Bourgeois contact structures: Tightness, fillability and applications

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Abstract Given a contact structure on a manifold V together with a supporting open book decomposition, Bourgeois gave an explicit construction of a contact structure on $V \times \mathbb{T}^2$. We prove that all such structures are universally tight in dimension 5, independent of whether the original contact manifold is itself tight or overtwisted. In arbitrary dimensions, we provide obstructions to the existence of strong symplectic fillings of Bourgeois manifolds. This gives a broad class of new examples of weakly but not strongly fillable contact 5-manifolds, as well as the first examples of weakly but not strongly fillable contact structures in all odd dimensions. These obstructions are particular instances of more general obstructions for $\mathbb{S}^1$-invariant contact manifolds. We also obtain a classification result in arbitrary dimensions, namely that the unit cotangent bundle of the n-torus has a unique symplectically aspherical strong filling up to diffeomorphism.

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1 Introduction

In [8], Bourgeois showed that, whenever (V, ξ) is a contact manifold endowed with a supporting open book decomposition (which always exist by work of Giroux [24]), then the manifold $V \times \mathbb{T}^2$ carries a natural contact structure. The main motivation behind such a construction was the problem of the existence of contact structures on higher-dimensional manifolds. For instance, it showed that every odd dimensional torus admits contact structures, a problem that had been open since Lutz [40] proved that $\mathbb{T}^5$ is contact, more than 20 years before.

It was not until recently that Borman-Eliashberg-Murphy [6] proved that contact structures in higher-dimensions actually exist in abundance (i.e. whenever the obvious topological obstructions vanish) by generalizing Eliashberg's [16] notion of *overtwistedness*, as well as the *h*-principle that comes with it, to higher dimensions. Overtwisted contact manifolds are topological/flexible in nature, and most of the associated contact-topological invariants (e.g. those coming from holomorphic curves) simply vanish. As a result, it has become relevant to find examples of high-dimensional contact structures beyond the overtwisted ones, which are more geometric/rigid and which potentially have rich associated invariants. Contact structures which are not overtwisted are usually referred to as *tight*.

The construction in [8] actually fits very well in this setting, as it is both very explicit and yields a very broad class of contact manifolds in arbitrary odd dimensions, with remarkable properties. For instance, [51] used it to construct the first examples of high dimensional (closed) contact manifolds admitting a

Plastikstufe, as defined in [47], which is subsequently equivalent to overtwistedness [13,33]. In [10], the authors also used it to construct contact structures on the product of a contact manifold with the 2-sphere. More recently, Lisi, Marinković and Niederkrüger [38] started the systematic study of the Bourgeois construction, in particular studying its fillability properties, and in this paper we continue this line of research.

In what follows we will use the following notation. Given an abstract open book (Σ^{2n}, ϕ) and the associated contact $(2n + 1)$-manifold $OBD(\Sigma, \phi)$, we denote by $BO(\Sigma, \phi)$ the contact manifold obtained via the Bourgeois construction [8]. Smoothly, $BO(\Sigma, \phi) = OBD(\Sigma, \phi) \times \mathbb{T}^2$ and we refer to Section 2 for further details.

Tightness in dimension 5 We begin by addressing the natural question of whether a given Bourgeois contact structure is tight or overtwisted.

In [38], the authors give examples, in every odd dimension, of an overtwisted (V, ξ) such that the associated Bourgeois contact manifold is tight. Moreover, in [22], it was shown that if V is a 3-manifold with non-zero first Betti number, then there exists a supporting open book such that the associated Bourgeois contact structure is (hyper)tight. In this paper, we prove that, at least in dimension 5, these are particular instances of a more general fact. Namely, 5-dimensional Bourgeois contact structures are rigid, inherently geometric objects, independently of the rigid or flexible nature of (V, ξ):

Theorem A (Tightness) *For every abstract open book (Σ^2, ϕ), the contact 5-manifold $BO(\Sigma, \phi)$ is (universally) tight.*

Recall that universally tight means that the universal cover is tight. In particular, universal tightness implies tightness. The fact that 5-dimensional Bourgeois contact structures are universally tight is a simple consequence of the fact that they are tight and that finite covers on either factor of the product again yield Bourgeois contact structures.

While there are many ways of making contact/symplectic manifolds more flexible (e.g. in dimension 3, by adding a Lutz twist; in any odd dimension, by taking the connected sum with an overtwisted sphere; or in dimension at least 6, by taking the flexibilization of a Weinstein manifold), Theorem A says that the Bourgeois construction can be interpreted as a "tightening" procedure. To our knowledge, there is currently no other procedure with an analogous property.

Moreover, the above result is sharp with respect to taking branched covers. Namely, if V is an overtwisted contact 3-manifold (or, more generally, $(2n+1)$-manifold), then branched covers $V \times \Sigma_g$ of $V \times \mathbb{T}^2$ are overtwisted; here, Σ_g is the orientable surface of genus $g \geq 1$, seen as a degree g branched cover of $\mathbb{T}^2$ over two points, and the branched cover is obtained by product with the identity on V. The fact that this is true for g big enough follows from the

results in [49]; that this holds already from $g = 2$ is moreover a consequence of an argument of Massot and Niederkrüger, based on ideas from [51] (cf. [48, Theorem I.5.1]; the interested reader can also consult [22, Observation 5.9] for details).

Symplectic fillability Another important problem in contact topology, that is related to the flexible/rigid classification of contact structures, is that of characterizing which contact structures admit symplectic fillings. Indeed, one of the first results of the theory of holomorphic curves is that symplectically fillable contact manifolds are tight.

Given Theorem A, it is natural to wonder whether Bourgeois contact structures are (at least weakly) symplectically fillable. While there exist partial results in this direction [38,42], a complete answer is yet to be found.

In order to find obstructions to strong fillings, we shall prove a more general statement concerning $\mathbb{S}^1$-invariant contact structures on manifolds of the form $V^{2n} \times \mathbb{S}^1$. Such a contact structure is determined by a splitting of the base V into *ideal* Liouville domains $V = V_+ \cup \overline{V}_-$ glued along a contact manifold $N := \partial V_+ = \partial V_-$ (see [25,42] or Section 6 below). The subsets $V_\pm$ correspond to the positive resp. negative regions when V is viewed as a convex hypersurface, and N to its dividing set. With this notation we then have the following homological and homotopical criteria in the case that the contact structure is strongly fillable.

Theorem B (Fillability of $\mathbb{S}^1$-invariant contact structures) *Suppose that $V^{2n} \times \mathbb{S}^1$ admits a strongly fillable $\mathbb{S}^1$-invariant contact structure with induced convex splitting $V = V_+ \cup \overline{V}_-$ and let $N := \partial V_+ = \partial V_-$ be the dividing set, which we assume to be connected. Let also (W^{2n+2}, ω) be a strong filling of $V \times \mathbb{S}^1$ and consider the natural inclusions and induced maps on homology:*

$$N \hookrightarrow V_\pm \hookrightarrow W \text{ and } H_*(N, \mathbb{Q}) \xrightarrow{I_\pm} H_*(V_\pm, \mathbb{Q}) \longrightarrow H_*(W, \mathbb{Q}).$$

Then the second inclusion induces an injection on the image $Im(I_\pm)$ in rational homology. In particular, if $V_\pm$ are Weinstein, then the inclusion $N \hookrightarrow W$ induces an injection in rational homology in all degrees strictly less than n.

In the case where W is semi-positive, we also have a surjection

$$\pi_1(V \times \mathbb{S}^1) \twoheadrightarrow \pi_1(W)$$

of fundamental groups.

Since Bourgeois contact structures are $\mathbb{T}^2$-invariant, they are in particular $\mathbb{S}^1$-invariant and one obtains a convex splitting with pieces of the form $V_\pm = \Sigma \times D^*\mathbb{S}^1$. Then applying Theorem B, we obtain the following result, which imposes strong topological restrictions on the filling as well as the original Bourgeois contact manifold:

Theorem C (Fillability of Bourgeois contact structures) *Suppose that a $(2n+1)$-dimensional Bourgeois contact manifold $BO(\Sigma, \phi)$ is strongly symplectically filled by (W, ω). Then the natural inclusion of a page into the filling, given by the composition of inclusions*

$$\Sigma \hookrightarrow OBD(\Sigma, \phi) \times \{pt\} \hookrightarrow BO(\Sigma, \phi) = \partial W \hookrightarrow W, \qquad (1)$$

induces an injection in rational homology.

Moreover, the composition of the natural inclusions $\mathbb{T}^2 \hookrightarrow BO(\Sigma, \phi) \hookrightarrow W$ also induces an injection in rational homology. Lastly, if W is semi-positive, then we have a surjection $\pi_1(\Sigma \times \mathbb{T}^2) \twoheadrightarrow \pi_1(W)$.

There are many topological situations where the above criteria can be immediately applied to obstruct strong fillability of certain Bourgeois contact structures, several of which are described below. We also have:

Remark 1.1 (Monodromy restrictions). The fact that $\Sigma \hookrightarrow W$ induces an injection in rational homology implies that the same is true for the natural inclusion $\Sigma \hookrightarrow OBD(\Sigma, \phi)$, as $\Sigma \hookrightarrow W$ can be written as the composition in Eq. (1). What is more, this also implies that $\phi_* = \text{Id}$ on $H_*(\Sigma; \mathbb{Q})$, i.e. ϕ lies in the Torelli group: indeed, as $\Sigma \hookrightarrow OBD(\Sigma, \phi)$ is the composition $\Sigma \hookrightarrow \Sigma_\phi \hookrightarrow OBD(\Sigma, \phi)$ where Σ_ϕ is the mapping torus part of the open book, one can appeal to the long exact sequence of a mapping torus [26, Example 2.48] to deduce that $\Sigma \hookrightarrow \Sigma_\phi$ is injective in rational homology if and only if $\phi_* = \text{Id}$ on $H_*(\Sigma; \mathbb{Q})$.

Remark 1.2 To deal with strong fillings in general, we need to use the polyfold machinery of Hofer–Wysocki–Zehnder [31] (for the case of *closed* spheres). However, this is not necessary if further technical assumptions on the filling are imposed, e.g. asphericity or semi-positivity.

Homology of aspherical fillings In the case where a Bourgeois manifold is assumed to admit a *symplectically aspherical* filling, we obtain the following significant strengthening of Theorem C, which determines the homology of any such filling:

Theorem D (Aspherical case) *Suppose that a $(2n + 1)$-dimensional Bourgeois contact manifold $BO(\Sigma, \phi)$ admits a symplectically aspherical filling (W, ω). Then the natural inclusion $\Sigma \times \mathbb{T}^2 \hookrightarrow W$ into the filling induces an isomorphism in integral homology.*

If ϕ is symplectically isotopic to Id, $OBD(\Sigma, \phi)$ admits a subcritical Stein filling. In this case the associated Bourgeois contact manifold admits a Stein filling according to [38, Theorem A.b]. Moreover, this filling is smoothly of the form $X \times \mathbb{T}^2$, where $X = \Sigma \times \mathbb{D}^2$ is the subcritical filling of the original

manifold. Theorem C, as well as Remark 1.1 and Theorem D, then suggest that perhaps it is always the case that strong fillability for Bourgeois manifolds implies $\phi = \mathrm{Id}$ (at least smoothly), and that the filling is (again, at least smoothly) the standard one, in arbitrary dimensions. The fact that the filling "remembers" both the $\mathbb{T}^2$-factor and the page Σ (at least homologically), as well as the fact that, in the aspherical case, the homology of the filling is the expected one are evidence in this direction. In particular, strong fillability for Bourgeois manifolds might be equivalent to Stein fillability; this is indeed true in several cases, as discussed below.

Fillability in dimension 5 Examples of weakly but not strongly fillable contact structures in dimension 3 are well-known. The first examples of such contact manifolds in higher dimensions were obtained in dimension 5 on manifolds also diffeomorphic to a product of a 3-manifold with a 2-torus cf. [42, Theorem E]. These examples are associated to contact 3-manifolds arising from so-called *Liouville pairs* and and are known to exist only on very specific 3-manifolds (see [42] and references therein). Below, we provide a large class of new 5-dimensional examples that arise via Bourgeois contact structures.

Indeed, as a first immediate corollary of Theorem C in dimension 5, we obtain:

Corollary E (Rational homology 3-spheres) *Suppose that $V = OBD(\Sigma, \phi)$ is a 3-dimensional rational homology sphere, i.e. $H_1(V; \mathbb{Q}) = 0$. Then the Bourgeois contact manifold $BO(\Sigma, \phi)$ is strongly symplectically fillable if and only if the page $\Sigma = \mathbb{D}^2$ is a disc (and $\phi = \mathrm{Id}$), in which case it is actually Stein fillable and V is $\mathbb{S}^3$.*

This then yields many examples of weakly but not strongly fillable contact manifolds in dimension 5. For example Legendrian surgery on any smoothly non-trivial Legendrian knot in $\mathbb{S}^3$ gives a Stein fillable contact structure on a rational homology sphere, smoothly different from the standard 3-sphere. The corresponding Bourgeois contact structure is weakly fillable by [38, Theorem A.a] (cf. [42, Example 1.1]).

We now consider the *planar* case, i.e. the case where the page of the open book of the original 3-manifold has genus zero. In the 3-dimensional situation, strong symplectic fillings of contact structures supported by planar open books exist in abundance, and in fact are in 1–1 correspondence with the factorizations of the monodromy into products of positive Dehn-twists [57]. Namely, contact 3-manifolds admitting a supporting open book with planar page and monodromy a product of positive Dehn twists are (precisely, by [57]) the convex boundaries of symplectic Lefschetz fibrations. In particular, in the planar case, strong fillability and Stein fillability are equivalent. However, if we apply the Bourgeois construction to a planar contact 3-manifold, and consider fillings of the resulting contact 5-manifold, the situation is surprisingly more rigid:

Theorem F (Planar case) *Let (Σ^2, ϕ) be an abstract open book with Σ of genus zero and ϕ a non-trivial product of Dehn twists, all of the same sign. Then, the contact 5-manifold $BO(\Sigma, \phi)$ is weakly but not strongly fillable.*

The fact that the examples in Theorem F are weakly fillable follows again from [38, Theorem A.a] or [42, Example 1.1]; and the fact that the conclusion also holds for products of all *negative* Dehn twists, from [38, Theorem B]. In the particular case of the annulus $\Sigma = D^*\mathbb{S}^1$, whose mapping class group is generated by the Dehn twist τ along the zero section, we get $BO(D^*\mathbb{S}^1, \tau^k)$ is strongly fillable if and only if $k = 0$; see Theorem K below for a higher-dimensional version.

Note that, in the planar case, the monodromy ϕ is necessarily the identity in homology of the page, since this is generated by the boundary loops, along which ϕ is trivial by assumption. In particular, the condition on ϕ given by Remark 1.1 is not restrictive. However, one can still prove the following, by applying Theorem C:

Corollary G *If Σ is planar and $BO(\Sigma, \phi)$ is strongly fillable, then ϕ lies in the commutator subgroup of the mapping class group (rel. boundary).*

This implies, for instance, that whenever Σ is a pair of pants (whose mapping class group is abelian) and $BO(\Sigma, \phi)$ is strongly fillable, then $\phi = \mathrm{Id}$. Note that whilst mapping class groups of higher genus surfaces are *perfect*, meaning that any element can be written as a product of commutators, this is not so in the planar case. In fact, any non-trivial product of positive Dehn twists will *not* lie in the commutator subgroup. This is because any product of positive Dehn twists gives a non-trivial positive braid after identifying all but one appropriately chosen boundary component to (marked) points, and positive braids survive in the abelianization of the mapping class group of the marked disk, which is infinite cyclic (cf. [18, p. 252]). In particular, one can then deduce Theorem F from Corollary G via this observation.

Fillability in higher dimensions As a further consequence of Theorem C we also obtain the following, which gives a plethora of weakly but not strongly fillable contact structures in arbitrary dimensions.

Corollary H (Stabilizations) *Let $OBD(\Sigma, \phi)$ be a $(2n - 1)$-dimensional contact manifold, and let $OBD(\Sigma_+, \phi_+)$ be obtained by a single positive stabilization. Then the Bourgeois manifold $BO(\Sigma_+, \phi_+)$ is not strongly symplectically fillable. In particular, if $OBD(\Sigma, \phi)$ is weakly fillable, then $BO(\Sigma_+, \phi_+)$ is weakly but not strongly fillable.*

The fact that such manifolds are not subcritically Stein fillable was already observed in [38, Corollary 1.4] and hence the above can be viewed as a significant strengthening of this. As a consequence we obtain the first known

examples of weakly but not strongly fillable contact structures in all dimensions.

See Theorems J and K below for more concrete examples.

Tight but non-fillable contact 5-folds Recall that an almost contact structure on V^{2n+1} is a hyperplane field ξ equipped with a complex structure $J: \xi \to \xi$. On a 3-manifold, this simply reduces to the data of a homotopy class of oriented 2-plane fields. Using Eliashberg's classification of overtwisted contact structures in dimension 3 [17], we can represent any almost contact structure on a 3-manifold by an overtwisted contact structure, which is supported by an open book by Giroux [24]. If we perform a single *positive* stabilization, the contact structure is unchanged up to isotopy, and hence the almost contact structure is also unchanged up to homotopy. Applying Corollary H to the stabilized open book, and combining with Theorem A, we then conclude:

Corollary I *For an almost contact structure (M^3, η) on a closed 3-manifold, there exists a universally tight but not strongly fillable contact structure on $M \times \mathbb{T}^2$ which is homotopic to the product almost contact structure $\eta \oplus T\mathbb{T}^2$.*

We remark that the contact structure in Corollary I can sometimes be weakly fillable (e.g. in the case where the monodromy of the planar open book is a product of all *negative* Dehn twists, for which we do not need to positively stabilize; cf. Theorem F).

Special higher-dimensional examples We also consider fillings for certain specific examples of higher dimensional Bourgeois contact manifolds.

The family $BO(D^*\mathbb{S}^n, \tau^k)$ We first consider the case of $BO(D^*\mathbb{S}^n, \tau)$, where τ is the Dehn–Seidel twist on $D^*\mathbb{S}^n$, and as a special case of Corollary H, or alternatively as a direct consequence of Theorem C, we give a negative answer to [38, Question 1.6]:

Theorem J *The Bourgeois contact manifold $BO(D^*\mathbb{S}^n, \tau)$ is weakly but not strongly symplectically fillable.*

More generally, from Theorem C one easily obtains that the Bourgeois contact manifold $BO(D^*\mathbb{S}^n, \tau^k)$ is not symplectically fillable for most values of k and n:

Theorem K *For $n \geq 1$, define the subset*

$$BOFill(n) = \left\{ k \in \mathbb{Z} : BO(D^*\mathbb{S}^n, \tau^k) \text{ is strongly fillable} \right\}.$$

Then $BOFill(n)$ is a subgroup of $\mathbb{Z}$. Denoting a generator by $k_0(n)$, we moreover have the following: if n is odd, then $k_0(n) = 0$, i.e. $BOFill(n)$ is the trivial group, and if n is even, then $k_0(n)$ is even.

Remark 1.3 The fact that $k_0(n)$ is even if n is, also is related to the fact that the Dehn-Seidel twist has finite order as a smooth map in this case; see [34] for the precise orders. We remark that it is conceivable that $BOFill(n)$ is always the trivial group, which would be a stronger result, although the group itself would no longer be interesting.

On the other hand, the contact manifolds $BO(D^*\mathbb{S}^n, \tau^k)$ admit weak fillings for *every* $k \in \mathbb{Z}$ [38, Theorem A.a]. We then obtain infinitely many weakly but not strongly fillable examples from this family alone; cf. Theorem J.

Unit cotangent bundle of $\mathbb{T}^n$ We determine the diffeomorphism type of the strong symplectically aspherical fillings of the unit cotangent bundle $S^*\mathbb{T}^n$ of $\mathbb{T}^n$ for $n \geq 2$ with its standard contact structure ξ_{std}, induced by the restriction of the standard Liouville form λ_{std} on the unit disc cotangent bundle $D^*\mathbb{T}^n$ to its boundary $S^*\mathbb{T}^n$.

In fact, $(S^*\mathbb{T}^n, \xi_{std})$ is none other than $BO(D^*\mathbb{T}^{n-2}, \mathrm{Id})$ To see this first observe that the contact manifold $OBD(D^*\mathbb{T}^{n-2}, \mathrm{Id})$ is the convex boundary of the subcritical Stein manifold

$$W = \underbrace{D^*\mathbb{S}^1 \times \cdots \times D^*\mathbb{S}^1}_{n-2 \text{ times}} \times \mathbb{D}^2 \, .$$

Then according to [38, Theorem A.b], $BO(T^*\mathbb{T}^{n-2}, \mathrm{Id})$ is then the convex boundary of the Stein manifold $\left(\prod_{i=1}^{n-2} D^*\mathbb{S}^1\right) \times D^*\mathbb{T}^2 = D^*\mathbb{T}^n$, with its split Stein structure, which is just the standard one.

We then prove the following uniqueness result, which is a strengthening of Theorem D for the case of $S^*\mathbb{T}^n$:

Theorem L *The contact manifold* $(S^*\mathbb{T}^n, \xi_{std})$, $n \geq 3$, *has a unique strong symplectically aspherical filling up to diffeomorphism.*

Theorem L is a smooth higher-dimensional version of a result by Wendl [57, Theorem 4] who classified symplectic fillings of $S^*\mathbb{T}^2$ up to symplectic deformation, which in turn generalized a weaker homeomorphism classification of Stipsicz [53, Theorem 1.6]. In dimension 5 (i.e. $n = 3$), Theorem F and Theorem L then give a complete *smooth* characterization of symplectically aspherical strong fillings for the Bourgeois contact 5-manifolds associated to any open book with page $D^*\mathbb{S}^1$; i.e. the only strongly fillable case is the trivial monodromy case, in which case the filling is smoothly unique. Note that, according to [38], all examples with page $D^*\mathbb{S}^1$ are weakly fillable.

Remark 1.4 Theorem L has also been independently obtained by Geiges–Kwon–Zehmisch [20]. While our original proof made use of punctured holomorphic curves, the current one uses now closed holomorphic spheres,
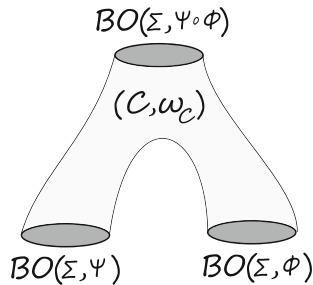
Fig. 1 The pseudo-Liouville cobordism (C, ω_C) given in Theorem 3.1

as kindly suggested to us by a referee in order to simplify the arguments. Hence it now follows a similar line of argument as the proof in [20].

Outline of the proofs For convenience of the reader, we outline the main arguments of the proofs of Theorems A, C and L.

TIGHTNESS IN DIMENSION 5 The proof of Theorem A involves some geometric group theory and hyperbolic geometry as well as some holomorphic curve techniques.

The first ingredient is the construction of a strong symplectic cobordism between Bourgeois contact structures; this is done in Section 3.1. More precisely, Theorem 3.1 is a "stabilized" version of the analogous result for open books, which was proven (independently) in [2,35]; see Figure 1. We point out that, while the symplectic form on the strong cobordism of Theorem 3.1 is exact, the Liouville vector field associated to the global primitive is *not* inwards pointing along the negative ends. We shall refer to a strong symplectic cobordism with an exact symplectic form as *pseudo-Liouville*.

For "most" cases of surfaces (the rest are dealt with case by case), standard results from low-dimensional topology then allow one to write any monodromy as a composition such that the contact structures on the negative ends of the cobordism in Figure 1 are hypertight; see Corollary 4.1. Then, a standard application of the holomorphic curve machinery à-la [1,28,47] gives a holomorphic plane in the symplectization of one of the negative ends starting with a Bishop family associated to a Plastikstufe in the positive end. While bubbles are ruled out by exactness, holomorphic caps at the negative ends are excluded via the explicit properties of the cobordism (C, ω_C) (see Theorem 3.1 for a precise statement) and via the specific Reeb dynamics at the negative ends; this is a subtle point. Now, the existence of such holomorphic plane contradicts hypertightness of each connected component of the concave boundary (C, ω_C), thus concluding the proof.

OBSTRUCTIONS TO FILLABILITY IN ARBITRARY DIMENSIONS The proof of Theorem B is mostly based on holomorphic curve techniques.

We first attach the symplectic cobordism as in [42, Section 6.1] to the original contact manifold to obtain a symplectic fibration over ∂V, with $\mathbb{S}^2$ as fiber; see Lemma 6.1. This cobordism can be seen as the attaching of a symplectic handle, having two distinguished symplectic co-cores $C_{\pm}$.

In the presence of a strong symplectic filling W, after attaching such a symplectic cobordism and obtaining a "capped" filling W_{cap} with boundary a fibration over a contact manifold with symplectic sphere fibers, we obtain an induced moduli space $\mathcal{M}$ of closed holomorphic spheres which "probes" the original filling W. A key point is that each of the two co-cores $C_{\pm}$ of the relevant handle of the capping are J-invariant, and every curve in the moduli space intersects it precisely once, by positivity of intersections. The same is true after considering the Gromov compactification $\overline{\mathcal{M}}$ obtained by adding nodal curves.

Assuming that there is a cycle σ in $\partial C_{\pm}$, which is non-trivial in $C_{\pm}$ and bounds a relative cycle b in the filling, this allows one to "pull-back" b to the moduli space, and push it to a relative cycle in the co-core bounding the original σ, thus contradicting the non-triviality of σ in $C_{\pm}$. Such pull-backs can be achieved via the theory of pseudo-cycles in the case that W is semi-positive; in general, we appeal to polyfold theory [31] as, for example, in [42, Section 7.2]. In other words, the homology of the co-core must then survive in the filling.

TOPOLOGY OF ASPHERICAL FILLINGS The proof of Theorem D relies on the fact that the moduli space considered above is automatically compact (see Proposition 7.1). One can then puncture the spheres by removing their intersection with (small open neighbourhoods of) the co-cores, obtaining a moduli space of cylinders. Since each cylinder retracts onto any of its boundary components, this moduli space also admits a retraction to a piece of its boundary, along which the evaluation map is a diffeomorphism. Using a pull-push argument as in the proof of Theorem B, one obtains the desired isomorphism in homology.

In the case of $S^*\mathbb{T}^n$ applying some standard algebraic topology, one can then prove that any symplectically apherical filling W is in fact homotopy equivalent to $D^*\mathbb{T}^n$. Then, using the s-cobordism theorem as in [3, Section 8], one concludes that W is diffeomorphic to $D^*\mathbb{T}^n$. For this final step it is essential that the fundamental group of $S^*\mathbb{T}^n$ is abelian.

2 The Bourgeois construction

Consider a closed, oriented, connected smooth manifold V^{2n-1} and an open book decomposition (B, θ), together with a defining map $\Phi : V \to \mathbb{D}^2$ having each $z \in \text{int}(\mathbb{D}^2)$ as regular value. Here, $B \subset V$ is a closed codimension-2

submanifold, $\theta = \Phi/|\Phi| : V \setminus B \to \mathbb{S}^1$ is a fiber bundle, and Φ is such that $\Phi^{-1}(0) = B$.

A 1-form α on V is said to be *adapted* to Φ if it induces a contact structure on the regular fibers of Φ and if $d\alpha$ is symplectic on the fibers of $\theta = \Phi/|\Phi|$. In particular, if ξ is a contact structure on V supported by (B, θ), in the sense of [24], then (by definition) there is such a pair (α, Φ) with α defining ξ.

Theorem 2.1 (Bourgeois [8]) *Consider an open book decomposition* (B, θ) *of* V^{2n-1}, *represented by a map* $\Phi = (\Phi_1, \Phi_2) : V \to \mathbb{R}^2$ *as above, and let* α *be a 1-form adapted to* Φ. *Then,* $\beta := \alpha + \Phi_1 dq_1 - \Phi_2 dq_2$ *is a contact form on* $M := V \times \mathbb{T}^2$, *where* (q_1, q_2) *are coordinates on* $\mathbb{T}^2$.

The contact form β on $M = V \times \mathbb{T}^2$ will be called a *Bourgeois form* associated to (α, Φ) in the following.

Remark 2.1 The contact structure on $M = V \times \mathbb{T}^2$ defined by β is actually independent, up to isotopy, on the pair (α, Φ) defining the contact open book (B, θ) on V. This can easily be seen in the case where $n = 2$, i.e. V is 3-dimensional, using the contractibility of the space of symplectic forms on the 2-dimensional page. In the higher dimensional setting, this is discussed in detail in [38, Section 2 and Appendix], using the reinterpretation of contact open books in terms of ideal Liouville structures [25].

Remark 2.2 The contact structure determined by a Bourgeois contact form is stable up to contactomorphism under finite covers of the torus factor. Indeed, up to precomposing by an automorphism of $\mathbb{T}^2$, any such cover is of the form

$$(q_1, q_2) \longmapsto (kq_1, q_2).$$

Pulling back gives a contact form $\beta_k = \alpha + k\Phi_1 dq_1 - \Phi_2 dq_2$ and a straightforward calculation shows that linear interpolation gives a family of contact forms.

Abstract open books and Bourgeois contact structures For the proof of Theorem A, it is also useful to interpret the Bourgeois construction in abstract terms. We briefly recall here the construction in order to fix some notation. The reader can consult for instance [19, Section 7.3] for further details.

Consider a Liouville domain (Σ^{2n-2}, λ), together with an exact symplecto-morphism ψ of $(\Sigma, d\lambda)$ (i.e. $\psi^*\lambda = \lambda - dh$, for some smooth $h : \Sigma \to \mathbb{R}^+$), fixing pointwise a neighborhood of the boundary $B := \partial\Sigma$. One can then consider the mapping torus Σ_ψ of (Σ, ψ), and the abstract open book

$$V_{\Sigma, \psi} := \left(B \times \mathbb{D}^2 \sqcup \Sigma_\psi\right) \big/ \sim \qquad (2)$$

where $\sim$ identifies $(p, \theta) \in \partial(B \times \mathbb{D}^2)$ with $[p, \theta] \in \partial\Sigma_\psi$. One can also construct a fiberwise Liouville form λ_ψ on the mapping torus $\pi_0 \colon \Sigma_\psi \to \mathbb{S}^1$ of (Σ, ψ). For large $K \gg 0$ the form $\alpha_K = K\pi_0^* d\theta + \lambda_\psi$ is contact on Σ_ψ. Moreover, it can be extended to a contact form on all of $V_{\Sigma,\psi}$ by $h_1(r)\lambda_B + h_2(r)d\theta$ on $B \times \mathbb{D}^2$, for a well chosen pair of functions (h_1, h_2), and $\lambda_B = \lambda|_B$.

We denote the resulting contact form on $V_{\Sigma,\psi}$ by $\alpha_{\Sigma,\psi}$. The contact manifold $(V_{\Sigma,\psi}, \ker(\alpha_{\Sigma,\psi}))$ will also be called an *abstract contact open book*, and denoted simply with $OBD(\Sigma, \psi)$. Sometimes, we will also use the contact form $\alpha_{\Sigma,\psi}$.

We point out that there is a well defined map $\Phi_{\Sigma,\psi} \colon V_{\Sigma,\psi} \to \mathbb{D}^2$ given by extending the projection to the circle on Σ_ψ by setting

$$\Phi_{\Sigma,\psi}|_{B \times \mathbb{D}^2}(p, r, \theta) = \rho(r)e^{i\theta} \in \mathbb{D}^2,$$

for some non-decreasing function ρ satisfying $\rho(r) = r$ near 0 and $\rho(r) = 1$ near $r = 1$. Notice also that $\alpha_{\Sigma,\psi}$ is adapted to $\Phi_{\Sigma,\psi}$ (as defined above).

We then denote by $\beta_{\Sigma,\psi}$ the Bourgeois form on $M_{\Sigma,\psi} := V_{\Sigma,\psi} \times \mathbb{T}^2$ associated to $(\alpha_{\Sigma,\psi}, \Phi)$ as in Theorem 2.1, and by $\xi_{\Sigma,\psi}$ the contact structure it defines. Finally we let $BO(\Sigma, \psi) := (M_{\Sigma,\psi}, \xi_{\Sigma,\psi})$.

Hypertightness for Bourgeois Contact Forms In the following sections, we will need a hypertightness criterion for $\alpha_{\Sigma,\psi}$. We first give a definition: Let $(V \times \mathbb{T}^2, \xi)$ be a contact manifold, and $\mathcal{B}$ any subset of the set of closed Reeb orbits of a contact form β. We say that β has $\mathbb{T}^2$-*trivial Reeb dynamics concentrated in* $\mathcal{B}$ if the image of every closed Reeb orbit not in $\mathcal{B}$ under the projection $V \times \mathbb{T}^2 \to \mathbb{T}^2$ is homotopically non-trivial.

A straightforward computation gives:

Observation 2.2 [7, Section 10.2] (cf. [22, Corollary 6.3]). *The Bourgeois contact form $\beta_{\Sigma,\psi}$ for $\xi_{\Sigma,\psi}$ has $\mathbb{T}^2$-trivial Reeb dynamics concentrated in the set $\mathcal{B}$ consisting of the submanifolds $\gamma_B \times \{q\} \subset V \times \mathbb{T}^2$, for all $q \in \mathbb{T}^2$ and all γ_B closed Reeb orbit of $(B, \alpha_{\Sigma,\psi}|_B)$. If the binding $(B, \alpha_{\Sigma,\psi}|_B)$ of the natural open book of $V_{\Sigma,\psi}$ admits no Reeb orbits that are contractible in $V_{\Sigma,\psi}$, then the Bourgeois contact structure $\xi_{\Sigma,\psi}$ is hypertight.*

Notice that, in the 3-dimensional case, Observation 2.2 implies that, if the binding consists of a collection of loops each having infinite order in $\pi_1(V)$, then the associated Bourgeois contact structure is hypertight.

We point out that we will not make use of Observation 2.2 in the proof of Theorem A, as we will apply it directly on a another contact form, which still defines the Bourgeois contact structure up to isotopy (see Lemma 5.1 below).

A supporting spinal open book decomposition We now present a geometric way of understanding the Bourgeois construction, via SOBDs (see also

Section 6 for an alternative SOBD). The notion of an SOBD, introduced in [39] in dimension 3 (see also [46, Appendix B] for a version in arbitrary dimensions), comes from the observation that an open book decomposition can be thought of as two fibrations glued together: the neighbourhood of the binding is a contact fibration over the 2-disc, while the mapping torus piece is a Liouville fibration over the circle. More generally, an SOBD consists of a contact fibration over a general Liouville domain, glued to a Liouville fibration over a contact manifold. The first fibration is called the *spine*; the second one, the *paper*. The base of the spine (the *vertebrae*) has boundary the base of the paper; the fibers of the paper are called the *pages*. The two fibrations are glued together along an *interface region*, which for our purposes we will think as a trivial product of a piece of the vertebrae (a collar neighbourhood of its boundary) and the contact fiber of the spine, and hence can be given the structure of both types of fibrations.

Let us see how this works in the case of Bourgeois manifolds. Consider $\Phi = \Phi_{\Sigma,\psi} = \rho e^{i\theta} = (\rho \cos(\theta), \rho \sin(\theta))$, a defining map for $V = V_{\Sigma,\psi} = OBD(\Sigma, \psi)$, together with the Giroux form $\alpha = \alpha_{\Sigma,\psi}$ and the associated Bourgeois form $\beta = \beta_{\Sigma,\psi}$. Let $\theta = \Phi/|\Phi|: V \setminus B \to \mathbb{S}^1$ be the open book coordinate. From Eq. 2, we obtain a decomposition

$$M = V \times \mathbb{T}^2 = B \times D^*\mathbb{T}^2 \cup \Sigma_\psi \times \mathbb{T}^2,$$

where we identify $D^*\mathbb{T}^2 \xrightarrow{\sim} \mathbb{D}^2 \times \mathbb{T}^2$ via $(q_1, p_1, q_2, p_2) \mapsto (p_1, -p_2, q_1, q_2)$.

We denote by $M_S := B \times D^*\mathbb{T}^2$, which we call the *spine*, and $M_P := \Sigma_\psi \times \mathbb{T}^2$, the *paper*. We also have the *interface region* $M_I \cong B \times [-\epsilon, \epsilon] \times \mathbb{T}^3$, corresponding to the region where M_S and M_P glue together. Observe that we have fibrations

$$\pi_S : M_S \to D^*\mathbb{T}^2, \quad \pi_P : M_P \to S^*\mathbb{T}^2 = \mathbb{T}^3,$$

where the monodromy of π_P coincides with ψ along the cotangent $\mathbb{S}^1$-direction, and is trivial along $\mathbb{T}^2$. The map π_S has contact fibers and Liouville base, whereas π_P has contact base and Liouville fibers. The interface region is a trivial product and hence fibers over B or over a collar neighbourhood of $\partial D^*\mathbb{T}^2$ inside $D^*\mathbb{T}^2$. This is then an SOBD for M. Observe that the fibers of π_P, the *pages* of the SOBD, coincide with the pages of the OBD for V. One may also view the SOBD as a fibration $\widehat{\pi}_P : M \setminus \mathcal{B} \to \mathbb{S}^1 \times \mathbb{T}^2$, where we define the *binding* of the SOBD as $\mathcal{B} = B \times \{0\} \times \mathbb{T}^2 \subset B \times \mathbb{D}^2 \times \mathbb{T}^2 = M_S$. This fibration has fibers which symplectically are copies of the Liouville completion of the page Σ, and has monodromy ψ along the first factor, and trivial monodromy along the second.

The Bourgeois contact structure $\xi = \xi_{\Sigma,\psi}$ is "supported" by the SOBD described above, in a sense which we now describe. Via the identification $D^*\mathbb{T}^2 \to \mathbb{D}^2 \times \mathbb{T}^2$ above, up to isotopy of contact forms, we have

$$\beta|_{M_S \setminus M_I} = \lambda_B + \lambda_{std},$$

where $\lambda_{std} = p_1 dq_1 + p_2 dq_2$ is the standard Liouville form on $D^*\mathbb{T}^2$. In other words, $\beta|_{M_S \setminus M_I}$ is a split contact form, having a Liouville and a contact summand. Note also that on $M_S \setminus M_I$ the Reeb vector field R_β of β agrees with R_B over the binding $\mathcal{B}$, and is transverse to the pages away from it. Similarly, up to isotopy

$$\beta|_{M_P \setminus M_I} = \lambda_\psi + \alpha_{std},$$

where $\alpha_{std} = \cos(\theta)dq_1 + \sin(\theta)dq_2$ is the standard contact form on $\mathbb{T}^3$, and so splits into a Liouville summand and a contact summand. In particular, the restriction of $d\beta$ to the pages of the SOBD is a positive symplectic form, and the Reeb vector field is transverse to the pages, agreeing with that of α_{std} and so tangent to the $\mathbb{T}^2$ factor. In other words, the contact structure, the contact form, as well as the Reeb dynamics of β are "compatible" with the underlying geometric decomposition. This interpretation also allows us to reobtain Observation 2.2.

Remark 2.3 The observation that Bourgeois contact manifolds are supported by the above SOBD, in the sense described above, should be attributed to Sam Lisi.

3 A cobordism of Bourgeois manifolds

We describe a strong (actually, pseudo-Liouville) cobordism between Bourgeois contact manifolds with the same page. Its purpose is to relate the Bourgeois manifold coming from two different monodromies to the one coming from their composition, and it will be used in the proof of Theorem A.

3.1 From disjoint union to composition of monodromies

Let $(\Sigma^{2n-2}, d\lambda)$ be a Liouville manifold, and let ϕ be an exact symplectomorphism relative to the boundary. Notice that the boundary $(B, \lambda_B) := (\partial \Sigma, \lambda|_{\partial \Sigma})$ can naturally be seen as the "binding" submanifold of the associated open book. For each $q \in \mathbb{T}^2$, we also let B_q be $B \times \{q\} \subset V_{\Sigma,\phi} \times \mathbb{T}^2 = M_{\Sigma,\phi}$.

The aim of this section is to give a proof of the following result (recall Figure 1):

Theorem 3.1 *There is a smooth cobordism C from $M_{\Sigma,\psi} \bigsqcup M_{\Sigma,\phi}$ to $M_{\Sigma,\psi\circ\phi}$. This cobordism is smoothly a product $C_0 \times \mathbb{T}^2$, where C_0 is a smooth cobordism from $V_{\Sigma,\psi} \bigsqcup V_{\Sigma,\phi}$ to $V_{\Sigma,\psi\circ\phi}$. Moreover, there is a symplectic form ω_C on C which satisfies the following properties:*

1. *ω_C admits local Liouville forms λ_+ and λ_- near $M_{\Sigma,\psi\circ\phi}$ and $M_{\Sigma,\psi} \bigsqcup M_{\Sigma,\phi}$ respectively, satisfying:*
   - (a) *λ_+ restricts on $M_{\Sigma,\psi\circ\phi}$ to the Bourgeois contact form $\beta_{\Sigma,\psi\circ\phi}$,*
   - (b) *λ_- restricts on $M_{\Sigma,\psi} \sqcup M_{\Sigma,\phi}$ to the Bourgeois contact forms $\beta_{\Sigma,\psi}$ and $\beta_{\Sigma,\phi}$ respectively; in particular, λ_- has (on each connected component) $\mathbb{T}^2$-trivial Reeb dynamics concentrated in $\{B_q\}_{q\in\mathbb{T}^2}$;*
2. *ω_C admits a global primitive ν which coincides with λ_+ at the convex boundary and such that $\nu|_{B_q} = \lambda_-|_{B_q}$ for each $B_q \subset M_{\Sigma,\psi} \bigsqcup M_{\Sigma,\phi}$.*

Item 1 means in particular that (C, ω_C) is a strong symplectic cobordism with convex boundary $BO(\Sigma, \phi \circ \psi)$ and concave boundary $BO(\Sigma, \phi) \sqcup BO(\Sigma, \psi)$. Notice however that we do *not* claim that the global 1-form ν in Item 2 defines a contact structure at the concave boundary; in other words, the cobordism we give is not claimed to be Liouville, but just pseudo-Liouville (as defined in the introduction). Lastly, we point out that Theorem 3.1 can be thought of as a "stabilized" version of [2, Proposition 8.3] and [35, Theorem 1]; in fact, smoothly (but not symplectically), the cobordism C is just the product of the cobordism from [2,35] with $\mathbb{T}^2$. For the reader's convenience, we start by giving a topological description of the cobordism C as obtained by gluing two "cobordisms with corners", C_{bot} and C_{top}, and then describe the symplectic structures on these pieces in more detail.

A topological description of C Let P be the pair of pants, i.e. the surface of genus 0 and with 3 boundary components. We view P as embedded in $\mathbb{R}^2$ as the (closed) disc D_+ of radius 1 with two smaller (open) disjoint discs $D_{-,1}$ and $D_{-,2}$, both of radius ϵ and centered at $-1/2$ and $+1/2$ respectively, removed from it. In cobordisms terms, P is seen as a smooth cobordism with concave boundary $\partial D_{-,1} \sqcup \partial D_{-,2}$ and convex boundary ∂D_+.

Consider then the fiber bundle $\pi_E : E \to P$, with fiber the page Σ, over the pair of pants P, where the monodromies along the two negative boundary components are given by ϕ and ψ respectively, and by their composition along the positive one. This can be realized for instance as follows: consider γ_1 and γ_2 disjoint arcs in P joining respectively $\partial D_{-,1}$ and $\partial D_{-,2}$ to ∂D_+, then cut $\Sigma \times P$ along $\Sigma \times \gamma_1$ and $\Sigma \times \gamma_2$ and glue them back respectively via $\phi \times \mathrm{Id}_{\gamma_1}$ and $\psi \times \mathrm{Id}_{\gamma_2}$.

The desired "bottom piece" C_{bot} of C is then the cobordism with corners $E \times \mathbb{T}^2$, which inherits a fibration $\pi = \pi_E \times id_{\mathbb{T}^2} : C_{bot} \to P \times \mathbb{T}^2$; cf. Figure 2. Notice that C_{bot} has the following distinguished boundary pieces:

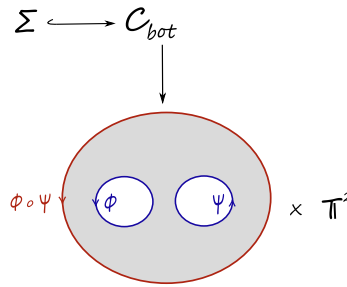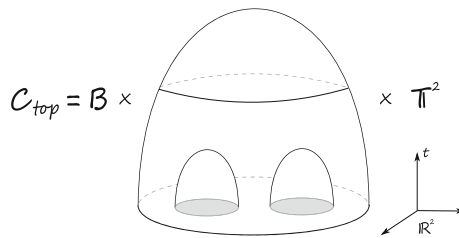**Fig. 2** Topological picture of C_{bot}

Fig. 3 Topological picture of the cobordism (with corners) $C_{top} \subset B \times [0, +\infty) \times \mathbb{R}^2 \times \mathbb{T}^2$. In the proof of Theorem 3.1, $\mathbb{R}^2 \times \mathbb{T}^2$ becomes $T^*\mathbb{T}^2$ and $[0, +\infty) \times B$ becomes the symplectization of B

- $\partial_{\pm} C_{bot}$ given by the preimage by π of $\partial D_+ \times \mathbb{T}^2$ and $\partial D_{-,1} \sqcup \partial D_{-,2} \times \mathbb{T}^2$, respectively;
- $\partial_0 C_{bot}$ given by $\partial C_{bot} \setminus \pi^{-1}(\partial P \times \mathbb{T}^2)$.

Furthermore, since ϕ and ψ are both identity near $\partial \Sigma$, one simply has that $\partial_0 C_{bot} = \partial \Sigma \times P \times \mathbb{T}^2$.

The "top piece" C_{top} of C is given as follows. Consider $P \subset \mathbb{R}^2$ as a subset of $\{0\} \times \mathbb{R}^2 \subset \mathbb{R}^3$. Let also S_+, $S_{-,1}$ and $S_{-,2}$ be three hemispheres in $[0, +\infty) \times \mathbb{R}^2$ centered respectively at 0, $-1/2$ and $+1/2$, and of radius 1, ϵ and ϵ respectively. Then, P together with these three hemispheres bounds a compact region $R \subset \mathbb{R}^3$ with piece-wise smooth boundary. The desired C_{top} is then just given by $C_{top} = B \times R \times \mathbb{T}^2$, where $B = \partial \Sigma$; cf. Figure 3. As was the case with C_{bot}, the manifold C_{top} also has the distinguished boundary pieces:

- $\partial_+ C_{top}$ given by $B \times S_+ \times \mathbb{T}^2$;
- $\partial_- C_{top}$ given by $B \times (S_{-,1} \sqcup S_{-,2}) \times \mathbb{T}^2$;
- $\partial_0 C_{top}$ given by $B \times P \times \mathbb{T}^2$.

The desired cobordism C is then topologically just obtained by gluing C_{top} with C_{bot} along $\partial_0 C_{top} = B \times P \times \mathbb{T}^2$ and $\partial_0 C_{bot} = B \times P \times \mathbb{T}^2$ (via the natural identification).

We now proceed to discuss the symplectic side of the construction, i.e. to give a detailed proof of Theorem 3.1. We will follow closely, with some adaptations, the proof given in [35].

A toroidal pair of pants cobordism in dimension In order to make the topological sketch above into a detailed proof taking care of the symplectic data, we need to utilise a symplectic counterpart of the product of the pair of pants P and $\mathbb{T}^2$ which has the right structure at the boundary. In other words, we want a strong 4-dimensional symplectic cobordism with concave end $(S^*\mathbb{T}^2, \xi_{std}) \sqcup (S^*\mathbb{T}^2, \xi_{std})$ and convex end $(S^*\mathbb{T}^2, \xi_{std})$, and having the topology of $P \times \mathbb{T}^2$. To this end, consider the unit disc cotangent bundle $D^*\mathbb{T}^2$ of $\mathbb{T}^2$, together with its standard symplectic structure $\omega_{std} = d\lambda_{std}$, where in coordinates $\lambda_{std} = p_1 dq_1 + p_2 dq_2$ is the standard Liouville form. To be precise, we need to work with scalar multiples $K\omega_{std}$ and $K\lambda_{std}$, where K is a positive real constant that will be determined later on in the proof. We also denote by X the Liouville vector field $p_1\partial_{p_1} + p_2\partial_{p_2}$.

Consider the submanifold $D^*_\epsilon\mathbb{T}^2$ of $D^*\mathbb{T}^2$ made of those covectors of norm less than a certain $\epsilon < 1/10$, and denote by $j_\pm\colon D^*_\epsilon\mathbb{T}^2 \to D^*\mathbb{T}^2$ the symplectomorphisms

$$(p_1, q_1, p_2, q_2) \overset{j_\pm}{\longmapsto} (p_1 \pm 1/2, q_1, p_2, q_2).$$

For ease of notation, we consider the inclusion

$$j = j_- \sqcup j_+\colon (D^*_\epsilon\mathbb{T}^2, K\omega_{std}) \sqcup (D^*_\epsilon\mathbb{T}^2, K\omega_{std}) \to (D^*\mathbb{T}^2, K\omega_{std}).$$

Then, the desired cobordism is $(Q, \omega_Q) := (D^*\mathbb{T}^2 \setminus j(D^*_\epsilon\mathbb{T}^2), K\omega_{std})$. Topologically this is just a product of the torus with a pair of pants $P \subset \mathbb{R}^2$ in the (p_1, p_2)-plane.

The Liouville field on a neighbourhood of the convex boundary is just given by X, whereas the one near the concave boundary is given by j_*X, which is the vector field $(p_1 \mp 1/2)\partial_{p_1} + p_2\partial_{p_2}$ on the image of $j_\pm$ respectively.

Lastly, we consider an auxiliary smooth function $f\colon T^*\mathbb{T}^2 \to \mathbb{R}$, which depends only on p_1, p_2, and satisfies:

1. $f = p_1^2 + p_2^2$ on $T^*\mathbb{T}^2 \setminus D^*\mathbb{T}^2$,
2. $f = (p_1 \mp 1/2)^2 + p_2^2$ on the image of $j_\pm$ respectively,
3. $\epsilon^2 < f < 1$ on the interior of Q.

Notice that Item 1 implies in particular that $df(X) > 0$ in a neighborhood of $\partial(D^*\mathbb{T}^2)$. Similarly, Item 2 implies that $df(j_*X) > 0$ on the image of j, except on $j(\{0\} \times \mathbb{T}^2 \sqcup \{0\} \times \mathbb{T}^2)$, where it vanishes.

Description of the symplectic cobordism Consider the bottom piece C_{bot} as in the topological sketch above. We now view C_{bot} as a fiber bundle with

fibers Σ over $Q = P \times \mathbb{T}^2 \subset T^*\mathbb{T}^2$, and want to prove that it admits a fiberwise Liouville form λ_0 which agrees with $e^t \lambda_B$ near $B \times Q \subset C_{bot}$. Here, $e^t \lambda_B$ is the normal form of λ on Σ on a sufficiently small neighborhood $(-\delta, 0] \times B$, with $t \in (-\delta, 0]$, of its contact boundary B given by the (globally defined) Liouville vector field Y for λ, which is outwards pointing along the boundary $\partial \Sigma = B$ of the page Σ.

Recall that C_{bot} is obtained from $\Sigma \times T^*\mathbb{T}^2$ by a cut and paste procedure along $\Sigma \times (\gamma_1 \cup \gamma_2) \times \mathbb{T}^2$, as described in the topological sketch above. Recall also that ϕ and ψ are exact symplectomorphisms of the page (Σ, λ), equal to the identity on a neighborhood of B; let then h_1 and h_2 be respectively the functions on Σ, constant near the boundary, such that $\phi^*\lambda = \lambda - dh_1$ and $\psi^*\lambda = \lambda - dh_2$. Taking normal coordinates $r_i \in (-\varepsilon, \varepsilon)$ to γ_i inside P, one can then consider smooth cutoff functions $\rho_i : (-\varepsilon, \varepsilon) \to [0, 1]$ equal to 0 near $r_i = -\varepsilon$ and to 1 near $r_i = 0$. It then follows that, in the cut and paste procedure to obtain C_{bot} from $\Sigma \times T^*\mathbb{T}^2$, the form $\lambda - d(\rho_i h_i)$ on $\Sigma \times (P \setminus (\gamma_1 \cup \gamma_2)) \times \mathbb{T}^2$ glues well under the identifications $\phi \times \mathrm{Id}_{\gamma_1}$ and $\psi \times \mathrm{Id}_{\gamma_2}$ along $\Sigma \times \gamma_1 \times \mathbb{T}^2$ and $\Sigma \times \gamma_2 \times \mathbb{T}^2$ respectively. The result is hence a fiberwise Liouville form λ_0 on the fiber bundle $\pi : C_{bot} \to Q$, which is equal to $e^t \lambda_B$ near $B \times Q$ as the h_i's are constant there.

We also point out that, for very large $K \gg 0$, the form $\Omega = d\lambda_0 + K\pi^* d\lambda_{std}$ on the total space C_{bot} of $\pi : C_{bot} \to Q$ is an exact symplectic form with primitive $\nu = \lambda_0 + K\pi^*\lambda_{std}$. Here, abusing notation sightly, we let λ_{std} denote the restriction of the canonical 1-form under the inclusion $Q \subset T^*\mathbb{T}^2$. This then gives the desired symplectic structure on C_{bot}; it only remains to describe the symplectic form on C_{top}.

For this, let τ be a function on $[0, \infty)$ vanishing at 0 with all its derivatives, and strictly monotone increasing on $(0, \infty)$. We then identify the C_{top} of the topological sketch above with the following set:

$$C_{top} = \left\{ (t, b, p, q) \in [0, \infty) \times B \times T^*\mathbb{T}^2 \mid \tau(t)^2 + f(p, q)^2 \geq \epsilon^2, \right.$$
$$\left. \tau(t)^2 + |p|^2 \leq 1 \right\}.$$

The advantage of this identification is that C_{top} inherits a natural Liouville structure $e^t \lambda_B + K\lambda_{std}$ from the symplectization $([0, \infty) \times B, d(e^t \lambda_B))$ and the cotangent structure $(T^*\mathbb{T}^2, \lambda_{std})$.

Now, C_{bot} and C_{top} can be naturally glued along the subsets $B \times Q \subset \partial C_{bot}$ and $\{\tau = 0\} = \{0\} \times B \times Q \subset C_{top}$ (corresponding to $\partial_0 C_{top}$ and $\partial_0 C_{bot}$ in the topological sketch). What is more, this gluing is compatible with the symplectic form $\Omega = d\lambda_0 + K\pi^* d\lambda_{std}$ on C_{bot} and $d(e^t \lambda_B) + K d\lambda_{std}$ on C_{top}. We thus obtain a symplectic cobordism (C, ω_C), and we now proceed to check that it satisfies the required properties. We will only do this for the negative ends, as the case of the positive end is entirely analogous.

Proof of Theorem 3.1 Notice first that by construction $C = C_0 \times \mathbb{T}^2$ is topologically a product. Furthermore, each boundary component of C determine $\mathbb{T}^2$-stabilised open books, which we think of as supporting SOBD's for the Bourgeois contact structures. Moreover, these SOBD's coincide, at least topologically, with those used to define the Bourgeois contact structure. From a symplectic point of view, we point out that the fiberwise Liouville form λ_0 on $C_{bot} \to Q$ glues well to the 1-form $e^t \lambda_B$ on C_{top} inherited from the ambient space $[0, \infty) \times B \times T^*\mathbb{T}^2$ to give a global 1-form denoted $\widehat{\lambda}_0$, which is a pullback of a form on C_0.

We now describe the primitives near the boundary in the bottom part C_{bot}. Consider the primitive $j_* \lambda_{std}$ of the standard symplectic form near the concave boundary of Q, which in coordinates is just $p_1 dq_1 + p_2 dq_2 \mp 1/2 dq_1$. The Liouville vector field Z associated to the local primitive $\lambda_- = \lambda_0 + K j_* \lambda_{std}$ is then defined on a neighborhood of the negative boundary components $\partial_- C_{bot}$ of C_{bot}. More precisely, an explicit computation shows that the vector field Z is, in the explicit pre-glued version $\Sigma \times (P \setminus (\gamma_1 \cup \gamma_2)) \times \mathbb{T}^2$ of C_{bot}, just $Y + Y' + j_* X + X'$, where Y is the Liouville form associated to λ on Σ, Y' is the $d\lambda$-dual of ρdh (hence tangent to the Σ factor), and X' is the $d\lambda_{std}$-dual of $\frac{1}{K} h d\rho$; here we define ρ to be ρ_i, respectively on each of the two negative ends. Hence, the vector field X' is tangent to the second factor of $\Sigma \times (P \setminus (\gamma_1 \cup \gamma_2)) \times \mathbb{T}^2$, and has no ∂_p component, using cotangent coordinates $(p, q) \in T^*\mathbb{T}^2$. In particular, the projection of Z on C_{bot} to Q via the fiber bundle map has the same ∂_p component as $j_* X$ near $\partial_- Q$, and is hence transverse to it and inwards pointing.

Now, via the natural (orientation preserving) diffeomorphisms

$$
\begin{aligned}
\mathbb{S}^1_\epsilon \times \mathbb{T}^2 &\to S^*_\epsilon \mathbb{T}^2 \subset D^*_\epsilon \mathbb{T}^2 \\
(\varphi, x, y) &\to (p_1 = -\cos(\varphi) \pm 1/2, q_1 = x, p_2 = \sin(\varphi), q_2 = y),
\end{aligned}
\tag{3}
$$

where $\mathbb{S}^1_\epsilon$ is the circle in $\mathbb{R}^2$ of radius ϵ, the restriction of the associated Liouville form on C_{bot} to the boundary is naturally pulled back to the Bourgeois contact form on the paper region of the SOBD, for a suitable choice of adapted contact form α and open book map Φ (cf. Remark 2.1).

In the top part C_{top} of the cobordism the Liouville vector field is just the linear combination $\partial_t + r \partial_r$, where r is the radial parameter $|p \mp 1/2|$ at the respective boundary components. This is then also transverse to the boundary.

By parametrizing the interior hemispherical caps using the canonical coordinates coming from the cotangent bundle of the torus, we obtain the following coordinate description on C_{top}:

$$
\lambda_- = e^{\sqrt{\epsilon^2 - |r|^2}} \lambda_B + K(p_1 \mp 1/2) dq_1 + K p_2 dq_2 .
$$

Now, via the (orientation preserving) diffeomorphisms

$$\mathbb{D}_\epsilon^2 \times \mathbb{T}^2 \to D_\epsilon^* \mathbb{T}^2$$

$$(r, \varphi, x, y) \to (p_1 = -r\cos(\varphi) \pm 1/2, q_1 = x, p_2 = r\sin(\varphi), q_2 = y),$$

naturally extending Eq. (3), where $\mathbb{D}_\epsilon^2$ is here the disc in $\mathbb{R}^2$ centered at the origin of radius ϵ, λ_- then pulls back on each negative boundary component of C_{top} to the Bourgeois contact form as in Theorem 2.1, for a suitable choice of adapted contact form α and open book map Φ, which can be taken compatible with that coming from the bottom boundary part. This concludes the proof of Item 1b.

Finally, notice that there is a global Liouville primitive ν for ω_C defined on all C, given by gluing $\lambda_0 + K\pi^*\lambda_{std}$ on $C_{bot} = E \times \mathbb{T}^2$ and $e^t \lambda_B + K\lambda_{std}$ on $C_{top} \subset [0, +\infty) \times B \times T^*\mathbb{T}^2$. By looking at the explicit expression for λ_- along C_{top}, we see that ν coincides with λ_- on the subsets of the form B_q defined before the statement of Theorem 3.1. Indeed, these are given by $B_q = B \times \{x_\pm\} \times \{q\}$, where $x_\pm = (t = \epsilon, p_1 = \pm 1/2, p_2 = 0) \in [0, +\infty) \times \mathbb{R}^2$ is the "origin" in the hemispheres $S_{-,1}, S_{-,2}$, i.e. the points of $S_{-,1}$ and $S_{-,2}$ with maximal t value. There, we just have $\lambda_-|_{B_q} = e^\epsilon \lambda_B = \nu|_{B_q}$. This concludes the proof. □

4 Factorizing the monodromy

Let Σ denote a connected orientable surface with boundary. We will denote the mapping class group as $MCG(\Sigma)$, which is defined to be the set of isotopy classes of orientation preserving diffeomorphisms of Σ; note that we do not require these diffeomorphisms to fix the boundary components. This group is naturally isomorphic to the group of isotopy classes of diffeomorphisms of the corresponding punctured surface. One may also consider $MCG(\Sigma, \partial\Sigma)$ of mapping classes fixing the boundary, and there is a natural forgetful map $MCG(\Sigma, \partial\Sigma) \to MCG(\Sigma)$ whose kernel is generated by boundary parallel Dehn twists.

We will refer to a surface as *sporadic* if it is either a disc, an annulus or a pair of pants. These cases correspond to the mapping class group being virtually abelian. The aim of this section is to prove the following:

Lemma 4.1 (Factorization Lemma) *Let $\phi \in MCG(\Sigma, \partial\Sigma)$ for a non-sporadic surface Σ. Then ϕ can be factored as $\phi = \phi_1 \circ \phi_2$, where, for each $i = 1, 2$, ϕ_i is such that each connected component of the binding of $V_i := OBD(\Sigma, \phi_i)$ has infinite order in $\pi_1(V_i)$.*

A direct consequence of Lemma 4.1 and Observation 2.2 is the following:

Corollary 4.1 *Let ϕ be a mapping class of a compact, orientable, non-sporadic surface Σ with boundary. Then ϕ can be factored as $\phi = \phi_1 \circ \phi_2$, with ϕ_1, ϕ_2 such that the Bourgeois contact manifolds $BO(\Sigma, \phi_1)$ and $BO(\Sigma, \phi_2)$ are hypertight.*

In order to prove Lemma 4.1, we start by recalling some results from geometric group theory and 3-dimensional hyperbolic geometry in Sections 4.1 and 4.2 respectively. The proof is then given in Section 4.3.

4.1 Dynamics of pseudo-Anosovs acting on the curve graph

Let Σ denote a compact, connected orientable surface (possibly with boundary). We recall that, by the Nielsen–Thurston classification theorem (see for instance [18, Theorem 13.2]), every element in $MCG(\Sigma)$ or $MCG(\Sigma, \partial\Sigma)$ is either pseudo-Anosov, reducible or of finite order. This characterization can also be extracted from the associated action on the *curve graph*, denoted $\mathcal{C}(\Sigma)$. This is the graph whose vertices are isotopy classes of essential, non boundary-parallel, simple closed curves on Σ so that there is an edge between two vertices if the corresponding curves can be made disjoint via isotopy. Endowing the edges to have length 1 we thus obtain a metric space on which the mapping class group acts via isometries.

The curve graph was famously shown to be δ-hyperbolic in the sense of Gromov by Masur-Minsky [43] in the case that Σ is neither the torus nor sporadic, which has many important implications. In particular, one has the following alternative description using this action that is again due to Masur-Minsky [43, Theorem 4.6].

Theorem 4.1 (Masur–Minsky) *Let $\phi \neq Id$ be an arbitrary mapping class on a non-sporadic compact, orientable, surface Σ with non-empty boundary. Then, we have the following trichotomy:*

- *(Finite Order): The action of ϕ on $\mathcal{C}(\Sigma)$ has finite orbits, but no fixed point.*
- *(Reducible): ϕ has a fixed point in $\mathcal{C}(\Sigma)$.*
- *(Pseudo-Anosov): ϕ has no finite orbits and in fact acts **hyperbolically** on $\mathcal{C}(\Sigma)$.*

In the following we will use some basic facts about Gromov-hyperbolic spaces and their boundaries at infinity, which can for example be found in the book of Bridson–Haefliger [11, Chapter 9].

We now collect some basic consequences of the fact that a pseudo-Anosov mapping class f acts hyperbolically on the curve graph. First we note that, for any choice of isotopy class α, which corresponds to a vertex in the curve graph $X = \mathcal{C}(\Sigma)$, the bi-infinite orbit $\beta = (f^n(\alpha))_{n \in \mathbb{Z}}$ is a *quasi-geodesic*. Moreover, such a quasi-geodesic determines two distinct points $p_\pm$ on the Gromov

boundary $\partial_\infty X$ of the curve graph, which correspond to the fixed points of the induced action on the boundary and form a repelling/attracting pair. Finally, the action on the boundary has *north-south dynamics*. More precisely, for *any* neighbourhoods $U_\pm$ around $p_\pm \in \partial_\infty X$ there is some (large) N so that

$$f^N(\partial_\infty X \setminus U_-) \subset U_+ \text{ and } f^{-N}(\partial_\infty X \setminus U_+) \subset U_-.$$

In fact, this north-south dynamics can also be seen using half spaces associated to the quasi-geodesic $\beta = (f^n(\alpha))_{n \in \mathbb{Z}}$, as described presently. First of all the nearest point projection $\Pi_\beta : X \to \beta$, mapping an arbitrary point to a point in β which is nearest to it, is well-defined in the coarse sense, meaning that any two nearest point projections are a uniformly bounded distance apart. We fix such a choice from now on; what follows is independent of this choice up to constants. We decompose $\beta = (f^n(\alpha))_{n \leq 0} \cup (f^n(\alpha))_{n \geq 0} = \beta_+ \cup \beta_-$ into two (quasi-)rays and define half spaces

$$H_\pm = \Pi_\beta^{-1}(\beta_\pm).$$

Taking closures in $X \cup \partial_\infty X$, the north-south dynamics implies that

$$\bigcap_{n \geq 0} f^n\left(\overline{H}_+\right) = p_+ \text{ and } \bigcap_{n \geq 0} f^{-n}\left(\overline{H}_-\right) = p_- .$$

For convenience we write $W_\pm^N = f^{\pm N}(H_\pm)$. Note that W_+^N is precisely the preimage of $\{f^n(\alpha)\}_{n \geq N}$ under the projection and similarly for W_-^N. In particular, W_+^N and W_-^M are disjoint for all $N, M \geq 1$. Moreover, these half spaces satisfy the following property:

Lemma 4.2 *There is a constant $C = C(\delta, \alpha)$, depending only on the hyperbolicity constant δ and the choice of curve α, such that every geodesic joining a point $z \in W_\pm^N$ to its nearest point projection lies in $W_\pm^{N-C}$.*

Proof We refer to Figure 4 for this proof. Let $z \in W_+^N$ have nearest point projection $f^L(\alpha)$ and let $[z, f^L(\alpha)]$ be a geodesic path from z to $f^L(\alpha)$. Assume that the nearest point projection $f^{N'}(\alpha)$ of some $x \in [z, f^L(\alpha)]$ sits outside W_+^{N-C}, i.e. $N' \leq N - C$.

As $(f^n(\alpha))_n$ is a quasi-geodesic, there is some $C' > 0$ so that, for any $N - C < M < N$, there is a point p on the geodesic segment $[f^{N'}(\alpha), f^L(\alpha)]$ which is at distance C' from $f^M(\alpha)$. Now, by the δ-slim property applied to the geodesic triangle $\{x, f^{N'}(\alpha), f^L(\alpha)\}$, there is a point y either on the geodesic segment $[x, f^{N'}(\alpha)]$ or on $[x, f^L(\alpha)]$ with distance at most δ from p. Hence, y is also at distance at most $C' + \delta$ from $f^M(\alpha)$.
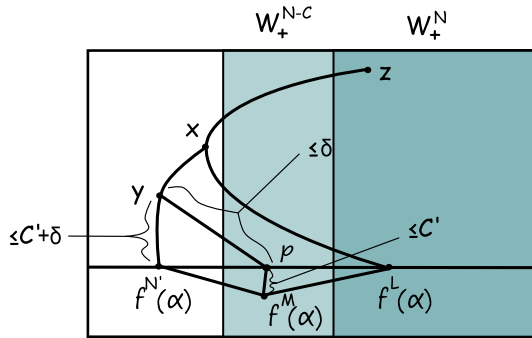
Fig. 4 The triangle in the proof of Lemma 4.2

Now, let us assume that $y \in [x, f^{N'}(\alpha)]$. Then, it follows that $[y, f^{N'}(\alpha)]$ has length at most $C'+\delta$, otherwise there would be a piecewise geodesic from x to $f^M(\alpha)$ shorter than the geodesic segment $[x, f^{N'}(\alpha)]$, contradicting the fact that $f^{N'}(\alpha)$ is the nearest point projection of x. But then the geodesic triangle $\{y, f^{N'}(\alpha), f^M(\alpha)\}$ violates the triangle inequality if C is taken big enough (this value only depends on δ and α, and not on the point y), as the two sides containing y are both of length at most $C'+\delta$, while the side $[f^{N'}(\alpha), f^M(\alpha)]$ has length growing linearly in $M - N' \geq C$.

If $y \in [x, f^L(\alpha)]$, the analogous argument shows that the geodesic triangle $\{y, f^L(\alpha), f^M(\alpha)\}$ violates the triangle inequality if C is big enough. This then proves the proposition, provided $C = C(\delta, \alpha)$ is taken to be large. □

We now prove the following, which only uses facts about the dynamics of isometries acting hyperbolically on δ-hyperbolic spaces. We are very thankful to Sebastian Hensel for pointing this out to us.

Proposition 4.1 *Let ϕ be a fixed mapping class on a non-sporadic compact, orientable, surface Σ with non-empty boundary. Then, there exists a pseudo-Anosov map f such that, for sufficiently large k, the mapping class $f^k \phi \in MCG(\Sigma, \partial \Sigma)$ is pseudo-Anosov.*

Proof Let ϕ be our given mapping class. We first claim that there is a pseudo-Anosov map f whose set of fixed points $\{q_\pm\}$ at infinity is mapped to a disjoint set under ϕ. To see this, first recall that, since Σ is non-sporadic, there exist at least two pseudo-Anosov mapping classes $\gamma, \eta \in MCG(\Sigma, \partial \Sigma)$ with disjoint sets of fixed points at infinity $\{p_\pm^\gamma\}$ and $\{p_\pm^\eta\}$[1]. Now, if for instance ϕ mapped the stable fixed point $p_+^\gamma \in \partial_\infty X$ of γ to its unstable fixed point p_-^γ (the other case is similar), then we can consider the conjugate $f = \gamma^N \eta \gamma^{-N}$ for large N

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[1] This follows, for example, from the fact that the action on the boundary of the curve complex is WPD in the sense of [4, Proposition 6].
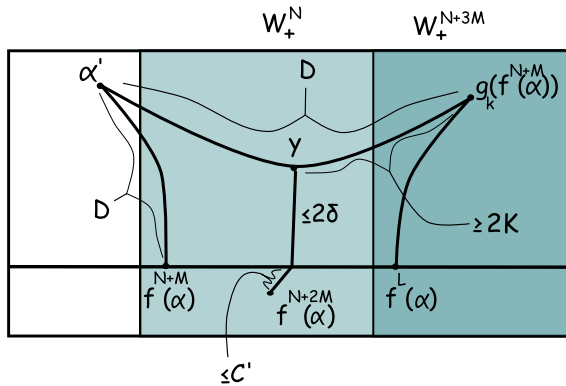
Fig. 5 The quadrilateral in the proof of Proposition 4.1

so that both fixed points $q_\pm$ of f lie in a small neighbourhood of the attracting fixed point p_+^γ of γ, which is then mapped under ϕ to a small neighbourhood of p_-^γ. In particular, the set $\{\phi(q_\pm)\}$ is disjoint from $\{q_\pm\}$ as claimed.

Consider now such a pseudo-Anosov f, and let $W_\pm^N = f^{\pm N}(H_\pm)$ as in the notation described before the statement of Proposition 4.1. Note that these subsets are defined with respect to some *a priori* fixed quasi-geodesic $(f^n(\alpha))_{n \in \mathbb{Z}}$. Observe that, in view of our assumption on f, we can use the north-south dynamics to assume that *both* $\phi^{-1}(W_-^N)$ and $\phi(W_+^N)$ are disjoint from W_-^N, for all large $N \gg 0$. In particular, for any given (large) M we can take k so that $g_k = f^k \phi$ satisfies

$$g_k(W_+^N) \subset W_+^{N+3M}.$$

Now suppose g_k is not pseudo-Anosov. In particular, up to taking powers, it has a fixed point, say α'. We claim that such a fixed point must belong to W_+^N. Assume not. Then consider the geodesic joining α' to $f^{N+M}(\alpha)$. This has one end point fixed by g_k, while the other gets mapped to $g_k(f^{N+M}(\alpha)) \in W_+^{N+3M}$ under g_k. Joining $g_k(f^{N+M}(\alpha))$ by a geodesic to its nearest point projection $f^L(\alpha)$, where $L \geq N + 3M$, we obtain a geodesic quadrilateral with vertices $g_k(f^{N+M}(\alpha))$, $f^L(\alpha)$, $f^{N+M}(\alpha)$, α' that has two sides of equal length $D = d(\alpha', f^{N+M}(\alpha))$ at the vertex α'. See Figure 5.

Then by the δ-slim triangle condition applied twice, the stability of quasi-geodesics and Lemma 4.2, we find a point y on one of the sides containing α' that has distance at most $K = C'(\delta, \alpha) + 2\delta$ from $f^{N+2M}(\alpha)$ where $C'(\delta, \alpha)$ is the uniform constant coming from quasi-geodesic stability that in turn depends only on δ and α. Moreover, we have (using that M is large) that y splits the geodesic it lies on into subarcs each of length at least $2K$, and hence of length at most $D - 2K$. This is also represented in Figure 5. Then the triangle inequality
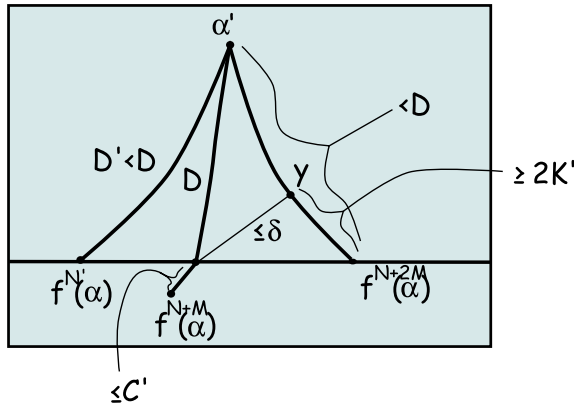
Fig. 6 The triangle $\{\alpha', f^{N'}(\alpha), f^{N+2M}(\alpha)\}$. (Here, recall that $K' = C' + \delta$)

implies that

$$d(\alpha', f^{N+2M}(\alpha)) < d(\alpha', f^{N+M}(\alpha)).$$

Now let $f^{N'}(\alpha)$ be the nearest point projection of α' to the quasi-geodesic $\beta = (f^n(\alpha))_{n \in \mathbb{Z}}$ so that $N' < N$ by the assumption that α' does not belong to W_+^N. We argue now as above. First apply the δ-slim triangle condition to the geodesic triangle with vertices $f^{N+2M}(\alpha)$, $f^{N'}(\alpha)$ and α', as well as the stability of quasi-geodesics to find a point y on the geodesic segments $[f^{N+2M}(\alpha), \alpha']$ or $[f^{N'}(\alpha), \alpha']$ that is distance at most $K' = C'(\delta, \alpha) + \delta$ to the point $f^{N+M}(\alpha)$ (see Figure 6). Then using that M is large, so that the distance from $f^{N+M}(\alpha)$ to $f^{N+2M}(\alpha)$ resp. $f^{N'}(\alpha)$ is larger than $2K'$, we deduce that y divides the geodesic segment it lies on into segments of length at least $2K'$, and hence of length at most $D - 2K'$. Thus we find a (piecewise geodesic) path joining α' to $f^{N+M}(\alpha)$ of length strictly less than

$$D' = d(\alpha', f^{N'}(\alpha)) \leq d(\alpha', f^{N+2M}(\alpha)) < d(\alpha', f^{N+M}(\alpha)).$$

This is a contradiction to the fact that $f^{N'}(\alpha)$ was a nearest point projection and we conclude that $\alpha' \in W_+^N$.

A similar argument (using $g_k^{-1} = \phi^{-1} f^{-k}$ instead of g_k and, for any $j \in \mathbb{Z}$, $\phi^{-1} \circ f^{-j}(\alpha)$ and $\phi^{-1}(W_-^j)$ instead of $f^j(\alpha)$ and W_+^j respectively) shows that any fixed point must also satisfy $\alpha' \in \phi^{-1}(W_-^N)$.

This however contradicts the fact that the two sets W_+^N and $\phi^{-1}(W_-^N)$ are disjoint. Thus no power of g_k can have fixed points in $X = \mathcal{C}(\Sigma)$ and, in view of Theorem 4.1, we deduce that g_k is pseudo-Anosov for all sufficiently large k. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Remark 4.1 One could also deduce Proposition 4.1 as a consequence of results of Bestvina–Fujiwara cf. [5, Theorem 1.1]. In particular, for non-sporadic surfaces, they constructed unbounded quasi-morphisms h on $MCG(\Sigma, \partial \Sigma)$ that are bounded on the stabilizer of any essential simple closed curve. Moreover, one can assume that h is **homogeneous** under (positive) powers, so that h then vanishes on all reducible classes. Consider any f so that $h(f) \neq 0$. Then given any reducible class ϕ, it follows from the quasi-morphism property that $h(f^k \phi) \neq 0$ for any sufficiently large k and hence $f^k \phi$ cannot be reducible or of finite order by the properties of h.

4.2 Some hyperbolic geometry

We recall the following theorem on hyperbolic mapping tori due to Thurston [56]:

Theorem 4.2 (Thurston [56]) *Let Σ be a compact, orientable surface with boundary and negative Euler characteristic. If ϕ is a pseudo-Anosov map on Σ, then the interior of the associated mapping torus has a complete hyperbolic structure of finite volume.*

We will also need another result due to Thurston on Dehn fillings of hyperbolic manifolds; an introductory account, as well as a detailed proof, can be found for instance in [41, Chapter 15]. For the readers' ease, we give here a statement of such a theorem which is adapted to the specific setting in which we will apply it.

Let N be an orientable 3-manifold with boundary ∂N a finite union $T_1 \sqcup \cdots \sqcup T_c$ of 2-dimensional tori. For each $i = 1, \ldots, c$, let also m_i, l_i be generators of $\pi_1(T_i)$. For any c-tuple $s = (s_1, \ldots, s_c)$ of *Dehn filling parameters*, i.e. of pairs $s_i = (p_i, q_i)$ of coprime integers, one can consider the compact (boundary-less) 3-manifold N_{fill} obtained by *Dehn filling* the boundary tori with parameters $s = (s_1, \ldots, s_c)$; more explicitly, for each $i = 1, \ldots, c$, a solid torus $P_i := \mathbb{D}^2 \times \mathbb{S}^1$ is glued to N via the (unique up to isotopy) gluing map $\partial P_i \to T_i$ sending a meridian of ∂P_i to a curve in the class $p_i m_i + q_i l_i \in \pi_1(T_i)$.

Theorem 4.3 (Thurston [55]) *In the setting described above, suppose moreover that the interior of N admits a complete hyperbolic metric of finite volume. Then, there is a compact set $K \subset \mathbb{R}^2$ such that, if every Dehn filling parameter s_i is in $\mathbb{R}^2 \setminus K$, the closed 3-manifold N_{fill} obtained by Dehn filling N with parameters $s = (s_1, \ldots, s_c)$ admits a finite-volume complete hyperbolic structure g. Moreover, the cores of the filling solid tori are closed geodesics of (N_{fill}, g_{hyp}).*

Notice that since each s_i is a pair of coprime integers, the theorem implies that one can ensure that a Dehn filling is hyperbolic by excluding finitely many values for *each* slope s_i.

Remark 4.2 Since the fundamental group of a closed hyperbolic manifold is torsion-free and its closed geodesics are all non-contractible, the cores of the Dehn filling tori will have infinite order in $\pi_1(N_{fill})$.

4.3 Proof of the Factorization Lemma

Proof (*Factorization Lemma*) Let f be a pseudo Anosov map on Σ as in Proposition 4.1. According to Proposition 4.1, $f^k\phi$ is pseudo Anosov on Σ for sufficiently large k. We then write $\phi = F \circ G$, where

$$F = f^{-k}, \ G = f^k\phi,$$

where both are pseudo Anosov for $k \gg 0$. By Theorem 4.2, the interiors of the mapping tori associated to (Σ, F) and (Σ, G) carry complete hyperbolic structures.

Let $\gamma_1, \ldots, \gamma_n$ be the components of the boundary $\partial\Sigma$. For each $i = 1, \ldots, n$, we then denote by c_i a curve in Σ which is parallel to γ_i and contained in the interior $\mathring{\Sigma}$; we can assume, up to isotopy, that they are pairwise disjoint. We also denote by $\tau_1, \ldots, \tau_n$ the corresponding right-handed Dehn twists, and $\tau := \tau_1 \ldots \tau_n$.

Observe that $\tau^r = \tau_1^r \ldots \tau_n^r$ for every $r \in \mathbb{Z}$, since the c_i's are disjoint.

Let $\phi_1 := F\tau^r$ and $\phi_2 := \tau^{-r}G$. It is easy to check that the 3-manifolds $OBD(\Sigma, \phi_1)$ and $OBD(\Sigma, \phi_2)$ correspond to Dehn fillings of, respectively, the mapping tori Σ_F and Σ_G with respect to Dehn filling parameters $s(r) = (s_1(r), \ldots, s_n(r))$ and $t(r) = (t_1(r), \ldots, t_n(r))$ such that $|s_i(r)|, |t_i(r)| \to +\infty$ for each $i = 1, \ldots, n$ as $r \to +\infty$. Thus, for sufficiently large r, the hyperbolic Dehn filling Theorem 4.3 implies that $OBD(\Sigma, \phi_1)$ and $OBD(\Sigma, \phi_2)$ carry hyperbolic structures and that the binding components (which coincide with the cores of the Dehn filling tori) are geodesics. In particular the latter have infinite order in the fundamental group (see Remark 4.2). In other words, we have found the desired decomposition $\phi = \phi_1 \circ \phi_2$ as posited in Lemma 4.1. □

5 Proof of tightness in dimension 5

The aim of this section is to prove Theorem A on tightness of the Bourgeois contact structures in dimension 5. For this, we use the following lemma, which is an analogue of the well-known fact that the convex end of a Liouville cobordism with hypertight concave end must be tight [1,28]:

Lemma 5.1 *Suppose the connected components of the bindings of $OBD(\Sigma^2, \phi)$ and $OBD(\Sigma^2, \psi)$ have infinite order in the corresponding fundamental groups. Then, $BO(\Sigma, \phi \circ \psi)$ is tight.*

Proof Let (C, ω_C) be the symplectic cobordism as in Theorem 3.1. According to Observation 2.2, Item 1b of Theorem 3.1 and our hypothesis on $OBD(\Sigma, \phi)$ and $OBD(\Sigma, \psi)$, the Reeb flow of $\lambda_-|_{\partial C_-}$ has no contractible periodic orbits. We now show that this implies that $BO(\Sigma, \phi \circ \psi)$ is tight.

We assume by contradiction that its convex boundary $BO(\Sigma, \phi \circ \psi)$ is overtwisted. According to [6], this implies the existence of an embedded Plastikstufe $\mathcal{PS}$, as defined in [47]. Up to attaching a topologically trivial Liouville cobordism to (C, ω_C) along its positive end, we may then assume that the induced contact form at the positive end is (a positive multiple of) a contact form α_{PS} which is "adapted" to $\mathcal{PS}$, i.e. it has the normal form described in [47, Proposition 4] near its core.

Attaching a cobordism at the negative ends using the local Liouville vector fields associated to λ_-, we obtain the negative Liouville completion $\widehat{C}$ of λ_-, with a symplectic form $\widehat{\omega}_C$ which coincides with $d(e^t \lambda_-)$ at the negative ends. We now apply the following standard argument. Take an $\widehat{\omega}_C$-compatible almost complex structure J, extending the local model of [47], and cylindrical in the cylindrical ends. We have a Bishop family of Fredholm regular J-holomorphic discs in $\widehat{C}$ with totally real boundary, stemming from the core of the Plastikstufe. Analogously to [47, Proposition 10], one can check that the exactness of the symplectic form near the positive end, and hence near the Plastikstufe, provides uniform bounds on the Hofer energy, defined as in [58, Page 115]. Provided that we rule out boundary bubbling, sphere bubbling and the appearance of symplectic caps, by standard bubbling analysis as in [1, 28] we can then obtain a finite energy plane in the negative ends. Note that there is no boundary bubbling, as shown in [47], nor sphere bubbling, by exactness of the symplectic form. We shall need to rule out holomorphic caps, and this can be done as follows (notice that this is not automatic from standard arguments, since $(\widehat{C}, \widehat{\omega}_C)$ is only pseudo-Liouville).

Assume the existence of a J-holomorphic cap c, considered as a map from $\mathbb{C}$ to $\widehat{C}$. The Hofer energy bounds on the Bishop family provide a periodic Reeb orbit of the Reeb flow of λ_-, to which c is negatively asymptotic. Now as γ is nullhomotopic in $\widehat{C}$, projecting to $\mathbb{T}^2$ via the globally defined projection, we see that the image of γ in $\mathbb{T}^2$ is also nullhomotopic. We conclude by Theorem 3.1 that the Reeb orbit γ must be a binding component B_q.

We now argue that such a cap cannot exist. To this end we let $\pi_\delta \colon \widehat{C} \to C$ be a (smooth) map that collapses the ends of the completed cobordism onto the boundary, depends only on the t-coordinate (monotonically) and is the identity on C away from a small δ-neighbourhood of ∂C.

Following [9, Section 6] we consider the ω-energy of a holomorphic plane

$$\mathbf{E}_\omega(c) = \int_{\mathbb{C}} c^* \overline{\omega}_\delta,$$

where $\overline{\omega}_\delta = \pi_\delta^* \omega_C$ is the 2-form which is explicitly given by

$$\overline{\omega}_\delta = \begin{cases} d\lambda_+ & \text{on } [0, \infty) \times \partial_+ C \\ \pi_\delta^* \omega_C & \text{on } C \\ d\lambda_- & \text{on } (-\infty, 0] \times \partial_- C. \end{cases}$$

In fact the resulting forms are all cohomologous relative to the ends for any $\delta > 0$ so that the precise value is irrelevant. Taking a limit as $\delta \searrow 0$, we obtain the (piecewise) description of the ω-energy of [9].

As the almost complex structure is cylindrical on the ends, it is easy to check that $\mathbf{E}_\omega(c)$ is non-negative, and strictly positive if c is not completely contained in an end. In particular, given the assumption that all Reeb orbits have infinite order in the fundamental group of the negative end, we deduce that the holomorphic cap cannot lie completely in the negative cylindrical end, and hence $\mathbf{E}_\omega(c) > 0$.

By Item 2 of Theorem 3.1, there is a primitive ν of ω_C on the initial cobordism that is positive along binding components. We remark that this primitive will not be of the form $d(e^t \lambda_-)$ near the negative end, as the cobordism is only pseudo-Liouville. Pulling back under the map π_δ that collapses each end in turn gives a primitive for $\overline{\omega}_\delta$ that naturally extends to the compactification given by adding $\{\infty\} \times M_\pm$, since it is invariant under positive translations in the t-direction. We denote the resulting primitive for $\overline{\omega}_\delta$ by $\overline{\nu}$.

Then integrating the exact form $\overline{\omega}_\delta = d\overline{\nu}$ along the holomorphic cap c, and using Stokes' theorem, we obtain:

$$0 < \int_c \overline{\omega}_\delta = \int_{-\gamma} \overline{\nu} < 0$$

and this contradiction finishes the proof. $\qquad\qquad\square$

We can now proceed to the proof of Theorem A:

Proof (Theorem A) We start by proving tightness in the "generic" case of non-sporadic page Σ^2. We then deal with sporadic pages on a case by case basis. Lastly, we explain how to deduce universal tightness by the proof of tightness.

Case 1: non-sporadic Σ By Lemma 4.1 we may factorise the monodromy $\phi = \phi_1 \circ \phi_2$, where the components of the bindings in $OBD(\Sigma, \phi_1)$ and $OBD(\Sigma, \phi_2)$ have infinite order. Then, according to Lemma 5.1, we conclude that $BO(\Sigma, \phi)$ is tight.

Case 2: Σ is a disc In this case, the monodromy ϕ is necessarily isotopic to the identity. In other words, the resulting contact 3-manifold is $(\mathbb{S}^3, \xi_{std})$ and the open book structure is the one induced by the subcritical Stein-filling $\mathbb{D}^4$. According to [38, Theorem A.(b)], the associated Bourgeois contact structure is Stein fillable, and hence tight.

Case 3: Σ is an annulus The mapping class group of the annulus is generated by a single positive Dehn twist around the core circle. If the monodromy is a non-negative power of such generator, then the resulting contact 3-manifold is Stein fillable; then, according to [42, Example 1.1], the associated Bourgeois contact structure is weakly fillable, and hence tight. If the power is negative, according to [38, Theorem B], the Bourgeois contact structure associated to $OBD(\Sigma, \phi)$ is contactomorphic to that associated to $OBD(\Sigma, \phi^{-1})$, so we obtain tightness for this case.

Case 4: Σ is a pair of pants For simplicity, enumerate from 1 to 3 the connected components of $\partial\Sigma$. For $i = 1, 2, 3$, let τ_i be a positive Dehn twist along the i-th connected component of $\partial\Sigma$; these give generators for the abelian mapping class group of the pair of pants (cf. [18, Section 3.6.4]). Thus we can write $\phi = \tau_1^{a_1} \circ \tau_2^{a_2} \circ \tau_3^{a_3}$. We then set $\tau := \tau_1 \circ \tau_2 \circ \tau_3$ and, for any $N \in \mathbb{N}_{>0}$, we can decompose ϕ as $\phi = F \circ G$, with $F := \phi \circ \tau^N = \prod_{i=1}^{3} \tau_i^{N+a_i}$ and $G := \tau^{-N} = \prod_{i=1}^{3} \tau_i^{-N}$. We then use the following result:

Lemma 5.2 *If $N > 0$ is big enough, each binding component of $OBD(\Sigma, F)$ is of infinite order in $\pi_1(OBD(\Sigma, F))$. The same is true for $OBD(\Sigma, G)$.*

Using Lemmas 5.1 and 5.2, we conclude that $BO(\Sigma, \phi)$ is tight, as desired. It's then only left to prove Lemma 5.2 in order to conclude the proof of tightness in the case of a pair of pants:

Proof (Lemma 5.2) We deal only with the case of $OBD(\Sigma, F)$; the proof for the manifold $OBD(\Sigma, G)$ is completely analogous.

We first point out that, as explained in detail for instance in [50, Section 3], the manifold $OBD(\Sigma, F)$ can be seen as obtained by Dehn surgery on the total space of $\mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^2$ along three $\mathbb{S}^1$-fibers, with coefficients $r_i := -\frac{1}{N+a_i}$, for each $i = 1, 2, 3$. In other words, $OBD(\Sigma, F)$ is the Seifert manifold

$$\{0, (o_1, 0); (N + a_1, -1), (N + a_2, -1), (N + a_3, -1)\}.$$

Moreover, the orbit space O of the Seifert fibration of $OBD(\Sigma, F)$ is a 2-dimensional orbifold, with underlying topological surface $\mathbb{S}^2$, and the binding B of $OBD(\Sigma, F)$ consists of a union of fibers of the Seifert fibration.

We recall that there is a notion of *orbifold Euler characteristic* χ_{orb} for orbifolds that behaves multiplicatively under finite covers of orbifolds. In our

special case of the base orbifold B of the Seifert fibered space $OBD(\Sigma, F)$, we have

$$\chi_{orb}(O) = \chi(\mathbb{S}^2) - \sum_{i=1}^{3} \left(1 + \frac{1}{N + a_i}\right) = -1 - \sum_{i=1}^{3} \frac{1}{N + a_i}.$$

From now on, let $N > 0$ be so big that $\chi_{orb}(O) < 0$. In particular, $OBD(\Sigma, F)$ is finitely covered by a circle bundle X over a hyperbolic surface S, in such a way that fibers of $X \to S$ are mapped to fibers of $OBD(\Sigma, F) \to O$ (see [52] for instance). Now, S being hyperbolic, the fibers of X are of infinite order in $\pi_1(X)$. As X covers $OBD(\Sigma, F)$ in a compatible way with their Seifert bundle structures, it follows the fibers of $OBD(\Sigma, F)$, hence its binding too, are of infinite order in its fundamental group, as desired.   □

Universal tightness Note that all the above arguments remain valid for the pull-back under any finite cover of $V \times \mathbb{T}^2$ induced by a finite cover of the first factor. Since finite covers over the second factor do not change the contact structure up to contactomorphism (cf. Remark 2.2), such covers also preserve tightness. Now *any* finite cover is itself covered by a composition of covers of the respective factors. Consequently, the contact structure remains tight under any finite cover on the first factor. Since the fundamental group of any closed 3-manifold is residually finite (cf. [27]) so is $\pi_1(V \times \mathbb{T}^2)$ and hence tightness on finite covers is equivalent to tightness on the universal cover of $V \times \mathbb{T}^2$ and universal tightness follows. This concludes the proof of Theorem A.   □

6 Obstructions to symplectic fillability

In this section we describe a general capping construction for $\mathbb{S}^1$-invariant contact structures by applying the handle attachments of Massot-Niederkrüger-Wendl [42], and use it to give new obstructions to symplectic fillability in all dimensions.

6.1 $\mathbb{S}^1$-invariant contact structures in terms of Giroux domains

As discussed in [15, Section 6] any $\mathbb{S}^1$-invariant contact structure induces a decomposition into so called "ideal Liouville domains" as defined in [42, Section 5]. We now describe how this decomposition arises. First of all we recall:

Definition 6.1 *(Giroux [25])* An *ideal Liouville domain* (V, ω) is a domain V with an exact symplectic form ω defined on the interior of W that admits a primitive λ, satisfying the following condition: for each (and hence any)

function $u\colon V \to \mathbb{R}_{\geq 0}$ with regular level set $\partial V = \{u = 0\}$, the form $u\lambda$ extends smoothly to V inducing a contact form on ∂V.

Notice that, by [25, Proposition 2], at the boundary of any ideal Liouville domain (V, ω) there is a contact *structure* ξ which is well defined, i.e. only depending on ω. Moreover, according to [25, Example 8], the complement of the dividing set of a convex hypersurface in a contact manifold is a disjoint union of ideal Liouville domains (the orientation of each of which can either correspond or be opposite to that induced by the ambient orientation and the transverse contact vector field on the hypersurface).

Now, given an ideal Liouville domain (V, ω), one can consider the associated *Giroux domain* as defined in [42, Section 5.3], namely the contact manifold $(V \times \mathbb{S}^1, \ker(udt + u\lambda))$; this is well defined in the sense that, up to isotopy relative boundary, it only depends on ω and not on the function u nor the primitive λ.

We now consider an $\mathbb{S}^1$-invariant contact structure on a product $V \times \mathbb{S}^1$. Note that the hypersurfaces $V \times \{pt\}$ are convex in the sense of Giroux [23], i.e. transverse to the contact vector field ∂_θ, where $\theta \in \mathbb{S}^1$. Now, according to what we pointed out above, this determines a decomposition $V = V_+ \cup \overline{V}_-$ with V_+, V_- ideal Liouville domains (where on V_- the ideal symplectic form is negative), where $V_\pm$ is the set where ∂_θ is positively/negatively tranverse to the contact structure. Moreover, by $\mathbb{S}^1$-invariance, the contact structure on $V \times \mathbb{S}^1$ is just obtained by gluing the two Giroux domains $V_+ \times \mathbb{S}^1$ and $\overline{V}_- \times \mathbb{S}^1$ associated to V_+ and $\overline{V}_-$ respectively along their common boundary $N \times \mathbb{S}^1$, with $N := \partial V_+ = \partial \overline{V}_-$, the dividing set, along which ∂_θ is tangent to the contact structure.

A capping cobordism Let now W^{2n+2} be a hypothetical strong symplectic filling of $V \times \mathbb{S}^1$. The aim of this subsection is to attach a symplectic cobordism on top of W which, near its positive boundary, splits as the product of a symplectic $\mathbb{S}^2$ and portion of the symplectization of ∂V_+, where $V = V_+ \cup \overline{V}_-$ is the decomposition in Liouville domains of the convex hypersurface $\{pt\} \times V$ in the $\mathbb{S}^1$-invariant contact manifold $V \times \mathbb{S}^1 = \partial W$. In order to achieve this, we use the symplectic cobordism described in [42, Section 6].

First, we describe what the cobordism should look like topologically (i.e. without specifying the symplectic form or dealing with details such as corner smoothings), as also done in [42, Section 6.1] in their analogous setting. For this, it is more convenient to see $V \times \mathbb{S}^1$ as obtained as the union $V_+ \times \mathbb{S}^1 \cup [-\delta, \delta] \times N \times \mathbb{S}^1 \cup \overline{V}_- \times \mathbb{S}^1$, as explained above. Now, the cobordism is obtained by attaching $V_\pm \times \mathbb{D}^2_\pm$ on top of $V_\pm \times \mathbb{S}^1 \subset V \times \mathbb{S}^1$ (and smoothing corners), where the latter is seen as the positive boundary of the trivial cobordism $[0, 1] \times V \times \mathbb{S}^1$. The positive boundary component of the resulting cobordism is then just $N \times \mathbb{S}^2$. Notice also that there are distinguished submanifolds $C_\pm :=$

$V_\pm \times \{0\} \subset V_\pm \times \mathbb{D}^2$ in this capping cobordism, with the property that, once removed, the cobordism deformation retracts onto the negative boundary.

Now that the topological picture has been described explicitly, we need to argue that the cobordism has a symplectic structure suited to our needs. In fact, this is precisely the content of [42, Section 6], where the smoothing procedure and the symplectic structure are described with great care. We hence refer the reader to that paper for details, and limit ourselves to stating the properties of the cobordism that we will need below.

Lemma 6.1 *There is a symplectic cobordism (C, ω_C) with negative (i.e. concave) contact boundary $\partial_- C = V \times \mathbb{S}^1$ and weakly convex positive boundary*

$$\partial_+ C = N \times \mathbb{S}^2$$

where $N = \partial V$. Moreover, there is a tubular neighborhood $(-\delta, 0] \times \partial_+ C$ such that ω_C is of the form $d(e^t \alpha_N) + \omega_S$, where $t \in (-\delta, 0]$ and

- *ω_S is an area form on $\mathbb{S}^2$,*
- *α_N is a contact form on N.*

Lastly, there are symplectic submanifolds $C_\pm$, diffeomorphic to $V_\pm$, such that $C \setminus C_\pm$ deformation retracts onto its negative boundary, and such that $C_\pm$ intersect transversely, positively and in exactly one point, each symplectic sphere in the previously described neighborhood of the positive boundary $\partial_+ C$.

In what follows, we denote by W_{cap} the result of stacking (C, ω_C) on top of the strong symplectic filling W along the common boundary $V \times \mathbb{S}^1$.

A moduli space of spheres in the capped filling We now consider a moduli space of pseudo-holomorphic spheres in W_{cap}. The setup and properties needed are essentially the same as those discussed in [42, Section 7.2]. For this reason, we limit ourselves to describing the situation and stating the necessary properties here, and refer the reader to [42] for further details. Note however that in their situation the positive boundary of the capped filling has two connected components, one of contact convex type and the other of stable Hamiltonian type, which leads to a contradiction; in our setting, we instead end up with one boundary component.

Now according to Lemma 6.1, the boundary ∂W_{cap} has a collar neighborhood in W_{cap} so that the symplectic form is split and of the following form:

$$\left((-\delta, 0] \times N \times \mathbb{S}^2, d(e^t \alpha) + \omega_S \right), \text{ where } t \in (-\delta, 0].$$

Hence, one can find an almost complex structure J compatible with the symplectic structure which, on this neighborhood, splits as a direct sum of an almost

complex structure J_S on the 2-sphere and some J_α on the symplectization factor $((-\epsilon, 0] \times N, d(e^t \alpha))$. In particular, for any point $(t, q) \in (-\epsilon, 0] \times N$, the sphere

$$u_{t,q} \colon \mathbb{S}^2 \to (-\epsilon, 0] \times N \times \mathbb{S}^2$$
$$y \mapsto (t, q, y)$$

is J-holomorphic. Moreover, as the co-cores $C_\pm$ from Lemma 6.1 are symplectic submanifolds of W_{cap}, one can choose such a J in such a way that $C_\pm$ are J-invariant in (W_{cap}, J); we hence assume this is the case. Lastly, in the complement of a neighborhood of $\partial W_{cap} \cup C_+ \cup C_-$, the J can then be chosen to be generic, so that every simple holomorphic sphere intersecting said region is regular.

We also have regularity near the boundary, which is proved in [42, Page 334] in a completely analogous setting:

Lemma 6.2 (Fredholm Regularity) *The spheres $u_{t,q}$ are Fredholm regular and have Fredholm index* $\dim(X) = 2n$.

One can then consider the connected component of the moduli space $\mathcal{M}$ of J-holomorphic spheres in W_{cap} containing the pseudo-holomorphic spheres $u_{t,q}$. We let $\mathcal{M}_*$ denote the corresponding marked moduli space on which there is a naturally defined evaluation map $ev \colon \mathcal{M}_* \to W_{cap}$. Lastly, we denote by $\overline{\mathcal{M}}, \overline{\mathcal{M}}_*$ the Gromov compactification of $\mathcal{M}, \mathcal{M}_*$ respectively; abusing notation slightly, we also continue to denote the evaluation map on the compactification by $ev \colon \overline{\mathcal{M}}_* \to W_{cap}$.

Now, we have the following important uniqueness property near the boundary, which can also be proven exactly as in [42, Pages 334 and 335]:

Lemma 6.3 (Local Uniqueness) *For sufficiently small $\epsilon > 0$, any curve u in the moduli space $\overline{\mathcal{M}}$, that intersects the collar $\mathcal{N} = (-\epsilon, 0] \times N \times \mathbb{S}^2$, is a reparametrization of one of the spheres $u_{t,q}$.*

In particular, $\overline{\mathcal{M}}, \overline{\mathcal{M}}_*$ are smooth manifolds near their boundary, and the restriction ev_∂ of $ev \colon \overline{\mathcal{M}}_* \to W_{cap}$ to a (sufficiently small) neighborhood of $\partial \overline{\mathcal{M}}_*$ is a diffeomorphism onto a neighborhood of ∂W_{cap}. We also have a diffeomorphism $\partial \overline{\mathcal{M}} \cong N$. It follows that each of the curves in $\mathcal{M}$ has to be simple, as they intersect each co-core geometrically once by positivity of intersections together with uniqueness near the boundary (as the latter implies that intersections cannot escape at the boundary).

Nodal stratification: Semi-positive case In this case the moduli space $\overline{\mathcal{M}}$ is a stratified space [2] (cf. [45, Section 6.5]):

$$\emptyset = \overline{\mathcal{M}}^{n+1} \subset \overline{\mathcal{M}}^{n} \subset \cdots \subset \overline{\mathcal{M}}^{1} \subset \overline{\mathcal{M}}^{0} = \overline{\mathcal{M}}.$$

Here, $\overline{\mathcal{M}}^{i}$ consists of nodal configurations of spheres which have at least i nodes, and the interior $\text{int}(\mathcal{M}^{i})$, consists of nodal configurations with precisely i nodes. For our choice of J, the top open stratum $\mathcal{M} = \text{int}(\overline{\mathcal{M}}^{0})$ is a smooth $2n$-dimensional manifold consisting of simply covered spheres. The elements of the i-th stratum $\text{int}(\overline{\mathcal{M}}^{i})$ contain *main* sphere components, which intersect at least one of the J-invariant co-cores $C_{\pm}$ at most once and so are again simply covered, while the rest are possibly multiply covered bubbles which intersect no co-cores. By the Uniqueness Lemma 6.3, no element in $\overline{\mathcal{M}}^{i}$ touches the boundary of W_{cap}, for $i \geq 1$.

Similar remarks hold for the marked moduli space, which is a stratified space

$$\emptyset = \overline{\mathcal{M}}_{*}^{n+1} \subset \overline{\mathcal{M}}_{*}^{n} \subset \cdots \subset \overline{\mathcal{M}}_{*}^{1} \subset \overline{\mathcal{M}}_{*}^{0} = \overline{\mathcal{M}}_{*},$$

and the forgetful map respects the stratification.

The semi-positive assumption implies that the dimension of the image of $\overline{\mathcal{M}}_{*}^{i}$ under the evaluation map is *at most* $2n + 2 - 2i$, in the sense of [45, Sec. 6.5]. More precisely, the image under the evaluation map of $\overline{\mathcal{M}}_{*}^{i}$ is covered by the images of underlying moduli spaces of *simple* stable maps, each of which is actually a smooth manifold for generic choice of almost complex structure. With a slight abuse of language, we shall occasionally say that the corresponding unmarked piece $\overline{\mathcal{M}}^{i}$ has dimension at most $2n - 2i$.

Remark 6.1 Since the semipositivity condition is automatically satisfied for every 6-dimensional symplectic manifold (see e.g. [45, Section 6.4]), in this case the evaluation map on the marked moduli space determines a pseudo-cycle on W_{cap} (see e.g. [45, Section 6.5]).

6.2 Proof of Theorem B

We are now ready to give a proof of Theorem B from the Introduction, i.e. of the fact that the inclusion of $N = \partial V_{\pm}$ induces an injection in rational homology

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[2] In this paper, by *stratified space*, we will mean a filtration $\emptyset = \mathcal{M}^{m+1} \subset \mathcal{M}^{m} \subset \cdots \subset \mathcal{M}^{0} = \mathcal{M}$ of a compact topological space $\mathcal{M}$, where the interior $\text{int}(\mathcal{M}^{i})$ of each $\mathcal{M}^{i}$ is called the *i-th stratum*, and the closure of each stratum satisfies $\overline{\mathcal{M}}^{i} = \bigcup_{j \geq i} \mathcal{M}^{j}$.

for classes that survive in $V_\pm$. We prove this first in the semi-positive setting, which uses the more standard language of pseudo-cycles as documented for instance in [45, Section 6.5]. Then, we prove the result in the general case using the polyfold technology developed by Hofer–Wysocki–Zehnder in the Gromov-Witten case in [31].

Proof of Theorem B in the semi-positive case We start by fixing some notation. Let i' be inclusion

$$N = N \times \{pt\} \hookrightarrow \partial W_{cap} = N \times \mathbb{S}^2$$

for pt a point in the equator of $\mathbb{S}^2$. We also let i denote the composition of i' with $\partial W_{cap} \hookrightarrow W_{cap}$. (Notice that $i(N)$ in fact lives in $\partial W \subset W_{cap}$.)

By Lemma 6.1, we have that each element in the moduli space $\overline{\mathcal{M}}$ intersects each co-core $C_\pm$ transversely in a single point. Taking the intersection of a curve with the co-core $C_\pm$ then gives a continuous map $\mathcal{I}^\pm : \overline{\mathcal{M}} \to C_\pm$, which, by uniqueness of the $u_{t,q}$'s near the boundary $N \times \mathbb{S}^2$ of W_{cap}, is a diffeomorphism near $\partial C_\pm$.

Define now the maps $\mathcal{I}_*^\pm$ and $f_\pm$ so that the following diagram is commutative:

$$
\begin{array}{c}
\xymatrix{
& & & i & \\
N \ar[r]^{i'} & N \times \mathbb{S}^2 = \partial W_{cap} \ar[r]^{\quad ev_\partial^{-1}}_{\quad \cong} & \partial\overline{\mathcal{M}}_* \ar@{^{(}->}[r]^{j} & \overline{\mathcal{M}}_* \ar[r]^{ev} & W_{cap} \\
& & & \pi \downarrow \ \ \mathcal{I}_*^\pm & \\
& & f_\pm & \overline{\mathcal{M}} & \\
& & & \mathcal{I}^\pm \downarrow & \\
& & & C_\pm \simeq V_\pm &
}
\end{array}
\tag{4}
$$

Here, j is the natural inclusion $\partial\overline{\mathcal{M}}_* \subset \overline{\mathcal{M}}_*$ and π is the map forgetting the marked point. (Recall also that ev is a diffeomorphism near the boundary, by the local uniqueness Lemma 6.3.) Moreover, by the explicit definitions of i' and $\mathcal{I}^\pm$, the map $f_\pm$ is actually homotopic to the natural inclusion of $N = \partial V_\pm$ into $C_\pm \simeq V_\pm$. We lastly note that, because we are in the semi-positive case, the evaluation map gives a pseudo-cycle representing a relative fundamental class that we denote $[\overline{\mathcal{M}}_*]$.

Recall that the statement of Theorem B claims that the natural composition $V_\pm \simeq V_\pm \times \{pt\} \hookrightarrow \partial W = V \times \mathbb{S}^1 \hookrightarrow W$ induces an injective map on rational homology once restricted to image$((I_\pm)_*)$, where $I_\pm : N = \partial V_\pm \hookrightarrow V_\pm$ is the natural injection, and $(I_\pm)_*$ is the corresponding map in rational homology. In

order to prove this, it is enough (and in fact stronger) to prove in fact that the same conclusion holds for the composition $V_\pm \hookrightarrow W \hookrightarrow W_{cap}$. We hence aim at proving the property for this last map then.

Notice that one can find a deformation retraction of W_{cap} minus slight push-offs of $C_\pm$ onto W which pushes the co-cores $C_\pm$ to $V_\pm \times \{pt\} \subset V \times \mathbb{S}^1 = \partial W$. In particular, in order to prove statement of Theorem B it is enough to prove the following: given a class $z \in H_*(N; \mathbb{Q})$ such that $(f_\pm)_*(z) \neq 0$ in $H_*(C_\pm; \mathbb{Q})$, then $i_*(z) \neq 0$ in $H_*(W_{cap}; \mathbb{Q})$. By looking at its contrapositive, we hence need to prove that, if $i_*(z) = 0$ in $H_*(W_{cap}; \mathbb{Q})$, then $(f_\pm)_*(z) = 0$ in $H_*(C_\pm; \mathbb{Q})$ as well.

Let then $(f_\pm)_*[\sigma] \in H_*(C_\pm; \mathbb{Q})$ be non-trivial, where σ is rational cycle in N, such that $i_*[\sigma] = 0$ in $H_*(W_{cap}; \mathbb{Q})$. We then write $\sigma = \partial b$ as a boundary in W_{cap}.

In the ideal case that all spaces and cycles are actually smooth manifolds one could simply finish the proof as follows. After a small perturbation to ensure transversality, the preimage $ev^{-1}(b)$ in $\overline{\mathcal{M}}_*$ is a chain with boundary $ev^{-1}(\sigma)$. As ev_∂ is a diffeomorphism and σ (seen in W_{cap}) lies in its image, this implies that $(\mathcal{I}_\pm)_*(ev^{-1}(b))$ has boundary σ in C_+, thus proving that $(f_\pm)_*[\sigma] = 0$ in $H_*(C_\pm; \mathbb{Q})$. However, since $\overline{\mathcal{M}}_*$ is not in general a manifold, we need rely on pseudo-cycles, as follows.

We first claim that the map $(ev, \mathcal{I}_*): \mathcal{M}_* \to W_{cap} \times C'_+$ is a $(2n+2)$-dimensional pseudo-cycle. Indeed, the closure of the image of $(ev, \mathcal{I}_*)$ is the image of the compactified moduli space $\overline{\mathcal{M}}_*$, on which the map $(ev, \mathcal{I}_*)$ is continuous. Moreover, the image of the nodal set $\overline{\mathcal{M}}_*^1$ for both maps agrees with the image of the space $\overline{\mathcal{M}}_*^s$ of underlying simple curves. As $\overline{\mathcal{M}}_*^s$ consists of components of dimension $\leq 2n$ by semi-positivity, this gives the claim.

Now, one can perturb b (relative to its boundary) so that $b \times C_+$ becomes weakly transverse to the pseudo-cycle $(ev, \mathcal{I}_*)$, in the sense of [45, Definition 6.5.10]. Strictly speaking, in order to apply their definition to our setting, the cycles b and σ, would need to be represented by embedded submanifolds (up to integer multiples). This will however only be true for (relative) cycles in degree $< n$. In the general case they can be represented, according to a result of Thom [54], by smooth maps from manifolds, possibly with boundary, to the target space. This being said, all the considerations in [45, Section 6.5], including the definition of weakly transverse and the proposition that we use below, carry on with the same proof to this more general setting. To ease the notation, we will however omit the explicit mention of these maps representing σ, b.

According to (a relative version of) [45, Proposition 6.5.17], the restriction of $\mathcal{I}_*$ to $ev^{-1}(b)$ gives a relative pseudo-cycle on C_+. (Here $ev^{-1}(b)$ should be interpreted as the preimage of the diagonal under the map $(b, ev): B \times \mathcal{M}_* \to W \times W$ in the case the the representing map $b: B \to W$ is not a submanifold.)

Moreover, the relative pseudo-cycle $\mathcal{I}_*(ev^{-1}(b))$ is a weak representative of a relative homology class (see [45, Discussion after Lemma 6.5.6]). Finally, as ev is a diffeomorphism near the boundary, the boundary of the relative pseudo-cycle $\mathcal{I}_*(ev^{-1}(b))$ in C_+ is the image of the original homology class $[\sigma]$ via $(f_\pm)_*\colon H_*(\Sigma; \mathbb{Q}) \to H_*(C_+; \mathbb{Q})$. In particular, $(f_\pm)_*[\sigma] = 0$ in $H_*(C_\pm; \mathbb{Q})$, as desired.

The statement about surjectivity at the level of the fundamental group follows, as pointed out to us by a referee, from the fact that bubbling occurs in codimension 2 for semi-positive fillings, again via a push-pull argument. This can be carried out by an argument as in the proof of Lemma 2.3 in Ghiggini-Niederkrüger-Wendl [21], to which we refer the reader for more details. □

We now need to deal with the general setting of possibly non semi-positive fillings. As in the proof in the semi-positive case above, we have that each element in the moduli space $\overline{\mathcal{M}}$ intersects each co-core $C_\pm$ transversely in a single point. This again gives a map $\mathcal{I}\colon \overline{\mathcal{M}} \to C_+$, which, by uniqueness of the $u_{t,q}$'s, is a diffeomorphism near the boundary. However, as we are in the higher dimensional setting, the evaluation map on the marked and compactified moduli space will not automatically be a pseudo-cycle (e.g. in dimensions $2n + 2 \geq 8$). In order to remedy this, we appeal to the polyfold technology of Hofer-Wysocki-Zehnder, which has been worked out in full detail in the Gromov-Witten case (i.e. for closed spheres) in [31]. In particular, before giving the proof, we need to recall the set up of the polyfold perturbation that we will use in order to prove our statement. The argument follows closely that of [42, Section 7.2].

More precisely, we view $\overline{\mathcal{M}}_*$ as lying inside a *Gromov–Witten polyfold* $\mathcal{B}_*$ [31, Section 2.2, Definition 2.29, Section 3.5] consisting of (not necessarily holomorphic) stable nodal configurations of spheres with one marked point and possibly multiple components. It comes with a natural evaluation map $ev : \mathcal{B}_* \to W_{cap}$, which extends the one of $\overline{\mathcal{M}}_*$. We view the nonlinear Cauchy-Riemann operator $\overline{\partial}_J$ as a *Fredholm section* [29, Definition 4.1] of a *strong polyfold bundle* $\mathcal{E}_* \to \mathcal{B}_*$ [31, Definition 2.37, 2.38, Section 3.6] with zero set $\overline{\partial}_J^{-1}(0) = \overline{\mathcal{M}}_*$. We also have a forgetful map $\pi : \mathcal{B}_* \to \mathcal{B}$, where similarly $\mathcal{B}$ is a polyfold containing $\overline{\mathcal{M}}$, given by forgetting the marked point, and a strong polyfold bundle $\mathcal{E} \to \mathcal{B}$ satisfying $\pi^*\mathcal{E} = \mathcal{E}_*$, with a corresponding Fredholm section $\overline{\partial}_J$ making the obvious diagram commute.

Since the subset $\overline{\mathcal{M}}^\partial := \{u_{t,q}\} \subset \overline{\mathcal{M}}$ consisting of the spheres in $(-\epsilon, 0] \times \partial X \times \mathbb{S}^2$ is transversely cut out, the section $\overline{\partial}_J$ is in *good position* [29, Definition 4.12] at the boundary. According to [29, Theorem 4.22], we may introduce an abstract perturbation $\mathbf{p}$, which is a multivalued section of $\mathcal{E}_*$ [29, Definition 3.35, Definition 3.43], so that:

- $\overline{\partial}_J + \mathbf{p}$ is transverse to the zero section of $\mathcal{E}_*$;

- The perturbed moduli space

$$\overline{\mathcal{M}}_*^{\mathbf{p}} = (\bar{\partial}_J + \mathbf{p})^{-1}(0) \subset \mathcal{B}_*$$

  is a $(2n + 2)$-dimensional compact, oriented, weighted branched orbifold with boundary and corners [29, Section 3.2, Definition 3.22]; and
- The perturbation $\mathbf{p}$ is supported away from a closed neighbourhood (in $\mathcal{B}_*$) of $\overline{\mathcal{M}}_*^{\partial} = \pi^{-1}(\overline{\mathcal{M}}^{\partial})$.

We then define $\overline{\mathcal{M}}^{\mathbf{p}} \subset \mathcal{B}$ to be the image of $\overline{\mathcal{M}}_*^{\mathbf{p}}$ under the forgetful map $\pi^{\mathbf{p}} := \pi|_{\overline{\mathcal{M}}_*^{\mathbf{p}}}$, which is a perturbation of $\overline{\mathcal{M}}$. Observe that the last condition implies that a collar neighbourhood of the boundary of $\overline{\mathcal{M}}^{\mathbf{p}}$ still consists of the holomorphic spheres from $\overline{\mathcal{M}}^{\partial}$, and local uniqueness still holds.

Moreover, for $\mathbf{p}$ sufficiently small, every element in $\overline{\mathcal{M}}^{\mathbf{p}}$ will be close to *some* element in $\overline{\mathcal{M}}$, in the Gromov–Hausdorff topology. Hence we can assume that all elements in $\overline{\mathcal{M}}^{\mathbf{p}}$ are transverse to the co-cores and have precisely one intersection point, as this intersection is purely topological for codimension reasons. Thus we obtain a well defined and smooth map $\mathcal{I}^{\mathbf{p}} : \overline{\mathcal{M}}^{\mathbf{p}} \to C'_+$, that agrees with $\mathcal{I}$ on $\overline{\mathcal{M}}^{\partial}$.

We are now in place to prove Theorem B in the non semi-positive case:

Proof of Theorem B in the general case Fix any sufficiently small abstract perturbation $\mathbf{p}$ as described above, and consider the map $\mathcal{I}_{\pm}^{\mathbf{p}} : \overline{\mathcal{M}}^{\mathbf{p}} \to C_{\pm}$. Recall also that in this setting one has a well defined notion of integrating over the moduli space: we refer to [30,31] for further details. For our purposes it suffices to use the fact that there is a notion of *sc-smooth differential forms* [30, Definition 1.8] in $\overline{\mathcal{M}}_*^{\mathbf{p}}$, such that Stokes' theorem holds [30, Theorem 1.11], as well as a *de-Rham cohomology group* $H_{dR}^*(\overline{\mathcal{M}}_*^{\mathbf{p}})$ [30, page 10], such that said differential forms can be pulled back under the evaluation map. In our setting, one then has a well-defined notion of degree of a map, which agrees with the usual notion at the boundary of the moduli space, where smoothness is built into the construction. We then formally define (relative) homology

$$H_*(\overline{\mathcal{M}}_*^{\mathbf{p}}, \partial\overline{\mathcal{M}}_*^{\mathbf{p}}) := \left(H_{dR}^*(\overline{\mathcal{M}}_*^{\mathbf{p}}) \right)^*,$$

as the dual (over $\mathbb{R}$) of the de-Rham cohomology group, with the obvious notions of push-forwards, i.e. as duals of pull-backs.

This allows us to define a shriek map on homology (with real coefficients). Formally one does this by defining maps on the duals of the associated de Rham cohomologies via the integration map, as a map $ev^! : H_*(W_{cap}, \partial W_{cap}) \to$

$H_*(\overline{\mathcal{M}}_*^{\mathbf{p}}, \partial\overline{\mathcal{M}}_*^{\mathbf{p}})$ given via the equation:

$$ev^!(a)(\beta) = \int_{\overline{\mathcal{M}}_*^{\mathbf{p}}} ev^* PD(a) \wedge \beta,$$

where we have first used Poincaré–Lefschetz duality in the target.

The remainder of the argument then goes through exactly as in the semi-positive setting and the theorem follows in the general case. More precisely, consider a rational cycle σ in N that is sent to 0 via the map induced in homology by the inclusion $i\colon N \to W'_{cap}$ as defined in Eq. (4). Consider $\overline{\mathcal{M}}_*^{\mathbf{p}}$, together with the perturbed intersection map $(\mathcal{I}_\pm^{\mathbf{p}})_*$. Then, if $i_*[\sigma]$ is boundary of a homology class b in W_{cap}, one can take its shriek $ev^!(b)$, seen as an element in the dual $(H^*_{dR}(\overline{\mathcal{M}}_*^{\mathbf{p}}))^*$, and push it, via the map induced in the dual of the cohomology by $\mathcal{I}_*^{\mathbf{p}}$, to a class c in the dual $(H^*_{dR}(C_\pm))^*$. By Poincaré duality on $C_\pm$, we now have that c can be seen as a class in $H_*(C_\pm; \mathbb{Q})$. Moreover, as ev at the boundary is actually a smooth map and, moreover, a diffeomorphism, we have $\partial c = (f_\pm)_*(\sigma)$. But this means that $(f_\pm)_*[\sigma] = 0$ in $H_*(C_\pm; \mathbb{Q})$, thus concluding the proof as in the semi-positive case. □

Remark 6.2 As an alternative to the above argument, one could appeal to the following. According to [32, Remark 15.8] (cf. also [44]) any compact oriented weighted branched orbifold admits a rational relative fundamental class. This then suffices to replicate the argument from the semi-positive case just using continuity of the maps, without the need to talk about sc-smoothness. Note, however, that our definition of relative homology bypasses the existence of this relative fundamental class.

6.3 The case of Bourgeois structures

Let $BO(\Sigma, \phi)$ be the Bourgeois manifold obtained from the abstract contact open book (Σ, ϕ). Denote more precisely $\beta = \alpha + \Phi_1 dx - \Phi_2 dy$ the contact form on $M \times \mathbb{T}^2$ as in Section 2, where (x, y) are coordinates on $\mathbb{T}^2$, $\Phi = (\Phi_1, \Phi_2)\colon M \to \mathbb{R}^2$ is a (properly chosen) function describing the open book decomposition $M = OBD(\Sigma, \phi)$ and α is a contact form on M supported by it.

Now, the vector field ∂_y corresponding to the second factor of $\mathbb{T}^2 = \mathbb{S}_x^1 \times \mathbb{S}_y^1$ of $M \times \mathbb{T}^2$ is contact. In particular, as described more generally in Section 6.1 for $\mathbb{S}^1$-equivariant contact structures, each hypersurface $M \times \mathbb{S}_x^1 \times \{y_0\}$ is convex, and has hence a natural splitting as $V_+ \cup \overline{V}_-$ as union of ideal Liouville domains. In fact, $V_\pm$ are smoothly just given by the product of half of the open book $OBD(\Sigma, \phi)$ with $\mathbb{S}_x^1$, i.e. $\Sigma \times D^*\mathbb{S}_x^1$ with corners smoothened. The Bourgeois contact manifold is then nothing else than the gluing of the two

Giroux domains $V_+ \times \mathbb{S}^1_y$ and $V_- \times \mathbb{S}^1_y$ along their boundary. (This decomposition is also explicitly described in [15, Section 5.3], without the use of this terminology.)

Remark 6.3 Since Bourgeois contact structures are $\mathbb{T}^2$-invariant and we are free to choose any coordinate system (x, y) on the $\mathbb{T}^2$ factor, we can assume that $\mathbb{S}^1_x$ represents any primitive homology class in $H_1(\mathbb{T}^2, \mathbb{Z})$. This observation will be needed below in proving part of Theorem C, namely that $\mathbb{T}^2$ injects in integer homology in the filling W.

We are now ready to prove Theorem C.

Proof of Theorem C Since the inclusion of $\Sigma \subseteq \partial V_+ = N$ into $V_+ \cong \Sigma \times D^*\mathbb{S}^1$ is homotopic to the inclusion of a fiber, it is injective on homology and the injectivity of the map induced by $\Sigma \hookrightarrow W$ on rational homology follows directly from Theorem B. Likewise, in the case of semi-positive filling, surjectivity at the level of fundamental groups of the inclusion $\Sigma \times \mathbb{T}^2 \hookrightarrow W$ also follows directly from Theorem B.

Furthermore, Theorem B in fact also implies that the inclusion of the $\mathbb{S}^1_x$-factor is injective in rational homology. Now taking $\mathbb{S}^1_x$ to represent an arbitrary (primitive) class in homology (c. f. Remark 6.3), we can then conclude that the whole first homology group $H_1(\mathbb{T}^2; \mathbb{Q})$ injects in $H_1(W; \mathbb{Q})$. Furthermore, the statement that $H_*(\mathbb{T}^2; \mathbb{Q})$ injects into $H_*(W; \mathbb{Q})$ is equivalent to the dual statement that the inclusion induces a surjection on cohomology $H^*(W, \mathbb{Q}) \to H^*(\mathbb{T}^2; \mathbb{Q})$ by the Universal Coefficient Theorem. Using the cup-product structure on $H^*(\mathbb{T}^2; \mathbb{Q})$, this follows from the injectivity in H_1, concluding the proof.   □

6.4 Applications to fillability

We now prove those corollaries of Theorem C that have been stated without proof in the Introduction. We start with Corollary E on the equivalence of strong symplectic and Stein fillability for Bourgeois manifolds associated to rational homology 3-spheres:

Proof of Corollary E If $V = OBD(\Sigma, \phi)$ is a rational homology 3-sphere, any inclusion of Σ in the 3-manifold V as a page of the given open book is null homologous with coefficients $\mathbb{Q}$. In particular, Theorem C tells that $BO(\Sigma, \phi)$ cannot be strongly fillable whenever $H_1(\Sigma; \mathbb{Q})$ is non-trivial, i.e. whenever Σ is not a disc. On the other hand, we know by [38, Theorem A.(b)] that $BO(\mathbb{D}^2, \mathrm{Id})$ is Stein fillable.   □

We now prove Theorem F, which states that Bourgeois 5-manifolds associated to 3-dimensional open books with planar pages and monodromies given

by products of Dehn twists all of the same sign are weakly but not strongly symplectically fillable.

Proof of Theorem F By [38, Theorems A.(a) and B] these contact structures are weakly fillable and, hence, it suffices to prove that these are not strongly fillable. According to Theorem C, it is hence enough to prove that the first Betti number of the manifold $b_1(OBD(\Sigma, \phi))$ is strictly smaller than that of the page $b_1(\Sigma) = \#\partial\Sigma - 1$. This can be seen as follows. Since for $\Sigma = \mathbb{D}^2$, the monodromy is necessarily trivial, we may assume that $\#\partial\Sigma \geq 2$. We can then attach $b_1(\Sigma) - 1 \geq 0$ discs to Σ, in order to cap all but 2 of its boundary components; as ϕ is a product of Dehn twist of all the same signs, this can moreover be done in such a way that the resulting monodromy $\widehat{\phi}$ in the resulting annular page $\widehat{\Sigma} = D^*\mathbb{S}^1$ is a power $\tau^{\pm K}$ of the positive Dehn twist τ, with $K > 0$.

Moreover, this 2-dimensional 2-handle attachment on the page can be realized by attaching a 4-dimensional 2-handle on the boundary component $OBD(\Sigma, \phi) \times \{1\}$ of a thickening $OBD(\Sigma, \phi) \times [0, 1]$, along neighborhoods, of the form $\mathbb{S}^1 \times \mathbb{D}^2$, of all but 2 binding components. The resulting positive boundary of these $b_1(\Sigma) - 1$ handle attachments is just $OBD(D^*\mathbb{S}^1, \tau^{\pm K})$ where τ is a Dehn twist and $K > 0$ as above. This manifold is then a Lens space and hence has $b_1 = 0$. Since attaching a 4-dimensional 2-handle reduces the first Betti number at the boundary by at most one, we deduce that $b_1(OBD(\Sigma, \phi)) \leq b_1(\Sigma) - 1$, as desired. □

Now, we give a proof of Corollary G, i.e. that if Σ is a planar surface and $BO(\Sigma, \phi)$ is strongly fillable, then ϕ lies in the commutator subgroup of the relative mapping class group:

Proof of Corollary G We identify the pure braid group with the mapping class group of the punctured surface obtained by collapsing all but one boundary component of Σ to a puncture. It is well known that abelianization of the pure braid group on n-strands is the free abelian group generated by $\binom{n-1}{2}$ Dehn twists around pairs of punctures, where $n = \#\partial\Sigma$ is the number of boundary components (see for instance [18, Chapter 9 and page 264]).

Also, the kernel of the map on mapping class groups induced by the collapsing map described above is generated by the n boundary parallel Dehn twists (see [18, Theorem 3.18 and Proposition 3.19]). Hence the abelianization of $MCG(\Sigma, \partial\Sigma)$ is generated by n boundary parallel Dehn twists and $\binom{n-1}{2}$ Dehn twists about pairs of boundary components different from a distinguished one; in total, it is hence generated by $\binom{n}{2} = n + \binom{n-1}{2}$ Dehn twists and has rank at most $\binom{n}{2}$.

Denote by $\widehat{\Sigma}_{i,j}$ the annulus obtained by capping off all but 2 boundary components γ_i, γ_j of Σ. Doing so for each pair of i, j with $i \neq j$, one obtains

a homomorphism

$$MCG(\Sigma, \partial\Sigma) \to \prod_{i \neq j} MCG(\widehat{\Sigma}_{i,j}, \gamma_i \cup \gamma_j) = \mathbb{Z}^{\binom{n}{2}}, \tag{5}$$

given by extending the element in the mapping class group of Σ relative $\partial\Sigma$ as Id over the caps. It is easily seen that this homomorphism is surjective. Moreover, by the explicit description of the $\binom{n}{2}$ generators for the abelianization of $MCG(\Sigma, \partial\Sigma)$ above, it induces an injective homomorphism after taking the abelianization. In particular, the map in Eq. (5) induces an isomorphism after abelianizing.

Hence, if the monodromy is not in the commutator subgroup, it is mapped to a non-trivial element in $\mathbb{Z}^{\binom{n}{2}}$, and so it is non-trivial in $MCG(\widehat{\Sigma}_{i_0, j_0}, \gamma_{i_0} \cup \gamma_{j_0}) \cong \mathbb{Z}$ after capping off all but 2 well-chosen boundary components $\gamma_{i_0}, \gamma_{j_0}$ of the page. We then deduce as in the proof of Theorem F that $b_1(OBD(\Sigma, \phi)) < b_1(\Sigma)$, and we conclude the argument by appealing to Theorem C. □

We now prove Corollary H about the fillability of Bourgeois structures arising from open books which are positive stabilizations.

Proof of Corollary H The abstract positive stabilization $OBD(\Sigma_+, \phi_+)$ of a contact open book $OBD(\Sigma, \phi) = (M, \xi)$ can be interpreted as the open book on $M \# \mathbb{S}^{2n+1}$ given by the Murasugi sum of $OBD(\Sigma, \phi)$ on M with the open book $OBD(D^*\mathbb{S}^n, \tau) = (\mathbb{S}^{2n+1}, \xi_{std})$, where τ is the positive Dehn-Seidel twist. More precisely, from the explicit construction of Murasugi sum (see for instance [12, Proposition 2.6]), one can see that the additional Lagrangian sphere S created in the stabilized page Σ_+ is actually contained in the contact ball $\mathbb{D}^{2n+1} = \mathbb{S}^{2n+1} \backslash D$, where D is the small Darboux ball taken out of the $\mathbb{S}^{2n+1}$ factor to build the connected sum $M \# \mathbb{S}^{2n+1}$. Hence, S is null-homologous in $M \# \mathbb{S}^{2n+1} = M$. In particular, as S is not null-homologous (by construction) in Σ_+, Theorem C gives that $BO(\Sigma_+, \phi_+)$ is not strongly fillable. □

We now give a proof of Theorem K on the fillability of $BO(D^*\mathbb{S}^n, \tau^k)$, of which Theorem J in the case $k = 1$ is an immediate consequence:

Proof of Theorem K It follows from Theorem 3.1 and [38, Theorems A.(b) and B] that $BOFill(n)$ is a subgroup of $\mathbb{Z}$. It is hence necessarily cyclic.

According to Theorem C, in order to determine the generator of $BOFill(n) \leq \mathbb{Z}$, it then suffices to check for which $k \geq 1$ the homology of the page injects in $\Sigma_{k,n} := OBD(D^*\mathbb{S}^n, \tau^k)$. According to [37, Proposition 4.10], for $n, k \geq 1$ the manifold $\Sigma_{k,n}$ is actually the $(2n + 1)$-dimensional Brieskorn manifold

$\Sigma(2,\dots,2,k)$. An explicit computation via Mayer–Vietoris, using the Heegaard splitting associated to the open book, gives the following: if n is odd, then $H_n(\Sigma_{k,n};\mathbb{Z}) = \mathbb{Z}_k$ for every k and so it vanishes over $\mathbb{Q}$; and $H_n(\Sigma_{k,n};\mathbb{Z}) = 0$ if n is even, and $k \geq 1$ is odd. We then see that $k_0(n)$ is even. Our method is inconclusive for n and $k \geq 2$ both even, since we have $H_n(\Sigma_{k,n};\mathbb{Q}) = \mathbb{Q}$, generated by the zero section of the page. Indeed, if k is such that τ^k is smoothly isotopic to Id, then $\Sigma_{k,n} \cong \mathbb{S}^n \times \mathbb{S}^{n+1} = \Sigma_{0,n}$ as smooth manifolds, hence the conclusion of Theorem C clearly holds, and we obtain no obstruction (according to [34], cf. [14,36], for arbitrary even $n \geq 2$, τ is known to have finite order as a smooth map). Alternatively, for odd n, one may observe that τ^k acts non-trivially in homology if $k \neq 0$ (by the Picard–Lefschetz formula), thus Remark 1.1 allows to conclude; however, τ^2 is homologically trivial if n is even, so this argument is inconclusive as well. □

7 Symplectically aspherical fillings of Bourgeois Contact Manifolds

Let W be a symplectically aspherical strong symplectic filling of $BO(\Sigma, \phi)$. The goal of this section is to prove Theorem D and Theorem L from the Introduction.

The symplectic manifold W_{cap} resulting from the handle attachment described in Lemma 6.1 then has boundary of the form $N \times \mathbb{S}^2$, where N is the contact boundary of $\Sigma \times D^*\mathbb{S}^1_x$ (smoothened at the corners). More precisely, in a neighborhood $(-\delta, 0] \times N \times \mathbb{S}^2$ of the boundary, the symplectic structure has normal form $d(e^t \alpha) + \omega_S$, with α a contact form on N and ω_S an area form on $\mathbb{S}^2$.

We thus obtain a moduli space $\mathcal{M}$ consisting of holomorphic spheres inside W_{cap}, arising from the $\mathbb{S}^2$-factor at the boundary. Under the asphericity assumption it turns out that there are no nodal degenerations in the Gromov compactification and we thus obtain:

Proposition 7.1 $\mathcal{M}$ *is a compact oriented smooth manifold (with boundary).*

Proof As the filling W is symplectically aspherical, there cannot be any nodal curve in $\overline{\mathcal{M}}$ which has one closed sphere component not intersecting either of the co-cores. Hence, the only nodal degeneration that could a priori occur is one separating a regular sphere in the moduli space in multiple spheres, each intersecting at least one co-core. However, this would imply, after puncturing the spheres at their intersection points with the co-cores, that one of the $\mathbb{S}^1$-factors in the $\mathbb{T}^2$-factor of $\partial W = OBD(\Sigma, \phi) \times \mathbb{T}^2$ is null-homologous in W. (Here, we use the fact that removing the co-cores from W_{cap} results in a manifold which deformation retracts onto W.) This would then contradict Theorem C, so that no nodal degenerations are possible, as desired. □

A moduli space of punctured curves As in Section 6, we can consider the marked moduli space $\mathcal{M}_*$ obtained by adding a marked point to the domains, which comes equipped with an evaluation map $ev : \mathcal{M}_* \to W_{cap}$, and a forgetful map $\pi : \mathcal{M}_* \to \mathcal{M}$.

We now consider a punctured version of the compact marked moduli space $\mathcal{M}_*$. More precisely, as each sphere in $\mathcal{M}$ intersects the co-cores $C_{\pm}$ positively and transversely in exactly one point, we may remove from each sphere in $\mathcal{M}_*$ small disc-like neighborhoods around its intersection with the two co-cores of Lemma 6.1. This results in a (compact) marked moduli space $\mathcal{M}_*^c$ of (compact) cylinders. Moreover, this has a natural fibration structure π over $\mathcal{M}^c := \mathcal{M}$ induced by the forgetful map $\mathcal{M}_* \to \mathcal{M}$, and an evaluation map $ev \colon \mathcal{M}_*^c \to W$, where we see here W_{cap} minus a small tubular neighborhood of the co-cores as naturally diffeomorphic to W. Because of the foliation property of the moduli space of spheres $\mathcal{M}_*$ near ∂W_{cap}, the evaluation map of the cylindrical counterpart $\mathcal{M}_*^c$ is a diffeomorphism from a neighborhood of the vertical boundary $\partial_v \mathcal{M}_*^c := \pi^{-1}(\partial \mathcal{M})$ to a neighborhood of $N \times D^* \mathbb{S}^1 \subset BO(\Sigma, \phi) = \partial W$ inside W, where N is the boundary of $V_{\pm} = \Sigma \times D^* \mathbb{S}_x^1$ with corners smoothened as in the description in the beginning of Section 6.3. Hence, the evaluation map has in particular degree 1.

What is more, it also sends the horizontal boundary $\partial_h \mathcal{M}_*^c := \overline{\partial \mathcal{M}_*^c \setminus \partial_v \mathcal{M}_*^c}$ into a small neighbourhood of the co-cores minus the co-cores themselves, i.e. into $N_\delta(C_{\pm}) \setminus C_{\pm}$. In particular, each cylinder in $\mathcal{M}_*^c$ has boundary components which have a natural sign $\pm$ according to where they are mapped to via ev; this gives a partition $\partial_h \mathcal{M}_*^c = \partial_h^+ \mathcal{M}_*^c \sqcup \partial_h^- \mathcal{M}_*^c$. We further note that the image of each of these components retracts onto the piece $V_{\pm} \times \mathbb{S}^1$ of $BO(\Sigma, \phi) = \partial W$ in the decomposition induced by the $\mathbb{S}^1$-invariant perspective (cf. Section 6.1).

Homology of aspherical fillings We now prove that the inclusion of each $V_{\pm} \times \mathbb{S}^1 = \Sigma \times D^* \mathbb{S}^1 \times \mathbb{S}^1 \simeq \Sigma \times \mathbb{S}^1 \times \mathbb{S}^1$ into an aspherical filling induces an isomorphism in integral homology.

Proof of Theorem D Recall that Lefschetz duality states that the cap product with the fundamental class living in the top degree homology relative to the boundary induces an isomorphism between relative cohomology groups and homology groups (with proper degree shifting). Hence, since the evaluation map ev of $\mathcal{M}_*^c$ has degree 1, naturality of the cap product implies that it induces a surjection on homology with $\mathbb{Z}$-coefficients. Next, note that the inclusion $\partial_h^{\pm} \mathcal{M}_*^c \to \mathcal{M}_*^c$ induces a homotopy equivalence, as each cylinder can be smoothly collapsed to either of its boundary components. Now, the image of $\partial_h^{\pm} \mathcal{M}_*^c$ under ev can be homotoped into $V_{\pm} \times \mathbb{S}^1$. Then, one can do the following: given any homology class in W, one can pull it back to the moduli space, homotope it to the positive boundary $\partial_h^+ \mathcal{M}^c$, and finally map it near

$V_{\pm} \times \mathbb{S}^1$ via ev. We hence obtain surjectivity of the inclusion $V_{\pm} \times \mathbb{S}^1 \hookrightarrow W$ as stated in Theorem D.

For injectivity, we note that the inclusion of $\Sigma \times \partial_+(D^*\mathbb{S}^1_x) \times \mathbb{S}^1_y \subseteq \partial V_+ \times \mathbb{S}^1_y$ into $V_+ \times \mathbb{S}^1_y$ is a homotopy equivalence, where $\partial_+(D^*\mathbb{S}^1)$ denotes the positive boundary component of $D^*\mathbb{S}^1$. Hence any non-trivial homology class $[c] \neq 0$ in $V_+ \times \mathbb{S}^1$ is represented by a class in $\partial V_+ \times \mathbb{S}^1$, along which the evaluation map is a diffeomorphism. Thus if $c = \partial b$ were the boundary of a singular chain, we could pull back, after perturbing the evaluation map (relative to the horizontal boundary), and then push the resulting chain $\bar{b} = ev^{-1}b$ into $\partial_h^+ \mathcal{M}_*^c$. Pushing forward by ev we would obtain that c is a boundary of $ev_*\bar{b}$ in $V_+ \times \mathbb{S}^1$, which is a contradiction. The case of V_- is completely analogous.

Finally observe that the inclusion of $\Sigma \times \partial_+(D^*\mathbb{S}^1_x) \times \mathbb{S}^1_y$ into ∂W is precisely the inclusion of $\Sigma \times \mathbb{T}^2 \hookrightarrow OBD(\Sigma, \phi) \times \mathbb{T}^2$, concluding the proof. $\quad\square$

7.1 Symplectically aspherical fillings of $S^*\mathbb{T}^n$

In this section, we consider $(S^*\mathbb{T}^n, \xi_{std})$, the unit cotangent bundle of $\mathbb{T}^n$, for $n \geq 2$, with its standard Stein fillable contact structure, and we prove Theorem L from the Introduction.

The Bourgeois contact manifold given by $BO(D^*\mathbb{T}^{n-2}, \mathrm{Id})$ is contacto-morphic to the contact boundary of $D^*\mathbb{T}^{n-2} \times D^*\mathbb{T}^2$ by [38, Theorem A.(b)], which can in turn be identified with the boundary of the unit cotangent bundle of the n-torus namely $(S^*\mathbb{T}^n, \xi_{st})$. In particular, ξ_{std} is an $\mathbb{S}^1$-invariant contact structure, with respect to the second $\mathbb{S}^1$-factor of the Bourgeois torus $\mathbb{T}^2$.

Symplectically aspherical fillings of $S^*\mathbb{T}^n$: homotopy type Recall that $S^*\mathbb{T}^n = BO(D^*\mathbb{T}^{n-2}, \mathrm{Id})$. According to the $\mathbb{S}^1$-invariant picture as in Section 6.1, this gives a decomposition

$$S^*\mathbb{T}^n = V_+ \times \mathbb{S}^1 \cup \overline{V}_- \times \mathbb{S}^1,$$

where $V_{\pm}$ are smoothly given by $D^*\mathbb{T}^{n-2} \times D^*\mathbb{S}^1$, i.e. $D^*\mathbb{T}^{n-1}$ (up to rounding corners). Denote then by $j_0 \colon \mathbb{T}^n \hookrightarrow S^*\mathbb{T}^n$ the inclusion given by the composition of inclusions $\mathbb{T}^n \hookrightarrow D^*\mathbb{T}^{n-1} \times \mathbb{S}^1 \hookrightarrow S^*\mathbb{T}^n$, where the first injection is just induced by the zero section $\mathbb{T}^{n-1} \to D^*\mathbb{T}^{n-1}$.

We next use the following result in order to conclude that the given aspherical filling (W, ω) is homotopy equivalent to $D^*\mathbb{T}^n$.

Proposition 7.2 *Let* $j \colon \mathbb{T}^n \hookrightarrow W$ *be given by the composition of j_0 defined above and the natural inclusion $S^*\mathbb{T}^n = \partial W$ into W. Consider also a lift $\widetilde{j}$ of j to the universal covers $\mathbb{R}^n$ and $\widetilde{W}$. Then:*

1. $H_0(\widetilde{W}; \mathbb{Z}) = \mathbb{Z}$ *and* $H_k(\widetilde{W}; \mathbb{Z}) = \{0\}$ *for* $k > 0$,

2. $j_*\colon \pi_1(\mathbb{T}^n) \to \pi_1(W)$ *is an isomorphism.*

According to [26, Section 4.2, Exercise 12]), this then implies that the inclusion j, and hence the inclusion of its (trivial) normal bundle $D^*\mathbb{T}^n$, induces a homotopy equivalence, as desired.

Remark 7.1 We point out that, as opposed to the simply connected case, it is not in general true that a map inducing an isomorphism on fundamental group and isomorphisms on homology is a homotopy equivalence. This is the reason why we need to consider the universal cover in Proposition 7.2.

Proof (Proposition 7.2) We consider the moduli space of punctured curves and its evaluation map $\mathcal{M}^c_* \to W$ considered as in the proof of Theorem D. Since the evaluation map ev of $\mathcal{M}^c_*$ has degree 1, it induces a surjection on π_1. Moreover, recall that the moduli space retracts onto its positive boundary, which is mapped onto a neighborhood of the positive component, that in this setting is just $D^*\mathbb{T}^{n-1} \times \{+1\} \times \mathbb{S}^1$. Arguing as in the proof of Theorem D, one concludes that $j\colon \mathbb{T}^n \hookrightarrow W$ also induces a surjection on fundamental group. In particular, we deduce that the fundamental group of the filling is abelian. The fact that j is H_*-injective follows from Theorem D; as $\pi_1(\mathbb{T}^n)$ and $j_*(\pi_1(\mathbb{T}^n)) = \pi_1(W)$ are abelian, this immediately implies that j is also π_1-injective.

The only thing left to prove is hence that the map $\widetilde{j}$ on the universal cover also induces an isomorphism in H_*. Injectivity follows trivially from the fact that the universal cover of $\mathbb{T}^n$ is contractible. We then prove surjectivity.

In order to do so, we first make the following observations. All holomorphic curves in $\mathcal{M}^c$ naturally lift to maps from the plane to the universal cover $\widetilde{W}$ of W. This gives a corresponding smooth moduli space $\widetilde{\mathcal{M}}$ of infinite holomorphic strips in $\widetilde{W}$, together with its marked version $\widetilde{\mathcal{M}}_*$ equipped with an evaluation map $\widetilde{ev}\colon \widetilde{\mathcal{M}}_* \to \widetilde{W}$. One can check that the map $\widetilde{ev}$ is proper and has degree 1 (using cohomology with compact supports), since the diffeomorphism on the vertical boundary $\partial_v \mathcal{M}^c_*$ of $\mathcal{M}^c_*$ induced by ev lifts to a diffeomorphism from a subset of the boundary of $\widetilde{\mathcal{M}}_*$ to a subset of the boundary of $\widetilde{W}$. Notice now that $\widetilde{\mathcal{M}}_*$ deformation retracts onto its positive horizontal boundary $\partial_h^+ \widetilde{\mathcal{M}}_*$, defined as the lift of $\partial_h^+ \mathcal{M}^c_*$, i.e. as the union of all the lifts of the positive boundaries of the cylinders in $\mathcal{M}^c_*$. As j is an isomorphism in π_1, using standard covering space arguments, one can check that this set is mapped, via the lifted evaluation map, to a lift $D^*\mathbb{R}^{n-1} \times \mathbb{R}$ of the piece $D^*\mathbb{T}^{n-1} \times \mathbb{S}^1$ of $\partial W \subset W$ to $\widetilde{W}$.

Using this setup, we can now prove that any homology class in $\widetilde{W}$ comes from $D^*\mathbb{R}^{n-1} \times \mathbb{R}$, which then concludes the proof of Proposition 7.2. More precisely, consider a class $x \in H_k(\widetilde{W}, \mathbb{Z})$, for $k \geq 1$. (The case $k = 0$ simply follows from the fact that $\widetilde{W}$ is connected.) As $\widetilde{\mathcal{M}}_*$ is an orientable manifold (with boundary) and $\widetilde{ev}$ has degree 1, there is a well defined $\widetilde{ev}^!(x)$ in $H_*(\widetilde{\mathcal{M}}_*)$

such that $\widetilde{ev}_* \widetilde{ev}^!(x) = x$, where $\widetilde{ev}^!$ is given, geometrically, by perturbing $\widetilde{ev}$ to be transverse to a cycle representing x and taking its preimage. Now, as $\widetilde{\mathcal{M}}_*$ retracts onto $\partial_h^+ \widetilde{\mathcal{M}}_*$, $\widetilde{ev}^!(x)$ can be homotoped to $\partial_h^+ \widetilde{\mathcal{M}}_*$. In particular, $x = \widetilde{ev}_* \widetilde{ev}^!(x)$ is homologous in $\widetilde{W}$ to a cycle σ in $\widetilde{ev}(\partial_h^+ \widetilde{\mathcal{M}}_*) = D^* \mathbb{R}^{n-1} \times \mathbb{R}$. As the latter is contractible, $[\sigma]$, and so x, is null-homologous in $\widetilde{W}$, as desired.
□

Symplectically aspherical fillings of $S^* \mathbb{T}^n$: diffeomorphism type Once the homotopy type is understood, the diffeomorphism type can be determined using the s-cobordism theorem. The argument below is just an adaptation of [3, Sections 5 and 8] to our setting. We thus give a sketch of the proof, referring to the proofs of the technical statements in [3]; for the readers' ease, we also adopt their notations.

We start by describing the spaces involved in the argument. Let W_1 be the result of attaching a topologically trivial cobordism $[0, 1] \times S^* \mathbb{T}^n$ to W along its boundary $M_0 := S^* \mathbb{T}^n = \{0\} \times S^* \mathbb{T}^n$.

Recall from the previous subsection that we have a natural inclusion $j_0 : \mathbb{T}^n \to \partial W = S^* \mathbb{T}^n$. As $j_0(\mathbb{T}^n)$ has trivial normal bundle in ∂W, there is a (smooth) copy of $W_0 := D^* \mathbb{T}^n$ which is entirely contained in the cobordism $[0, 1] \times S^* \mathbb{T}^n$. Let then $X := W_1 \setminus W_0$ and $M_1 := \partial W_1$; notice that $\partial X = M_1 \cup (-M_0)$. The aim is now to prove that X is diffeomorphic to a cylinder $[0, 1] \times S^* \mathbb{T}^n$, so that W_1 is actually diffeomorphic to W_0, as desired.

As explained in [3, Lemmas 5.1 and 5.2], the fact that W is homotopy equivalent to $D^* \mathbb{T}^n$ implies that the inclusions $M_0, M_1 \hookrightarrow X$ induce isomorphisms on π_1 and on H_*. Moreover, as $S^* \mathbb{T}^n$ is a *simple space* (i.e. the action of its π_1 on every homotopy group is trivial), arguing exactly as in [3, Lemmas 8.1 and 8.2] one can show that $M_0, M_1 \hookrightarrow X$ actually induce isomorphisms on all homotopy groups. This proves that X is an h-cobordism between M_0 and M_1.

Now, as $M_0 = S^* \mathbb{T}^n$, the Whitehead group $\mathrm{Wh}(\pi_1(M_0))$ vanishes, so that the Whitehead torsion of the inclusion $M_0 \hookrightarrow X$ is necessarily zero. The s-cobordism theorem then tells that X is diffeomorphic to $[0, 1] \times S^* \mathbb{T}^n$, as desired.

8 Further discussion and open questions

Our results are, together with [38], among the first steps in understanding the nature of the contact structures given by Bourgeois' construction, and several open questions remain. Firstly:

Question 8.1 Are Bourgeois contact structures tight in *all* odd dimensions? Moreover, is every Bourgeois contact structure *weakly* fillable, at least in dimension 5?

It is an important problem to understand more precisely the dependence of the Bourgeois structure on the starting open book decomposition. By a direct consequence of their definition, all Bourgeois contact structures are contact deformations of the almost contact structure $\xi_V \oplus T\mathbb{T}^2$ (i.e. the endpoint η_1 of a path $(\eta_t)_{t \in [0,1]}$ of hyperplane fields starting at $\eta_0 = \xi_V \oplus T\mathbb{T}^2$ and such that η_t is contact for $t > 0$). One can then construct, as in [42, Example 1.1], weak cobordisms between $BO(\Sigma, \phi)$ and $BO(\Sigma', \phi')$ for any $OBD(\Sigma', \phi')$ and $OBD(\Sigma, \phi)$ supporting the same contact structure. What's more, besides sharing the formal homotopy class, Theorem A in this paper shows in particular that the tight vs overtwisted classification type of any 5-dimensional $BO(\Sigma, \phi)$ is independent of the open book.

On the other hand, in [7, Corollaries 10.6 and 10.8], Bourgeois used cylindrical contact homology, with respect to noncontractible homotopy classes of Reeb orbits, in order to distinguish infinitely many Bourgeois contact manifolds arising from open books supporting the standard contact structure on $\mathbb{S}^3$; and similarly for $\mathbb{T}^3$. Further instances of different open books supporting the same contact structure that induce non-contactomorphic Bourgeois contact manifolds can be found in [38, Example 1.5]; in the same spirit, other examples also come from Theorem C proved above.

Question 8.2 Can we find further contactomorphisms of Bourgeois contact manifolds, beyond the inversion of the monodromy from [38]? More ambitiously, can we classify the contactomorphism type of all the Bourgeois contact manifolds arising from some fixed contact structure, especially via rigid holomorphic curves invariants?

In general, Theorem C imposes strong constraints on the monodromy. This suggests the following:

Question 8.3 If $BO(\Sigma, \phi)$ is strongly fillable, is ϕ (at least smoothly) trivial?

References

1. Albers, P., Hofer, H.: On the Weinstein conjecture in higher dimensions. Comment. Math. Helv. **84**(2), 429–436 (2009). https://doi.org/10.4171/CMH/167
2. Avdek, R.: Liouville hypersurfaces and connect sum cobordisms. J. Symplectic Geom. **19**(4), 865–957 (2021). https://doi.org/10.4310/JSG.2021.v19.n4.a2
3. Barth, K., Geiges, H., Zehmisch, K.: The diffeomorphism type of symplectic fillings. J. Symplectic Geom. **17**(4), 929–971 (2019). https://doi.org/10.4310/JSG.2019.v17.n4.a1
4. Bestvina, M., Fujiwara, K.: Bounded cohomology of subgroups of mapping class groups. Geom. Topol. **6**, 69–89 (2002). https://doi.org/10.2140/gt.2002.6.69
5. Bestvina, M., Fujiwara, K.: Quasi-homomorphisms on mapping class groups. Glas. Mat. Ser. III **42(62)**(1), 213–236 (2007). https://doi.org/10.3336/gm.42.1.15
6. Borman, M., Eliashberg, Y., Murphy, E.: Existence and classification of overtwisted contact structures in all dimensions. Acta Math. **215**(2), 281–361 (2015). https://doi.org/10.1007/s11511-016-0134-4
7. Bourgeois, F.: A Morse-Bott approach to contact homology. ProQuest LLC, Ann Arbor, MI (2002). Thesis (Ph.D.)–Stanford University
8. Bourgeois, F.: Odd dimensional tori are contact manifolds. Int. Math. Res. Not. **30**, 1571–1574 (2002). https://doi.org/10.1155/S1073792802205048
9. Bourgeois, F., Eliashberg, Y., Hofer, H., Wysocki, K., Zehnder, E.: Compactness results in symplectic field theory. Geom. Topol. **7**, 799–888 (2003). https://doi.org/10.2140/gt.2003.7.799
10. Bowden, J., Crowley, D., Stipsicz, A.I.: Contact structures on $M \times S^2$. Math. Ann. **358**(1–2), 351–359 (2014). https://doi.org/10.1007/s00208-013-0963-9
11. Bridson, M.R., Haefliger, A.: Metric spaces of non-positive curvature, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. **319**. Springer-Verlag, Berlin (1999). https://doi.org/10.1007/978-3-662-12494-9
12. Casals, R., Murphy, E.: Contact topology from the loose viewpoint. In: Proceedings of the Gökova Geometry-Topology Conference 2015, pp. 81–115. Gökova Geometry/Topology Conference (GGT), Gökova (2016)
13. Casals, R., Murphy, E., Presas, F.: Geometric criteria for overtwistedness. J. Amer. Math. Soc. **32**(2), 563–604 (2019). https://doi.org/10.1090/jams/917
14. Cerf, J.: La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudo-isotopie. Inst. Hautes Études Sci. Publ. Math. **39**, 5–173 (1970)
15. Ding, F., Geiges, H.: Contact structures on principal circle bundles. Bull. Lond. Math. Soc. **44**(6), 1189–1202 (2012). https://doi.org/10.1112/blms/bds042
16. Eliashberg, Y.: Classification of overtwisted contact structures on 3-manifolds. Invent. Math. **98**(3), 623–637 (1989). https://doi.org/10.1007/BF01393840

17. Eliashberg, Y.: Filling by holomorphic discs and its applications. In: Geometry of low-dimensional manifolds, 2 (Durham, 1989), London Math. Soc. Lecture Note Ser., vol. **151**, pp. 45–67. Cambridge Univ. Press, Cambridge (1990)

18. Farb, B., Margalit, D.: A primer on mapping class groups, Princeton Mathematical Series, vol. 49. Princeton University Press, Princeton, NJ (2012)

19. Geiges, H.: An introduction to contact topology, Cambridge Studies in Advanced Mathematics, vol. 109. Cambridge University Press, Cambridge (2008). https://doi.org/10.1017/CBO9780511611438 . (**An introduction to contact topology, Cambridge Studies in Advanced Mathematics**)

20. Geiges, H., Kwon, M., Zehmisch, K.: Diffeomorphism type of symplectic fillings of unit cotangent bundles. arXiv e-prints arXiv:1909.13586, To appear in J. Topol, Anal (2019)

21. Ghiggini, P., Niederkrüger, K., Wendl, C.: Subcritical contact surgeries and the topology of symplectic fillings. J. Éc. polytech. Math. **3**, 163–208 (2016). https://doi.org/10.5802/jep.31

22. Gironella, F.: On some examples and constructions of contact manifolds. Math. Ann. (2019). https://doi.org/10.1007/s00208-019-01895-4

23. Giroux, E.: Convexité en topologie de contact. Comment. Math. Helv. **66**(4), 637–677 (1991). https://doi.org/10.1007/BF02566670

24. Giroux, E.: Géométrie de contact: de la dimension trois vers les dimensions supérieures. In: Proceedings of the International Congress of Mathematicians, Vol. **II** (Beijing, 2002), pp. 405–414. Higher Ed. Press, Beijing (2002)

25. Giroux, E.: Ideal Liouville domains, a cool gadget. J. Symplectic Geom. **18**(3), 769–790 (2020). https://doi.org/10.4310/JSG.2020.v18.n3.a5

26. Hatcher, A.: Algebraic topology. Cambridge University Press, Cambridge (2002)

27. Hempel, J.: Residual finiteness for 3-manifolds. In: Combinatorial group theory and topology (Alta, Utah, 1984), Ann. of Math. Stud., vol. **111**, pp. 379–396. Princeton Univ. Press, Princeton, NJ (1987)

28. Hofer, H.: Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three. Invent. Math. **114**(3), 515–563 (1993). https://doi.org/10.1007/BF01232679

29. Hofer, H., Wysocki, K., Zehnder, E.: A general Fredholm theory. III. Fredholm functors and polyfolds. Geom. Topol. **13**(4), 2279–2387 (2009). https://doi.org/10.2140/gt.2009.13.2279

30. Hofer, H., Wysocki, K., Zehnder, E.: Integration theory on the zero sets of polyfold Fredholm sections. Math. Ann. **346**(1), 139–198 (2010). https://doi.org/10.1007/s00208-009-0393-x

31. Hofer, H., Wysocki, K., Zehnder, E.: Applications of polyfold theory I: The polyfolds of Gromov-Witten theory. Mem. Amer. Math. Soc. **248**(1179), v+218 (2017). https://doi.org/10.1090/memo/1179

32. Hofer, H., Wysocki, K., Zehnder, E.: Polyfold and Fredholm Theory. arXiv e-prints arXiv:1707.08941 (2017)

33. Huang, Y.: On plastikstufe, bordered Legendrian open book and overtwisted contact structures. J. Topol. **10**(3), 720–743 (2017). https://doi.org/10.1112/topo.12020

34. Keating, A., Randal-Williams, O.: On the order of dehn twists. ArXiv Mathematics e-prints 2104.13154 (2021)

35. Klukas, M.: Open books and exact symplectic cobordisms. Internat. J. Math. **29**(4), 1850026, 19 (2018). https://doi.org/10.1142/S0129167X1850026X

36. Krylov, N.A.: Relative mapping class group of the trivial and the tangent disk bundles over the sphere. Pure Appl. Math. Q. **3**(3, Special Issue: In honor of Leon Simon. Part 2), 631–645 (2007). https://doi.org/10.4310/PAMQ.2007.v3.n3.a1

37. Kwon, M., van Koert, O.: Brieskorn manifolds in contact topology. Bull. Lond. Math. Soc. **48**(2), 173–241 (2016). https://doi.org/10.1112/blms/bdv088

38. Lisi, S., Marinković, A., Niederkrüger, K.: On properties of Bourgeois contact structures. Algebr. Geom. Topol. **19**(7), 3409–3451 (2019). https://doi.org/10.2140/agt.2019.19.3409
39. Lisi, S., Van Horn-Morris, J., Wendl, C.: On symplectic fillings of spinal open book decompositions I: Geometric constructions. arXiv e-prints arXiv:1810.12017 (2018)
40. Lutz, R.: Sur la géométrie des structures de contact invariantes. Ann. Inst. Fourier (Grenoble) **29**(1), xvii, 283–306 (1979). http://www.numdam.org/item?id=AIF_1979__29_1_283_0
41. Martelli, B.: An Introduction to Geometric Topology. arXiv e-prints arXiv:1610.02592 (2016)
42. Massot, P., Niederkrüger, K., Wendl, C.: Weak and strong fillability of higher dimensional contact manifolds. Invent. Math. **192**(2), 287–373 (2013). https://doi.org/10.1007/s00222-012-0412-5
43. Masur, H.A., Minsky, Y.N.: Geometry of the complex of curves. I. Hyperbolicity. Invent. Math. **138**(1), 103–149 (1999). https://doi.org/10.1007/s002220050343
44. McDuff, D.: Groupoids, branched manifolds and multisections. J. Symplectic Geom. **4**(3), 259–315 (2006). http://doi.org/euclid.jsg/1180135649
45. McDuff, D., Salamon, D.: J-holomorphic curves and symplectic topology, American Mathematical Society Colloquium Publications, vol. **52**, second edn. American Mathematical Society, Providence, RI (2012)
46. Moreno, A.: Algebraic Torsion in higher-dimensional contact manifolds. PhD thesis, HU-Berlin edoc server, https://doi.org/10.18452/19849 (2017)
47. Niederkrüger, K.: The plastikstufe–a generalization of the overtwisted disk to higher dimensions. Algebr. Geom. Topol. **6**, 2473–2508 (2006). https://doi.org/10.2140/agt.2006.6.2473
48. Niederkrüger, K.: Higher dimensional contact topology via holomorphic disks. In: Contact and symplectic topology, Bolyai Soc. Math. Stud., vol. **26**, pp. 173–244. János Bolyai Math. Soc., Budapest (2014). https://doi.org/10.1007/978-3-319-02036-5_5
49. Niederkrüger, K., Presas, F.: Some remarks on the size of tubular neighborhoods in contact topology and fillability. Geom. Topol. **14**(2), 719–754 (2010). https://doi.org/10.2140/gt.2010.14.719
50. Ozbagci, B.: Explicit horizontal open books on some Seifert fibered 3-manifolds. Topology Appl. **154**(4), 908–916 (2007). https://doi.org/10.1016/j.topol.2006.10.004
51. Presas, F.: A class of non-fillable contact structures. Geom. Topol. **11**, 2203–2225 (2007). https://doi.org/10.2140/gt.2007.11.2203
52. Scott, P.: The geometries of 3-manifolds. Bull. London Math. Soc. **15**(5), 401–487 (1983). https://doi.org/10.1112/blms/15.5.401
53. Stipsicz, A.I.: Gauge theory and Stein fillings of certain 3-manifolds. Turkish J. Math. **26**(1), 115–130 (2002)
54. Thom, R.: Quelques propriétés globales des variétés différentiables. Comment. Math. Helv. **28**, 17–86 (1954). https://doi.org/10.1007/BF02566923
55. Thurston, W.P.: Three-dimensional geometry and topology. Vol. **1**, Princeton Mathematical Series, vol. **35**. Princeton University Press, Princeton, NJ (1997). Edited by Silvio Levy
56. Thurston, W.P.: Hyperbolic Structures on 3-manifolds, II: Surface groups and 3-manifolds which fiber over the circle. arXiv Mathematics e-prints math/9801045 (1998)
57. Wendl, C.: Strongly fillable contact manifolds and J-holomorphic foliations. Duke Math. J. **151**(3), 337–384 (2010). https://doi.org/10.1215/00127094-2010-001
58. Wendl, C.: Lectures on symplectic field theory. arXiv e-prints arXiv:1612.01009 (2016)