Correction to: Depth functions of symbolic powers of homogeneous ideals

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The original proof of Theorem 3.3 incorrectly claims that $\overline{I^{(t)}} = \bigcap_{F \in \mathcal{F}(I)} \overline{I_F^t}$. We have found a counter-example to this claim. The proof remains correct if we replace $\overline{I^{(t)}}$ by $\bigcap_{F \in \mathcal{F}(I)} \overline{I_F^t}$ for all $t \geq 1$; see the corrected proof below. The correction concerns only this proof and does not affect any result of the paper.

Theorem 3.3 *Let I be a monomial ideal in R such that $I^{(t)}$ is integrally closed for $t \gg 0$. Then $\text{depth } R/I^{(t)}$ is a convergent function with*

$$\lim_{t \rightarrow \infty} \text{depth } R/I^{(t)} = \dim R - \dim F_s(I),$$

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which is also the minimum of depth $R/I^{(t)}$ among all integrally closed symbolic powers $I^{(t)}$.

Proof Let m be the minimum of depth $R/I^{(t)}$ among all integrally closed symbolic powers $I^{(t)}$. Choose an integrally closed symbolic power $I^{(s)}$ such that $\text{depth } R/I^{(s)} = m$. By Theorem 2.6(ii), there exists an integer a such that $\text{depth } R/I^{(s)} \geq \text{depth } R/I^{(t)}$ for $t \geq as^2$. This implies $\text{depth } R/I^{(t)} = m$ for all integrally closed symbolic powers $I^{(t)}$ with $t \geq as^2$. Since $I^{(t)}$ is integrally closed for $t \gg 0$, we get $\text{depth } R/I^{(t)} = m$ for $t \gg 0$.

Let $I_t = \bigcap_{F \in \mathcal{F}(I)} \overline{I_F^t}$. We will show that $m = \min_{t \geq 1} \text{depth } R/I_t$. Since $I_t = I^{(t)}$ for $t \gg 0$ by Proposition 2.2,

$$m \geq \min_{t \geq 1} \text{depth } R/I_t.$$

By Proposition 2.3, we have

$$\text{depth } R/I_s \geq \text{depth } R/(I_s)^{(t)}$$

for all $s, t \geq 1$. For $t \gg 0$, $I^{(st)}$ is integrally closed and so is I_F^{st} for all $F \in \mathcal{F}(I)$ by Proposition 2.2. This implies $I_F^{st} \subseteq (\overline{I_F^s})^t \subseteq \overline{I_F^{st}} = I_F^{st}$. Hence, $(\overline{I_F^s})^t = I_F^{st}$. So we get

$$(I_s)^{(t)} = \bigcap_{F \in \mathcal{F}(I)} I_F^{st} = I^{(st)}.$$

Therefore,

$$\text{depth } R/I_s \geq \text{depth } R/I^{(st)} \geq m$$

for all $s \geq 1$. Now, we can conclude that

$$m = \min_{t \geq 1} \text{depth } R/I_t.$$

It remains to show that $\min_{t \geq 1} \text{depth } R/I_t = \dim R - \dim F_s(I)$. For that, we need the following auxiliary observation (cf. [41, Proposition 2.5]).

Let $\mathcal{F}$ denote the filtration of the ideals I_t , $t \geq 0$. Let $R(\mathcal{F}) = \bigoplus_{t \geq 0} I_t y^t$. Then $R(\mathcal{F})$ is an algebra generated by monomials in $k[x_1, \dots, x_n, y]$. We have

$$R(\mathcal{F}) = \bigcap_{F \in \mathcal{F}(I)} \bigoplus_{t \geq 0} \overline{I_F^t} y^t.$$

For each $F \in \mathcal{F}(I)$, the algebra $\bigoplus_{t \geq 0} \overline{I_F^t} y^t$ is the normalization of the finitely generated algebra $\bigoplus_{t \geq 0} I_F^t y^t$. Hence, $\bigoplus_{t \geq 0} \overline{I_F^t} y^t$ is a finitely generated algebra. The monomials of $\bigoplus_{t \geq 0} \overline{I_F^t} y^t$ form a finitely generated semigroup. Since the semigroup of the monomials of $R(\mathcal{F})$ is the intersections of these semigroups, it is also finitely generated [14, Corollary 1.2]. From this, it follows that $R(\mathcal{F})$ is a finitely generated algebra. Moreover, as an intersection of normal rings, $R(\mathcal{F})$ is a normal ring. By [20, Theorem 1], this implies that $R(\mathcal{F})$ is Cohen–Macaulay.

Let $G(\mathcal{F}) = \bigoplus_{t \geq 0} I_t / I_{t+1}$. Then $G(\mathcal{F})$ is a factor ring of $R(\mathcal{F})$ by the ideal $\bigoplus_{t \geq 0} I_{t+1} y^t$. Hence, $G(\mathcal{F})$ is a finitely generated algebra. By [4, Theorem 4.5.6(b)], we have $\dim G(\mathcal{F}) = \dim R$. By the proof of the necessary part of [40, Theorem 1.1], the Cohen–Macaulayness of $R(\mathcal{F})$ implies that of $G(\mathcal{F})$. By [5, Theorem 9.23], these facts imply

$$\begin{aligned} \min_{t \geq 1} \text{depth } R/I_t &= \text{grade } \mathfrak{m}G(\mathcal{F}) = \text{ht } \mathfrak{m}G(\mathcal{F}) \\ &= \dim G(\mathcal{F}) - \dim G(\mathcal{F})/\mathfrak{m}G(\mathcal{F}) \\ &= \dim R - \dim G(\mathcal{F})/\mathfrak{m}G(\mathcal{F}). \end{aligned}$$

We have $G(\mathcal{F})/\mathfrak{m}G(\mathcal{F}) = \bigoplus_{t \geq 0} I_t / (\mathfrak{m}I_t + I_{t+1})$. Since $F_s(I) = \bigoplus_{t \geq 0} I^{(t)} / \mathfrak{m}I^{(t)}$, $I^{(t+1)} \subseteq \mathfrak{m}I^{(t)}$ [10, Proposition 9] and $I_t = I^{(t)}$ for $t \gg 0$, the graded algebras $G(\mathcal{F})/\mathfrak{m}G(\mathcal{F})$ and $F_s(I)$ share the same Hilbert quasi-polynomial [4, Theorem 4.4.3]. From this, it follows that $\dim G(\mathcal{F})/\mathfrak{m}G(\mathcal{F}) = \dim F_s(I)$. Therefore,

$$\min_{t \geq 1} \text{depth } R/I_t = \dim R - \dim F_s(I).$$

□

Moreover, the reference [40] lists the wrong year. It has to be 1989 instead of 1997.

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