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Domains of holomorphy for irreducible unitary representations of simple Lie groups

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1. Introduction

Let us consider a unitary irreducible representation $(\pi, \mathcal{H})$ of a simple, non-compact and connected algebraic Lie group G. Let us denote by K a maximal compact subgroup of G. According to Harish-Chandra, the Lie algebra submodule $\mathcal{H}_K$ of K-finite vectors of π consists of analytic vectors for the representation, i.e. for all $v \in \mathcal{H}_K$ the orbit map

$$f_v : G \to \mathcal{H}, \quad g \mapsto \pi(g)v$$

is real analytic. For these functions f_v we determine, and in full generality, their natural domain of definition as holomorphic functions (see Theorem 5.1 below):

Theorem 1.1. *Let $(\pi, \mathcal{H})$ be a unitary irreducible representation of G. Let $v \in \mathcal{H}$ be a non-zero K-finite vector and f_v be the corresponding orbit map. Then there exists a unique maximal $G \times K_{\mathbb{C}}$-invariant domain $D_\pi \subseteq G_{\mathbb{C}}$, independent of v, to which f_v extends holomorphically. Explicitly:*

(i) *$D_\pi = G_{\mathbb{C}}$ if π is the trivial representation.*
(ii) *$D_\pi = \Xi^+ K_{\mathbb{C}}$ if G is Hermitian and π is a non-trivial highest weight representation.*
(iii) *$D_\pi = \Xi^- K_{\mathbb{C}}$ if G is Hermitian and π is a non-trivial lowest weight representation.*
(iv) *$D_\pi = \Xi K_{\mathbb{C}}$ in all other cases.*

In the theorem above Ξ, Ξ^+, Ξ^- are certain G-domains in $X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$ over $X = G/K$ with proper G-action. These domains are studied in this paper because of their relevance for the theorem above (see [KO]). Let us mention that Ξ is the familiar crown domain and that the inclusion $\Xi K_{\mathbb{C}} \subset D_\pi$ traces back to our joint work with Robert Stanton [KSI,KSII].

Acknowledgement: I am happy to point out that this paper is related to joint work with Eric M. Opdam [KO]. Also I would like to thank Joseph Bernstein who, over the years, helped me with his comments to understand the material much better.

Finally I appreciate the work of a very good referee who made many useful remarks on style and organization of the paper.

2. Notation

Throughout this paper G shall denote a connected simple non-compact Lie group. We denote by $G_{\mathbb{C}}$ the universal complexification of G and suppose:

- $G \subseteq G_{\mathbb{C}}$;
- $G_{\mathbb{C}}$ is simply connected.

We fix a maximal compact subgroup $K < G$ and form

$$X = G/K\,,$$

the associated Riemannian symmetric space of the non-compact type. The universal complexification $K_{\mathbb{C}}$ of K will be realized as a subgroup of $G_{\mathbb{C}}$. We set

$$X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$$

and call $X_{\mathbb{C}}$ the *affine complexification* of X. Note that

$$X \hookrightarrow X_{\mathbb{C}}, \quad gK \mapsto gK_{\mathbb{C}}$$

defines a G-equivariant embedding which realizes X as a totally real form of the Stein symmetric space $X_{\mathbb{C}}$. We write $x_0 = K_{\mathbb{C}} \in X_{\mathbb{C}}$ for the standard base point in $X_{\mathbb{C}}$.

However, the natural complexification of X is not $X_{\mathbb{C}}$, but the *crown domain* $\Xi \subsetneq X_{\mathbb{C}}$ whose definition we recall now. We shall provide the standard definition of Ξ, see [AG].

Lie algebras of subgroups $L < G$ will be denoted by the corresponding lower case German letter, i.e. $\mathfrak{l} < \mathfrak{g}$; complexifications of Lie algebras are marked with a $\mathbb{C}$-subscript, i.e. $\mathfrak{l}_{\mathbb{C}}$ is the complexification of $\mathfrak{l}$.

Let us denote by $\mathfrak{p}$ the orthogonal complement to $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan–Killing form. We set

$$\hat{\Omega} = \{Y \in \mathfrak{p} \mid \operatorname{spec}(\operatorname{ad} Y) \subset (-\pi/2, \pi/2)\}\,.$$

Then

$$\Xi = G \exp(i\hat{\Omega}) \cdot x_0 \subset X_{\mathbb{C}}$$

is a G-invariant neighborhood of X in $X_{\mathbb{C}}$, commonly referred to as *crown domain*. Sometimes it is useful to have an alternative, although less invariant picture of the crown domain: if $\mathfrak{a} \subset \mathfrak{p}$ is a maximal abelian subspace and $\Omega := \hat{\Omega} \cap \mathfrak{p}$, then

$$\Xi = G \exp(i\Omega) \cdot x_0 . \tag{2.1}$$

The set Ω is nicely described through the restricted root system $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$:

$$\Omega = \{Y \in \mathfrak{a} \mid \alpha(Y) < \pi/2 \ \forall \alpha \in \Sigma\} .$$

If $\mathcal{W}$ is the Weyl group of Σ, then we note that Ω is $\mathcal{W}$-invariant.

Sometimes we will employ the root space decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}^\alpha$ with $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ as usual. We choose a positive system $\Sigma^+ \subset \Sigma$ and form the nilpotent subalgebra $\mathfrak{n} = \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}^\alpha$.

2.1. The example of $G = \mathrm{Sl}(2, \mathbb{R})$. For illustration and later use we will exemplify the above notions at the basic case of $G = \mathrm{Sl}(2, \mathbb{R})$.

We let $K = \mathrm{SO}(2, \mathbb{R})$ be our choice for the maximal compact subgroup and identify $X = G/K$ with the upper half plane $D^+ := \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$. We recall that

$$X_\mathbb{C} = \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \setminus \mathrm{diag}[\mathbb{P}^1(\mathbb{C})]$$

with $G_\mathbb{C}$ acting diagonally by fractional linear transformations. The G-embedding of $X = D^+$ into $X_\mathbb{C}$ is given by

$$z \mapsto (z, \bar{z}) \in X_\mathbb{C} .$$

If D^- denotes the lower half plane, then the crown domain is given by

$$\Xi = D^+ \times D^- \subseteq X_\mathbb{C} .$$

In addition we record two G-domains in $X_\mathbb{C}$ which sit above Ξ, namely:

$$\Xi^+ = D^+ \times \mathbb{P}^1(\mathbb{C}) \setminus \mathrm{diag}[\mathbb{P}^1(\mathbb{C})] , \tag{2.2}$$

$$\Xi^- = \mathbb{P}^1(\mathbb{C}) \times D^- \setminus \mathrm{diag}[\mathbb{P}^1(\mathbb{C})] . \tag{2.3}$$

Observe that $\Xi = \Xi^+ \cap \Xi^-$.

3. Remarks on G-invariant domains in $X_\mathbb{C}$ with proper action

One defines elliptic elements in $X_\mathbb{C}$ by

$$X_{\mathbb{C},\mathrm{ell}} = G \exp(i\mathfrak{p}) \cdot x_0 = G \exp(i\mathfrak{a}) \cdot x_0 .$$

The main result of [AG] was to show that Ξ is a maximal domain in $X_{\mathbb{C},\mathrm{ell}}$ with G-action proper. In particular, G acts properly on Ξ.

It was found in [KO] that Ξ in general is not a maximal domain in $X_\mathbb{C}$ for proper G-action: the domains Ξ^+ and Ξ^- from (2.2)–(2.3) yield counterexamples. To know all maximal domains is important for the theory of representations [KO, Sect. 4].

That Ξ in general is not maximal for proper action is related to the unipotent model for the crown which was described in [KO]. To be more precise, we showed that there exists a domain $\hat{\Lambda} \subseteq \mathfrak{n}$ containing 0 such that

$$\Xi = G \exp(i\hat{\Lambda}) \cdot x_0 \,. \tag{3.1}$$

Now there is a big difference between the unipotent parametrization (3.1) and the elliptic parametrization (2.1): If we enlarge Ω the result is no longer open; in particular, $X_{\mathbb{C},\mathrm{ell}}$ is not a domain. On the other hand, if we enlarge the open set $\hat{\Lambda}$ the resulting set is still open; in particular $X_{\mathbb{C},\mathrm{u}} := G\exp(i\mathfrak{n}) \cdot x_0$ is a domain. Thus, if there were a bigger domain than Ξ with proper action, then it is likely by enlargement of $\hat{\Lambda}$.

We need some facts on the boundary of Ξ.

3.1. Boundary of Ξ. Let us denote by $\partial\Xi$ the topological boundary of Ξ in $X_{\mathbb{C}}$. One shows that

$$\partial_{\mathrm{ell}}\Xi := G\exp(i\partial\Omega)\cdot x_0 \subseteq \partial\Xi$$

(cf. [KSII]) and calls $\partial_{\mathrm{ell}}\Xi$ the *elliptic part* of $\partial\Xi$. We define the *unipotent part* $\partial_{\mathrm{u}}\Xi$ of $\partial\Xi$ to be the complement to the elliptic part:

$$\partial_{\mathrm{u}}\Xi = \partial\Xi \setminus \partial_{\mathrm{ell}}\Xi \,.$$

The relevance of $\partial_{\mathrm{u}}\Xi$ is as follows. Let $X \subset D \subseteq X_{\mathbb{C}}$ denote a G-domain with proper G-action. Then $D \cap \partial_{\mathrm{ell}}\Xi = \emptyset$ by the above cited result of [AG]. Thus if $D \not\subset \Xi$, then one has

$$D \cap \partial_{\mathrm{u}}\Xi \neq \emptyset \,.$$

Let us describe $\partial_{\mathrm{u}}\Xi$ in more detail. For $Y \in \mathfrak{a}$ we define a reductive subalgebra of $\mathfrak{g}_{\mathbb{C}}$ by

$$\mathfrak{g}_{\mathbb{C}}[Y] = \{Z \in \mathfrak{g}_{\mathbb{C}} \mid e^{-2i\,\mathrm{ad}(Y)} \circ \sigma(Z) = Z\}$$

with σ the Cartan involution on $\mathfrak{g}_{\mathbb{C}}$ which fixes $\mathfrak{k} + i\mathfrak{p}$. Then there is a partial result on $\partial_{\mathrm{u}}\Xi$, for instance stated in [FH]:

$$\partial_{\mathrm{u}}\Xi \subseteq \{G\exp(e)\exp(iY)\cdot x_0 \mid Y \in \partial\Omega, \tag{3.2}$$

$$0 \neq e \in \mathfrak{g}_{\mathbb{C}}[Y] \cap i\mathfrak{g} \text{ nilpotent}\} \,. \tag{3.3}$$

If Y is such that only one root, say α, attains the value $\pi/2$, then we call Y and as well the elements in the boundary orbit $G\exp(e)\exp(iY)\cdot x_0$ *regular*. Accordingly we define the *regular unipotent* boundary $\partial_{\mathrm{u,reg}}\Xi = \{z \in \partial_{\mathrm{u}}\Xi \mid z \text{ regular}\}$. Note that $\mathfrak{g}_{\mathbb{C}}[Y]$ is of especially simple form for regular Y, namely

$$\mathfrak{g}_{\mathbb{C}}[Y] = i\mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{g}[\alpha]^{-\theta} \oplus i\mathfrak{g}[\alpha]^{\theta}$$

where $\mathfrak{g}[\alpha] = \mathfrak{g}^{\alpha} \oplus \mathfrak{g}^{-\alpha}$. Hence, in the regular situation, one can choose e above to be in $i\mathfrak{g}[\alpha]^{\theta} + i\mathfrak{a}$. We summarize our discussion:

Proposition 3.1. *Let $X \subset D \subseteq X_{\mathbb{C}}$ be a G-invariant domain with proper G-action which is not contained in Ξ. Then $D \cap \partial_{\mathrm{u,reg}} \Xi \neq \emptyset$. More precisely, there exists $Y \in \partial\Omega$ regular (with $\alpha \in \Sigma$ the unique root attaining $\pi/2$ on Y) and a non-zero nilpotent element $e \in i\mathfrak{g}[\alpha]^{\theta} + i\mathfrak{a}$ such that*

$$\exp(e)\exp(iY) \cdot x_0 \in \partial_{\mathrm{u,reg}} \Xi \cap D\,.$$

4. Maximal domains for proper action

The aim of this section is to classify all maximal G-domains in $X_{\mathbb{C}}$ which contain X and maintain proper action. The answer will depend whether G is of Hermitian type or not.

4.1. Non-Hermitian groups. The objective is to prove the following theorem:

Theorem 4.1. *Suppose that G is not of Hermitian type. If $X \subset D \subset X_{\mathbb{C}}$ is a G-invariant domain with proper G-action, then $D \subset \Xi$.*

Before we can give the proof of the theorem some preparation is needed. The proof relies partly on a structural fact characterizing non-Hermitian groups (see Lemma 4.4 below) and on a precise knowledge of the basic case of $G = \mathrm{Sl}(2, \mathbb{R})$.

Let us begin with the relevant facts for $G = \mathrm{Sl}(2, \mathbb{R})$. With $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ our choices for $\mathfrak{a}$ and $\mathfrak{n}$ are

$$\mathfrak{a} = \mathbb{R} \cdot T \quad \text{and} \quad \mathfrak{n} = \mathbb{R} \cdot E\,.$$

Note that $\Omega = (-\pi/4, \pi/4)T$.

Then a slight modification of results in [KO, Sects. 3 and 4] yield:

Lemma 4.2. *Let $G = \mathrm{Sl}(2, \mathbb{R})$ and $\mathcal{J} \subset \mathbb{R}$ be an open subset. Then*

$$\Xi_{\mathcal{J}} := G\exp(i\mathcal{J} \cdot E) \cdot x_0$$

is a G-invariant open subset of $X_{\mathbb{C}}$ and the following holds:

(i) *G does not act properly if $\{-1, 1\} \subset \mathcal{J}$.*
(ii) $\Xi = \Xi_{(-1,1)}$.
(iii) $\Xi^+ = \Xi_{(-1,\infty)}$.
(iv) $\Xi^- = \Xi_{(-\infty,1)}$.

We also need that $\partial\Xi$ is a fiber bundle over the affine symmetric space G/H where $H = \mathrm{SO}_e(1,1)$. Notice that H is the stabilizer of the boundary point

$$z_H := \exp(-i\pi T/4)\cdot x_0 = (1,-1) \in \partial_{\mathrm{ell}}\Xi\,.$$

Write τ for the involution on G, resp. $\mathfrak{g}$, fixing H, resp. $\mathfrak{h}$, and denote by $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ the corresponding eigenspace decomposition. The $\mathfrak{h}$-module $\mathfrak{q}$ breaks into two eigenspaces $\mathfrak{q} = \mathfrak{q}^+ \oplus \mathfrak{q}^-$ with

$$\mathfrak{q}^\pm = \mathbb{R}\cdot e^\pm \quad \text{where} \quad e^\pm = \begin{pmatrix} 1 & \mp 1 \\ \pm 1 & -1 \end{pmatrix}.$$

Finally write

$$\mathcal{C} = \mathbb{R}_{\geq 0}\cdot e^+ \cup \mathbb{R}_{\geq 0}\cdot e^-$$

and $\mathcal{C}^\times = \mathcal{C}\setminus\{0\}$. Note that both $\mathcal{C}$ and $\mathcal{C}^\times$ are H-stable. We cite [KO, Th. 3.1]:

Lemma 4.3. *Let* $G = \mathrm{Sl}(2,\mathbb{R})$. *Then the map*

$$G\times_H \mathcal{C} \to \partial\Xi, \quad [g,e]\mapsto g\exp(ie)\cdot z_H$$

is a G*-equivariant homeomorphism. Moreover,*

(i) $\partial_{\mathrm{ell}}\Xi = G\cdot z_H \simeq G/H$,
(ii) $\partial_{\mathrm{u}}\Xi = G\exp(i\mathcal{C}^\times)\cdot z_H \simeq G\times_H \mathcal{C}^\times$,
(iii) $\partial_{\mathrm{u}}\Xi = G\exp(iE)\cdot x_0 \amalg G\exp(-iE)\cdot x_0$.

As a last piece of information we need a structural fact which is only valid for non-Hermitian groups.

Lemma 4.4. *Suppose that* G *is not of Hermitian type. Then for all* $\alpha\in\Sigma$ *and* $E\in\mathfrak{g}^\alpha$ *there exists an* $m\in M = Z_K(\mathfrak{a})$ *such that*

$$\mathrm{Ad}(m)E = -E\,.$$

Proof. Let us remark first that we may assume that G is of adjoint type. If G is complex, then the assertion is clear as $T := \exp(i\mathfrak{a})\subset M$ provides us with the elements we are looking for. More generally for $\dim\mathfrak{g}^\alpha > 1$ one knows (Kostant) that $M_0 = \exp(\mathfrak{m})$ acts transitively on the unit sphere in $\mathfrak{g}^\alpha$ (cf. [Kos]).

In the sequel we use the terminology and tables of the classification of real simple Lie algebras as found in the monograph [K, App. C]. As G is not Hermitian, Kostant's result leaves us with the following cases for $\mathfrak{g}$: $\mathfrak{sl}(n,\mathbb{R})$ for $n\geq 3$, $\mathfrak{so}(p,q)$ for $0, 2\neq p,q$ and $p+q>2$, $E\ I$, $E\ II$, $E\ V$, $E\ VI$, $E\ VIII$, $E\ IX$, $F\ I$ and G.

Now we make the following observation. The lemma is true for $G = \mathrm{Sl}(3,\mathbb{R})$ as a simple matrix computation shows. Suppose that α is such that it can be put into an A_2-subsystem of Σ. As $\dim\mathfrak{g}^\alpha$ is one-dimensional (by our reduction) this means that we can put $E \in \mathfrak{g}^\alpha$ in a subalgebra isomorphic to $\mathfrak{sl}(3, \mathbb{R})$. Now it is important to recall the nature of the component group of M, see [K, Th. 7.55]. It follows that the M-group of $\mathrm{Sl}(3, \mathbb{R})$ (isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$) embeds into the M-group of G.

The A_2-reduction described above deletes most of the cases in our list. We remain with the orthogonal cases $\mathfrak{so}(p, q)$ for $0, 2 \neq p, q$ and $p \neq q$. A simple matrix computation, which we leave to the reader, finishes the proof. □

Proof of Theorem 4.1. Suppose that G is not of Hermitian type. Let $X \subset D \subset \Xi$ be a G-invariant domain with proper G-action which is not contained in Ξ. We shall show that D does not exist.

According to Proposition 3.1 we find a regular $Y \in \partial\Omega$ and a non-zero nilpotent $e \in \mathfrak{g}_\mathbb{C}[Y] \cap i\mathfrak{g}$ such that

$$\exp(e)\exp(iY) \cdot x_0 \in \partial_{\mathrm{u,reg}}\Xi \cap D\,.$$

Let $\alpha \in \Sigma$ be the root corresponding to Y. Write $Y = Y^\alpha + Y'$ with Y^α, $Y' \in \mathfrak{a}$ such that $\alpha(Y') = 0$. It is known that $Y^\alpha \in \partial\Omega$ and $Y' \in \Omega$. Hence we may use $\mathfrak{sl}(2)$-reduction which in conjunction with Lemma 4.3 implies the existence of $E^\alpha \in \mathfrak{g}^\alpha$ such that:

- $\{E^\alpha, \theta(E^\alpha), [E^\alpha, \theta(E^\alpha)]\}$ is an $\mathfrak{sl}(2)$-triple,
- $\exp(iE^\alpha)\exp(iY') \cdot x_0 \in \partial_{\mathrm{u,reg}}\Xi \cap D$.

Now, as G is not of Hermitian type, Lemma 4.4 implies that there exists an element $m \in M$ such that $\mathrm{Ad}(m)E^\alpha = -E^\alpha$. Hence

$$\exp(-iE^\alpha)\exp(iY') \cdot x_0 \in \partial_{\mathrm{u,reg}}\Xi$$

as well. But this contradicts Lemma 4.2(i). □

4.2. Hermitian groups. Let now G be of Hermitian type and $G \subseteq P^- K_\mathbb{C} P^+$ be a Harish-Chandra decomposition of G in $G_\mathbb{C}$. We define flag varieties

$$F^+ = G_\mathbb{C}/K_\mathbb{C}P^+ \quad \text{and} \quad F^- = G_\mathbb{C}/K_\mathbb{C}P^-$$

and inside of them we declare the flag domains

$$D^+ = GK_\mathbb{C}P^+/K_\mathbb{C}P^+ \quad \text{and} \quad D^- = GK_\mathbb{C}P^-/K_\mathbb{C}P^-\,.$$

Then in the

$$X_\mathbb{C} \hookrightarrow F^+ \times F^-, \quad gK_\mathbb{C} \mapsto (gK_\mathbb{C}P^+, gK_\mathbb{C}P^-) \tag{4.1}$$

identifies $X_\mathbb{C}$ as a Zariski open affine piece of $F^+ \times F^-$. In more detail: As G is of Hermitian type, there exist $w_0 \in N_{G_\mathbb{C}}(K_\mathbb{C})$ such that $w_0 P^\pm w_0^{-1} = P^\mp$.

In turn, this element induces a $G_{\mathbb{C}}$-equivariant biholomorphic map:

$$\phi : F^+ \to F^-, \quad gK_{\mathbb{C}}P^+ \mapsto gw_0K_{\mathbb{C}}P^- .$$

With that the embedding (4.1) gives the following identification of $X_{\mathbb{C}}$:

$$X_{\mathbb{C}} = \{(z, w) \in F^+ \times F^- \mid \phi(z) \top w\}, \tag{4.2}$$

where $\top$ stands for the transversality notion in the flag variety F^-. We recall what it means to be transversal. First note that the notion is $G_{\mathbb{C}}$-invariant, i.e. for $z, w \in F^-$ and $g \in G_{\mathbb{C}}$ one has $z \top w$ if and only if $gz \top gw$. Now for the base point $z^- = K_{\mathbb{C}}P^- \in F^-$ one has $z^- \top w$ if and only if $w \in P^-w_0z^-$.

We keep the realization of $X_{\mathbb{C}}$ in $F^+ \times F^-$ (cf. (4.1)) in mind and recall the description of Ξ:

$$\Xi = D^+ \times D^-$$

(see [KSII]).

For subsets $X^\pm \subset F^\pm$ we write $X^+ \times_\top X^-$ for those elements $(x^+, x^-) \in X^+ \times X^-$ which are transversal, i.e. $\phi(x^+) \top x^-$. With this terminology in mind we finally define

$$\Xi^+ = D^+ \times_\top F^-,$$
$$\Xi^- = F^+ \times_\top D^- .$$

4.2.1. Basic structure theory of Ξ^+ and Ξ^-. It is obvious that both Ξ^+ and Ξ^- are open and G-invariant. However, as was pointed out by the referee, it is a priori not clear that they are connected. In order to see this let $p_+ : \Xi^+ \to D^+$ be the projection onto the first factor. Likewise we define $p_- : \Xi^- \to D^-$.

Proposition 4.5. *Let $\epsilon \in \{-, +\}$. The map $p_\epsilon : \Xi^\epsilon \to D^\epsilon$ induces the structure of a holomorphic fiber bundle with fiber isomorphic to P^ϵ.*

Proof. We confine ourselves with the case $\epsilon = +$.

As p_+ is G-equivariant and D^+ is G-homogeneous, it is sufficient to determine the fiber $p_+^{-1}(z^+)$. Recall that $z^+ = K_{\mathbb{C}}P^+ \in F^+$ is the base point. Now

$$p_+^{-1}(z^+) = \{(z^+, w) \in F^+ \times F^- \mid \phi(z^+) \top w\}.$$

Observe that $\phi(z^+) = w_0z^-$ and that $w_0z^- \top w$ is equivalent to $z^- \top w_0^{-1}w$. By the definition of transversality this means that $w_0^{-1}w \in P^-w_0z^-$ or $w \in w_0P^-w_0z^-$. It is no loss of generality to assume that $w_0 = w_0^{-1}$. So we arrive at $w \in P^+z^-$ and this concludes the proof of the proposition. □

Corollary 4.6. *Both Ξ^+ and Ξ^- are contractible.*

It was observed by the the referee that Proposition 4.5 allows the following interesting reformulation.

Corollary 4.7. *The map*

$$G \times_K P^+ \to \Xi^+, \quad [g, p] \mapsto (gz^+, gpz^-)$$

is a G-equivariant diffeomorphism. In particular Ξ^+ is G-biholomorphic to $T^{0,1}D^+$, the antiholomorphic tangent bundle of D^+. Likewise, Ξ^- is G-biholomorphic to $T^{0,1}D^-$.

Corollary 4.7 combined with the Harish–Chandra decomposition implies that $\Xi^\epsilon \simeq D^\epsilon \times P^\epsilon$ as complex manifolds. In particular Ξ^ϵ is Stein.

The fact that $K_\mathbb{C}$ normalizes P^ϵ allows us to speak of $G \times P^\epsilon$-invariant domains in $X_\mathbb{C}$. It follows from (4.1) and Corollary 4.7 that Ξ^ϵ is $G \times P^\epsilon$-invariant.

Proposition 4.8. *Let $\epsilon \in \{-, +\}$. The real group G acts properly on Ξ^ϵ. Moreover Ξ^ϵ is a maximal $G \times P^\epsilon$-invariant domain in $X_\mathbb{C}$ for proper G-action.*

Proof. As the G-action is proper on D^ϵ, it follows that G acts properly on Ξ^ϵ. In the sequel we deal with $\epsilon = +$ only. It remains to show that Ξ^+ is a maximal $G \times P^+$-invariant domain in $X_\mathbb{C}$ for proper G-action. We argue by contradiction and suppose that $D \supsetneq \Xi^+$ is a $G \times P^+$-domain in $X_\mathbb{C}$ with proper G-action. Then $D = (D_0 \times F^-) \cap X_\mathbb{C}$ with $D_0 \supsetneq D^+$ a G-domain with proper action. Now recall the following facts:

- There are only finitely many G-orbits in F^+.
- There are precisely two orbits with proper G-action: D^+ and $\phi^{-1}(D^-)$.

The assertion follows. □

Remark 4.9. Suppose that G is of Hermitian type. Then it can be shown that if $X \subseteq D \subseteq X_\mathbb{C}$ is a G-invariant domain with proper G-action, then $D \subseteq \Xi^+$ or $D \subseteq \Xi^-$.

As we will not need this fact, we refrain from a proof.

If $D \subseteq X_\mathbb{C}$ is a subset, then we write $DK_\mathbb{C}$ for its preimage in $G_\mathbb{C}$ under the canonical projection $G_\mathbb{C} \to X_\mathbb{C}$.

Proposition 4.10. *The following assertions hold:*

(i) $\Xi^+ K_\mathbb{C} = GK_\mathbb{C}P^+$,
(ii) $\Xi^- K_\mathbb{C} = GK_\mathbb{C}P^-$.

Proof. It suffices to prove (i). Recall the embedding (4.1), and the definition of transversality condition. We deduce that $P^+ \subset \Xi^+ K_\mathbb{C}$. As $\Xi^+ K_\mathbb{C}$ is $G \times K_\mathbb{C}$-invariant, it follows that $GP^+K_\mathbb{C} = GK_\mathbb{C}P^+ \subset \Xi^+ K_\mathbb{C}$.

Conversely, Corollary 4.7 implies that GP^+ maps onto Ξ^+ and thus $\Xi^+ \subset GP^+K_\mathbb{C}$. □

We conclude this subsection with some easy facts on the structure of Ξ^+ and Ξ^- which will be used later on.

4.2.2. Unipotent model for Ξ^+ and Ξ^-. We begin with the unipotent parameterization of Ξ^+ and Ξ^-. Some terminology is needed.

According to C. Moore, Σ is of type C_n or BC_n. Hence we find a subset $\{\gamma_1, \dots, \gamma_n\}$ of long strongly orthogonal restricted roots. We fix $E_j \in \mathfrak{g}^{\gamma_j}$ such that $\{E_j, \theta(E_j), [E_j, \theta E_j]\}$ becomes an $\mathfrak{sl}(2)$-triple. Set $T_j := 1/2[E_j, \theta E_j]$ and note that

$$\Omega = \bigoplus_{j=1}^{n} (-\pi/2, \pi/2) T_j .$$

We set $V = \bigoplus_{j=1}^{n} \mathbb{R} \cdot E_j$ and take a cube inside V by

$$\Lambda = \bigoplus_{j=1}^{n} (-1, 1) E_j .$$

In [KO, Sect. 8], we have shown that

$$\Xi = G \exp(i\Lambda) \cdot x_0 .$$

In this parametrization of Ξ the unipotent boundary piece has a simple description:

$$\partial_{\mathrm{u}} \Xi = G \exp(i \partial \Lambda) \cdot x_0 . \tag{4.3}$$

The strategy now is to enlarge Ξ by enlarging Λ while maintaining that the object stays a domain on which G acts properly. But now we have to be a little bit careful with our choice of E_j. Replacing E_j by $-E_j$ has no effect for the matters cited above, but for the sequel. Our choice is such that $\gamma_1, \dots, \gamma_n$ are positive roots (this determines the non-compact roots in Σ^+ uniquely). We set

$$\Lambda^+ = \bigoplus_{j=1}^{n} (-1, \infty) E_j \quad \text{and} \quad \Lambda^- = \bigoplus_{j=1}^{n} (-\infty, 1) E_j .$$

Then, a direct generalization of Lemma 4.2(iii), (iv) yields:

Proposition 4.11. *The following assertions hold:*

(i) $\Xi^+ = G \exp(i\Lambda^+) \cdot x_0$,
(ii) $\Xi^- = G \exp(i\Lambda^-) \cdot x_0$.

Remark 4.12. If we define subcones of the nilcone $\mathcal{N} \subseteq \mathfrak{g}$ by

$$\mathcal{N}^+ = \mathrm{Ad}(K) \left[\bigoplus_{j=1}^{n} [0, \infty) E_j \right] \quad \text{and} \quad \mathcal{N}^- = -\mathcal{N}^+ ,$$

then one can show that the maps

$$G \times_K \mathcal{N}^{\pm} \to \Xi^{\pm}, \quad [g, Y] \mapsto g \exp(iY) \cdot x_0$$

are homeomorphic.

5. Representation theory

Let $(\pi, \mathcal{H})$ be a unitary representation of G and $\mathcal{H}_K$ the underlying Harish-Chandra module of K-finite vectors. Notice that $\mathcal{H}_K$ is naturally a module for $K_{\mathbb{C}}$.

We say that $(\pi, \mathcal{H})$ is a highest, resp. lowest, weight representation if G is of Hermitian type and $\mathfrak{p}^+ = \mathrm{Lie}(P^+)$, resp. $\mathfrak{p}^-$, acts on $\mathcal{H}_K$ in a finite manner.

We turn to the main result of this paper.

Theorem 5.1. *Let $(\pi, \mathcal{H})$ be a unitary irreducible representation of G. Let $v \in \mathcal{H}$ be a non-zero K-finite vector and*

$$f_v : G \to \mathcal{H}, \quad g \mapsto \pi(g)v$$

the corresponding orbit map. Then there exists a unique maximal $G \times K_{\mathbb{C}}$-invariant domain $D_\pi \subseteq G_{\mathbb{C}}$, independent of v, to which f_v extends holomorphically. Explicitly:

(i) *$D_\pi = G_{\mathbb{C}}$ if π is the trivial representation.*
(ii) *$D_\pi = \Xi^+ K_{\mathbb{C}}$ if G is Hermitian and π is a non-trivial highest weight representation.*
(iii) *$D_\pi = \Xi^- K_{\mathbb{C}}$ if G is Hermitian and π is a non-trivial lowest weight representation.*
(iv) *$D_\pi = \Xi K_{\mathbb{C}}$ in all other cases.*

Proof. If π is trivial, then the assertion is clear. So let us assume that π is non-trivial in the sequel. Fix a nonzero K-finite vector v and consider the orbit map $f_v : G \to \mathcal{H}$. We recall the following two facts:

- f_v extends to a holomorphic G-equivariant map $f_v : \Xi K_{\mathbb{C}} \to \mathcal{H}$ (see [KSII, Th. 1.1]).
- If $D_v \subseteq G_{\mathbb{C}}$ is a $G \times K_{\mathbb{C}}$-invariant domain to which f_v extends holomorphically, then G acts properly on $D_v/K_{\mathbb{C}}$ (see [KO, Th. 4.3]).

We begin with the case where G is not of Hermitian type. Here the assertion follows from the bulleted items above in conjunction with Theorem 4.1.

So we may assume for the remainder that G is of Hermitian type. If π is a highest weight representation, then it is clear that f_v extends to a holomorphic map $GK_{\mathbb{C}}P^+ \to \mathcal{H}$. Thus, in this case $\Xi^+ K_{\mathbb{C}} = GK_{\mathbb{C}}P^+$ (cf. Proposition 4.10) is a maximal domain of definition for f_v by Proposition 4.8 and the second bulleted item from above. Likewise, if $(\pi, \mathcal{H})$ is a lowest weight representation, then $\Xi^- K_{\mathbb{C}}$ is a maximal domain of definition of f_v. As both Ξ^+ and Ξ^- are simply connected with sufficiently regular boundary, it follows that these maximal domains are in fact unique.

It remains to show:

- If f_v extends holomorphically on a domain $D \supset \Xi$ such that $D \cap [\Xi^+ \setminus \Xi] \neq \emptyset$, then $(\pi, \mathcal{H})$ is a highest weight representation.
- If f_v extends holomorphically p on a domain $D \supset \Xi$ such that $D \cap [\Xi^- \setminus \Xi] \neq \emptyset$, then $(\pi, \mathcal{H})$ is a lowest weight representation.

It is sufficient to deal with the first case. So suppose that f_v extends to a bigger domain D such that $D \cap [\Xi^+ \setminus \Xi] \neq \emptyset$. Taking derivatives and applying the fact that $d\pi(\mathcal{U}(\mathfrak{g}_{\mathbb{C}}))v = \mathcal{H}_K$, we see that f_u extends to D for all $u \in \mathcal{H}_K$. By Proposition 3.1, (4.3) and our assumption we find $1 \leq j \leq n$ be such that $\exp(iE_j)\exp(iY) \cdot x_0 \in D$ for some $Y \in \Omega$ with $\gamma_j(Y) = 0$. Let $G_j < G$ be the analytic subgroup corresponding to the $\mathfrak{sl}(2)$-triple $\{E_j, \theta(E_j), [E_j, \theta(E_j)]\}$. Basic representation theory of type I-groups in conjunction with [KO, Th. 4.7], yields that $\pi|_{G_j}$ breaks into a direct sum of highest weight representations. Applying $N_K(\mathfrak{a})$ (which in particular permutes the G_k and preserves $\mathcal{H}_K$) we see that above matters hold for any other G_k as well (note that Y might change but this does not matter as Ω is $N_K(\mathfrak{a})$-invariant). It follows that π is a highest weight representation and completes the proof of the theorem. □

Remark 5.2. The domains Ξ, Ξ^+ and Ξ^- are independent of the choice of the connected group G. Accordingly, the above theorem holds for all simple connected non-compact Lie groups G, i.e. we can drop the assumption that $G \subseteq G_{\mathbb{C}}$ and $G_{\mathbb{C}}$ simply connected.

Problem 5.3. The above theorem should hold true for all irreducible admissible Banach representations of G under the reservation that (i) gets modified to: $D_\pi = G_{\mathbb{C}}$ if π is finite dimensional.

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