal entropy. As an application, we obtain a resonance-free region for the resolvent of the Laplacian on $\Gamma\backslash\mathbb{H}^{d+1}$. Our approach is to construct a coding for the geodesic flow and then prove a Dolgopyat-type spectral estimate for the corresponding transfer operator.

1 Introduction

1.1 Exponential mixing of the geodesic flow

Let $\mathbb{H}^{d+1}$ be the hyperbolic $(d+1)$-space. Let $G = \mathrm{SO}(d+1,1)^\circ$, which is the group of orientation preserving isometries of $\mathbb{H}^{d+1}$. Let $\Gamma < G$ be a non-elementary, torsion-free, geometrically finite discrete subgroup with parabolic elements. Denote by δ the critical exponent of Γ, which is defined as the abscissa of convergence of the Poincaré series $\sum_{\gamma\in\Gamma} e^{-sd(o,\gamma o)}$. Set

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$M = \Gamma \backslash \mathbb{H}^{d+1}$, so M contains cusps. We consider the geodesic flow $(\mathcal{G}_t)_{t \in \mathbb{R}}$ acting on the unit tangent bundle $T^1(M)$ over M. The invariant measure for the flow we will work with is the Bowen–Margulis–Sullivan measure m^{BMS}, which is supported on the non-wandering set of the geodesic flow and is known to be the unique probability measure with maximal entropy δ [37].

Our main result is establishing exponential mixing of the geodesic flow.

Theorem 1.1 *The geodesic flow is exponentially mixing with respect to m^{BMS}: there exists $\eta > 0$ such that for any functions $\phi, \psi \in C^1(T^1(M))$ and any $t > 0$, we have*

$$\int_{T^1(M)} \phi \cdot \psi \circ \mathcal{G}_t \, dm^{\text{BMS}} = m^{\text{BMS}}(\phi) m^{\text{BMS}}(\psi) + O(\|\phi\|_{C^1} \|\psi\|_{C^1} e^{-\eta t}),$$

where $\| \cdot \|_{C^1}$ is the C^1-norm with respect to the Riemannian metric on $T^1(M)$.

For a geometrically finite discrete subgroup Γ, Sullivan [49] proved the ergodicity of the geodesic flow with respect to m^{BMS} and Rudolph [43] proved that the geodesic flow is mixing with respect to m^{BMS}. When $\delta > d/2$, Theorem 1.1 was proved by Mohammadi–Oh [33] and Edwards–Oh [19] using the representation theory of $L^2(M)$ and the spectral gap of Laplace operator [28]. When Γ is convex cocompact, i.e., geometrically finite without parabolic elements, Theorem 1.1 and its corollaries were proved by Naud [34], Stoyanov [47] and Sarkar–Winter [44] building on the work of Dolgopyat [18]. Therefore, the main contribution of our work lies in the groups with small critical exponent and with parabolic elements, completing the story of exponential mixing of the geodesic flow on a geometrically finite hyperbolic manifold.

Using Roblin's transverse intersection argument [35,36,42], we obtain the decay of matrix coefficients (Theorem 9.1) from Theorem 1.1. Theorem 1.1 and 9.1 are known to have many immediate applications in number theory and geometry. To name a few, see [32] for counting closed geodesics, [25] for shrinking target problems and [7] for some general counting results.

1.2 Resonance-free region

Recall $M = \Gamma \backslash \mathbb{H}^{d+1}$. Consider the Laplace operator Δ_M on M. Lax and Phillips completely described its spectrum on $L^2(M)$ when M has infinite volume [28]. The half line $[d^2/4, \infty)$ is the continuous spectrum and it contains no embedded eigenvalues. The rest of the spectrum (point spectrum) is finite and starting at $\delta(d/2 - \delta)$ if $\delta > d/2$ and is empty if $\delta \le d/2$. Let S be the set of eigenvalues of Δ_M. The resolvent of the Laplacian

$$R_M(s) = (\Delta_M - s(d - s))^{-1} : L^2(M) \to L^2(M)$$

is well-defined and analytic on $\{\Re s > d/2, \ s(d-s) \notin S\}$. Guillarmou and Mazzeo showed that $R_M(s)$ has a meromorphic continuation to the whole complex plane as an operator from $C_c^\infty(M)$ to $C^\infty(M)$ with poles of finite rank [21]. These poles are called resonances. Patterson showed that on the line $\Re s = \delta$, the point $s = \delta$ is the unique pole of $\Gamma(s - \frac{d}{2} + 1)R_M(s)$ and it is a simple pole [39]. We use Theorem 1.1 to further obtain a resonance-free region.

Theorem 1.2 *There exists $\eta > 0$ such that on the half-plane $\Re s > \delta - \eta$, $s = \delta$ is the only resonance for the resolvent $R_M(s)$ if $\delta \notin d/2 - \mathbb{N}_{\geq 1}$; otherwise, $R_M(s)$ is analytic on $\Re s > \delta - \eta$.*

In the convex cocompact case, a resonance-free region of the resolvent is closely related to a zero free region of the Selberg zeta function. But in the geometrically finite case, such relation is not well understood except for the surface case.

1.3 On the proof of the main theorem

The proof of Theorem 1.1 can be reduced to the case when Γ is Zariski dense and then the proof falls into two parts: we code the geodesic flow and prove a Dolgopyat-type spectral estimate for the corresponding transfer operator. Ultimately, the obstructions to applying Dolgopyat's original argument in our context are purely technical, but to overcome these obstructions in any context is the heart of the matter.

To prove exponential mixing using the symbolic-dynamic approach of Dolgopyat, one approach is to construct a section to the flow. In sum, one seeks a $2d$ submanifold S in $\mathrm{T}^1(M)$ transversal to the geodesic flow which is a Poincaré section, on which the return map can be tightly organized. The challenge lies in that it is required to find a return map R defined on a full measure subset S_0 of S, such that the map $F(v) := \mathcal{G}_{R(v)}(v)$, $v \in S_0$, on S_0 is hyperbolic and can be modelled on a full shift of countable many symbols.

We overcome this difficulty by connecting the return map on S_0 to an expanding map on the boundary $\partial\mathbb{H}^{d+1}$. The precise description of the expanding map on the boundary is as follows. We consider the upper-half space model for $\mathbb{H}^{d+1}$ and without loss of generality, we may assume that ∞ is a parabolic fixed point of Γ. Let $\mathrm{Stab}_\infty(\Gamma)$ be the group of stabilizers of ∞ in Γ and Γ_∞ be a maximal normal abelian subgroup in $\mathrm{Stab}_\infty(\Gamma)$. Set $\Delta_0 := \Delta_\infty$ to be a fundamental domain of Γ_∞ in $\partial\mathbb{H}^{d+1}\backslash\{\infty\}$ (see Sect. 2.3 for details). Denote by Λ_Γ the limit set of Γ and μ the Patterson–Sullivan measure, which is a finite measure supported on Λ_Γ.

Proposition *4.1 There are constants $C_1 > 0$, λ, $\epsilon_0 \in (0, 1)$, a countable collection of disjoint, open subsets $\{\Delta_j\}_{j \in \mathbb{N}}$ in Δ_0 and an expanding map $T : \sqcup_j \Delta_j \to \Delta_0$ such that:*

1. *$\sum_j \mu(\Delta_j) = \mu(\Delta_0)$.*
2. *For each j, there is an element $\gamma_j \in \Gamma$ such that $\Delta_j = \gamma_j \Delta_0$ and $T|_{\Delta_j} = \gamma_j^{-1}$.*
3. *Each γ_j is a uniform contraction: $|\gamma_j'(x)| \leq \lambda$ for all $x \in \Delta_0$.*
4. *For each γ_j, $|D(\log |\gamma_j'|)(x)| < C_1$ for all $x \in \Delta_0$, where $D(\log |\gamma_j'|)(x)$ is the differential of the map $z \mapsto \log |\gamma_j'(z)|$ at x.*
5. *Let R be the function on $\sqcup_j \Delta_j$ given by $R(x) = \log |DT(x)|$. Then $\int e^{\epsilon_o R} d\mu < \infty$.*

The last property is known as the exponential tail property. Moreover, we show that the coding satisfies the **uniform nonintegrable condition (UNI)** (Lemma 4.5). We use Proposition 4.15 as a bridge to connect the geodesic flow $(\mathcal{G}_t)_{t \in \mathbb{R}}$ on $T^1(M)$ and the expanding map T on Δ_0. We show that the geodesic flow is a factor of a hyperbolic skew product flow constructed using T.

The construction of the coding starts with the following observation: locally, in a neighborhood of a parabolic fixed point, we can use the structure of the parabolic fixed points to find a "flower" centered at this parabolic fixed point, and the "flower" can be partition into a countable union of open sets of the form $\gamma(\Delta_0)$ for some $\gamma \in \Gamma$. Once we have the algorithm to find the partition for local regions, we still face the question of how to patch these flowers together. We introduce an inductive algorithm to find pairwise disjoint flowers.

But the bulk of the work lies in proving the exponential tail property. We show that this follows from Proposition 6.15 which says that the measure of the set that has not been partitioned at time n decays exponentially. At the time n, the remaining part is a sheet with many holes, consisting of "flowers"; while the Patterson–Sullivan measure is a measure supported on the fractal limit set, and we have limited knowledge of the regularity of this measure. It is interesting to figure out how to use minimal tools to get the required estimate.

When the non-wandering set of the geodesic flow is compact, the coding is well-studied and we have, for example, the Bowen–Series' coding [6], Bowen's coding [9] and Ratner's coding [41]. When manifolds contain cusps, only some partial knowledge is available. Dal'bo–Peigné [15,16] and Babillot–Peigné [3] provided the coding for generalized Schottky groups. Stadlbauer [45] and Ledrappier-Sarig [31] provided the coding for non-uniform lattices in $SO(2, 1)^\circ$. They made use of the fact that such a discrete subgroup is a free group and has a nice fundamental domain in $\mathbb{H}^2$. Our coding works for general geometrically finite discrete subgroups with parabolic elements and is partly

inspired by the works of Lai–Sang Young [51] and Burns–Masur–Matheus–Wilkinson [11].

In a forthcoming joint work with Sarkar [29], we establish the exponential mixing for frame flows on geometrically finite hyperbolic manifolds with cusps. We prove this by using the coding of the geodesic flow constructed in this paper and then performing a frame flow version of Dolgopyat's method. The crucial cancellations of the summands of the transfer operators twisted by holonomy are obtained from the local non-integrability condition and the non-concentration property of Sarkar–Winter [44]. But the challenge in the presence of cusps is that the latter holds only on a certain *good* subset. This is resolved by a large deviation property for symbolic recurrence to the good subset, which is inspired by the work of Tsujii–Zhang [50]. It is proved by studying the combinatorics of cusp excursions and showing an effective renewal theorem, as in the work of Li [30], which uses the spectral gap of the transfer operator for the geodesic flow in Proposition 7.3.

In [22], using the coding of Schottky groups, Guillopé–Lin–Zworski were able to study the Selberg zeta function through a dynamical zeta function. They gave a simple proof of the analytic continuation and a growth estimate of the Selberg zeta function. Hopefully, the coding constructed in our work will be helpful in the study of the Selberg zeta function for higher dimensional geometrically finite manifold.

Other applications include the Fourier decay of the Patterson–Sullivan measure. In [4], Bourgain–Dyatlov proved Fourier decay of Patterson-Sullivan measures for convex cocompact Fuchsian groups. The first step of their proof is to use the coding of the limit set to construct an appropriate transfer operator. With our coding available, it is very likely to generalize the Fourier decay to geometrically finite discrete subgroups with parabolic elements.

Our proof of obtaining a Dolgopyat-type spectral estimate is influenced by the one in [1,2,5,18,34,47]. The key of Dolgopyat's approach is to estimate the decay of certain oscillatory integrals against the *fractal* Patterson-Sullivan measure: for function f of the form $\sum_{j \in J} \exp(ib\tau_j(x))$, where b is a real number, $\tau_j \in C^2(\Delta_0)$ and J is some index set, we have $|\int f \, d\mu|$ is bounded by some negative power of $|b|$. We successfully attain this estimate by combining dynamics and the regularity properties of the Patterson–Sullivan measure, which we think are the essential ingredients to gain the decay.

Another possible argument is to analyze each $\int \exp(ib\tau_j) \, d\mu$ and show the decay. Such a result is known as the Fourier decay of the Patterson–Sullivan measure. This is especially challenging when the critical exponent δ is small. In [23], Jordan–Sahlsten proved the Fourier decay of some fractal measures. Their idea is to approximate the fractal measure by the Lebesgue measure and use the Fourier decay of the Lebesgue measure, which is well-studied. But this approximation is sensitive to the Hausdorff dimension of the fractal sets.

A similar idea also appears in a preprint by Kahlil [26]. It is unclear whether their approach provides an alternative way to establish the Fourier decay of fractal measures without dimension restriction.

There are works trying to use anisotropic Banach spaces to prove exponential mixing. The key is to show there exists $\epsilon > 0$ such that the strip $\{-\epsilon < \Re s < 0\}$ is free of *Pollicott–Ruelle resonances*. For a geometrically finite discrete subgroup with the critical exponent $\delta < d/2$, it might happen that there are resonances with large imaginary parts and real parts close to zero. Recently, there is a work in progress of Gouëzel–Tapie–Schapira on the Pollicott–Ruelle resonances for SPR manifolds, which include geometrically finite manifolds. They show the resonances are discrete, but it is not clear whether one can use this property to attain the required resonance-free region.

Organization of the paper.

- In Sect. 2, we gather the basic facts and preliminaries about hyperbolic spaces, geometrically finite discrete subgroups, the structure of cusps, Patterson-Sullivan measure, and Bowen–Margulis–Sullivan measure.
- In Sect. 3, we prove that Theorem 1.1 can be reduced to Zariski dense case.
- In Sect. 4, we state the results of the coding (Proposition 4.1, Lemma 4.5, 4.8). We construct a hyperbolic skew product flow and state the result that it is exponential mixing (Theorem 4.13). We show that the geodesic flow on $T^1(M)$ is a factor of this hyperbolic skew product flow (Proposition 4.15) and deduce the exponential mixing of the geodesic flow from Theorem 4.13.
- In Sect. 5, we provide an explicit description of the action of an element $\gamma \in \Gamma$ on $\partial \mathbb{H}^{d+1}$ and the estimate on the norm of the derivative of γ (Sect. 5.1). We list the basics for the multi-cusp case (Sect. 5.2). The doubling property and the friendliness of Patterson-Sullivan measure are proved in Sects. 5.3 and 5.4.
- In Sect. 6, we start with the construction of the coding for one cusp case, which is also the first step for multi-cusp case. The main result is exponential decay of the remaining set (Proposition 6.15). Section 6.3–6.6 are devoted to the proof the Proposition 6.15. The coding for the multi-cusp case will be provided in Sects. 6.7 and 6.8. The results of the coding (Proposition 4.1, Lemma 4.5, 4.8) will be proved in Sect. 6.7-6.9.
- In Sect. 7, we prove a Dolgopyat-type spectral estimate for the corresponding transfer operator and the main result is an L^2-contraction proposition (Proposition 7.3).
- In Sect. 8, we finish the proof of Theorem 4.13.
- In Sect. 9, we prove the application of obtaining a resonance-free region for the resolvent $R_M(s)$ (Theorem 1.2).

Notation. In the paper, given two real functions f and g, we write $f \ll g$ if there exists a constant $C > 0$ only depending on Γ such that $f \leq Cg$. We write $f \approx g$ if $f \ll g$ and $g \ll f$.

2 Preliminary of hyperbolic spaces and PS measure

2.1 Hyperbolic spaces

We will use the upper-half space model for $\mathbb{H}^{d+1}$:

$$\mathbb{H}^{d+1} = \{x = (x_1, \ldots, x_{d+1}) \in \mathbb{R}^{d+1} : x_{d+1} > 0\}.$$

Let $o = (0, \ldots, 0, 1) \in \mathbb{H}^{d+1}$. For $x \in \mathbb{H}^{d+1}$, write $h(x)$ for the height of the point x, which is the last coordinate of x. The Riemannian metric on $\mathbb{H}^{d+1}$ is given by

$$ds^2 = \frac{dx_1^2 + \cdots + dx_{d+1}^2}{x_{d+1}^2}.$$

Let $\partial \mathbb{H}^{d+1}$ be the visual boundary. On $\partial \mathbb{H}^{d+1} = \mathbb{R}^d \cup \{\infty\}$, we have the spherical metric, denoted by $d_{\mathbb{S}^d}(\cdot, \cdot)$. We also have the Euclidean metric, denoted by $d_E(x, x')$ or $|x - x'|$ for any $x, x' \in \partial \mathbb{H}^{d+1}$. This metric will be used most frequently; we will simply write $d(\cdot, \cdot)$ when there is no confusion.

For $g \in G$, it acts on $\partial \mathbb{H}^{d+1}$ conformally. For $x \in \partial \mathbb{H}^{d+1}$, let $|g'(x)|$ be the linear distortion of the conformal action of g at x with respect to the Euclidean metric. It is also the norm of the derivative seen as a linear map on tangent spaces. Let $|g'(x)|_{\mathbb{S}^d}$ be the norm with respect to the spherical metric. We have the relation

$$|g'(x)|_{\mathbb{S}^d} = \frac{1 + |x|^2}{1 + |gx|^2} |g'(x)|. \tag{2.1}$$

Another formula for $|g'(x)|_{\mathbb{S}^d}$ is

$$|g'(x)|_{\mathbb{S}^d} = e^{-\beta_x(g^{-1}o, o)},$$

where $\beta_x(\cdot, \cdot)$ is the Busemann function given by $\beta_x(z, z') = \lim_{t \to +\infty} d(z, x_t) - d(z', x_t)$ with x_t an arbitrary geodesic ray tending to x.

We denote $\mathbb{H}^{d+1} \cup \partial \mathbb{H}^{d+1}$ by $\overline{\mathbb{H}^{d+1}}$.

2.2 Geometrically finite discrete subgroups

Let Γ be a torsion-free, non-elementary discrete subgroup in G. We list some basics of geometrically finite discrete subgroups.

The **limit set** of Γ is the set Λ_Γ of all the accumulation points of an orbit Γx for some $x \in \mathbb{H}^{d+1}$. As we assume Γ is torsion-free, Λ_Γ is contained in $\partial \mathbb{H}^{d+1}$. The convex hull, hull($\Lambda_\Gamma$), of Λ_Γ is the smallest convex subset in $\mathbb{H}^{d+1}$ which contains all the geodesics connecting any two distinct points of Λ_Γ. The convex core of M is $C(M) = \Gamma \backslash \text{hull}(\Lambda_\Gamma) \subset M$.

A limit point $x \in \Lambda_\Gamma$ is called **conical** if there exists a geodesic ray tending to x and a sequence of elements $\gamma_n \in \Gamma$ such that $\gamma_n o$ converges to x, and the distance between $\gamma_n o$ and the geodesic ray is bounded. A subgroup Γ' of Γ is called parabolic if Γ' fixes only one point in $\partial \mathbb{H}^{d+1}$. A point $x \in \Lambda_\Gamma$ is called a **parabolic fixed point** if its stabilizer in Γ, $\text{Stab}_\Gamma(x)$, is parabolic. A parabolic fixed point is called **bounded parabolic** if the quotient $\text{Stab}_\Gamma(x) \backslash (\Lambda_\Gamma - \{x\})$ is compact.

A **horoball** based at $x \in \partial \mathbb{H}^{d+1}$ is the set $\{y \in \mathbb{H}^{d+1} : \beta_x(y, o) < t\}$ for some $t \in \mathbb{R}$. The boundary of a horoball is called a horosphere. We call a horoball H based at a parabolic fixed point $x \in \Lambda_\Gamma$ a horocusp region, if we have $\gamma H \cap H = \emptyset$ for any $\gamma \in \Gamma - \text{Stab}_\Gamma(x)$. Then the image of H in M under the quotient map, $\Gamma \backslash \Gamma H$, is isometric to $\text{Stab}_\Gamma(x) \backslash H$ and is called a **proper horocusp** of M.

Definition 2.2 (*Geometrically finite discrete subgroup* [8,40]) A non-elementary discrete subgroup $\Gamma < \text{SO}(d + 1, 1)^\circ$ is called **geometrically finite** if it satisfies one of the following equivalent conditions:

(i) There is a (possibly empty) finite union V of proper horocusps of M, with disjoint closures, such that $C(M) - V$ is compact.
(ii) Every limit point of Γ is either conical or bounded parabolic.

2.3 Structure of cusps

Assume that Γ is a geometrically finite discrete subgroup with parabolic elements and ∞ is a parabolic fixed point of Γ. Let $\Gamma'_\infty = \text{Stab}_\Gamma(\infty)$ be the parabolic subgroup of Γ fixing ∞. Then Γ'_∞ acts on $\mathbb{R}^d$, part of $\partial \mathbb{H}^{d+1}$, isometrically with respect to the Euclidean metric. The following is a result of Bieberbach (see [21, Page 5] or [8, Section 2.2]).

Lemma 2.3 (Bieberbach) *Consider the action of Γ'_∞ on $\mathbb{R}^d$. Then there exist a maximal normal abelian subgroup $\Gamma_\infty \subset \Gamma'_\infty$ of finite index and an affine subspace $Z \subset \mathbb{R}^d$ of dimension k, invariant under Γ'_∞, such that Γ_∞ acts as a group of translations of rank k on Z. If $\mathbb{R}^d = Y \times Z$ is an orthogonal*

decomposition, with $Y \simeq \mathbb{R}^{d-k}$ and associated coordinates (y, z), then we can write each element $\gamma \in \Gamma'_\infty$ in the form

$$\gamma(y, z) = (A_\gamma y, R_\gamma z + b_\gamma), \quad b_\gamma \in \mathbb{R}^k, \quad A_\gamma \in O(n - k), \quad R_\gamma \in O(k)$$

where for each γ, $R_\gamma^m = \text{Id}$ for some $m \in \mathbb{N}$, with $m = 1$ if $\gamma \in \Gamma_\infty$.

The dimension k is called the **rank** of the parabolic fixed point ∞.

Fix an orthogonal decomposition $\mathbb{R}^d = Y \times Z \simeq \mathbb{R}^{d-k} \times \mathbb{R}^k$. As Γ_∞ acts on $\mathbb{R}^k$ as a group of translations, it admits a fundamental region Δ'_∞ which is an open k-dimensional parallelotope in $\mathbb{R}^k$. Since Γ is geometrically finite, ∞ is a bounded parabolic fixed point. By definition, the quotient $\Gamma'_\infty \backslash (\Lambda_\Gamma - \{\infty\})$ is compact; the quotient $\Gamma_\infty \backslash (\Lambda_\Gamma - \{\infty\})$ is also compact as Γ_∞ is a finite index subgroup of Γ'_∞. Therefore, there exists a constant $C > 0$ such that the set $B_Y(C) = \{y \in \mathbb{R}^{d-k} : |y| < C\}$ in $\mathbb{R}^{d-k}$ has the property that Fig. 1

$$\Lambda_\Gamma \subset \{\infty\} \cup \left(\cup_{\gamma \in \Gamma_\infty} \gamma \left(\overline{B_Y(C/2) \times \Delta'_\infty} \right) \right).$$

Definition 2.4 We call the open set $\Delta_\infty := B_Y(C) \times \Delta'_\infty$ a **fundamental region** for the parabolic fixed point ∞.

2.4 PS measure and BMS measure

Patterson–Sullivan measure. Recall δ is the critical exponent of Γ. Patterson [38] and Sullivan [48] constructed a Γ-invariant conformal density $\{\mu_y\}_{y \in \mathbb{H}^{d+1}}$ of dimension δ on Λ_Γ, which is a set of finite Borel measures such that for any $y, z \in \mathbb{H}^{d+1}$, $x \in \partial \mathbb{H}^{d+1}$ and $\gamma \in \Gamma$,

$$\frac{d\mu_y}{d\mu_z}(x) = e^{-\delta \beta_x(y,z)} \quad \text{and} \quad (\gamma)_* \mu_y = \mu_{\gamma y}, \tag{2.5}$$

where $\gamma_* \mu_y(E) = \mu_y(\gamma^{-1} E)$ for any Borel subset E of $\partial \mathbb{H}^{d+1}$. This family of measures is unique up to homothety, and the action of Γ on $\partial \mathbb{H}^{d+1}$ is ergodic relative to the measure class defined by these measures (Fig. 1).

As μ_y's are absolutely continuous with respect to each other, for most of the paper, we will consider μ_o and denote it by μ for short. We call it the Patterson–Sullivan measure (or PS measure). The following **quasi-invariance** property of the PS measure will be frequently used: for any Borel subset E of $\partial \mathbb{H}^{d+1}$ and any $\gamma \in \Gamma$,

$$\mu(\gamma E) = \int_E |\gamma'(x)|^\delta_{\mathbb{S}^d} d\mu(x). \tag{2.6}$$
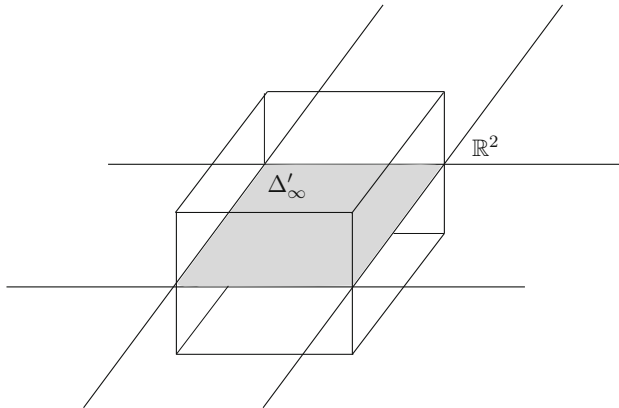
Fig. 1 Here ∞ is a parabolic fixed point of rank 2 in $\partial \mathbb{H}^4$. The intersection $\Lambda_\Gamma \cap \mathbb{R}^3$ has bounded distance to $\mathbb{R}^2$

Bowen–Margulis–Sullivan measure. Let $\partial^2(\mathbb{H}^{d+1}) = \partial \mathbb{H}^{d+1} \times \partial \mathbb{H}^{d+1} - $ Diagonal. The Hopf parametrization of $T^1(\mathbb{H}^{d+1})$ as $\partial^2(\mathbb{H}^{d+1}) \times \mathbb{R}$ is given by

$$v \mapsto (x, x_-, s = \beta_x(o, v_*)),$$

where x (resp. x_-) is the forward endpoint (resp. backward) endpoint of v under the geodesic flow, and $v_* \in \mathbb{H}^{d+1}$ is the based point of v. The geodesic flow on $T^1(\mathbb{H}^{d+1})$ is represented by the translation on $\mathbb{R}$-coordinate.

The Bowen–Margulis–Sullivan measure (or BMS measure) on $T^1(\mathbb{H}^{d+1})$ is defined by

$$d\tilde{m}^{\mathrm{BMS}}(x, x_-, s) = e^{\delta \beta_x(o, x_*)} e^{\delta \beta_{x_-}(o, x_*)} d\mu(x) d\mu(x_-) ds,$$

where x_* is the based point of the unit tangent vector given by (x, x_-, s). It is invariant under the geodesic flow $\mathcal{G}_t$ from the definition. The group Γ acts on $\partial^2(\mathbb{H}^{d+1}) \times \mathbb{R}$ by

$$\gamma(x, x_-, s) = (\gamma x, \gamma x_-, s - \beta_x(o, \gamma^{-1} o)).$$

This formula, together with (2.5), implies that $\tilde{m}^{\mathrm{BMS}}$ is left Γ-invariant; hence $\tilde{m}^{\mathrm{BMS}}$ induces a measure m^{BMS} on $T^1(M)$, which is the Bowen–Margulis–Sullivan measure on $T^1(M)$. For geometrically finite discrete subgroups, Sullivan showed that m^{BMS} is finite and ergodic with respect to the action of the geodesic flow [49]. Otal and Peigné showed that m^{BMS} is the unique measure supported on the non-wandering set of the geodesic flow with maximal entropy δ [37]. After normalization, we suppose that m^{BMS} is a probability measure.

3 Reduction to Zariski dense case

The group $SO(d+1,1)$ is Zariski closed and connected and the subgroup $SO(d+1,1)^\circ$ is its analytic connected component containing identity. For a subgroup Γ of $SO(d+1,1)^\circ$, it is said to be Zariski dense in $SO(d+1,1)^\circ$ if it is Zariski dense in $SO(d+1,1)$. The proof of Theorem 1.1 can be reduced to Zariski dense case.

Theorem 3.1 *Assume that $\Gamma < SO(d+1,1)^\circ$ is a Zariski dense, torsion-free, geometrically finite subgroup with parabolic elements. The geodesic flow $(\mathcal{G}_t)_{t\in\mathbb{R}}$ on $T^1(M)$ is exponentially mixing with respect to m^{BMS}: there exists $\eta > 0$ such that for any functions $\phi, \psi \in C^1(T^1(M))$ and any $t > 0$, we have*

$$\int_{T^1(M)} \phi \cdot \psi \circ \mathcal{G}_t \, dm^{BMS} = m^{BMS}(\phi) m^{BMS}(\psi) + O(\|\phi\|_{C^1} \|\psi\|_{C^1} e^{-\eta t}).$$

***From Theorem* 3.1 *to Theorem* 1.1** Suppose Γ is not Zariski dense. Let H be the Zariski closure of Γ in $SO(d+1,1)$ and let H_1 be the Zariski connected component of H containing the identity. Let $\Gamma_1 = \Gamma \cap H_1$. Then Γ_1 is a finite index subgroup of Γ and the Zariski closure of Γ_1 is H_1. We will only consider Γ_1 because the exponential mixing of Γ follows from the same statement for Γ_1 by taking covering space.

Let H_o be the analytic connected component of H_1 containing identity. Since Γ is non-elementary, the group H_o doesn't fix any point on the boundary. By a classic result (see [14] for example), up to conjugacy, H_o preserves a hyperbolic subspace $\mathbb{H}^m$ with $m \leq d$ and the restriction of H_o to $\mathbb{H}^m$ contains $SO(m,1)^\circ$ with compact kernel. Preserving subspace is a Zariski closed condition, we know that H_1 also preserves $\mathbb{H}^m$ and the restriction of H_1 to $\mathbb{H}^m$ satisfies the same properties as H_o. Since Γ_1 is a torsion free discrete subgroup, the restriction map $\Gamma_1 \to \Gamma_1|_{\mathbb{H}^m}$ is injective. Then the Zariski closure of $\Gamma_1|_{\mathbb{H}^m}$ also contains $SO(m,1)^\circ$. At most passing to an index 4 subgroup, we can suppose that $\Gamma_1|_{\mathbb{H}^m}$ is a subgroup of $SO(m,1)^\circ$. Hence $\Gamma_1|_{\mathbb{H}^m}$ is Zariski dense in $SO(m,1)^\circ$ and geometrically finite. (Definition 2.2 (2) implies that $\Gamma_1|_{\mathbb{H}^m}$ is still geometrically finite.) The BMS measure m^{BMS} of Γ_1 on the unit tangent bundle $\Gamma_1 \backslash T^1 \mathbb{H}^{d+1}$ is actually supported on $\Gamma_1 \backslash T^1 \mathbb{H}^m$, which is the Zariski dense case. $\qquad\square$

4 The geodesic flow and the boundary map

For the rest of the paper, our standing assumption is

$\Gamma < G$ Zariski dense, torsion-free, geometrically finite with parabolic elements and ∞ is a parabolic fixed point of Γ.

Let $\Delta_0 := \Delta_\infty$ be a fundamental region for the parabolic fixed point ∞ described in Sect. 2.3. In Sect. 6, we will construct a coding of the limit set satisfying the following properties.

Proposition 4.1 *There are constants* $C_1 > 0$, λ, $\epsilon_0 \in (0, 1)$, *a countable collection of disjoint, open subsets* $\{\Delta_j\}_{j \in \mathbb{N}}$ *in* Δ_0 *and an expanding map* $T : \sqcup_j \Delta_j \to \Delta_0$ *such that:*

1. $\sum_j \mu(\Delta_j) = \mu(\Delta_0)$.
2. *For each* j, *there is an element* $\gamma_j \in \Gamma$ *such that* $\Delta_j = \gamma_j \Delta_0$ *and* $T|_{\Delta_j} = \gamma_j^{-1}$.
3. *Each* γ_j *is a uniform contraction:* $|\gamma_j'(x)| \leq \lambda$ *for all* $x \in \Delta_0$.
4. *For each* γ_j, $|D(\log |\gamma_j'|)(x)| < C_1$ *for all* $x \in \Delta_0$, *where* $D(\log |\gamma_j'|)(x)$ *is the differential of the map* $z \mapsto \log |\gamma_j'(z)|$ *at* x.
5. *Let* R *be the function on* $\sqcup_j \Delta_j$ *given by* $R(x) = \log |DT(x)|$. *Then* $\int e^{\epsilon_0 R} d\mu < \infty$.

Denote by $\mathcal{H} = \{\gamma_j\}_{j \in \mathbb{N}}$ the set of inverse branches of T. The last property is known as the exponential tail property and we will prove another form instead:

$$\sum_{\gamma \in \mathcal{H}} |\gamma'|_\infty^{\delta - \epsilon_0} < \infty, \tag{4.2}$$

where $|\gamma'|_\infty = \sup_{x \in \Delta_0} |\gamma'(x)|$. Proposition 4.1 (5) can be deduced from (4.2) by separating the integral to the sum of integrals over Δ_j and using quasi-invariance of PS measure.

Using Proposition 4.1, it can be shown that there exists a T-invariant ergodic probability measure ν on Δ_0 which is absolutely continuous with respect to PS measure and the density function $\bar{f}_0$ is a positive Lipschitz function bounded away from 0 and ∞ on $\Delta_0 \cap \Lambda_\Gamma$ (see for example [51, Lemma 2]).

The coding satisfies **uniform nonintegrable condition (UNI)**. Let

$$R_n(x) := \sum_{0 \leq k \leq n-1} R(T^k(x)) \text{ for } x \text{ with } T^k(x) \in \sqcup_j \Delta_j$$

$$\text{for all } 0 \leq k \leq n-1,$$

$$\mathcal{H}^n = \{\gamma_{j_1} \cdots \gamma_{j_n} : \gamma_{j_k} \in \mathcal{H} \text{ for } 1 \leq k \leq n\}.$$

For $\gamma \in \mathcal{H}^n$, we have $R_n(\gamma x) = -\log |\gamma'(x)|$. Set

$$C_2 = C_1/(1 - \lambda). \tag{4.3}$$

Then by Proposition 4.1 (3) and (4), we obtain for any $\gamma \in \mathcal{H}_n$,

$$\sup_{x \in \Delta_0} |D(\log |\gamma'|)(x)| \leq C_2. \tag{4.4}$$

Lemma 4.5 (UNI) *There exist $r > 0$ and $\epsilon_0 > 0$ such that for any $C > 1$ the following holds for any large n_0. There exist $j_0 \in \mathbb{N}$ and $\{\gamma_{mj} : 1 \leq m \leq 2, 1 \leq j \leq j_0\}$ in $\mathcal{H}_{n_0}$ such that for any $x \in \Lambda_\Gamma \cap \overline{\Delta}_0$ and any unit vector $e \in \mathbb{R}^d$ there exists $j \leq j_0$ such that for all $y \in B(x, r)$*

$$|\partial_e (\tau_{1j} - \tau_{2j})(y)| \geq \epsilon_0, \tag{4.6}$$

where $\tau_{mj}(x) = R_{n_0}(\gamma_{mj} x)$. Moreover, for all m, j,

$$|\mathrm{D}\tau_{mj}|_\infty \leq C_2, \quad |\gamma'_{mj}|_\infty \leq \epsilon_0/C. \tag{4.7}$$

The expanding map in the coding gives a contracting action in a neighborhood of ∞.

Lemma 4.8 *There exist $0 < \lambda < 1$ and a neighbourhood Λ_- of ∞ in Λ_Γ such that Λ_- is disjoint from $\overline{\Delta}_0$ and for any $\gamma \in \mathcal{H}$ and any $y, y' \in \Lambda_-$,*

$$\gamma^{-1}(\Lambda_-) \subset \Lambda_-, \quad d_{\mathbb{S}^d}(\gamma^{-1}y, \gamma^{-1}y') \leq \lambda d_{\mathbb{S}^d}(y, y'). \tag{4.9}$$

The proofs of these results will be postponed to Sect. 6. Proposition 4.1 and Lemma 4.8 will be proved at the end of Sect. 6.8 and Lemma 4.5 will be proved in Sect. 6.9.

4.1 A semiflow over hyperbolic skew product

Hyperbolic skew product. We construct a hyperbolic skew product using Lemma 4.8. Let $\Lambda_+ = \Lambda_\Gamma \cap \left(\sqcup_j \Delta_j\right)$ and Λ_- be given as Lemma 4.8. Define the map $\hat{T}$ on $\Lambda_+ \times \Lambda_-$ by

$$\hat{T}(x, x_-) = (\gamma_j^{-1} x, \gamma_j^{-1} x_-) \text{ for } (x, x_-) \in \Lambda_+ \times \Lambda_- \text{ with } x \in \Delta_j, \tag{4.10}$$

where γ_j is given as in Proposition 4.1 (2). Lemma 4.8 implies $\gamma^{-1}\Lambda_- \subset \Lambda_-$ for any $\gamma \in \mathcal{H}$. So $\hat{T}$ is well-defined.

Let $p : \Lambda_+ \times \Lambda_- \to \Lambda_+$ be the projection to the first coordinate. This gives rise to a semiconjugacy between $\hat{T}$ and T. We equip $\Lambda_+ \times \Lambda_-$ with the metric

$$d((x, x_-), (x', x'_-)) = d_E(x, x') + d_{\mathbb{S}^d}(x_-, x'_-).$$

(4.9) implies that the action of $\hat{T}$ on the fibre $\{x\} \times \Lambda_-$ is contracting. Using this observation, we obtain

Proposition 4.11 *1. There exists a unique $\hat{T}$-invariant, ergodic probability measure $\hat{v}$ on $\Lambda_+ \times \Lambda_-$ whose projection to Λ_+ is v.*
2. We have a disintegration of $\hat{v}$ over v: for any continuous function w on $\Lambda_+ \times \Lambda_-$,

$$\int_{\Lambda_+ \times \Lambda_-} w \mathrm{d}\hat{v} = \int_{\Lambda_+} \int_{\Lambda_-} w \mathrm{d}v_x(x_-)\mathrm{d}v(x).$$

Moreover, there exists $C > 0$ such that for any Lipschitz function w on $\Lambda_+ \times \Lambda_-$, defining $\bar{w}(x) = \int w \mathrm{d}v_x$, we have

$$\|\bar{w}\|_{\mathrm{Lip}} \le C \|w\|_{\mathrm{Lip}}.$$

Proof For the first statement, see [27, Theorem A] or [10, Proposition 1]. For the second statement, see [10, Proposition 3, Proposition 6], where they consider Riemannian manifold case but the same proofs also work in our fractal case. □

Remark The measure $\hat{v}$ is actually independent of the choice of the stable direction Λ_-: any Λ_- satisfying Lemma 4.8 will lead to the same measure $\hat{v}$.

Hyperbolic skew product flow. Let $R : \Lambda_+ \to \mathbb{R}_+$ be the function given in Proposition 4.1. By abusing notation, define $R : \Lambda_+ \times \Lambda_- \to \mathbb{R}_+$ by setting $R(x, x_-) = R(x)$. Define the space

$$\Lambda^R = \{(x, x_-, s) \in \Lambda_+ \times \Lambda_- \times \mathbb{R} : 0 \le s < R(x, x_-)\}.$$

Let $R_n = \sum_{j=0}^{n-1} R \circ \hat{T}^j$. The hyperbolic skew product flow $\{\hat{T}_t\}_{t \ge 0}$ over Λ^R is defined by $\hat{T}_t(x, x_-, s) = (\hat{T}^n(x, x_-), s + t - R_n(x, x_-))$ for $\hat{v}$-almost every x, where n is the nonnegative integer such that $0 \le s + t - R_n(x, x_-) < R(\hat{T}^n(x, x_-))$. We equip Λ^R with the measure $\mathrm{d}\hat{v}^R := \mathrm{d}\hat{v} \times \mathrm{d}t/\bar{R}$, where dt is Lebesgue measure on $\mathbb{R}_+$ and $\bar{R} = \int_{\Lambda_+ \times \Lambda_-} R \mathrm{d}\hat{v}$. This is a $\hat{T}_t$-invariant ergodic measure.

Remark We don't use the commonly used "suspension space" construction to construct Λ^R. The reason is that we will use a cutoff function in the proof of Theorem 1.1 and such cutoff functions are ill-defined in the suspension space, which is a quotient space of $\Lambda_+ \times \Lambda_- \times \mathbb{R}$.

For any L^∞ function $w : \Lambda^R \to \mathbb{R}$, the Lipschitz norm of w is defined by

$$\|w\|_{\mathrm{Lip}} = |w|_\infty + \sup_{(y,a) \ne (y',a') \in \Lambda^R} \frac{|w(y, a) - w(y', a')|}{d(y, y') + |a - a'|}. \qquad (4.12)$$

In Sect. 8, we will prove that $\hat{T}_t$ is exponential mixing with respect to $\hat{v}^R$.

Theorem 4.13 *There exist $\epsilon_1 > 0$ and $C > 1$ such that for any Lipschitz functions u, w on Λ^R and any $t > 0$, we have*

$$\left| \int u\, w \circ \hat{T}_t \mathrm{d}\hat{v}^R - \int u \mathrm{d}\hat{v}^R \int w \mathrm{d}\hat{v}^R \right| \leq C e^{-\epsilon_1 t} \|u\|_{\mathrm{Lip}} \|w\|_{\mathrm{Lip}}.$$

4.2 Exponential mixing of geodesic flow

The map from Λ^R to $\mathrm{T}^1(M)$. We construct a map from Λ^R to $\mathrm{T}^1(M)$ which allows us to deduce the exponential mixing of the geodesic flow from that of $\hat{T}_t$.

Recall the Hopf parametrization in Sect. 2.4. We introduce the following time change map to have the function R given by derivative (see Proposition 4.1):

$$\tilde{\Phi} : \Lambda^R \to \partial^2(\mathbb{H}^{d+1}) \times \mathbb{R}, \quad (x, x_-, s) \mapsto (x, x_-, s - \log(1 + |x|^2)).$$

The map $\tilde{\Phi}$ induces a map $\Phi : \Lambda^R \to \mathrm{T}^1(M)$, where we use the Hopf parametrization to identify $\mathrm{T}^1(M)$ with $\Gamma \backslash \partial^2(\mathbb{H}^{d+1}) \times \mathbb{R}$. Note that $\Lambda_+ \times \{\infty\} \times \{0\}$ is mapped to the unstable horosphere based at ∞ and passing o. The map Φ defines a semiconjugacy between two flows:

$$\Phi \circ \hat{T}_t = \mathcal{G}_t \circ \Phi, \quad \text{for } t \geq 0. \tag{4.14}$$

To see this, note that for any $(x, x_-, s) \in \Lambda^R$, we have the expresssion

$$\hat{T}_t(x, x_-, s) = (\hat{T}^n(x, x_-), s + t - R_n(x, x_-)),$$
$$\hat{T}^n(x, x_-) = \gamma^{-1}(x, x_-) \text{ for some } \gamma \in \mathcal{H}^n.$$

By straightforward computation, we obtain

$$\tilde{\Phi} \circ \hat{T}_t(x, x_-, s) = \mathcal{G}_t \circ \gamma^{-1} \tilde{\Phi}(x, x_-, s),$$

which leads to (4.14) by passing to the quotient space.

Relating $\hat{v}^R$ with m^{BMS}. The map Φ is not injective in general. Nevertheless, we are able to use $(\Lambda^R, \hat{T}_t, \hat{v}^R)$ to study $(\mathrm{T}^1(M), \mathcal{G}_t, m^{\mathrm{BMS}})$. The main result is the following proposition.

Proposition 4.15 *The map $\Phi : (\Lambda^R, \hat{T}_t, \hat{v}^R) \to (\mathrm{T}^1(M), \mathcal{G}_t, m^{\mathrm{BMS}})$ is a factor map, i.e.,*

$$\Phi_* \hat{v}^R = m^{\mathrm{BMS}} \text{ and } \Phi \circ \hat{T}_t = \mathcal{G}_t \circ \Phi \text{ for all } t \geq 0.$$

We need two lemmas to prove this proposition.

Lemma 4.16 *There exists a measurable subset U in Λ^R such that by setting $V = \Phi(U)$ in $\mathrm{T}^1(M)$, the restriction map of Φ on U gives a bijection between U and V. Moreover, the set V is of positive BMS measure.*

Proof We make use of the following commutative diagram

$$
\begin{array}{ccc}
\Lambda^R & \xrightarrow{\tilde{\Phi}} & \partial^2(\mathbb{H}^{d+1}) \times \mathbb{R} \\
{\scriptstyle \Phi}\downarrow & \swarrow{\scriptstyle \pi} & \\
\mathrm{T}^1(M) & &
\end{array}
$$

where π is the covering map. Let $\epsilon > 0$ be a number such that $\epsilon < \inf_{(x, x_-) \in \Lambda_+ \times \Lambda_-} R(x, x_-)$. Set $S = \Lambda_+ \times \Lambda_- \times [0, \epsilon)$. The restriction map $\tilde{\Phi}|_S$ gives a bijection between S and its image. Pick any $x \in S$. As π is a covering map, there exists an open set $W \subset \partial^2(\mathbb{H}^{d+1}) \times \mathbb{R}$ containing $\tilde{\Phi}(x)$ such that the restriction map $\pi|_W$ is a bijection. The sets $U = \tilde{\Phi}^{-1}(W \cap \tilde{\Phi}(S))$ in Λ^R and $V = \pi(W \cap \tilde{\Phi}(S))$ satisfy the proposition. $\square$

Lemma 4.17 *Let $\mathcal{Q}'$ be any subset in Λ^R with full $\hat{\nu}^R$ measure and $\mathcal{Q}$ be any subset in $\mathrm{T}^1(M)$ with full m^{BMS} measure. Then there exist $x \in \mathcal{Q}'$ and $y \in \mathcal{Q}$ such that $\Phi(x)$ and y are in the same stable leaf.*

Proof The idea of the proof is straightforward: we make use of the local product description of $\hat{\nu}^R$ and m^{BMS}.

Let Φ_U be the restriction of Φ on U. In view of Lemma 4.16, we can consider the measure $\Phi_U^*(m^{\mathrm{BMS}}|_V)$ on U, the pull back of $m^{\mathrm{BMS}}|_V$, and denote it by m for simplicity. We can choose U and V sufficiently small so that m is given by

$$
\mathrm{d}m(x, x_-, t) = cD(x, x_-)^{-2\delta} \mathrm{d}\mu(x) \mathrm{d}\mu(x_-) dt,
$$

where c is a positive constant and $D(x, x_-) = e^{\beta_x(o, x_*)/2} e^{\beta_{x_-}(o, x_*)/2}$ known as the visual distance. Let $p : \Lambda^R \to \Lambda_+ \times \mathbb{R}$ be the projection map, forgetting the Λ_--coordinate. Then the pushforward measure $p_* m$ is given by

$$
\mathrm{d}p_* m(x, t) = c \mathrm{d}\mu(x) \mathrm{d}t \int_{\{x\} \times \Lambda_- \times \{t\} \cap U} D(x, x_-)^{-2\delta} \mathrm{d}\mu(x_-).
$$

So it is absolutely continuous with respect to the measure $\mathrm{d}\nu \otimes \mathrm{d}t$.

We can find a set of the form $B = B_+ \times \Lambda_- \times (t_1, t_2)$ such that $\nu(B_+) > 0$ and $m(B \cap U) > 0$. The pushforward measure $p_*(\hat{\nu}^R|_B)$ is given by

$$
dp_*(\hat{\nu}^R|_B) = \mathrm{d}\nu \otimes \mathrm{d}t. \tag{4.18}
$$

On the one hand, we have that $p(Q' \cap B)$ is a conull set in $p(B)$ with respect to $p_*(\hat{v}^R|_B)$.

On the other hand, we consider $B' := \Phi_U^{-1}(Q \cap V) \cap B$. It is a set with positive m measure and hence $p_*(B')$ is of positive $p_*(m)$ measure. The fact that $dp_*(m)$ is absolutely continuous with respect to $dv \otimes dt$ and (4.18) imply that $p_*(B')$ is of positive $p_*(\hat{v}^R|_B)$ measure. Therefore,

$$p(Q' \cap B) \cap p(B') \neq \emptyset.$$

Let (x, t) be a point in the intersection. Then the points $(x, x_-, t) \in Q'$ and $\Phi(x, x'_-, t) \in Q$ satisfy the conditions of the lemma. □

Proof of Proposition 4.15 Let f be a C^1 function on $T^1(M)$ with finite C^1-norm. Since m^{BMS} is ergodic [49], by Birkhoff ergodic theorem, for m^{BMS}-a.e. y in $T^1(M)$

$$\lim_{T \to +\infty} \frac{1}{T} \int_{0 \leq t \leq T} f(\mathcal{G}_t y) dt = \int f dm^{\text{BMS}}. \tag{4.19}$$

Let Q be the set of points at which (4.19) hold and it is a set of full m^{BMS} measure.

We consider $f \circ \Phi$, which can be thought as the lifting of f to Λ^R. It is $\hat{v}^R$-integrable. Since $\hat{T}_t$ is mixing with respect to $\hat{v}^R$, by Birkhoff ergodic theorem, for $\hat{v}^R$-a.e. x,

$$\lim_{T \to +\infty} \frac{1}{T} \int_{0 \leq t \leq T} f \circ \Phi(\hat{T}_t x) dt = \int f \circ \Phi d\hat{v}^R.$$

Using the semiconjugacy $\Phi \circ \hat{T}_t = \mathcal{G}_t \circ \Phi$, we actually have

$$\lim_{T \to +\infty} \frac{1}{T} \int_{0 \leq t \leq T} f(\mathcal{G}_t \Phi x) dt = \int f \circ \Phi d\hat{v}^R. \tag{4.20}$$

Let Q' be the set of points at which (4.20) hold and it is a set of full $\hat{v}^R$ measure.

By Lemma 4.17, there exist points $x \in \Lambda^R$ and $y \in T^1(M)$ such that $\Phi(x)$ and y are in the same stable leaf. Due to $d(\mathcal{G}_t y, \mathcal{G}_t \Phi x) \to 0$ as $t \to +\infty$ and the uniform continuity of f,

$$\lim_{T \to +\infty} \left(\frac{1}{T} \int_{0 \leq t \leq T} f(\mathcal{G}_t y) dt - \frac{1}{T} \int_{0 \leq t \leq T} f(\mathcal{G}_t \Phi x) dt \right) = 0.$$

Therefore, we can deduce that

$$\int f \, dm^{\mathrm{BMS}} = \int f \circ \Phi \, d\hat{v}^R.$$

The above equation holds for every C^1 function on $\mathrm{T}^1(M)$. The proof is complete. $\qquad\square$

Proof of Theorem 3.1. We are ready to prove Theorem 3.1. With Theorem 4.13 and Proposition 4.15 available, the work lies in the comparing the norm of the functions on Λ^R with that on $\mathrm{T}^1(M)$. This is not obvious. Consider two points of the form (y, a) and (y', a) in Λ^R. By (4.12), $d((y, a), (y', a))$ remains the same when a changes. But if these two points are projected to $\mathrm{T}^1(M)$, changing a means flowing these two points by the geodesic flow and $d(\Phi((y, a), \Phi(y', a)))$ will change. Moreover, the function R used to define Λ^R is unbounded, making the argument more complex.

Proof of Theorem 3.1 Let u, v be any two C^1-functions on $\mathrm{T}^1(M)$ with finite C^1-norm. Without loss of generality, we may assume that $m^{\mathrm{BMS}}(u) = 0$. Set $U = u \circ \Phi$ and $W = w \circ \Phi$. Using the semiconjugacy of Φ, we obtain

$$\int u \cdot w \circ \mathcal{G}_t \, dm^{\mathrm{BMS}} = \int U \cdot W \circ \hat{T}_t \, d\hat{v}^R.$$

We use a cutoff function to relate the norms of U, W with those of u, w. Let $\epsilon > 0$ be a constant less than $\epsilon_1/2$. Let τ_t be a decreasing Lipschitz function on $[0, \infty)$ such that $\tau_t = 1$ on $[0, \epsilon t]$, $\tau_t = 0$ on $(\epsilon t + 1, \infty)$ and $|\tau_t|_{\mathrm{Lip}} < 2$. Set $U_t = U \cdot \tau_t$ and $W_t = W \cdot \tau_t$. For any two points (y, a) and (y', a') (we may assume $a \geq a'$), we have

$$\begin{aligned}
&|U_t(y, a) - U_t(y', a')| \\
&\quad \leq |U_t(y, a) - U_t(y, a')| + |U_t(y, a') - U_t(y', a')| \\
&\quad \leq \tau_t(a')|U(y, a) - U(y, a')| + |u|_\infty|\tau_t(a) - \tau_t(a')| \\
&\qquad + \tau_t(a')|U(y, a') - U(y', a')| \\
&\quad \ll |u|_{C^1}|a - a'| + |u|_\infty|a - a'| + e^{\epsilon t}|u|_{C^1}d(y, y'),
\end{aligned}$$

where to obtain the last inequality, we use the fact $d(\Phi(y, a'), \Phi(y', a')) \leq e^{a'}d(y, y')$ and $\tau_t \neq 0$ only on $[0, \epsilon t + 1]$. Therefore, we have

$$\|U_t\|_{\mathrm{Lip}} \ll e^{\epsilon t}\|u\|_{C^1}. \tag{4.21}$$

A verbatim of the above argument also implies $\|W_t\|_{\mathrm{Lip}} \ll e^{\epsilon t}\|w\|_{C^1}$.

We also need the following L^1-estimate. Using the exponential tail condition (Proposition 4.1 (5)), we obtain

$$|U_t - U|_{L^1(\hat{v}^R)} \leq |u|_\infty \int \max\{R(x) - \epsilon t, 0\}d\nu(x)$$

$$\ll |u|_\infty \int e^{\epsilon_o(R(x)-\epsilon t)}d\nu(x) \ll e^{-\epsilon_o \epsilon t}|u|_\infty. \tag{4.22}$$

The similar estimate holds for $W_t - W$. As $m^{\mathrm{BMS}}(u) = 0$, we have

$$|\int U_t d\hat{v}^R| \ll e^{-\epsilon_o \epsilon t}|u|_\infty. \tag{4.23}$$

Using Theorem 4.13 together with (4.21), (4.22) and (4.23), we obtain

$$|\int U \cdot W \circ \hat{T}_t d\hat{v}^R|$$

$$\leq |\int U_t \cdot W_t \circ \hat{T}_t d\hat{v}^R|$$

$$+ |\int (U - U_t) \cdot W_t \circ \hat{T}_t d\hat{v}^R| + |\int U \cdot (W - W_t) \circ \hat{T}_t d\hat{v}^R|$$

$$\ll |\int U_t d\hat{v}^R| \cdot |\int W_t d\hat{v}^R|$$

$$+ e^{-\epsilon_1 t}\|U_t\|_{\mathrm{Lip}}\|W_t\|_{\mathrm{Lip}} + |w|_\infty|U - U_t|_{L^1(\hat{v}^R)}$$

$$+ |u|_\infty|W - W_t|_{L^1(\hat{v}^R)}$$

$$\ll (e^{-(\epsilon_1-2\epsilon)t} + e^{-\epsilon_o \epsilon t})|u|_{C^1}|w|_{C^1}.$$

Due to $\epsilon < \epsilon_1/2$, the proof is complete. □

5 Parabolic fixed points and measure estimate

In this section, we provide a detailed description of the Γ-action on $\partial\mathbb{H}^{d+1}$ and different types of estimate for the PS measure.

5.1 Explicit computation

Let H_∞ be the horoball based at ∞ given by $\mathbb{R}^d \times \{x \in \mathbb{R} : x > 1\}$. For a horoball H, we define the height of the horoball by

$$h(H) := \sup_{y \in H} h(y).$$

Lemma 5.1 *Suppose* $g \in G$ *is not in* $Stab_G(\infty)$. *Let* $p = g\infty$ *and* $p' = g^{-1}\infty$. *Then we have*

●

$$h(g H_\infty) = h(g^{-1} H_\infty). \tag{5.2}$$

● *For any* $x \in \mathbb{H}^{d+1} \cup \partial\mathbb{H}^{d+1}$, *we have*

$$gx = h(g H_\infty)\frac{x - (p', 0)}{|x - (p', 0)|^2}\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} + (p, 0)$$

$$g^{-1}x = h(g H_\infty)\frac{x - (p, 0)}{|x - (p, 0)|^2}\begin{pmatrix} A^{-1} & 0 \\ 0 & 1 \end{pmatrix} + (p', 0), \tag{5.3}$$

where $A \in SO(d)$, *and we view* x, $(p, 0)$ *and* $(p', 0)$ *as row vectors in* $\mathbb{R}^{d+1}$.

Proof By Proposition A.3.9 (2) in [12], the action of g on the upper half space is given by

$$gx = \lambda\iota(x)\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} + (b, 0),$$

where A is in $SO(d)$, $\lambda \in \mathbb{R}^+$, $b \in \mathbb{R}^d$ and $\iota(x)$ either equals x or is given by an inversion with respect to a unit sphere centered at $\mathbb{R}^d \times \{0\}$. In fact, this is the Bruhat decomposition of G. Since g does not fix ∞, $\iota(x)$ is an inversion. We have for any $x \in \mathbb{H}^{d+1}$

$$gx = \lambda\frac{x - (x', 0)}{|x - (x', 0)|^2}\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} + (b, 0)$$

with $x' \in \mathbb{R}^d$. Hence $b = g\infty = p$ and $x' = g^{-1}\infty = p'$.
Note that

$$h(gx) = \lambda h(x)/|x - (p', 0)|^2.$$

Since g maps the original horoball $H_\infty = \mathbb{R}^d \times \{x > 1\}$ to the horoball $g H_\infty$ based at p, it follows from the above formula that

$$h(g H_\infty) = \sup_{x \in \mathbb{R}^d \times \{1\}} h(gx) = \lambda.$$

To obtain the formula for $g^{-1}x$, note that

$$x = g(g^{-1}x) = h(gH_\infty) \frac{g^{-1}x - (p', 0)}{|g^{-1}x - (p', 0)|^2} \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} + (p, 0).$$

This yields

$$\frac{1}{h(gH_\infty)} (x - (p, 0)) \begin{pmatrix} A^{-1} & 0 \\ 0 & 1 \end{pmatrix} + (p', 0) = \frac{g^{-1}x - (p', 0)}{|g^{-1}x - (p', 0)|^2} + (p', 0).$$

Applying both sides the inversion with respect to the unit sphere centered at $(p', 0)$ and using the fact that $A \in SO(d)$, we obtain the formula for $g^{-1}x$. Meanwhile, we apply to $g^{-1}x$ the argument used to get the formula for gx. Comparing the formulas given by these two methods, we have that $h(gH_\infty) = h(g^{-1}H_\infty)$. □

Lemma 5.4 *For a horoball H based at $p \neq \infty$ and $g \in G$ not in $Stab_G(\infty)$, we have*

$$h(gH) \geq \frac{h(g^{-1}H_\infty)h(H)}{d(g^{-1}\infty, p)^2 + h(H)^2}, \tag{5.5}$$

$$h(gH) \leq \frac{h(g^{-1}H_\infty)h(H)}{(d(g^{-1}\infty, p) - h(H)/2)^2}. \tag{5.6}$$

Proof Using (5.3), we obtain

$$h(gH) \geq h(g(p, h(H))) = \frac{h(g^{-1}H_\infty)h(H)}{d(p, g^{-1}\infty)^2 + h(H)^2}.$$

For (5.6), we have

$$h(gH) = \sup_{y \in \partial H} h(gy) = \sup_{y \in \partial H} \frac{h(g^{-1}H_\infty)h(y)}{|y - (g^{-1}\infty, 0)|^2}.$$

Note that for every $y \in \partial H$, we have $|y - (g^{-1}\infty, 0)|^2 \geq d_E(y', g^{-1}\infty)^2 \geq (d(g^{-1}\infty, p) - h(H)/2)^2$, where y' is the projection of y to $\partial \mathbb{H}^{d+1}$ and this yields (5.6). □

Let $\mathcal{P}$ be the set of parabolic fixed points in $\partial \mathbb{H}^{d+1}$. Two parabolic fixed points are called equivalent if they are in the same Γ-orbit. Let $\mathbf{P}$ be a complete set of inequivalent parabolic fixed points. As Γ is geometrically finite, the set $\mathbf{P}$ is finite. Suppose that $\mathbf{P} = \{p_1, \ldots, p_j\}$ for some $j \in \mathbb{N}$, a complete set of inequivalent parabolic fixed points and set $p_1 = \infty$.

Fix a collection of pairwise disjoint horoballs based at parabolic fixed points as follows. Without loss of generality, we may assume that H_∞ is a horocusp region for ∞. For the parabolic fixed point $\gamma\infty$, we attach the horoball $H_{\gamma\infty} := \gamma H_\infty$ to it. For other parabolic fixed point p_i in $\mathbf{P}$, we fix a horoball H_{p_i} based at p_i which is a horocusp region for p_i. For other parabolic fixed points γp_i in the Γ orbit of p_i, we attach the horoball $H_{\gamma p_i} := \gamma H_{p_i}$ to it. By Definition 2.2, we can choose the horoballs in such a way that they are pairwise disjoint. For $p \in \mathcal{P}$, we define the **height** function $h(p)$ of p as the height of H_p based at p, that is

$$h(p) := h(H_p).$$

For $x \in \partial\mathbb{H}^{d+1}$ and $r > 0$, set $B(x, r)$ to be the ball centered at x of radius r in $\partial\mathbb{H}^{d+1}$ with respect to the Euclidean metric.

Lemma 5.7 (Explicit computation) *Suppose γ is not in $Stab_\infty(\Gamma)$. Then for any $r > 0$ and $x \in \partial\mathbb{H}^{d+1}$,*

- $\gamma^{-1}B(p, r) = B(p', h(p)/r)^c,$
- $|(\gamma^{-1})'(x)| = h(p)/d(x, p)^2, \quad |\gamma'(x)| = h(p)/d(x, p')^2,$

where $p = \gamma\infty$ and $p' = \gamma^{-1}\infty$.

Proof The first equation follows from (5.3) easily.

In view of Lemma 5.1, the computation of the derivative of inversion maps gives the expression of $|\gamma'(x)|$ and $|(\gamma^{-1})'(x)|$. $\qquad\square$

5.2 Multi-cusps

Recall that $\mathbf{P} = \{p_1, \ldots, p_j\}$. For each p_i, we consider a coordinate change transformation: let g_i be an element in G such that $g_i p_i = \infty$. This g_i is not unique and we can choose a g_i such that $g_i H_{p_i} = H_\infty = \mathbb{R}^d \times \{x > 1\}$. We will frequently make use of the following commutative diagram:

$$\begin{array}{ccc} \mathbb{H}^{d+1} & \xrightarrow{g_i} & \mathbb{H}^{d+1} \\ \Gamma\downarrow & & \downarrow g_i\Gamma g_i^{-1} \\ \mathbb{H}^{d+1} & \xrightarrow{g_i} & \mathbb{H}^{d+1}. \end{array} \qquad (5.8)$$

On the right hand side of the diagram, the acting group is $g_i\Gamma g_i^{-1}$ and ∞ is a parabolic fixed point of the group.

Once g_i's are fixed, we consider the action of $g_i\Gamma g_i^{-1}$ and its parabolic fixed points $g_i\mathcal{P}$. We think they are in i-th hyperbolic space. Set the horoball

$H_{g_i p}(i) = g_i H_p$ for $p \in \mathcal{P}$ and define the height

$$h(g_i p, i) := h(H_{g_i p}(i)).$$

If there is no confusion, we always abbreviate $H_{g_i p}(i)$ and $h(g_i p, i)$ to $H_{g_i p}$ and $h(g_i p)$.

The results in Sect. 2.3 hold for each p_i. We have the group $(g_i \Gamma g_i^{-1})_\infty$, which is a maximal normal abelian subgroup in $\text{Stab}_{g_i \Gamma g_i^{-1}}(\infty)$. Write

$$\Gamma_{p_i} = g_i^{-1} (g_i \Gamma g_i^{-1})_\infty g_i. \tag{5.9}$$

Let Δ'_{p_i} be a fundamental region for the parabolic fixed point ∞ under the $g_i \Gamma g_i^{-1}$-action. We can choose Δ'_{p_i} in such a way that for $\Delta_{p_i} = g_i^{-1} \Delta'_{p_i}$

$$\{p_i, \ 1 \le i \le j\} \cap (\cup_{1 \le k \le j} \overline{\Delta}_{p_k}) = \emptyset. \tag{5.10}$$

This choice is possible because, for each p_i, we can find a Δ'_{p_i} such that Δ_{p_i} sufficiently close to p_i in $\partial \mathbb{H}^{d+1}$ under the spherical metric. Set

$$\Delta = \cup_{1 \le k \le j} \Delta_{p_k}.$$

By (5.10), we have $\overline{\Delta} \cap \{\infty\} = \emptyset$. So the set $\overline{\Delta}$ is compact.

Consider any parabolic fixed point $p = \gamma p_i$ with $\gamma \in \Gamma$. We know that such γ is not unique (any element in $\gamma \text{Stab}_\Gamma(p_i)$ also works) and we fix a choice of γ such that $\gamma^{-1} p_i \in \overline{\Delta}_{p_i}$. We call γ **the representation** of p. Set

$$x_p := \gamma^{-1} p_i.$$

Lemma 5.11 *There exists $C > 1$ such that for $1 < i \le j$ and for any parabolic fixed point p in Δ, we have*

$$1/C \le h(g_i p)/h(p) \le C.$$

Proof Consider the action of g_i on $\partial \mathbb{H}^{d+1}$. Notice that $p_i = g_i^{-1} \infty$ and $h(p_i) = h(H_{p_i}) = h(g_i^{-1} H_\infty)$. Applying (5.5) to the horoball H_p based at p and the element g_i, we obtain

$$h(g_i p) = h(g_i H_p) \ge \frac{h(p_i) h(p)}{d(p_i, p)^2 + h(p)^2}.$$

It follows from $\overline{\Delta}$ compact that $d(p_i, p)$ is bounded for $p \in \Delta$. Then $h(g_i p) \ge h(p)/C$.

For the other inequality, notice that $g_i p_1 = g_i \infty$ and $h(g_i p_1) = h(g_i H_\infty)$. Applying (5.5) with with the horoball $H_{g_i p}$ based at $g_i p$ and g_i^{-1} we have

$$h(p) = h(g_i^{-1} H_{g_i p}) \geq \frac{h(g_i p_1) h(g_i p)}{d(g_i p_1, g_i p)^2 + h(g_i p)^2}.$$

It follows from (5.10) that $g_i \overline{\Delta} \cap \{\infty\} = g_i \overline{\Delta} \cap \{g_i p_i\} = \emptyset$. So $g_i \overline{\Delta}$ is compact and $d(g_i p_1, g_i p)$ is bounded for $p \in \Delta$. Then $h(p) \geq h(g_i p)/C$. $\square$

Lemma 5.12 *For $1 < i \leq j$, the map $g_i : \Delta \to g_i \Delta$ is bi-Lipschitz.*

Proof By (5.3), we have

$$d(g_i x, g_i y) = h(p_i) d \left(\frac{x - p_i}{|x - p_i|^2}, \frac{y - p_i}{|y - p_i|^2} \right) \leq C |x - y| = C d(x, y),$$

where the inequality due to (5.10).

For the other direction, we use (5.3) to obtain

$$d(g_i^{-1} x, g_i^{-1} y) = h(p_i) d \left(\frac{x - g_i p_1}{|x - g_i p_1|^2}, \frac{y - g_i p_1}{|y - g_i p_1|^2} \right)$$
$$\leq C |x - y| = C d(x, y),$$

where the inequality is due to that $d(x, g_i p_1)$ is bounded below for $x \in g_i \Delta$ by (5.10). $\square$

Patterson–Sullivan measure under conjugation. In the presence of multi-cusps, we need to consider the Patterson–Sullivan measure for the conjugation of Γ. Recall that $\{\mu_y\}_{y \in \mathbb{H}^{d+1}}$ is the Γ-invariant conformal density of dimension δ and we denoted μ_o by μ for short. For each g_i with $1 < i \leq j$, set $\Gamma_i = g_i \Gamma g_i^{-1}$. The limit set Λ_{Γ_i} is $g_i \Lambda_\Gamma$ and the critical exponent of Γ_i equals δ. For every $y \in \mathbb{H}^{d+1}$, define the following measure

$$\tilde{\mu}_y := (g_i)_* \mu_{g_i^{-1} y},$$

where $(g_i)_* \mu_{g_i^{-1} y}(E) = \mu_{g_i^{-1} y}(g_i^{-1} E)$ for any Borel subset E of $\partial \mathbb{H}^{d+1}$. It is easy to check that $\tilde{\mu}_y$ is supported on Λ_{Γ_i} and for any $y, z \in \mathbb{H}^{d+1}, x \in \partial \mathbb{H}^{d+1}$ and $\gamma \in \Gamma_i$,

$$\frac{d\tilde{\mu}_y}{d\tilde{\mu}_z}(x) = e^{-\delta \beta_x(y,z)} \quad \text{and} \quad (\gamma)_* \tilde{\mu}_y = \tilde{\mu}_{\gamma y}.$$

It follows from the uniqueness of Γ_i-invariant conformal density that this construction gives exactly the Γ_i-invariant conformal density on Λ_{Γ_i} of dimension

δ. In later sections, we will denote $\tilde{\mu}_o$ by μ_{Γ_i} and the above analysis yields that for any Borel subset E of $\partial \mathbb{H}^{d+1}$

$$e^{-\delta d(o, g_i o)} \mu(g_i^{-1} E) \leq \mu_{\Gamma_i}(E) \leq e^{\delta d(o, g_i o)} \mu(g_i^{-1} E). \tag{5.13}$$

5.3 Doubling property of PS measure

We start with two results: Proposition 5.14 and Lemma 5.15, deduced from [46, Theorem 2]. They used spherical metric, but locally it is equivalent to euclidean metric.

Proposition 5.14 • *(Doubling property) For every $C > 1$, there exists $\epsilon < 1$ such that for every $x \in \Lambda_\Gamma \cap \Delta$ and $1/C \geq r > 0$,*

$$\mu(B(x, r)) > \epsilon \mu(B(x, Cr)).$$

• *(Growth of measure) There exists $C_3 > 1$, such that for every $x \in \Lambda_\Gamma \cap \Delta$ and $r < 1/C_3$,*

$$2\mu(B(x, r)) < \mu(B(x, C_3 r)).$$

Lemma 5.15 *Let p be a parabolic fixed point in Δ of rank k. For $0 < r \leq h(p)$,*

$$\mu(B(p, r)) \approx r^{2\delta - k} h(p)^{k - \delta}. \tag{5.16}$$

Lemma 5.17 *For every $C > 1$, there exists $C' > 1$ such that for every parabolic fixed point $p = \gamma \infty \in \Delta$ with γ the representation, for any Borel subset $E \subset B(p, Ch(p))$, we have*

$$h(p)^\delta \mu(\gamma^{-1} E)/C' \leq \mu(E) \leq C' h(p)^\delta \mu(\gamma^{-1} E).$$

Proof As the PS measure is quasi-invariant, we have

$$\mu(\gamma^{-1} E) = \int_{x \in E} |(\gamma^{-1})' x|^\delta \left(\frac{1 + |x|^2}{1 + |\gamma^{-1} x|^2} \right)^\delta \, d\mu(x).$$

By Lemma 5.7, we have $|(\gamma^{-1})' x| = h(p)/d(x, p)^2$. We also have

$$|d(\gamma^{-1} \infty, 0) - h(p)/d(x, p)| \leq |\gamma^{-1} x| \leq d(\gamma^{-1} \infty, 0) + h(p)/d(x, p).$$

Due to $\gamma^{-1} \infty \in \overline{\Delta}$, if $h(p)/d(x, p) > \max\{2d(\gamma^{-1} \infty, 0), 1\}$, then $1 + |\gamma^{-1} x|^2 \approx (h(p)/d(x, p))^2$. Otherwise, due to $h(p)/d(x, p) \geq 1/C$, we

also have $1 + |\gamma^{-1}x|^2 \approx (h(p)/d(x, p))^2$. The lemma follows from these bounds on $|(\gamma^{-1})'x|$ and $1 + |\gamma^{-1}x|^2$. The computation also makes sense even if $x = p$, because the ratio $|(\gamma^{-1})'x|/(1 + |\gamma^{-1}x|^2)$ is always bounded in $B(p, Ch(p)) - \{p\}$ and we can extend it continous to p. □

Recall $\Delta_0 := \Delta_\infty = \Delta_{p_1}$.

Lemma 5.18 *There exists $C > 0$ such that for any Borel set E of diameter less than the diameter of Δ_0 and $\gamma_1 \in \Gamma_\infty$, we have that for any $x \in E$*

$$\frac{\mu(\gamma_1 E)}{\mu(E)} \in (1/C, C) \left(\frac{1 + |x|^2}{1 + |\gamma_1 x|^2} \right)^\delta.$$

Proof Due to the derivative of γ_1 and the quasi-invariance of PS measure, we obtain

$$\mu(\gamma_1 E) = \int_E \left(\frac{1 + |x|^2}{1 + |\gamma_1 x|^2} \right)^\delta d\mu(x).$$

Now for any x, y in E, we have

$$\frac{1 + |x|^2}{1 + |y|^2} = 1 + \frac{(|x| - |y|)(|x| + |y|)}{1 + |y|^2} \le C', \tag{5.19}$$

with $C' > 1$ only depending on the diameter of E. The same argument also gives the same upper bound for $(1 + |y|^2)/(1 + |x|^2)$. The set $\gamma_1 E$ is a set with the same diameter as E. So we also have

$$\frac{1 + |\gamma_1 x|^2}{1 + |\gamma_1 y|^2} \in (1/C', C') \tag{5.20}$$

for any x, y in E. The proof is complete by applying (5.19) and (5.20) to the formula of $\mu(\gamma_1 E)$. □

Lemma 5.21 *There exist constants $c > 0$ and $C > 1$ such that for every parabolic fixed point $p \neq \infty$, if $r \le h(p)/C$, then*

$$\mu(B(p, r) - B(p, r/\sqrt{e})) \ge c\mu(B(p, r)).$$

Proof Consider $p \in \overline{\Delta}_0$. Indeed, since $\Gamma_\infty \overline{\Delta}_0$ covers the intersection $\mathbb{R}^d \cap \Lambda_\Gamma$, we can always find a γ_1 in Γ_∞ such that $\gamma_1 p \in \overline{\Delta}_0$. Then applying Lemma 5.18 to $E = B(p, r)$ and $E = B(p, r) - B(p, r/\sqrt{e})$, we have

$$\frac{\mu(B(p, r) - B(p, r/\sqrt{e}))}{\mu(B(p, r))} \approx \frac{\mu(B(\gamma_1 p, r) - B(\gamma_1 p, r/\sqrt{e}))}{\mu(B(\gamma_1 p, r))}.$$

We only need to give a lower bound to $\mu(B(p, r) - B(p, r/\sqrt{e}))$ and then use (5.16) to obtain Lemma 5.21.

Assume $p \in \overline{\Delta_0}$ is of rank k. Consider the case when $p = \gamma\infty$ with $\gamma \in \Gamma$ the representation of p. We claim that there exists a constant $C > 1$ such that for $\gamma_1 \in \Gamma_\infty$, with $\gamma\gamma_1\Delta_0 \subset B(p, r) - B(p, r/\sqrt{e})$, we have

$$\mu(\gamma\gamma_1\Delta_0) \gg \frac{h(p)^\delta}{(d(\gamma_1\Delta_0, x_p) + C)^{2\delta}}. \tag{5.22}$$

Proof of the claim: By Lemma 5.17, we have $\mu(\gamma\gamma_1\Delta_0) \approx h(p)^\delta \mu(\gamma_1\Delta_0)$. By Lemma 5.18, we have

$$\mu(\gamma_1\Delta_0) \approx \left(\frac{1 + |x|^2}{1 + |\gamma_1 x|^2}\right)^\delta \mu(\Delta_0),$$

for any $x \in \Delta_0$. Now, since $|\gamma_1 x| \leq d(\gamma_1\Delta_0, x_p) + |x_p| + C'$, so

$$\mu(\gamma_1\Delta_0) \geq \mu(\Delta_0)/(d(\gamma_1\Delta_0, x_p) + C)^{2\delta}$$

for some constant $C > 1$.

By Lemma 5.7, we have $\gamma^{-1}(B(p, r) - B(p, r/\sqrt{e})) = B(x_p, \sqrt{e}h(p)/r) - B(x_p, h(p)/r)$. Let $C' = \text{diam}(\Delta_0)$. Let $\mathbb{R}^k$ be the subspace described in Lemma 2.3. For a set E in $\mathbb{R}^d$, we define $\text{Vol}_{\mathbb{R}^k}(E)$ as $\text{Vol}(E \cap \mathbb{R}^k)$. Since $x_p \in \Lambda_\Gamma \cap \mathbb{R}^d$ has bounded distance to $\mathbb{R}^k$, the number of $\gamma_1\Delta_0$'s in such region is at least

$$\text{Vol}_{\mathbb{R}^k}\left(B\left(x_p, \sqrt{e}h(p)/r - C'\right) - B\left(x_p, h(p)/r + C'\right)\right) / \text{Vol}_{\mathbb{R}^k}(\Delta_0)$$
$$\gg h(p)^k r^{-k}. \tag{5.23}$$

Then (5.23) and (5.22) imply

$$\mu(B(p, r) - B(p, r/\sqrt{e}))$$
$$\geq \sum_{\gamma_1\Delta_0 \subset B(x_p, \sqrt{e}h(p)/r) - B(x_p, h(p)/r)} \mu(\gamma\gamma_1\Delta_0)$$
$$\gg h(p)^{k-\delta} r^{2\delta - k}.$$

Consider general case when $p = \gamma p_i$ with $\gamma \in \Gamma$ the representation of p. We estimate the measure $\mu(\gamma\gamma_1\Delta_{p_i})$ for any $\gamma_1 \in \Gamma_{p_i}$ satisfying $\gamma\gamma_1\Delta_{p_i} \subset B(p, r) - B(p, r/\sqrt{e})$. Using (5.13), we have

$$\mu(\gamma\gamma_1\Delta_{p_i}) \approx \mu_{\Gamma_i}(g_i\gamma\gamma_1\Delta_{p_i}),$$

where $\Gamma_i := g_i \Gamma g_i^{-1}$. Lemma 5.12 yields

$$g_i \gamma \gamma_1 \Delta_{p_i} \subset g_i(B(p, r) - B(p, r/\sqrt{e})) \subset B(g_i p, Cr) - B(g_i p, r/(C\sqrt{e})). \tag{5.24}$$

So we can use the argument for the previous case to obtain

$$\mu_{\Gamma_i}(g_i \gamma \gamma_1 \Delta_{p_i}) \approx r^{2\delta} h(g_i p)^{-\delta}. \tag{5.25}$$

Then we count the number of $\gamma \gamma_1 \Delta_{p_i}$'s in $B(p, r) - B(p, r/\sqrt{e})$. It equals the number of $g_i \gamma \gamma_1 \Delta_{p_i}$'s in $g_i(B(p, r) - B(p, r/\sqrt{e}))$. The map g_i maps $B(p, r)$ and $B(p, r/\sqrt{e})$ to two spheres and the distance between $g_i B(p, r)$ and $g_i B(p, r/\sqrt{e})$ is at least $(1 - 1/\sqrt{e})r/C$. The map $g_i \gamma^{-1} g_i^{-1}$ maps $g_i B(p, r)$ and $g_i B(p, r/\sqrt{e})$ to two spheres and let R and p' be the radius and the center of the outer sphere respectively. Using (5.24) and Lemma 5.7, we have

$$R \in (h(g_i p)/(Cr), C\sqrt{e}h(g_i p)/r). \tag{5.26}$$

For every $x \in B(g_i p, Cr) - B(g_i p, r/(C\sqrt{e}))$, we have $|(g_i \gamma^{-1} g_i^{-1})'(x)| \in (h(g_i p)/(C^2 r^2), C^2 e h(g_i p)/r^2)$. So the distance between $g_i \gamma^{-1} B(p, r)$ and $g_i \gamma^{-1} B(p, r/\sqrt{e})$ is at least $(1 - 1/\sqrt{e})h(g_i p)/(C^3 r)$. This distance estimate together with (5.26) implies there exists some constant $c \in (0, 1)$ such that

$$g_i \gamma^{-1}(B(p, r) - B(p, r/\sqrt{e})) \supset B(p', R) - B(p', cR).$$

The number of $g_i \gamma_1 \Delta_{p_i}$ in $g_i \gamma^{-1}(B(p, r) - B(p, r/\sqrt{e}))$ is at least

$$\begin{aligned} \mathrm{Vol}_{\mathbb{R}^k}\left(B\left(p', R - C''\right) - B\left(p', cR + C''\right)\right) / \mathrm{Vol}_{\mathbb{R}^k}(g_i \Delta_{p_i}) \\ \gg R^k \gg h(g_i p)^k r^{-k}, \end{aligned} \tag{5.27}$$

where $C'' = \mathrm{diam}(g_i \Delta_{p_i})$. A lower bound for $\mu(B(p, r) - B(p, r/\sqrt{e}))$ can be obtained using Lemma 5.11, (5.25) and (5.27). □

5.4 Friendliness of PS measure

For any $r > 0$, set

$$N_r(\Delta_0) := \{x \in \Delta_0 : \ d(x, \partial \Delta_0) \leq r\}. \tag{5.28}$$

Lemma 5.29 *There exist $0 < \epsilon_0 < 1$ such that for all $0 < \epsilon < \epsilon_0$ there exists* $\lambda = \lambda(\epsilon) \in (0, 1)$ *for all $r < 1$*

$$\mu(N_{\epsilon r}(\Delta_0)) \leq \lambda \mu(N_r(\Delta_0)). \tag{5.30}$$

Moreover, the constant $\lambda(\epsilon)$ tends to zero as ϵ tends to zero.

Recall from Sect. 2.3 that $\Delta_0 = B_Y(C) \times \Delta_0'$. Recall that the set Δ_0' is a parallelotope. Let l' be a facet of Δ_0' and $l = B_Y(C) \times l'$. Let γ be the element in Γ_∞ identifying l' with the opposite facet l'' so γ also identifies $B_Y(C) \times l'$ with $B_Y(C) \times l''$. Set

$$N_r(l) := \{x \in \Delta_0 \cup \gamma^{-1}\Delta_0 : d(x, l) \leq r\}.$$

Lemma 5.29 is deduced from the following lemma.

Lemma 5.31 *There exist $0 < \epsilon, \lambda < 1$ such that for all $r < 1$*

$$\mu(N_{\epsilon r}(l)) \leq \lambda \mu(N_r(l)).$$

Proof of Lemma 5.29 Assume that ∞ is a rank k cusp. If ∞ is not a cusp of maximal rank, then note that $(\partial B_Y(C)) \times \Delta_0' = \{|y| = C\} \times \Delta_0'$ does not intersect Λ_Γ. A small neighborhood of this boundary has zero PS measure. Therefore, we just need to consider the neighborhood of l's. Using Lemma 5.31, we obtain

$$\mu(N_{\epsilon r}(\Delta_0)) \leq \sum_l \mu(N_{\epsilon r}(l)) \leq \lambda \sum_l \mu(N_r(l)).$$

Each $N_r(l)$ is covered by Δ_0 and one of its translates $\gamma \Delta_0$. By Lemma 5.18, there exists $C' > 0$ such that

$$\lambda \sum_l \mu(N_r(l)) \leq \lambda C' 2k \mu(N_r(\Delta_0)).$$

We can replace ϵ by ϵ^n and using Lemma 5.31 repeatedly, which will yield an arbitrary small λ in Lemma 5.29. $\qquad\square$

Proof of Lemma 5.31 The proof is similar to the argument of using Lemma 3.11 to deduce Lemma 3.10 in [17]. Let L be the hyperplane containing l and $N_r(L)$ be the r-neighborhood of L. [17, Lemma 3.11] is stated in spherical metric but locally spherical metric is equivalent to the euclidean metric. So [17, Lemma 3.11] implies that there exists $\epsilon > 0$ such that for every $\xi \in E := \Lambda_\Gamma \cap N_{\epsilon r}(l)$, there exists $0 < \rho_\xi < 1$ satisfying

$$\mu(B(\xi, \rho_\xi) \cap (N_r(L) - N_{\epsilon r}(L))) \geq c\mu(B(\xi, \rho_\xi)), \tag{5.32}$$

where $0 < c < 1$ is a constant only depending on Γ. The family $\{B(\xi, \rho_\xi)\}_{\xi \in E}$ forms a covering of E.

It follows from Vitali covering Lemma that there exists a disjoint subcollection $\{B(\xi, \rho_\xi)\}_{\xi \in I}$ with $I \subset E$ countable, such that

$$\cup_{\xi \in I} B(\xi, 5\rho_\xi) \supset \cup_{\xi \in E} B(\xi, \rho_\xi) \supset E.$$

The set $B(\xi, \rho_\xi) \cap (N_r(L) - N_{\epsilon r}(L))$ may not be contained in $N_r(l) - N_{\epsilon r}(l)$, but we can cover it by some translations of $N_r(l) - N_{\epsilon r}(l)$. By elementary computation, we can use no more than k_0 number of elements γ_j's in Γ_∞ with k_0 depending on Δ_0 such that

$$\cup_j \gamma_j (N_r(l) - N_{\epsilon r}(l)) \supset B(\xi, \rho_\xi) \cap (N_r(L) - N_{\epsilon r}(L)).$$

Using this inclusion, Lemma 5.18 and disjointness of $B(\xi, \rho_\xi)$'s for $\xi \in I$, we obtain

$$C' k_0 \mu(N_r(l) - N_{\epsilon r}(l)) \geq \mu(\cup_j \gamma_j (N_r(l) - N_{\epsilon r}(l)))$$
$$\geq \sum_{\xi \in I} \mu(B(\xi, \rho_\xi) \cap (N_r(L) - N_{\epsilon r}(L))$$

Using (5.32) and doubling property in Proposition 5.14, we have

$$\sum_{\xi \in I} \mu(B(\xi, \rho_\xi) \cap (N_r(L) - N_{\epsilon r}(L)) \geq c \sum_{\xi \in I} \mu(B(\xi, \rho_\xi))$$
$$\geq c\epsilon' \sum_{\xi \in I} \mu(B(\xi, 5\rho_\xi)) \geq c\epsilon' \mu(N_{\epsilon r}(l)),$$

Combining the above two formulas, we conclude that there exists $0 < \lambda < 1$ such that

$$\mu(N_{\epsilon r}(l)) \leq \lambda \mu(N_r(l)).$$

$\square$

6 Coding of limit set

In this section, we construct the coding and prove Proposition 4.1, Lemma 4.5, and Lemma 4.8. At first reading, the reader might want to concentrate on the case when there is one cusp in the manifold, i.e., $\mathbf{P} = \{p_1\}$ and the coordinate change of transformation g_1 is the identity. This will significantly reduce the notational burden while not sacrificing too much of the main results.

6.1 Coding for local regions

We introduce "flower" J_p, the building block for the coding. Actually, J_p's are almost the union of a countable subcollection of open subsets Δ_j in the coding. The advantage of considering J_p is that J_p has a clean boundary which makes it possible to estimate the measure.

We first consider the case when $p = \gamma\infty$ **is a parabolic fixed point in** Δ **with** $\gamma \in \Gamma$ **the representation of** p and $x_p = \gamma^{-1}\infty$. Let $\eta \in (0, 1)$. We define the set $J_{p,\eta}$ as follows. By Lemma 5.7, we have

$$\gamma^{-1} B(p, \eta h(p)) = B(x_p, 1/\eta)^c.$$

Suppose that ∞ is a parabolic fixed point of maximal rank. Then $\mathbb{R}^d \subset \partial\mathbb{H}^{d+1}$ is tessellated by the translations of $\overline{\Delta}_0$. Take $R_{p,\eta}$ to be the smallest parallelotope tiled by the translations of $\overline{\Delta}_0$ such that it contains $B(x_p, 1/\eta)$. Let

$$J_{p,\eta} = \gamma R_{p,\eta}^c, \tag{6.1}$$
$$N_p = \{\gamma_1 \in \Gamma_\infty : \gamma_1 \Delta_0 \subset R_{p,\eta}^c\} = \{\gamma_1 \in \Gamma_\infty : \gamma\gamma_1 \Delta_0 \subset J_{p,\eta}\}. \tag{6.2}$$

Suppose ∞ is a parabolic fixed point of rank k in general. Let Z be the affine subspace in $\partial\mathbb{H}^{d+1}$ described in Lemma 2.3 where elements in Γ_∞ act as translations, and $\Delta_0 = B_Y(C) \times \Delta_0'$. So Z is tessellated by the translations of $\overline{\Delta}_0'$. Take $R_{p,\eta}$ in Z to be the smallest parallelotope tiled by the translations of $\overline{\Delta}_0'$ such that $B_Y(2/\eta) \times R_{p,\eta}$ contains $B(x_p, 1/\eta)$. Set

$$J_{p,\eta} = \gamma (B_Y(2/\eta) \times R_{p,\eta})^c \subset B(p, \eta h(p)), \tag{6.3}$$
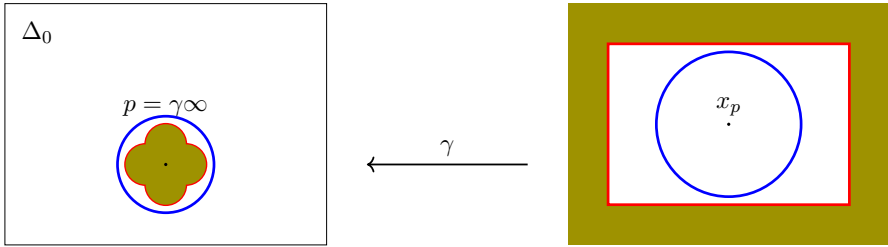$$N_p = \{\gamma_1 \in \Gamma_\infty : \gamma_1 \Delta_0 \subset (B_Y(2/\eta) \times R_{p,\eta})^c\}$$
$$= \{\gamma_1 \in \Gamma_\infty : \gamma\gamma_1 \Delta_0 \subset J_{p,\eta}\}. \tag{6.4}$$

The set $J_{p,\eta}$ enjoys the following property

$$J_{p,\eta} \cap \Lambda_\Gamma = \gamma \left(\bigcup_{\gamma_1 \in N_p} \gamma_1 \left(\overline{\Delta}_0 \cap \Lambda_\Gamma\right) \right), \tag{6.5}$$

that is to say, the countable disjoint union $\bigsqcup_{\gamma_1 \in N_p} \gamma\gamma_1 \Delta_0$ is a conull set in $J_{p,\eta}$.
The open sets $\gamma\gamma_1 \Delta_0$ **with** $\gamma_1 \in N_p$ **are the ones described in Proposition 4.1** in $J_{p,\eta}$, and on each $\gamma\gamma_1 \Delta_0$, the expanding map T is given by $T|_{\gamma\gamma_1 \Delta_0} = (\gamma\gamma_1)^{-1}$ (Fig. 2).

Fig. 2 The shaded region on the left hand side is $J_{p,\eta}$, which is the image under the action of γ on the complement of the white rectangle on the right hand side

We also have the following distance relation:

$$d((\gamma\gamma_1)^{-1}\infty, \Delta_0) = d(\gamma_1^{-1}x_p, \Delta_0) \geq 1/\eta \text{ for any } \gamma_1 \in N_p. \quad (6.6)$$

Lemma 6.7 *There exists* $0 < c_4 < 1$ *such that for any* $\eta \in (0, 1)$

$$B(p, c_4\eta h(p)) \subset J_{p,\eta} \subset B(p, \eta h(p)),$$
$$B(x_p, 1/\eta) \subset (\gamma^{-1}J_{p,\eta})^c \subset B(x_p, 1/(c_4\eta)).$$

Proof Due to the compactness of Δ_0, there exists c_4 such that $(\gamma^{-1}J_{p,\eta})^c = (B_Y(2/\eta) \times R_{p,\eta}) \subset B(x_p, 1/(c_4\eta))$. The first statement can be deduce from the second using Lemma 5.7. $\qquad\square$

In the following, we abbreviate $J_{p,\eta}$ **to** J_p. For $r > 0$, let

$$N_r(\partial J_p) := \{x \in J_p^c : d(x, \partial J_p) \leq r\},$$
$$N_r(\partial\gamma^{-1}J_p) := \{x \in (\gamma^{-1}J_p)^c : d(x, \partial\gamma^{-1}J_p) \leq r\}. \quad (6.8)$$

Lemma 6.9 *Fix* $C > 1$. *For every* $0 < \eta < 1/4C^2$, *there exists* $0 < c = c(\eta) < 1$ *depending on* η *such that for any* $r < h(p)$,

$$\mu(N_{C\eta r}(\partial J_p)) \leq c\mu(N_r(\partial J_p)). \quad (6.10)$$

Moreover, $c(\eta)$ *tends to zero as* η *tends to zero.*

The proof of Lemma 6.9 will be given in the "Appendix".

We consider the general case. **Let** p **be any parabolic fixed point in** Δ. **Write** $p = \gamma p_i$ **with** $\gamma \in \Gamma$ **the representation of** p. If $p_i = \infty$, let J_p and N_p be defined as (6.3) and (6.4) respectively. Otherwise, we use the following

commutative diagram to define J_p:

$$
\begin{array}{ccc}
\mathbb{H}^{d+1} & \xrightarrow{\;g_i\;} & \mathbb{H}^{d+1} \\
{\scriptstyle\Gamma}\big\downarrow & & \big\downarrow{\scriptstyle g_i\Gamma g_i^{-1}} \\
\mathbb{H}^{d+1} & \xrightarrow{\;g_i\;} & \mathbb{H}^{d+1}.
\end{array}
$$

Note that $g_i p = (g_i \gamma g_i^{-1})\infty \in g_i \Delta$. So for the action of $g_i \Gamma g_i^{-1}$ on $\partial\mathbb{H}^{d+1}$, we can define $J_{i,p}$ for the parabolic fixed point $g_i p$ as (6.3). Set

$$
J_p := g_i^{-1} J_{i,p}, \quad N_p = \{\gamma_1 \in \Gamma_{p_i} : \gamma\gamma_1\Delta_{p_i} \subset J_p\}, \tag{6.11}
$$

where Γ_{p_i} is a subgroup of Γ defined in (5.9). The set J_p enjoys the property

$$
J_p \cap \Lambda_\Gamma = \bigsqcup_{\gamma_1 \in N_p} \left(\gamma\gamma_1\overline{\Delta}_{p_i} \cap \Lambda_\Gamma \right). \tag{6.12}
$$

On each set $\gamma\gamma_1\Delta_{p_i}$, we have an expanding map given by $(\gamma\gamma_1)^{-1}$ which maps this set to Δ_{p_i}.

The following lemma is an analog of Lemma 6.7.

Lemma 6.13 *There exists some constant $C_5 > 1$ such that for any $\eta \in (0, 1)$*

$$
B(g_i x_p, 1/\eta) \subset (g_i \gamma^{-1} J_p)^c \subset B(g_i x_p, 1/(c_4\eta)),
$$
$$
B\left(p, \eta h(p)/C_5\right) \subset g_i^{-1} B\left(g_i p, c_4\eta h(g_i p)\right) \subset J_p \subset g_i^{-1} B\left(g_i p, \eta h(g_i p)\right)
$$
$$
\subset B\left(p, C_5\eta h(p)\right),
$$

where $x_p = \gamma^{-1} p_i$.

Proof We use Lemma 6.7, 5.11 and 5.12 to obtain the lemma. □

6.2 Coding for Δ_0

The construction of the coding for the whole Δ_0 is by induction. Let $\Omega_0 := \Delta_0$.

Since we already have a nice coding for flowers J_p, the idea is to find a collection of pairwise disjoint flowers J_p to cover the intersection $\Lambda_\Gamma \cap \Omega_0$. Here p is a parabolic fixed point. From the construction of J_p (Lemma 6.7), we know that the higher the height $h(p)$ is, the larger J_p is. So we start with parabolic fixed points with large heights. We want that the full flower J_p is inside Ω_0. Hence we only take parabolic fixed points p away from the boundary of Ω_0.

Take

$$h_n = e^{-n},$$

$\eta \in (0, 1)$ a sufficiently small constant to be specified at the end of the proof of Proposition 6.15.

All the constants appearing later will be independent of η unless we state it explicitly.

- For $n \in \mathbb{N}$, let

$$P_{n+1} = \{p \in \mathcal{P} : \ \eta h(p) \in (h_{n+1}, h_n], \ B(p, h_n/(4\eta)) \subset \Omega_n\}. \quad (6.14)$$

- For any $p \in P_{n+1}$, write $p = \gamma p_i$ with $\gamma \in \Gamma$ the representation of p. Construct J_p and N_p as in the previous section.
- Set

$$\Omega_{n+1} = \Omega_n - D_{n+1} = \Omega_n - \cup_{p \in P_{n+1}} J_p.$$

Using the definition of J_p, Lemma 6.13 and the separation property (Lemma 6.16), it can be shown that the sets J_p's with $p \in P_n$ and $n \in \mathbb{N}$ are mutually disjoint (Lemma 6.17) and inside Δ_0. In Proposition 6.15, it will be shown that the union $\cup_n \cup_{p \in P_n} J_p$ is conull in Δ_0 with respect to the PS measure μ. By (6.12), the countable disjoint union

$$\bigcup_{n \in \mathbb{N}} \bigcup_{p=\gamma p_i \in P_n} \bigcup_{\gamma_1 \in N_p} \gamma \gamma_1 \Delta_{p_i}$$

is also conull in Δ_0 with respect to PS measure. On each set $\gamma \gamma_1 \Delta_{p_i}$ we have an expanding map given by $(\gamma \gamma_1)^{-1}$ which maps this set to Δ_{p_i}. For one cusp case, these are the countable collection of disjoint open subsets and the expanding map. When there are multi-cusps, this is the first step to construct the coding and the rest will be provided in Sects. 6.7 and 6.8.

The main result of this section is the following proposition.

Proposition 6.15 *There exist $\epsilon_0 > 0$ and $N > 0$ such that for all $n > N$, we have*

$$\mu(\Omega_n) \leq (1 - \epsilon_0)^n.$$

For one cusp case, this yields Proposition 4.1 (1). Moreover, the exponential tail property (4.2) will follow from Proposition 6.15 rather directly and it will be proved in Sect. 6.7. To prove this proposition, we need a lot of preparations and we postpone its proof to the end of Sect. 6.6.

6.3 Separation

Lemma 6.16 (Separation property) *For any two different parabolic fixed points p, p', we have*

$$d(p, p') > \sqrt{h(p)h(p')}.$$

Proof Let x, x' be the euclidean centers of H_p and $H_{p'}$ repectively. By disjointness of horoballs, then $d_E(x, x') \geq (h(p) + h(p')/2$. By the Pythagoras' theorem, we obtain

$$d(p, p') \geq \sqrt{d_E(x, x')^2 - ((h(p) - h(p'))/2)^2} \geq \sqrt{h(p)h(p')}.$$

$\square$

This property plays a key role in the construction of the coding and the proof of Proposition 6.15.

Lemma 6.17 *If $\eta < 1/(4eC_5)$, then the sets J_p's with $p \in P_n$ and $n \in \mathbb{N}$ are mutually disjoint, and the distance between any two connected components of $\partial\Omega_n$ is strictly greater than $h_n/(2\eta)$.*

Proof Notice that $\Omega_n = \Omega_{n-1} - \cup_{p \in P_n} J_p$. By induction, we only need to prove two cases.

Case 1: We consider J_p, $J_{p'}$ with distinct p, p' in P_n. Using Lemma 6.13 and Lemma 6.16, we obtain

$$\sqrt{h(p)h(p')} - C_5\eta(h(p) + h(p')) \geq \frac{h_n}{\eta} - 2C_5h_{n-1} \geq \frac{h_n}{2\eta}.$$

Therefore, two sets J_p, $J_{p'}$ are disjoint and the distance between them is at least $h_n/(2\eta)$.

Case 2: We consider J_p with $p \in P_n$ and Ω_{n-1}. By (6.14) and Lemma 6.13, the distance between J_p and $\partial\Omega_{n-1}$ is also greater than

$$\frac{h_{n-1}}{4\eta} - C_5\eta h(p) \geq \frac{h_{n-1}}{4\eta} - C_5h_{n-1} > \frac{h_n}{2\eta}.$$

Hence J_p is inside Ω_{n-1}.

By induction, for different connected components of Ω_n, their distance is at least $h_n/(2\eta)$. $\square$

6.4 Equivalence classes in Q_n

Motivation of equivalence classes. We introduce the notion of equivalence classes to attain the exponential tail property.

Let's start with some definitions. The visual map $\pi : T^1(\mathbb{H}^{d+1}) \to \partial\mathbb{H}^{d+1}$ is defined by

$$\pi(x) = \lim_{t \to \infty} \mathcal{G}_t(x),$$

which maps x to the forward endpoint in $\partial\mathbb{H}^{d+1}$ of the geodesic defined by x.

Recall that we fix p_1 as ∞ and H_∞ is the horoball based at ∞ given by $\mathbb{R}^d \times \{x \in \mathbb{R} : x > 1\}$. Let $\widetilde{H_\infty}$ be the corresponding unstable horosphere. More precisely, let x_o be the unit tangent vector based at $(0, 1) \in \mathbb{R}^d \times \mathbb{R}$ with $\pi(x_o) = 0$. Then $\widetilde{H_\infty}$ is the set of x in $T^1(\mathbb{H}^{d+1})$ such that $d(\mathcal{G}_{-t}x_o, \mathcal{G}_{-t}x) \to 0$ as $t \to +\infty$. For a set $E \subset \partial\mathbb{H}^{d+1} - \{\infty\} \simeq \mathbb{R}^d$, let

$$\widetilde{E} = \pi|_{\widetilde{H_\infty}}^{-1}(E)$$

be the preimage of E under the map π restricted on $\widetilde{H_\infty}$.

Let

$H_p(\eta)$ be the horoball based at p with height equal to $\eta h(p)$,

$\mathcal{C}_\eta = \cup_{p \in \mathcal{P}} \Gamma \, T^1(H_p(\eta))$.

Then $\mathcal{C}_\eta$ is the lift of the unit tangent bundle over proper horocusps of M.

At the time n, the set $\mathcal{G}_n\widetilde{\Omega_n}$ is a large sheet with many holes, consisting of "flowers" of different sizes, corresponding to different $\mathcal{G}_n\widetilde{J_p}$. Here is the source of the exponential tail: for $x \in \mathcal{G}_n\widetilde{\Omega_n}$ with $\pi(x) \in \Lambda_\Gamma \cap \Omega_0$ and x not in the cusp region $\mathcal{C}_\eta$, the recurrence of geodesic flow implies that there exits a new flower $\mathcal{G}_n\widetilde{J_p}$ inside the neighborhood of x with size bounded below (Lemma 6.55). So a fixed proportion of the neighborhood of x will be coded in a fixed time (Lemma 6.53).

Lemma 6.55 doesn't apply to the set $\mathcal{G}_n\widetilde{\Omega_n} \cap \mathcal{C}_\eta$. Points in the set $\pi(\mathcal{G}_n\widetilde{\Omega_n} \cap \mathcal{C}_\eta)$ are contained in the balls centered at certain parabolic fixed points. We want to argue that from step n to step $n + 1$, the points near the outer edge of the balls will escape the cusps. We illustrate the scheme of the proof. For $n \in \mathbb{N}$, define

$$Q_{n+1} = \{p \in \mathcal{P} : \eta h(p) \in (h_{n+1}, h_n], \ B(p, \eta h(p)) \cap \Omega_n \neq \emptyset,$$
$$d(p, \partial\Omega_n) \leq h_n/(4\eta)\}.$$

Then we can cover $\mathcal{C}_\eta \cap \mathcal{G}_n \widetilde{\Omega}_n$ by the union of balls centered at parabolic fixed points in Q_m with $m \leq n$ (Lemma 6.37, 6.38):

$$\mathcal{C}_\eta \cap \mathcal{G}_n \widetilde{\Omega}_n = \mathcal{G}_n(\cup_{m \leq n} \cup_{p \in Q_m} \widetilde{B}(p, r(p, n))) \cap \mathcal{G}_n \widetilde{\Omega}_n,$$

where $r(p, n) = \sqrt{\eta h(p) h_n}$. From step n to step $n + 1$, the part

$$\mathcal{G}_{n+1}(\widetilde{B}(p, r(p, n)) - \widetilde{B}(p, r(p, n + 1)))$$

will leave the cusp region $\mathcal{C}_\eta$. We want a lower bound for the measure of $(B(p, r(p, n)) - B(p, r(p, n + 1))) \cap \Omega_n$. The main difficulty is that $B(p, r(p, n)) \cap \Omega_n$ may not be a full ball, in which case its PS measure is hard to estimate.

We introduce the notion of equivalence classes to resolve this issue. Consider a subset of Q_n:

$$Q'_n = \{p \in Q_n : B(p, r(p, n)) \cap \partial \Omega_{n-1} \neq \emptyset\}.$$

Pick any $p \in Q'_n$. As the ball $B(p, r(p, n)) \cap \Omega_{n-1}$ is not a full ball, we will pair it with another partial ball and use the doubling property of the PS measure. Notice that there is a unique component in $\partial \Omega_{n-1}$ closest to p (Lemma 6.17). If it is $\partial \Omega_0$, note that $\Lambda_\Gamma \cap \mathbb{R}^d$ is covered by the translations of Δ_0, so the symmetry property of these translations gives the point p' to pair with p (if p is around the corners of $\partial \Omega_0$, we may need more than one point to pair with p). If it is some ∂J_q, write $q = \gamma^{-1} p_i$ with $\gamma^{-1} \in \Gamma$ the representation of q. We map $B(p, r(p, n))$ and ∂J_q by $g_i \gamma$ and get a picture similar to the previous case. We find the paring point for $g_i \gamma p$ and map it back to get the one for p (Fig. 3). The work lies in modifying the radius $r(p, n)$: $r(p, n)$ is defined to $\sqrt{\eta h_n h(p)}$ depending on $h(p)$, and it may happen that the horosphere attached to the pairing point of p has a different height.
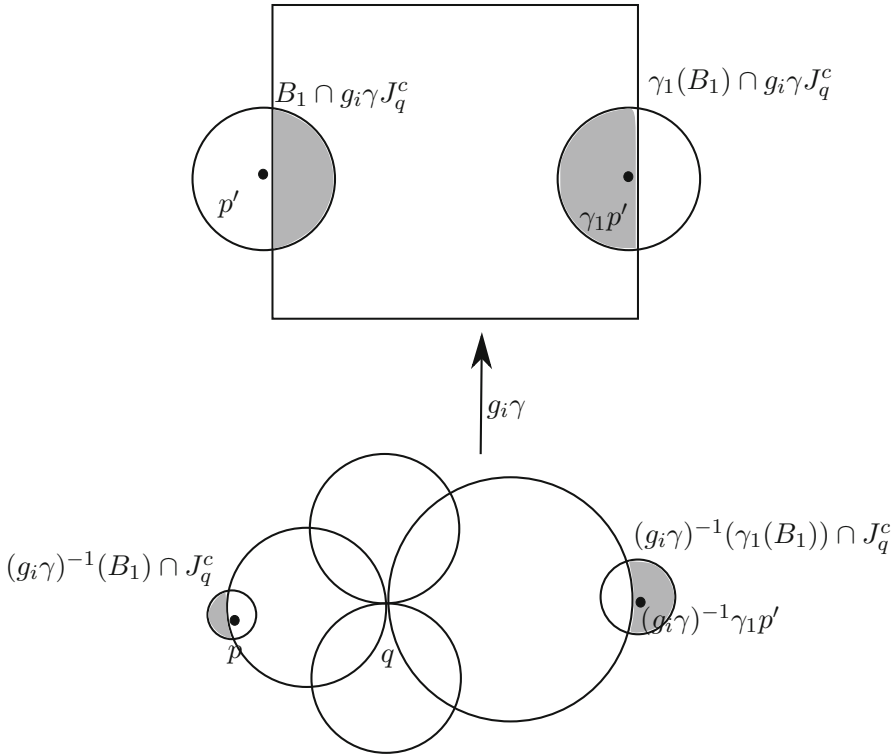
Finding the radius. For Lemma 6.18 - Lemma 6.23, we consider $p \in Q'_n$ such that the component in $\partial \Omega_{n-1}$ closest to p is ∂J_q with $q \in \cup_{l=1}^{n-1} P_l$. Write $q = \gamma^{-1} p_i$ for some $\gamma \in \Gamma$ the representation of q (Fig. 3).

Lemma 6.18 *There exists $C > 1$ such that we have*

$$\eta h(p) \leq h_{n-1} \leq C \eta^3 h(q), \quad \frac{\eta h(g_i p)}{C} \leq h_{n-1} \leq C \eta^3 h(g_i q).$$

Proof It follows from Lemma 6.16 that

$$d(p, q) \geq \sqrt{h(p) h(q)} \geq \sqrt{h_{n-1} h(q)/(e\eta)}.$$

Fig. 3 Pairing partial balls: $q = \gamma^{-1}p_i$, $p' = g_i\gamma p$, $B_1 = B(g_i\gamma p, \tilde{r}_{p,m}/C_6)$, $\gamma_1 \in (g_i\Gamma g_i^{-1})_\infty$

Meanwhile by Lemma 6.13, we have

$$d(p, q) \leq d(p, \partial J_q) + \max_{y \in \partial J_q} d(y, q) \leq r(n, p)$$
$$+ C_5\eta h(q) \leq h_{n-1} + C_5\eta h(q) \leq (1 + C_5)\eta h(q).$$

So the above two inequalities lead to first statement. The second statement follows easily from the first statement and Lemma 5.11. □

Lemma 6.19 *There exists $C > 1$ such that*

$$B(g_i p, h(g_i p)) \subset B(g_i q, C\eta h(g_i q)) - B(g_i q, \eta h(g_i q)/C).$$

Proof For any $\xi \in \partial B(g_i p, h(g_i p))$, an upper bound for $d(\xi, g_i q)$ is given by

$$d(\xi, g_i q) \leq d(\xi, g_i p) + d(g_i p, \partial J_{g_i q}) + \max_{y \in \partial J_{g_i q}} d(y, g_i q)$$
$$\leq h(g_i p) + Ch_{n-1} + \eta h(g_i q) \leq C\eta h(g_i q).$$

A lower bound for $d(\xi, g_i q)$ is given by

$$
\begin{aligned}
d(\xi, g_i q) &\geq d(g_i q, \partial J_{g_i q}) - d(g_i p, \partial J_{g_i q}) - h(g_i p) \\
&\geq c_4 \eta h(g_i q) - C h_{n-1} - h(g_i p) \geq c_4 \eta h(g_i q) - C \eta^2 h(g_i q).
\end{aligned}
$$

Hence by taking η sufficiently small, we reach the conclusion. $\qquad\square$

Lemma 6.20 *There exists $C > 1$ such that we have*

$$
\frac{h(g_i p)}{C \eta^2 h(g_i q)} \leq h(g_i \gamma p) \leq \frac{C h(g_i p)}{\eta^2 h(g_i q)}.
$$

Proof Apply Lemma 5.4 to the horoball $H = H_{g_i p}$ based at $g_i p$ and the element $g = g_i \gamma g_i^{-1}$. Notice that $g_i q = (g_i \gamma g_i^{-1})^{-1} \infty = g^{-1} \infty$ and $h(g_i q) = h(H_{g_i q}) = h(g^{-1} H_\infty)$. We have

$$
\frac{h(g_i q) h(g_i p)}{d(g_i q, g_i p)^2 + h(g_i p)^2} \leq h(g_i \gamma p) = h(gH) \leq \frac{h(g_i q) h(g_i p)}{(d(g_i q, g_i p) - h(g_i p)/2)^2},
$$

By Lemma 6.18 and 6.19, we obtain the lemma. $\qquad\square$

For any $m \in \mathbb{N}$, set

$$
\tilde{r}_{p,m} = \sqrt{\frac{h_m h(g_i \gamma p)}{\eta h(g_i q)}}. \tag{6.21}
$$

At the point $g_i \gamma p$, we will consider ball $B(g_i \gamma p, \tilde{r}_{p,m}/C_6)$, where $C_6 > 1$ is a constant given in Lemma 6.23 such that $\tilde{r}_{p,m}/C_6$ guarantees the equivalence classes we introduce later are well-defined. Once we have chosen the ball $B(g_i \gamma p, \tilde{r}_{p,m}/C_6)$, we will map it back by $(g_i \gamma)^{-1}$ to attain the "correct" ball at p.

By Lemma 6.20, 5.11 and 6.18, we have

$$
\tilde{r}_{p,n} \leq \frac{C \sqrt{\eta h_n h(g_i p)}}{\eta^2 h(g_i q)} \leq \frac{C h(g_i p)}{\eta h(g_i q)} \leq C \eta h(g_i \gamma p). \tag{6.22}
$$

Lemma 6.23 *There exists $C_6 > 1$ such that for any point p', if $d(g_i \gamma p', g_i \gamma \partial J_q) \leq \tilde{r}_{p,n}/C_6$, then $d(p', \partial J_q) \leq h_n$.*

Proof It follows from Lemma 6.7 applying $g_i \gamma g_i^{-1}$ that for any $C_6 > 1$, if $d(g_i \gamma p', g_i \gamma \partial J_q) \leq \tilde{r}_{p,n}/C_6$, then

$$
g_i \gamma p' = g_i \gamma g_i^{-1}(g_i p') \in B(g_i x_q, 2/(c_4 \eta)) - B(g_i x_q, 1/(2\eta)),
$$

where $g_i x_q = g_i \gamma p_i = g_i \gamma g_i^{-1} \infty$. For any x in the line segment between $g_i \gamma p'$ and $g_i \gamma \partial J_q$, we use Lemma 5.7 to obtain $|(g_i \gamma^{-1} g_i^{-1})' x| \leq 4\eta^2 h(g_i q)$.

By Lemma 5.12 and (6.22), we obtain

$$d(p', \partial J_q) \leq C d(g_i p', g_i \partial J_q) \leq C\eta^2 h(g_i q) d(g_i \gamma p', g_i \gamma \partial J_q)$$
$$\leq C\eta^2 \tilde{r}_{p,n} h(g_i q)/C_6 \leq C\eta h(g_i p)/C_6.$$

By taking $C_6 > 1$ large enough, we have $d(p', \partial J_q) \leq h_n$.   □

Definition of equivalence classes. Now we define equivalence classes in Q_n. We define them by induction. For Q_1,

- for $p \in Q_1 - Q_1'$, set the equivalence class $C(p)$ of p to be $\{p\}$.
- for $p \in Q_1'$, set

$$C(p) = \{\gamma_1 p : \gamma_1 B(p, \sqrt{\eta h_1 h(p)}) \cap \partial\Omega_0 \neq \emptyset, \ \gamma_1 \in \Gamma_\infty\}.$$

Set

$$Q_1'' := \cup_{p \in Q_1} C(p),$$

and for any $p' \in Q_1''$ and $m \geq 1$, define

$$r_{p',m} = \sqrt{\eta h_m h(p')}, \quad B_{p',m} = B(p, r_{p',m}).$$

Suppose we have defined Q_n''. We define the equivalence classes in Q_{n+1} and the set Q_{n+1}'' as follows:

Case 1: for $p \in Q_{n+1}' - \cup_{l \leq n} Q_l''$ such that the component in $\partial\Omega_n$ closest to p is $\partial\Omega_0$, set

$$C(p) = \{\gamma_1 p : \gamma_1 B(p, \sqrt{\eta h_{n+1} h(p)}) \cap \partial\Omega_0 \neq \emptyset, \ \gamma_1 \in \Gamma_\infty\}.$$

For any $p' \in C(p)$ and $m \geq n+1$, define

$$r_{p',m} = \sqrt{\eta h_m h(p')}, \quad B_{p',m} = B(p', r_{p',m}).$$

Case 2: for $p \in Q_{n+1}' - \cup_{l \leq n} Q_l''$ such that the component in $\partial\Omega_n$ closest to p is some J_q, write $q = \gamma^{-1} p_i$ with γ^{-1} the representation of q. Let $\tilde{r}_{p,n}$ and C_6 be as given in (6.21) and Lemma 6.23 respectively. If $B(g_i \gamma p, \tilde{r}_{p,n}/C_6) \cap g_i \gamma \partial J_q \neq \emptyset$, set

$$C(p) = \{(g_i\gamma)^{-1}\gamma_1 g_i\gamma p : \gamma_1 B(g_i\gamma p, \tilde{r}_{p,n}/C_6) \cap g_i\gamma \partial J_q \neq \emptyset,$$
$$\gamma_1 \in (g_i\Gamma g_i^{-1})_\infty\}. \tag{6.24}$$

Otherwise, set $C(p) = \{p\}$.
For any $p' \in C(p)$ and $m \geq n+1$, define

$$r_{p',m} = \frac{1}{C_6}\sqrt{\frac{h_m h(g_i\gamma p')}{\eta h(g_i q)}} \text{ (which equals } r_{p,m}), \tag{6.25}$$

$$B_{p',m} = (g_i\gamma)^{-1} B(g_i\gamma p', r_{p',m}). \tag{6.26}$$

Case 3: for $p \in Q_{n+1} - \cup_{l \leq n} Q_l''$ such that p does not belong to the union of equivalence classes defined in the previous two cases, set $C(p) = \{p\}$ and for any $m \geq n+1$, define

$$r_{p,m} = \sqrt{\eta h_m h(p)}, \quad B_{p,m} = B(p, r_{p,m}).$$

Set

$$Q_{n+1}'' = \bigcup_{p \in Q_{n+1} - \cup_{l \leq n} Q_l''} C(p).$$

Then $\cup_{1 \leq l \leq (n+1)} Q_l'' \supset \cup_{1 \leq l \leq (n+1)} Q_l$.

It is worthwhile to point out that under our definition of equivalence classes, it may happen that for $p \in Q_n - \cup_{l < n} Q_l''$, its equivalence class $C(p)$ may contain points p' whose associated horospheres don't appear in the time interval $[n-1, n)$. This is our motivation to establish results like Lemmas 6.27 and 6.40.

In the following discussion of the points p in Q_n'', if the definition of p involves a boundary component of $\partial \Omega_{n-1}$, we will need to consider this boundary component a lot of the times. For simplicity, we call the boundary component used to define $p \in Q_n''$ **the associated boundary component of p**.

Uniformity among equivalence classes. For $p \in Q_n - \cup_{l < n} Q_l''$, we show that, up to some constant, the points in the equivalence class $C(p)$ are "uniform".

Lemma 6.27 *There exists $C_7 > 1$ such that for any $p \in Q_n - \cup_{l < n} Q_l''$ and any $p' \in C(p)$ we have*

$$1/C_7 \leq h(p)/h(p') \leq C_7.$$

It suffices to prove Lemma 6.27 for the case when $\#C(p) \geq 2$ and the associated component of p in $\partial \Omega_{n-1}$ is some ∂J_q. Write $q = \gamma^{-1} p_i$ with γ^{-1}

the representation of q. Let $r_{p,m}$ and $B_{p,m}$ be defined as in (6.25) and (6.26) respectively. We first show the following estimate.

Lemma 6.28 (Location of balls) *There exists a constant $C > 1$ such that for $C(p)$ with the associated boundary component ∂J_q and for $p' \in C(p)$*

$$B(g_i \gamma p', r_{p',m}) \subset B\left(g_i x_q, C/\eta\right) - B(g_i x_q, 1/(C\eta)), \tag{6.29}$$
$$g_i B_{p',m} \subset B(g_i q, C\eta h(g_i q)) - B(g_i q, \eta h(g_i q)/C). \tag{6.30}$$

Proof By Lemma 6.13, we have

$$g_i \gamma \partial J_q \subset B(g_i x_q, 1/(c_4 \eta)) - B(g_i x_q, 1/\eta).$$

We also have $r_{p,n} \le C\eta$ by (6.22). Meanwhile, the construction of the equivalence class $C(p)$ implies that $r_{p',m} = r_{p,m}$. Hence we obtain (6.29). We use Lemma 5.7 to obtain (6.30) from (6.29).   □

Proof of Lemma 6.27 We prove the following explicit estimate:

$$h(g_i p') \approx \eta^2 h(g_i q) h(g_i \gamma p'). \tag{6.31}$$

This together with Lemma 5.11 and $h(g_i \gamma p') = h(g_i \gamma p)$ will lead to Lemma 6.27. Note that $h(g_i \gamma p') \le C$, with C a constant depending on Γ. We apply Lemma 5.4 to the horoball $H = H_{g_i \gamma p'}$ based at $g_i \gamma p'$ and the element $g = g_i \gamma^{-1} g_i^{-1}$. Notice that $g_i x_q = (g_i \gamma^{-1} g_i^{-1})^{-1} \infty = g^{-1} \infty$ and $h(g_i x_q) = h(g^{-1} H_\infty)$. We obtain

$$\frac{h(g_i x_q) h(g_i \gamma p')}{d(g_i x_q, g_i \gamma p')^2 + h(g_i \gamma p')^2} \le h(g_i p')$$
$$= h(gH) \le \frac{h(g_i x_q) h(g_i \gamma p')}{(d(g_i x_q, g_i \gamma p') - h(g_i \gamma p')/2)^2}.$$

Due to (5.2), we have $h(g_i x_q) = h(g^{-1} H_\infty) = h(g H_\infty) = h(g_i q)$. By (6.29), we have $d(g_i \gamma p', g_i x_q) \approx 1/\eta$. Therefore, we obtain

$$\frac{\eta^2 h(g_i q) h(g_i \gamma p')}{C} \le h(g_i p') \le C\eta^2 h(g_i q) h(g_i \gamma p').$$

                                                                                                                          □

Lemma 6.32 *There exists $C_8 > 1$ such that for any $p \in Q_n''$ and any $m \ge n$, the ball $B_{p,m}$ satisfies*

$$B(p, \sqrt{\eta h(p) h_m}/C_8) \subset B_{p,m} \subset B(p, C_8 \sqrt{\eta h(p) h_m}).$$

Proof It is enough to prove the case when the associated component of p in $\partial\Omega_{n-1}$ is some ∂J_q. Write $q = \gamma^{-1}p_i$ with γ^{-1} the representation of q. By definition,

$$B_{p,m} = (g_i\gamma)^{-1}B(g_i\gamma p, r_{p,m})$$

with $r_{p,m} = \frac{1}{C_6}\sqrt{\frac{h_m h(g_i\gamma p)}{\eta h(g_i q)}}$.

Consider the action of $g_i\gamma^{-1}g_i^{-1}$ on $B(g_i\gamma p, r_{p,m})$. By Lemma 5.7 and (6.29), we have

$$|(g_i\gamma^{-1}g_i^{-1})'x| \approx \eta^2 h(g_i q) \text{ for any } x \in B(g_i\gamma p, r_{p,m}). \quad (6.33)$$

Meanwhile, there exists a point $p' \in C(p)$ such that p' satisfies Lemma 6.20. We have $h(g_i\gamma p) = h(g_i\gamma p')$ and $h(p) \approx h(p')$ (Lemma 6.27). As a result, we obtain

$$h(g_i\gamma p) \approx \frac{h(g_i p)}{\eta^2 h(g_i q)}. \quad (6.34)$$

(6.33) and (6.34) yield there exists $C > 1$ such that

$$B(g_i p, \sqrt{\eta h(g_i p)h_m}/C) \subset g_i B_{p,m} \subset B(g_i p, C\sqrt{\eta h(g_i p)h_m}).$$

We use Lemma 5.11 and 5.12 to finish the proof. $\qquad\square$

Well-definedness of equivalence classes

Lemma 6.35 *For any two equivalence classes $C(p')$ and $C(p'')$, they are either the same or disjoint.*

Proof Assume that these two equivalence classes are not the same and the intersection is nonempty.

Case 1: Suppose one of these two equivalence classes just consists of one point, say $\#C(p') = 1$ and $\#C(p'') \geq 2$. We may assume that $p'' \in Q'_n$ for some n. If the associated component of p'' is $\partial\Omega_0$, then we are in Case 1 of the definition of equivalence classes. For $p' \in C(p'')$, we obtain that $h(p') = h(p'') \in (h_n, h_{n-1}]/\eta$. As $\#C(p') = 1$, we know that p' is contained in $Q_l - \cup_{i<l}Q''_i$ for some $l < n$, which contradicts $h(p') \in (h_n, h_{n-1}]/\eta$.

In the sequel, we assume further that the associated component of p'' is some ∂J_q. Write $q = \gamma^{-1}p_i$. As p' belongs to the equivalence class $C(p'')$, it follows from the definition that

$$B(g_i \gamma p', r_{p',n}) \cap g_i \gamma \partial J_q \neq \emptyset, \quad B(g_i \gamma p'', r_{p'',n}) \cap g_i \gamma \partial J_q \neq \emptyset,$$

$$(6.36)$$

where $r_{p',n}$ is defined as in (6.25) and equals $r_{p'',n}$.

The fact that $C(p')$ just consists of p' implies $p' \in Q_l - \cup_{i<l} Q_i''$ for some $l < n$. Meanwhile, as ∂J_q is the associated component of p'', by Lemma 6.18 and 6.27, we have

$$h(q) \geq h(p'')/(C\eta^2) \geq h(p')/(C\eta^2) \geq h_l/(C\eta^3).$$

Hence $\partial J_q \subset \partial \Omega_l$.

(6.36) allows us to apply Lemma 6.23 to p', and we obtain

$$d(p', \partial J_q) \leq h_n < \sqrt{\eta h_l h(p')}.$$

So $p' \in Q_l' - \cup_{i<l} Q_i''$. (6.36) yields

$$B(g_i \gamma p', r_{p',l}) \cap g_i \gamma \partial J_q \neq \emptyset, \quad B(g_i \gamma p'', r_{p'',l}) \cap g_i \gamma \partial J_q \neq \emptyset.$$

As $l < n$, $C(p')$ contains p'', which is a contradiction.

Case 2: Suppose that $\#C(p'), \#C(p'') \geq 2$. Without loss of generality, we may assume that $p' \in Q_m' - \cup_{l<m} Q_l''$ and $p'' \in Q_n' - \cup_{l<n} Q_l''$ and $m \leq n$.

Let $p \in C(p') \cap C(p'')$. Then it follows from the construction of equivalence classes and Lemma 6.23 that there are boundary components ∂_1 and ∂_2 in $\partial \Omega_{n-1}$ such that

$$d(p, \partial_1) \leq h_{m-1}, \quad d(p, \partial_2) \leq h_{n-1}.$$

On the one hand, as ∂_1 and ∂_2 are in $\partial \Omega_{n-1}$, if they are distinct, Lemma 6.17 states that their distance is greater than $h_{n-1}/(2\eta)$. On the other hand, using Lemma 6.27, we obtain

$$h_n/h_m \geq h(p'')/(eh(p')) = (h(p'')/h(p))(h(p)/(eh(p'))) \geq 1/(eC_7^2).$$

Then

$$d(\partial_1, \partial_2) \leq h_{m-1} + h_{n-1} \leq (1 + eC_7^2)h_{n-1} < h_{n-1}/(2\eta).$$

We conclude that $\partial_1 = \partial_2$.

There are two possibilities for ∂_1. One possibility is that ∂_1 is some ∂J_q. Write $q = \gamma^{-1} p_i$ with γ^{-1} the representation of q. As $g_i \gamma p$ is related with

$g_i \gamma p'$ and $g_i \gamma p''$ by elements in $(g_i \Gamma g_i^{-1})_\infty$, we have $\gamma_1 g_i \gamma p'' = g_i \gamma p'$ for some $\gamma_1 \in (g_i \Gamma g_i^{-1})_\infty$. As $m \leq n$, we have

$$\emptyset \neq B(g_i p'', r_{p'',n}) \cap g_i \gamma \partial J_q \subset B(g_i p'', r_{p'',m}) \cap g_i \gamma \partial J_q.$$

As a result, we have $C(p') = C(p'')$. The other possibility is that $\partial_1 = \partial \Omega_0$. It follows directly from the construction of equivalence classes that

$$\emptyset \neq B(p'', r_{p'',n}) \cap \partial \Omega_0 \subset B(p'', r_{p'',m}) \cap \partial \Omega_0.$$

Hence $C(p') = C(p'')$. $\square$

6.5 Auxiliary sets A_n and B_n in Ω_n

We introduce auxiliary sets A_n and B_n in Ω_n. By Lemma 6.35, the set Q_n'' is disjoint with $\cup_{1 \leq l \leq (n-1)} Q_n''$. For any $p \in Q_n''$ and any $m \geq n$, we have defined the ball $B_{p,m}$. Note that it follows from the construction of Q_n'' that if $\#C(p) = 1$, then the full ball $B_{p,n}$ is contained in Ω_n. For each n, we define

$$B_n = \Omega_n \cap \bigcup_{p \in \cup_{1 \leq l \leq n} Q_l''} B_{p,n} \quad \text{and} \quad A_n = \Omega_n - B_n.$$

In the followings, we will show how to use the set B_n to detect whether a point is in the cusps of the manifold at time $t = n$ or not.

B_n and cusps

Lemma 6.37 *For $x \in \widetilde{\Omega}_0$, if $\mathcal{G}_n x \in \mathcal{C}_\eta$, then there exists a parabolic fixed point p with $\eta h(p) > h_n$ such that*

$$d(\pi(x), p) < \sqrt{\eta h(p) h_n}.$$

Proof By assumption, in the universal cover $\mathrm{T}^1(\mathbb{H}^{d+1})$, the point $\mathcal{G}_n x$ is contained in a horoball $H_p(\eta)$. Hence $h_n < \eta h(p)$. If $h_n \leq \eta h(p)/2$, then we can use Pythagorean's theorem to conclude that $d(\pi(x), p) \leq \sqrt{\eta h(p) h_n}$ (see Figure 4). If $h_n \geq \eta h(p)/2$, then $d(\pi(x), p) \leq \eta h(p)/2 \leq \sqrt{\eta h(p) h_n}$. $\square$

Lemma 6.38 *Fix $c_9 < \min\{1/C_5, 1/C_8^2\}$, where C_5 and C_8 are constants given in Lemma 6.13 and Lemma 6.32 respectively. For any $x \in \widetilde{\Omega}_n$, if $\mathcal{G}_n x \in \mathcal{C}_{c_9 \eta}$, then $\pi(x) \in B_n$.*
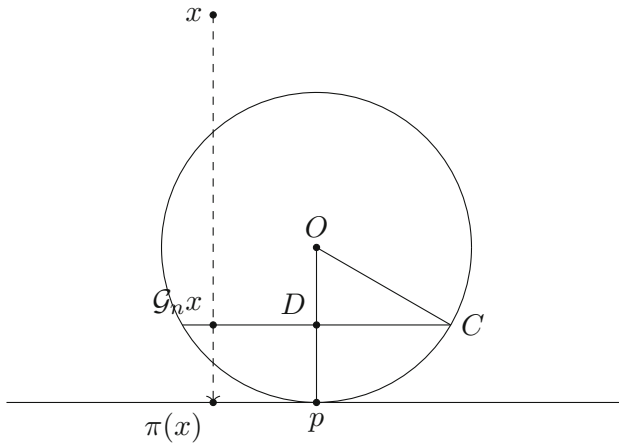
Fig. 4 Radius

Proof For $x \in \widetilde{\Omega_n}$, if $\mathcal{G}_n x \in \mathcal{C}_{c_9\eta}$, then it follows from Lemma 6.37 that there exists a parabolic fixed point p with $c_9\eta h(p) > h_n$ such that

$$d(\pi(x), p) < \sqrt{c_9\eta h(p)h_n} \leq c_9\eta h(p). \tag{6.39}$$

By the definition of P_n and Q_n, this p must belong to $\bigcup_{j<n}(P_j \cup Q_j)$. If p is in some P_j, then by Lemma 6.13 we have

$$\eta h(p)/C_5 < d(\pi(x), p),$$

contradicting the assumption that $c_9 < 1/C_5$. So p must be in some Q_j. We use the construction of B_n, Lemma 6.32, that is $B_n \supset B(p, \sqrt{\eta h(p)h_n}/C_8)$, and (6.39) to conclude that $\pi(x) \in B_n$. $\qquad \square$

Remark If $\pi(x) \in B_n$, then the point $\mathcal{G}_n x$ is contained in $\mathcal{C}_{C\eta}$. So the set $\mathcal{G}_n \widetilde{B_n}$ is almost the same as the set of points in the cusps at time $t = n$, i.e. $\mathcal{C}_\eta \cap \mathcal{G}_n \widetilde{\Omega_n}$.

Parabolic fixed points, B_n and different generations

Lemma 6.40 *We have $P_n \cap (\cup_{l \leq n} Q''_l) = \emptyset$.*

Proof If not, suppose $p \in P_n$ is also contained in an equivalence class $C(p')$ with $p' \in Q_m - (\cup_{1 \leq l \leq m-1} Q''_l)$ and $m \leq n$. Due to $\#C(p) \geq 2$, by the construction of equivalence classes, we must have $p' \in Q'_m$. Let ∂ be the associated boundary component of p'. Recall the construction of the equivalence classes. If ∂ is $\partial\Omega_0$, it is easy to obtain $d(p, \partial\Omega_0) < h_m$. If ∂ is some ∂J_q, due to $\#C(p') \geq 2$, we use Lemma 6.23 to deduce that $d(p, J_q) \leq h_m$. By

Lemma 6.27, we have $h_m/h_n \leq h(p')/h(p) \leq C_7$. Hence by the definition of P_n

$$d(p, \partial) \geq h_{n-1}/(4\eta) \geq h_m/(4eC_7\eta) > h_m,$$

which is a contradiction.   □

Lemma 6.41 *There exists a constant* $0 < c_{10} < 1$ *such that for any* $p \in P_{n+1} \cup Q''_{n+1}$, *we have*

$$d(p, B_n) \geq c_{10}h_n/\eta. \tag{6.42}$$

Proof Let $p \in P_{n+1} \cup Q''_{n+1}$ and $B_{q,n}$ be a ball in B_n. By Lemma 6.40, p and q are two different parabolic fixed points. We have

$$
\begin{aligned}
d(p, B_{q,n}) &\geq d(p, q) - C\sqrt{\eta h(q)h_n} \qquad \text{(by Lemma 6.32)} \\
&\geq \sqrt{h(p)h(q)} - C\sqrt{\eta h(q)h_n} \qquad \text{(by Lemma 6.16)} \\
&= \sqrt{h(q)}(\sqrt{h(p)} - C\sqrt{\eta h_n}) \\
&\geq \sqrt{h_n/C\eta}\left(\sqrt{h_n/(Ce\eta)} - C\sqrt{\eta h_n}\right) \geq h_n/(C\eta).
\end{aligned}
$$

□

Recall that $D_{n+1} = \cup_{p \in P_{n+1}} J_p$ and $\Omega_{n+1} = \Omega_n - D_{n+1}$.

Lemma 6.43 *If* $\eta < \frac{c_{10}}{C_5C_8}$, *then:*
1. We have the followings:

$$D_{n+1} \cap B_n = \emptyset \quad \text{and} \quad \left(\cup_{p \in Q''_{n+1}} B_{p,n+1}\right) \cap B_n = \emptyset,$$
$$A_{n+1} = (A_n - D_{n+1} - A_n \cap B_{n+1}) \cup (A_{n+1} \cap B_n), \tag{6.44}$$
$$A_n \cap B_{n+1} = \cup_{p \in Q''_{n+1}}(B_{p,n+1} \cap \Omega_{n+1}), \quad A_{n+1} \cap B_n = B_n - B_{n+1}. \tag{6.45}$$

2. For $p \in Q''_n$ *and* $m \geq n$, *we have* $B_{p,m} \cap \Omega_m = B_{p,m} \cap \Omega_n$.

Proof For $p \in P_{n+1}$, by Lemma 6.13, we have $J_p \subset B(p, C_5h(p))$. Then $C_5\eta h(p) \leq C_5h_n \leq c_{10}h_n/\eta$. By Lemma 6.41, we have $J_p \cap B_n = \emptyset$. For $p \in Q''_{n+1}$, by Lemma 6.32, we have $B_{p,n+1} \subset B(p, C_8\sqrt{\eta h(p)h_n})$. Then $C_8\sqrt{\eta h(p)h_n} \leq C_8h_n < c_{10}h_n/\eta$.

By Lemma 6.41, we have $B_{p,n+1} \cap B_n = \emptyset$.

The rest of the first statement can be obtained easily.

For $m > l \geq n$, by $D_{l+1} \cap B_l = \emptyset$ and $B_{p,m} \cap \Omega_l \subset B_l$ when $p \in Q_n''$ we know that

$$B_{p,m} \cap \Omega_{l+1} = B_{p,m} \cap (\Omega_l - D_{l+1}) = B_{p,m} \cap \Omega_l,$$

which implies the second part of the statement. $\square$

6.6 Energy exchange argument

We are ready to prove Proposition 6.15.

Lemma 6.46 *There exists $c_{11} > 0$ such that*

$$\mu(B_n \cap A_{n+1}) > c_{11}\mu(B_n). \tag{6.47}$$

The definition of equivalence classes is mainly used in this lemma. The idea is that the left hand side of (6.47) can be expressed as a sum over equivalence classes. Over one equivalence class, we obtain a full ball whose measure we are able to estimate (Fig. 5).

Proof of Lemma 6.46 We claim that for any distinct $p, p' \in \cup_{1 \leq l \leq n} Q_l''$, we have $B_{p,n} \cap B_{p',n} = \emptyset$. The first equation in Lemma 6.43 verifies the case when $p \in Q_l''$ and $p' \in Q_j''$ with $l \neq j$.

When $p, p' \in Q_l''$, using Lemma 6.16, and 6.32, we have

$$\begin{aligned}
d(B_{p,l}, B_{p',l}) &\geq d(p, p') - C\sqrt{\eta h(p)h_l} - C\sqrt{\eta h(p')h_l} \\
&\geq \sqrt{h(p)h(p')} - C\sqrt{\eta h(p)h_l} - C\sqrt{\eta h(p')h_l} \\
&\geq h_l/(C\eta) - 2Ch_{l-1} > 0,
\end{aligned}$$

showing the claim.

So $\mu(B_n \cap A_{n+1})$ can be divided into the sum over $p \in \cup_{1 \leq l \leq n} Q_l''$. Using Lemma 6.35, we can further group the sum into equivalence classes. Due to (6.45), $\mu(B_n \cap A_{n+1}) = \mu(B_n - B_{n+1})$. Then the proof of (6.47) is reduced to proving that there exists $c_{11} > 0$ such that for each equivalence class $C(p)$, we have

$$\sum_{p' \in C(p)} \mu((B_{p',n} - B_{p',n+1}) \cap \Omega_n) \geq c_{11} \sum_{p' \in C(p)} \mu(B_{p',n} \cap \Omega_n).$$

We first consider the equivalence classes defined in Case 1 and Case 3 in page 28. Then by the definition of equivalence class and Lemma 5.18, we

obtain

$$\frac{\sum_{p' \in C(p)} \mu((B_{p',n} - B_{p',n+1}) \cap \Omega_n)}{\sum_{p' \in C(p)} \mu(B_{p',n} \cap \Omega_n)}$$

$$\geq \frac{\mu(B_{p,n} - B_{p,n+1})}{C\mu(B_{p,n})} = \frac{\mu(B(p, r_{p,n}) - B(p, r_{p,n+1}))}{C\mu(B(p, r_{p,n}))} \geq \frac{c}{C},$$

where the last inequality follows from Lemma 5.21 and $r_{p,n} = \sqrt{\eta h_m h_p} \ll \eta h(p)$.

Next we consider the equivalence classes defined in Case 2 (Fig. 5). Suppose the associated boundary component of p is ∂J_q with $q = \gamma^{-1} p_i$ and γ^{-1} is the representation of q. We first assume that $p_i = \infty$. By Lemma 5.17 and (6.30) for any Borel subset $E \subset B_{p,n}$, we have

$$h_q^\delta \mu(\gamma E)/C \leq \mu(E) \leq C h_q^\delta \mu(\gamma E).$$
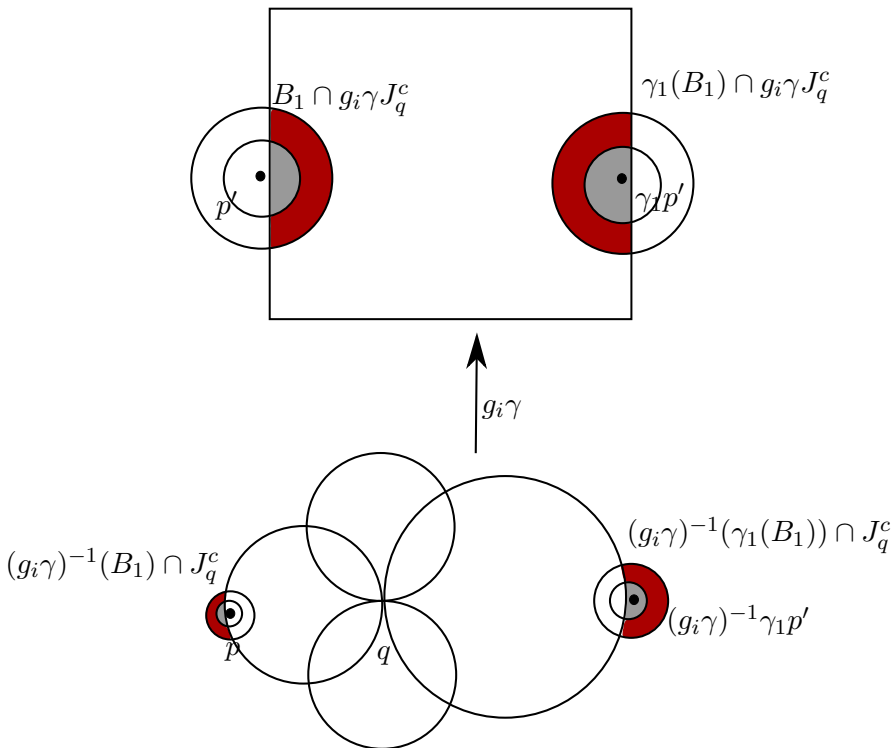
Fig. 5 Pairing partial balls: $q = \gamma^{-1} p_i$, $p' = g_i \gamma p$, $B_1 = B(g_i \gamma p, \tilde{r}_{p,m}/C_6)$, $\gamma_1 \in (g_i \Gamma g_i^{-1})_\infty$

We have

$$\frac{\sum_{p' \in C(p)} \mu((B_{p',n} - B_{p',n+1}) \cap \Omega_n)}{\sum_{p' \in C(p)} \mu(B_{p',n} \cap \Omega_n)}$$
$$\geq \frac{\sum_{p' \in C(p)} \mu((B(\gamma p', r_{p',n}) - B(\gamma p', r_{p',n+1})) \cap \gamma J_q^c)}{C \sum_{p' \in C(p)} \mu(B(\gamma p', r_{p',n}) \cap \gamma J_q^c)}. \qquad (6.48)$$

For each $p' \in C(p)$, we can write $p' = \gamma^{-1} \gamma_1 \gamma p$ with $\gamma_1 \in \Gamma_\infty$. We have

$$\mu(B(\gamma p', r_{p',n}) \cap \gamma J_q^c) = \mu(\gamma_1 B(\gamma p, r_{p,n}) \cap \gamma J_q^c)$$
$$\approx \mu(B(\gamma p, r_{p,n}) \cap \gamma_1^{-1} \gamma J_q^c),$$

where we use Lemma 5.18 and (6.29) to compute $|\gamma_1(x)|$ and $|x|$ for x in $B(\gamma p, r_{p,n})$. Summing over $p' \in C(p)$, we can get a full ball. Similarly, we have

$$\mu((B(\gamma p', r_{p',n}) - B(\gamma p', r_{p',n+1})) \cap \gamma J_q^c)$$
$$\approx \mu((B(\gamma p, r_{p,n}) - B(\gamma p, r_{p,n+1})) \cap \gamma_1^{-1} \gamma J_q^c).$$

We use these two observations and Lemma 5.21, $r_{p,n} \ll \eta h(\gamma p)$ (6.22) to conclude

$$(6.48) \geq \frac{\mu(B(\gamma p, r_{p,n}) - B(\gamma p, r_{p,n+1}))}{C \mu(B(\gamma p, r_{p,n}))} \geq \frac{c}{C}.$$

For general p_i, let $g_i p_i = \infty$ and $\Gamma_i = g_i \Gamma g_i^{-1}$. Using (5.13), we obtain

$$\frac{\sum_{p' \in C(p)} \mu((B_{p',n} - B_{p',n+1}) \cap \Omega_n)}{\sum_{p' \in C(p)} \mu(B_{p',n} \cap \Omega_n)}$$
$$\approx \frac{\sum_{p' \in C(p)} \mu_{\Gamma_i}(g_i(B_{p',n} - B_{p',n+1}) \cap g_i \Omega_n)}{\sum_{p' \in C(p)} \mu_{\Gamma_i}(g_i B_{p',n} \cap g_i \Omega_n)}. \qquad (6.49)$$

This fraction can be estimated the same way as we estimate (6.48). So

$$(6.49) \geq c/C.$$

$\square$

Let $C_{12} = 2C_5 C_3 + 4C_8$, where C_3, C_5 and C_8 are constants given by Proposition 5.14, Lemma 6.13 and Lemma 6.32 respectively. Let

$$\Omega_n' = \{x \in \Omega_n : d(x, \partial \Omega_n) \leq C_{12} h_n\}.$$

This is the set of points with distance less than $C_{12}h_n$ to the boundary of Ω_n.

Lemma 6.50 (Boundary estimate) *There exists $c_{13} > 0$ depending on $C_{12}\eta$ such that*

$$\mu(\Omega'_n) \leq c_{13}\mu(\Omega_n)$$

and c_{13} tends to 0 as $C_{12}\eta$ tends to 0.

Proof The boundary $\partial\Omega_n$ consists of $\partial\Omega_0$ and ∂J_p with $p \in \cup_{1\leq l\leq n}P_l$. For any $p \in \cup_{1\leq l\leq n}P_l$, write $p = \gamma p_i$ with $\gamma \in \Gamma$ the representation of p and $\Gamma_i = g_i\Gamma g_i^{-1}$. Recall the definitions (5.28) and (6.8). Note that $h_n/(4\eta) \leq h(p)$. It follows from Lemmas 5.11 and 5.12 that there exists $C > 1$ such that $h_n/(C\eta) < h(g_i p)$ and

$$g_i N_{C_{12}h_n}(\partial J_p) \subset N_{CC_{12}h_n}(\partial J_{i,p}), \quad N_{h_n/(C\eta)}(\partial J_{i,p}) \subset g_i N_{h_n/(4\eta)}(\partial J_p),$$

where $J_{i,p}$ is defined as in (6.11). It follows from (5.13), Lemma 5.29 and 6.9 that there exists $c > 0$ such that

$$
\begin{aligned}
\mu(\Omega'_n) &= \mu(N_{C_{12}h_n}(\partial\Omega_0)) + \sum_{p\in\cup_{1\leq l\leq n}P_l}\mu(N_{C_{12}h_n}(\partial J_p)) \\
&\leq c\mu(N_{h_n/(4\eta)}(\partial\Omega_0)) + C'\sum_{p\in\cup_{1\leq l\leq n}P_l}\mu_{\Gamma_i}(N_{CC_{12}h_n}(\partial J_{i,p})) \\
&\leq c\mu(N_{h_n/(4\eta)}(\partial\Omega_0)) + cC'\sum_{p\in\cup_{1\leq l\leq n}P_l}\mu_{\Gamma_i}(N_{h_n/(C\eta)}(\partial J_{i,p})) \\
&\leq c\mu(N_{h_n/(4\eta)}(\partial\Omega_0)) + cC'^2\sum_{p\in\cup_{1\leq l\leq n}P_l}\mu(N_{h_n/(4\eta)}(\partial J_p)) \\
&\leq cC'^2\mu(\Omega_n),
\end{aligned}
$$

where the last inequality is due to Lemma 6.17 and $C' = \max_i e^{\delta d(o,g_i o)}$. □

Lemma 6.51 *There exists $0 < c_{14} < 1$ such that*

$$\mu(A_n \cap (D_{n+1} \cup B_{n+1})) \leq c_{14}\mu(A_n) + \mu(\Omega'_n).$$

Proof By Lemma 6.43, we have

$$A_n \cap B_{n+1} \cap (\Omega_n - \Omega'_n) \subset \bigcup_{p\in Q''_{n+1}} B_{p,n+1}.$$

We consider the points $p \in Q''_{n+1}$ such that $B_{p,n+1}$ intersects the set on the left. Denote the set of such points by Q'''_{n+1}. By Lemma 6.32, we have

$$B_{p,n+1} \subset B(p, C_8\sqrt{\eta h(p)h_{n+1}}) \subset B(p, C_8 h_n).$$

Then its distance to $\partial\Omega_n$ is greater than $(C_{12} - 2C_8)h_n \geq C_{12}h_n/2$. So Q'''_{n+1} must be a subset of points in Case 3 in page 28, and $B_{p,n+1} = B(p, \sqrt{\eta h(p)h_{n+1}}) \subset B(p, h_n)$.

For $p \in P_{n+1}$, by Lemma 6.13, we have $J_p \subset B(p, C_5\eta h(p)) \subset B(p, C_5 h_n)$.

By (6.42), for $p \in P_{n+1} \cup Q''_{n+1}$

$$d(p, B_n) \geq c_{10}h_n/\eta \geq C_{12}h_n/2. \tag{6.52}$$

Hence

$$B(p, C_5 h_n) \subset B(p, C_{12}h_n/2) \subset A_n.$$

Then by doubling property in Proposition 5.14

$$\mu(B(p, C_5 h_n)) \leq c_{14}\mu(B(p, C_{12}h_n/2)).$$

By Lemma 6.16, the points in the set $P_{n+1} \cup Q'''_{n+1}$ are of distance h_{n+1}/η apart from each other. Hence the balls $B(p, C_{12}h_n/2)$ are disjoint. Adding them together, we obtain

$$\mu(A_n \cap (D_{n+1} \cup B_{n+1}) - \Omega'_n) \leq \sum_{p \in P_{n+1} \cup Q'''_{n+1}} \mu(B(p, C_5 h_n))$$

$$\leq c_{14} \sum_{p \in P_{n+1} \cup Q'''_{n+1}} \mu(B(p, C_{12}h_n/2)) \leq c_{14}\mu(A_n).$$

$\square$

Set $A'_n = A_n - \Omega'_n$ which is the set of points in A_n with distance at least $C_{12}h_n$ to the boundary $\partial\Omega_n$.

Lemma 6.53 *There exist $N \in \mathbb{N}$ and $c_{15} > 0$ depending on η such that*

$$\mu(\cup_{l=1}^{N} D_{n+l}) \geq c_{15}\mu(A'_n).$$

Let $\widetilde{A_n}$ be the subset of $\widetilde{\Omega_n}$ such that $\pi(\widetilde{A_n}) = A_n$. The key point of the proof is that we can use the recurrence property of the geodesic flow on $\mathcal{G}_n(\widetilde{A_n})$, since Lemma 6.38 tells us that $\mathcal{G}_n(\widetilde{A_n})$ stays in a compact subset. Recall that

we introduce some notations. We assumed that there are j cusps in M and $\{p_i\}_{1 \le i \le j}$ is a complete set of inequivalent parabolic fixed points. We used the notation H_{p_i} to denote the horoball based at p_i. Now let $H_{p_i}^s \subset \mathrm{T}^1(\mathbb{H}^{d+1})$ be the strong stable horosphere, that is,

$$H_{p_i}^s := \{x \in \mathrm{T}^1(H_{p_i}) : \text{ the basepoint of } x \text{ is at } \partial H_{p_i} \text{ and } \pi(x) = p_i\}.$$

By abusing the notation, we also use $H_{p_i}^s$ to denote its image in the quotient space $\mathrm{T}^1(M)$.

For every $x \in \mathrm{T}^1(M)$ and $\epsilon > 0$, set $W^u(x, \epsilon)$ to be the local strong unstable manifold at x, that is,

$$W^u(x, \epsilon) := \{y \in \mathrm{T}^1(M) : \lim_{t \to -\infty} d(\mathcal{G}_t x, \mathcal{G}_t y) = 0, \ d^u(y, x) \le \epsilon\},$$

where $d(\cdot, \cdot)$ is the Riemannian metric on $\mathrm{T}^1(M)$ and $d^u(\cdot, \cdot)$ is the Riemannian metric restricted on the strong unstable manifold.

Denote by W in $\mathrm{T}^1(M)$ the non-wandering set of the geodesic flow.

Lemma 6.54 *Let K be any compact subset in W. Then there exists $U_0 > 0$ such that for every x in K and every $H_{p_i}^s$ in $\mathrm{T}^1(M)$, there exists a time $t \in [0, U_0]$ such that $\mathcal{G}_t(W^u(x, 1))$ meets $H_{p_i}^s$.*

Proof Let $\epsilon < 1/10$ and consider $Z_1 = \cup_{x \in \partial H_{p_i}^s} W^u(x, \epsilon)$ and $Z_2 = \cup_{x \in \partial H_{p_i}^s} W^u(x, 5\epsilon)$ in $\mathrm{T}^1(M)$. Then Z_1 is a transversal section to the geodesic flow. By ergodicity of geodesic flow on non-wandering set W, there exists a point y such that its negative time orbit is dense and there exists $t_0 \ge 0$ such that $\mathcal{G}_{t_0} y \in Z_1$. We can cover the compact set K with a finite number of balls of radius ϵ. There exists $t_1 > 0$ such that $\mathcal{G}_{[-t_1, 0]} y$ intersects every ball.

Fix any x in K. There exists $x' \in W^u(x, \epsilon)$ and $-s \in [-t_1, 0]$ such that $d(x', \mathcal{G}_{-s} y) \le 2\epsilon$ and $\mathcal{G}_{-s} y$ are in the same strong stable manifold (that is to say, $\lim_{t \to \infty} d(\mathcal{G}_t x', \mathcal{G}_t(\mathcal{G}_{-s} y)) = 0$). Therefore

$$d(\mathcal{G}_{s+t_0} x', \mathcal{G}_{t_0} y) \le 2\epsilon.$$

Using $\mathcal{G}_{t_0} y \in Z_1$ and local product structure, we have $\mathcal{G}_{s+t_0} x' \in \mathcal{G}_{s_1} Z_2$ for some $s_1 \in [-\epsilon, \epsilon]$. Due to the definition of Z_2, we can find $x'' \in W^u(x, 6\epsilon)$ such that $\mathcal{G}_{s+t-s_1} x'' \in H_{p_i}^s$. $\qquad\square$

The following lemma is a straightforward corollary of Lemma 6.54. Recall that $c_9 > 0$ is the constant given in Lemma 6.38. Let $K_{c_9 \eta} = W - \Gamma \backslash \mathcal{C}_{c_9 \eta}$. The base of non-wandering set in M is the convex core $C(M)$ and the base of $\Gamma \backslash \mathcal{C}_{c_9 \eta}$ is a union of proper horocusps. By Definition 2.2, we know that $K_{c_9 \eta}$ is compact.

Lemma 6.55 *Let U_0 be the constant in Lemma 6.54 with $K = K_{c9\eta}$. For every x in $\Delta_0 \cap \Lambda_\Gamma$ and $n \in \mathbb{N}$, if $\mathcal{G}_n \tilde{x}$ is in $K_{c9\eta}$, where $\tilde{x}$ is the point in $\widetilde{\Omega}_0$ such that $\pi(\tilde{x}) = x$, then the ball $B(x, h_n)$ contains a parabolic fixed point with height in $h_n[e^{-U_0}, 1]$.*

Proof Let $\tilde{B}$ be the set in $\widetilde{\Omega}_0$ such that $\pi(\tilde{B}) = B(x, h_n)$. We have $\mathcal{G}_n \tilde{B} = W^u(\mathcal{G}_n \tilde{x}, 1)$. As $\mathcal{G}_n \tilde{x} \in K_{c9\eta}$, by Lemma 6.54, there exists $t \in [0, U_0]$ such that $\mathcal{G}_t W^u(\mathcal{G}_n \tilde{x}, 1)$ intersects some $H^s_{p_i}$. Hence in the universal cover $T^1(\mathbb{H}^{d+1})$, the unstable leaf $\mathcal{G}_t W^u(\mathcal{G}_n \tilde{x}, 1)$ is tangent to a horoball. Let q be the basepoint of the horoball. Then q is in $B(x, h_n)$.   □

Proof of Lemma 6.53 Set $N = U_0 + 2\lfloor -\log \eta \rfloor + 2$. We claim that: There exists $C' > 1$ depending on η such that $\cup_{1 \leq l \leq N} P_{n+l}$ is a $C'h_n$ dense set in $A'_n \cap \Lambda_\Gamma$. That is to say, for every $x \in A'_n \cap \Lambda_\Gamma$, there exists some $p \in \bigcup_{1 \leq l \leq N} P_{n+l}$ such that $d(x, p) \leq C'h_n$.

Let $k = \lfloor -\log \eta \rfloor$. Fix any point $x \in A'_n \cap \Lambda_\Gamma$. We consider the position of x in Ω_{n+k}.

Case 1 Suppose $x \notin \Omega_{n+k}$. Then $x \in J_p$ for some $p \in \cup_{1 \leq l \leq k} P_{n+l}$. So $d(x, p) \leq C_5 \eta h(p) \leq C_5 h_n$.

Case 2 Suppose $x \in \Omega_{n+k}$ and $d(x, \partial \Omega_{n+k}) < 3h_{n+k}/\eta$. As $x \notin \Omega'_n$, we have $d(x, \partial \Omega_n) \geq C_{12} h_n$. Meanwhile, we have $3h_{n+k}/\eta < C_{12} h_n$. Consequently, the connected component in $\partial \Omega_{n+k}$ closest to x is some ∂J_p with $p \in \cup_{1 \leq l \leq k} P_{n+l}$. Hence

$$d(x, p) \leq d(x, \partial J_p) + d(\partial J_p, p) \leq 3h_{n+k}/\eta + C_5 \eta h(p) \leq Ch_n.$$

Case 3 Suppose $x \in A_{n+k} \cap \Lambda_\Gamma$ and $d(x, \partial \Omega_{n+k}) > 2h_{n+k}/\eta$. By Lemma 6.38, $\mathcal{G}_{n+k} \tilde{x} \in K_{c9\eta}$. It then follows from Lemma 6.55 that $B(x, h_{n+k})$ contains a parabolic fixed point p with height in $h_{n+k}[e^{-U_0}, 1]$. Let $j = \lfloor -\log(\eta h(p)) \rfloor$, then $j \in n+2k+[0, U_0+1]$. Let's consider the position of p.

– Suppose $p \in P_{j+1}$. Then $d(x, p) \leq h_{n+k}$.
– Suppose $p \notin P_{j+1}$ and $p \notin \Omega_j$. Note that the conditions on x and $p \in B(x, h_{n+k})$ imply that $p \in \Omega_{n+k}$. So there exists some $q \in \cup_{l=1}^{j-n-k} P_{n+k+j}$ such that $p \in J_q$. We obtain

$$d(x, q) \leq d(x, p) + d(p, q) \leq h_{n+k} + C_5 \eta h(q) \leq Ch_{n+k}.$$

– Suppose $p \notin P_{j+1}$ and $p \in \Omega_j$. Because $\eta h(p) \in (h_{j+1}, h_j]$, we must have $p \in Q_{j+1}$. By the definition of Q_{j+1}, we have $d(p, \partial \Omega_j) \leq h_j/\eta$. Observe that

$$d(p, \partial \Omega_{n+k}) \geq d(x, \partial \Omega_{n+k}) - d(x, p) > 2h_{n+k}/\eta - h_{n+k} > h_j/\eta.$$

So there exists $q \in \cup_{l=1}^{j-n-k} P_{n+k+l}$ such that $d(p, J_{q,\eta}) \le h_j/\eta$. This implies

$$d(x, q) \le d(x, p) + d(p, q) \le h_{n+k} + h_j/\eta + C_5 \eta h(q) \le C h_{n+k}.$$

Case 4 Suppose $x \in B_{n+k}$ and $d(x, \partial \Omega_{n+k}) \ge 3h_{n+k}/\eta$. As $x \in A_n$, we have $x \in B_{n+k} - B_n$. So there exists $p \in \cup_{1 \le l \le k} Q''_{n+l}$ such that $x \in B_{p,n+k}$. By Lemma 6.32, we have

$$B_{p,n+k} \subset B(p, C_8\sqrt{\eta h(p)h_{n+k}}) \subset B(p, C_8\sqrt{h_n h_{n+k}}).$$

Since $h_k \ge \eta$, for any $y \in B_{p,n+k}$, using Lemma 6.32, we have

$$d(y, \partial\Omega_{n+k}) \ge d(x, \partial\Omega_{n+k}) - d(x, y) \ge 3h_{n+k}/\eta - 2C_8\sqrt{h_n h_{n+k}} \ge 2h_{n+k}/\eta.$$

So the full ball $B_{p,n+k}$ is contained in Ω_{n+k}. By a similar argument as in the proof of Lemma 6.46, we have $\mu(B_{p,n+k} - B_{p,n+k+1}) > 0$. We can find a point $y \in \Lambda_\Gamma \cap (B_{p,n+k} - B_{p,n+k+1})$. By (6.45), we know that in fact y is in $A_{n+k+1} \cap \Lambda_\Gamma$.

– If $d(y, \partial\Omega_{n+k+1}) > 2h_{n+k+1}/\eta$, the point y belongs to Case 3.
– Otherwise, $d(y, \partial\Omega_{n+k+1}) \le 2h_{n+k+1}/\eta$. But $d(y, \partial\Omega_{n+k}) > 2h_{n+k}/\eta$, there exists J_q with $q \in P_{n+k+1}$ such that $d(y, \partial J_q) \le 2h_{n+k+1}/\eta$.

It follows that there exists $q \in \cup_{1 \le l \le N} P_{n+l}$ such that

$$d(x, q) \le d(x, y) + d(y, q) \le 2C_8\sqrt{h_n h_{n+k}} + d(y, q) \le C h_n.$$

Finally, by Lemma 6.13 we know that for $p \in \cup_{1 \le l \le N} P_{n+l}$, the balls $B(p, \eta h(p)/C_5)$ are disjoint. Using the claim and doubling property (Proposition 5.14),

$$\mu(\cup_{l=1}^N D_{n+l}) \ge \sum_{p \in \cup_{1 \le l \le N} P_{n+l}} \mu(B(p, \eta h(p)/C_5))$$

$$\ge c_{15} \sum_{p \in \cup_{1 \le l \le N} P_{n+l}} \mu(B(p, C' h_n)) \ge c_{15}\mu(A'_n \cap \Lambda_\Gamma) = c_{15}\mu(A'_n),$$

finishing the proof. □

Proof of Proposition 6.15 We will prove the following statement and Proposition 6.15 is a direct consequence of this: for η sufficiently small, there exist N and $c_0 > 0$ depending on η such that

$$\mu(\cup_{l=1}^N D_{n+l}) \ge c_0\mu(\Omega_n).$$

Recall that c_{11}, c_{13} and c_{14} are the constants given in Lemma 6.46, 6.50 and 6.51 respectively. We can take c_{13} small enough such that $c_{13} < c_{11}$ and $c_{13} + c_{14} < 1$. Write $t_n = \frac{\mu(A_n)}{\mu(B_n)}$, which makes sense even if $\mu(B_n) = 0$. Then by Lemma 6.43, 6.46, 6.51, and 6.50

$$
\begin{aligned}
t_{n+1} &= \frac{\mu(A_{n+1})}{\mu(B_{n+1})} = \frac{\mu(A_n) + \mu(B_n \cap A_{n+1}) - \mu(A_n \cap (D_{n+1} \cup B_{n+1}))}{\mu(B_n) - \mu(B_n \cap A_{n+1}) + \mu(A_n \cap B_{n+1})} \\
&\geq \frac{\mu(A_n) + c_{11}\mu(B_n) - (c_{14}\mu(A_n) + c_{13}\mu(\Omega_n))}{\mu(B_n) - c_{11}\mu(B_n) + (c_{14}\mu(A_n) + c_{13}\mu(\Omega_n))} \\
&= \frac{t_n - (c_{14} + c_{13})t_n + (c_{11} - c_{13})}{1 + (c_{14} + c_{13})t_n - (c_{11} - c_{13})} = f(t_n).
\end{aligned}
$$

Here f is fractional function and of the form $f(t) = \frac{a_1 t + a_2}{b_1 t + b_2}$ with $a_i, b_i > 0$, which is a monotone function. Hence

$$
\begin{aligned}
\inf_{t \in \mathbb{R}^+} f(t) &\geq \min \left\{ \frac{a_1}{b_1}, \frac{a_2}{b_2} \right\} \\
&= \min \left\{ \frac{1 - (c_{14} + c_{13})}{c_{14} + c_{13}}, \frac{c_{11} - c_{13}}{1 - (c_{11} - c_{13})} \right\} = q(c_{13}).
\end{aligned}
$$

By $t_n > 0$, there is a uniform lower bound of t_n for all $n \in \mathbb{N}$.

Then use Lemma 6.53 to obtain the desired statement:

$$
\begin{aligned}
\mu(\cup_{l=1}^N D_{n+l}) &\geq c_{15}\mu(A_n') \geq c_{15}(\mu(A_n) - \mu(\Omega_n')) \\
&\geq c_{15}(\mu(A_n) - c_{13}\mu(\Omega_n)) = c_{15}\left(\frac{t_n}{1 + t_n} - c_{13} \right) \mu(\Omega_n).
\end{aligned}
$$

If c_{13} is small enough, then $\frac{t_n}{1+t_n} \geq \frac{q(c_{13})}{1+q(c_{13})} > c_{13}$. Then we can fix a small η in Lemma 6.50 such that c_{13} satisfies these restriction. □

6.7 Exponential tail

For one cusp case, we have described how to construct the countable collection of disjoint open subsets in Δ_0 and the expanding map in Sect. 6.2. When there are multi-cusps, the coding is constructed in two steps and we describe the first step here and finish the rest in Sect. 6.8.

Suppose that there are j cusps. Recall the regions Δ_{p_i} introduced in Sect. 5.2. We claim: there is a countable collection of disjoint open subsets $\sqcup_{i,k} \Delta_{p_i,k}$ in $\Delta (= \sqcup_i \Delta_{p_i})$ and an expanding map $T_0 : \sqcup_{i,k} \Delta_{p_i,k} \to \Delta$ such that

- $\sum_{i,k} \mu(\Delta_{p_i,k}) = \mu(\Delta)$.

- For each $\Delta_{p_i,k}$, it is a subset in Δ_{p_i}. And there exists an element $\gamma_0 \in \Gamma$ such that $\Delta_{p_i,k} = \gamma_0 \Delta_{p_l}$ for some $1 \leq l \leq j$ and $T|_{\Delta_{p_i,k}} = \gamma_0^{-1}$.

Denote by $\mathcal{H}_0$ the set of inverse branches of T_0.

The construction is as follows: for each Δ_{p_i}, we apply the construction in Sect. 6.2 to the group $\Gamma_i = g_i \Gamma g_i^{-1}$ and the region $g_i \Delta_{p_i} = \Delta'_{p_i}$. In particular, we attain a countable collection of disjoint open sets $\Delta'_{p_i,k}$. Moreover, Proposition 6.15 holds for Δ'_{p_i}. We set $\Delta_{p_i,k} = g_i^{-1} \Delta'_{p_i,k}$.

For an element γ_0 in $\mathcal{H}_0$, if γ_0 maps Δ_{p_l} into some $\Delta_{p_i,k}$, then we define

$$|\gamma_0'|_\infty = \sup_{x \in \Delta_{p_l}} |\gamma_0'(x)|. \qquad (6.56)$$

The infinity norm of the derivative of a composition map is defined similarly. We prove the following.

Lemma 6.57 *There exists $\epsilon > 0$ such that*

$$\sum_{\gamma_0 \in \mathcal{H}_0} |\gamma_0'|_\infty^{\delta-\epsilon} < \infty. \qquad (6.58)$$

For one cusp case, this gives the exponential tail (4.2). When there are multi-cusps, (6.58) can be understood as that the map T_0 satisfies the exponential tail property.

We start the proof of Lemma 6.57 with the following result. Denote by $\cup_n P_n$ the set of "good parabolic fixed points" which appear in the first step of the construction of the coding for multi-cusp case and are defined similarly as (6.14).

Lemma 6.59 *There exists $C > 0$ such that for any parabolic fixed point $p = \gamma p_i \in \Delta_0 \cap \cup_n P_n$, we have for any $\epsilon \in (0, \delta - k/2)$,*

$$\sum_{\gamma_1 \in N_p} |(\gamma \gamma_1)'|_\infty^{\delta-\epsilon} \leq C(2\delta - k - 2\epsilon)^{-1} h(p)^{-\epsilon} \eta^{-2\epsilon} \mu(J_p),$$

where k is the rank of the parabolic fixed point p and N_p is defined in (6.4).

Proof We first consider the case when $p = \gamma \infty$. By Lemma 5.7, we have for every $x \in \Delta_0$ and every $\gamma_1 \in N_p$,

$$|(\gamma \gamma_1)'(x)| = |\gamma'(\gamma_1 x)| = \frac{h(p)}{d(\gamma_1 x, x_\gamma)^2}.$$

As $\cup_{\gamma_1 \in N_p} \gamma_1 \Delta_0 \subset B(x_\gamma, 1/\eta)^c$ where $x_\gamma = \gamma^{-1}\infty$, we use general polar coordinates to obtain

$$\sum_{\gamma_1 \in N_p} |(\gamma\gamma_1)'|_\infty^{\delta-\epsilon} \ll h(p)^{\delta-\epsilon} \sum_{\gamma_1 \in N_p} \frac{1}{d(\gamma_1 \Delta_0, x_\gamma)^{2\delta-2\epsilon}} \ll \frac{h(p)^{\delta-\epsilon}\eta^{2\delta-2\epsilon-k}}{2\delta - 2\epsilon - k}.$$
(6.60)

Meanwhile, by the quasi-invariance of PS measure and (2.1), we have for every $\gamma_1 \in N_p$

$$\mu(\gamma\gamma_1 \Delta_0) = \int_{x \in \Delta_0} |(\gamma\gamma_1)'(x)|_{\mathbb{S}^n}^\delta d\mu(x) \approx \int_{x \in \Delta_0} |(\gamma\gamma_1)'(x)|^\delta d\mu(x)$$
$$\approx \frac{\mu(\Delta_0)h(p)^\delta}{d(\gamma_1 \Delta_0, x_\gamma)^{2\delta}}.$$

Therefore,

$$\mu(\cup_{\gamma_1 \in N_p} \gamma\gamma_1 \Delta_0) \gg \mu(\Delta_0)h(p)^\delta \sum_{\gamma_1 \in N_p} \frac{1}{d(\gamma_1 \Delta_0, x_\gamma)^{2\delta}} \gg \frac{h(p)^\delta\eta^{2\delta-k}}{2\delta - k}.$$
(6.61)

Hence (6.60) and (6.61) together yield the statement for the case when $p = \gamma\infty$.

For the general case when $p = \gamma p_i$ with $g_i p_i = \infty$. Note that for every $\gamma_1 \in N_p$, we have $\gamma\gamma_1 = g_i^{-1}(g_i\gamma\gamma_1 g_i^{-1})g_i$. Hence by Lemma 5.11 and 5.12

$$h(g_i p) \approx h(p), \quad |(\gamma\gamma_1)'|_\infty = \sup_{x \in \Delta_{p_i}} |(\gamma\gamma_1)'(x)|$$
$$\approx \sup_{x \in g_i \Delta_{p_i}} |(g_i\gamma\gamma_1 g_i^{-1})'(x)| = |(g_i\gamma\gamma_1 g_i^{-1})'|_\infty.$$

Write $\Gamma_i = g_i \Gamma g_i^{-1}$. Using (5.13), we obtain

$$\mu(\gamma\gamma_1 \Delta_{p_i}) \approx \mu_{\Gamma_i}(g_i\gamma\gamma_1 g_i^{-1}(g_i \Delta_{p_i})).$$

We have $g_i x_p = (g_i\gamma g_i^{-1})^{-1}\infty \in g_i \overline{\Delta}_{p_i}$. Because $g_i p = g_i\gamma g_i^{-1}\infty$ and $g_i\gamma_1 g_i^{-1}(g_i \Delta_{p_i}) \subset B(g_i x_p, 1/\eta)^c$ for any $\gamma_1 \in N_p$, we are able to compare $\sum_{\gamma_1} |(g_i\gamma\gamma_1 g_i^{-1})'|_\infty^{\delta-\epsilon}$ with $\mu_{\Gamma_i}(\cup_{\gamma_1} g_i\gamma\gamma_1 g_i^{-1}(g_i \Delta_{p_i}))$ as above and this will prove Lemma 6.59 for the general case. □

Proof of Lemma 6.57 We only need to sum the inverse branches in $\mathcal{H}_0$ whose images are in Δ_0. For a general inverse branches whose image is in Δ_{p_j}, we consider the group $g_j \Gamma g_j^{-1}$ and the inequality can be proved in the same fashion. By Lemma 6.59 and Proposition 6.15, for any sufficiently small $\epsilon \in (0, 1)$,

$$\sum_{n \in \mathbb{N}} \sum_{p = \gamma p_i \in P_n \cap \Delta_0} \sum_{\gamma_1 \in N_p} |(\gamma \gamma_1)'|_\infty^{\delta - \epsilon} \ll \sum_{n \in \mathbb{N}} \sum_{p \in P_n} \mu(J_p) h(p)^{-\epsilon} \eta^{-2\epsilon}$$

$$\leq \eta^{-2\epsilon} \sum_{n \in \mathbb{N}} \mu(\Omega_n) e^{\epsilon(n+1)}$$

$$\leq \eta^{-2\epsilon} \sum_{n \geq N} (1 - \epsilon_0)^n e^{\epsilon(n+1)} + \eta^{-2\epsilon} \sum_{n < N} \mu(\Omega_n) e^{\epsilon(n+1)}.$$

By choosing an ϵ small enough such that $(1 - \epsilon_0)e^\epsilon < 1$, the above sum is finite. □

6.8 Coding for multi-cusps

We caution the readers that the symbol γ will be used to denote an inverse branch in this section.

In Sect. 6.7, we have found a countable collection of disjoint open subsets $\sqcup_{i,k} \Delta_{p_i,k}$ in $\Delta(= \sqcup_i \Delta_{p_i})$ and an expanding map $T_0 : \sqcup_{i,k} \Delta_{p_i,k} \to \Delta$. Without loss of generality, we may suppose that T_0 is irreducible, which means there doesn't exist a nonempty subset of $I_1 \subsetneq \{1, \ldots, j\}$ such that

$$T_0(\cup_{i \in I_1} \Delta_{p_i}) \subset \cup_{i \in I_1} \Delta_{p_i}.$$

Otherwise, we can consider the restriction of T_0 to the union $\cup_{i \in I_1} \Delta_{p_i}$.

For $x \in \Delta_0 = \Delta_{p_1}$, define the first return time

$$n(x) = \inf\{n \in \mathbb{N} : T_0^n(x) \in \Delta_{p_1}\}.$$

Set $n(x) = \infty$ if $T_0^n(x)$ doesn't come back to Δ_{p_1} for all $n \in \mathbb{N}$ or $T_0^n(x)$ lies outside of the domain of definition of T_0 for some n.

The expanding map T in Proposition 4.1 is defined by

$$T(x) = T_0^{n(x)}(x) \text{ for } x \text{ such that } n(x) < \infty.$$

By the definition of T_0, we have

- either $T(x) = \gamma^{-1}x$ with $\gamma \in \mathcal{H}_0$ and $\gamma : \Delta_{p_1} \to \Delta_{p_1}$,

- or $T(x) = \gamma_{n(x)}^{-1} \cdots \gamma_1^{-1} x$ with $\gamma_l \in \mathcal{H}_0$ for $l = 1, \ldots, n(x)$, where γ_l maps $\Delta_{p_{k(l+1)}}$ to $\Delta_{p_{k(l)}}$ with $p_{k(1)} = p_{k(n(x)+1)} = p_1$ and $p_{k(l)} \neq p_1$ for $1 < l \leq n(x)$.

The string $\gamma_{n(x)}^{-1} \cdots \gamma_1^{-1}$ gives an open subset $\gamma_1 \cdots \gamma_{n(x)} \Delta_{p_1} \subset \Delta_{p_1}$. They consist of open subsets described in Proposition 4.1.

To prove (1), (3) and (4) in Proposition 4.1, we start with a preliminary version of Lemma 4.8. Define

$$U_i = g_i^{-1} B(g_i \Delta_{p_i}, 1/(2\eta))^c \text{ for } 1 \leq i \leq j.$$

Lemma 6.62 *If γ is an inverse branch in $\mathcal{H}_0$ which maps Δ_{p_i} into Δ_{p_l}, then $\gamma^{-1} U_l \subset U_i$.*

Proof Due to the construction, we know that γp_i is a parabolic fixed point inside Δ_{p_l}. The definition of U_l implies

$$g_l U_l \subset B(g_l \gamma p_i, 1/(2\eta))^c.$$

Because the maps g_i's are bi-Lipschitz (Lemma 5.12), we obtain

$$g_i U_l \subset B(g_i \gamma p_i, 1)^c = B(g_i \gamma g_i^{-1} \infty, 1)^c. \tag{6.63}$$

By Lemma 5.7, we have

$$\begin{aligned}(g_i \gamma g_i^{-1})^{-1} B(g_i \gamma g_i^{-1} \infty, 1)^c &= B((g_i \gamma g_i^{-1})^{-1} \infty, h(g_i \gamma p_i)) \\ &\subseteq B((g_i \gamma g_i^{-1})^{-1} \infty, 1).\end{aligned} \tag{6.64}$$

By (6.6), we obtain

$$d(g_i \Delta_{p_i}, (g_i \gamma g_i^{-1})^{-1} \infty) \geq 1/\eta. \tag{6.65}$$

Combining (6.63)-(6.65) together, we conclude that

$$g_i \gamma^{-1} U_l \subset B((g_i \gamma g_i^{-1})^{-1} \infty, 1) \subset B(g_i \Delta_{p_i}, 1/(2\eta))^c.$$

$\square$

We prove Proposition 4.1 (1) and (4.2). The proof is to consider an induced map and reduce the number of cusps by 1 at a time.

Let $q = p_j$. Denote $\cup_{1 \leq i \leq j-1} \Delta_{p_i} = \Delta - \Delta_q$ by X_1 and for $x \in X_1$, define

$$n_1(x) = \inf\{n \in \mathbb{N} : T_0^n(x) \in X_1\}.$$

The map T_1 is given by $T_1(x) = T_0^{n_1(x)}(x)$ for x such that $n_1(x) < \infty$. Since T_0 is irreducible, this induced system is also irreducible on X_1. Write

$\mathcal{H}_q :=$ the set of the inverse branches of T_0 which are from Δ_q to Δ_q,

$\mathcal{H}_p :=$ the set of the inverse branches of T_0 which are from X_1 to X_1,

$\mathcal{H}_{pq} :=$ the set of the inverse branches of T_0 which are from X_1 to Δ_q,

$\mathcal{H}_{qp} :=$ the set of the inverse branches of T_0 which are from Δ_q to X_1.

As T_1 is a composition of multiples of T_0's, we have

- either $T_1(x) = \gamma^{-1}x$ with $\gamma \in \mathcal{H}_p$,
- or $T_1(x) = \gamma_{n_1(x)}^{-1} \cdots \gamma_1^{-1}x$ with $\gamma_1 \in \mathcal{H}_{qp}$, $\gamma_{n_1(x)} \in \mathcal{H}_{pq}$ and $\gamma_l \in \mathcal{H}_q$ for $l = 2, \ldots, n_1(x) - 1$.

The string $\gamma_1 \cdots \gamma_{n_1(x)}$ is an inverse branch of T_1. Set

$\mathcal{H}_1 :=$ the set of all inverse branches of T_1,

$\mathcal{H}_q^n := \{\gamma_1 \cdots \gamma_n : \gamma_i \in \mathcal{H}_q \text{ for } 1 \leq i \leq n\}$ for every $n \in \mathbb{N}$.

Lemma 6.66 *There exists $C > 0$ such that for every $n \in \mathbb{N}$ and for every $\gamma \in \mathcal{H}_q^n$, we have*

$$|\gamma'(x)| \geq |\gamma'|_\infty/C \text{ for any } x \in \Delta_q.$$

Proof We first notice that $|\gamma'(x)| \approx |(g_j\gamma g_j^{-1})'(g_jx)|$. Write $p = \gamma p_j$. By Lemma 5.7, we have

$$|(g_j\gamma g_j^{-1})'(y)| = \frac{h(g_jp)}{d(y, g_j\gamma^{-1}p_j)^2}.$$

By (6.6), we have $d(g_j\Delta_q, g_j\gamma^{-1}p_j) = d(g_j\Delta_q, (g_j\gamma g_j^{-1})^{-1}\infty) > 1/(2\eta)$. Then for every $y \in g_j\Delta_{p_j} = g_j\Delta_q$, the distance $d(y, g_j\gamma^{-1}p_j) \in [d(g_j\Delta_q, g_j\gamma^{-1}p_j) \pm \text{diam}(g_j\Delta_q)]$, which implies the lemma. $\square$

Lemma 6.67 *There exist $C > 0$, $\epsilon > 0$ such that for every $l \in \mathbb{N}$*

$$\sum_{1 \leq k \leq l, \ \gamma_k \in \mathcal{H}_q} |(\gamma_1 \cdots \gamma_l)'|_\infty^\delta < C(1 - \epsilon)^l.$$

Proof Claim: there exists $\epsilon > 0$ such that for every $n \in \mathbb{N}$ and for any $h \in \mathcal{H}_q^n$, we have

$$\sum_{\gamma \in \mathcal{H}_q} \mu(h\gamma \Delta_q) \leq (1 - \epsilon)\mu(h\Delta_q). \tag{6.68}$$

Proof of the claim: for a measurable set $E \subset \Delta_q$, by Lemma 6.66

$$\mu(hE) = \int_E |h'(x)|_{\mathbb{S}^d}^\delta d\mu(x) \approx \int_E |h'(x)|^\delta d\mu(x) \in \mu(E)|h'|_\infty^\delta [1/C, 1]. \tag{6.69}$$

Write $F = \cup_{\gamma \in \mathcal{H}_q} \gamma \Delta_q$. Since T_0 is irreducible, we have $\mu(F) < \mu(\Delta_q)$. By (6.69)

$$\sum_{\gamma \in \mathcal{H}_q} \mu(h\gamma \Delta_q) = \mu(hF) \leq |h'|_\infty^\delta \mu(F)$$
$$= \frac{\mu(F)}{\mu(\Delta_q - F)} |h'|_\infty^\delta \mu(\Delta_q - F) \leq C'\mu(h(\Delta_q - F)).$$

So we have

$$(1 + 1/C') \sum_{\gamma \in \mathcal{H}_q} \mu(h\gamma \Delta_q) \leq \mu(hF) + \mu(h(\Delta_q - F)) \leq \mu(h\Delta_q).$$

Using (6.68), Lemma 6.66 and (6.69) with $E = \Delta_q$, we obtain

$$\sum_{1 \leq k \leq l, \; \gamma_k \in \mathcal{H}_q} |(\gamma_1 \cdots \gamma_l)'|_\infty^\delta \leq C \sum_{1 \leq k \leq l, \; \gamma_k \in \mathcal{H}_q} \mu(\gamma_1 \cdots \gamma_l \Delta_q)$$
$$\leq C \sum_{1 \leq k \leq l-1, \; \gamma_k \in \mathcal{H}_q} (1 - \epsilon)\mu(\gamma_1 \cdots \gamma_{l-1} \Delta_q) \leq C(1 - \epsilon)^l.$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Proof of Proposition 4.1 (1) and (4.2) We first use Proposition 6.15, (6.58), Lemma 6.66 and 6.67 to prove that for the expanding map T_1, we have

1. There exists $\epsilon_1 > 0$ such that

$$\sum_{\gamma \in \mathcal{H}_1} |\gamma'|_\infty^{\delta - \epsilon_1} < \infty, \tag{6.70}$$

where $|\gamma'|_\infty$ is defined as in (6.56).

2. There exists $\epsilon > 0$ such that for every $n \in \mathbb{N}$,

$$\mu(\{x \in X_1 : n_1(x) > n + 1\}) \ll (1 - \epsilon)^n. \qquad (6.71)$$

The second statement in particular implies that the map T_1 is defined almost everywhere in $\Delta - \Delta_q$.

Due to Lemma 6.67, we can find a large l_0 such that $\sum_{1 \leq k \leq l_0, \ \gamma_k \in \mathcal{H}_q} |(\gamma_1 \cdots \gamma_{l_0})'|_\infty^\delta < 1$. Then using (6.58) and submultiplicativity $|(\gamma_1 \gamma_2)'|_\infty \leq |(\gamma_1)'|_\infty |(\gamma_2)'|_\infty$, we obtain

$$\sum_{1 \leq k \leq l_0, \ \gamma_k \in \mathcal{H}_q} |(\gamma_1 \cdots \gamma_{l_0})'|_\infty^{\delta - \epsilon} < \infty,$$

where $\epsilon > 0$ is the constant given by (6.58). Hence we can find $0 < \epsilon_1 < \epsilon$ small such that

$$\sum_{1 \leq k \leq l_0, \ \gamma_k \in \mathcal{H}_q} |(\gamma_1 \cdots \gamma_{l_0})'|_\infty^{\delta - \epsilon_1} < 1.$$

Using submultiplicativity, we obtain constants $C > 0, \rho < 1$ such that for $l \in \mathbb{N}$

$$\sum_{1 \leq k \leq l, \ \gamma_k \in \mathcal{H}_q} |(\gamma_1 \cdots \gamma_l)'|_\infty^{\delta - \epsilon_1} \leq C\rho^l. \qquad (6.72)$$

Denote $\sum_{\gamma \in \mathcal{H}_1} |\gamma'|_\infty^{\delta - \epsilon_1}$ by E_q. For every inverse branch of T_1, it can be uniquely decomposed as $\gamma_0 \gamma_1 \cdots \gamma_l \gamma_{l+1}$ with $\gamma_{l+1} \in \mathcal{H}_{pq}, \gamma_i \in \mathcal{H}_q$ with $i = 1, \cdots l$ and $\gamma_0 \in \mathcal{H}_{qp}$. Using this expression and submultiplicativity, we obtain

$$E_q \leq \sum_{\gamma \in \mathcal{H}_p} |\gamma'|_\infty^{\delta - \epsilon_1} + \sum_{l \geq 1} \left(\sum_{\gamma_0 \in \mathcal{H}_{qp}} |\gamma_0'|_\infty^{\delta - \epsilon_1} \right) \left(\sum_{\gamma_{l+1} \in \mathcal{H}_{pq}} |\gamma_{l+1}'|_\infty^{\delta - \epsilon_1} \right)$$
$$\left(\sum_{\gamma_i \in \mathcal{H}_q, 1 \leq i \leq l} |(\gamma_1 \cdots \gamma_l)'|_\infty^{\delta - \epsilon_1} \right). \qquad (6.73)$$

Therefore E_p is also finite due to (6.73), (6.72) and (6.58).

The set of x such that $T_0^n(x)$ is outside of domain of definition of T_0 for some n has zero PS measure by Proposition 6.15. We only need to consider the set of x such that $T_0^n(x)$ is in the domain of definition for every n. If $x \in X_1$

with $n_1(x) > n + 1$, then x must be in $\gamma_0 \gamma_1 \cdots \gamma_n \Delta_q$ with $\gamma_0 \in \mathcal{H}_{qp}$ and $\gamma_i \in \mathcal{H}_q$ for $1 \leq i \leq n$. Therefore Lemma 6.67 implies

$$\mu(\{x \in X_1 : n_1(x) > n + 1\}) \leq \sum_{\gamma_0 \in \mathcal{H}_{qp}} \sum_{\gamma_i \in \mathcal{H}_q, 1 \leq i \leq n} \mu(\gamma_0 \gamma_1 \cdots \gamma_n \Delta_q)$$

$$\leq \left(\sum_{\gamma_0 \in \mathcal{H}_{qp}} |\gamma_0'|_\infty^\delta \right) \left(\sum_{\gamma_i \in \mathcal{H}_q, 1 \leq i \leq n} |(\gamma_1 \cdots \gamma_n)'|_\infty^\delta \right) \ll (1 - \epsilon)^n.$$

We keep reducing the number of cusps by considering the set $X_2 := X_1 - \Delta_{p_{j-1}}$ and the induced map $T_2 : X_2 \to X_2$ which is constructed similar to T_1: in particular, the inverse branches of T_2 are compositions of elements in $\mathcal{H}_1$. Analogs of Lemma 6.66 and 6.67 for T_2 also hold. The replacements of Proposition 6.15 and (6.58) are (6.71) and (6.70) respectively. Using these three ingredients, we can show the properties like (6.70) and (6.71) also hold for T_2. The proof of Proposition 4.1(1) and (4.2) will be finished by repeating this.   □

Now, we will finish proving the rest of results for the coding except Lemma 4.5 (UNI).

Proof of Lemma 4.8 Take

$$\Lambda_- = \Lambda_\Gamma \cap \{|x| > 1/(2\eta)\} = \Lambda_\Gamma \cap U_1. \tag{6.74}$$

The contracting map γ from Δ_0 to Δ_0 is a composition of maps in $\mathcal{H}_0$, so the inclusion follows directly from Lemma 6.62. Write $p = \gamma\infty$. By Lemma 5.7,

$$|(\gamma^{-1})'(x)|_{\mathbb{S}^d} = \frac{h(p)}{d(x, p)^2} \frac{1 + |x|^2}{1 + |\gamma^{-1}x|^2}.$$

For $x \in \Lambda_-$, as $p = \gamma\infty \in \Delta_0$, we have

$$\frac{1 + |x|^2}{d(x, p)^2} \leq \frac{1 + |x|^2}{(|x| - \mathrm{diam}(\Delta_0))^2}.$$

The right hand side of the inequality is around 1 as $|x| \geq 1/(2\eta)$. For $\gamma^{-1}x \in \Lambda_-$, we have $|\gamma^{-1}x| \geq 1/(2\eta)$. Hence $|(\gamma^{-1})'(x)|_{\mathbb{S}^d} \leq \lambda$ for some λ independent of γ.   □

Proof of Proposition 4.1 (3) By Lemma 5.7, we have

$$|\gamma'(x)| = \frac{h(p)}{d(x, \gamma^{-1}\infty)^2}.$$

By Lemma 4.8, we have $\gamma^{-1}\infty \in \Lambda_-$, which implies $d(\gamma^{-1}\infty, x) \geq 1/(2\eta)$. Hence

$$|\gamma'(x)| \leq (2\eta)^2 h(p) \leq 4\eta^2.$$

□

Proof of Proposition 4.1 **(4).** We need the following lemma, which will also be needed in later sections.

Lemma 6.75 *Let γ be any element in Γ which does not fix ∞. For any $x \in \Delta_0$ and any unit vector $e \in \mathbb{R}^d$, we have*

$$\partial_e \log |\gamma'(x)| = -\frac{2\langle x - \xi, e\rangle}{|x - \xi|^2}$$

where $\xi = \gamma^{-1}\infty$.

Proof It can be shown using Lemma 5.7 and elementary computation. □

For $\gamma \in \mathcal{H}$, Proposition 4.1 (4) can be deduced using Lemma 6.75 and the observation that $|\gamma^{-1}\infty| \geq 1/(2\eta)$ (Lemma 4.8 and (6.74)).

6.9 Verifying UNI

We prove Lemma 4.5 in this part. Let Γ_f be the semigroup generated by γ in $\mathcal{H}$ and Γ_b be the semigroup generated by γ^{-1} with $\gamma \in \mathcal{H}$. Let Λ_f and Λ_b be the limit set of Γ_f and Γ_b on $\partial\mathbb{H}^{d+1}$, that is the set of accumulation points of orbit $\Gamma_f o$ and $\Gamma_b o$ for some $o \in \mathbb{H}^{d+1}$ respectively. It follows from the definition that the limit set Λ_f is Γ_f-invariant and Λ_b is Γ_b-invariant. Due to [24, Proposition 3.19] (convergence property of Möbius transformation), we have that Λ_f is a Γ_f-minimal set and Λ_b is a Γ_b-minimal set.

Lemma 6.76 *The limit set Λ_b is not contained in an affine subspace in $\mathbb{R}^d \cup \{\infty\}$ or a sphere in $\mathbb{R}^d$.*

Proof Let A be an affine subspace or a sphere with minimal dimension which contains Λ_b. Because Λ_b is Γ_b invariant, the semigroup Γ_b must preserve A, so does the Zariski closure of Γ_b. The Zariski closure of a semigroup is a group (see for example [13, Lemma 6.15]). The Zariski closures of Γ_f and Γ_b are the same. Hence Γ_f also preserves A and Λ_f is in A. **We claim:** $\mu(\Lambda_f) = \mu(\Lambda_\Gamma \cap \Delta_0) > 0$. Then because Γ is Zariski dense, by [20, Corollary 1.4], we conclude that $\mu(A)$ is non zero if and only if $A = \mathbb{R}^d$, finishing the proof.

Proof of the claim: Let x be any point in $\Lambda_\Gamma \cap \Delta_0$ such that $T^n x \in \Lambda_\Gamma \cap \Delta_0$ for every $n \in \mathbb{N}$. We can write $x = \gamma_n T^n(x) \in \gamma_n \Delta_0$ for some $\gamma_n \in \mathcal{H}^n$. Fix any $y \in \Lambda_f$, it follows from Proposition 4.1 (3) that $d(\gamma_n y, \gamma_n T^n(x)) \to 0$. So $\gamma_n y \to x$ and $x \in \Lambda_f$. Due to Proposition 4.1 (1), the set of x's such that $T^n x \in \Lambda_\Gamma \cap \Delta_0$ for every $n \in \mathbb{N}$ is a conull set in $\Lambda_\Gamma \cap \Delta_0$. Hence $\mu(\Lambda_f) = \mu(\Lambda_\Gamma \cap \Delta_0)$. □

Lemma 6.77 *For every $x \in \Lambda_\Gamma \cap \overline{\Delta}_0$, there exist pairs of points (ξ_{1m}, ξ_{2m}), $m = 1, \ldots, k_x$ in the limit set Λ_b and $\epsilon'_x > 0$ such that for every unit vector $e \in \mathbb{R}^d$ there exists m,*

$$\left| \left\langle \frac{x - \xi_{1m}}{|x - \xi_{1m}|^2} - \frac{x - \xi_{2m}}{|x - \xi_{2m}|^2}, e \right\rangle \right| > 2\epsilon'_x > 0.$$

Proof The map $inv_x : \xi \mapsto \frac{x-\xi}{|x-\xi|^2}$ is an inversion and this map is injective. If there exists a unit vector $e \in \mathbb{R}^d$ such that

$$\left\langle \frac{x - \xi_1}{|x - \xi_1|^2} - \frac{x - \xi_2}{|x - \xi_2|^2}, e \right\rangle = 0$$

for all ξ_1, ξ_2 in Λ_b, then $inv_x(\Lambda_b)$ is contained in an affine subspace parallel to $e^\perp$. Hence Λ_b itself is contained in an affine subspace in $\mathbb{R}^d \cup \{\infty\}$ or a sphere in $\mathbb{R}^d$, which contradicts Lemma 6.76. Therefore, for every unit vector $e \in \mathbb{R}^d$, there exist ξ_1, ξ_2 in Λ_b such that

$$\left\langle \frac{x - \xi_1}{|x - \xi_1|^2} - \frac{x - \xi_2}{|x - \xi_2|^2}, e \right\rangle \neq 0.$$

We use continuity and compactness to finish the proof. □

Lemma 6.78 *Let ξ be any point in Λ_b. For any $\epsilon_2, \epsilon_3 > 0$, there exists $n_\xi \in \mathbb{N}$ such that for any $n \geq n_\xi$, there exists γ in $\mathcal{H}^n$ satisfying*

$$d_{\mathbb{S}^d}(\gamma^{-1} \infty, \xi) \leq \epsilon_2, \quad |\gamma'|_\infty \leq \epsilon_3.$$

Proof Since Λ_b is Γ_b minimal, for any point $\xi' \in \Lambda_b$, there exists a sequence $\{\gamma_n^{-1}\}$ in Γ_b such that $\gamma_n^{-1} \xi'$ converges to ξ and $|\gamma_n'|_\infty$ tends to zero. By Lemma 4.8, we know that $\gamma_n^{-1} \Lambda_-$ also converges to ξ. Hence we can always find a γ in Γ_f with $|\gamma'|_\infty \leq \epsilon_3$ and $d_{\mathbb{S}^d}(\gamma^{-1} \Lambda_-, \xi) \leq \epsilon_2$. Let n_ξ be the unique number such that $\gamma \in \mathcal{H}^{n_\xi}$.

For any $\gamma_1 \in \cup_{n \geq 1} \mathcal{H}^n$, we have $|(\gamma_1 \gamma)'|_\infty \leq |\gamma_1'|_\infty |\gamma'|_\infty \leq |\gamma'|_\infty$ and

$$d_{\mathbb{S}^d}((\gamma_1 \gamma)^{-1} \infty, \xi) = d_{\mathbb{S}^d}(\gamma^{-1}(\gamma_1^{-1} \infty), \xi) \leq d_{\mathbb{S}^d}(\gamma^{-1} \Lambda_-, \xi) \leq \epsilon_2.$$

Therefore, for any $m = n_\xi + n$, choose any $\gamma_1 \in \mathcal{H}^n$ and then $\gamma_1 \gamma \in \mathcal{H}^m$ and it satisfies Lemma 6.78. □

Combining the above two lemmas, by Lemma 6.75 and the formula $R_n(\gamma x) = -\log|\gamma'(x)|$ for $\gamma \in \mathcal{H}^n$, we have

$$|\partial_e(R_n \circ \gamma_{1m} - R_n \circ \gamma_{2m})(y)| = \left| \left\langle \frac{y - \gamma_{1m}^{-1}\infty}{|y - \gamma_{1m}^{-1}\infty|^2} - \frac{y - \gamma_{2m}^{-1}\infty}{|y - \gamma_{2m}^{-1}\infty|^2}, e \right\rangle \right|.$$

Using this expression, Lemma 6.77, 6.78 and continuity, we obtain

Lemma 6.79 *For every $x \in \Lambda_\Gamma \cap \overline{\Delta}_0$, there exist $\epsilon_x, \epsilon_x' > 0$ such that for any $\epsilon_3 > 0$, there exists $n_x \in \mathbb{N}$ such that the following holds for any $n \geq n_x$. There exist $k_x \in \mathbb{N}$, $\gamma_{im} \in \mathcal{H}^n$ with $i = 1, 2$ and $m = 1, \ldots, k_x$ satisfying*

- *$|\gamma_{im}'| < \epsilon_3$ for every $i = 1, 2$ and $m = 1, \ldots, k_x$.*
- *for any unit vector $e \in \mathbb{R}^d$, there exists $m \in \{1, \ldots, k_x\}$ such that for any $y \in B(x, \epsilon_x)$,*

$$|\partial_e(R_n \circ \gamma_{1m} - R_n \circ \gamma_{2m})(y)| \geq \epsilon_x' > 0.$$

Proof of Lemma 4.5 For every $x \in \Lambda_\Gamma \cap \overline{\Delta}_0$, we apply Lemma 6.79 to x and get two constants $\epsilon_x, \epsilon_x' > 0$. Since $\Lambda_\Gamma \cap \overline{\Delta}_0$ is compact, we can find a finite set $\{x_1, \ldots, x_l\}$ such that $\cup B(x_j, \epsilon_{x_j}/2) \supset \Lambda_\Gamma \cap \overline{\Delta}_0$. Let $\epsilon_0 = \inf\{\epsilon_{x_j}'\}$ and $r = \inf\{\epsilon_{x_j}/2\}$. Take $\epsilon_3 = \epsilon_0/C$ and $n_0 \geq \sup_{1 \leq j \leq l}\{n_{x_j}\}$. Then for every x_j, there exists a finite set $\{\gamma_{im}\}$ in $\mathcal{H}^{n_0}$ satisfying results in Lemma 6.79. We put all these γ_{im}'s together and this is the finite set in $\mathcal{H}^{n_0}$ described in Lemma 4.5. For any $x \in \Lambda_\Gamma \cap \overline{\Delta}_0$, it is contained in some $B(x_j, \epsilon_{x_j}/2)$. Then $B(x, r) \subset B(x_j, \epsilon_{x_j})$. The family $\{\gamma_{im}\}$ for x_j will satisfy nonvanishing condition on $B(x, r)$, that is for every unit vector $e \in \mathbb{R}^d$ there exists m such that for any $y \in B(x, r)$

$$|\partial_e(R_{n_0} \circ \gamma_{1m} - R_{n_0} \circ \gamma_{2m})(y)| \geq \epsilon_0 > 0.$$

Finally, the inequality $|D\tau_{im}|_\infty \leq C_2$ is due to (4.4). □

7 Spectral gap and Dolgopyat-type spectral estimate

In this section, we prove a Dolgopyat-type spectral estimate and the main result is Proposition 7.3. Our argument is influenced by the one in [1,2,5,18, 34,47] and there is some technical variation in the current setting. The proof involves proving a cancellation lemma (Lemma 7.14) and using it to obtain

L^2 contraction. The rough idea is as follows. Denote the set $\Delta_0 \cap \Lambda_\Gamma$ by Λ_0. With the UNI property (Lemma 4.5) available, for each ball $B(y, r)$ with $y \in \Lambda_0$, one uses the doubling property of the PS measure to find a point $x \in B(y, r) \cap \Lambda_0$ such that cancellation happens on $B(x, r')$. Then, to run the classical argument, one needs to find finitely many such pairwise disjoint balls $B(x_i, r')$'s contained in Δ_0 such that $\sqcup B(x_i, Dr')$ covers Δ_0 for some $D > 1$. The difficulty lies in that the balls $B(x_i, r)$'s are produced using PS measure so the position of $B(x_i, r')$'s is in some sense random and some $B(x_i, r')$ may not be fully contained in Δ_0. To overcome this, we find $B(x_i, r')$'s which only cover a subset of Δ_0 and divide the proof of Proposition 7.3 into the cases when the iteration is small and when the iteration is large.

7.1 Twisted transfer operators

For $s \in \mathbb{C}$, let L_s be the twisted transfer operator defined by

$$L_s(u)(x) = \sum_{\gamma \in \mathcal{H}} |\gamma'(x)|^{\delta+s} u(\gamma x). \tag{7.1}$$

For $u : \Delta_0 \to \mathbb{C}$, define

$$\|u\|_{\mathrm{Lip}} = \max\{|u|_\infty, |u|_{\mathrm{Lip}}\},$$

where $|u|_{\mathrm{Lip}} = \sup_{x \neq y} |u(x) - u(y)|/d(x, y)$, where $d(\cdot, \cdot)$ is the Euclidean distance. Denote by $\mathrm{Lip}(\Delta_0)$ the space of functions $u : \Delta_0 \to \mathbb{C}$ with $\|u\|_{\mathrm{Lip}} < \infty$. We also introduce a family of equivalent norms on $\mathrm{Lip}(\Delta_0)$:

$$\|u\|_b = \max\{|u|_\infty, |u|_{\mathrm{Lip}}/(1 + |b|)\}, \quad b \in \mathbb{R}.$$

With Proposition 4.1 available, we obtain the following lemma by a verbatim of the proof of [1, Proposition 2.5].

Lemma 7.2 *Write $s = \sigma + ib$. The family $s \mapsto L_s$ of operators on $\mathrm{Lip}(\Delta_0)$ is continuous on $\{s \in \mathbb{C} : \sigma > -\epsilon_o\}$, where ϵ_o is given as in Proposition 4.1 (4). Moreover, $\sup_{|\sigma| < \epsilon_o} \|L_s\|_b < \infty$.*

Define the PS measure μ_E on Δ_0 with respect to the Euclidean metric by

$$d\mu_E(x) = (1 + |x|^2)^\delta d\mu(x).$$

Using the quasi-invariance of the PS measure μ, we obtain that the dual operator of L_0 preserves the measure μ_E by a straightforward computation. Our main result of Dolgopyat argument is the following L^2 contracting proposition.

Proposition 7.3 *There exist $C > 0$, $\beta < 1$, $\epsilon > 0$ and $b_0 > 0$ such that for all v in $\mathrm{Lip}(\Delta_0)$, $m \in \mathbb{N}$ and $s = \sigma + ib$ with $|\sigma| \leq \epsilon$ and $|b| > b_0$, we have*

$$\int |L_s^m v|^2 \mathrm{d}\mu_E \leq C\beta^m \|v\|_b^2.$$

The proof will be given at the end of Sect. 7.5.

Recall that ν is the unique T-invariant ergodic probability measure on $\Delta_0 \cap \Lambda_\Gamma$ which is absolutely continuous with respect to the PS measure μ with a positive Lipschitz density function $\bar{f}_0$. So ν is also absolutely continuous with respect to μ_E with a positive Lipschitz density function f_0. Based on these, it is a classical result that the operator L_0 acts on $\mathrm{Lip}(\Delta_0)$ and has a spectral gap and a simple isolated eigenvalue at 1 with f_0 the corresponding eigenfunction.

For $\sigma \in \mathbb{R}$ close enough to 0, L_σ acting on $\mathrm{Lip}(\Delta_0)$ is a continuous perturbation of L_0 (see Lemma 7.2). Hence, it has a unique eigenvalue λ_σ close to 1, and the corresponding eigenfunction f_σ (normalized so that $\int f_\sigma = 1$) belongs to $\mathrm{Lip}(\Delta_0)$, strictly positive, and tends to f_0 in $\mathrm{Lip}(\Delta_0)$ as $\sigma \to 0$. Choose a sufficiently small $\epsilon \in (0, \epsilon_o)$ such that for $\sigma \in (-\epsilon, \epsilon)$, f_σ is well defined and

$$1/2 \leq \lambda_\sigma \leq 2, \quad f_0/2 \leq f_\sigma \leq 2f_0, \quad |f_0|_{\mathrm{Lip}}/2 \leq |f_\sigma|_{\mathrm{Lip}} \leq 2|f_0|_{\mathrm{Lip}}.$$

For $s = \sigma + ib$ with $|\sigma| < \epsilon$ and $b \in \mathbb{R}$, define a modified transfer operator $\tilde{L}_s$ by

$$\tilde{L}_s(u) = (\lambda_\sigma f_\sigma)^{-1} L_s(f_\sigma u). \tag{7.4}$$

It satisfies $\tilde{L}_\sigma 1 = 1$, and $|\tilde{L}_s u| \leq \tilde{L}_\sigma |u|$.

Lemma 7.5 (Lasota-Yorke inequality) *There is a constant $C_{16} > 1$ such that*

$$|\tilde{L}_s^n v|_{\mathrm{Lip}} \leq C_{16}(1 + |b|)|v|_\infty + C_{16}\lambda^n |v|_{\mathrm{Lip}} \tag{7.6}$$

holds for any $s = \sigma + ib$ with $|\sigma| < \epsilon$, and all $n \geq 1$, $v \in \mathrm{Lip}(\Delta_0)$, where λ is given as in Proposition 4.1.

The proof of this lemma is a verbatim of proof of [1, Lemma 2.7]. The following lemma can be deduced from Lemma 7.5 by a straightforward computation.

Lemma 7.7 *We have $\|\tilde{L}_s^n\|_b \leq 2C_{16}$ for all $s = \sigma + ib$ with $|\sigma| < \epsilon$ and all $n \geq 1$.*

7.2 Cancellation lemma

The main result of this subsection is the cancellation lemma (Lemma 7.14) and the proof is inspired by the proof of analogous results in [34] and [47]. We start with detailing all the constants.

Let C_{17} be the constant which will be specified in (7.28). We define the cone

Definition 7.8 For $b \in \mathbb{R}$, let

$$\mathcal{C}_b = \{(u, v) : u, v \in \mathrm{Lip}(\Delta_0), \ u > 0, \ 0 \le |v| \le u, \ |\log u|_{\mathrm{Lip}} \le C_{17}|b|,$$
$$|v(x) - v(y)| \le C_{17}|b|u(y)d(x, y) \ \text{ for all } \ x, y \in \Delta_0\}.$$

Let $r > 0$ and $\epsilon_0 > 0$ be the same constants as the ones in Lemma 4.5. We apply Lemma 4.5 with $C = 16C_{17}$. Let n_0 be a sufficiently large integer which satisfies Lemma 4.5 and the inequality

$$\lambda^{n_0} C_{17}(1 + \mathrm{diam}(\Delta_0)) \le 1. \tag{7.9}$$

Let γ_{mj}, with $m = 1, 2$, $j = 1, \ldots, j_0$ be the inverse branches given by Lemma 4.5.

Let $k \in \mathbb{N}$ be such that

$$k\epsilon_0 > 16(C_2 + \epsilon_0), \tag{7.10}$$

where C_2 is given in (4.3).

Note that the measure ν is absolutely continuous with respect to the PS measure μ. Let $D > 0$ be such that for all $x \in \Lambda_\Gamma \cap \Delta_0$ and $r' \le 1/C_3$ with C_3 given in Proposition 5.14

$$\nu(B(x, Dr')) > \nu(B(x, (k + 2)r')). \tag{7.11}$$

Let $\epsilon_2 > 0$ be such that

$$(2C_{17}\epsilon_2 + 1/4)e^{2C_{17}\epsilon_2} \le 3/4. \tag{7.12}$$

Let $\epsilon_3 > 0$ be such that

$$\epsilon_3(D + 2) < \min\{\epsilon_2, \ r, \ 1/C_3\}, \ \epsilon_3(D + 2)(C_2 + \epsilon_0) < 3\pi/2, \ \epsilon_3 k\epsilon_0 < \pi. \tag{7.13}$$

Recall the notation τ_{mj} introduced in Lemma 4.5. For $s = \sigma + ib \in \mathbb{C}$, define

$$A_{s,\gamma_{mj}}(v)(x) = e^{(s+\delta)\tau_{mj}(x)} f_\sigma(\gamma_{mj}x)v(\gamma_{mj}x).$$

Lemma 7.14 *There exists* $0 < \eta_0 < 1$ *such that the following holds. For* $s = \sigma + ib$ *with* $|\sigma| \leq \epsilon$, $|b| > 1$, *for* $(u, v) \in C_b$, *and for any* $y \in \Lambda_0$, *there exists* $x \in B(y, \epsilon_3 D/|b|) \cap \Lambda_\Gamma$ *such that we have the following: there exists* $j \in \{1, \ldots, j_0\}$ *such that one of the following inequalities holds on* $B(x, \epsilon_3/|b|)$:

> ***type*** γ_{1j} : $|A_{s,\gamma_{1j}}(v) + A_{s,\gamma_{2j}}(v)| \leq \eta_0 A_{\sigma,\gamma_{1j}}(u) + A_{\sigma,\gamma_{2j}}(u)$,
>
> ***type*** γ_{2j} : $|A_{s,\gamma_{1j}}(v) + A_{s,\gamma_{2j}}(v)| \leq A_{\sigma,\gamma_{1j}}(u) + \eta_0 A_{\sigma,\gamma_{2j}}(u)$.

We first prove a quick estimate.

Lemma 7.15 *Let* ϵ_2 *be the constant defined in* (7.12). *For any* $|b| > 1$, *for* $(u, v) \in C_b$ *and for a ball* Z *of radius* $\epsilon_2/|b|$, *we have*

1. $\inf_Z u \geq e^{-2C_{17}\epsilon_2} \sup_Z u$;
2. *either* $|v| \leq \frac{3}{4}u$ *for all* $x \in Z$ *or* $|v| \geq \frac{1}{4}u$ *for all* $x \in Z$.

Proof The first inequality is due to $|\log u(x) - \log u(y)| \leq C_{17}|b||x - y|$ for every $x, y \in \Delta_0$.

Suppose there exists $x_0 \in Z$ such that $|v(x_0)| \leq \frac{1}{4}u(x_0)$. Then by (7.12)

$$|v(x)| \leq |v(x) - v(x_0)| + \frac{1}{4}u(x_0) \leq C_{17}|x - x_0||b| \sup_Z u + \frac{1}{4} \sup_Z u$$

$$\leq \left(2C_{17}\epsilon_2 + \frac{1}{4}\right) \sup_Z u \leq (2C_{17}\epsilon_2 + \frac{1}{4})e^{2C_{17}\epsilon_2} \inf_Z u \leq \frac{3}{4}u(x).$$

$\square$

Proof of Lemma 7.14 It follows (7.11) that there exists $x_0 \in (B(y, \epsilon_3 D/|b|) - B(y, (k + 2)\epsilon_3/|b|)) \cap \Lambda_\Gamma$. Let $B_1 = B(y, \epsilon_3/|b|)$, $B_2 = B(x_0, \epsilon_3/|b|)$ and $\hat{B}$ the smallest ball containing $B_1 \cup B_2$. For all $x \in B_1$, $x' \in B_2$, we have

$$d(x, x') \in \frac{\epsilon_3}{b}[k, D + 2]. \tag{7.16}$$

In view of (7.13), the radius of $\hat{B}$ is smaller than $\epsilon_2/|b|$ and it is contained in $B(y, r)$. Let $e_0 = (y - x_0)/|y - x_0|$.

By Lemma 4.5 for the point y there exists j in $\{1, \ldots, j_0\}$ such that (4.6) holds for $B(y, r)$ with $e = e_0$. From now on, j is fixed, so we abbreviate $(\gamma_{1j}, \gamma_{2j})$ to (γ_1, γ_2) and (τ_{1j}, τ_{2j}) to (τ_1, τ_2).

Due to $|\gamma'_m|_\infty \leq \lambda \leq 1$, the radius of $\gamma_m \hat{B}$ is smaller than $\epsilon_2/|b|$. So we can apply Lemma 7.15 to $\gamma_m \hat{B}$ and we have that either $|v(\gamma_m x)| \geq \frac{1}{4}u(\gamma_m x)$ for all $x \in \hat{B}$ or $|v(\gamma_m x)| \leq \frac{3}{4}u(\gamma_m x)$ for all $x \in \hat{B}$. Suppose that

$$|v(\gamma_m x)| \leq \frac{3}{4}u(\gamma_m x)$$

holds for some $m \in \{1, 2\}$ for all $x \in \hat{B}$. Then Lemma 7.14 can be proved by a straightforward computation.

Suppose that for $x \in \hat{B}$ and $m = 1, 2$

$$|v(\gamma_m x)| \geq \frac{1}{4} u(\gamma_m x). \tag{7.17}$$

Claim Under the assumption of (7.17), there exists $C_{18} > 0$ independent of b and (u, v) such that for $l \in \{1, 2\}$, we have

$$\text{either } \left| \frac{A_{s,\gamma_1}(v)}{A_{s,\gamma_2}(v)} \right| \leq C_{18} \text{ for all } x \in B_l \text{ or } \left| \frac{A_{s,\gamma_2}(v)}{A_{s,\gamma_1}(v)} \right| \leq C_{18} \text{ for all } x \in B_l. \tag{7.18}$$

Proof of the claim Fix any $x_0 \in \Delta_0$. Due to $|\tau_m'|_\infty \leq C_2$ (see (4.4)), we have for any $x \in \hat{B}$,

$$|\tau_1(x) - \tau_2(x)| \leq |\tau_1(x_0) - \tau_2(x_0)| + 2C_2|x - x_0|.$$

Hence there exists a constant $C(\tau_1, \tau_2)$ depending on τ_1, τ_2 such that

$$\left| \frac{A_{s,\gamma_1}(v)}{A_{s,\gamma_2}(v)} \right| \leq C(\tau_1, \tau_2) \frac{f_\sigma(\gamma_1 x) u(\gamma_1 x)}{f_\sigma(\gamma_2 x) u(\gamma_2 x)}.$$

For the middle term,

$$\frac{f_\sigma(\gamma_1 x)}{f_\sigma(\gamma_2 x)} \leq 4 \frac{\sup f_0}{\inf f_0}.$$

Since the radius of $\gamma_2 B_l$ is less than $\epsilon_2/|b|$, using Lemma 7.15, we have for every x in B_l

$$\frac{u(\gamma_1 x)}{u(\gamma_2 x)} \leq \frac{\sup_{B_l} u(\gamma_1)}{\inf_{B_l} u(\gamma_2)} \leq e^{2C_{17}\epsilon_2} \frac{\sup_{B_l} u(\gamma_1)}{\sup_{B_l} u(\gamma_2)}.$$

Putting these together, we have

$$\left| \frac{A_{s,\gamma_1}(v)}{A_{s,\gamma_2}(v)} \right| \leq C_{18} \frac{\sup_{B_l} u(\gamma_1)}{\sup_{B_l} u(\gamma_2)}$$

where $C_{18} = 4C(\tau_1, \tau_2) e^{2C_{17}\epsilon_2} \frac{\sup f_0}{\inf f_0}$. We have a similar inequality for $\left| \frac{A_{s,\gamma_2}(v)}{A_{s,\gamma_1}(v)} \right|$. Note that either $\frac{\sup_{B_l} u(\gamma_1)}{\sup_{B_l} u(\gamma_2)} \leq 1$ or $\frac{\sup_{B_l} u(\gamma_2)}{\sup_{B_l} u(\gamma_1)} \leq 1$. The proof of the claim finishes. $\square$

Now we start to compute the angle and our definitions are only for $x \in \hat{B}$. The function $\arg(v(\gamma_m x))$ is well defined because $|v(\gamma_m x)| \geq u(\gamma_m x)/4 > 0$. Let

$$\Theta(x) = b(\tau_1(x) - \tau_2(x)), \quad V(x) = \arg(v(\gamma_1 x)) - \arg(v(\gamma_2 x)),$$

and let

$$\Phi(x) = \Theta(x) + V(x).$$

We apply Lemma 4.5 to $\hat{B}$ and obtain that for $x \in \hat{B}$,

$$|\partial_{e_0} \Theta(x)| \geq |b|\epsilon_0, \quad |\Theta'(x)| \leq |b|C_2.$$

For the angle function,
by (7.17) and (4.7), we have for $i \in \{1, 2\}$ and $x, x' \in \hat{B}$

$$
\begin{aligned}
|\arg v(\gamma_i x) - \arg v(\gamma_i x')| &= |\operatorname{Im}(\log v(\gamma_i x) - \log v(\gamma_i x'))| \\
&\leq \frac{|v(\gamma_i x) - v(\gamma_i x')|}{|v(\gamma_i x)|} \\
&\leq C_{17}|b| \frac{u(\gamma_i x)|\gamma_i x - \gamma_i x'|}{|v(\gamma_i x)|} \leq |b|\epsilon_0|x - x'|/4.
\end{aligned}
$$

This implies that for $x, x' \in \hat{B}$

$$|V(x) - V(x')| \leq |b|\epsilon_0|x - x'|/2.$$

Combining the estimates for Θ and V, we obtain for $x, x' \in \hat{B}$

$$|\Phi(x) - \Phi(x')| \leq b(C_2 + \epsilon_0)|x - x'|, \qquad (7.19)$$

and for $x, x + te_0 \in \hat{B}$ with $t \in \mathbb{R}^+$,

$$|\Phi(x) - \Phi(x + te_0)| \geq b\epsilon_0 t/2.$$

Hence for $x_1 = y$, $x_2 = x_0$ which are the centers of B_1 and B_2 respectively, by (7.16),

$$|\Phi(x_1) - \Phi(x_2)| \in \epsilon_3[k\epsilon_0/2, (D + 2)(C_2 + \epsilon_0)]. \qquad (7.20)$$

Let $\epsilon_4 = \epsilon_3 k\epsilon_0/8$. We claim that there exists $l \in \{1, 2\}$ such that

$$d(\Phi(x_l), 2\pi\mathbb{Z}) > \epsilon_4. \qquad (7.21)$$

If not so, then both the distance from $\Phi(x_1)$ to $2\pi\mathbb{Z}$ and that from $\Phi(x_2)$ to $2\pi\mathbb{Z}$ are less than ϵ_4. By (7.20) and (7.13)

$$|\Phi(x_1) - \Phi(x_2)| \leq \epsilon_3(D+2)(C_2 + \epsilon_0) \leq 3\pi/2 < 2\pi - 2\epsilon_4.$$

Hence $\Phi(x_1)$, $\Phi(x_2)$ are in a ball $(2n\pi - \epsilon_4, 2n\pi + \epsilon_4)$ with $n \in \mathbb{Z}$. This implies that

$$|\Phi(x_1) - \Phi(x_2)| \leq 2\epsilon_4 = \epsilon_3 k\epsilon_0/4,$$

contradicting with (7.20).

Without loss of generality, we may assume (7.21) holds for x_1. For any x in the ball B_1, by (7.19) and (7.10)

$$|\Phi(x) - \Phi(x_l)| \leq (C_2 + \epsilon_0)\epsilon_3 \leq k\epsilon_3\epsilon_0/16 = \epsilon_4/2.$$

Combined with (7.21), we have

$$d(\Phi(x), 2\pi\mathbb{Z}) \geq \epsilon_4/2. \tag{7.22}$$

In conclusion, there exists $l \in \{1, 2\}$ such that for all $x \in B_l$, $d(\Phi(x), 2\pi\mathbb{Z}) > \epsilon_4/2$ and (7.18) holds. Without loss of generality, we may assume $|A_{s,\gamma_1}(v)(x)/A_{s,\gamma_2}(v)(x)| \leq C_{18}$ for all $x \in B_l$. By an elementary inequality [34, Lemma 5.12], there exists $0 < \eta_0 < 1$ depending on ϵ_4 and C_{18} such that on B_l

$$|A_{s,\gamma_1}(v) + A_{s,\gamma_2}(v)| \leq \eta_0|A_{s,\gamma_1}(v)| + |A_{s,\gamma_2}(v)| \leq \eta_0 A_{\sigma,\gamma_1}(u) + A_{\sigma,\gamma_2}(u).$$

$\square$

For b with $|b| \geq 1$, let

$$\Delta_b = \left\{ x \in \Delta_0 \mid d(x, \partial\Delta_0) > \frac{\epsilon_3(D+1)}{|b|} \right\}. \tag{7.23}$$

For any $(u, v) \in \mathcal{C}_b$, we can find $\{x_i\}_{1 \leq i \leq l_0} \subset \Lambda_0 := \Lambda_\Gamma \cap \Delta_0$ such that $B(x_i, \epsilon_3/|b|)$'s are disjoint balls contained in Δ_0,

$$\Lambda_0 \cap \Delta_b \subset \cup_{1 \leq i \leq l_0} B(x_i, 2\epsilon_3 D/|b|),$$

and on each $B(x_i, \epsilon_3/|b|)$ one of the $2j_0$ inequalities in Lemma 7.14 holds. In fact, suppose we have already found some points x_i's but $\cup B(x_i, 2\epsilon_3 D/|b|)$ don't cover the set $\Lambda_0 \cap \Delta_b$. Then for a point $y \in \Lambda_0 \cap \Delta_b - \cup B(x_i, 2\epsilon_3 D/|b|)$, we apply Lemma 7.14 to y and obtain a point $x \in B(y, \epsilon_3 D/|b|) \cap \Lambda_0$ such

that Lemma 7.14 holds on $B(x, \epsilon_3/|b|)$. Moreover, the ball $B(x, \epsilon_3/|b|)$ is contained in Δ_0 and it is disjoint from $\cup B(x_i, \epsilon_3/|b|)$.

Let $B_i = B(x_i, \epsilon_3/|b|)$ and $\tilde{B}_i = B(x_i, \epsilon_3/(3|b|))$ for $i = 1, \ldots, l_0$. Let $\eta \in [\eta_0, 1)$ and define a C^1 function $\chi : \Delta_0 \to [\eta, 1]$ as follows: it equals 1 outside of $\cup_{m,j,i} \gamma_{mj} B_i$; for each B_i, if B_i is of type γ_{mj}, let $\chi(\gamma_{mj}(y)) = \eta$ for $y \in \tilde{B}_i$ and $\chi \equiv 1$ on other $\gamma_{m'j'} B_i$. We can choose η close to 1 and independent of b such that $|\chi'(x)| \leq |b|$ for all $x \in \Delta_0$.

Corollary 7.24 *Under the same assumptions as in Lemma 7.14, for $(u, v) \in \mathcal{C}_b$ and $\chi = \chi(b, u, v)$ a C^1 function described as above, we have*

$$|\tilde{L}_s^{n_0} v| \leq \tilde{L}_\sigma^{n_0}(\chi u).$$

Define $J_i = B(x_i, 2\epsilon_3 D/|b|)$ for $i = 1, \ldots, l_0$ and let $\tilde{B} = \cup \tilde{B}_j$.

Proposition 7.25 *Suppose that w is a positive Lipschitz function with $|\log w(x) - \log w(y)| \leq K|b||x - y|$ for some $K > 0$. Then*

$$\int_{\tilde{B}} w \mathrm{d}\nu \geq \epsilon_4 \int_{\Delta_b} w \, \mathrm{d}\nu, \qquad (7.26)$$

with $\epsilon_4 = \epsilon_5 e^{-4\epsilon_3 DK}$, where ϵ_5 comes from doubling property only depending on D and ν.

Proof Since $\cup_i J_i$ covers Δ_b, it is sufficient to prove for each i we have a similar inequality. Due to hypothesis, we obtain $\inf_{\tilde{B}_i} w \geq e^{-4\epsilon_3 DK} \sup_{J_i} w$. By doubling property, there exists ϵ_5 depending on D such that

$$\nu(\tilde{B}_i) \geq \epsilon_5 \nu(J_i).$$

Therefore

$$\int_{\tilde{B}_i} w \, \mathrm{d}\nu \geq \nu(\tilde{B}_i) \inf_{\tilde{B}_i} w \geq \epsilon_5 \nu(J_i) e^{-4\epsilon_3 DK} \sup_{J_i} w \geq \epsilon_4 \int_{J_i} w \, \mathrm{d}\nu.$$

$\square$

7.3 Invariance of cone condition

We define the constants

$$C'_{17} = 16(\delta + \epsilon)C_2 |f_0|_\infty |f_0^{-1}|_\infty + 16|f_0^{-1}|_\infty |f_0|_{\mathrm{Lip}} + 4C_2 + 2, \qquad (7.27)$$

$$C_{17} = \max\{8|f_0^{-1}|_\infty |f_0|_{\mathrm{Lip}} + (\delta + 3)C_2 + 1 + 4|f_0|_\infty |f_0^{-1}|_\infty C'_{17}, 6C_{16}\}. \qquad (7.28)$$

Lemma 7.29 *Let $C_{17} > 0$ be the constant defined in (7.28) and n_0 be the constant defined in (7.9). For $s = \sigma + ib$ with $|\sigma| < \epsilon$ and $|b| > 1$, for $(u, v) \in \mathcal{C}_b$, we have*

$$(\tilde{L}_\sigma^{n_0}(\chi u), \, \tilde{L}_s^{n_0} v) \in \mathcal{C}_b, \tag{7.30}$$

where $\chi = \chi(b, u, v)$ is the same as the one in Corollary 7.24.

The proof is a verbatim of the proof of [1, Lemma 2.12].

7.4 L^2 contraction for bounded iterations

In this part, we will prove Proposition 7.3 for the case when m bounded by $\log|b|$. Compared with [2], where they can finish the proof of an analog of Proposition 7.3 at this stage, we have the difficulty about the boundary. More precisely, Proposition 7.25 is one of the ingredients to obtain Proposition 7.3. Now the integration region of the right hand side of (7.26) is Δ_b, which is smaller than Δ_0, so it just enables us to obtain L^2 contraction for bounded iterations. For large iteration, we will use a Lipschitz contraction lemma (Lemma 7.39) to obtain L^2 contraction in the next subsection.

Lemma 7.31 *For $|b| > 1$ and $v \in \mathrm{Lip}(\Delta_0)$, if $|v|_{\mathrm{Lip}} \geq C_{17}|b||v|_\infty$, then*

$$\|\tilde{L}_s^{n_0} v\|_b \leq \frac{9}{10}\|v\|_b.$$

Proof We have

$$|\tilde{L}_s^{n_0} v|_\infty \leq |v|_\infty \leq \frac{1}{C_{17}|b|}|v|_{Lip} \leq \frac{2}{C_{17}}\|v\|_b.$$

By Lemma 7.5, we obtain

$$\begin{aligned}
|\tilde{L}_s^{n_0} v|_{\mathrm{Lip}} &\leq C_{16}(1 + |b|)|v|_\infty + C_{16}\lambda^{n_0}|v|_{\mathrm{Lip}} \\
&\leq (1 + |b|)\left(\frac{C_{16}(1 + |b|)}{C_{17}|b|} + C_{16}\lambda^{n_0}\right)\|v\|_b \\
&\leq (1 + |b|)\left(\frac{1}{3} + \frac{1}{6}\right)\|v\|_b = (1 + |b|)\frac{1}{2}\|v\|_b,
\end{aligned}$$

where the last inequality is due to $C_{17} \geq 6C_{16}$ and $\lambda^{n_0} C_{17} \leq 1$. □

Lemma 7.32 *There exist $C_{19} > 0$ and $\beta < 1$ such that for all $s = \sigma + ib$ with $|\sigma| < \epsilon$ and $|b|$ large enough and $m \leq [C_{19} \log |b|]$*

$$\int |\tilde{L}_s^{mn_0} v|^2 dv \leq \beta^m \|v\|_b^2. \tag{7.33}$$

Proof If for all $0 \leq p \leq m - 1$, we have $|\tilde{L}_s^{pn_0} v|_{\text{Lip}} \geq C_{17} |b| |\tilde{L}_s^{pn_0} v|_\infty$, then by Lemma 7.31,

$$\int |\tilde{L}_s^{mn_0} v|^2 dv \leq \|\tilde{L}_s^{mn_0} v\|_b^2 \leq (\frac{9}{10})^m \|v\|_b^2.$$

Otherwise, suppose p is the smallest integer such that $|\tilde{L}_s^{pn_0} v|_{\text{Lip}} \leq C_{17} |b| |\tilde{L}_s^{pn_0} v|_\infty$. We consider $v' = \tilde{L}_s^{pn_0} v$. Then Lemma 7.31 implies $\|v'\|_b \leq (\frac{9}{10})^p \|v\|_b$. We only need to show that

$$\int |\tilde{L}_s^{(m-p)n_0} v'|^2 dv \leq \beta^{m-p} \|v'\|_b^2.$$

We reduce to the case when $p = 0$, that is $|v|_{\text{Lip}} \leq C_{17} |b| |v|_\infty$. Define $u_0 \equiv 1$, $v_0 = v/|v|_\infty$ and induitively,

$$u_{m+1} = \tilde{L}_\sigma^{n_0}(\chi_m u_m), \quad v_{m+1} = \tilde{L}_s^{n_0}(v_m),$$

where $\chi_m = \chi(b, u_m, v_m)$. It is immediate that $(u_0, v_0) \in \mathcal{C}_b$, and it follows from Lemma 7.29 that $(u_m, v_m) \in \mathcal{C}_b$ for all m. Hence in particular the χ_m's are well defined.

We will show that there exist $\beta_1 \in (0, 1)$, $\kappa > 0$ and $C > 0$ such that for all m

$$\int u_{m+1}^2 dv \leq \beta_1 \int u_m^2 dv + C|b|^{-\kappa}. \tag{7.34}$$

Then note that

$$|\tilde{L}_s^{mn_0} v| = |v|_\infty |\tilde{L}_s^{mn_0} v_0| = |v|_\infty |v_m| \leq |v|_\infty u_m.$$

As a result,

$$\int |\tilde{L}_s^{mn_0} v|^2 dv \leq |v|_\infty^2 \int u_m^2 dv$$

$$\leq |v|_\infty^2 (\beta_1^m \int u_0^2 dv + C|b|^{-\kappa} \sum_{0 \leq l \leq m-1} \beta_1^l)$$

$$\leq (\beta_1^m + C|b|^{-\kappa}/(1 - \beta_1))|v|_\infty^2.$$

We can find $C_{19} > 0$ and $\beta < 1$ such that for any large enough $|b|$, (7.33) holds for all $m \leq [C_{19} \log |b|]$.

Now we prove (7.34). By definition

$$u_{m+1}(x) = \lambda_\sigma^{-n_0} f_\sigma^{-1}(x) \sum_{\gamma \in \mathcal{H}^{n_0}} |\gamma'(x)|^{\delta+\sigma} f_\sigma(\gamma x) \chi_m(\gamma x) u_m(\gamma x)$$

$$= \lambda_\sigma^{-n_0} f_\sigma^{-1}(x) \sum_{\gamma \in \mathcal{H}^{n_0}} \left(|\gamma'(x)|^{\delta/2} f_\sigma^{1/2}(\gamma x) u_m(\gamma x) \right)$$

$$\left(|\gamma'(x)|^{\delta/2+\sigma} f_\sigma^{1/2}(\gamma x) \chi_m(\gamma x) \right),$$

so by Cauchy-Schwarz

$$u_{m+1}^2(x) \leq (\lambda_\sigma^{-n_0} f_\sigma(x))^{-2} \left(\sum_{\gamma \in \mathcal{H}^{n_0}} |\gamma'(x)|^{\delta} f_\sigma(\gamma x) u_m^2(\gamma x) \right)$$

$$\left(\sum_{\gamma \in \mathcal{H}^{n_0}} |\gamma'(x)|^{\delta+2\sigma} f_\sigma(\gamma x) \chi_m^2(\gamma x) \right)$$

$$\leq \xi(\sigma) \tilde{L}_0^{n_0}(u_m^2) \tilde{L}_{2\sigma}^{n_0}(\chi_m^2)$$

where (noting that $\lambda_0 = 1$)

$$\xi(\sigma) = (\lambda_\sigma^{-2} \lambda_{2\sigma})^{n_0} \left| \frac{f_0}{f_\sigma} \right|_\infty \left| \frac{f_{2\sigma}}{f_\sigma} \right|_\infty \left| \frac{f_\sigma}{f_0} \right|_\infty \left| \frac{f_\sigma}{f_{2\sigma}} \right|_\infty.$$

As in Proposition 7.25, we write $\Delta_0 = \tilde{B} \sqcup \tilde{B}^c$. Let $\mathcal{H}_c$ be the set of inverse branches given by Lemma 4.5. If $y \in \tilde{B}$, then there exists $\gamma_i \in \mathcal{H}_c$ such that

$$\tilde{L}_{2\sigma}^{n_0}(\chi_m^2)(y) \leq \lambda_{2\sigma}^{-n_0} f_{2\sigma}(y)^{-1} \left(\eta^2 |\gamma_i'(y)|^{\delta+2\sigma} f_{2\sigma}(\gamma_i y) \right.$$

$$\left. + \sum_{\gamma \in \mathcal{H}^{n_0} \setminus \{\gamma_i\}} |\gamma'(y)|^{\delta+2\sigma} f_{2\sigma}(\gamma y) \right)$$

$$= \tilde{L}_{2\sigma}^{n_0}(1)(y) - (1-\eta^2) \lambda_{2\sigma}^{-n_0} f_{2\sigma}(y)^{-1} |\gamma_i'(y)|^{\delta+2\sigma} f_{2\sigma}(\gamma_i y)$$

$$\leq 1 - (1-\eta^2) 2^{-(n_0+2)} \inf f_0 \cdot |f_0|_\infty^{-1} \cdot \inf_{\{\gamma_i \in \mathcal{H}_c\}} |\gamma_i'|^{\delta+2\sigma} = \eta_1 < 1,$$

In this way we obtain that there exists $\eta_1 < 1$ such that

$$u^2_{m+1}(y) \le \begin{cases} \eta_1 \xi(\sigma) \tilde{L}^{n_0}_0(u^2_m)(y), & y \in \tilde{B}, \\ \xi(\sigma) \tilde{L}^{n_0}_0(u^2_m)(y), & y \in \tilde{B}^c. \end{cases}$$

Since $(u_m, v_m) \in \mathcal{C}_b$, it follows in particular that $|\log u_m|_{\mathrm{Lip}} \le C_{17}|b|$. Hence by (7.9),

$$u^2_m(\gamma x)/u^2_m(\gamma y) \le \exp(2C_{17}\lambda^{n_0}|b|d(x, y)) \le \exp(2|b|d(x, y)).$$

Let $w = \tilde{L}^{n_0}_0(u^2_m)$. Then

$$\frac{w(x)}{w(y)} = \frac{f_0(y) \sum_{\gamma \in \mathcal{H}^{n_0}} |\gamma'(x)|^\delta f_0(\gamma x)u^2_m(\gamma x)}{f_0(x) \sum_{\gamma \in \mathcal{H}^{n_0}} |\gamma'(y)|^\delta f_0(\gamma y)u^2_m(\gamma y)}$$

$$\le \exp\left(\left(2|f_0^{-1}|_\infty |f_0|_{\mathrm{Lip}} + \delta C_2 + 2|b|\right) d(x, y)\right).$$

Hence $|\log w|_{\mathrm{Lip}} \le K|b|$ with $K = 2|f_0^{-1}|_\infty |f_0|_{\mathrm{Lip}} + \delta C_2 + 2$. Using Proposition 7.25, we have

$$(1 - \eta_1) \int_{\tilde{B}} w \mathrm{d}\nu \ge \epsilon_4(1 - \eta_1) \int_{\Delta_b} w \mathrm{d}\nu.$$

Setting $\beta' = 1 - \epsilon_4(1 - \eta_1)$, we can further write

$$\eta_1 \int_{\tilde{B}} w \mathrm{d}\nu + \int_{\Delta_b - \tilde{B}} w \mathrm{d}\nu \le \beta' \int_{\Delta_b} w \mathrm{d}\nu \le \beta' \int_{\Delta_0} w \mathrm{d}\nu.$$

Hence

$$\int_{\Delta_b} u^2_{m+1} \mathrm{d}\nu \le \xi(\sigma) \left(\eta_1 \int_{\tilde{B}} \tilde{L}^{n_0}_0(u^2_m) \mathrm{d}\nu + \int_{\Delta_b - \tilde{B}} \tilde{L}^{n_0}_0(u^2_m) \mathrm{d}\nu\right)$$

$$\le \xi(\sigma)\beta' \int_{\Delta_0} \tilde{L}^{n_0}_0(u^2_m) \mathrm{d}\nu = \xi(\sigma)\beta' \int_{\Delta_0} u^2_m \mathrm{d}\nu. \qquad (7.35)$$

By (5.30), (7.23) and $|u_{m+1}| \le 1$,

$$\int_{\Delta_0 - \Delta_b} u^2_{m+1} \mathrm{d}\nu \le \nu(\Delta_0 - \Delta_b) \le C|b|^{-\kappa}. \qquad (7.36)$$

Finally we can shrink ϵ if necessary so that $\xi(\sigma)\beta' < 1$ for $|\sigma| < \epsilon$ and then (7.35) and (7.36) imply (7.34). $\square$

7.5 Proof of Proposition 7.3

Lemma 7.37 *There exist $\epsilon \in (0, 1)$, $\tau \in (0, 1)$ and $C_{20} > 0$ such that for all $s = \sigma + ib$ with $|\sigma| < \epsilon$, $n \geq 1$ and $v \in \mathrm{Lip}(\Delta_0)$, we have*

$$|\tilde{L}_s^n v|_\infty^2 \leq C_{20}(1 + |b|)\tau^n |v|_\infty \|v\|_b + C_{20}B^n |v|_\infty \int |v|\mathrm{d}\nu$$

where $B > 1$ is a constant depending on ϵ and it tends to 1 as $\epsilon \to 0$.

Proof We have

$$|\tilde{L}_s^n v(x)| \leq \lambda_\sigma^{-n} f_\sigma^{-1}(x) \sum_{\gamma \in \mathcal{H}^n} |\gamma'(x)|^{\delta+\sigma} f_\sigma(\gamma x)|v|(\gamma x)$$

$$= \lambda_\sigma^{-n} f_\sigma^{-1}(x) \sum_{\gamma \in \mathcal{H}^n} \left(|\gamma'(x)|^{\delta/2+\sigma} f_\sigma^{1/2}(\gamma x)|v|^{1/2}(\gamma x)\right)$$

$$\left(|\gamma'(x)|^{\delta/2} f_\sigma^{1/2}(\gamma x)|v|^{1/2}(\gamma x)\right).$$

Using Cauchy-Schwarz, we obtain

$$|\tilde{L}_s^n v(x)|^2 \leq (\lambda_\sigma^{-2}\lambda_{2\sigma})^n \xi(\sigma) \tilde{L}_{2\sigma}^n(|v|)(x) \cdot \tilde{L}_0^n(|v|)(x),$$

where $\xi(\sigma) = |f_0/f_\sigma|_\infty |f_{2\sigma}/f_\sigma|_\infty |f_\sigma/f_0|_\infty |f_\sigma/f_{2\sigma}|_\infty \leq 64$. Hence

$$|\tilde{L}_s^n v|_\infty^2 \leq 64 B^n |v|_\infty |\tilde{L}_0^n(|v|)|_\infty, \tag{7.38}$$

where $B > 1$ is a constant depending on ϵ with $B \to 1$ as $\epsilon \to 0$.

Since $\tilde{L}_0$ is a normalized transfer operator for the uniformly expanding map T, there exists $\tau_1 \in (0, 1)$ such that $|\tilde{L}_0^n v|_\infty \leq C\tau_1^n \|v\|_{\mathrm{Lip}}$ for all $v \in \mathrm{Lip}(\Delta_0)$ with $\int v\mathrm{d}\nu = 0$. (This is a consequence of spectral gap of quasi-compact operator $\tilde{L}_0$.) Hence by decomposing $|v|$ into $(|v| - \int |v|\mathrm{d}\nu) + \int |v|\mathrm{d}\nu$, we obtain

$$|\tilde{L}_0^n(|v|)|_\infty \leq 2C\tau_1^n \|v\|_{\mathrm{Lip}} + \int |v|\mathrm{d}\nu.$$

Substituting into (7.38), we have

$$|\tilde{L}_s^n v|_\infty^2 \leq 128C(B\tau_1)^n(1 + |b|)|v|_\infty \|v\|_b + 64B^n |v|_\infty \int |v|\mathrm{d}\nu.$$

Finally, shrink ϵ if necessary so that $\tau = B\tau_1 < 1$.   □

Lemma 7.39 *There exist $C > 0$, $\epsilon \in (0, 1)$, $A > 0$ and $\beta \in (0, 1)$ such that*

$$\|\tilde{L}_s^{mn_0} v\|_b \leq C\beta^m \|v\|_b$$

for all $m \geq A \log |b|$, $s = \sigma + ib$ with $|\sigma| < \epsilon$ and $|b|$ large enough, and all $v \in \mathrm{Lip}(\Delta_0)$.

Proof Let $N = [C_{19} \log |b|] n_0$. Using Lemma 7.37 for $\tilde{L}_s^N v$ and $n = lN$, Lemma 7.7 and (7.33), we obtain

$$
\begin{aligned}
|\tilde{L}_s^{(l+1)N} v|_\infty^2 &\leq C_{20}(1 + |b|)\tau^{lN}|\tilde{L}_s^N v|_\infty \|\tilde{L}_s^N v\|_b \\
&\quad + C_{20}B^{lN}|\tilde{L}_s^N v|_\infty \left(\int |\tilde{L}_s^N v|^2 \mathrm{d}\nu\right)^{1/2}. \\
&\leq 2C_{16}C_{20}(1 + |b|)\tau^{lN}|v|_\infty \|v\|_b + 2C_{16}C_{20}B^{lN}|v|_\infty \beta^{N/2}\|v\|_b.
\end{aligned}
$$

We fix l depending on τ, C_{19} and n_0 such that $(1 + |b|)\tau^{lN/2} \leq 1$. Then by shrinking B if necessary, there exists $\beta_1 < 1$, such that

$$|\tilde{L}_s^{(l+1)N} v|_\infty \leq \beta_1^{(l+1)N} \|v\|_b. \tag{7.40}$$

For Lipschitz norm, we have

$$
\begin{aligned}
|\tilde{L}_s^{(l+2)N} &v|_{\mathrm{Lip}} \\
&\leq C_{16}(1 + |b|)|\tilde{L}_s^{(l+1)N} v|_\infty + C_{16}\lambda^N |\tilde{L}_s^{(l+1)N} v|_{\mathrm{Lip}} \\
&\leq C_{16}(1 + |b|)\beta_1^{(l+1)N}\|v\|_b + C_{16}^2\lambda^N((1 + |b|)|v|_\infty + \lambda^{(l+1)N}|v|_{\mathrm{Lip}}) \\
&\leq C_{16}^2(1 + |b|)\|v\|_b(\beta_1^{(l+1)N} + \lambda^N + \lambda^{(l+2)N}) \leq 3C_{16}^2(1 + |b|)\beta_2^N\|v\|_b,
\end{aligned}
$$

for some $\beta_2 < 1$, where we use Lemma 7.5 to get the first inequality and (7.40) to get the second one. For the infinity norm, by (7.40) and Lemma 7.7, we obtain

$$|\tilde{L}_s^{(l+2)N} v|_\infty \leq 2C_{16}\beta_1^{(l+1)N}\|v\|_b.$$

Combining these two norm estimates, we obtain

$$\|\tilde{L}_s^{(l+2)N} v\|_b \leq C_{16}^2(2\beta_1^{(l+1)N} + 3\beta_2^N)\|v\|_b \leq \beta_3^{(l+2)N/n_0}\|v\|_b, \tag{7.41}$$

for some $\beta_3 < 1$ if $|b|$ is large enough to absorb the constant $6C_{16}^2$.

Let $A = 2(l + 2)C_{19}$ and $N_1 = (l + 2)N/n_0 = (l + 2)[C_{19} \log |b|] \leq A \log |b|$. For $m \geq A \log |b|$, we can write $m = dN_1 + r$ with $r \in \mathbb{N}$ and

$r < N_1$. Therefore by (7.41) and Lemma 7.7,

$$\|\tilde{L}_s^{mn_0}v\|_b = \|\tilde{L}_s^{dN_1n_0}(\tilde{L}_s^{rn_0}v)\|_b \leq \beta_3^{dN_1}\|\tilde{L}_s^{rn_0}v\|_b$$
$$\leq 2C_{16}\beta_3^{dN_1}\|v\|_b \leq 2C_{16}(\sqrt{\beta_3})^m\|v\|_b.$$

$\square$

Proof of Proposition 7.3 It is sufficient to prove that for all $m \in \mathbb{N}$,

$$\int |\tilde{L}_s^{mn_0}v|^2 d\nu \leq C\beta^m\|v\|_b^2. \tag{7.42}$$

Then for any $k \in \mathbb{N}$, suppose $k = mn_0 + r$ with $0 \leq r < n_0$. We have

$$\int |L_s^k v|^2 d\mu_E \leq C\lambda_\sigma^{2k} \int |\tilde{L}_s^k(f_\sigma^{-1}v)|^2 d\nu \leq C\lambda_\sigma^{2k}\beta^m\|\tilde{L}_s^r(f_\sigma^{-1}v)\|_b^2$$
$$\leq C\lambda_\sigma^{2k}\beta^m\|f_\sigma^{-1}v\|_b^2 \leq C\lambda_\sigma^{2k}\beta^m\|v\|_b^2.$$

By choosing ϵ small such that $\lambda_\sigma^{2n_0}\beta < 1$ for any $|\sigma| < \epsilon$, we obtain Proposition 7.3.

It remains to prove (7.42). For $m > A \log |b|$, by Lemma 7.39, we obtain

$$\int |\tilde{L}_s^{mn_0}v|^2 d\nu \leq \|\tilde{L}_s^{mn_0}v\|_b^2 \leq C\beta^m\|v\|_b^2.$$

For $A \log |b| \geq m \geq C_{19} \log |b|$, by (7.33) and Lemma 7.7, we know

$$\int |\tilde{L}_s^{mn_0}v|^2 d\nu \leq \beta^{[C_{19}\log|b|]}\|\tilde{L}_s^{(m-[C_{19}\log|b|])n_0}v\|_b^2$$
$$\leq 2C_{16}\beta^{[C_{19}\log|b|]}\|v\|_b^2 \leq 2C_{16}\beta_1^m\|v\|_b^2$$

for some $\beta_1 = \beta^{C_{19}/A} < 1$.

The case when $m \leq C_{19} \log |b|$ has been verified in Lemma 7.32. $\square$

8 Exponential mixing

In this section, we prove Theorem 4.13. As a first step, an analogous result concerning expanding semiflow will be proved. Let $T : \Lambda_+ \to \Lambda_+$ be the uniformly expanding map and $R : \Lambda_+ \to \mathbb{R}_+$ be the roof function as defined in Proposition 4.1. Set $\Lambda_+^R = \{(x, t) \in \Lambda_+ \times \mathbb{R} : 0 \leq t < R(x)\}$. We define a semi-flow $T_t : \Lambda_+^R \to \Lambda_+^R$ by $T_s(x, t) = (T^n x, t + s - R_n(x))$ where n is the unique integer satisfying $R_n(x) \leq t + s < R_{n+1}(x)$. Recall that ν is

the unique T-invariant ergodic probability measure on Λ_+. Then the flow T_t preserves the probability measure $\nu^R = \nu \times \text{Leb} / (\nu \times \text{Leb})(\Lambda_+^R)$. We will also use the probability measure $\mu_E^R = \mu_E \times \text{Leb} / (\mu_E \times \text{Leb})(\Lambda_+^R)$ on Λ_+^R. We show that T_t is exponentially mixing.

For a bounded function on Λ_+^R, we define two norms. Set

$$\|U\|_{\mathcal{B}_0} = |U|_\infty + \sup_{(x,a)\neq(x',a')\in\Lambda_+^R} \frac{|U(x,a) - U(x',a')|}{d(x,x') + |a - a'|},$$

$$\|V\|_{\mathcal{B}_1} = |V|_\infty + \sup_{x\in\Lambda_+} \frac{\text{Var}_{(0,R(x))}\{t \mapsto V(x,t)\}}{R(x)},$$

where $\text{Var}_{(0,R(x))}\{t \mapsto V(x,t)\}$ is the total variation of the function $t \mapsto V(x,t)$ on the interval $(0, R(x))$.

Theorem 8.1 *There exist $C > 0$, $\epsilon > 0$ such that for all $t > 0$ and for any two functions U, V on Λ_+^R with $\|U\|_{\mathcal{B}_0}$, $\|V\|_{\mathcal{B}_1}$ finite, we have*

$$\left| \int U \cdot V \circ T_t \mathrm{d}\mu_E^R - \left(\int U \mathrm{d}\mu_E^R \right) \left(\int V \mathrm{d}\nu^R \right) \right| \leq Ce^{-\epsilon t} \|U\|_{\mathcal{B}_0} \|V\|_{\mathcal{B}_1}.$$

Remark 8.2 Applying this theorem to the function $U(x,t) \frac{\mathrm{d}\nu^R}{\mathrm{d}\mu_E^R}(x)$, we obtain

$$\left| \int U \cdot V \circ T_t \mathrm{d}\nu^R - \left(\int U \mathrm{d}\nu^R \right) \left(\int V \mathrm{d}\nu^R \right) \right| \leq Ce^{-\epsilon t} \|U\|_{\mathcal{B}_0} \|V\|_{\mathcal{B}_1}.$$

(8.3)

With Proposition 7.3 available, Theorem 8.1 can be proved essentially along the same lines as the proof of [2, Theorem 7.3] (see also [2, Section 7.5]). We provide a sketch of the proof here. For a pair of functions U, V, let $\rho(t) = \int U \cdot V \circ T_t \mathrm{d}\mu_E^R$ be the correlation function and the observation is that the Laplace transform of ρ, denoted by $\hat{\rho}$, can be expressed as a sum of twisted transfer operators L_s [2, Lemma 7.17]. One shows that $\hat{\rho}$ admits an analytic continuation to a neighborhood of each point $s = ib$ and this part of the argument uses the quasi-compactness of the twisted transfer operators [2, Lemma 7.21, 7.22]. When $|b|$ is large, the Dolgopyat-type estimate (Proposition 7.3), which is a replacement of [2, Proposition 7.7] in the current setting, is used to imply that $\hat{\rho}$ admits an analytic extension to a strip $\{s = \sigma + ib \in \mathbb{C} : |\sigma| < \sigma_0\}$ for all sufficiently small σ_0 [2, Corollary 7.20]. The result of exponential mixing then follows from the classical Paley-Wiener theorem [2, Theorem 7.23].

The difference between our result and that in [2] is the classes of functions in concern. The only adjustment we need to make is [2, Lemma 7.18], which is

a norm estimate for C^1 functions in their paper, but for functions with finite $\mathcal{B}_0$ norm in the current setting. The precise statement is as follows. For a function $U : \Lambda_+ \to \mathbb{R}$ with $\|U\|_{\mathcal{B}_0} < \infty$ and $s \in \mathbb{C}$, set $\hat{U}_s(x) = \int_0^{R(x)} e^{-ts} U(x,t) \mathrm{d}t$.

Lemma 8.4 *There exists $C > 0$ such that for $s = \sigma + ib$ with $|\sigma| \leq \epsilon_o/4$ (ϵ_o is given as in Proposition 4.1 (5)), the function $L_s \hat{U}_{-s}$ is Lipschitz on Δ_0 and*

$$\|L_s \hat{U}_{-s}\|_b \leq \frac{C\|U\|_{\mathcal{B}_0}}{\max\{1, |b|\}}.$$

Proof We first prove for $x \in \Lambda_+$ we have

$$|\hat{U}_{-s}(x)| \leq \frac{Ce^{\epsilon_o R(x)/2}}{\max\{1, |b|\}} \|U\|_{\mathcal{B}_0}.$$

By definition, we have

$$\hat{U}_{-s}(x) = \int_0^{R(x)} U(x,t) e^{ts} \mathrm{d}t.$$

The case when $|b| \leq 1$ is easy. When $|b| > 1$, one uses integration by parts and the fact that U is Lipschitz with respect to t to obtain

$$|\hat{U}_{-s}(x)| \leq (2|U|_\infty e^{\epsilon_o R(x)/4} + |U|_{\mathrm{Lip}} R(x) e^{\epsilon_o R(x)/4})/\max\{1, |b|\}.$$

Then

$$|L_s \hat{U}_{-s}| \leq \frac{C\|U\|_{\mathcal{B}_0}}{\max\{1, |b|\}} L_\sigma(e^{\epsilon_o R/2}).$$

Observe that by (4.2)

$$L_\sigma(e^{\epsilon_o R/2}) = \sum_{\gamma \in \mathcal{H}} |\gamma'(x)|^{\delta+\sigma} e^{\epsilon_o R(\gamma x)/2} \leq \sum_{\gamma \in \mathcal{H}} |\gamma'(x)|^{\delta - 3\epsilon_o/4} < \infty. \tag{8.5}$$

So $|L_s \hat{U}_{-s}| \leq \frac{C\|U\|_{\mathcal{B}_0}}{\max\{1, |b|\}}$.

We estimate the Lipschitz norm of $L_s \hat{U}_{-s}$. We have

$$L_s \hat{U}_{-s}(x) - L_s \hat{U}_{-s}(y) = \sum_{\gamma \in \mathcal{H}} |\gamma'(x)|^{\delta+s} (\hat{U}_{-s}(\gamma x) - \hat{U}_{-s}(\gamma y))$$

$$+ (|\gamma'(x)|^{\delta+s} - |\gamma'(y)|^{\delta+s}) \hat{U}_{-s}(\gamma y).$$

The term $|\gamma'(x)|^{\delta+s} - |\gamma'(y)|^{\delta+s}$ can be estimated using Proposition 4.1 (4). For the term $\hat{U}_{-s}(\gamma x) - \hat{U}_{-s}(\gamma y)$, suppose that $R(\gamma x) \geq R(\gamma y)$, we use Proposition 4.1 (4) again and get

$$
\begin{aligned}
|\hat{U}_{-s}(\gamma x) - \hat{U}_{-s}(\gamma y)| &\leq |R(\gamma x) - R(\gamma y)||U|_\infty e^{\sigma R(\gamma x)} \\
&\quad + \int_0^{R(\gamma y)} |U(\gamma x, t) - U(\gamma y, t)|e^{t\sigma}\,\mathrm{d}t \\
&\leq (C_1 e^{\sigma R(\gamma x)} + R(\gamma y)e^{\sigma R(\gamma y)})\|U\|_{\mathcal{B}_0}d(x, y).
\end{aligned}
$$

Then we use (8.5) to conclude that there exists some C (independent of U) such that

$$
|L_s\hat{U}_{-s}|_{\mathrm{Lip}} \leq C\|U\|_{\mathcal{B}_0}.
$$

$\square$

Proof of Theorem 4.13 Now Theorem 4.13 can be proved using the same lines as the proof of [2, Theorem 2.7] (see also [2, Section 8.2]). In particular, in the proof of [2, Lemma 8.3], we use (8.3) to replace [2, Theorem 7.3] and Proposition 4.11 (2) to relate the measures $\hat{v}^R$ and v^R. $\square$

9 Resonance-free region

Recall that Γ is a geometrically finite discrete subgroup in $G = \mathrm{SO}(d+1, 1)^\circ$. We begin by defining the measures $m^{\mathrm{BR}}, m^{\mathrm{BR}_*}$ and m^{Haar}. Recall the definition of the BMS measure on $\mathrm{T}^1(\mathbb{H}^{d+1}) \cong \partial^2(\mathbb{H}^{d+1}) \times \mathbb{R}$:

$$
\mathrm{d}\tilde{m}^{\mathrm{BMS}}(x, x_-, s) = e^{\delta\beta_x(o, x_*)}e^{\delta\beta_{x_-}(o, x_*)}\mathrm{d}\mu(x)\mathrm{d}\mu(x_-)\mathrm{d}s,
$$

where x_* is the based point of the unit tangent vector given by (x, x_-, s). We define the measures $\tilde{m}^{\mathrm{BR}}, \tilde{m}^{\mathrm{BR}_*}$ and $\tilde{m}^{\mathrm{Haar}}$ on $\mathrm{T}^1(\mathbb{H}^{d+1}) \cong \partial^2(\mathbb{H}^{d+1}) \times \mathbb{R}$ similarly as follows:

$$
\begin{aligned}
\mathrm{d}\tilde{m}^{\mathrm{BR}}(x, x_-, s) &= e^{d\beta_x(o, x_*)}e^{\delta\beta_{x_-}(o, x_*)}\mathrm{d}m_o(x)\mathrm{d}\mu(x_-)\mathrm{d}s; \\
\mathrm{d}\tilde{m}^{\mathrm{BR}_*}(x, x_-, s) &= e^{\delta\beta_x(o, x_*)}e^{\delta\beta_{x_-}(o, x_*)}\mathrm{d}\mu(x)\mathrm{d}m_o(x_-)\mathrm{d}s; \\
\mathrm{d}\tilde{m}^{\mathrm{Haar}}(x, x_-, s) &= e^{d\beta_x(o, x_*)}e^{d\beta_{x_-}(o, x_*)}\mathrm{d}m_o(x)\mathrm{d}m_o(x_-)\mathrm{d}s,
\end{aligned}
$$

where m_o is the unique probability measure on $\partial(\mathbb{H}^{d+1})$ which is invariant under the stabilizer of o in G.

These measures are all left Γ-invariant and induce measures on $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^{d+1})$, which we will denote by $m^{\mathrm{BR}}, m^{\mathrm{BR}_*}$ and m^{Haar} respectively. Here we do not

normalize the BMS measure to a probability measure, which is different from the previous part.

By [36, Theorem 5.8], Theorem 1.1 implies exponential decay of matrix coefficients.

Theorem 9.1 *There exists $\eta > 0$ such that for any compactly supported functions $\phi, \psi \in C^1(T^1(M))$, we have*

$$e^{(d-\delta)t} \int_{T^1(M)} \phi \cdot \psi \circ \mathcal{G}_t \, dm^{\mathrm{Haar}} = \frac{m^{\mathrm{BR}_*}(\phi) m^{\mathrm{BR}}(\psi)}{m^{\mathrm{BMS}}(T^1(M))} + O(\|\phi\|_{C^1} \|\psi\|_{C^1} e^{-\eta t})$$

for all $t > 0$, where O depends on the supports of ϕ, ψ.

For $x, y \in \mathbb{H}^{d+1}$ and $T > 0$, let

$$N(T, x, y) = \#\{\gamma \in \Gamma \mid d(x, \gamma y) \leq T\},$$

where d is the hyperbolic distance on $\mathbb{H}^{d+1}$. In [33], it was shown that Theorem 9.1 implies the following:

Corollary 9.2 *There exists $\eta > 0$ such that for any $x, y \in \mathbb{H}^{d+1}$ and $T > 0$, we have*

$$N(T, x, y) = c_{x,y} e^{\delta T} + O(e^{(\delta - \eta)T}),$$

where $c_{x,y} > 0$ is a constant depending on x, y.

Proof of Corollary 1.2 For $x, y \in \mathbb{H}^{d+1}$ and $s \in \mathbb{C}$ with $\Re s > \delta$, let $P_s(x, y)$ be the Poincaré series defined by

$$P_s(x, y) = \sum_{\gamma \in \Gamma} e^{-sd(x, \gamma y)}.$$

We first prove that $P_s(x, y)$ is meromorphic on $\Re s > \delta - \eta$ with a unique pole $s = \delta$. By Fubini's theorem

$$P_s(x, y) = \int_0^\infty \frac{1}{s} e^{-sT} N(T, x, y) dT = \int_0^\infty \frac{1}{s} e^{-(s-\delta)T} c_{x,y} dT$$

$$+ \int_0^\infty \frac{1}{s} e^{-sT} (N(T, x, y) - c_{x,y} e^{\delta T}) dT.$$

The first part is a meromorphic function of s having a unique pole at $s = \delta$. The second part, it follows from Corollary 9.2 that it is absolutely convergence if $\Re s > \delta - \eta$, hence it is analytic on $\Re s > \delta - \eta$. Then we use [21, Theorem 7.3] to deduce that the resolvent $R_M(s)$ is also analytic on $\{s \in \mathbb{C} : \delta - \eta < \Re s < \delta\}$. □

10   Appendix: Proof of Lemma 6.9

Proof of Lemma 6.9 We divide into the cases when r lies in different intervals. Let $\beta = C\sqrt{\eta}$.

- Case A: $r \leq \eta h(p)$ By Lemma 5.7, we have $|(\gamma^{-1})'x| = h(p)/d(x, p)^2$. Using Lemma 6.7, we have

$$N_r(\partial J_p) \subset B(p, 2\eta h(p)) - B(p, c_4 \eta h(p)). \tag{10.1}$$

Hence for $x \in N_r(\partial J_p)$

$$|(\gamma^{-1})'x| \in [1/(4\eta^2 h(p)), 1/(c_4^2 \eta^2 h(p))].$$

Then

$$N_{r/(4\eta^2 h(p))}(\partial \gamma^{-1} J_p) \subset \gamma^{-1} N_r(\partial J_p) \subset N_{r/(c_4^2 \eta^2 h(p))}(\partial \gamma^{-1} J_p). \tag{10.2}$$

Notice that $\partial \gamma^{-1} J_p = \partial(B_Y(2/\eta) \times R_{p,\eta})$ and $R_{p,\eta}$ is a parallelotope tiled by the translations of $\overline{\Delta}_0'$.

- Case A_1: $r \leq \eta^2 h(p)$
  Recall that Lemma 5.29 is proved using Lemma 5.31. Using the same argument, we obtain an analog of Lemma 5.29 for $N_r(\partial \gamma^{-1} J_p)$. Using this version of Lemma 5.29 with $\epsilon = 4\beta/c_4^2$, the inequality $r/(4\eta^2 h(p)) < 1$ and (10.2), we have

$$\mu(\gamma^{-1} N_{\beta r}(\partial J_p)) \leq \mu(N_{\beta r/(c_4^2 \eta^2 h(p))}(\partial \gamma^{-1} J_p))$$

$$\leq \lambda \mu(N_{r/(4\eta^2 h(p))}(\partial \gamma^{-1} J_p)) \leq \lambda \mu(\gamma^{-1} N_r(\partial J_p)).$$

Using Lemma 5.17, we obtain

$$\frac{\mu(N_{\beta r}(\partial J_p))}{\mu(N_r(\partial J_p))} \leq C \frac{\mu(\gamma^{-1} N_{\beta r}(\partial J_p))}{\mu(\gamma^{-1} N_r(\partial J_p))} \leq C\lambda,$$

where λ tends to zero as η tends to zero.

– Case A_2: $\eta^{3/2} h(p) < r \leq \eta h(p)$
  We compute the measure by counting the number of translations of Δ_0.
  By (10.1) and Lemma 5.7, we obtain $\gamma^{-1} N_r(\partial J_p) \subset B(p', 1/(c_4\eta)) - B(p', 1/(2\eta))$. Let $\gamma' \Delta_0$ be any fundamental domain contained in $\gamma^{-1} N_r(\partial J_p)$ with $\gamma' \in \Gamma_\infty$. Using Lemma 5.18, we obtain that $\mu(\gamma' \Delta_0) \approx \eta^{2\delta} \mu(\Delta_0)$. By Lemma 5.17, (10.2)

$$\mu(N_{\beta r}(\partial J_p)) \ll h(p)^\delta \mu(N_{\beta r/(c_4^2 \eta^2 h(p))}(\partial \gamma^{-1} J_p)).$$

By counting the number of fundamental domains, we obtain that the region $N_{\beta r/(c_4^2 \eta^2 h(p))}(\partial \gamma^{-1} J_p)$ can be covered by $(1/\eta)^{k-1} \cdot (\beta r/(c_4^2 \eta^2 h(p)))$ disjoint rectangles $\gamma' \Delta_0$ with $\gamma' \in \Gamma_\infty$. So we have

$$\mu(N_{\beta r}(\partial J_p)) \ll h(p)^\delta \cdot (1/\eta)^{k-1} \cdot (\beta r/(c_4^2 \eta^2 h(p))) \cdot \eta^{2\delta} \mu(\Delta_0).$$

By Lemma 5.17, (10.2)

$$\mu(N_r(\partial J_p)) \gg h(p)^\delta \mu(N_{r/(4\eta^2 h(p))}(\partial \gamma^{-1} J_p)).$$

Meanwhile, as $r/(4\eta^2 h(p)) \geq 1/4\eta^{1/2}$, the number of rectangles $\gamma' \Delta_0$ inside $N_{r/(4\eta^2 h(p))}(\partial \gamma^{-1} J_p)$ is greater than $(1/\eta)^{k-1} \cdot (r/(4\eta^2 h(p)))$. Hence we have

$$\mu(N_r(\partial J_p)) \gg h(p)^\delta \cdot (1/\eta)^{k-1} \cdot (r/(4\eta^2 h(p))) \cdot \eta^{2\delta} \mu(\Delta_0).$$

Therefore,

$$\mu(N_{\beta r}(\partial J_p)) \ll \beta \mu(N_r(\partial J_p)).$$

• Case B: $\eta^{1/2} h(p) \leq r \leq h(p)$
  We handle this case using (5.16). By Lemma 6.7 and the inequality $h(p)/2 > \beta r \geq \eta h(p)$, we have

$$\mu(N_{\beta r}(\partial J_p)) \leq \mu(J_p \cup N_{\beta r}(\partial J_p)) \leq \mu(B(p, \eta h(p) + \beta r))$$

$$\leq \mu(B(p, 2\beta r)) \ll (2\beta r)^{2\delta - k} h(p)^{k-\delta}.$$

Meanwhile, we have

$$\mu(J_p \cup N_r(\partial J_p)) \geq \mu(B(p, r)) \gg r^{2\delta - k} h(p)^{k-\delta}.$$

Hence

$$\frac{\mu(N_r(\partial J_p) - N_{\beta r}(\partial J_p))}{\mu(N_{\beta r}(\partial J_p))} = \frac{\mu(J_p \cup N_r(\partial J_p)) - \mu(J_p \cup N_{\beta r}(\partial J_p))}{\mu(N_{\beta r}(\partial J_p))}$$
$$\gg \frac{r^{2\delta - k} h(p)^{k-\delta} - (2\beta r)^{2\delta - k} h(p)^{k-\delta}}{(2\beta r)^{2\delta - k} h(p)^{k-\delta}}.$$

Therefore

$$\mu(N_{\beta r}(\partial J_p)) \ll \beta^{2\delta - k} \mu(N_r(\partial J_p)).$$

Now to prove (6.10) we consider $\eta^{1/2} r$ and r. Then one of them belongs to $(0, \eta^2 h(p)) \cup [\eta^{3/2} h(p), \eta h(p)] \cup [\eta^{1/2} h(p), h(p)]$. Inequality (6.10) follows from the observation that

$$\frac{\mu(N_{C\eta r}(\partial J_p))}{\mu(N_r(\partial J_p))} \leq \min \left\{ \frac{\mu(N_{C\eta r}(\partial J_p))}{\mu(N_{\beta r}(\partial J_p))}, \frac{\mu(N_{\beta r}(\partial J_p))}{\mu(N_r(\partial J_p))} \right\}.$$

$\square$

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