Correction to: Generic family with robustly infinitely many sinks

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All the main results of [2] are correct, but this would need a substantial variation of the proof as done in [3]. In this corrigendum, we prefer to change the topologies considered in all the statements of [2]. Also we generalize and correct the fundamental property of parablender.

Correction of the statements For $d, r, k \geq 0$, and M, N Riemannian manifolds, two different spaces of C^d -families $(f_a)_a$ of C^r -maps $f_a \in C^r(M, N)$ parametrized by $\mathbb{R}^k$, can be defined as follows:

$$C_A^{d,r}(\mathbb{R}^k, M, N) = \{(f_a)_a : \partial_a^i \partial_z^j f_a(z) \text{ exists continuously for all } i \leq d, i + j \leq r \text{ and } (a, z) \in \mathbb{R}^k \times M\}$$

$$C_{PS}^{d,r}(\mathbb{R}^k, M, N) = \{(f_a)_a : \partial_a^i \partial_z^j f_a(z) \text{ exists continuously for all } j \leq r, i \leq d \text{ and } (a, z) \in \mathbb{R}^k \times M\}$$

We endow these spaces with the compact open topology w.r.t. the considered derivatives.

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We notice that in the important case $r = \infty$ these two spaces are equal. In general, it holds:

$$C_A^{d,d+r} \subset C_{PS}^{d,r} \subset C_A^{d,r}.$$

The previous article dealt with the space $C_{PS}^{d,r}$; the space $C_A^{d,r}$ should be considered instead of $C_{PS}^{d,r}$. Then the whole article is correct, but the statement of the fundamental property of the parablender which needs $1 \leq d < r \leq \infty$ (the case $d = r$ does not seem to work).

Therefore, the topology involved in the statement of Theorems A, C (and facts 4.2, 4.3, 4.4 of its proof) must be corrected to $C_A^{d,r}$ for any $1 \leq d < r \leq \infty$. Likewise, the topology involved in the statement of Theorems B must be corrected to $C_A^{d,r}$ for any $1 \leq d < r < \infty$.

This correction removes the case $d = r \geq 2$. However, the variation [3] of [2] gives the case $d = r \geq 1$ and also $d \geq r$ for the topologies $C_A^{d,r}$ and $C_{PS}^{d,r}$. In particular the statements of the main theorems of the article under correction are correct.

Here is the mistake I made. The space $C_{PS}^{d,r}$ is actually not stable by composition. For instance, if $N = M$, there exists $(f_a)_a \in C_{PS}^{d,r}$ such that $(f_a \circ f_a)_a$ does not belong to $C_{PS}^{d,r}$. That is why we correct it by the space $C_A^{d,r}$ which is stable by composition.

I am grateful to S. Crovisier for valuable suggestions on the presentation of the following section.

Correction and generalization of the fundamental property

Let us fix $k \geq 0$, $0 \leq d < r \leq \infty$. Given a Riemannian manifold M , and C^d -families of points $(z_a)_{a \in \mathbb{R}^k}$, its C^d -jet at $a = 0$ is denoted by $J_0^d(z_a)_a = \sum_{j=0}^d \frac{\partial_a^j z_a}{j!} a^{\otimes j}$. Let $J_0^d M$ be the space of C^d -jets of C^d -families of points in M at $a = 0$.

We notice that any $C_A^{d,r}$ -family $(f_a)_a$ of C^r -maps f_a of M acts canonically $J_0^d M$ as the map:

$$J_0^d(f_a)_a: J_0^d(z_a)_a \in J_0^d M \mapsto J_0^d(f_a(z_a))_a \in J_0^d M$$

Let us define a category of C^d -parablenders containing those of [1, 2]. To this end, put $I_e = [-1, 1]$, $Y_e = I_e \times I_e$, $\partial^s Y_e = \partial I_e \times I_e$ and $\partial^u Y_e = I_e \times \partial I_e$.

Definition 1 An affine C^d -para-IFS of $\mathbb{R}^2$ is a finite set of family of maps $\{(\hat{g}_a^\delta)_{a \in \mathbb{R}^k} : \delta \in \Delta\}$ from planar subsets $Y^\delta(a)$ onto $Y_\delta(a)$ of the form:

$$\hat{g}_a^\delta : (x, y) \in Y^\delta(a) = I_e \times I^\delta(a) \mapsto \left(\Lambda_\delta(a) \cdot x + p_\delta(a), \frac{y - q_\delta(a)}{\lambda_\delta(a)} \right) \in Y_\delta(a) = I_\delta(a) \times I_e,$$

where $I_\delta := [p_\delta - \Lambda_\delta, p_\delta + \Lambda_\delta]$, $I^\delta := [q_\delta - \lambda_\delta, q_\delta + \lambda_\delta]$, the functions $\lambda_\delta, \Lambda_\delta, p_\delta, q_\delta$ are polynomial such that $|\Lambda_\delta(0)| < 1/2 < |\lambda_\delta(0)| < 1$ and such that, with $\partial^u Y_\delta := I_\delta \times \partial I_e$ and $\partial^s Y^\delta := \partial I_e \times I^\delta$, there exist compact neighborhoods $A' \Subset A \subset J_0^d \mathbb{R}^2$ of 0 satisfying:

- (i) for every $J_0^d(z_a)_a \in A$, there exists $\delta \in \Delta$ such that $J_0^d(\hat{g}_a^\delta(z_a))_a$ is in A' .
- (ii) For every $\delta \in \Delta$, for every a small, the subsets $Y_\delta(a)$ and $Y^\delta(a)$ are included in Y_e , with $\partial^u Y_\delta(a) \Subset \partial^u Y_e$ and $\partial^s Y^\delta(a) \Subset \partial^s Y_e$.
- (ii) for every $Z = \sum_{i=0}^d z_i a^{\otimes i} \in A$, the value z_0 belongs to the interior of $Y^\delta(0)$ for every $\delta \in \Delta$.

Definition 2 Let $r \geq d$. A family $(f_a)_a$ of local diffeomorphisms of $\mathbb{R}^2$ defines an *affine-like C^d -parablender* if a finite set of its inverse branches $\{(g_a^\delta)_a : \delta \in \Delta\}$ is a $C_A^{d,r}$ -perturbation of an affine C^d -para-IFS $\{(\hat{g}_a^\delta)_a : \delta \in \Delta\}$.

Then, for small $a \in \mathbb{R}^k$ and $\delta \in \Delta$, with $\hat{I}^\delta$ a small neighborhood of I^δ , the image by g_a^δ of $[-1, 1] \times \hat{I}^\delta$ intersects Y_e at a set $Y_\delta(f_a)$ close to $Y_\delta(a)$. The set $Y_\delta(f_a)$ is bounded by two segments $\partial^u Y_\delta(f_a)$ of $\partial^u Y_e$, and two curves $\partial^s Y_\delta(f_a)$ close to $\partial^s Y_\delta(a)$. The image by f_a of $Y_\delta(f_a)$ is denoted by $Y^\delta(f_a)$. It is a filled square close to $Y^\delta(a)$. The set $Y^\delta(f_a)$ is bounded by two segments $\partial^s Y^\delta(f_a)$ of $\partial^s Y_e$ and two segments $\partial^u Y^\delta(f_a)$ close to $\partial^u Y^\delta(a)$.

We notice that (i)-(ii)-(iii) are still satisfied by $(g_a^\delta)_a$ instead of $(\hat{g}_a^\delta)_a$ and $Y_\delta(f_a)$ instead of $Y_\delta(a)$.

Then, for every $\underline{\delta} = (\delta_i)_{i \leq -1} \in \Delta^{\mathbb{Z}^-}$, we define the following local unstable manifold:

$$W_{loc}^u(\underline{\delta}; f_a) := \bigcap_{n \geq 1} f_a^n(Y_{\delta_{-n}}(f_a)).$$

Example 3 In [1], we showed an example of affine-like C^d -parablender with $\text{Card } \Delta = 2$. It is precisely for this example that we consider the topology on the inverse branches rather than on the dynamics, since the degenerate case $\Lambda_\delta = 0$ does occur in the limit of a renormalization process.

Example 4 In [2], we defined in §2.2, the family of maps $(f_{a\epsilon})_a$ for $f \in U_0$ and $\epsilon > 0$ small enough. The covering property (i) is shown in section §2.3.2 $A = \{P \in J_0^2 \mathbb{R}^2 : P(0) \in [-1, 1] \times [-2/3, 2/3] \text{ } \partial^i P(0) \in [-1, 1] \times [-2\epsilon, 2\epsilon]\}$ and A' a neighborhood of $\{P \in J_0^2 \mathbb{R}^2 : P(0) \in [-1/2, 1/2] \times [-1/2, 1/2] \text{ } \partial^i P(0) \in [-1/2, 1/2] \times [-\frac{3}{2}\epsilon, \frac{3}{2}\epsilon]\}$.

Let us fix an affine C^d -para-IFS $((\hat{g}_a^\delta)_{a \in \mathbb{R}^k})_{\delta \in \Delta}$. Let $\hat{\gamma} : x \in [-1, 1] \mapsto (x, x^2)$.

Fundamental property of the parablender *If $\infty \geq r > d \geq 1$ and $|\Lambda_\delta|(0) < |\lambda_\delta|^d(0)$ for every $\delta \in \Delta$, there exist a $C_A^{d,r}$ -neighborhood V_γ of the constant family of functions $(\hat{\gamma})_a$ and a $C_A^{d,r}$ -neighborhood V_g of $((\hat{g}_a^\delta)_a)_\delta$ such that for every affine-like parablender $(f_a)_a$ with inverse branches $((g_a^\delta)_a)_\delta \in V_g$, every $(\gamma_a)_a \in V_\gamma$ has its image $(\Gamma_a = \gamma_a([-1, 1]))_a$ which is C^d -paratangent at $a = 0$ to a local unstable manifold of $(f_a)_a$.*

We recall that $(\Gamma_a)_a$ is C^d -paratangent to a local unstable manifold $(W_{loc}^u(\underline{\delta}; f_a))_a$ if there are C^d -families of points $(C_a)_a$ in $(\Gamma_a)_a$ and $(Q_a)_a$ in $(W_{loc}^u(\underline{\delta}; f_a))_a$ such that:

$$J_0^d(C_a)_a = J_0^d(Q_a)_a \quad \text{and} \quad J_0^d(T_{C_a} \Gamma_a)_a = J_0^d(T_{Q_a} W_{loc}^u(\underline{\delta}; f_a))_a.$$

Remark 5 The fundamental property of parablender cannot be satisfied in the topology $C_A^{d,d}$ since the map $a \mapsto T_{Q_a} W_{loc}^u(\underline{\delta}; f_a)$ is in general not of class C^d .

To prove the fundamental property, we are going to define a sequence of symbols $(\delta_k)_{k \leq -1}$ and a C^d -family of points $(C_a)_a$ of $(\Gamma_a)_a$ satisfying the following property:

- (H₁) for every $k \leq -1$, $J_0^d(G_a^k \circ C_a)_a$ is in A , with $G_a^k = g_a^{\delta_k} \circ \dots \circ g_a^{\delta_{-1}}$,
- (H₂) for every $k \leq -1$, $J_0^d(DG_a^k \circ T_{C_a} \Gamma_a)_a$ is small.

Proof that (H₁) – (H₂) implies the fundamental property By proceeding as in the proof of Thm B. [1], (H₁) implies the existence of a C^d -curve of points $(Q_a)_a$ in $(W_{loc}^u(\underline{\delta}; f_a))_a$ such that:

$$J_0^d(C_a)_a = J_0^d(Q_a)_a.$$

Note that $J_0^d(G_a^k \circ Q_a)_a$ is equal to $J_0^d(G_a^k \circ C_a)_a$ for every k and so it is in A , for every $k \leq -1$.

As $(g_a)_a$ is a $C_A^{d,d+1}$ -perturbation of $(\hat{g}_a)_a$, $(W_{loc}^u(\underline{\delta}; f_a))_a$ is $C_A^{d,d+1}$ -close to a be horizontal, by Prop. 1.6 [2]. The same holds for $(W_{loc}^u(\sigma^k(\underline{\delta}); f_a))_a$, with $\sigma^k(\underline{\delta}) = (\delta_{i+k})_{i \leq -1}$. Hence, with $L_a \in \mathbb{P}^1$ the line tangent to $W_{loc}^u(\underline{\delta}; f_a)$ at Q_a , it holds that $J_0^d(L_a)_a$ is small, and $J_0^d(D_{Q_a} G_a^k \circ L_a)_a$ as well. Consequently, by (H₂), $J_0^d(DG_a^k \circ T_{C_a} \Gamma_a)_a$ is close to $J_0^d(D_{C_a} G_a^k \circ L_a)_a$ for every $k \leq -1$.

Let us notice that the action of $T_{Q_a} G_a^k$ of DG_a^k on $\mathbb{P}^1$ is exponentially expanding at the neighborhood of L_a . The same holds for $J_0^d(T_{Q_a} G_a^k)_a$: it is exponentially expanding at a ball centered at $J_0^d(L_a)_a$ and which contains $J_0^d(T_{C_a} \Gamma_a)_a$. Thus they are equal. $\square$

The proof of $(H_1) - (H_2)$ is done by constructing by induction on $n \leq 0$ a sequence of symbols $\delta_{n-1}, \dots, \delta_{-1} \in \Delta$ such that there exists a C^d -curve $(C_a^n)_a \in (\Gamma_a)_a$ satisfying:

- (a) C_0^n is in the interior of the domain of G_0^n .
- (b) $G_a^n(C_a^n)$ is the point of $G_a^n(\Gamma_a)$ with the minimal y -coordinate.
- (c) $J_0^d(G_a^k(C_a^n))_a$ is in A for every $n \leq k \leq 0$ and $J_0^d(G_a^{n-1}(C_a^n))_a$ is in A' .

We observe that any C^d -curve $(C_a)_a$ in $(\Gamma_a)_a$, such that $J_0^d(C_a)_a$ is a cluster value of $(J_0^d(C_a^n)_a)_n$ satisfies (H_1) by (c). To see that it satisfies also (H_2) , let us bring some materials.

Let $\kappa < 1$ and let $\eta > 0$ be small and such that:

$$1 > \kappa \cdot \exp(\eta), \quad 2\eta \exp(\eta) + \eta/(1 - \kappa \exp(\eta))\eta < d(A', A^c) \quad \text{and} \quad (\star)$$

$$1 > \kappa > \max_{\delta \in \Delta} \frac{\Lambda_\delta}{\lambda_\delta^d} (0) \exp(\eta).$$

Let $p_y: \mathbb{R}^2 \rightarrow \mathbb{R}$ be the 2^{sd} coordinate projection. For any $n \leq -1$, and $\delta'_n, \dots, \delta'_{-1} \in \Delta$, we define the line field:

$$L(\delta'_n \cdots \delta'_{-1}, f_a) := \ker D(p_y \circ g_a^{\delta'_n} \circ \cdots \circ g_a^{\delta'_{-1}}) \quad \text{and}$$

$$L(\emptyset, f_a) := \ker p_y = \mathbb{R} \times \{0\}.$$

We will define a complete distance on the space of $C_A^{d,d}$ -line field families and show the following below:

Lemma 6 *For V_g sufficiently small, there exists a small neighborhood V of $0 \in \mathbb{R}^k$ such that for all $n < 0$ and $\delta'_{n-1}, \dots, \delta'_{-1} \in \Delta_d$, the $C_A^{d,d}$ -distance between the families $(L(\delta'_{n-1} \cdots \delta'_{-1}, f_a))_{a \in V}$ and $(L(\delta'_n \cdots \delta'_{-1}, f_a))_{a \in V}$ (restricted to the intersection of their definition domains) is at most $\eta \kappa^{|n|} (\lambda_{\delta'_n} \cdots \lambda_{\delta'_{-1}})$. In particular, $(L(\delta'_n \cdots \delta'_{-1}, f_a))_{a \in V}$ is $\eta(1 - \kappa)^{-1} \cdot C_A^{d,d}$ -close to the horizontal line field $(L(\emptyset, f_a))_{a \in V}$.*

Proof that $(a - b - c)$ implies (H_2) By (b), the curve Γ_a is tangent to $L(\delta_n \cdots \delta_{-1}, f_a)$ at C_a^n , for every a small. Note also that $DG_a^k \circ T\Gamma_a$ is equal to $L(\delta_n \cdots \delta_{k-1}, f_a) \circ G_a^k$ at C_a^n for every $n \leq k \leq -1$. Thus $J_0^d(DG_a^k \circ T\Gamma_a)_a$ is equal to $J_0^d(L(\delta_n \cdots \delta_{k-1}, f_a) \circ G_a^k)_a$ at $J_0^d(C_a^n)_a$.

By (c), $J_0^d(G_a^k \circ C_a)_a$ is in the compact set A for every $k \leq -1$. Also $J_0^d(L(\delta_n \cdots \delta_{k-1}, f_a))_a$ is $\eta(1 - \kappa)^{-1}$ -small by Lemma 6.

Thus, $J_0^d(DG_a^k \circ T\Gamma_a)_a$ is uniformly dominated by $\eta(1 - \kappa)^{-1} \text{diam } A$ at $J_0^d(C_a^n)_a$, among $n \leq k \leq -1$. Hence (H_2) holds true at the cluster value $J_0^d(C_a)_a$ of $(J_0^d(C_a^n)_a)_n$.

Proof of the induction hypothesis (a-b-c) Let $n = 0$. Let C_a^0 be the point which realizes the y -minimum of Γ_a . As it is C^d -close to 0 for V_γ small, its C^d -jet $J_0^d(C_a^0)_a$ is in A . Hence by (i), there exists a symbol δ_{-1} such that $G_a^{-1} \circ C_a^0 = g_a^{\delta_{-1}} \circ C_a^0$ has its C^d -jets at $a = 0$ in A' .

Let $n \leq -1$. Let us assume $\delta_{n-1}, \dots, \delta_{-1} \in \Delta$ constructed such that $(C_a^m)_a$ satisfies $(a - b - c)$ for every $n \leq m \leq 0$. We put $L_{m,a} := L(\delta_m \cdots \delta_{-1}, f_a)$ for every $n - 1 \leq m \leq 0$.

For every $n - 1 < m \leq 0$, we can extend $(L_{m,a})_a$ on Y_e such that nearby $a = 0$, the line fields $(L_{m,a})_a$ and $(L_{n-1,a})_a$ are $\eta \sum_{j=m}^n \kappa^{|j|} \lambda_{\delta_j} \cdots \lambda_{\delta_{-1}} C^{d,d}$ -close by Lemma 6.

Therefore, there is a unique point $C_{n-1}(a)$ at which $L_{n-1,a}$ and $T\Gamma_a$ are equal, this proves (b). Moreover, $J_0^d(C_a^{n-1})_a$ and $J_0^d(C_a^m)_a$ are $\eta \sum_{j=m}^n \kappa^{|j|} \lambda_{\delta_j} \cdots \lambda_{\delta_{-1}} C^{d,d}$ -close.

We will define the norm involved, and we will prove the following below:

Lemma 7 *For V_g sufficiently small, for every $((g_a^\delta)_a)_\delta \in V_g$, and for every $\delta \in \Delta$, the following map is $\exp(\eta)/\lambda_\delta(0)$ Lipschitz:*

$$J^d(g_a^\delta)_a: J_0^d(z_a)_a \in A \mapsto J_0^d(g_a^\delta \circ z_a)_a \in J_0^d \mathbb{R}^2.$$

Hence, $J_0^d(G_a^{m-1}(C_a^{n-1}))_a$ and $J_0^d(G_a^{m-1}(C_a^m))_a$ is less than:

$$\frac{\eta \kappa^{|m|}}{\lambda_{\delta_{m-1}}} \exp(|m-1|\eta) + \eta \sum_{j=m-1}^n \kappa^{|j|} \lambda_{\delta_j} \cdots \lambda_{\delta_{m-2}} \exp(|m-1|\eta)$$

By the first inequality of $(\star)$ and since $1 > |\lambda_{\delta_{m-1}}(0)| \geq 1/2$, the above sum is at most $2\eta \exp(\eta) + \eta/(1 - \kappa \exp(\eta))$.

By the second inequality of $(\star)$ and the second part of (c) at step m , $J_0^d(G_a^{m-1}(C_a^{n-1}))_a$ is in A for every $n \leq m \leq 0$. This proves the first part of (c) at step $n - 1$.

Note that for $m = n$, we obtained that $J_0^d(G_a^{n-1}(C_a^{n-1}))_a$ is in A and such that $G_0^{n-1}(C_0^{n-1})$ belongs to the domain of any g_0^δ , hence (a) is satisfied.

Let $\delta_{n-2} \in \Delta$ be such that $J_0^d(g_a^{\delta_{n-2}})_a$ sends $J_0^d(G_a^{n-2}(C_a^{n-1}))_a$ into A' . Note that the second part of (c) is satisfied. $\square$

Proof of Lemma 7 We only need to prove this proposition for $(\hat{g}_a)$ since the Lipschitz constant depends continuously on the $C^{d,d}$ -perturbation.

With $J_0^d(z_a)_a = \sum_{j=0}^d z_j a^{\otimes j}$, and since $\partial_z^j \hat{g}_0^\delta = 0$ for every $j \geq 2$, it holds:

$$\begin{aligned}
J^d(\dot{g}_a^\delta(z_a))_a &= \sum_{n=0}^d \sum_{i+k_1+\dots+k_j=n} \frac{(\partial_a^i \partial_z^j \dot{g}_0^\delta)}{i!j!} \cdot (\otimes_l z_{k_l}) \otimes a^{\otimes n} \\
&= \sum_{n=0}^d \sum_{i+k=n} \frac{(\partial_a^i \partial_z \dot{g}_0^\delta)}{i!} \cdot z_k \otimes a^{\otimes n}
\end{aligned}$$

Hence $J^d(\dot{g}_a^\delta)_a$ is a linear map with upper triangular matrix in the base $(z_n)_{1 \leq n \leq d}$ and with diagonal equal to $\partial_z \dot{g}_0^\delta \cdot id$ which is $|\lambda_\delta^{-1}(0)|$ -Lipschitz. Hence there exists $(c_i)_{i \leq d}$ with c_i small w.r.t. c_j whenever $i < j$, such that for the norm $\sum_{j=0}^d z_j a^{\otimes j} \in J_0^d \mathbb{R}^2 \mapsto \sum_{j \leq d} c_j \|z_j\|$, the map $J^d(\dot{g}_a^\delta)_a$ is $\lambda_\delta^{-1}(0) \exp(\eta/2)$ -Lipschitz. By taking V_g small in function of η , we get the lemma. $\square$

To prove Lemma 6, let us order the set $\{(i, j) : i + j \leq d\}$ by:

$$(i, j) \succ (i', j') \text{ iff } i > i' \text{ or } i = i' \text{ and } j > j'.$$

Proof of Lemma 6. Let $\delta \in \Delta$, and let $\mathcal{L}(\Omega)$ be the space of $C^{d,d}$ -families of line fields $(L_a)_{a \in V}$ over a set $\Omega \subset \mathbb{R}^2$. We notice that the following map fixes the horizontal line field $H : (a, z) \mapsto \mathbb{R} \times \{0\}$.

$$\phi : (L_a)_a \in \mathcal{L}(Y^\delta) \mapsto ((D\dot{g}_a^\delta)^{-1} \circ L_a \circ \dot{g}_a^\delta)_a \in \mathcal{L}(Y_\delta).$$

In the identification of $\mathbb{P}\mathbb{R}^1$ which associates with a line its slope, H is the zero section and the map $(D\dot{g}_a^\delta)^{-1}$ is the multiplication by $\Lambda_\delta(a)\lambda_\delta(a)$. Thus the map ϕ is linear:

$$\phi(L_a)_a = (\Lambda_\delta(a)\lambda_\delta(a) \cdot L_a \circ \dot{g}_a^\delta)_a.$$

Note that Lemma 6 is proved if we show that ϕ is $\exp(3\eta/4)\Lambda_\delta(0)\lambda_\delta^{1-d}(0)$ -contracting for a $C_A^{d,d}$ -equivalent norm, independent of δ .

Then, there exist continuous functions $(C_{i',j'}^{i,j})_{(i',j') < (i,j)}$ such that, for every small $(L_a)_a \in \mathcal{L}(Y^\delta)$:

$$\begin{aligned}
\partial_a^i \partial_z^j \phi(L_a)_a &= \Lambda_\delta(a)\lambda_\delta(a) \cdot \partial_a^i \partial_z^j (L_a) \circ \dot{g}_a^\delta \cdot (\partial_z \dot{g}_a^\delta)^i \\
&\quad + \sum_{(i',j') < (i,j)} C_{i',j'}^{i,j}(z, a) \partial_a^{i'} \partial_z^{j'} (L_a) \circ \dot{g}_a^\delta
\end{aligned}$$

Thus the derivatives $[\partial_a^i \partial_z^j]_{i+j \leq d}$ are bounded from above by an upper triangular matrix with diagonal coefficients at most $\|\Lambda_\delta \lambda_\delta \cdot (\partial_z \dot{g}_a^\delta)^d\|$. For V

sufficiently small, the latter is at most $\exp(\eta/2)|\Lambda_\delta(0)| \cdot |\lambda_\delta(0)^{d-1}|$. Hence, there exists $(c_{i,j})_{i+j \leq d}$ with $c_{i,j}$ small w.r.t. $c_{i',j'}$ whenever $(i, j) < (i', j')$, such that the map ϕ is $\exp(3\eta/4)|\Lambda_\delta(0)| \cdot |\lambda_\delta(0)^{d-1}|$ -contracting for the norm $\|(L_a)_a\| = \sum_{i+j \leq d} c_{i,j} \|\partial_a^i \partial_z^j L_a\|_{C^0}$. $\square$

References

1. Berger, P., Crovisier, S., Pujals, E.: Iterated functions systems, blenders, and parablenders. In: Conference of Fractals and Related Fields, pp. 57–70. Springer, (2015)
2. Berger, P.: Generic family with robustly infinitely many sinks. *Invent. Math.* **205**(1), 121–172 (2016)
3. Berger, P.: Emergence and non-typicality of the finiteness of the attractors in many topologies. *Proc. Steklov Inst. Math.* **297**(1), 1–27 (2017)

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