Correction to: Supersingular K3 surfaces are unirational

Christian Liedtke¹

Published online: 1 February 2021
© Springer-Verlag GmbH Germany, part of Springer Nature 2021

Correction to: Invent. math. (2015) 200:979–1014
<https://doi.org/10.1007/s00222-014-0547-7>

As was pointed out by Bragg and Lieblich in [1], the proof of [2, Proposition 3.5] contains a mistake, which has the effect that the main result, the unirationality of supersingular K3 surfaces [2, Theorem 5.3], remains a conjecture.

Degree p multisections

More precisely, the proof only shows the following:

Proposition 0.1 *We keep the notations and assumptions of [2, Proposition 3.2] and assume that $\widehat{\mathrm{Br}}(X)$ is not p -divisible. Set $R := k[[t]]$ and $S := \mathrm{Spec} R$, and let*

$$\overline{A} \rightarrow Y \times_k S \rightarrow S$$

The original article can be found online at <https://doi.org/10.1007/s00222-014-0547-7>.

✉ Christian Liedtke
liedtke@ma.tum.de

¹ TU München, Zentrum Mathematik, M11, Boltzmannstr. 3, 85748 Garching bei München, Germany

be a family of A -torsors associated to a non-trivial p -torsion element of $\widehat{\mathrm{Br}}(X)(R)$ as in [2, Proposition 3.2]. Then, after possibly replacing S by some finite flat cover

- (1) *There exists a degree p multisection $D_\xi \subset \overline{A}_\xi$ such that the induced morphism*

$$D_\xi \rightarrow \xi \times_k S$$

is finite and flat of degree p .

- (2) *More precisely, $D_\xi \rightarrow \xi \times_k S$ arises from a family of $A_\xi[p]$ -torsors.*

In particular, contrary to what was claimed in [2, Proposition 3.5], one may not be able to choose the degree p multisection D_ξ to be radicial, that is, purely inseparable, over $\xi \times_k S$.

The strategy of establishing [2, Theorem 5.3] consisted in showing that any two supersingular K3 surfaces in characteristic $p \geq 5$ can be connected by one-dimensional families of the type as constructed in [2, Section 3] and then, that unirationality is preserved in these families. This reduced the unirationality conjecture to the showing that there exists at least one unirational K3 surface in each characteristic, where one can use Kummer surfaces, whose unirationality was established by Shioda [3].

Now, in order to show that unirationality is preserved in these one-dimensional families, one needs Assertion (4) of [2, Theorem 3.6], whose proof requires that one can find degree p multisections in [2, Proposition 3.5] that are radicial over the base. Without this, one can only conclude that any two supersingular K3 surfaces in characteristic $p \geq 5$ are connected by a sequence of very special degree p correspondences (as in Assertion (3) of [2, Theorem 3.6]), which is not sufficient for unirationality.

Fibration structure

Moreover, the proof of [2, Theorem 4.3] contains a mistake. In fact, the proof only shows the following:

Theorem 0.2 *Let N and N_+ be the supersingular K3 lattices in odd characteristic p of Artin-invariants σ_0 and $\sigma_0 + 1$, respectively. Then, there exists a rank 20 lattice Λ , and isometries*

$$N \cong U \oplus \Lambda \quad \text{and} \quad N_+ \cong U(p) \oplus \Lambda,$$

where U denotes the hyperbolic plane. This decomposition gives rise to a dominant rational map ϖ_N of moduli spaces of rigidified K3 crystals with a section σ_N

$$\begin{array}{c} \mathcal{M}_{N+1} \\ \sigma_N \uparrow \downarrow \varpi_N \\ \mathcal{M}_N \end{array}$$

More precisely,

- (1) *this map is defined on the open set of $\mathcal{M}_{N+1}$ of rigidified K3 crystals with Artin invariant $\sigma_0 + 1$ and maps surjectively onto the open set of rigidified K3 crystals of Artin invariant σ_0 .*
- (2) *The normalization of the geometric generic fibre is $\mathbb{P}^1$. (In general, ϖ_N may not be generically smooth.)*

In the proof, the map is only defined on geometric points and extension to a morphism would require more arguments. In particular, the computations in Step 6 of the proof only show that the normalisation of the geometric generic fibre is isomorphic to $\mathbb{P}^1$.

In particular, the table [2, Remark 4.4] has to be corrected as follows: for $\sigma_0(N) = 3$, the moduli space $\mathcal{M}_N$ consists geometrically of two copies of the Fermat hypersurface of degree $(p + 1)$ in $\mathbb{P}^3$. In this case, ϖ_N arises geometrically by choosing a pencil of planes through a line of this surface and then, one obtains a fibration over $\mathbb{P}^1$, whose geometric generic fibre is a singular rational curve that is not smooth. This gives an example, where the geometric generic fibre of ϖ_N is not smooth, and in particular, not isomorphic to $\mathbb{P}^1$.

References

1. Bragg, D., Lieblich, M.: Perfect points on genus one curves and consequences for supersingular K3 surfaces. [arXiv:1904.04803](https://arxiv.org/abs/1904.04803) (2019)
2. Liedtke, C.: Supersingular K3 surfaces are unirational. *Invent. math.* **200**, 979–1014 (2015)
3. Shioda, T.: Some results on unirationality of algebraic surfaces. *Math. Ann.* **230**, 153–168 (1977)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.