Correction to: A polyhedron comparison theorem for 3-manifolds with positive scalar curvature

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Published online: 8 February 2022
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Correction to: Invent. math. (2020) 219:1–37
<https://doi.org/10.1007/s00222-019-00895-0>

We use the notations from [Li20] in this erratum. There is a mistake in the proof of Lemma 3.1 of [Li20], affecting Theorem 1.4 and Theorem 1.5 of the paper. Indeed, the assumption that $\angle(F_{j,\infty}, F_{j+1,\infty}) < \angle(F'_j, F'_{j+1})$ does not guarantee that $T_p M$ can be placed strictly within the tangent cone of P at its vertex. It turns out that Lemma 3.1 requires assumptions (1.2) and (1.3), which we state here. Let $P \subset \mathbf{R}^3$ be a cone type polyhedron with a vertex p' , side faces F'_j and base face B' , and γ_j be the dihedral angle between F'_j and B' . In the sequel we allow triangles to be degenerate, namely two sides of a triangle can be parallel—in which case the angle between those two sides is zero.

$$\gamma_j \leq \pi/2, j = 1, 2, \dots, k. \quad (1.2)$$

The online version of the original article can be found under <https://doi.org/10.1007/s00222-019-00895-0>.

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On B' , there exists a vertex A' such that the following holds: let A'_1, A'_2 be the adjacent vertices of A' . Then for all sides L' of B' not containing A' , the line through L' forms a triangle (or a degenerate triangle) Δ with the rays given by $\overrightarrow{A'A'_1}$ and $\overrightarrow{A'A'_2}$, such that inner angles of Δ , except the one at A' , are not larger than $\pi/2$. (1.3)

Remark 0.1 Assumption 1.3 is satisfied by all triangles. In general, it is equivalent to the existence of three consecutive vertices A'_1, A', A'_2 , at which the inner angles satisfy:

$$\angle A'_1 \leq \frac{\pi}{2}, \quad \angle A'_2 \leq \frac{\pi}{2}, \quad \frac{\pi}{2} \leq \angle A' + \angle A'_1 \leq \pi, \quad \frac{\pi}{2} \leq \angle A' + \angle A'_2 \leq \pi.$$

Lemma 3.1 Let $P \subset \mathbf{R}^3$ be a polyhedron of cone type, (M, g) be of P -type. Assume P satisfies (1.2) and (1.3), and $\angle_{ij}(M) < \angle_{ij}(P)$. Then the infimum $\mathcal{I}$ appeared in (1.4) is negative.

Proof Consider the tangent cone of M at the vertex p , and the side faces $F_{j,\infty}$. Denote by F'_j the side faces of P . In the base face of P , take the vertex A' satisfying (1.3), and without loss of generality assume A' lies on the intersection of F'_1, F'_2 . Take $A \in F_{1,\infty} \cap F_{2,\infty}$, $A \neq p$. Take a plane $\pi \subset \mathbf{R}^3$ through A , such that the dihedral angle between π and $F_{j,\infty}$ equals γ_j , $j = 1, 2$ (such a plane π exists since M satisfies assumption (1.1)). Denote by $E_\infty \subset T_p M$ the compact closed region separated by π . We will prove that for any $l \geq 3$, the dihedral angle between $F_{l,\infty}$ and π is strictly greater than γ_l , and thus

$$\mathcal{H}^2(\pi \cap \partial E_\infty) - \sum_{j=1}^k (\cos \gamma_j) \mathcal{H}^2(F_{j,\infty} \cap \partial E_\infty) < 0,$$

finishing the proof.

For simplicity of notations we prove the statement for $l = 3$. (In general, take a face $F_{j,\infty}$ and consider the cone formed by the planes through $F_{1,\infty}, F_{2,\infty}, F_{j,\infty}$, and compare to the cone formed by the planes through F'_1, F'_2, F'_j with exactly the same argument.) We fix some notations. Denote by $B = F_{1,\infty} \cap \pi \cap F_{3,\infty}$, $C = F_{2,\infty} \cap \pi \cap F_{3,\infty}$, ν the unit normal vector of π pointing to p . For $j = 1, 2, 3$, denote by ν_j the unit normal vector of $F_{j,\infty}$ pointing out of $T_p M$, $\beta_j = \angle(\nu, \nu_j)$ (thus $\beta_1 = \gamma_1$, $\beta_2 = \gamma_2$), and $\alpha_1 = \angle(\overrightarrow{AB}, \overrightarrow{AC})$, $\alpha_2 = \angle(\overrightarrow{BA}, \overrightarrow{BC})$, $\alpha_3 = \angle(\overrightarrow{CA}, \overrightarrow{CB})$. Denote by θ the dihedral angle between $F_{1,\infty}, F_{2,\infty}$, and θ_j the dihedral angle between $F_{3,\infty}$ and $F_{j,\infty}$, $j = 1, 2$. We then denote the corresponding geometric quantities

in P by v', v'_1 , etc. In particular, since the side faces of P meet the base face at γ_j , we have $\beta'_j = \gamma_j$, $j = 1, 2, 3$.

By the spherical law of cosines, applied to the spherical triangle cut out by E_∞ from a small 2-sphere centered at A , we have

$$\begin{aligned}\cos \alpha_1 \sin \beta_1 \sin \beta_2 &= (v \times v_1) \cdot (v_2 \times v) = (v \cdot v_2)(v_1 \cdot v) - v_1 \cdot v_2 \\ &= \cos \beta_1 \cos \beta_2 + \cos \theta.\end{aligned}\quad (0.1)$$

And the similar equation holds on P . Since $\beta_1 = \beta'_1$, $\beta_2 = \beta'_2$, $\theta < \theta'$, we have $\alpha_1 < \alpha'_1$. As $\sum_{j=1}^3 \alpha_j = \sum_{j=1}^3 \alpha'_j = \pi$, we have either $\alpha_2 > \alpha'_2$ or $\alpha_3 > \alpha'_3$. Assume that $\alpha_2 > \alpha'_2$. Analogous to (0.1), we have

$$\begin{aligned}\cos \alpha_2 \sin \beta_2 \sin \beta_3 &= \cos \beta_2 \cos \beta_3 + \cos \theta_1, \\ \cos \alpha'_2 \sin \beta'_2 \sin \beta'_3 &= \cos \beta'_2 \cos \beta'_3 + \cos \theta'_1.\end{aligned}$$

By assumption, $\theta_1 < \theta'_1$. Hence

$$\cos \alpha_2 \sin \beta_2 \sin \beta_3 - \cos \beta_2 \cos \beta_3 > \cos \alpha'_2 \sin \beta'_2 \sin \beta'_3 - \cos \beta'_2 \cos \beta'_3. \quad (0.2)$$

Note that $\cos \beta'_2 = \cos \beta_2 \geq 0$ (by (1.2)). We claim that (0.2) implies that $\beta_3 > \beta'_3$ as follows. Suppose the contrary, that $\beta_3 \leq \beta'_3$. Then $\cos \beta_2 \cos \beta_3 \geq \cos \beta'_2 \cos \beta'_3$. Thus (0.2) implies that $\cos \alpha_2 \sin \beta_3 > \cos \alpha'_2 \sin \beta'_3$. However, (1.3) implies $\alpha'_2 \leq \pi/2 \Rightarrow \cos \alpha'_2 \geq 0$. This is a contradiction. $\square$

We now reformulate the main theorems of the paper. Other steps of their proofs are unchanged.

Theorem 1.4 *Let (M^3, g) be a Riemannian polyhedron of P -type with side faces $F_1, \dots, F_k$, where $P \subset \mathbf{R}^3$ is a cone or prism with side faces $F'_1, \dots, F'_k$. Denote by γ_j the angle between F'_j and the base face of P . Assume that everywhere along $F_j \cap F_{j+1}$,*

$$|\pi - (\gamma_j + \gamma_{j+1})| < \angle(F_j, F_{j+1}), \quad (1.1)$$

and when P is a cone, P satisfies (1.2) and (1.3).

Then the strict comparison statement holds for (M, g) . Namely, if $R(g) \geq 0$, and each F_j is mean convex, then the dihedral angles of M cannot be everywhere less than those of P .

Theorem 1.5 *Under the same assumptions of Theorem 1.4 and the extra assumption (when P is a prism) that*

$$\gamma_j \leq \pi/2, j = 1, 2, \dots, k, \quad \text{or} \quad \gamma_j \geq \pi/2, j = 1, 2, \dots, k,$$

then we have the rigidity statement. Namely, if $R(g) \geq 0$, each F_j is mean convex, and $\angle_{ij}(M, g) \leq \angle_{ij}(P, g_{Euclid})$, then (M, g) is isometric to a flat polyhedron in $\mathbf{R}^3$.

Acknowledgements The author wishes to thank Bernhard Hanke for pointing out the mistake in Lemma 3.1.

Reference

- [Li20] Li, C.: A polyhedron comparison theorem for 3-manifolds with positive scalar curvature. *Invent. Math.* **219**, 1–37 (2020)

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