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G_2-instantons on Resolutions of G_2-orbifolds

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Abstract: We explain a construction of G_2-instantons on manifolds obtained by resolving G_2-orbifolds. This includes the case of G_2-instantons on resolutions of T^7/Γ as a special case. The ingredients needed are a G_2-instanton on the orbifold and a Fueter section over the singular set of the orbifold which are used in a gluing construction. In the general case, we make the very restrictive assumption that the Fueter section is pointwise rigid. In the special case of resolutions of T^7/Γ, improved control over the torsion-free G_2-structure allows to remove this assumption. As an application, we construct a large number of G_2-instantons on the simplest example of a resolution of T^7/Γ. We also construct one new example of a G_2-instanton on the resolution of $(T^3 \times \mathrm{K3})/\mathbb{Z}_2^2$.

Contents

1. Introduction

In [Ber55], Berger presented a list of groups which can possibly occur as the holonomy groups of Riemannian manifolds. However, constructing manifolds which realise these holonomy groups remained a wide-open problem for decades. A milestone in this direction was the formulation and proof of the Calabi conjecture in [Cal54, Cal57] and [Yau77, Yau78] respectively. Among other things, the proof of this conjecture gives a powerful characterisation of manifolds admitting a metric with holonomy $\mathrm{SU}(n)$, giving rise to a wealth of examples of such manifolds. Another entry on Berger's list is the exceptional holonomy group G_2. The first compact examples of Riemannian manifolds with holonomy equal to G_2 were constructed in [Joy96] by resolving an orbifold of the form T^7/Γ, where Γ is a finite group of isometries of T^7. In [JK21], this construction was extended to resolutions of orbifolds of the form Y/Γ, where Y is a manifold with holonomy contained in G_2, but not necessarily flat, and Γ is a finite group of G_2-involutions. However, an analogue of the Calabi conjecture for the holonomy group G_2 remains out of reach, and not much is known about which 7-manifolds admit torsion-free G_2-structures, and if they do, how many.

In the seminal article [Don83], the moduli space of anti-self-dual connections was used to define invariants of smooth 4-manifolds. Following this, a rich theory of gauge theoretical invariants and their relations to other manifold invariants in 4 dimensions was developed. The article [DT98] then recognised some of the 4-dimensional phenomena in dimension 7, for example the existence of a functional whose critical points are instantons. With great optimism, one may hope to recreate the four-dimensional success story in dimension 7, and use the moduli space of G_2-instantons to define invariants of G_2-manifolds that do not change when the G_2-structure is deformed. This may shed some light on how many G_2-structures a 7-manifold admits. For example, if two G_2-structures on the same manifold with different gauge theoretical invariants exist, one cannot be deformed into the other.

There are analytic difficulties present in dimension 7 that were not there in dimension 4, and therefore the study of G_2-instantons has mainly focused on the construction of examples. The examples that have appeared in the literature so far are [Wal13a], using a gluing construction on Joyce's Generalised Kummer construction, [SEW15, MNSE21, Wal16] using a gluing construction on the Twisted Connected Sum construction, and [GN95, LO18, LO20] using cohomogeneity one methods.

We add to this by generalising the results from [Wal13a]: we prove a gluing theorem that can be used to construct G_2-instantons on the G_2-manifolds from [JK21]. This manifold construction begins with a G_2-manifold Y and a group of involutions Γ on it. (In many situations) The set of fixed points L of Γ is a smooth three-dimensional submanifold. Therefore, Y/Γ is an orbifold whose singular set is three-dimensional, and a fibre of the normal bundle of L in Y/Γ looks like $\mathbb{C}^2/\{\pm 1\}$. The four-dimensional Eguchi–Hanson space is a resolution of the orbifold $\mathbb{C}^2/\{\pm 1\}$. Hence, replacing every fibre of the normal bundle of L in $M/\langle\iota\rangle$ by an Eguchi–Hanson space yields some smooth manifold N. Using formidable analysis, [JK21] then constructs a torsion-free G_2-structure on N.

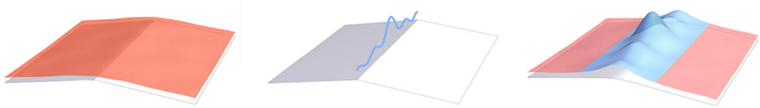

Fig. 1. Illustration of the gluing construction. Left: a G_2-instanton θ (in red) over a G_2-orbifold (in gray). Middle: a Fueter section s (in blue) over the singular set of the orbifold. Right: the underlying smooth manifold is a resolution of the orbifold from the beginning. The gluing construction provides a connection over this smooth manifold which looks like s close to the resolution locus, and looks like θ far away from the resolution locus

Given a G_2-instanton θ on Y/Γ one may be able to construct from it a G_2-instanton on N. To do this, one needs a connection on each of the glued in Eguchi–Hanson spaces. One way to get such a connection is by taking a family, say s, of anti-self-dual instantons over Eguchi–Hanson space. If θ and s satisfy a simple topological compatibility condition, one can glue them together to a one-parameter family of connections A_t. This is in general not a G_2-instanton, but in this article we prove theorems showing that one can perturb A_t to genuine G_2-instantons in some situations. A simple choice for a family s is to choose the same infinitesimally rigid instanton on each of the Eguchi–Hanson spaces, which is the setting of our first theorem (cf. Theorem 4.133):

Theorem. *Assume that the section s is given by a rigid ASD-instanton in every point $x \in L$, and assume that the connection θ used to define the approximate G_2-instanton A_t is infinitesimally rigid.*
There exists $c > 0$ such that for small t there exists $\underline{a}_t = (a_t, \xi_t) \in C^{1,\alpha}(\Omega^0 \oplus \Omega^1(\mathrm{Ad}\, E_t))$ such that $\widetilde{A}_t := A_t + a_t$ is a G_2-instanton. Furthermore, $\underline{a}_t$ satisfies $||\underline{a}_t||_{C^{1,\alpha}_{-1,\delta;t}} \leq ct^{1/18}$.

Here, $\alpha \in (0, 1)$ must be a small number and $|| \cdot ||_{C^{1,\alpha}_{-1,\delta;t}}$ denotes a weighted Hölder norm. We use this theorem to construct two types of examples of G_2-instantons. First, on the Generalised Kummer Construction [Joy96], viewed as a special case of the extended construction from [JK21] (cf. Corollary 5.4):

Corollary. *Let Γ act on T^7 as defined in Eq. (5.1) and let N'_t denote the one parameter family of resolutions of T^7/Γ from Sect. 3.1. Then, for t small enough, there exist* 2205 *non-flat, irreducible G_2-instantons with structure group* SO(3) *over N' which are pairwise not gauge equivalent.*

Second, in [JK21, Section 7.3] a resolution of $(T^3 \times \mathrm{K3})/\mathbb{Z}_2^2$ was constructed, and we construct a G_2-instanton on it (cf. Corollary 5.15):

Corollary. *Let N_t denote the one parameter family of resolutions of $(T^3 \times X)/\Gamma$ from Sect. 5.2.1. Then, for t small enough, there exists an irreducible G_2-instanton with structure group* SO(3) *over the resolution N_t.*

Thanks to the improved control over the torsion-free G_2-structure on resolutions of T^7/Γ from [Pla20, Theorem 4.58] we have an even stronger gluing theorem on the Generalised Kummer Construction from [Joy96]. On these manifolds, we need not require that the family s is given by rigid instantons. That said, an arbitrary family s may behave very wildly in the direction of the three-manifold L, and our construction does not produce a G_2-instanton in this case. However, if s satisfies a first order equation in the direction of L, called the Fueter equation, then the construction still works (cf. Theorem 4.134):

Theorem. *Let $N \to Y'$ be the resolution of the orbifold $Y' = T^7/\Gamma$ from before. Assume that the connection θ used to define the approximate G_2-instanton A_t is infinitesimally rigid and that s is an infinitesimally rigid Fueter section.*
There exists $c > 0$ such that for small t there exists an $\underline{a}_t = (a_t, \xi_t) \in C^{1,\alpha}(\Omega^0 \oplus \Omega^1(\mathrm{Ad}\, E_t))$ such that $\widetilde{A}_t := A_t + a_t$ is a G_2-instanton. Furthermore, $\underline{a}_t$ satisfies $||\underline{a}_t||_{\mathfrak{X}_t} \leq ct^{2-2\alpha}$.

Here, $|| \cdot ||_{\mathfrak{X}_t}$ denotes a complicated composite norm. It consists of a part that is harmonic in the Eguchi–Hanson directions in the gluing region and a rest, and the two parts are scaled differently.
Unfortunately, no genuine examples of these more general ingredients are known. That is: all known rigid Fueter sections are actually sections of rigid instantons.
The article is structured as follows: in Sect. 2 we prepare some facts that are used later on. Notably, in Proposition 2.22 we give a proof of the folklore result that the moduli spaces of framed instantons over an ALE space and over its compactification are in bijection. Following this, in Sect. 3, we review the two construction methods for torsion-free G_2-structures from [Joy96] and [JK21]. In Sect. 4 we prove our gluing theorem for G_2-instantons: we first construct the approximate solution A_t, then construct the perturbation to a genuine G_2-instanton. Last, in Sect. 5, we apply this construction method to the construction new G_2-instantons.

2. Background

In this section we briefly provide the necessary background in G_2-geometry, as well as gauge theory in dimensions 4 and 7.

The material in Sect. 2.1 about G_2-geometry is completely standard, but is important for us to fix notations.
Section 2.2 reviews gauge theory in dimension 4, and its content is also well known, but we provide proofs of some statements for which we could not locate a proof in the literature: first, the auxiliary Proposition 2.13 constructing a trivialisation of a principal bundle around an orbifold singularity of a certain nice form; second, the folklore result Proposition 2.22 that the moduli spaces of instantons on an ALE space and on its one point compactification are in bijection; third, the auxiliary Proposition 2.38, showing that the action of the isometry group of Eguchi–Hanson space lifts to an action on the tautological bundle defined over the product of Eguchi–Hanson space and the moduli space of instantons on it, roughly speaking.
The very short section Sect. 2.3 about gauge theory in dimension 7 is again completely standard.

2.1. G_2-structures We now introduce G_2-structures and their torsion, following the treatment in [Joy00].

Definition 2.1 (*Definition 10.1.1 in* [Joy00]). Let $(x_1, \dots, x_7)$ be coordinates on $\mathbb{R}^7$. Write $\mathrm{d}x_{ij\dots l}$ for the exterior form $\mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \dots \wedge \mathrm{d}x_l$. Define $\varphi_0 \in \Omega^3(\mathbb{R}^7)$ by

$$\varphi_0 = \mathrm{d}x_{123} + \mathrm{d}x_{145} + \mathrm{d}x_{167} + \mathrm{d}x_{246} - \mathrm{d}x_{257} - \mathrm{d}x_{347} - \mathrm{d}x_{356}. \tag{2.2}$$

The subgroup of $\mathrm{GL}(7, \mathbb{R})$ preserving φ_0 is the exceptional Lie group G_2. It also fixes the Euclidean metric $g_0 = \mathrm{d}x_1^2 + \dots + \mathrm{d}x_7^2$, the orientation on $\mathbb{R}^7$, and $*\varphi_0 \in \Omega^4(\mathbb{R}^7)$.

Definition 2.3. The skew-symmetric bilinear map $\times : \mathbb{R}^7 \to \mathbb{R}^7$ defined by

$$\varphi_0(u, v, w) = g_0(u \times v, w)$$

for $u, v, w \in \mathbb{R}^7$ is called the *cross product induced by* φ.

Theorem 2.4 (Theorem 8.5 in [SW17]). *Let* $\psi = *\varphi_0$. *Then* $\Lambda^*(\mathbb{R}^7)^*$ *splits into irreducible representations of* G_2 *as follows:*

$$\begin{aligned}\Lambda^1 V^* &= \Lambda^1_7,\\ \Lambda^2 V^* &= \Lambda^2_7 \oplus \Lambda^2_{14},\\ \Lambda^3 V^* &= \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}\end{aligned}$$

and correspondingly for $\Lambda^k(\mathbb{R}^7)^* \simeq \Lambda^{7-k}(\mathbb{R}^7)^*$ *with* $k = 4, 5, 6$. *Here,* $\dim \Lambda^k_d = d$ *and*

$$\begin{aligned}\Lambda^2_7 &:= \{\alpha : *(\alpha \wedge \varphi_0) = 2\alpha\} = \{i(u)\varphi_0 : u \in \mathbb{R}^7\} \simeq \Lambda^1_7,\\ \Lambda^2_{14} &:= \{\alpha : *(\alpha \wedge \varphi_0) = -\alpha\} = \{\alpha : \alpha \wedge \psi = 0\} \simeq \mathfrak{g}_2,\\ \Lambda^3_1 &:= \langle \varphi_0 \rangle,\\ \Lambda^3_7 &:= \{i(u)\psi : u \in \mathbb{R}^7\} \simeq \Lambda^1_7, \ \textit{and}\\ \Lambda^3_{27} &:= \{\alpha : \alpha \wedge \varphi_0 = 0 \ \textit{and} \ \alpha \wedge \psi = 0\} \simeq \mathrm{Sym}_0(\mathbb{R}^7)\end{aligned}$$

Definition 2.5. Let M be an oriented 7-manifold. A principal subbundle Q of the bundle of oriented frames with structure group G_2 is called a *G_2-structure*. Viewing Q as a set of linear maps from tangent spaces of M to $\mathbb{R}^7$, there exists a unique $\varphi \in \Omega^3(M)$ such that Q identifies φ with $\varphi_0 \in \Omega^3(\mathbb{R}^7)$ at every point.
Such G_2-structures are in 1-1 correspondence with 3-forms on M for which there exists an oriented isomorphism mapping them to φ_0 at every point. We will therefore also refer to such 3-forms as G_2-structures.

Let M be a manifold with G_2-structure φ. We call $\nabla\varphi$ the *torsion* of a G_2-structure $\varphi \in \Omega^3(M)$. Here, ∇ denotes the Levi-Civita induced by φ in the following sense: we have $G_2 \subset \mathrm{SO}(7)$, so φ defines a Riemannian metric g on M, which in turn defines a Levi-Civita connection. As a shorthand, we also use the following notation: write $\Theta(\varphi) = *\varphi$, where "$*$" denotes the Hodge star defined by g. Using this, the following theorem gives a characterisation of torsion-free G_2-manifolds:

Theorem 2.6 (Propositions 10.1.3 and 10.1.5 in [Joy00]). *Let M be an oriented* 7-*manifold with G_2-structure φ with induced metric g. The following are equivalent:*

(i) $\mathrm{Hol}(g) \subseteq G_2$,
(ii) $\nabla\varphi = 0$ *on M, where ∇ is the Levi-Civita connection of g, and*
(iii) $\mathrm{d}\varphi = 0$ *and* $\mathrm{d}\Theta(\varphi) = 0$ *on M.*

If these hold then g is Ricci-flat.

The goal of Sect. 3 will be to construct G_2-structures that induce metrics with holonomy *equal* to G_2. A torsion-free G_2-structure alone only guarantees holonomy *contained* in G_2, but in the compact setting a characterisation of manifolds with holonomy equal to G_2 is available:

Theorem 2.7 (Proposition 10.2.2 and Theorem 10.4.4 in [Joy00]). *Let M be a compact oriented manifold with torsion-free G_2-structure φ and induced metric g. Then* $\mathrm{Hol}(g) = G_2$ *if and only if* $\pi_1(M)$ *is finite. In this case the moduli space of metrics with holonomy G_2 on M, up to diffeomorphisms isotopic to the identity, is a smooth manifold of dimension $b^3(M)$.*

Note that this theorem makes no statement about the existence of a torsion-free G_2-structure in the first place. Finding a characterisation of manifolds which admit a torsion-free G_2-structure and even the construction of examples remain challenging problems in the field.
Later on, we will investigate perturbations of G_2-structures and analyse how that changes their torsion. To this end, we will use the following estimates for the map Θ defined before:

Proposition 2.8 (Proposition 10.3.5 in [Joy00] and eqn. (21) of part I in [Joy96]). *There exists $\epsilon > 0$ and $c > 0$ such that whenever M is a 7-manifold with G_2-structure φ satisfying* $\mathrm{d}\varphi = 0$*, then the following is true. Suppose $\chi \in C^\infty(\Lambda^3 T^*M)$ and $|\chi| \leq \epsilon$. Then $\varphi + \chi$ is a G_2-structure, and*

$$\Theta(\varphi + \chi) = *\varphi - T(\chi) - F(\chi), \tag{2.9}$$

where "$*$" denotes the Hodge star with respect to the metric induced by φ, $T : \Omega^3(M) \to \Omega^4(M)$ is a linear map (depending on φ), and F is a smooth function from the closed ball of radius ϵ in $\Lambda^3 T^*M$ to $\Lambda^4 T^*M$ with $F(0) = 0$. Furthermore,

$$\begin{aligned}
|F(\chi)| &\leq c\,|\chi|^2, \\
|\mathrm{d}(F(\chi))| &\leq c\left\{|\chi|^2\left|\mathrm{d}^*\varphi\right| + |\nabla\chi|\,|\chi|\right\}, \\
[\,\mathrm{d}(F(\chi))]_\alpha &\leq c\left\{[\chi]_\alpha\,||\chi||_{L^\infty}\left|\left|\mathrm{d}^*\varphi\right|\right|_{L^\infty} + ||\chi||^2_{L^\infty}\,[\mathrm{d}^*\varphi]_\alpha\right. \\
&\qquad \left. +[\nabla\chi]_\alpha\,||\chi||_{L^\infty} + ||\nabla\chi||_{L^\infty}\,[\chi]_\alpha\right\},
\end{aligned}$$

as well as

$$\begin{aligned}
|\nabla(F(\chi))| &\leq c\left\{|\chi|^2\,|\nabla\varphi| + |\nabla\chi|\,|\chi|\right\}, \\
[\nabla(F(\chi))]_{C^{0,\alpha}} &\leq c\left\{[\chi]_\alpha\,||\chi||_{L^\infty}\,||\nabla\varphi||_{L^\infty} + ||\chi||^2_{L^\infty}\,[\nabla\varphi]_\alpha\right. \\
&\qquad \left. +[\nabla\chi]_\alpha\,||\chi||_{L^\infty} + ||\nabla\chi||_{L^\infty}\,[\chi]_\alpha\right\}.
\end{aligned}$$

Here, $|\cdot|$ denotes the norm induced by φ, ∇ denotes the Levi-Civita connection of the metric induced by φ, and $[\cdot]_{C^{0,\alpha}}$ denotes the unweighted Hölder semi-norm induced by this metric.

Finally, the landmark result on the existence of torsion-free G_2-structures is the following theorem. It first appeared in [Joy96, part I, Theorem A], and we present a rewritten version in analogy with [JK21, Theorem 2.7]:

Theorem 2.10. *Let α, K_1, K_2, K_3 be any positive constants. Then there exist $\epsilon \in (0, 1]$ and $K_4 > 0$, such that whenever $0 < t \leq \epsilon$, the following holds.*
Let M be a compact oriented 7-manifold, with G_2-structure φ with induced metric g satisfying $\mathrm{d}\varphi = 0$*. Suppose there is a closed 3-form ψ on M such that* $\mathrm{d}^*\varphi = \mathrm{d}^*\psi$ *and*

(i) $||\psi||_{C^0} \leq K_1 t^\alpha$, $||\psi||_{L^2} \leq K_1 t^{7/2+\alpha}$, *and* $||\psi||_{L^{14}} \leq K_1 t^{-1/2+\alpha}$.
(ii) The injectivity radius inj *of* g *satisfies* $\mathrm{inj} \geq K_2 t$.
(iii) The Riemann curvature tensor Rm *of* g *satisfies* $||\mathrm{Rm}||_{C^0} \leq K_3 t^{-2}$.

Then there exists a smooth, torsion-free G_2*-structure* $\widetilde{\varphi}$ *on* M *such that* $||\widetilde{\varphi} - \varphi||_{C^0} \leq K_4 t^\alpha$ *and* $[\widetilde{\varphi}] = [\varphi]$ *in* $H^3(M, \mathbb{R})$. *Here all norms are computed using the original metric* g.

On $\mathbb{H}$ with coordinates (y_0, y_1, y_2, y_3) we have the three symplectic forms $\omega_1, \omega_2, \omega_3$ given as

$$\omega_0 = \mathrm{d}y_0 \wedge \mathrm{d}y_1 + \mathrm{d}y_2 \wedge \mathrm{d}y_3,$$
$$\omega_1 = \mathrm{d}y_0 \wedge \mathrm{d}y_2 - \mathrm{d}y_1 \wedge \mathrm{d}y_3,$$
$$\omega_2 = \mathrm{d}y_0 \wedge \mathrm{d}y_3 + \mathrm{d}y_1 \wedge \mathrm{d}y_2.$$

Identify $\mathbb{R}^7$ with coordinates $(x_1, \ldots, x_7)$ with $\mathbb{R}^3 \oplus \mathbb{H}$ with coordinates $((x_1, x_2, x_3), (y_1, y_2, y_3, y_4))$. Then we have for $\varphi_0, *\varphi_0$ from Definition 2.1:

$$\varphi_0 = \mathrm{d}x_{123} - \sum_{i=1}^{3} \mathrm{d}x_i \wedge \omega_i, \quad *\varphi_0 = \mathrm{vol}_{\mathbb{H}} - \sum_{\substack{(i,j,k)=(1,2,3) \\ \text{and cyclic permutation}}} \omega_i \wedge \mathrm{d}x_{jk}. \tag{2.11}$$

This linear algebra statement easily extends to product manifolds in the following sense: if X is a Hyperkähler 4-manifold, and $\mathbb{R}^3$ is endowed with the Euclidean metric, then $\mathbb{R}^3 \times X$ has a G_2-structure. The G_2-structure is given by the same formula as in the flat case, namely Eq. (2.11), after replacing $(\omega_1, \omega_2, \omega_3)$ with the triple of parallel symplectic forms defining the Hyperkähler structure on X. This *product* G_2*-structure* will be glued into G_2-orbifolds Sect. 3.

2.2. Gauge theory in dimension 4 In this part we briefly review the theory of ASD instantons on ALE spaces. We follow the treatment of [Nak90]. A treatment of compact 4-manifolds can be found in [DK90].

Let $\Gamma \subset \mathrm{SU}(2)$ be a finite subgroup and let X be an ALE 4-manifold asymptotic to $\mathbb{C}^2/\Gamma$. Even though X is non-compact, some of the results from gauge theory on compact manifolds carry over to this setting. First, we explain a correspondence between gauge equivalence classes of connections on X and on its one point compactification $\hat{X} = X \cup \{\infty\}$. For $\hat{X}$, we have:

Proposition 2.12 (p.687 in [Kro89] and Proposition 2.36 in [Wal13b]). *Let* (X, g) *be an ALE manifold asymptotic to* $\mathbb{C}^2/\Gamma$ *and let* $\hat{X} = X \cup \{\infty\}$ *be the one point compactification of* X.

1. *The topological space* $\hat{X}$ *is an orbifold and there exist a neighbourhood* V *of* ∞ *and an orbifold chart* $f : B^4/\Gamma \to V$, *where* B^4 *is the unit ball in* $\mathbb{R}^4$.
2. *The orbifold* $\hat{X}$ *carries an orbifold metric* $\hat{g}$ *of regularity* $C^{3,\alpha}$ *for any* $\alpha \in (0, 1)$ *such that the restriction of* $\hat{g}$ *to* $X \subset \hat{X}$ *is conformally equivalent to* g.

Let G be a compact connected Lie group with a faithful representation $G \to \mathrm{GL}(V)$. Let $\hat{P}$ be an orbifold G-bundle over $\hat{X}$ and denote its restriction to X by P, i.e. $P = \hat{P}|_X$. That is, $\hat{P}$ restricted to $V \simeq B^4/\Gamma$ from Proposition 2.12 is the trivial bundle $B^4 \times G$ together with a fixed lift of the action of Γ on B^4 to $B^4 \times G$. Over the point $0 \in B^4$, this defines a homomorphism $\rho : \Gamma \to G$. The following proposition states that this homomorphism essentially characterises the orbifold bundle over B^4 completely.

Proposition 2.13. *There exists a trivialisation* $\kappa : \hat{P}|_{B^4} \to B^4 \times G$ *such that* Γ *acts through left multiplication by* ρ*:*

$$\gamma \cdot \kappa^{-1}(b, g) = \kappa^{-1}(\gamma \cdot b, \rho(\gamma)g) \textit{ for } \gamma \in \Gamma, (b, g) \in B^4 \times G. \tag{2.14}$$

Proof. The lift of the action of Γ to $B^4 \times G$ can be viewed as an element $w \in C^\infty(B^4, \mathrm{Hom}(\Gamma, G))$ via $\gamma \cdot (b, g) = (\gamma \cdot b, w(b)(\gamma) \cdot g)$. The space B^4 is connected, so by Corollary A.9 the conjugacy class of w does not change over B^4. That is, there exists $\sigma \in C^\infty(B^4, G)$ such that $l_\sigma r_{\sigma^{-1}} w \in C^\infty(B^4, \mathrm{Hom}(\Gamma, G))$ is constant and $l_\sigma r_{\sigma^{-1}} w(0) = \rho$. Thus σ defines a trivialisation of $B^4 \times G$ in which Γ acts through left multiplication via ρ. □

Because of Proposition 2.13 we can fix a trivialisation of $\hat{P}$ over B^4 such that Γ acts through left multiplication by ρ. Then denote by A_0 any extension of the product connection with respect to this trivialisation to all of $\hat{P}$. Different choices of extension will give rise to the very same spaces in Eq. (2.17). We identify $[R, \infty) \times S^3/\Gamma \simeq X \backslash K$ for some $R > 0$ big enough and a compact set $K \subset X$. Then the monodromy representation of A_0 restricted to $\{t\} \times S^3/\Gamma$, say $h : \pi_1(\{t\} \times S^3/\Gamma) \to G$, satisfies

$$h = \rho \tag{2.15}$$

under the canonical identification $\Gamma \simeq \pi_1(\{t\} \times S^3/\Gamma)$. Extend the projection onto the first component $X\backslash K \simeq [R, \infty) \times S^3 \to [R, \infty)$ to a smooth positive function r on all of X. For a non-negative integer l, a weight $\delta \in \mathbb{R}$, and $p \geq 1$ define the weighted Sobolev norm on the k-forms with values in the adjoint bundle with compact support $\Omega_0^k(\mathrm{Ad}\, P)$ via

$$||\alpha||_{L^p_{l,\delta}} = \sum_{j=0}^{l} \left(\int_X |\nabla^j_{A_0} \alpha|^p r^{-(\delta - j)p - 4} \, \mathrm{d}V \right)^{1/p}, \tag{2.16}$$

and denote by $L^p_{l,\delta}(\Lambda^k \otimes \mathrm{Ad}\, P)$ the completion of $\Omega_0^k(\mathrm{Ad}\, P)$ with respect to the norm $||\cdot||_{L^p_{l,\delta}}$.

As before, set $E = P \times_G V$ and for $l \geq 3$ define

$$\begin{aligned}
\mathscr{A}^{l,\delta} &= \{A_0 + \alpha : \alpha \in L^2_{l,\delta}(\Lambda^1 \otimes \mathrm{Ad}\, P)\}, \\
\mathscr{G}_0^{l+1,\delta+1} &= \{s \in L^2_{l+1,\mathrm{loc}}(\mathrm{End}(E)) : s(x) \in G \text{ for all } x \in G, ||s - \mathrm{Id}||_{L^2_{l+1,\delta+1}} < \infty\}, \\
G_\rho &= \{s \in G : s\rho s^{-1} = \rho\}, \\
\mathscr{G}^{l+1,\delta+1} &= \{s \in L^2_{l+1,\mathrm{loc}}(\mathrm{End}(E)) : s(x) \in G \text{ for all } x \in G, \\
&\qquad ||s - s_\infty||_{L^2_{l+1,\delta+1}} < \infty \text{ for some } s_\infty \in G_\rho\}.
\end{aligned} \tag{2.17}$$

In the definition of $\mathscr{G}^{l+1,\delta+1}$ we regarded $s_\infty \in G_\rho$ as an element in $C^\infty(\mathrm{End}(E))$ as follows: consider $\hat{P}$ over B^4 defined by the orbifold chart around ∞. Using the trivialisation from Proposition 2.13, this canonically defines a gauge transformation over B^4. (It is the same to say that we obtain a gauge transformation by parallel transport with respect to A_0.) This gauge transformation is Γ-equivariant by definition of G_ρ and Proposition

2.13. We then extend it arbitrarily on the rest of $\hat{X}$ to an element in $C^\infty(\mathrm{End}(E))$. The choice of the extension does not matter for the condition $||s - s_\infty||_{L^2_{l+1,\delta+1}} < \infty$.

The gauge groups $\mathscr{G}_0^{l+1,\delta+1}$ and $\mathscr{G}^{l+1,\delta+1}$ both act on $\mathscr{A}^{l,\delta}$, and the quotient spaces $\mathscr{A}^{l,\delta}/\mathscr{G}_0^{l+1,\delta+1}$ and $\mathscr{A}^{l,\delta}/\mathscr{G}^{l+1,\delta+1}$ are called the moduli space of framed connections and the moduli space of unframed connections, respectively. We can restrict to anti-self-dual connections:

$$\mathscr{A}^{l,\delta}_{\mathrm{asd}} = \{A \in \mathscr{A}^{l,\delta} : A \text{ is anti-self-dual}\}$$

and obtain the *moduli space of framed ASD connections* $M^{l,\delta} := \mathscr{A}^{l,\delta}_{\mathrm{asd}}/\mathscr{G}_0^{l+1,\delta+1}$ and the *moduli space of ASD connections* $\mathscr{A}^{l,\delta}_{\mathrm{asd}}/\mathscr{G}^{l+1,\delta+1}$.

The four quotient spaces $\mathscr{A}^{l,\delta}/\mathscr{G}_0^{l+1,\delta+1}$, $\mathscr{A}^{l,\delta}/\mathscr{G}^{l+1,\delta+1}$, $M^{l,\delta}$, and $\mathscr{A}^{l,\delta}_{\mathrm{asd}}/\mathscr{G}^{l+1,\delta+1}$ are topological spaces. For $M^{l,\delta}$ we will observe explicitly (cf. Theorem 2.23) that it is metrisable and therefore Hausdorff, and the same argument works for the other three quotient spaces, cf. [DK90, Lemma 4.2.4].

Moving on to the orbifold, we define:

Definition 2.18. For $l \geq 3$ let

$$\begin{aligned}
\mathscr{A}^{l,\mathrm{orb}}_{\mathrm{asd}} &= \{A_0 + \alpha : \alpha \in L^2_l(\Lambda^1 \otimes \mathrm{Ad}\,\hat{P}))\},\\
\mathscr{G}^{l+1,\mathrm{orb}} &= \{s \in L^2_{l+1}(\mathrm{End}\,V) : s(x) \in G \text{ for all } x \in \hat{X}, s(\infty) \in G_\rho\},\\
\mathscr{G}_0^{l+1,\mathrm{orb}} &= \{s \in \mathscr{G}^{l+1,\mathrm{orb}} : s(\infty) = \mathrm{Id}\}.
\end{aligned}$$

Then $\mathscr{G}^{l+1,\mathrm{orb}}$ and $\mathscr{G}_0^{l+1,\mathrm{orb}}$ both act on $\mathscr{A}^{l,\mathrm{orb}}_{\mathrm{asd}}$ and we can form the quotient spaces $\mathscr{A}^{l,\mathrm{orb}}_{\mathrm{asd}}/\mathscr{G}^{l+1,\mathrm{orb}}$ and $M^{l,\mathrm{orb}} = \mathscr{A}^{l,\mathrm{orb}}_{\mathrm{asd}}/\mathscr{G}_0^{l+1,\mathrm{orb}}$. Here, $M^{l,\mathrm{orb}}$ is called the *moduli space of framed ASD connections* on $\hat{X}$.

We have that these definitions are essentially independent of the chosen regularity l:

Proposition 2.19. *For $3 \leq l_1 < l_2$, the inclusion maps*

$$M^{l_1,orb} \hookrightarrow M^{l_2,orb}, \qquad M^{l_1,-2} \hookrightarrow M^{l_2,-2}$$

are homeomorphisms.

The proof of Proposition 2.19 works the same as in the compact case, i.e. the proof of [DK90, Proposition 4.2.16]. The only difference is that in the non-compact case, i.e. for the claim $M^{l_1,-2} \hookrightarrow M^{l_2,-2}$, one has to take the weighted Sobolev norms from Eq. (2.16). These have their own versions of the Sobolev embedding theorem and, if the weight is non-positive, the multiplication theorem for Sobolev norms also holds. These properties of weighted Sobolev norms are proved in [Pac13, Corollary 6.8].

Proposition 2.20. *For any $A \in \mathscr{A}^{l,-2}_{\mathrm{asd}}$ there exists a connection $\hat{A} \in \mathscr{A}(\hat{P})$ satisfying $\hat{A}|_P = A$.*

Proof. Corollary A.14 gives a bundle P' over $\hat{X}$ with connection A' together with an injective bundle homomorphism $\xi : P \to P'$. After fixing a trivialisation of $\hat{P}$ around ∞, this canonically defines an isomorphism of orbifold G-bundles $h : \hat{P} \to P'$, and $\hat{A} := h^*(A')$ satisfies $\hat{A}|_P = A$. ◻

Definition 2.21. Define the map

$$\Psi : M^{3,-2} \to M^{3,\mathrm{orb}}$$

as follows: for $[A_0 + a] \in M^{3,-2}$ let $\hat{A} \in \mathscr{A}(\hat{P})$ be the induced connection from Proposition 2.20 and set $\Psi([A_0 + a]) := [\hat{A}]$.

Proposition 2.22. *The function* Ψ *from Definition 2.21 is bijective.*

Proof. **Ψ is injective:** let $[A_0 + a], [A_0 + \widetilde{a}] \in M^{3,-2}$ such that $\Psi([A_0 + a]) = [\hat{A}]$ as well as $\Psi([A_0 + \widetilde{a}]) = [\hat{A}']$. If $[\hat{A}] = [\hat{A}']$, then $\hat{A}' = s\hat{A}$ for some $s \in \mathscr{G}_0^{4,\mathrm{orb}}$. We have $s(\infty) = \mathrm{Id}$, so $(s - \mathrm{Id}) = \mathcal{O}(|x|)$ and $\nabla^k_{A_0}(s - \mathrm{Id}) = \mathcal{O}(1)$ for $k \in \{1, 2, 3, 4\}$. Here, $\nabla^k_{A_0}$ includes terms containing the Levi-Civita connection for the orbifold metric $\hat{g}$ on $\hat{X}$ for $k > 1$, and $|x|$ denotes the distance from $\infty \in \hat{X}$ in this metric. In particular, $\nabla^k_{A_0}(s - \mathrm{Id}) = \mathcal{O}(|x|^{1-k})$. We have

$$\left|\nabla^k_{A_0}(s - \mathrm{Id})\right|_g = (1 + r^2)^{-k} \left|\nabla^k_{A_0}(s - \mathrm{Id})\right|_{\hat{g}} = \mathcal{O}(r^{-2k}\,|x|^{1-k}) = \mathcal{O}(r^{-1-k}),$$

where g denotes the ALE metric, in the first step we used the definition of $\hat{g}$ from the proof of Proposition 2.12 and the fact that we are measuring a tensor with k covariant indices and 0 contravariant indices. Thus, $s \in \mathscr{G}_0^{4,-1}$. Therefore, $[A_0 + a] = [A_0 + \widetilde{a}]$ as elements in $M^{3,-2}$, which shows the claim.
Ψ is surjective: Let $[A_0 + a] \in M^{3,\mathrm{orb}}$, i.e. $A_0 + a \in \mathscr{A}^{3,\mathrm{orb}}_{\mathrm{asd}}$. Similar to the previous point we find that $\nabla^k_{A_0} a = \mathcal{O}(r^{-2-k})$. By construction $\Psi([(A_0 + a)|_X]) = [A_0 + a]$, which proves the claim. □

Because of Proposition 2.19 we will drop the regularity and decay from the notation of our moduli spaces most of the time. That is, we will often write M for $M^{l,\delta}$ with any $l \geq 3$ and $\delta = -2$. Likewise for $\mathscr{A}, \mathscr{G}, \mathscr{G}_0, \mathscr{A}^{\mathrm{orb}}, M^{\mathrm{orb}}, \mathscr{G}^{\mathrm{orb}}$, and $\mathscr{G}_0^{\mathrm{orb}}$.
The important results about the local structure of M are the following:

Theorem 2.23 (Theorem 2.4 and Proposition 5.1 in [Nak90]). *M is a nonsingular C^∞-manifold and for $[A] \in M$ its tangent space is isomorphic to*

$$H^1_{A,-2} := \{\alpha \in L^2_{l,-2}(\Lambda^1 \otimes \mathrm{Ad}\, P) : \delta_A(\alpha) = 0\},$$

where

$$\begin{aligned} \delta_A : \Omega^1(\mathrm{Ad}\, P) &\to (\Omega^0 \oplus \Omega^2_+)(\mathrm{Ad}\, P) \\ \alpha &\mapsto (\mathrm{d}^*_A \alpha, \mathrm{d}^+_A \alpha). \end{aligned} \tag{2.24}$$

For the linear operator δ_A we have the following analytic result:

Proposition 2.25 (Proposition 5.10 in [Wal13a]). *Let $A \in \mathscr{A}(E)$ be a finite energy ASD instanton on E. Then the following holds:*

1. *If $a \in \mathrm{Ker}\,\delta_A$ decays to zero at infinity, i.e., $\lim_{r\to\infty} \sup_{\rho(x)=r} |a|(x) = 0$, then $\nabla^k_A a = \mathcal{O}(|\pi|^{-3-k})$ for all $k \geq 0$.*
2. *If $(\xi, \omega) \in \mathrm{Ker}\,\delta^*_A$ decays to zero at infinity, then $(\xi, \omega) = 0$.*

The Hyperkähler triple of X acts on the 1-form part of $\Omega^1(\mathrm{Ad}\,P)$. It is checked in [Ito88, Section 4] together with [Ito85, Proposition 2.4] that this action restricts to $H^1_{A,-2}$ for all $[A] \in M$. We thus have a triple of complex structures on M. The following theorem states that this defines a Hyperkähler structure with respect to the standard metric on M:

Theorem 2.26 (Theorem 2.6 and Proposition 5.1 in [Nak90]). *The metric g_M defined by*

$$g_M(\alpha, \beta) = \int_X g(\alpha, \beta)\,\mathrm{vol}_X \qquad \textit{for } \alpha, \beta \in H^1_{A,-2}$$

and the Hyperkähler triple defined by acting with the Hyperkähler triple of X on the 1-form part of $\Omega^1(\mathrm{Ad}\,P)$ is well-defined on M and defines a Hyperkähler structure on M.

Theorem 2.27 (Theorem 2.47 in [Wal13b]). *Let $\rho : \Gamma \to G$ be a homomorphism, A_0 a connection on a bundle P that is flat at infinity as in Proposition 2.13 whose holonomy representation is equal to ρ in the sense of Eq. (2.15). Let $\delta \in (-3, -1)$ and $A = A_0 + \alpha$ for some $\alpha \in L^2_{1,\delta}(\Lambda^1 \otimes \mathrm{Ad}\,P)$. Then the L^2 index of δ_A, defined as*

$$\begin{aligned}&\dim\{a \in L^2(\Lambda^1 \otimes \mathrm{Ad}\,P) \cap C^\infty(\Lambda^1 \otimes \mathrm{Ad}\,P) : \delta_A(a) = 0\}\\ &\quad - \dim\{\underline{a} \in L^2((\Lambda^0 \oplus \Lambda^2_+) \otimes \mathrm{Ad}\,P) \cap C^\infty((\Lambda^0 \oplus \Lambda^2_+) \otimes \mathrm{Ad}\,P) : \delta^*_A(\underline{a}) = 0\},\end{aligned}$$

is given by

$$\mathrm{ind}\,\delta_A = -2\int_X p_1(\mathrm{Ad}\,P) + \frac{2}{|\Gamma|}\sum_{g\in\Gamma\setminus\{e\}} \frac{\chi_{\mathfrak{g}}(g) - \dim\mathfrak{g}}{2 - \mathrm{tr}\,g}. \tag{2.28}$$

Here $p_1(\mathrm{Ad}\,P)$ is the Chern-Weil representative of the first Pontrjagin class of P and $\chi_{\mathfrak{g}}$ is the character of g acting on $\mathfrak{g}$, the Lie algebra associated with G, via ρ, and $\mathrm{tr}\,g$ *is the trace of g acting on $\mathfrak{g}$. Moreover, if A is an ASD instanton, then* $\mathrm{ind}\,\delta_A = \dim\mathrm{Ker}\,\delta_A = \dim M$.

We will now explain one example of an ASD-instanton on Eguchi–Hanson space that will be needed later. To this end, we recall the construction of it as a Hyperkähler quotient as explained in [GRG97]. Let $\mathcal{M} = \mathbb{H}^2$ with quaternionic coordinates q_a, $a \in \{1, 2\}$, and let $\mathrm{U}(1)$ act on $\mathcal{M}$ via

$$q_a \mapsto q_a e^{it}, \quad t \in (0, 2\pi]. \tag{2.29}$$

A Hyperkähler moment map for this action is given by

$$\begin{aligned}\mu : \mathcal{M} &\to \mathrm{Im}(\mathbb{H}) \simeq \mathbb{R}^3 \otimes \mathfrak{u}(1)\\ (q_1, q_2) &\mapsto \frac{1}{2}\sum_{a\in\{1,2\}} q_a i \overline{q}_a.\end{aligned} \tag{2.30}$$

Let $\zeta = \frac{i}{2} \in \mathrm{Im}(\mathbb{H})$. The group $\mathrm{U}(1)$ acts freely on $\mu^{-1}(\zeta)$ and the general theory of Hyperkähler reduction gives rise to a Hyperkähler structure on the four-dimensional manifold $\mathcal{M}/\!/\!/\mathrm{U}(1) := \mu^{-1}(\zeta)/\,\mathrm{U}(1)$.

Definition 2.31. The Hyperkähler space $X_{\mathrm{EH}} = \mathcal{M}/\!/\!/\mathrm{U}(1)$ is called *Eguchi–Hanson space*.

From this, we get our first example of an ASD-instanton on Eguchi–Hanson space:

Proposition 2.32 (Section 2 in [GN92]). *The* $\mathrm{U}(1)$*-bundle* $\mathcal{R} := \mu^{-1}(i/2) \to X_{EH} = \mu^{-1}(i/2)/\mathrm{U}(1)$ *admits a non-flat finite energy ASD instanton A asymptotic to the representation* $\rho : \mathbb{Z}_2 \to \mathrm{U}(1)$ *determined by* $\rho(-1) = -1$ *in the sense of Eq. (2.15).*

There exists an ADHM-type construction of ASD-instantons on ALE spaces which generalises this example, cf. [KN90], but we will not need this here.
An additional property of $\mathcal{R}$ that we will need later is the following:

Proposition 2.33. *There exists a lift of the action of the holomorphic isometry group* $\mathrm{U}(2)/\{\pm 1\}$ *of* X_{EH} *to* $\mathcal{R}$.

Proof. We show in Proposition A.1 that the holomorphic isometry group $\mathrm{U}(2)/\{\pm 1\}$ is realised as an action of $\mathrm{U}(2)/\{\pm 1\}$ on $\mu^{-1}(i/2)$ that commutes with the action of $\mathrm{U}(1)$ on $\mu^{-1}(i/2)$. The action of $\mathrm{U}(2)/\{\pm 1\}$ on $\mu^{-1}(i/2)$ is the desired lift of the action of $\mathrm{U}(2)/\{\pm 1\}$ on X_{EH}. □

Remark 2.34. We can apply Theorem 2.27 to the $\mathrm{U}(1)$-bundle over X_{EH} defined before to find that it is rigid. As $\mathrm{Ad}\,\mathcal{R}$ has rank 1, we have that $p_1(\mathrm{Ad}\,\mathcal{R}) = c_2(\mathrm{Ad}\,\mathcal{R}^{\mathbb{C}}) = 0$, and plugging this into the index formula from Theorem 2.27 proves the claim.

Remark 2.35. On simply connected compact manifolds it is the case that any $\mathrm{U}(1)$-bundle admits an ASD-instanton that is unique up to the action of the gauge group. This is a consequence of the Hodge theorem. On non-compact manifolds a variation of the Hodge theorem for L^2-forms holds, see [Loc87, Example 0.15], and can be used to give an alternative proof of Remark 2.34 without the use of the index formula.

Before ending the section we will state two results about universal bundles that will be needed later. The following proposition is proved in [DK90, Proposition 5.2.17] for compact manifolds, but the proof carries over to the ALE setting with small alterations.

Proposition 2.36. *There exist*

- *a G-bundle* $\widetilde{\mathbb{P}}$ *over* $M \times \hat{X}$ *with a natural action of* $G_\rho \simeq \mathscr{G}/\mathscr{G}_0$ *on* $\widetilde{\mathbb{P}}$ *covering the action of* G_ρ *on M,*
- *a connection* $\widetilde{\mathbb{A}} \in \mathscr{A}(\widetilde{\mathbb{P}})$ *that is invariant under the action of* $G_\rho \simeq \mathscr{G}/\mathscr{G}_0$*, and*
- *for each choice of* $\phi \in \mathrm{Iso}_\Gamma(G, P_\infty)$ *a canonical isomorphism of G-bundles with* Γ *left action* $\underline{\phi} : \widetilde{\mathbb{P}}|_{M\times\{\infty\}} \to G \times M$

satisfying:

- *for any element* $[A] \in M$ *there exists an isomorphism* $\widetilde{\mathbb{P}}|_{\{[A]\}\times\hat{X}} \simeq \hat{P}$ *such that under this isomorphism* $\widetilde{\mathbb{A}}|_{\{[A]\}\times X}$ *and A agree up to the action of* $\mathscr{G}_0$*.*
- *if we decompose the curvature of* $\widetilde{\mathbb{A}}$ *over* $M \times X$ *according to the bi-grading on* $\Lambda^* T^*(M \times X)$ *induced by* $T^*(M \times X) = \pi_1^* T^* M \oplus \pi_2^* T^* X$*, then its components satisfy the following:*
  - $F^{1,1}_{\widetilde{\mathbb{A}}} \in \Gamma(\mathrm{Hom}(\pi_1^* T^* M, \pi_2^* T^* X \otimes \mathrm{Ad}\, P))$ *at* $([A], x)$ *is the evaluation of* $a \in T_{[A]}M$ *at x,*
  - $F^{0,2}_{\widetilde{\mathbb{A}}} \in \Gamma(\pi_2^* \Lambda^-(X)^* \otimes \mathrm{Ad}\, P)$*, where* Λ^- *is defined using the ALE metric on X,*
- $\underline{\phi}^* A_{product} = \widetilde{\mathbb{A}}|_{M\times\{\infty\}}$*, where* $A_{product} \in \mathscr{A}(G \times M)$ *denotes the product connection.*

By Proposition A.1, the group of holomorphic isometries acting on X_{EH} is $\mathrm{U}(2)/\{\pm 1\}$. This induces a non-effective action of $\mathrm{U}(2)$ on $\hat{X}_{\text{EH}}$ by demanding that each group element fixes $\infty \in \hat{X}_{\text{EH}}$. Then $\mathrm{U}(2)$ acts from the left on M (and equally M^{orb}) as follows: $\mathrm{U}(2)$ is connected, so $(u^{-1})^*E$ and E are homotopic bundles and in particular isomorphic. Different choices of isomorphism give rise to gauge equivalent connections, so $[(u^{-1})^*A] \in M$ is well-defined.
Later on (cf. Definition 4.9) we will need the following assumption:

Assumption 2.37. The action of $\mathrm{U}(2)$ on $M \times \hat{X}_{\text{EH}}$ can be lifted to an action on $\widetilde{\mathbb{P}}$ that preserves $\widetilde{\mathbb{A}}$.

In the examples constructed in Sect. 5 this assumption will be satisfied because of the following proposition:

Proposition 2.38. *Let $\widetilde{\mathbb{P}} \to M \times \hat{X}_{EH}$ be the tautological bundle with tautological connection $\widetilde{\mathbb{A}}$ from Proposition 2.36.*
If the action of $\mathrm{U}(2)$ on $\hat{X}_{EH}$ can be lifted to an action on $\hat{P}$, then the action of $\mathrm{U}(2)$ on $M \times \hat{X}_{EH}$ can be lifted to an action on $\widetilde{\mathbb{P}}$. If it exists, this lift can be chosen to preserve $\widetilde{\mathbb{A}}$.

Proof. First, assume that the action of $\mathrm{U}(2)$ on $\hat{X}_{\text{EH}}$ can be lifted to an action on $\hat{P}$. This is equivalent to saying that for all $g \in G$ there exists a bundle isomorphism $\xi_g : \hat{P} \to \hat{P}$ covering $g : \hat{X}_{\text{EH}} \to \hat{X}_{\text{EH}}$. The bundle $\widetilde{\mathbb{P}}$ is defined as $\widetilde{\mathbb{P}} = \pi_2^* \hat{P}/\mathscr{G}_0^{\text{orb}}$, where $\pi_2 : \mathscr{A}_{\text{asd}}^{\text{orb}} \times \hat{X}_{\text{EH}} \to \hat{X}_{\text{EH}}$ is the projection onto the second factor. Let $([A], x) \in M \times \hat{X}_{\text{EH}}$ and $[u] \in \widetilde{\mathbb{P}}_{([A],x)}$ where $u \in \left(\pi_2^* \hat{P}\right)_{(A,x)} \simeq \hat{P}_x$. We define $\kappa_g : \widetilde{\mathbb{P}} \to \widetilde{\mathbb{P}}$ covering $g : M \times \hat{X}_{\text{EH}} \to M \times \hat{X}_{\text{EH}}$ via $\kappa_g[u] := [\xi_g(u)]$. The check that this is well-defined and that this lift preserves $\widetilde{\mathbb{A}}$ is straightforward, using the definition of $\widetilde{\mathbb{A}}$ from [DK90, Section 5.2.3]. □

2.3. Gauge theory on G_2-manifolds

Definition 2.39. Let (Y, φ) be a G_2-manifold, $\psi = *_\varphi \varphi$, and E be a principal bundle over Y. A connection $A \in \mathscr{A}(E)$ is called a *G_2-instanton*, if $F_A \in \Gamma(\Lambda^2_{14} \otimes \operatorname{Ad} E)$, i.e. (by Theorem 2.4)

$$F_A \wedge \psi = 0, \tag{2.40}$$

where the wedge product is taken in the 2-form part of $\Lambda^2 \otimes \operatorname{Ad} E$.

Example 2.41. Let A be an ASD instanton on a bundle E over a Hyperkähler 4-fold X. Denote by $p_X : \mathbb{R}^3 \times X \to X$ the projection onto the second factor. Then $\mathbb{R}^3 \times X$ carries the torsion-free G_2-structure φ from Eq. (2.11), and $p_X^* A$ is a G_2-instanton on the bundle $p_X^* E$ with respect to this G_2-structure. To see this, let $\omega_1, \omega_2, \omega_3 \in \Omega^2(X)$ denote a Hyperkähler triple on X. These 2-forms are self-dual, thus A being ASD is equivalent to $F_A \wedge \omega_i = 0$ for $i \in \{1, 2, 3\}$. Recall that for the product G_2-structure, we have that

$$*\varphi = \psi = \frac{1}{2}\omega_1^2 - \mathrm{d}x_{12} \wedge \omega_3 - \mathrm{d}x_{23} \wedge \omega_1 - \mathrm{d}x_{31} \wedge \omega_2$$

and therefore

$$F_{p_X^* A} \wedge \psi = p_X^*(F_A) \wedge \psi = 0.$$

The linearisation of the G_2-instanton equation Eq. (2.40) is not elliptic. This problem is overcome in the following proposition:

Lemma 2.42 (Proposition 1.98 in [Wal13b]). *Let (Y, φ) be a compact G_2-manifold, $\psi = *_\varphi \varphi$, and E be a principal bundle over Y, and $A \in \mathscr{A}(E)$. Then A is a G_2-instanton if and only if there exists $\xi \in \Omega^0(Y, \mathrm{Ad}\, E)$ such that*

$$*(F_A \wedge \psi) + \mathrm{d}_A \xi = 0. \tag{2.43}$$

Adding the Coulomb gauge condition to this, we consider for a fixed connection $A \in \mathscr{A}(E)$, $\xi \in \Omega^0(Y, \mathrm{Ad}\, E)$, and $a \in \Omega^1(Y, \mathrm{Ad}\, E)$ the system

$$\begin{aligned} *(F_{A+a} \wedge \psi) + \mathrm{d}_{A+a}\xi &= 0 \\ \mathrm{d}_A^* a &= 0. \end{aligned} \tag{2.44}$$

Here, every solution (ξ, a) defines the G_2-instanton $A + a$ which is in Coulomb gauge with respect to A. A computation in coordinates shows that the linearisation of Eq. (2.44) is an elliptic operator:

Proposition 2.45. *The linearisation of Eq. (2.44) is*

$$\begin{aligned} L_A : (\Omega^0 \oplus \Omega^1)(Y, \mathrm{Ad}\, E) &\to (\Omega^0 \oplus \Omega^1)(Y, \mathrm{Ad}\, E) \\ \begin{pmatrix} \xi \\ a \end{pmatrix} &\mapsto \begin{pmatrix} 0 & \mathrm{d}_A^* \\ \mathrm{d}_A & *(\psi \wedge \mathrm{d}_A) \end{pmatrix} \begin{pmatrix} \xi \\ a \end{pmatrix} \end{aligned} \tag{2.46}$$

which is a self-adjoint elliptic operator if $\mathrm{d}^*\varphi = 0$.

Remark 2.47. A coordinate-free proof for the ellipticity of operator L_A is given in [RC98, Section 3, Lemma 4].

3. Resolutions of G_2-orbifolds

In this section we review the two manifold constructions featuring in this article: the Generalised Kummer Construction from [Joy96] and its generalisation from [JK21]. The most important notations are collected in Table 2.

3.1. Torsion-free G_2-structures on resolutions of T^7/Γ In the two articles [Joy96], Joyce constructed the first compact examples of manifolds with holonomy equal to G_2. One starts with the flat 7-torus T^7, which carries the flat G_2-structure φ_0. A quotient of the torus by a finite group of maps Γ preserving the G_2-structure still carries a flat G_2-structure, but has *singularities*. The singularities are modelled on $\mathbb{R}^3 \times (\mathbb{C}^2/\{\pm 1\})$ and admit a resolution $\mathbb{R}^3 \times X_{\mathrm{EH}}$. Gluing this resolution into T^7/Γ, we have two different G_2-structures near the resolution locus: the flat G_2-structure φ_0 and the product G_2-structure φ_t^P which depends on one real parameter that is proportional to the size of the unique minimal 2-sphere in X_{EH}. Gluing φ_0 and φ_t^P, one obtains a 1-parameter family of smooth manifolds N_t and G_2-structures $\varphi^t \in \Omega^3(N)$ depending on a single real parameter $t \in (0, 1)$. While φ_0 and φ_t^P are torsion-free, the glued structure φ^t is not, because of the error introduced by the gluing. However, one can check that

$$\left|\left|\varphi_t^P - \varphi^t\right|\right|_{C^k} \leq ct^4 \tag{3.1}$$

Table 1. Most important notation from Sect. 3

Notation	Description	References
$(T^7/\Gamma, \varphi_0)$, $(Y/\langle\iota\rangle, \varphi)$	G_2-orbifolds	
φ_t^P	Product structure on glued in part	Eqs. (3.1) and (3.14)
N_t	Resolution of G_2-orbifold	Theorem 3.2 and Definition 3.27
φ^t	Small torsion G_2-structure on resolution of T^7/Γ	
$\widetilde{\varphi}^t$	Torsion-free G_2-structure on resolution of T^7/Γ	
ν	Normal bundle of $\text{fix}(\iota)$	
φ_t^ν	G_2-structure on ν	Section 3.2.2
$\widetilde{\varphi}_t^\nu$	Improvement of φ_t^ν	Section 3.2.2
P	Eguchi–Hanson bundle glued into orbifold	Eq. (3.11)
$\widetilde{\varphi}_t^P$	Improvement of φ_t^P	Proposition 3.15
φ_t^N	Small torsion G_2-structure on N_t	Eq. (3.30)
$\widetilde{\varphi}_t^N$	Torsion-free G_2-structure on N_t	Eq. (3.33)
$\eta, \zeta, \xi_{1,2}, \xi_{0,3}, \tau_{1,1}, \chi_{1,3}, \theta_{3,1}, \theta_{2,2}, \upsilon_{1,2}, \alpha_{0,2}, \alpha_{2,0}, \beta_{0,3}, \beta_{2,1}$	Correction terms of secondary importance	Propositions 3.8, 3.15 and 3.19 and Theorem 3.26

for any k and a constant $c > 0$ independent of t. This implies that for small t, Theorem 2.10 can be applied and one obtains a torsion-free G_2-structure $\widetilde{\varphi}^t \in \Omega^3(N)$ satisfying

$$\left|\left|\widetilde{\varphi}^t - \varphi^t\right|\right|_{C^0} \leq ct^{1/2}.$$

We need an improved version of this estimate using weighted Hölder norms denoted by $||\cdot||_{C^{2,\alpha/2}_{\beta;t}}$ from [Pla20]. At this point, we will not reproduce the definition of these norms from the reference, because we will define more general norms in Definition 4.19.

Theorem 3.2 (Theorem 4.58 in [Pla20]). *Let N_t be the resolution of T^7/Γ and $\varphi^t \in \Omega^3(N_t)$ the G_2-structure with small torsion from Sect. 3.1. There exists $c > 0$ independent of t such that the following is true: for t small enough, there exists $\eta^t \in \Omega^2(N_t)$ such that $\widetilde{\varphi} = \varphi^t + \mathrm{d}\eta^t$ is a torsion-free G_2-structure, and η^t satisfies*

$$\left|\left|\eta^t\right|\right|_{C^{2,\alpha/2}_{\beta;t}} \leq ct^{7/2-\beta}.$$

In particular,

$$\begin{aligned}\left|\left|\widetilde{\varphi} - \varphi^t\right|\right|_{L^\infty} \leq ct^{5/2} \text{ and } \left|\left|\widetilde{\varphi} - \varphi^t\right|\right|_{C^{0,\alpha/2}} \\ \leq ct^{5/2-\alpha/2} \text{ as well as } \left|\left|\widetilde{\varphi} - \varphi^t\right|\right|_{C^{1,\alpha/2}} \leq ct^{3/2-\alpha/2}.\end{aligned}$$

3.2. Torsion-free G_2-structures on Joyce–Karigiannis manifolds In [JK21], the authors constructed new examples of compact manifolds with holonomy G_2. They first used a gluing procedure to construct a G_2-structure with small torsion and then applied Theorem 2.10 to perturb this G_2-structure into a torsion-free G_2-structure.

The main difference to Joyce's original construction is the following: if one uses the cutoff procedure from the T^7/Γ case in the new setting, one produces a G_2-structure that does not satisfy the necessary estimates to apply Theorem 2.10. The authors of [JK21] overcome this problem by constructing a G_2-structure with *even* smaller torsion, to which Theorem 2.10 *can* be applied.

3.2.1. Ingredients for the construction Let Y be a compact manifold endowed with a torsion-free G_2-structure φ. Write g for the metric induced by φ. Let $\iota : Y \to Y$ be a G_2-involution, i.e. satisfying $\iota^2 = \mathrm{Id}$, $\iota \neq \mathrm{Id}$, $\iota^*\varphi = \varphi$. We then have:

Proposition 3.3 (Proposition 2.13 in [JK21]). *Let $L = \mathrm{fix}(\iota)$ and assume $L \neq \emptyset$. Then L is a smooth, orientable 3-dimensional compact submanifold of Y which is totally geodesic, and, with respect to a canonical orientation, is associative.*

Assumption 3.4. We assume that L is nonempty, and we assume we are given a closed, coclosed, nowhere vanishing 1-form λ on L.

Such a 1-form need not exist, and cases in which its existence can be guaranteed are discussed in [JK21, Section 7.1].

3.2.2. G_2-structures on the normal bundle ν of L The metric defined by φ defines a splitting

$$TY|_L \simeq \nu \oplus TL, \tag{3.5}$$

which is orthogonal with respect to g. Write g_L for the metric on L induced by g and $g|_L = h_\nu \oplus g_L$. Write $\widetilde{\nabla}^\nu$ for some connection on ν. For now, we may think of $\widetilde{\nabla}^\nu$ as being the restriction of the Levi-Civita connection of g to $\nu \to L$, but later we will need the freedom to choose another connection. We write elements in ν as (x, α), where $x \in L, \alpha \in \nu_x$. For $R > 0$ let

$$U_R = \{(x, \alpha) \in \nu : |\alpha|_{h_\nu} < R\}.$$

Write $\pi : U_R \to L$ for the projection $(x, \alpha) \mapsto x$. We will make use of a map $\Upsilon : U_R \to Y$ satisfying the following:

1. Υ is a diffeomorphism onto its image,
2. $\Upsilon(x, 0) = x$ for $x \in L$,
3. $\Upsilon(x, -\alpha) = \iota \circ \Upsilon(x, \alpha)$ for $(x, \alpha) \in U_R$,
4. the induced pushforward $\Upsilon_* : TU_R \to TY$ restricted to the zero section of TU_R is the identity map on $T_x L \oplus \nu_x$.

For example, $\Upsilon = \exp$ would satisfy these four conditions for small R. But later on we require Υ to satisfy an extra condition that exp need not satisfy.
Write $(\cdot t) : \nu \to \nu$ for the dilation map $(x, \alpha) \mapsto (x, t\alpha)$, and for $t \neq 0$, define $\Upsilon_t = \Upsilon \circ (\cdot t) : U_{|t|^{-1}R} \to Y$.
The connection $\widetilde{\nabla}^\nu$ defines a splitting

$$T\nu = V \oplus H, \quad \text{where } V \simeq \pi^*(\nu) \text{ and } H \simeq \pi^*(TL), \tag{3.6}$$

where V and H are the vertical and horizontal subbundles of the connection. Combining Eqs. (3.5) and (3.6), we have that $T\nu \simeq \pi^*(TY|_L)$. Denote by

$$\varphi^\nu \in \Omega^3(\nu), \psi^\nu \in \Omega^4(\nu), \text{ and } g^\nu \in S^2(\nu) \tag{3.7}$$

the structures obtained from φ, ψ, and g via this isomorphism and for $t > 0$ write $\varphi^\nu_t = (\cdot t)^*\varphi^\nu$, as well as $\psi^\nu_t = (\cdot t)^*\psi^\nu$, and $g^\nu_t = (\cdot t)^*g^\nu$. Note that this definition implicitly depends on the choice of $\widetilde{\nabla}^\nu$. The main result of [JK21, Section 3] is then:

Proposition 3.8. *There exist $R > 0$, a connection $\widetilde{\nabla}^\nu$ on ν and a map $\Upsilon : U_R \to M$ satisfying*

1. *Υ is a diffeomorphism onto its image,*
2. *$\Upsilon(x, 0) = x$ for $x \in L$,*
3. *$\Upsilon(x, -\alpha) = \iota \circ \Upsilon(x, \alpha)$ for $(x, \alpha) \in U_R$,*
4. *the induced pushforward $\Upsilon_* : TU_R \to TY$ restricted to the zero section of TU_R is the identity map on $T_x L \oplus \nu_x$,*

and for $t > 0$ a closed G_2-structure $\widetilde{\varphi}^\nu_t$ on $\nu/\{\pm 1\}$ and closed 4-form $\widetilde{\psi}^\nu_t \in \Omega^4(\nu/\{\pm 1\})$ satisfying the following properties: first,

$$\varphi^\nu_t - \widetilde{\varphi}^\nu_t = \mathcal{O}(t^2 r^2) \quad \text{and} \quad \psi^\nu_t - \widetilde{\psi}^\nu_t = \mathcal{O}(t^2 r^2). \tag{3.9}$$

Second, there exist $\eta \in \Omega^2(\nu)$, $\zeta \in \Omega^3(\nu)$ such that

$$\begin{aligned} |\eta|_{g^\nu} &= \mathcal{O}(r^3) && \text{and} && |\,\mathrm{d}\eta|_{g^\nu} = \left|\Upsilon^*\varphi - \widetilde{\varphi}^\nu_1|_{U_R}\right|_{g^\nu} = \mathcal{O}(r^2), \\ |\zeta|_{g^\nu} &= \mathcal{O}(r^3) && \text{and} && |\,\mathrm{d}\zeta|_{g^\nu} = \left|\Upsilon^*\psi - \widetilde{\psi}^\nu_1|_{U_R}\right|_{g^\nu} = \mathcal{O}(r^2). \end{aligned}$$

3.2.3. G_2-structures on the resolution P of $\nu/\{\pm 1\}$ The G_2-structure $\varphi \in \Omega^3(Y)$ defines for all $x \in Y$ a cross product $\times : T_x Y \times T_x Y \to T_x Y$ as in Definition 2.3. We then have a complex structure $I \in \mathrm{End}(\nu)$ given by

$$I(V) = \frac{\lambda}{|\lambda|} \times V \text{ for } V \in \nu_x, x \in L. \tag{3.10}$$

Recall the metric h_ν on ν defined by $g|_L = h_\nu \oplus g_L$, cf. Section 3.2.2. Then I and h_ν together define a $\mathrm{U}(2)$-reduction of the frame bundle of ν. Denote by X_{EH} the Eguchi–Hanson space with Hyperkähler triple $\widetilde{\omega}_1, \widetilde{\omega}_2, \widetilde{\omega}_3$. Denote by $\rho : X_{\mathrm{EH}} \to \mathbb{C}^2/\{\pm 1\}$ the blowup map of the blowup with respect to the complex structure induced by $\widetilde{\omega}_1$ and let

$$P = \mathrm{Fr} \times_{\mathrm{U}(2)} X_{\mathrm{EH}}. \tag{3.11}$$

Denote by $\sigma : P \to L$ the projection of this bundle. Analogously, we have

$$\nu/\{\pm 1\} = \mathrm{Fr} \times_{\mathrm{U}(2)} \mathbb{C}^2/\{\pm 1\}.$$

Let $L' \subset L$ be a nonempty, open set on which we can extend $e_1 := \frac{\lambda}{|\lambda|} \in T^*(L')$ to an orthonormal basis (e_1, e_2, e_3). Then there exist $\hat{\omega}^I, \hat{\omega}^J, \hat{\omega}^K \in \Omega^2((\nu/\{\pm 1\})|_{L'})$ such that φ^ν from Eq. (3.7) has the form

$$\varphi^\nu = e_1 \wedge e_2 \wedge e_3 - \hat{\omega}^I \wedge e_1 - \hat{\omega}^J \wedge e_2 - \hat{\omega}^K \wedge e_3. \tag{3.12}$$

We define $\check{\omega}^I, \check{\omega}^J, \check{\omega}^K \in \Omega^2(P|_{L'})$ as follows: For $x \in L'$, let $f \in \mathrm{Fr}_x$ such that $f : (\nu/\{\pm 1\})_x \to \mathbb{C}^2/\{\pm 1\}$ satisfies

$$f^*(\omega_1, \omega_2, \omega_3) = (\hat{\omega}^I|_{\nu_x}, \hat{\omega}^J|_{\nu_x}, \hat{\omega}^K|_{\nu_x}),$$

where $(\omega_1, \omega_2, \omega_3)$ denotes the flat Hyperkähler triple on $\mathbb{C}^2/\{\pm 1\}$. This choice of f defines isomorphisms of complex surfaces $P_x \simeq X_{\mathrm{EH}}$ and $(\nu/\{\pm 1\})_x \simeq \mathbb{C}^2/\{\pm 1\}$. Let

$\check{\omega}^I, \check{\omega}^J, \check{\omega}^K \in \Omega^2(P_x)$ be the pullback of $\widetilde{\omega}_1, \widetilde{\omega}_2, \widetilde{\omega}_3 \in \Omega^2(X_{\mathrm{EH}})$ under this isomorphism. This is independent of the choice of f, and therefore defines $\check{\omega}^I, \check{\omega}^J, \check{\omega}^K \in \Omega^2(P_x)$. The following diagram sums up the situation:

$$\begin{array}{ccc} (P_x, \check{\omega}^I|_{P_x}, \check{\omega}^J|_{P_x}, \check{\omega}^K|_{P_x}) & \xrightarrow{\quad\simeq\quad} & (X_{\mathrm{EH}}, \widetilde{\omega}_1, \widetilde{\omega}_2, \widetilde{\omega}_3) \\ \downarrow\rho & & \downarrow\rho \\ (\nu_x/\{\pm 1\}, \hat{\omega}^I|_{\nu_x/\{\pm 1\}}, \hat{\omega}^J|_{\nu_x/\{\pm 1\}}, \hat{\omega}^K|_{\nu_x/\{\pm 1\}}) & \xrightarrow{\simeq} & (\mathbb{C}^2/\{\pm 1\}, \omega_1, \omega_2, \omega_3) \end{array} \tag{3.13}$$

Here, by abuse of notation we denoted the map $P_x \to \nu_x/\{\pm 1\}$ which makes the diagram commutative also by ρ. Horizontal arrows pull Hyperkähler triples back to one another, Hyperkähler triples connected by vertical arrows are asymptotically close to each other. A complicated point is the actual definition of $\check{\omega}^I, \check{\omega}^J, \check{\omega}^K$ as 2-forms on $P|_{L'}$. Equation (3.13) tells us what they look like fibrewise. To make sense of them as global objects on P, one needs to choose a connection on P. In [JK21], the horizontal subspaces $\check{H}$ were defined to this end which allows us to decompose forms on P into vertical and horizontal components, much like for forms on ν. There are then unique vertical 2-forms which restrict to $\check{\omega}^I|_{P_x}, \check{\omega}^J|_{P_x}, \check{\omega}^K|_{P_x}$ on every fibre.
We are now ready to define $\varphi_t^P \in \Omega^3(P|_{L'})$, $\psi_t^P \in \Omega^4(P|_{L'})$ via

$$\begin{aligned} \varphi_t^P &:= \check{\varphi}_{0,3} + t^2 \check{\varphi}_{2,1} \\ &:= \sigma^*(e_1 \wedge e_2 \wedge e_3) - t^2 \left(\sigma^*(e_1) \wedge \check{\omega}^I - \sigma^*(e_2) \wedge \check{\omega}^J - \sigma^*(e_3) \wedge \check{\omega}^K \right), \\ \psi_t^P &:= t^4 \check{\psi}_{4,0} + t^2 \check{\psi}_{2,2} \\ &:= \frac{1}{2} \check{\omega}^I \wedge \check{\omega}^I - \sigma^*(e_2 \wedge e_3) \wedge \check{\omega}^I - \sigma^*(e_3 \wedge e_1) \wedge \check{\omega}^J - \sigma^*(e_1 \wedge e_2) \wedge \check{\omega}^K. \end{aligned} \tag{3.14}$$

These expressions are independent of the choice of (e_2, e_3), and therefore define forms $\varphi_t^P \in \Omega^3(P)$, $\psi_t^P \in \Omega^4(P)$, not just forms over $L' \subset L$. Let also g_t^P denote the metric induced by φ_t^P.
As in the previous section, we add terms to φ_t^P and ψ_t^P to define *closed* forms on P, and we have the following control over how they are asymptotic to forms on $\nu/\{\pm 1\}$:

Proposition 3.15 (Section 4.5 in [JK21]). *There exist* $\xi_{1,2}, \xi_{0,3} \in \Omega^3(P)$, $\tau_{1,1} \in \Omega^2(\{x \in P : \check{r}(x) > 1)$, *such that*

$$\widetilde{\varphi}_t^P := \varphi_t^P + t^2 \xi_{1,2} + t^2 \xi_{0,3}$$

is closed and satisfies

$$\widetilde{\varphi}_t^P = \rho^* \widetilde{\varphi}_t^\nu + t^2 \, \mathrm{d}\tau_{1,1} \tag{3.16}$$

where $\check{r} > 1$. *These forms satisfy the following estimates:*

$$\begin{aligned} \left| \nabla^k (t^2 \xi_{1,2}) \right|_{g_t^P} &= \begin{cases} \mathcal{O}(t^{1-k}), & \check{r} \le 1, \\ \mathcal{O}(t^{1-k} \check{r}^{-3-k}), & \check{r} > 1, \end{cases} \\ \left| \nabla^k (t^2 \xi_{0,3}) \right|_{g_t^P} &= \begin{cases} \mathcal{O}(t^{2-k}), & \check{r} \le 1, \\ \mathcal{O}(t^{2-k} \check{r}^{2-k}), & \check{r} > 1, \end{cases} \end{aligned} \tag{3.17}$$

$$\left|\nabla^k(t^2\tau_{1,1})\right|_{g_t^P} = \mathcal{O}(t^{1-k}\check{r}^{-3-k}). \tag{3.18}$$

Proposition 3.19 (Section 4.5 in [JK21]). *There exist* $\chi_{1,3}, \theta_{3,1}, \theta_{2,2} \in \Omega^4(P)$, $\upsilon_{1,2} \in \Omega^3(\{x \in P : \check{r}(x) > 1)$, *such that*

$$\widetilde{\psi}_t^P := \psi_t^P + t^2\chi_{1,3} + t^4\theta_{3,1} + t^4\theta_{2,2} \tag{3.20}$$

is closed and satisfies

$$\widetilde{\psi}_t^P = \rho^*\widetilde{\psi}_t^\nu + t^2\,\mathrm{d}\upsilon_{1,2} \tag{3.21}$$

where $\check{r} > 1$. *These forms satisfy the following estimates:*

$$\left|\nabla^k(t^2\chi_{1,3})\right|_{g_t^P} := \begin{cases} \mathcal{O}(t^{1-k}), & \check{r} \le 1, \\ \mathcal{O}(t^{1-k}\check{r}^{-3-k}), & \check{r} > 1, \end{cases} \tag{3.22}$$

$$\left|\nabla^k(t^4\theta_{3,1})\right|_{g_t^P} := \begin{cases} \mathcal{O}(t^{1-k}), & \check{r} \le 1, \\ 0, & \check{r} > 1, \end{cases} \tag{3.23}$$

$$\left|\nabla^k(t^4\theta_{2,2})\right|_{g_t^P} := \begin{cases} \mathcal{O}(t^{2-k}), & \check{r} \le 1, \\ \mathcal{O}(t^{2-k}\check{r}^{2-k}), & \check{r} > 1, \end{cases} \tag{3.24}$$

$$\left|\nabla^k(t^2\upsilon_{1,2})\right|_{g_t^P} := \mathcal{O}(t^{1-k}\check{r}^{-3-k}). \tag{3.25}$$

3.2.4. Correcting for the leading-order errors on P Armed with the G_2-structures φ on Y and $\widetilde{\varphi}_t^P$ on P, we could define a glued together G_2-structure just as is done in the case of resolutions of T^7/Γ. However, in this case it would turn out that the torsion of the glued together G_2-structure is too big and Theorem 2.10 cannot be applied. We thus make use of the following correction terms which will make the torsion of the glued together G_2-structure small enough.

Theorem 3.26 (Theorem 5.1 in [JK21]). *There exist* $\alpha_{0,2}, \alpha_{2,0} \in \Omega^2(P)$, $\beta_{0,3}, \beta_{2,1} \in \Omega^3(P)$, *satisfying for all* $t > 0$ *the equation*

$$\begin{aligned}(D_{\varphi_t^P}\Theta)&\left(t^2[\mathrm{d}\alpha_{0,2}]_{1,2} + t^4[\mathrm{d}\alpha_{2,0}]_{3,0} + t^2\xi_{1,2}\right)\\ &= t^2\,\mathrm{d}\beta_{0,3} + t^4[\mathrm{d}\beta_{2,1}]_{3,1} + t^2\chi_{1,3} + t^4\theta_{3,1}.\end{aligned}$$

Moreover, for $\gamma > 0$ *sufficiently small and for all* $k \ge 0$, *these forms satisfy the following estimates*

$$\left|\nabla^k(t^2\alpha_{0,2})\right|_{g_t^P} = \begin{cases} \mathcal{O}(t^{2-k}), & \check{r} \le 1, \\ \mathcal{O}(t^{2-k}\check{r}^{-2-k+\gamma}), & \check{r} \ge 1, \end{cases}$$

$$\left|\nabla^k(t^4\alpha_{2,0})\right|_{g_t^P} = \begin{cases} \mathcal{O}(t^{2-k}), & \check{r} \le 1, \\ \mathcal{O}(t^{2-k}\check{r}^{-2-k+\gamma}), & \check{r} \ge 1, \end{cases}$$

$$\left|\nabla^k(t^2\beta 0,3)\right|_{g_t^P} = \begin{cases} \mathcal{O}(t^{2-k}), & \check{r} \le 1, \\ \mathcal{O}(t^{2-k}\check{r}^{-2-k+\gamma}), & \check{r} \ge 1, \end{cases}$$

$$\left|\nabla^k(t^4\beta 2,1)\right|_{g_t^P} = \begin{cases} \mathcal{O}(t^{2-k}), & \check{r} \le 1, \\ \mathcal{O}(t^{2-k}\check{r}^{-2-k+\gamma}), & \check{r} \ge 1, \end{cases}$$

3.2.5. G_2-structures on the resolution N_t of $Y/\langle\iota\rangle$ We are now ready to glue together P and $Y/\langle\iota\rangle$ to a manifold, and define a G_2-structure with small torsion on it.

Definition 3.27. Define

$$N_t := \left[\rho^{-1}(U_{t^{-1}R}/\{\pm 1\}) \coprod (Y \setminus L)/\langle\iota\rangle\right] / \sim, \tag{3.28}$$

where $x \sim \Upsilon_t \circ \rho(x)$ for $x \in \rho^{-1}(U_{t^{-1}R}/\{\pm 1\})$.

Definition 3.29. Let $a : [0, \infty) \to \mathbb{R}$ be a smooth function with $a(x) = 0$ for $x \in [0, 1]$, and $a(x) = 1 \in [2, \infty)$. Define then

$$\varphi_t^N = \begin{cases} \widetilde{\varphi}_t^P + \mathrm{d}[t^2\alpha_{0,2} + t^4\alpha_{2,0}], & \text{if } \check{r} \le t^{-1/9}, \\ \widetilde{\varphi}_t^P + \mathrm{d}[t^2\alpha_{0,2} + t^4\alpha_{2,0} + a(t^{1/9}\check{r}) \cdot \Upsilon_*\eta], & \text{if } t^{-1/9} \le \check{r} \le 2t^{-1/9}, \\ \widetilde{\varphi}_t^P + \mathrm{d}[t^2\alpha_{0,2} + t^4\alpha_{2,0} + \Upsilon_*\eta], & \text{if } 2t^{-1/9} \le \check{r} \le t^{-4/5}, \\ \widetilde{\varphi}_t^\nu + \mathrm{d}[(1 - a(t^{4/5}\check{r}))(t^2\tau_{1,1} + t^2\alpha_{0,2} + t^4\alpha_{2,0}) + \Upsilon_*\eta], & \text{if } t^{-4/5} \le \check{r} \le 2t^{-4/5}, \\ \varphi, & \text{elsewhere,} \end{cases} \tag{3.30}$$

$$\psi_t^N = \begin{cases} \widetilde{\psi}_t^P + \mathrm{d}[t^2\beta_{0,3} + t^4\beta_{2,1}], & \text{if } \check{r} \le t^{-1/9}, \\ \widetilde{\psi}_t^P + \mathrm{d}[t^2\beta_{0,3} + t^4\beta_{2,1} + a(t^{1/9}\check{r}) \cdot \Upsilon_*\zeta], & \text{if } t^{-1/9} \le \check{r} \le 2t^{-1/9}, \\ \widetilde{\psi}_t^P + \mathrm{d}[t^2\beta_{0,3} + t^4\beta_{2,1} + \Upsilon_*\zeta], & \text{if } 2t^{-1/9} \le \check{r} \le t^{-4/5}, \\ \widetilde{\psi}_t^\nu + \mathrm{d}[(1 - a(t^{4/5}\check{r}))(t^2 v_{1,2} + t^2\beta_{0,3} + t^4\beta_{2,1}) + \Upsilon_*\zeta], & \text{if } t^{-4/5} \le \check{r} \le 2t^{-4/5}, \\ \psi, & \text{elsewhere,} \end{cases} \tag{3.31}$$

The important properties of these forms are that φ_t^N and ψ_t^N are closed, and that ψ_t^N is close to being the Hodge dual of φ_t^N. That is, the 3-form $\varphi_t^N - *_{\varphi_t^N}\psi_t^N$ satisfies the assumption of Theorem 2.10 and φ_t^N can be perturbed to a torsion-free G_2-structure. This yields the following theorem:

Theorem 3.32 (Theorem 6.4 in [JK21]). *For small t there exists $\eta_t \in \Omega^2(N_t)$ such that $\widetilde{\varphi}_t^N := \varphi_t^N + \mathrm{d}\eta_t$ is a torsion-free G_2-structure, and*

$$\left\|\widetilde{\varphi}_t^N - \varphi_t^N\right\|_{L^\infty} \le ct^{1/18} \tag{3.33}$$

for some constant $c > 0$ independent of t.

4. The Gluing Construction for Instantons

We now turn to constructing G_2-instantons on the resolutions of $Y/\langle\iota\rangle$ explained in the previous section. As is common in gluing constructions in differential geometry, we obtain this result by following the three step procedure of (1) constructing an approximate solution, (2) estimating the linearisation of the equation to be solved, (3) perturbing the approximate solution to a genuine solution.

This method was first employed in [Tau82] for the construction of anti-self-dual connections over 4-manifolds. A similar but simpler proof of the same results is given in [DK90, Section 7.2].

In Sect. 4.1 we explain how a section s of a moduli bundle gives rise to a connection $s(A)$ on the bundle of Eguchi–Hanson spaces P from Eq. (3.11), cf. Theorem 4.15. If the topological compatibility condition Assumption 4.1 is satisfied, we can glue $s(A)$ to a G_2-instanton θ on the orbifold $Y/\langle\iota\rangle$. The resulting connection A_t is close to being a G_2-instanton and in Sect. 4.2 we will quantify this. We will see that this error is small in a suitable norm if s satisfies a first order partial differential equation, the Fueter equation. Section 4.3 is the difficult part of the analysis, where we give an estimate for the inverse of the linearised instanton operator. In Sects. 4.4 and 4.5 we complete the argument and construct the perturbation that turns the approximate solution from before into a genuine solution to the G_2-instanton equation.
Throughout we will use the notation from the previous section. That is, Y is a G_2-manifold with G_2-involution $\iota : Y \to Y$, and N_t is the resolution of $Y/\langle\iota\rangle$. The resolution N_t is obtained by gluing in the Eguchi–Hanson bundle P over the singular locus $L = \mathrm{fix}(\iota)$. On P we have the G_2-structures φ_t^P and $\widetilde{\varphi}_t^P$, and on N_t we have the G_2-structure φ_t^N with small torsion and the torsion-free G_2-structure $\widetilde{\varphi}_t^N$. In the case that N_t is a resolution of T^7/Γ, we also defined the G_2-structures φ^t and $\widetilde{\varphi}^t$. These two will also be denoted by φ_t^N and $\widetilde{\varphi}_t^N$ respectively and the special case of T^7/Γ will need no special treatment most of the time. The exception is the pre-gluing estimate for resolutions of T^7/Γ, Corollary 4.58, which is better than in the general case. See Table 2 for a reminder of the most important notation from the previous section.
In the case of resolutions of T^7/Γ, our main result is Theorem 4.134, in the general case it is Theorem 4.133. We will use both theorems in Sect. 5 to construct new examples of G_2-instantons on the resolution of T^7/Γ and the resolution of $(T^3 \times \mathrm{K3})/\mathbb{Z}_2^2$.

4.1. The pregluing construction

4.1.1. Moduli bundles of ASD-instantons Let $\pi : E_0 \to Y/\langle\iota\rangle$ be an orbifold G-bundle with connection θ, i.e. a G-bundle with connection over Y together with a lift $\hat{\iota}$ of ι such that $\hat{\iota}^2 = \mathrm{Id}$ and such that $\hat{\iota}^*\theta = \theta$. As before, $\mathrm{fix}(\iota) = L$ and we now set $E_\infty = E_0|_L$, which is a G-bundle with $\mathbb{Z}_2$-action, and $A_\infty = \theta|_{E_\infty}$. For each connected component of L choose a framed moduli space of ASD instantons M on a bundle E over Eguchi–Hanson space X_{EH}, cf. Section 2.2. The homomorphism $\rho : \mathbb{Z}_2 \to G$ used in the definition of M defines a $\mathbb{Z}_2$ left action on G. We then ask for E_0 and M to be compatible in the following sense:

Assumption 4.1. For all $l \in L$ there exists an isomorphism of manifolds with G right action and $\mathbb{Z}_2$ left action $\phi : E_\infty|_l \to G$.

Proposition 4.2. *Let $G_\rho \subset G$ be the stabiliser of ρ as in Eq. (2.17). Then there exists a G_ρ-reduction $\check{E}$ of E_∞ such that A_∞ reduces to $\check{E}$.*

Proof. As before, let $\rho : \mathbb{Z}_2 \to G$ be the representation that defines the asymptotic limit for connections in M. Define

$$\check{E} := \{u \in E_\infty : u \cdot \rho(-1) = \hat{\iota}(u)\}. \tag{4.3}$$

To see that this is a G_ρ-bundle, fix $l \in L$ and let $\phi : E_\infty|_l \to G$ be the isomorphism from Assumption 4.1. Then $u \in \check{E}|_l$ if and only if $\phi(u) \in G_\rho$.

Table 2. Most important notation from Sect. 4

Notation	Description	References
E_0	Orbifold bundle over $Y/\langle\iota\rangle$	
θ	Connection on E_0	
(E_∞, A_∞)	$= (E_0, \theta)\|_{\mathrm{fix}(\iota)}$	
$\check{E}$	Reduction of E_∞	Eq. (4.3)
M	Moduli space of instantons on X_{EH}	Section 2.2
$\mathfrak{M}$	Moduli bundle over $\mathrm{fix}(\iota)$	Eq. (4.7)
$\mathfrak{F}$	Fueter equation	Eq. (4.14)
s	Section of $\mathfrak{M}$	Section 4.1.2
$s(E)$	Bundle on P induced by s	Theorem 4.15
$s(A)$	Connection on A induced by s	Theorem 4.15
$\|\|\cdot\|\|_{C^{k,\alpha}_{l,\delta;t}(U)}$	Weighted Hölder norm on N_t	Definition 4.19
$\hat{X}_{\mathrm{EH}}$	One point compactification of X_{EH}	Proposition 2.12
$\hat{P}$	$\hat{X}_{\mathrm{EH}}$-bundle over $\mathrm{fix}(\iota)$	Theorem 4.15
$\overline{A_\infty}$	A_∞ extended to a neighbourhood of points at infinity in $\hat{P}$	Definition 4.23
$\underline{A_\infty}$	A_∞ extended to a neighbourhood	Definition 4.23
	of $\mathrm{fix}(\iota)$ in $Y/\langle\iota\rangle$	Eq. (4.31)
σ, b	Terms for comparing θ, $s(A)$	Eq. (4.31)
A_t	Glued connection on N_t	Proposition 4.28
L_t	$= L_{A_t}$, linearised instanton equation	Eq. (2.44)
e_t	Pregluing error	Eq. (4.62)
Q_t	Quadratic part of instanton equation	Eq. (4.62)
ι_t	Maps sections of $V\mathfrak{M}$ to 1-forms on N_t	Definition 4.63
π_t	Map in the opposite direction	Definition 4.63
$\overline{\pi}_t$	$= \iota_t \pi_t$	Definition 4.63
η_t	$= \mathrm{Id} - \overline{\pi}_t$	Definition 4.63
$\|\|\cdot\|\|_{\mathfrak{X}_t}$, $\|\|\cdot\|\|_{\mathfrak{Y}_t}$	Norms with different scaling for ι_t and π_t parts	Definition 4.76
$\|\|\cdot\|\|_{C^{0,\alpha}_\beta}$	Weighted norms on model spaces	Definition 4.87
$V^P_{\epsilon_1,\epsilon_2;t}(y)$	Neighborhood in P	Section 4.3.3
$U^P_{\epsilon_1/t,\epsilon_2/t;t}$	Neighborhood in $\mathbb{R}^3 \times X_{\mathrm{EH}}$	Section 4.3.3
$s^P = s^{P,\epsilon_1,\epsilon_2}_{d,y;t}$	Maps 1-forms from P to $\mathbb{R}^3 \times X_{\mathrm{EH}}$	Section 4.3.3
$V^\nu_{\epsilon_1,\epsilon_2,\epsilon_3;t}(y)$	Neighborhood in ν	Section 4.3.3
$U^\nu_{\epsilon_1/t,\epsilon_2/t,\epsilon_3/t;t}$	Neighborhood in $\mathbb{R}^3 \times (\mathbb{C}^2/\{\pm 1\})$	Section 4.3.3
$s^\nu = s^{\nu,\epsilon_1,\epsilon_2,\epsilon_3}_{d,y;t}$	Maps 1-forms from ν to $\mathbb{R}^3 \times (\mathbb{C}^2/\{\pm 1\})$	Section 4.3.3

It remains to check that A_∞ reduces to $\check{E}$. To this end, let $\gamma : I \to \check{E}$ be a curve. Then

$$\begin{aligned} A_\infty(\dot{\gamma}(0)) &= \hat{\iota}^* A_\infty(\dot{\gamma}(0)) \\ &= A_\infty\left(\frac{d}{dt}(\gamma(t) \cdot \rho(-1))|_{t=0}\right) \\ &= \mathrm{Ad}(\rho(-1))\,(A_\infty(\dot{\gamma}(0)))\,. \end{aligned} \tag{4.4}$$

In the first step we used $\hat{\iota}^*\theta = \theta$. The second step is the defining property of $\check{E}$ from Eq. (4.3). Now, for any subgroup $H \subset G$ we define the *centraliser of* H *in* G as $Z(H) = \{g \in G : hgh^{-1} = g \text{ for all } h \in H\}$. Then

$$\mathrm{Lie}(Z(H)) = \mathfrak{z}_H := \{V \in \mathfrak{g} : \mathrm{Ad}(h)V = V \text{ for all } h \in H\}. \tag{4.5}$$

This equality holds, because for $X = \dot{g}(0) \in \mathrm{Lie}(Z(H))$, where $g : I \to Z(H)$ is a curve, we have that $\mathrm{Ad}(h)X = \frac{d}{dt}(hg(t)h^{-1})|_{t=0} = X$ by definition of $Z(H)$.

Conversely, for $V \in \mathfrak{z}_H$, we have that $g(t) := \exp(tV)$ is a curve with $\dot{g}(0) = V$ in $Z(H)$, because $hg(t)h^{-1} = \exp(t \cdot \mathrm{Ad}(h)V) = \exp(tV) = g(t)$ for all $h \in H$. Therefore, by Eqs. (4.4) and (4.5), we have that $A_\infty|_{\check{E}}$ takes values in $\mathrm{Lie}(G_\rho)$, i.e. restricts to a connection on $\check{E}$. □

Definition 4.6. Define the *moduli bundle*

$$\mathfrak{M} := (\mathrm{Fr} \times \check{E}) \times_{\mathrm{U}(2)\times G_\rho} M \tag{4.7}$$

and its *vertical tangent space*

$$V\mathfrak{M} := (\mathrm{Fr} \times \check{E}) \times_{\mathrm{U}(2)\times G_\rho} TM. \tag{4.8}$$

4.1.2. Fueter sections and connections on bundles over P In the following, we will study sections $s : L \to \mathfrak{M}$. It will turn out that such a section s gives rise to a connection that is almost a G_2-instanton, if it satisfies a first order differential equation, the *Fueter equation* (cf. Definition 4.13).

Definition 4.9. Let $s : L \to \mathfrak{M}$ be a section. We define its covariant derivative $\nabla s \in \Omega^1(L, V\mathfrak{M})$ as follows: for $x \in L$, $X \in T_xL$ let $f \in C^\infty(\mathrm{Fr})$ and $e \in C^\infty(\check{E})$ be local sections around x such that $A^{\mathrm{LC}}\,\mathrm{d}f(x) = 0$ and $A_\infty(\mathrm{d}e(X)) = 0$, where A^{LC} is the Levi-Civita connection of Y. Let $B : L \to M$ be a local section around x such that $s = [(f, e), B]$. Then

$$\nabla_X(s) = [(f, e), dB(X)] \in (\mathrm{Fr} \times \check{E}) \times_{\mathrm{U}(2)\times G_\rho} TM.$$

Definition 4.10. Let $s : L \to \mathfrak{M}$ be a section. Fix $x \in L$ and let e_1, e_2, e_3 be an orthonormal basis of T_xL. The G_2-structure on Y defines a map

$$\begin{aligned} \Lambda^1(T_xL) &\to \Lambda^+ P_x \\ e_i &\mapsto \check{\omega}_i|_{P_x} =: \omega_i. \end{aligned} \tag{4.11}$$

The ω_i correspond to complex structures on P_x and therefore, by Theorem 2.26, to elements $I_i \in \mathrm{End}(V_x\mathfrak{M})$. We thus have a Clifford multiplication given by

$$\begin{aligned} e_i \cdot : V_x\mathfrak{M} &\to V_x\mathfrak{M} \\ a &\mapsto I_i(a). \end{aligned} \tag{4.12}$$

Definition 4.13. A section $s : L \to \mathfrak{M}$ is called a *Fueter section* if

$$\mathfrak{F}s := \sum_{i=1}^{3} e_i \cdot \nabla_{e_i} s = 0 \in \Gamma(s^* V\mathfrak{M}), \tag{4.14}$$

where (e_1, e_2, e_3) is a local orthonormal frame.

The following is an extension of [DS11, Theorem 1]:

Theorem 4.15. *Denote by $\widetilde{\mathbb{P}} \to M \times \hat{X}_{EH}$ the tautological bundle with tautological connection $\widetilde{\mathbb{A}}$ over $M \times X_{EH}$ from Proposition 2.36 and assume that there exists a lift of the* $\mathrm{U}(2)$*-action on $M \times \hat{X}_{EH}$ to $\widetilde{\mathbb{P}}$ preserving $\widetilde{\mathbb{A}}$. Let $s \in C^\infty(\mathfrak{M})$, and denote $\hat{P} = \mathrm{Fr} \times_{\mathrm{U}(2)} \hat{X}_{EH}$. Then there exists a natural G-bundle $s(E)$ over $\hat{P}$ with connection $s(A) \in \mathscr{A}(s(E)|_P)$ together with an isomorphism of G-bundles with $\mathbb{Z}_2$ left action $\Phi : s(E)|_{\hat{P}\setminus P} \to E_\infty$ so that:*

(i) The pair $(s(E), s(A))|_{P_x}$ *represents* $s(x)$. *That means: if* $s(x) = [(f, e), [B]]$ *for* $f \in \mathrm{Fr}_x$, $e \in (E_0)_x$, $[B] \in M$, *then under the diffeomorphism* $X_{EH} \simeq P_x$, $y \mapsto [f, y]$, *the* G*-bundles* $s(E)|_{P_x}$ *and* E *are isomorphic, and* B *and* $s(A)$ *are gauge equivalent.*

(ii) The map Φ *identifies* A_∞ *and* $s(A)$ *over the fibre at infinity, i.e.* $\Phi^* A_\infty = s(A)|_{\hat{P} \setminus P}$.

(iii) The connection $s(A)|_P$ *is a* $(\psi_t^P)^*$*-instanton if and only if* s *is a Fueter section. Here,* $s(A)$ *being a* $(\psi_t^P)^*$*-instanton means that* $F_{s(A)} \wedge (\psi_t^P)^* = 0$, *where* $(\psi_t^P)^* = \sum \sigma^*(e^i) \wedge \sigma^*(e^j) \wedge \check{\omega}^k$ *and* $\sigma : P \to L$ *is the projection of the bundle* P *(cf. Equation (3.11)).*

Proof. **Construction of** $s(E)$, $s(A)$, **and** Φ: together with the connections ∇^{LC} on Fr and A_∞ on $\check{E}$, the connection $\widetilde{\mathbb{A}}$ induces a connection α on the principal G-bundle $(\mathrm{Fr} \times \check{E}) \times_{\mathrm{U}(2) \times G_\rho} \widetilde{\mathbb{P}} \to (\mathrm{Fr} \times \check{E}) \times_{\mathrm{U}(2) \times G_\rho} (M \times \hat{X}_{\mathrm{EH}})$ via the formula

$$\alpha([(U, V), T]) := \widetilde{\mathbb{A}}(T), \tag{4.16}$$

where $U \in T\,\mathrm{Fr}$, $V \in T\check{E}$ are horizontal vectors and $T \in T\widetilde{\mathbb{P}}$. By assumption, $\widetilde{\mathbb{A}}$ is left-invariant, which makes the definition of α independent of the chosen representative. Consider the map

$$\begin{aligned}(s \times \mathrm{Id}) : \hat{P} = \mathrm{Fr} \times_{\mathrm{U}(2)} \hat{X}_{\mathrm{EH}} &\to (\mathrm{Fr} \times \check{E}) \times_{\mathrm{U}(2) \times G_\rho} (M \times \hat{X}_{\mathrm{EH}}) \\ [f, y] &\mapsto [(f, e), (B, y)],\end{aligned}$$

where $s(\sigma(e)) = [(f, e), B] \in \mathfrak{M}_{\pi(e)}$. Then define

$$s(E) := (s \times \mathrm{Id})^*((\mathrm{Fr} \times \check{E}) \times_{\mathrm{U}(2) \times G_\rho} \widetilde{\mathbb{P}}), \quad s(A) := (s \times \mathrm{Id})^* \alpha$$

and the trivialisation $\underline{\phi} : \widetilde{\mathbb{P}}|_{M^{\mathrm{orb}} \times \{\infty\}} \to G \times M^{\mathrm{orb}}$ from Proposition 2.36 induces an isomorphism

$$\begin{aligned}\Phi : s(E)|_{\hat{P} \setminus P} &\\ &\simeq (s \times \mathrm{Id}\,|_{\hat{X}_{\mathrm{EH}} \setminus X_{\mathrm{EH}}})^* \left((\mathrm{Fr} \times \check{E}) \times_{\mathrm{U}(2) \times G_\rho} \widetilde{\mathbb{P}}|_{M \times \{\infty\}} \right) \\ &\to s^* \left((\mathrm{Fr} \times \check{E}) \times_{\mathrm{U}(2) \times G_\rho} G \times M \right) \\ &\simeq \check{E} \times_{G_\rho} G \simeq E_\infty.\end{aligned} \tag{4.17}$$

The last point of Proposition 2.36 states that $\underline{\phi}^* A_{\mathrm{product}} = \widetilde{\mathbb{A}}|_{M \times \{\infty\}}$ which implies that $\Phi^* A_\infty = s(A)|_{\hat{P} \setminus P}$.

$s(A)$ **is a** $(\psi_t^P)^*$**-instanton if and only if** s **is a Fueter section:** for easier notation, assume that the bundle Fr is trivial and ∇^{LC} is the product connection. The proof of the general case works the same. In this case, $L \times \hat{X}_{\mathrm{EH}} = \hat{P}$ and $s(E) = (s \times \mathrm{Id})^*(\check{E} \times_{G_\rho} \widetilde{\mathbb{P}})$. Then fix $(l, x) \in L \times \hat{X}_{\mathrm{EH}} = \hat{P}$, an orthonormal basis (e_1, e_2, e_3) of $T_l L$ and denote by (e^1, e^2, e^3) its dual basis. Around l, write $s(x) = [e, B]$ with the property that $\mathrm{d}e(V)$ is parallel for all $V \in T_l L$. Then, for $Z \in T_x \hat{X}_{\mathrm{EH}}$:

$$\begin{aligned}F_{s(A)}(e_i, Z) &= \left((s \times \mathrm{Id})^* F_\alpha\right)(e_i, Z) \\ &= F_\alpha\left([\mathrm{d}e(e_i), (\mathrm{d}B(e_i), 0)], [\mathrm{d}e(e_i), (0, Z)]\right) \\ &= F_{\widetilde{\mathbb{A}}}(\mathrm{d}B(e_i), Z) \\ &= \mathrm{d}B(e_i)(Z).\end{aligned} \tag{4.18}$$

In the first step we used that the curvature of a pullback connection is the pullback of its curvature. The third step is the definition of α from Eq. (4.16), and in the last step we used the curvature properties of the tautological connection $\tilde{\mathbb{A}}$ from Proposition 2.36. As before, denote by I_1, I_2, I_3 the Hyperkähler triple of complex structures on X_{EH} and $\omega_1, \omega_2, \omega_3$ the corresponding symplectic forms. The Fueter condition from Definition 4.13 for s is equivalent to the following equation of elements in $\Omega^1(X_{\mathrm{EH}}, \mathrm{Ad}\, P)$:

$$0 = \sum_{i=1}^{3} I_i(\mathrm{d}B(e_i)) = \sum_{i=1}^{3} \omega_i(\mathrm{d}B(e_i), \cdot) = \sum_{i=1}^{3} \omega_i(F_{s(A)}(e_i, \cdot), \cdot)$$
$$= * \left(\sum_{i=1}^{3} \omega_i \wedge F_{s(A)}(e_i, \cdot) \right)$$

where $*$ denotes the Hodge star on X_{EH}. The first equality is the Fueter equation, the third equality is Eq. (4.18), and the second and fourth equality are linear algebra computations that can be computed in standard coordinates.
Applying $*$ to both sides gives

$$0 = \left(\sum_{i=1}^{3} \omega_i \wedge F_{s(A)}(e_i, \cdot) \right)$$

which in turn implies

$$0 = \sum_{i,j,k \text{ cyclic}} \omega_i \wedge e^j \wedge e^k \wedge [F_{s(A)}]_{(1,1)},$$

where $[F_{s(A)}]_{(1,1)}$ denotes the $(1, 1)$-component of $F_{s(A)}$ according to the bi-grading on $\Lambda^* T^*(L \times X_{\mathrm{EH}})$ induced by $T^*(L \times X_{\mathrm{EH}}) = T^*L \oplus T^*X_{\mathrm{EH}}$. On the other hand, $[F_{s(A)}]_{(0,2)} \in \Omega^2(X_{\mathrm{EH}}, \mathrm{Ad}\, P)$ is anti-self-dual by Proposition 2.36, thus

$$0 = \sum_{i,j,k \text{ cyclic}} \omega_i \wedge e^j \wedge e^k \wedge [F_{s(A)}]_{(0,2)}.$$

Last, $0 = \sum_{i,j,k \text{ cyclic}} \omega_i \wedge e^j \wedge e^k \wedge [F_{s(A)}]_{(2,0)}$, because this is a sum of forms of type $(2, 4)$ which must vanish as L has dimension 3. □

4.1.3. Gluing connections over P and $Y/\langle \iota \rangle$ Throughout the rest of the article, we will use weighted the Hölder norms from [Wal17, Section 6]:

Definition 4.19. For $\delta, l \in \mathbb{R}$, let

$$\begin{aligned} w_{l,\delta;t} &: N_t \to \mathbb{R} \\ x &\mapsto \begin{cases} t^\delta (t + r_t(x))^{-l-\delta}, & \text{if } r_t(x) \le \sqrt{t} \\ r_t^{-l+\delta} & \text{if } r_t(x) > \sqrt{t} \end{cases} \end{aligned} \tag{4.20}$$

and by slight abuse of notation use the same symbol to denote $w_{l,\delta;t} : N_t \times N_t \to \mathbb{R}$ given by $w_{l,\delta;t}(x, y) = \min\{w(x), w(y)\}$. Let $U \subset N_t$. For $\alpha \in (0, 1)$, $\beta \in \mathbb{R}$, $k \in \mathbb{N}$, and f a tensor field on N_t define the *weighted Hölder norm of* f via

$$[f]_{C^{0,\alpha}_{l,\delta;t}(U)} := \sup_{\substack{x,y \in U,\, x \neq y \\ d(x,y) \le t + \min\{r_t(x), r_t(y)\}}} w_{l-\alpha,\delta;t}(x, y) \frac{|f(x) - f(y)|}{d(x, y)^\alpha},$$

$$||f||_{L^\infty_{l,\delta;t}(U)} := \left|\left|w_{l,\delta;t} f\right|\right|_{L^\infty(U)},$$

$$||f||_{C^{k,\alpha}_{l,\delta;t}(U)} := \sum_{j=0}^{k} \left|\left|\nabla^j f\right|\right|_{L^\infty_{l-j,\delta;t}(U)} + \left[\nabla^j f\right]_{C^{0,\alpha}_{l-j,\delta;t}(U)}.$$

The term $f(x)-f(y)$ in the first line denotes the difference between $f(x)$ and the parallel transport of $f(y)$ to the fibre over x along one of the shortest geodesics connecting x and y. When U is not specified, take $U = N_t$. We use the notation $||\cdot||_{C^{k,\alpha}_{l,\delta;t}(U),g}$ for the weighted Hölder norm with respect to the metric g, i.e. use parallel transport with respect to the Levi-Civita connection induced by the metric g, and measure vectors in g. If no metric g is specified, we take $g = g_t^N$. For our analysis, we need $\delta \in (-1, 0)$, $\alpha \in (0, 1)$, $\alpha \ll |\delta|$, for example $\delta = -1/64$, $\alpha = 1/256$ will work.

Remark 4.21. Note that $w_{l,\delta;t}$ is not continuous, but that does not cause any problems.

Proposition 4.22 (Proposition 6.2 in [Wal17]). *If $(f, g) \mapsto f \cdot g$ is a bilinear form satisfying $|f \cdot g| \le |f|\,|g|$, then*

$$||f \cdot g||_{C^{k,\alpha}_{l_1+l_2,\delta_1+\delta_2;t}} \le ||f||_{C^{k,\alpha}_{l_1,\delta_1;t}} \cdot ||g||_{C^{k,\alpha}_{l_2,\delta_2;t}}.$$

We have shown that $s(A)$ is a $(\psi_t^P)^*$-instanton. It is, however, in general not a G_2-instanton with respect to ψ_t^P because of the $(2, 0)$ part of its curvature. We will later estimate the failure of $s(A)$ of being a G_2-instanton.

Definition 4.23. For $l \in L$ choose a neighbourhood $l \in V_l \subset L$ over which E_∞ is trivial. Use the identification $\Phi : s(E)|_{\hat{P}\setminus P} \to E_\infty$ and parallel transport with respect to $s(A)$ to get a trivialisation of $s(E)$ around $\hat{P}|_{V_l} \setminus P|_{V_l}$, say on a neighbourhood $U_l \subset \hat{P}$. Using this, we can view the pullback of $s(A)|_{\hat{P}\setminus P}$ under the projection $U_l \to V_l$ as a connection $\overline{A_\infty}^l \in \mathscr{A}(s(E)|_{U_l})$. This definition is independent of the choice of $l \in L$, and therefore defines some connection $\overline{A_\infty} \in \mathscr{A}(s(E)|_U)$, where $U \subset \hat{P}$ is a neighbourhood of the points at infinity $\hat{P}\setminus P$.

Now is the first time we cite a non-trivial result from [Wal17]. Therein, Fueter sections into a moduli bundle of ASD-instantons on $\mathbb{R}^4$ were considered, while in this section ASD-instantons on X_{EH} are considered. At some points this changes the analysis, and these results are reproved in this new setting in the coming sections. At some points, results carry over without having to change the proof. The following proposition is the first such result:

Proposition 4.24 (Proposition 7.4 in [Wal17]). *There exists $c > 0$ such that for all $t \in (0, T)$:*

$$\left|\left|[F_{s(A)}]_{2,0} - F_{\overline{A_\infty}}\right|\right|_{C^{0,\alpha}_{-2,0;t}(U),g_t^P} \le ct^2, \tag{4.25}$$

$$\left|\left|[F_{s(A)}]_{1,1}\right|\right|_{C^{0,\alpha}_{-3,0;t}(U),g_t^P} \le ct^2, \text{ and} \tag{4.26}$$

$$\left|\left|[F_{s(A)}]_{0,2}\right|\right|_{C^{0,\alpha}_{-4,0;t}(U),g_t^P} \le ct^2. \tag{4.27}$$

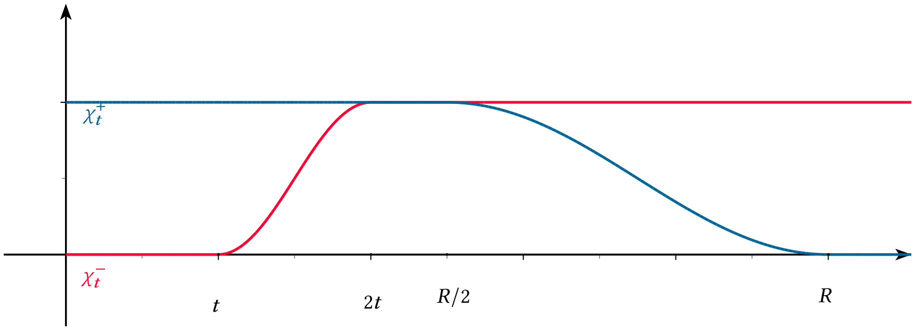

Fig. 2. The cut-off functions χ_t^- and χ_t^+ from Eq. (4.30) for small t

Proposition 4.28. *Let $E_0 \to Y/\langle\iota\rangle$ be an orbifold bundle with connection θ satisfying Assumption 4.1, $L = \text{fix}(\iota)$, and $s : L \to \mathfrak{M}$ be a Fueter section. Then there exists a G-bundle E_t over N_t and a connection A_t on E_t such that*

$$(E_t, A_t)|_{N_t \setminus \Upsilon_t(U_{t^{-1}R})} \simeq (E_0, \theta)|_{N_t \setminus \Upsilon_t(U_{t^{-1}R})} \quad \textit{and}$$
$$(E_t, A_t)|_{\Upsilon_t(U_1)} \simeq (s(E), s(A))|_{\rho^{-1}(U_1)}.$$

Proof. **Construction of E_t**: By Theorem 4.15 we have a bundle isomorphism $\Phi : E_\infty \to s(E)|_{\hat{P}\setminus P}$. Let $U \subset \hat{P}$ be a neighbourhood of $\hat{P}\setminus P$. Now use radial parallel transport with respect to θ on E_0 and parallel transport with respect to $\overline{A_\infty}$ (the pullback of $\Phi^* A_\infty$ to a neighbourhood of $\hat{P} \setminus P$ defined in Proposition 4.24) to extend Φ to the neighbourhood $\Upsilon(U_R) \subset Y$ of L, denote the extension by Ψ. The conditions $\hat{\iota}^*\theta = \theta$ and Assumption 4.1 ensure that this is well-defined.
As in Sect. 3.2.3 we use the symbol ρ to denote the map $\rho : P \to \nu/\{\pm 1\}$ induced by the blowup map $X_{\text{EH}} \to \mathbb{C}^2/\{\pm 1\}$ on Eguchi–Hanson space. For small enough t we have that the overlap $O := U_{t^{-1}R} \cap \rho(U)$ is non-empty. Use this to define E_t by gluing together E_0 and $s(E)$ via Ψ over O, i.e.

$$E_t := E_0|_{Y \setminus \Upsilon_t(U_{t^{-1}R} \setminus O)} \cup s(E)|_{\rho^{-1}(U_{t^{-1}R})} / \sim, \tag{4.29}$$

where $v \sim \Psi(v)$ for $v \in E_0|_{\Upsilon_t(O)}$.
Construction of A_t: Let $\chi_t^- : N_t \to [0,1]$ and $\chi_t^+ : N_t \to [0,1]$ be rescalings of a smooth cut-off function such that

$$\begin{aligned} \chi_t^-|_{\{r_t \le t\}} &\equiv 0 \text{ and } \chi_t^-|_{\{r_t \ge 2t\}} \equiv 1, \\ \chi_t^+|_{\{r_t \le R/2\}} &\equiv 1 \text{ and } \chi_t^+|_{\{r_t \le R\}} \equiv 0. \end{aligned} \tag{4.30}$$

Similar to the definition of $\overline{A_\infty} \in \mathscr{A}(s(E)|_U)$, define $\underline{A_\infty} \in \mathscr{A}\left(E_0|_{\Upsilon_t(U_{t^{-1}R})}\right)$ by pulling back $A_\infty \in \mathscr{A}(E_\infty)$. By definition of E_t, we have that $\overline{A_\infty}$ and $\underline{A_\infty}$ are both connections on E_t. The map Φ identifies A_∞ and $s(A)$ by the second point of Theorem 4.15. Because Ψ is an extension of Φ defined by radial parallel transport, and $\overline{A_\infty}$ and $\underline{A_\infty}$ are also defined via radial parallel transport, we have that $\overline{A_\infty} = \underline{A_\infty}$ as connections on $E_t|_{\Upsilon_t(O)}$.
We then have $\sigma \in \Omega^1(\text{Ad}\, s(E)|_O)$ and $b \in \Omega^1(\text{Ad}\, E_0|_O)$ such that

$$s(A) = \overline{A_\infty} + \sigma, \qquad \theta = \underline{A_\infty} + b \quad \text{over } O. \tag{4.31}$$

Define then

$$A_t := \begin{cases} s(A) & \text{on } r_t < t \\ \underline{A_\infty} + \chi_t^- b + \chi_t^+ \sigma & \text{on } t \leq r_t \leq R \\ \theta & \text{on } r_t > R. \end{cases} \tag{4.32}$$

□

The following proposition follows immediately from Definition 4.19.

Proposition 4.33. *Let χ_t^- and χ_t^+ as in Eq. (4.30). Then there exists $c > 0$ such that for all $t \in (0, T)$:*

$$\begin{aligned} ||\chi_t^-||_{C^{0,\alpha}_{0,0;t}} + ||\mathrm{d}\chi_t^-||_{C^{0,\alpha}_{-1,0;t}} &\leq c, \\ ||\chi_t^+||_{C^{0,\alpha}_{0,0;t}} + ||\mathrm{d}\chi_t^+||_{C^{0,\alpha}_{0,0;t}} &\leq c. \end{aligned}$$

The following proposition is proved like Proposition 4.24 with the proof from [Wal17] directly carrying over to this setting. The estimate for σ holds because of the fast decay of the curvature of ASD connections on ALE spaces, see Proposition 2.19. The estimate for b holds because over L we have that $\underline{A_\infty} = \theta$, not just in the L-direction. That is because $\underline{A_\infty}$ is defined using parallel transport with respect to θ as in Definition 4.23.

Proposition 4.34 (Proposition 7.6 in [Wal17]). *Let $\sigma \in \Omega^1(\mathrm{Ad}\, s(E)|_O)$ and $b \in \Omega^1(\mathrm{Ad}\, E_0|_O)$ as defined in Eq. (4.31). Then there exists $c > 0$ such that for all $t \in (0, T)$:*

$$\begin{aligned} ||\sigma||_{C^{0,\alpha}_{-3,0;t}(t \leq r_t \leq R)} + \left|\left|\mathrm{d}_{\overline{A_\infty}}\sigma\right|\right|_{C^{0,\alpha}_{-4,0;t}(t \leq r_t \leq R)} &\leq ct^2 \text{ and} \\ ||b||_{C^{0,\alpha}_{1,0;t}(r_t \leq R)} + \left|\left|\mathrm{d}_{\underline{A_\infty}} b\right|\right|_{C^{0,\alpha}_{0,0;t}(r_t \leq R)} &\leq ct^2. \end{aligned}$$

4.2. Pregluing estimate The goal of this section is to derive an estimate for $F_{A_t} \wedge \widetilde{\psi}_t^N$. This is achieved in Corollary 4.55 in the general case, and in Corollary 4.58 in the special case of resolutions of T^7/Γ.

4.2.1. Estimates for the G_2-structures involved We have constructed a connection A_t that looks like $s(A)$ near L and looks like θ far away from L. The connection $s(A)$ is close to being a G_2-instanton with respect to ψ_t^P, so in order to control the pregluing error, we will need to estimate the difference $\psi_t^N - \varphi_t^P$. This will be done in Propositions 4.35 and 4.38.
On the other hand, θ is a G_2-instanton with respect to ψ, so we will need to estimate the difference $\psi_t^N - \psi$. This will be done in Proposition 4.40.

Proposition 4.35. *There exists $c > 0$ independent of t such that*

$$\left|\left|\psi_t^N - \psi_t^P\right|\right|_{C^{0,\alpha}_{2,0;t}(U_R)} \leq ct^{-1}. \tag{4.36}$$

Proof. We have

$$
\begin{aligned}
&|\psi_t^N - \psi_t^P|_{g_t^N} \\
&= \begin{cases}
\mathrm{d}[t^2\beta_{0,3} + t^4\beta_{2,1} +]t^2\chi_{1,3} + t^4\theta_{3,1} + t^4\theta_{2,2} & \text{if } \check{r} \leq t^{-1/9} \\
\mathrm{d}[t^2\beta_{0,3} + t^4\beta_{2,1} + a(t^{1/9}\check{r}) \cdot \Upsilon_*\zeta] + t^2\chi_{1,3} + t^4\theta_{3,1} + t^4\theta_{2,2} & \text{if } t^{-1/9} \leq \check{r} \leq 2t^{-1/9} \\
\mathrm{d}[t^2\beta_{0,3} + t^4\beta_{2,1} + \Upsilon_*\zeta] + t^2\chi_{1,3} + t^4\theta_{3,1} + t^4\theta_{2,2} & \text{if } 2t^{-1/9} \leq \check{r} \leq t^{-4/5} \\
\mathrm{d}[(1 - a(t^{4/5}\check{r}))(t^2\beta_{0,3} + t^4\beta_{2,1}) + \Upsilon_*\zeta] + t^2\chi_{1,3} + t^4\theta_{3,1} + t^4\theta_{2,2} - a(t^{4/5}\check{r})t^2 v_{1,2} & \text{if } t^{-4/5} \leq \check{r} \leq 2t^{-4/5} \\
\mathrm{d}(\Upsilon_*\zeta) + t^2\chi_{1,3} + t^4\theta_{3,1} + t^4\theta_{2,2} - t^2 v_{1,2} & \text{if } 2t^{-4/5} \leq \check{r}
\end{cases} \\
&= \begin{cases}
\mathcal{O}(t) & \text{if } \check{r} \leq t \\
\mathcal{O}(t\check{r}^{-3}) & \text{if } t \leq \check{r} \leq t^{-1/9} \\
\mathcal{O}(t\check{r}^{-3} + t^2\check{r}^2) & \text{if } t^{-1/9} \leq \check{r} \leq 2t^{-1/9} \\
\mathcal{O}(t\check{r}^{-3} + t^2\check{r}^2) & \text{if } 2t^{-1/9} \leq \check{r} \leq t^{-4/5} \\
\mathcal{O}(t^2\check{r}^2 + \check{r}^{-4}) & \text{if } t^{-4/5} \leq \check{r} \leq 2t^{-4/5} \\
\mathcal{O}(t^2\check{r}^2 + \check{r}^{-4}) & \text{if } 2t^{-4/5} \leq \check{r},
\end{cases}
\end{aligned}
\tag{4.37}
$$

where we used Propositions 3.19 and 3.8 and Theorem 3.26 in the second step. Multiplying with the weight function $(t + r_t)^{-2}$ gives the estimate for the $L^\infty_{2,0;t}$-norm, and the estimate for the $C^{0,\alpha}_{2,0;t}$-norm is proved analogously. □

Proposition 4.38. *Let N_t be the resolution of T^7/Γ from Sect. 3.1. There exists $c > 0$ independent of t such that*

$$
\left|\left|\psi_t^N - \psi_t^P\right|\right|_{C^{0,\alpha}_{2,0;t}(U_R)} \leq ct^4. \tag{4.39}
$$

Proof. This is a restatement of Eq. (3.1). In the case that N_t is the resolution of T^7/Γ we have that ψ_t^P is closed, so the forms $t^2\chi_{1,3}, t^4\theta_{3,1}, t^4\theta_{2,2}$ from Proposition 3.19 can be chosen to be 0. Furthermore, in this case $\widetilde{\psi}_t^\nu = \Upsilon_t^*(*\varphi)$, so $\zeta = 0$. Using this and that the cut-off happens where $\zeta t^{-1}/2 \leq \check{r} \leq \zeta t^{-1}$, the same proof as for Eq. (4.36) shows the claim. □

The following estimate holds in general, not just for resolutions of T^7/Γ:

Proposition 4.40. *There exists $c > 0$ independent of t such that*

$$
\left|\left|\psi_t^N - \psi\right|\right|_{C^{0,\alpha}_{-2,0;t}(\{x \in N_t : \check{r}(x) \geq 1\})} \leq ct^2. \tag{4.41}
$$

Proof. Using Propositions 3.8 and 3.19 and Theorem 3.26, the proof is analogous to Proposition 4.35. □

Last we need an estimate comparing $\widetilde{\psi}_t^N$ and ψ_t^N in a Hölder norm. In Theorem 3.26 we had this estimate for the L^∞-norm, but not for the $C^{0,\alpha}_{0,0;t}$-norm. Going through the proof of 2.10, one can improve this to a $C^{0,\alpha}_{0,0;t}$-estimate as stated in the following proposition. For the case of resolutions of T^7/Γ, this was done in [Wal13a, Proposition 4.20], and the proof carries over to resolutions of $Y/\langle\iota\rangle$.

Proposition 4.42. *There exists $c > 0$ independent of t such that*

$$\left|\left|\widetilde{\psi}_t^N - \psi_t^N\right|\right|_{C^{0,\alpha}_{0,0;t}} \leq ct^{1/18}. \tag{4.43}$$

4.2.2. Principal bundle curvature estimates For our pregluing estimate we will want to estimate $*(F_{A_t} \wedge \widetilde{\psi}_t^N)$. This is done in Corollaries 4.55 and 4.58. Most of the heavy lifting is done by the following Proposition 4.44: here we get an estimate for $*(F_{A_t} \wedge \psi_t^N)$ which then is combined with the estimate for $\widetilde{\psi}_t^N - \psi_t^N$.

Proposition 4.44. *There exists $c > 0$ such that for all $t \in (0, T)$ we have*

$$\left|\left|*(F_{A_t} \wedge \psi_t^N)\right|\right|_{C^{0,\alpha}_{-2,0;t}} \leq ct. \tag{4.45}$$

Proof. We will estimate $*(F_{A_t} \wedge \psi_t^N)$ separately on some regions:

1. On $r_t \leq 2t$ we have

$$F_{A_t} = F_{s(A)} + \chi_t^- \mathrm{d}_{A_\infty} b + \chi_t^- [\sigma, b] + \frac{1}{2}(\chi_t^-)^2 [b, b] + \mathrm{d}\chi_t^- \wedge b.$$

Thus by Proposition 4.22, Proposition 4.33, and Proposition 4.34:

$$\begin{aligned}
&||F_{A_t} - F_{s(A)}||_{C^{0,\alpha}_{-2,0;t}(r_t \leq 2t)} \\
&\leq ||1||_{C^{0,\alpha}_{-2,0;t}(r_t \leq 2t)} \left|\left|\chi_t^-\right|\right|_{C^{0,\alpha}_{0,0;t}(r_t \leq 2t)} \left|\left|\mathrm{d}_{A_\infty} b\right|\right|_{C^{0,\alpha}_{0,0;t}(r_t \leq 2t)} \\
&\quad + \left|\left|\chi_t^-\right|\right|_{C^{0,\alpha}_{0,0;t}(r_t \leq 2t)} ||\sigma||_{C^{0,\alpha}_{-3,0;t}(r_t \leq 2t)} ||b||_{C^{0,\alpha}_{1,0;t}(r_t \leq 2t)} \\
&\quad + \frac{1}{2} ||1||_{C^{0,\alpha}_{-3,0;t}(r_t \leq 2t)} \left|\left|\chi_t^-\right|\right|^2_{C^{0,\alpha}_{0,0;t}(r_t \leq 2t)} ||b||^2_{C^{0,\alpha}_{1,0;t}(r_t \leq 2t)} \\
&\quad + ||1||_{C^{0,\alpha}_{-2,0;t}(r_t \leq 2t)} \left|\left|\mathrm{d}\chi_t^-\right|\right|_{C^{0,\alpha}_{-1,0;t}(r_t \leq 2t)} ||b||_{C^{0,\alpha}_{1,0;t}(r_t \leq 2t)} \\
&\leq ct^2
\end{aligned} \tag{4.46}$$

where we also used the fact that $||1||_{C^{0,\alpha}_{-l,0;t}(r_t \leq 2t)} \leq ct^l$ if $l > 0$, which follows from Definition 4.19 using $r_t \leq 2t$.
Remember that $[F_{s(A)}]_{2,0} \wedge \psi_t^P = 0$ by the ASD condition and $[F_{s(A)}]_{1,1} \wedge \psi_t^P = 0$ by the Fueter condition (cf. Theorem 4.15). By Proposition 4.24, we therefore have:

$$\begin{aligned}
&\left|\left|F_{s(A)} \wedge \psi_t^P\right|\right|_{C^{0,\alpha}_{-2,0;t}(r_t \leq 2t)} \\
&\leq \left|\left|[F_{s(A)}]_{(0,2)} \wedge \psi_t^P\right|\right|_{C^{0,\alpha}_{-2,0;t}(r_t \leq 2t)} \\
&\leq \left|\left|[F_{s(A)} - F_{\theta|_L}]_{(0,2)}\right|\right|_{C^{0,\alpha}_{-2,0;t}(r_t \leq 2t)} \cdot \left|\left|\psi_t^P\right|\right|_{C^{0,\alpha}_{0,0;t}(r_t \leq 2t)} + \\
&\left|\left|F_{\theta|_L}\right|\right|_{C^{0,\alpha}_{0,0;t}(r_t \leq 2t)} \cdot \left|\left|\psi_t^P\right|\right|_{C^{0,\alpha}_{0,0;t}(r_t \leq 2t)} \cdot ||1||_{C^{0,\alpha}_{-2,0;t}(r_t \leq 2t)} \\
&\leq ct^2,
\end{aligned} \tag{4.47}$$

where we again used Proposition 4.22. Last, note that by Proposition 4.24 and Eq. (4.46) we have $\left|\left|F_{A_t}\right|\right|_{C^{0,\alpha}_{-4,0;t}(r_t\leq 2t)} \leq ct^2$. Thus, by Proposition 4.22 and Eq. (4.36):

$$\begin{aligned}\left|\left|F_{A_t}\wedge(\psi_t^N-\psi_t^P)\right|\right|_{C^{0,\alpha}_{-2,0;t}(r_t\leq 2t)} &\leq \left|\left|F_{A_t}\right|\right|_{C^{0,\alpha}_{-4,0;t}(r_t\leq 2t)}\left|\left|\psi_t^N-\psi_t^P\right|\right|_{C^{0,\alpha}_{2,0;t}(r_t\leq 2t)}\\ &\leq ct.\end{aligned} \tag{4.48}$$

Putting the estimates from Eqs. (4.46) to (4.48) together, we get

$$\begin{aligned}&\left|\left|*(F_{A_t}\wedge\psi_t^N)\right|\right|_{C^{0,\alpha}_{-2,0;t}(r_t\leq 2t)}\\ &\leq \left|\left|F_{s(A)}\wedge\psi_t^P)\right|\right|_{C^{0,\alpha}_{-2,0;t}(r_t\leq 2t)}+\left|\left|(F_{s(A)}-F_{A_t})\wedge\psi_t^P\right|\right|_{C^{0,\alpha}_{-2,0;t}(r_t\leq 2t)}\\ &\quad+\left|\left|F_{A_t}\wedge(\psi_t^N-\psi_t^P)\right|\right|_{C^{0,\alpha}_{-2,0;t}(r_t\leq 2t)}\\ &\leq c(t^2+t^2+t)\leq ct.\end{aligned}$$

2. On $2t\leq r_t\leq R/2$ we have $A_t=A_\infty+\sigma+b$ and therefore

$$F_{A_t}=F_\theta+[\sigma,b]+F_{s(A)}-F_{A_\infty}. \tag{4.49}$$

First,

$$\begin{aligned}&\left|\left|(F_{s(A)}-F_{A_\infty})\wedge\psi_t^P\right|\right|_{C^{0,\alpha}_{-2,0;t}(2t\leq r_t\leq R/2)}\\ &\leq \left|\left|[F_{s(A)}-F_{A_\infty}]_{2,0}\wedge\psi_t^P\right|\right|_{C^{0,\alpha}_{-2,0;t}(2t\leq r_t\leq R/2)}\\ &\leq \left|\left|[F_{s(A)}-F_{A_\infty}]_{2,0}\right|\right|_{C^{0,\alpha}_{-2,0;t}(2t\leq r_t\leq R/2)}\left|\left|\psi_t^P\right|\right|_{C^{0,\alpha}_{0,0;t}(2t\leq r_t\leq R/2)}\\ &\leq ct^2,\end{aligned} \tag{4.50}$$

where we used point (ii) of Theorem 4.15 in the first step and Proposition 4.24 in the last step. We also have

$$\begin{aligned}&\left|\left|(F_{s(A)}-F_{A_\infty})\wedge(\psi_t^N-\psi_t^P)\right|\right|_{C^{0,\alpha}_{-2,0;t}(2t\leq r_t\leq R/2)}\\ &\leq \left|\left|(F_{s(A)}-F_{A_\infty})\right|\right|_{C^{0,\alpha}_{-4,0;t}(2t\leq r_t\leq R/2)}\left|\left|\psi_t^N-\psi_t^P\right|\right|_{C^{0,\alpha}_{2,0;t}(2t\leq r_t\leq R/2)}\\ &\leq ct\end{aligned} \tag{4.51}$$

where we used Proposition 4.24 and Eq. (4.36), therefore

$$\begin{aligned}&\left|\left|(F_{s(A)}-F_{A_\infty})\wedge\psi_t^N\right|\right|_{C^{0,\alpha}_{-2,0;t}(2t\leq r_t\leq R/2)}\\ &\leq \left|\left|(F_{s(A)}-F_{A_\infty})\wedge\psi_t^P\right|\right|_{C^{0,\alpha}_{-2,0;t}(2t\leq r_t\leq R/2)}\\ &\quad+\left|\left|(F_{s(A)}-F_{A_\infty})\wedge(\psi_t^N-\psi_t^P)\right|\right|_{C^{0,\alpha}_{-2,0;t}(2t\leq r_t\leq R/2)}\\ &\leq ct.\end{aligned} \tag{4.52}$$

Second,

$$
\begin{aligned}
&\left|\left|[\sigma, b] \wedge \psi_t^N\right|\right|_{C^{0,\alpha}_{-2,0;t}(2t \le r_t \le R/2)} \\
&\le c\,||\sigma||_{C^{0,\alpha}_{-3,0;t}(2t \le r_t \le R/2)}\,||b||_{C^{0,\alpha}_{1,0;t}(2t \le r_t \le R/2)}\left|\left|\psi_t^N\right|\right|_{C^{0,\alpha}_{0,0;t}(2t \le r_t \le R/2)} \\
&\le ct^4
\end{aligned}
\tag{4.53}
$$

by Proposition 4.34.
Third,

$$
\begin{aligned}
&\left|\left|F_\theta \wedge \psi_t^N\right|\right|_{C^{0,\alpha}_{-2,0;t}(2t \le r_t \le R/2)} \\
&\le ||F_\theta \wedge \psi||_{C^{0,\alpha}_{-2,0;t}(2t \le r_t \le R/2)} \\
&\quad + ||F_\theta||_{C^{0,\alpha}_{0,0;t}(2t \le r_t \le R/2)}\left|\left|\psi_t^N - \psi\right|\right|_{C^{0,\alpha}_{-2,0;t}(2t \le r_t \le R/2)} \\
&\le ct^2
\end{aligned}
\tag{4.54}
$$

where we used the fact that θ is a G_2-instanton with respect to ψ as well as Eq. (4.41) in the second step. So, altogether

$$
\begin{aligned}
\left|\left|*(F_{A_t} \wedge \psi_t^N)\right|\right|_{C^{0,\alpha}_{-2,0;t}(2t \le r_t \le R/2)} &\le \left|\left|F_\theta \wedge \psi_t^N\right|\right|_{C^{0,\alpha}_{-2,0;t}(2t \le r_t \le R/2)} \\
&\quad + \left|\left|[\sigma, b] \wedge \psi_t^N\right|\right|_{C^{0,\alpha}_{-2,0;t}(2t \le r_t \le R/2)} \\
&\quad + \left|\left|(F_{s(A)} - F_{A_\infty}) \wedge \psi_t^N\right|\right|_{C^{0,\alpha}_{-2,0;t}(2t \le r_t \le R/2)} \\
&\le ct
\end{aligned}
$$

by combining Eqs. (4.49) and (4.52) to (4.54).

3. On $R/2 \le r_t \le R$ we have $A_t = \theta + \chi_t^+\sigma$ and therefore

$$
F_{A_t} = F_\theta + \chi_t^+\,\mathrm{d}_\theta\sigma + \frac{1}{2}(\chi_t^+)^2[\sigma, \sigma] + \mathrm{d}\chi_t^+ \wedge \sigma.
$$

Therefore, we find that

$$
\begin{aligned}
\left|\left|F_{A_t} - F_\theta\right|\right|_{C^{0,\alpha}_{-2,0;t}(R/2 \le r_t)} &\le \left|\left|\chi_t^+\right|\right|_{C^{0,\alpha}_{0,0;t}(R/2 \le r_t)}\,||\mathrm{d}_\theta\sigma||_{C^{0,\alpha}_{-4,0;t}(R/2 \le r_t)}\,||1||_{C^{0,\alpha}_{2,0;t}(R/2 \le r_t)} \\
&\quad + \frac{1}{2}\left|\left|\chi_t^+\right|\right|^2_{C^{0,\alpha}_{0,0;t}(R/2 \le r_t)}\,||\sigma||^2_{C^{0,\alpha}_{-3,0;t}(R/2 \le r_t)}\,||1||_{C^{0,\alpha}_{4,0;t}(R/2 \le r_t)} \\
&\quad + \left|\left|\mathrm{d}\chi_t^+\right|\right|_{C^{0,\alpha}_{0,0;t}(R/2 \le r_t)}\,||\sigma||_{C^{0,\alpha}_{-3,0;t}(R/2 \le r_t)}\,||1||_{C^{0,\alpha}_{1,0;t}(R/2 \le r_t)} \\
&\le ct^2
\end{aligned}
$$

where we used Propositions 4.22,4.33 and 4.34 in the second step. Using this, we see

$$
\left|\left|F_{A_t} \wedge \psi_t^N\right|\right|_{C^{0,\alpha}_{-2,0;t}(R/2 \le r_t)} \le \left|\left|(F_{A_t} - F_\theta) \wedge \psi_t^N\right|\right|_{C^{0,\alpha}_{-2,0;t}(R/2 \le r_t)}
$$

$$+ \left|\left| F_\theta \wedge \psi_t^N \right|\right|_{C^{0,\alpha}_{-2,0;t}(R/2 \leq r_t)}$$
$$\leq ct^2,$$

where we used the fact that $\psi_t^N = \psi$ where $r_t \geq R/2$ and that θ is a G_2-instanton with respect to ψ.

We have that $F_{A_t} \wedge \psi_t^N = 0$ outside the three considered regions, which proves the claim. □

Corollary 4.55. *There exists $c > 0$ such that*

$$\left|\left| *(F_{A_t} \wedge \widetilde{\psi}_t^N) \right|\right|_{C^{0,\alpha}_{-2,0;t}} \leq ct^{1/18}. \tag{4.56}$$

Proof. First, observe that

$$\left|\left| F_{A_t} \right|\right|_{C^{0,\alpha}_{-2,0;t}} \leq c. \tag{4.57}$$

This follows from estimating F_{A_t} separately on the three regions from the proof of Proposition 4.44. Then

$$\begin{aligned} \left|\left| *(F_{A_t} \wedge \widetilde{\psi}_t^N) \right|\right|_{C^{0,\alpha}_{-2,0;t}} &\leq \left|\left| *(F_{A_t} \wedge \psi_t^N) \right|\right|_{C^{0,\alpha}_{-2,0;t}} + \left|\left| *(F_{A_t} \wedge (\widetilde{\psi}_t^N - \psi_t^N)) \right|\right|_{C^{0,\alpha}_{-2,0;t}} \\ &\leq \left|\left| *(F_{A_t} \wedge \psi_t^N) \right|\right|_{C^{0,\alpha}_{-2,0;t}} + \left|\left| F_{A_t} \right|\right|_{C^{0,\alpha}_{-2,0;t}} \left|\left| \widetilde{\psi}_t^N - \psi_t^N \right|\right|_{C^{0,\alpha}_{0,0;t}} \\ &\leq c(t + t^{1/18}) \leq ct^{1/18} \end{aligned}$$

where we used Proposition 4.44 to estimate the first summand in the last step, and Eqs. (4.43) and (4.57) to estimate the second summand in the last step. □

As promised, we now turn to the special case of resolutions of T^7/Γ, rather than general G_2-orbifolds. We get a better pregluing estimate here, which is due to the following two facts: first, we get a better estimate for $*(F_{A_t} \wedge \psi_t^N)$ on the resolution of T^7/Γ, because near the associative, A_t is close to $s(A)$, which is close to being a G_2-instanton with respect to ψ_t^P, and Proposition 4.38 says that $\psi_t^N - \psi_t^P$ is small. Second, the difference $\widetilde{\psi}_t^N - \psi_t^N$ is smaller on resolutions of T^7/Γ than in the general case.

Corollary 4.58. *Let N_t be the resolution of T^7/Γ from Theorem 3.2. Then there exists $c > 0$ such that for all $t \in (0, T)$ we have*

$$\left|\left| *(F_{A_t} \wedge \widetilde{\psi}_t^N) \right|\right|_{C^{0,\alpha}_{-2,0;t}} \leq ct^2. \tag{4.59}$$

Proof. We first prove

$$\left|\left| *(F_{A_t} \wedge \psi_t^N) \right|\right|_{C^{0,\alpha}_{-2,0;t}} \leq ct^2. \tag{4.60}$$

as in Proposition 4.44, the only difference being that Eq. (4.39) in Eqs. (4.48) and (4.51) gives a factor of t^2 rather than t, yielding Eq. (4.60). For small enough $\alpha \in (0, 1)$ we have that

$$\left|\left| \widetilde{\psi}_t^N - \psi_t^N \right|\right|_{C^{0,\alpha}_{0,0;t}} \leq ct^{5/2} \tag{4.61}$$

by Theorem 3.2. Taking Eqs. (4.60) and (4.61) together gives Eq. (4.59) as in the proof of Corollary 4.55. □

4.3. Linear estimates We now arrived in the second step of the three step process of (1) constructing an approximate solution, (2) estimating the linearisation of the instanton equation, and (3) perturbing the approximate solution to a genuine solution. The estimate in question is Proposition 4.78. It makes use of the norms $||\cdot||_{\mathfrak{X}_t}$ and $||\cdot||_{\mathfrak{Y}_t}$ that are defined in Sect. 4.3.1.

The idea of the proof is this: near the resolution locus of the associative L, the linearisation of the instanton equation is approximately equal to the linearisation of the Fueter equation. Deformations of the approximate solution and deformations of the Fueter section live in different spaces, so some work will need to go into making this statement precise.

Over the course of Sects. 4.3.3 to 4.3.5 we work out an estimate for the linearised operator modulo deformations of the approximate instanton that come from deformations of the Fueter section. This estimate is given in Proposition 4.108. We use a Schauder estimate for the linearised operator, which is given in section Sect. 4.3.4, together with analysis on the local models $\mathbb{R}^3 \times X_{\mathrm{EH}}$ and $\mathbb{R}^3 \times \mathbb{C}^2/\{\pm 1\}$, which is explained in Sect. 4.3.3.

So we have estimates for the linearised operator on instanton deformations that come from deformations of the Fueter section from Sect. 4.3.2 and on the other instanton deformations from Sect. 4.3.5. In Sects. 4.3.6 and 4.3.7 we combine both and complete the proof of Proposition 4.108.

4.3.1. Stating the estimate In the previous section, we constructed a connection $A_t \in \mathscr{A}(E_t)$. The linearisation of the G_2-instanton equation together with the Coulomb gauge condition is

$$\begin{aligned} L_t := L_{A_t} : (\Omega^0 \oplus \Omega^1)(M, \operatorname{Ad} E) &\to (\Omega^0 \oplus \Omega^1)(M, \operatorname{Ad} E) \\ \begin{pmatrix} \xi \\ a \end{pmatrix} &\mapsto \begin{pmatrix} 0 & \mathrm{d}^*_{A_t} \\ \mathrm{d}_{A_t} & *(\widetilde{\psi}^N_t \wedge \mathrm{d}_{A_t}) \end{pmatrix} \begin{pmatrix} \xi \\ a \end{pmatrix}, \end{aligned}$$

cf. Equation (2.44). We introduce the following notation for the constant part and the quadratic part of the G_2-instanton equation: for $\underline{a} = (\xi, a) \in (\Omega^0 \oplus \Omega^1)(N_t, \operatorname{Ad} E_t)$ define e_t as well as $Q_t(\underline{a}) \in \Omega^0(N_t, \operatorname{Ad} E_t)$ via

$$\begin{aligned} &* (F_{A_t+a} \wedge \widetilde{\psi}^N_t) + \mathrm{d}_{A_t+a}\xi \\ &\quad = \underbrace{*(F_{A_t} \wedge \widetilde{\psi}^N_t)}_{=:e_t} + *(\mathrm{d}_{A_t} a \wedge \widetilde{\psi}^N_t) + \mathrm{d}_{A_t}\xi + \underbrace{\frac{1}{2} * ([a \wedge a] \wedge \widetilde{\psi}^N_t) + [\xi, a]}_{=:Q_t(\underline{a})}. \end{aligned} \tag{4.62}$$

In this section we will study the operator L_t and derive an estimate for the operator norm of the inverse of L_t. This operator norm will be taken with respect to the complicated norms $||\cdot||_{\mathfrak{X}}$ and $||\cdot||_{\mathfrak{Y}}$, taken from [Wal17, Section 8], which we will explain now.

We need a way to decompose elements in $\Omega^1(N_t, \operatorname{Ad} E_t)$ into a part coming from a section of $s^*(V\mathfrak{M})$, which is nonzero only near the gluing area, and a rest:

Definition 4.63. The section s gives rise to a connection $s(A) \in \mathscr{A}(s(E))$ by Theorem 4.15. A section $f \in \Gamma(s^* V\mathfrak{M})$ analogously defines an element in $T_{s(A)}\mathscr{A}(s(E)) = \Omega^1(P, \operatorname{Ad} s(E))$, say $i_* f$. Use this to define

$$\begin{aligned} \iota_t : \Gamma(s^* V\mathfrak{M}) &\to \Omega^1(N_t, \operatorname{Ad} E_t) \\ f &\mapsto \chi^+_t \cdot i_* f. \end{aligned} \tag{4.64}$$

Further define $\pi_t : \Omega^1(N_t, \operatorname{Ad} E_t) \to \Gamma(s^* V\mathfrak{M})$ for $a \in \Omega^1(N_t, \operatorname{Ad} E_t)$ and $x \in L$ by

$$(\pi_t a)(x) := \sum_\kappa \int_{P_x} \langle a, \iota_t \kappa \rangle_{g_t^P} \operatorname{vol}_{g_t^P|_{P_x}} \cdot \kappa, \tag{4.65}$$

where κ runs through an orthonormal basis of $(V\mathfrak{M})_{s(x)}$ with respect to the inner product $\langle \iota_t \cdot, \iota_t \cdot \rangle_{g_P^t}$. Here the integral is taken with respect to the metric induced by φ_t^P restricted to P_x. Let further $\overline{\pi}_t := \iota_t \pi_t$ and $\eta_t := \operatorname{Id} - \overline{\pi}_t$.

The following proposition states that ι_t and π_t are bounded operators. The proof of these estimates is similar to the proof of [Wal17, Proposition 6.4].

Proposition 4.66. *For $l \leq -1$ and $\delta \in \mathbb{R}$ such that $l - \alpha + \delta > -3$ and $l + \delta < -1$ there is a constant $c > 0$ such that for all $t \in (0, T)$ we have*

$$\begin{aligned} ||\iota_t f||_{C^{0,\alpha}_{l,\delta;t}} &\leq c t^{-1-l} \, ||f||_{C^{0,\alpha}} \text{ and} \\ ||\pi_t a||_{C^{0,\alpha}} &\leq c t^{1+l-\alpha} \, ||a||_{C^{0,\alpha}_{l,\delta;t}(V_{[0,R),t})} . \end{aligned}$$

Proof. The proof of the first inequality is the same as the proof of [Wal17, Proposition 6.4].
To prove the second inequality, note that by Proposition 2.25 we have for $x \in L, \kappa \in (V\mathfrak{M})_{s(x)}$

$$|i_* \kappa|_{g_1^P} \leq c_\kappa (1 + \check{r})^{-3}$$

for a constant c_κ depending on $x \in L$ and on κ. Because $(V\mathfrak{M})_{s(x)}$ is a finite-dimensional vector space we can take $c = \max_{||\kappa||_{L^2, g_1^P} = 1} c_\kappa$ to get the estimate

$$|i_* \kappa|_{g_1^P} \leq c (1 + \check{r})^{-3} \, ||\kappa||_{g_1^P, L^2} \tag{4.67}$$

for a constant c independent of κ. By compactness of L, we can assume c to also be independent of $x \in L$. By measuring in g_t^P instead of g_1^P we get from Eq. (4.67):

$$|i_* \kappa|_{g_t^P} = t^{-1} |i_* \kappa|_{g_1^P} \leq c t^2 (t + t\check{r})^{-3} \, ||\kappa||_{g_1^P, L^2} . \tag{4.68}$$

For some interval $J \subset \mathbb{R}$ and $x \in L$ we denote $P_{x,J} := \left\{ u \in P_x : \check{r}(u) \in J \right\}$ and similarly for $(\nu/\{\pm 1\})_{x,J}$. By abuse of notation we write $\operatorname{vol}_{g_t^P}$ for $\operatorname{vol}_{g_t^P|_{P_x}} \in \Omega^4(P_x)$ and similarly for $\operatorname{vol}_{g_t^\nu}$.

$$\begin{aligned} \int_{P_x} \langle a, \chi_t^+ \cdot i_* \kappa \rangle_{g_t^P} \operatorname{vol}_{g_t^P} &\leq \int_{P_x} |a|_{g_t^P} |\chi_t^+ \cdot i_* \kappa|_{g_t^P} \operatorname{vol}_{g_t^P} \\ &\leq c \int_{P_{x,[0,1]}} \frac{t^2}{(t + t\check{r})^3} w_{l,\delta;t}^{-1} \operatorname{vol}_{g_t^P} \; ||a||_{L^\infty_{l,\delta;t}, g_t^P} \, ||\kappa||_{L^2, g_1^P} \\ &\quad + c \int_{P_{x,\left[1, Rt^{-1}\right]}} \frac{t^2}{(t + t\check{r})^3} w_{l,\delta;t}^{-1} \operatorname{vol}_{g_t^P} \; ||a||_{L^\infty_{l,\delta;t}, g_t^P} \, ||\kappa||_{L^2, g_1^P} \\ &\leq c \operatorname{vol}_{g_t^P}(P_x, [0,1]) \cdot t^{l-1} \; ||a||_{L^\infty_{l,\delta;t}, g_t^P} \, ||\kappa||_{L^2, g_1^P} \\ &\quad + c \int_{(\nu/\{\pm 1\})_{x,\left[0, Rt^{-1}\right]}} \frac{t^2}{(t + t\check{r})^3} w_{l,\delta;t}^{-1} \operatorname{vol}_{g_t^\nu} \; ||a||_{L^\infty_{l,\delta;t}, g_t^P} \, ||\kappa||_{L^2, g_1^P} \end{aligned}$$

$$
\begin{aligned}
&\leq ct^{l+3} \; ||a||_{L^{\infty}_{l,\delta;t}, g_t^P} \, ||\kappa||_{L^2, g_1^P} \\
&\quad + c \int_0^{\sqrt{t}} t^{2-\delta}(t+r)^{l+\delta-3} r^3 \, \mathrm{d}r \cdot ||a||_{L^{\infty}_{l,\delta;t}, g_t^P} \, ||\kappa||_{L^2, g_1^P} \\
&\quad + c \int_{\sqrt{t}}^R t^2 r^{l-\delta}(t+r)^{-3} r^3 \, \mathrm{d}r \cdot ||a||_{L^{\infty}_{l,\delta;t}, g_t^P} \, ||\kappa||_{L^2, g_1^P} \, .
\end{aligned}
\tag{4.69}
$$

Here we used Eq. (4.68) in the second step. In the third step, we switched from integrating over $P_{x,[1,Rt^{-1}]}$ to integrating over $\nu_{x,[1,Rt^{-1}]}$. We could do this because $t\check{r}$ on P corresponds to the radius function r on ν, and $g_t^P|_{P_{x,[1,Rt^{-1}]}} - \rho^* g_t^\nu|_{P_{x,[1,Rt^{-1}]}} \to 0$ measured in g_t^ν as $t \to 0$ by Eqs. (3.9) and (3.16). The latter implies that we can change $\mathrm{vol}_{g_t^P}$ to $\mathrm{vol}_{g_t^\nu}$. We used the definition of $w_{l,\delta;t}$ and changing into sphere coordinates in the fourth step.
We now treat the two integrals separately.

$$
\begin{aligned}
\int_0^{\sqrt{t}} (t+r)^{l+\delta-3} r^3 \, \mathrm{d}r &= \Bigg[(r+t)^{\delta+l} \Bigg(-\frac{3t}{\delta+l} - \frac{t^3}{(-2+\delta+l)(r+t)^2} \\
&\qquad + \frac{3t^2}{(-1+\delta+l)(r+t)} + \frac{r+t}{1+\delta+l} \Bigg) \Bigg]_0^{\sqrt{t}} \\
&\leq c(t^{\delta+l+1} + t^{\delta/2+l/2+1/2}) \\
&\leq ct^{\delta+l+1},
\end{aligned}
\tag{4.70}
$$

where we used a computer algebra system to compute the integral in the first step and used $\delta + l + 1 < 0$ in the third step. For the second integral we find that

$$
\begin{aligned}
\int_{\sqrt{t}}^R r^{l-\delta}(t+r)^{-3} r^3 \, \mathrm{d}r &\leq \int_{\sqrt{t}}^R r^{l+1-\delta} \, \mathrm{d}r \\
&\leq \left[r^{l+1-\delta} \right]_{\sqrt{t}}^R \\
&\leq t^l \cdot t^{-l/2-\delta/2-1/2} \cdot t^1 + c \\
&\leq ct^{l+1}
\end{aligned}
\tag{4.71}
$$

where we used the fact that $-l - \delta - 1 > 0$ to estimate the first summand in the last step, and the fact that $l \leq -1$ to estimate the second summand in the last step.
Combining Eqs. (4.69) to (4.71) we get

$$
\int_{P_x} \langle a, \chi_t \cdot i_* \kappa \rangle_{g_t^P} \, \mathrm{vol}_{g_t^P} \leq ct^{3+l} \, ||a||_{L^{\infty}_{l,\delta;t}} \, ||\kappa||_{L^2, g_1^P} \, .
\tag{4.72}
$$

If $\kappa_1, \kappa_2 \in (V\mathfrak{M}_t)_{s(x)}$, then

$$
\begin{aligned}
\langle \chi_t^+ \cdot i_* \kappa_1, \chi_t^+ \cdot i_* \kappa_2 \rangle_{L^2, g_t^P} &\sim \langle i_* \kappa_1, i_* \kappa_2 \rangle_{L^2, g_t^P} \\
&\sim t^2 \langle i_* \kappa_1, i_* \kappa_2 \rangle_{L^2, g_1^P},
\end{aligned}
\tag{4.73}
$$

where $\sim$ means comparable uniformly in t. Here, in the second step we used the fact that $\mathrm{vol}_{g_t^P|_{P_x}} = t^4 \, \mathrm{vol}_{g_1^P|_{P_x}}$ and $\langle \kappa_1(y), \kappa_2(y)\rangle_{g_t^P} = t^{-2}\langle \kappa_1(y), \kappa_2(y)\rangle_{g_1^P}$ for $y \in P_x$. Equation (4.73) implies that if κ has unit length with respect to the inner product $\langle \iota_t \cdot, \iota_t \cdot\rangle_{g_t^P}$, then

$$||\kappa||_{L^2, g_1^P} \leq ct^{-1}. \tag{4.74}$$

Because $||\cdot||_{L^2, g_1^P}$ and $||\cdot||_{L^\infty, g_1^P}$ are norms on a finite-dimensional vector space, they are equivalent, and thus

$$||\kappa||_{L^\infty, g_1^P} \leq ct^{-1}. \tag{4.75}$$

Combining Eqs. (4.72) and (4.74) to (4.75) and recalling the definition of π_t from Definition 4.63 gives

$$\begin{aligned} ||\pi_t a||_{L^\infty} &\leq \left| \sum_\kappa \int_{P_x} \langle a, \iota_t \kappa\rangle_{g_t^P} \, \mathrm{vol}_{g_t^P|_{P_x}} \right| \cdot ||\kappa||_{L^\infty, g_1^P} \\ &\leq ct^{1+l} \, ||a||_{L^\infty_{l,\delta;t}} . \end{aligned}$$

The estimate for the $||\cdot||_{C^{0,\alpha}}$ Hölder norm follows analogously. □

We are now ready to define the norms which we will use to prove estimates for the operator L_t:

Definition 4.76. Denote by $\mathfrak{X}_t$ and $\mathfrak{Y}_t$ the Banach spaces $C^{1,\alpha}(N_t, (\Lambda^0 \oplus \Lambda^1) \otimes \mathrm{Ad}\, E_t)$ and $C^{0,\alpha}(N_t, (\Lambda^0 \oplus \Lambda^1) \otimes \mathrm{Ad}\, E_t)$ equipped with the norms

$$\begin{aligned} \left|\left|\underline{a}\right|\right|_{\mathfrak{X}_t} &:= t^{-\delta/2} \left|\left|\eta_t \underline{a}\right|\right|_{C^{1,\alpha}_{-1,\delta;t}} + t \left|\left|\pi_t \underline{a}\right|\right|_{C^{1,\alpha}} \quad \text{and} \\ \left|\left|\underline{a}\right|\right|_{\mathfrak{Y}_t} &:= t^{-\delta/2} \left|\left|\eta_t \underline{a}\right|\right|_{C^{0,\alpha}_{-2,\delta;t}} + t \left|\left|\pi_t \underline{a}\right|\right|_{C^{0,\alpha}} \end{aligned} \tag{4.77}$$

respectively.

Using these norms, we can now state the main result of this section:

Proposition 4.78. *Let N_t be the resolution of T^7/Γ from Sect. 3.1. Let s be the Fueter section and θ be the G_2-instanton used in the construction of A_t (cf. Proposition 4.28). If s is infinitesimally rigid and θ is infinitesimally rigid and irreducible, then there exists a constant $c > 0$ which is independent of t such that for small enough t and all $\underline{a} \in (\Omega^0 \oplus \Omega^1)(N_t, \mathrm{Ad}\, E_t)$:*

$$\left|\left|\underline{a}\right|\right|_{\mathfrak{X}_t} \leq c \left|\left|L_t \underline{a}\right|\right|_{\mathfrak{Y}_t} . \tag{4.79}$$

Unfortunately, we are restricted to the case where N_t is a resolution of T^7/Γ. The reason for this is the following: our gluing construction will always produce a connection A_t that is close to being a G_2-instanton with respect to φ_t^N, that is the G_2-structure with small torsion. Hence, if the difference $\widetilde{\varphi}_t^N - \varphi_t^N$ is small, then A_t is also close to being a G_2-instanton with respect to $\widetilde{\varphi}_t^N$, the torsion-free G_2-structure that we are mainly interested in.

For resolutions of general $Y/\langle\iota\rangle$ we have the estimate Proposition 4.42, which states $\widetilde{\varphi}_t^N - \varphi_t^N = \mathcal{O}(t^{1/18})$, measured in a suitable norm. For resolutions of T^7/Γ we have

the much stronger estimate Theorem 3.2 which gives $\mathcal{O}(t^{5/2})$ roughly speaking. Thus, we have a much better *approximate G_2-instanton* on T^7/Γ than on $Y/\langle\iota\rangle$.
It will turn out that we can always perturb the approximate G_2-instanton on T^7/Γ to a genuine instanton, but can only perturb the approximate G_2-instanton on $Y/\langle\iota\rangle$ to a genuine G_2-instanton in very special cases (namely when s is constant). We will revisit this issue in Sect. 4.5, where we construct the perturbation to a genuine G_2-instanton.

4.3.2. Comparison with the Fueter operator Given an element $v \in \Gamma(s^* V\mathfrak{M})$ one may do two different things to it: either embed it into $\Omega^1(N_t, \mathrm{Ad}\, E_t)$ first, and then apply L_t. Or apply the linearised Fueter operator first, and then embed it into $\Omega^1(N_t, \mathrm{Ad}\, E_t)$. It will turn out that the two are the same, up to a small error. In [Wal17], Fueter sections into a moduli bundle of ASD-instantons on $\mathbb{R}^4$ were considered, and the following proposition was proved in that setting. In this article, ASD-instantons on X_{EH} are considered, but the proof works essentially the same way. That said, we do need that $\widetilde{\psi}_t^N - \psi_t^P$ is small. This is true on resolutions of T^7/Γ by Proposition 4.38 and Theorem 3.2 but not proved for general resolutions of G_2-orbifolds. Consequently, we only know the following two propositions to hold on resolutions of T^7/Γ.

Proposition 4.80 (Proposition 8.26 in [Wal17]). *Let N_t be the resolution of T^7/Γ from Sect. 3.1. There exists a constant $c > 0$ such that for all $t \in (0, T)$ and all $v \in \Gamma(s^* V\mathfrak{M})$ the following estimate holds:*

$$||L_t \iota_t v - \iota_t \, \mathrm{d}_s \mathfrak{F} v||_{C^{0,\alpha}_{-2,0;t}} \leq c t^2 \, ||v||_{C^{1,\alpha}} . \tag{4.81}$$

The following proposition is then a simple consequence. It essentially provides the estimate for the inverse of L_t on the space $\mathrm{Im}\, \overline{\pi}_t \subset \Omega^1(N_t, \mathrm{Ad}\, E_t)$.

Proposition 4.82. *Let N_t be the resolution of T^7/Γ from Sect. 3.1. If s is infinitesimally rigid, then there exists a constant $c > 0$ such that for all $t \in (0, T)$ and all $v \in \Gamma(s^* V\mathfrak{M})$ the following estimate holds:*

$$||v||_{C^{1,\alpha}} \leq c \, ||\pi_t L_t \iota_t v||_{C^{0,\alpha}} . \tag{4.83}$$

Proof. We have

$$\begin{aligned} ||v||_{C^{1,\alpha}} &\leq c \, ||\mathrm{d}_s \mathfrak{F} v||_{C^{0,\alpha}} \\ &= c \, ||\pi_t \iota_t \, \mathrm{d}_s \mathfrak{F} v||_{C^{0,\alpha}} \\ &\leq c \left(||\pi_t L_t \iota_t v||_{C^{0,\alpha}} + ||\pi_t (L_t \iota_t v - \iota_t \, \mathrm{d}_s \mathfrak{F} v)||_{C^{0,\alpha}} \right) \\ &\leq c \left(||\pi_t L_t \iota_t v||_{C^{0,\alpha}} + t^{1-\alpha} \, ||v||_{C^{0,\alpha}} \right), \end{aligned}$$

where we used the fact that s is infitesimally rigid in the first step, and Propositions 4.66 and 4.80 in the last step. For small t, we can then absorb the factor $t^{1-\alpha} \, ||v||_{C^{0,\alpha}}$ into the left hand side. □

Remark 4.84. Apart from the connection to [Wal17], the situation in this subsection is also very similar to, but more complicated than, the situation in [Pla20, Propositions 4.29 and 4.35].

4.3.3. The model operators on $\mathbb{R}^3 \times X_{EH}$ *and* $\mathbb{R}^3 \times \mathbb{C}^2/\{\pm 1\}$ As before, let X_{EH} be the Eguchi–Hanson space. To prove the estimate in Proposition 4.78, we will compare the operator L_t with the linearised instanton equation in the model case of a pulled back ASD instanton on $\mathbb{R}^3 \times X_{\mathrm{EH}}$.

Properties of the Model Operator

Let A be a finite energy ASD instanton on a G-bundle E over X_{EH}. The infinitesimal deformations of A are then governed by the operator δ_A from Eq. (2.24). Denote by $p_{X_{\mathrm{EH}}} : \mathbb{R}^3 \times X_{\mathrm{EH}} \to X_{\mathrm{EH}}$ the projection onto the second factor. By a slight abuse of notation we denote the pullbacks of A and E to $\mathbb{R}^3 \times X_{\mathrm{EH}}$ under $p_{X_{\mathrm{EH}}}$ by A and E as well.

Denote by L_A be the linearised G_2-instanton operator from Eq. (2.46). We can define the map $(\,\cdot\,)^\sharp \lrcorner \varphi : p_{\mathbb{R}^3}^* T^*\mathbb{R}^3 \xrightarrow{\simeq} p_{X_{\mathrm{EH}}}^* \Lambda^+ T^* X_{\mathrm{EH}}$, which takes a 1-form, dualises it, and plugs it into the product G_2-structure φ from Eq. (2.11). It maps $\mathrm{d}x_i$ to $-\omega_i$. Using it, we can relate δ_A and L_A as follows:

Proposition 4.85 (Proposition 2.70 in [Wal13b]). *Under the identification*

$$(\,\cdot\,)^\sharp \lrcorner \varphi : p_{\mathbb{R}^3}^* T^*\mathbb{R}^3 \xrightarrow{\simeq} p_{X_{EH}}^* \Lambda^+ T^* X_{EH}$$

and accordingly

$$\Omega^0 \oplus \Omega^1(\mathbb{R}^3 \times X_{EH}, \mathrm{Ad}\, E) \simeq \Omega^0(\mathbb{R}^3 \times X_{EH}, p_{X_{EH}}^*[(\mathbb{R} \oplus \Lambda^+ T^* X_{EH} \oplus T^* X_{EH}) \otimes \mathrm{Ad}\, E])$$

the operator L_A *can be written as* $L_A = F + D_A$ *where*

$$F(\xi, \omega, a) = \sum_{i=1}^{3} (-\langle \partial_i \omega, \omega_i \rangle, \partial_i \xi \cdot \omega_i, I_i \partial_i a) \quad \textit{and} \quad D_A = \begin{pmatrix} 0 & \delta_A \\ \delta_A^* & 0 \end{pmatrix}.$$

Moreover,

$$L_A^* L_A = \Delta_{\mathbb{R}^3} + \begin{pmatrix} \delta_A \delta_A^* & \\ & \delta_A^* \delta_A \end{pmatrix}. \tag{4.86}$$

We define the following weighted norms on the model spaces $\mathbb{R}^3 \times X_{\mathrm{EH}}$ and $\mathbb{R}^3 \times (\mathbb{C}^2/\{\pm 1\})$:

Definition 4.87. For $\beta \in \mathbb{R}$, let

$$\begin{aligned} w_\beta : \mathbb{R}^3 \times X_{\mathrm{EH}} &\to \mathbb{R} \\ x &\mapsto (1 + \check{r}(x))^{-\beta} \end{aligned} \qquad \begin{aligned} w_\beta : \mathbb{R}^3 \times (\mathbb{C}^2/\{\pm 1\}) &\to \mathbb{R} \\ x &\mapsto (1 + d(x, 0))^{-\beta} \end{aligned}$$

and define the weighted Hölder norms $||\cdot||_{C^{0,\alpha}_\beta}$ as in Definition 4.19, but using this new weight function.

We then have the following result:

Proposition 4.88 (Proposition 2.74 in [Wal13b]). *Let* $\widetilde{X}$ *be an ALE space. Let* $\beta \in (-3, 0)$. *Then* $\underline{a} \in C^{1,\alpha}_\beta$ *is in the kernel of* $L_I : C^{1,\alpha}_\beta \to C^{0,\alpha}_{\beta-1}$ *if and only if it is given by the pullback of an element of the* L^2 *kernel of* δ_I *to* $\mathbb{R}^3 \times \widetilde{X}$.

Comparison with L_t

We now explain two maps s^P and s^ν: the first for "zooming into" the resolution locus of the associative L, the second for "zooming into" the gluing region of N_t. Fix a point $y \in L$, a scaling parameter $d \in \mathbb{Z}$, a gluing parameter $t \in (0, T)$, and two positive real numbers ϵ_1, ϵ_2 defining the scale of the region into which to zoom in.
Let

$$V^P_{\epsilon_1,\epsilon_2;t}(y) := \{x \in P : \sigma(x) \in \mathrm{Im}(\exp_y|_{(-\epsilon_1,\epsilon_1)^3}), \check{r}(x)t < \epsilon_2\} \subset P,$$
$$U^P_{\epsilon_1/t,\epsilon_2/t;t} := \{(x,z) \in \mathbb{R}^3 \times X_{\mathrm{EH}} : x \in (-\epsilon_1/t, \epsilon_1/t)^3, \rho(z) < \epsilon_2/t\}.$$

Here we implicitly used an identification $T_yL \simeq \mathbb{R}^3$ to have $\exp_y$ acting on $(-\epsilon_1, \epsilon_1)^3$. Choose this identification so that it maps the orthonormal basis $e_1(y), e_2(y), e_3(y) \in T^*_yL$ from Sect. 3.2.3 to the standard basis $\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3 \in \Lambda^1((\mathbb{R}^3)^*)$. Fix an element $f \in \mathrm{Fr}_y$ of the unitary frame bundle of ν around $y \in L$. It induces an isometry $X_{\mathrm{EH}} \simeq P_y$, and assume that f is chosen so that $\widetilde{\omega}_i$ is sent to $\check{\omega}_i|_{P_y}$ under this map for $i \in \{1,2,3\}$. Then, for small ϵ_1, we define

$$\begin{aligned} E^P : U^P_{\epsilon_1/t,\epsilon_2/t;t} &\to V^P_{\epsilon_1,\epsilon_2;t}(y) \\ (x,z) &\mapsto \mathcal{P}_{s\mapsto \exp_y(stx)}(f(z)) \in P. \end{aligned} \tag{4.89}$$

Here, $s \mapsto \exp_y(stx)$ denotes the unique shortest geodesic from y to $\exp(tx)$ in L, and $\mathcal{P}_{s\mapsto\exp_y(stx)}$ denotes parallel transport in P with respect to $\check{H}$ along this curve, cf. the paragraph before Eq. (3.14). For ϵ_1 small enough, this is a diffeomorphism. The reason for this definition is the following: because of our choices of identifications $T_yL \simeq \mathbb{R}^3$ and $P_y \simeq X_{\mathrm{EH}}$ we have that $(E^P)^*(\varphi^P_t)(0,z)$ is the standard G_2-structure on $\mathbb{R}^3 \times X_{\mathrm{EH}}$ for all $z \in X_{\mathrm{EH}}$, cf. Eq. (3.14). Let a be a tensor field of valence (p,q), i.e. in index notation p lower indices and q upper indices, on $V^P_{\epsilon_1,\epsilon_2;t}(y)$. We then define

$$s^P(a) := s^{P,\epsilon_1,\epsilon_2}_{d,y;t}(a) := t^{d+p-q}(E^P)^*a, \tag{4.90}$$

which is a tensor on $U_{\epsilon_1/t,\epsilon_2/t;t}$. The point of this is the following proposition:

Proposition 4.91. *There are constants $c > 0$ and $\epsilon > 0$ such that for small t the following holds: for all $\epsilon_1, \epsilon_2 \in (0,\epsilon)$ and for all $\underline{a} \in (\Omega^0 \oplus \Omega^1)(N_t, E_t)$:*

$$\left|\left|s^{P,\epsilon_1,\epsilon_2}_{d,t;y}\underline{a}\right|\right|_{L^\infty_{l+\delta}(U^P_{\epsilon_1/t,\epsilon_2/t;t})} \sim t^{d+l}\left|\left|\underline{a}\right|\right|_{L^\infty_{l,\delta;t}(V^P_{\epsilon_1,\epsilon_2}(y))}, \tag{4.92}$$

$$\left|\left|s^{P,\epsilon_1,\epsilon_2}_{d,t;y}\underline{a}\right|\right|_{C^{k,\alpha}_{l+\delta}(U^P_{\epsilon_1/t,\epsilon_2/t;t})} \sim t^{d+l}\left|\left|\underline{a}\right|\right|_{C^{k,\alpha}_{l,\delta;t}(V^P_{\sqrt{t},\sqrt{t}}(y))}, \tag{4.93}$$

where $\sim$ means comparable independently of t. Furthermore, using the Hyperkähler isomorphism $P_y \simeq X_{EH}$ induced by f, we can view the connection $s(A)$ over P_y as a connection over X_{EH}, denoted by $f_*(s(y))$. Then

$$\begin{aligned} &\left|\left|L_t\underline{a} - \left(s^{P,\sqrt{t},\sqrt{t}}_{2,t;y}\right)^{-1} L_{p^*_{X_{EH}}f_*(s(y))} s^{P,\sqrt{t},\sqrt{t}}_{1,t;y}\underline{a}\right|\right|_{C^{0,\alpha}_{-2,\delta;t}(V^P_{\sqrt{t},\sqrt{t}}(y))} \\ &\leq c\sqrt{t}\left|\left|\underline{a}\right|\right|_{C^{1,\alpha}_{-1,\delta;t}(V^P_{\sqrt{t},\sqrt{t}}(y))}. \end{aligned} \tag{4.94}$$

Proof. We first prove Eq. (4.92): for $(0, z) \in U_{\epsilon_1/t,\epsilon_2/t;t}$ the map $\mathrm{d}_{(0,z)} E^P$ (cf. Equation (4.89)) is an isometry for the metric $t^2(g_{\mathbb{R}^3} \oplus g_{(1)})$ on $T_{(0,z)}(\mathbb{R}^3 \times X_{\mathrm{EH}})$ and the metric on $T_{E^P(0,z)}P$ induced by g_t^P. Because of the scaling factor t^{d+p-q} from Eq. (4.99) we have that

$$|s^{P,\epsilon_1,\epsilon_2}_{d,t;y}\underline{a}(0,z)|_{g_{\mathbb{R}^3}\oplus g_{(1)}} = t^d |\underline{a}(E^P(0,z))|_{g_t^P}. \tag{4.95}$$

The map E^P is not, in general, an isometry away from this one point, as $\exp_y$ need not be an isometry. Thus, Eq. (4.95) need not hold in points different from $(0, z)$. However, using Taylor expansions in a neighbourhood of y in L for $\underline{a}$ and g_t^P we get

$$\left|\left|s^{P,\epsilon_1,\epsilon_2}_{d,t;y}\underline{a}\right|\right|_{L^\infty_{l+\delta}(U_{\epsilon_1/t,\epsilon_2/t;t})} \sim t^{d+l}\left|\left|\underline{a}\right|\right|_{L^\infty_{l,\delta;t}(V_{\epsilon_1,\epsilon_2}(y)),g_t^P}.$$

Now Eqs. (4.37) and Proposition 4.42 give the claim for the metric $\tilde{g}_t^N$ instead of g_t^P, which is Eq. (4.92). Equation 4.93 is proved analogously.
Now to prove Eq. (4.94): as in Eq. (4.95), we see that for $x \in P_y$, $\check{r}(x) < 1/\sqrt{t}$,

$$L_{s(A)}\underline{a}(x) - \left(\left(s^{P,\sqrt{t},\sqrt{t}}_{2,t;y}\right)^{-1} L_{p^*_{X_{\mathrm{EH}}} f_*(s(y))} s^{P,\sqrt{t},\sqrt{t}}_{1,t;y}\underline{a}\right)(x) = 0. \tag{4.96}$$

And $A_t - s(A) = \mathcal{O}(1)$ on P_y, so

$$\begin{aligned}
&\left|\left|L_t\underline{a} - \left(\left(s^{P,\sqrt{t},\sqrt{t}}_{2,t;y}\right)^{-1} L_{p^*_{X_{\mathrm{EH}}} f_*(s(y))} s^{P,\sqrt{t},\sqrt{t}}_{1,t;y}\underline{a}\right)\right|\right|_{C^{0,\alpha}_{-2,\delta;t}(\{u\in P_y:\check{r}(u)<1/\sqrt{t}\})} \\
&\leq c\left|\left|[A_t - s(A), \underline{a}]\right|\right|_{C^{0,\alpha}_{-2,\delta;t}(\{u\in P_y:\check{r}(u)<1/\sqrt{t}\})} \\
&\leq c\left|\left|\underline{a}\right|\right|_{C^{0,\alpha}_{-1,\delta;t}(\{u\in P_y:\check{r}(u)<1/\sqrt{t}\})}\left|\left|A_t - s(A)\right|\right|_{C^{0,\alpha}_{-1,0;t}(\{u\in P_y:\check{r}(u)<1/\sqrt{t}\})} \\
&\leq c\sqrt{t}\left|\left|\underline{a}\right|\right|_{C^{0,\alpha}_{-1,\delta;t}(\{u\in P_y:\check{r}(u)<1/\sqrt{t}\})} \\
&\leq c\sqrt{t}\left|\left|\underline{a}\right|\right|_{C^{1,\alpha}_{-1,\delta;t}(\{u\in P_y:\check{r}(u)<1/\sqrt{t}\})}
\end{aligned} \tag{4.97}$$

where in the third step we used $A_t - s(A) = \mathcal{O}(1)$ to estimate the second factor as $\sqrt{t}$. This was possible because the weight function is bounded by $\sqrt{t}$ on $\{u \in P_y : \check{r}(u) < 1/\sqrt{t}\}$.
Eq. 4.94 now follows from using Taylor expansions for $\underline{a}$, g_t^P, and s around y, and comparing g_t^P and $\tilde{g}_t^N$ as in the proof of Eq. (4.92). □

We now define s^ν: let $\epsilon_1 > 0$, $\epsilon_2 > \epsilon_3 > 0$, and

$$\begin{aligned}
V^\nu_{\epsilon_1,\epsilon_2,\epsilon_3;t}(y) &:= \{x \in \nu/\{\pm 1\} : \sigma(x) \in \mathrm{Im}(\exp_y|_{(-\epsilon_1,\epsilon_1)^3}), \epsilon_3 < r(x) < \epsilon_2\}, \\
U^\nu_{\epsilon_1/t,\epsilon_2/t,\epsilon_3/t;t} &:= \{(x,z) \in \mathbb{R}^3 \times \mathbb{C}^2/\{\pm 1\} : x \in (-\epsilon_1/t, \epsilon_1/t)^3, \epsilon_3/t < |\rho(z)| < \epsilon_2/t\}.
\end{aligned}$$

Just as in the definition of $V^P_{\epsilon_1,\epsilon_2;t}$, we implicitly used an identification $T_yL \simeq \mathbb{R}^3$ so that e^i is sent to $\mathrm{d}x^i$ for $i \in \{1, 2, 3\}$. Recall also the frame f that sends $\tilde{\omega}_i$ to $\check{\omega}_i|_{P_y}$ for $i \in \{1, 2, 3\}$ under the isometry $X_{\mathrm{EH}} \simeq P_y$ induced by f. We see from Eq. (3.13) that

ω_i is sent to $\hat{\omega}_i|_{\nu_y}$ under the isometry $\mathbb{C}^2/\{\pm 1\} \simeq (\nu/\{\pm 1\})_y$ induced by f. For small $\epsilon_1, \epsilon_2, \epsilon_3$, the map

$$\begin{aligned} E^\nu : U^\nu_{\epsilon_1/t,\epsilon_2/t,\epsilon_3/t;t} &\to V^\nu_{\epsilon_1,\epsilon_2,\epsilon_3;t}(y) \\ (x,z) &\mapsto \mathcal{P}^\nu_{s\mapsto \exp_y(tsx)}(f(z)) \in \nu/\{\pm 1\} \end{aligned} \tag{4.98}$$

is a diffeomorphism, where $\mathcal{P}^\nu$ denotes parallel transport in ν with respect to the connection $\widetilde{\nabla}^\nu$ from Proposition 3.8. Because of our choices of identifications $T_y L \simeq \mathbb{R}^3$ and $(\nu/\{\pm 1\})_y \simeq \mathbb{C}^2/\{\pm 1\}$ we have that $(E^P)^*(\varphi^\nu_t)(0,z)$ is the standard G_2-structure on $\mathbb{R}^3 \times \mathbb{C}^2/\{\pm 1\}$, for all $z \in \mathbb{C}^2/\{\pm 1\}$, cf. Equation (3.12). We now define s^ν just as we defined s^P in Eq. (4.99), only exchanging E^P for E^ν: for a tensor field a of valence (p,q) on $V^\nu_{\epsilon_1,\epsilon_2,\epsilon_3;t}(y)$ set

$$s^\nu(a) := s^{\nu,\epsilon_1,\epsilon_2,\epsilon_3}_{d,y;t}(a) := t^{d+p-q}(E^\nu)^* a. \tag{4.99}$$

In the following we use the norms from Definition 4.87. So, the notation $C^{0,\alpha}_0$ does not mean zero boundary condition, but means that the weight function appears with powers of 0 and $0+\alpha$ in the two summands of the definition $||\cdot||_{C^{0,\alpha}_0}$. We have the following analogue of Proposition 4.91:

Proposition 4.100. *There are constants $c > 0$ and $\epsilon > 0$ such that for small t the following holds: for all $\epsilon_1, \epsilon_2 \in (0,\epsilon)$, $\epsilon_3 \in (t,\epsilon)$ and for all $\underline{a} \in (\Omega^0 \oplus \Omega^1)(N_t, E_t)$:*

$$\left|\left| w^\nu_{l,\delta;t} s^{\nu,\epsilon_1,\epsilon_2,\epsilon_3}_{d,t;y} \underline{a} \right|\right|_{L^\infty_0(U^\nu_{\epsilon_1/t,\epsilon_2/t,\epsilon_3/t;t})} \sim t^{d+l} \left|\left| \underline{a} \right|\right|_{L^\infty_{l,\delta;t}(V^\nu_{\epsilon_1,\epsilon_2,\epsilon_3}(y))}, \tag{4.101}$$

$$\left|\left| w^\nu_{l,\delta;t} s^{\nu,\epsilon_1,\epsilon_2,\epsilon_3}_{d,t;y} \underline{a} \right|\right|_{C^{k,\alpha}_0(U^\nu_{\epsilon_1/t,\epsilon_2/t,\epsilon_3/t;t})} \sim t^{d+l} \left|\left| \underline{a} \right|\right|_{C^{k,\alpha}_{l,\delta;t}(V^\nu_{\epsilon_1,\epsilon_2,\epsilon_3}(y))}, \tag{4.102}$$

where $\sim$ means uniformly comparable in t and

$$w^\nu_{l,\delta;t} = \begin{cases} r^{-l-\delta} & \text{if } r \leq 1/\sqrt{t} \\ r^{-l+\delta} t^\delta & \text{if } r > 1/\sqrt{t}. \end{cases}$$

Furthermore, using the Hyperkähler isomorphism $P_y \simeq X_{EH}$ induced by f, we can view the connection $s(A)$ over P_y as a connection over X_{EH}. By Eqs. (2.15) and (2.17), this connection is asymptotic to a flat connection, say A_0, on the orbifold $\mathbb{C}^2/\{\pm 1\}$ with monodromy representation ρ. Then

$$\begin{aligned} &\left|\left| L_t \underline{a} - \left(s^{\nu,\epsilon_1,\epsilon_2,\epsilon_3}_{2,t;y} \right)^{-1} L_{p^*_{\mathbb{C}^2} A_0} s^{\nu,\epsilon_1,\epsilon_2,\epsilon_3}_{1,t;y} \underline{a} \right|\right|_{C^{0,\alpha}_{-2,\delta;t}(V^\nu_{\epsilon_1,\epsilon_2,\epsilon_3}(y))} \\ &\leq c(\epsilon_1 + \epsilon_2 + (t/\epsilon_3)^2) \left|\left| \underline{a} \right|\right|_{C^{1,\alpha}_{-1,\delta;t}(V^\nu_{\epsilon_1,\epsilon_2,\epsilon_3}(y))}, \end{aligned} \tag{4.103}$$

where $p_{\mathbb{C}^2} : \mathbb{R}^3 \times \mathbb{C}^2/\{\pm 1\} \to \mathbb{C}^2/\{\pm 1\}$ denotes the projection onto the second factor.

Proof. Equations (4.101) and (4.102) are proved as in Proposition 4.91.
We now prove Eq. (4.103). Adapting Eq. (4.97) to the area $\{u \in P_y : \epsilon_3/t < \check{r}(u) < \epsilon_2/t\}$ we get

$$\begin{aligned}&\left|\left|L_t \underline{a} - \left(\left(s_{2,t;y}^{P,\epsilon_1,\epsilon_2}\right)^{-1} L_{p_{X_{\mathrm{EH}}}^* f_*(s(y))} s_{1,t;y}^{P,\epsilon_1,\epsilon_2} \underline{a}\right)\right|\right|_{C^{0,\alpha}_{-2,\delta;t}(\{u\in P_y:\epsilon_3/t<\check{r}(u)<\epsilon_2/t\})} \\ &\leq c\epsilon_2 \left|\left|\underline{a}\right|\right|_{C^{1,\alpha}_{-1,\delta;t}(\{u\in P_y:\epsilon_3/t<\check{r}(u)<\epsilon_2/t\})} .\end{aligned} \tag{4.104}$$

We have $\left|\left|p_{X_{\mathrm{EH}}}^* f_*(s(y)) - A_0\right|\right|_{C^{0,\alpha}_{0;0}} = \mathcal{O}((\rho \circ p_{X_{\mathrm{EH}}})^{-2})$ by Eq. (2.17) and the fact that we use $\delta = -2$ in our definition of moduli space (cf. Proposition 2.19). Thus, for $x \in P_y$ with $\epsilon_3/t < \check{r}(x)t < R$,

$$\left|\left(s_{2,t;y}^{P,\sqrt{t},\sqrt{t}}\right)^{-1} \left[L_{p_{X_{\mathrm{EH}}}^* f_*(s(y))} - L_{p_{X_{\mathrm{EH}}}^* A_0}\right] s_{1,t;y}^{P,\sqrt{t},\sqrt{t}} \underline{a}\right|_{\tilde{g}_t^N}(x) \leq c(t/\epsilon_3)^2. \tag{4.105}$$

Combining Eq. (4.104) and (4.105) we get the desired Eq. (4.103) on $P_y \cap V^\nu_{\epsilon_1,\epsilon_2,\epsilon_3}(y)$. Equation 4.103 then follows like Eq. (4.94) by taking Taylor expansions around y, and this time comparing g_t^ν and $\tilde{g}_t^N$ using Eq. (3.9) and Propositions 3.15, 4.35 and 4.42. □

4.3.4. Schauder estimate On $Y/\langle\iota\rangle$ we have the estimate

$$\left|\left|\underline{a}\right|\right|_{C^{1,\alpha}} \leq c\left(\left|\left|L_\theta \underline{a}\right|\right|_{C^{0,\alpha}} + \left|\left|\underline{a}\right|\right|_{L^\infty}\right)$$

from standard elliptic theory, e.g. [Bes87, Section H]. With some additional work, we get an estimate for weighted norms on $\mathbb{R}^3 \times X_{\mathrm{EH}}$ (see [Wal17, Proposition 8.15]), and can then glue these two estimates together to obtain:

Proposition 4.106 (Proposition 8.15 in [Wal17]). *There exists $c > 0$ such that for all $t \in (0, T)$ the following estimate holds:*

$$\left|\left|\underline{a}\right|\right|_{C^{1,\alpha}_{-1,\delta;t}} \leq c\left(\left|\left|L_t \underline{a}\right|\right|_{C^{0,\alpha}_{-2,\delta;t}} + \left|\left|\underline{a}\right|\right|_{L^\infty_{-1,\delta;t}}\right). \tag{4.107}$$

4.3.5. Estimate of $\eta_t \underline{a}$ The following proposition is the crucial ingredient in the construction of solutions to the instanton equation:

Proposition 4.108. *There exists a constant $c > 0$ independent of t such that for t small enough and for all $\underline{a} \in (\Omega^0 \oplus \Omega^1)(N_t, \mathrm{Ad}\, E_t)$ the following estimate holds:*

$$||a||_{L^\infty_{-1,\delta;t}} \leq c\left(\left|\left|L_t \underline{a}\right|\right|_{C^{0,\alpha}_{-2,\delta;t}} + \left|\left|\overline{\pi}_t \underline{a}\right|\right|_{L^\infty_{-1,\delta;t}}\right). \tag{4.109}$$

Proof. Assume not, then there exist $t_i \to 0$ and $\underline{a}_i$ such that

$$\left|\left|\underline{a}_i\right|\right|_{L^\infty_{-1,\delta;t_i}} \equiv 1, \tag{4.110}$$

$$\lim_{i\to\infty} \left|\left|L_{t_i} \underline{a}\right|\right|_{C^{0,\alpha}_{-2,\delta;t_i}} = 0, \tag{4.111}$$

$$\lim_{i\to\infty} \left|\left|\overline{\pi}_{t_i} \underline{a}\right|\right|_{L^\infty_{-1,\delta;t_i}} = 0. \tag{4.112}$$

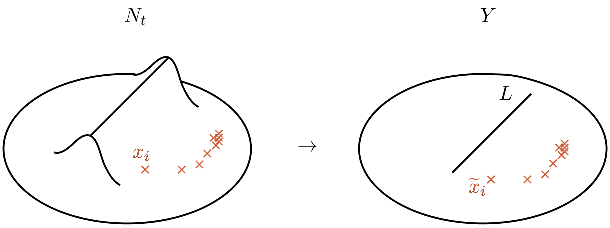

Fig. 3. Blowup analysis away from the associative is reduced to the analysis of θ on Y

It follows from Proposition 4.106 that

$$\left|\left|\underline{a}_i\right|\right|_{C^{1,\alpha}_{-1,\delta;t}} \leq c. \tag{4.113}$$

Let $x_i \in N_{t_i}$ such that

$$w_{-1,\delta;t}(x_i)\left|\underline{a}_i\right|(x_i) = 1. \tag{4.114}$$

Without loss of generality we can assume to be in one of three following cases, and we will arrive at a contradiction in each of them.

Case 1. "$\underline{a}_i$ goes to zero near L and on the neck", i.e. if $z_i \in N_{t_i}$ such that $r_{t_i}(z_i) \to 0$, then $w_{-1,\delta;t}(z_i)\left|\underline{a}_i\right|(z_i) \to 0$.
Without loss of generality, the sequence (x_i) accumulates away from L, i.e. $\lim_{i\to\infty} r_{t_i}(x_i) > 0$ (see Fig. 3).
Without loss of generality assume that $x_i \to x^* \in Y/\langle\iota\rangle$, where we used that $(Y\backslash L)/\langle\iota\rangle \subset N_{t_i}$, cf. Definition 3.27. Now, using a diagonal argument and the Arzelà-Ascoli theorem, we find that a subsequence of $\underline{a}_i$ converges to a limit $\underline{a}^* \in \Omega^1((Y \setminus L)/\langle\iota\rangle, \mathrm{Ad}\, E_0)$ in $C^{1,\alpha/2}_{\mathrm{loc}}$. Denote by $\pi_\iota : Y \to Y/\langle\iota\rangle$ the quotient map, and denote by $\widetilde{x}_i$ an arbitrary lift of x_i, i.e. $\pi_\iota(\widetilde{x}_i) = x_i$. By passing to a subsequence we still have $\widetilde{x}_i \to \widetilde{x}^*$ for some $\widetilde{x}^* \in Y$. Denote also $\widetilde{\underline{a}}^* := \pi_\iota^* \underline{a}^* \in (\Omega^0 \oplus \Omega^1)(\mathrm{Ad}\, E_0|_{Y\backslash L})$.
Equation (4.111) implies that this limit satisfies $L_\theta \widetilde{\underline{a}}^* = 0$ on $Y\backslash L$. We can extend $\widetilde{\underline{a}}^*$ to all of Y as a distribution, and we find that then $L_\theta \widetilde{\underline{a}}^* = 0$ on Y in the sense of distributions. By elliptic regularity, e.g. [Fol95, Theorem 6.33], we have that $\widetilde{\underline{a}}^*$ is smooth.
Lastly, we note that Eq. (4.114) implies $\widetilde{\underline{a}}^*(\widetilde{x}^*) \neq 0$. By assumption, θ is infinitesimally rigid and irreducible, which is a contradiction.

Case 2. "The sequence does not go to zero near L", i.e. there exists $y_i \in N_{t_i}$ such that $t_i^{-1} r_{t_i}(y_i)$ is bounded, but $w_{-1,\delta;t}(y_i)\left|\underline{a}_i\right|(y_i) \nrightarrow 0$.
Without loss of generality assume that this is the sequence (x_i), i.e. $\lim_{i\to\infty} t_i^{-1} r_{t_i}(x_i) < \infty$ (see Fig. 4).
For $\underline{a}_i = (\xi_i, a_i) \in (\Omega^0 \oplus \Omega^1)(N_t, \mathrm{Ad}\, E_t)$, let

$$\underline{b}_i := \left(s^{P,\sqrt{t_i},\sqrt{t_i}}_{1,\sigma(x_i);t_i}(\xi_i), s^{P,\sqrt{t_i},\sqrt{t_i}}_{1,\sigma(x_i);t_i}(a_i)\right).$$

Proposition 4.91 then gives

$$\left|\left|\underline{b}_i\right|\right|_{C^{1,\alpha}_{-1+\delta}(U^P_{1/\sqrt{t_i},1/\sqrt{t_i}})} \leq c \text{ and } \lim_{i\to\infty}\left|\left|L_{p^*_{X_{\mathrm{EH}}} f_* s(\sigma(x_i))}\underline{b}_i\right|\right|_{C^{0,\alpha}_{-2+\delta}} = 0.$$

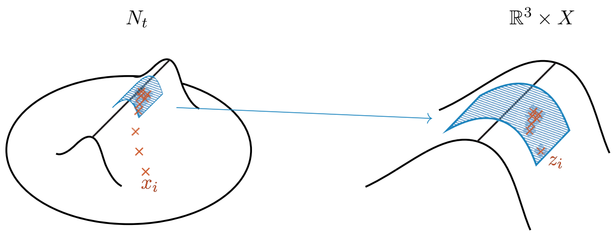

Fig. 4. Blowup analysis near the associative is, by means of the map s^P, reduced to the analysis of the pull-back of the ASD instanton defined by $s(\sigma(y^*))$ to $\mathbb{R}^3 \times X_{\text{EH}}$

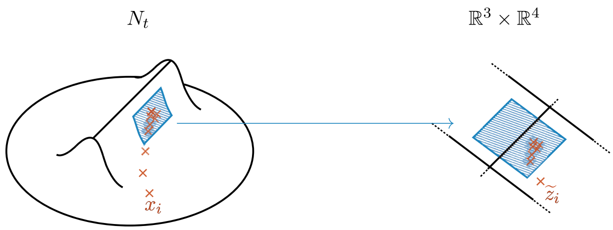

Fig. 5. Blowup analysis in the neck region is reduced to the analysis of the flat G_2-instanton defined on the pull-back of the framing at infinity defined by $s(\sigma(y^*))$ to $\mathbb{R}^3 \times \mathbb{R}^4$

Without loss of generality we can assume $\sigma(x_i) \to y^* \in L$. By a diagonal argument and the Arzelà-Ascoli theorem, we have $\underline{b}_i \to \underline{b}^* \in (\Omega^0 \oplus \Omega^1)(\mathbb{R}^3 \times X_{\text{EH}}, \text{Ad}\, p^*_{X_{\text{EH}}} f_* s(\sigma(y^*)))$ in $C^{1,\alpha/2}_{\text{loc}}$, satisfying $L_{p^*_{X_{\text{EH}}} f_* s(\sigma(y^*))} \underline{b}^* = 0$. Proposition 4.88 implies that $\underline{b}^* = p^*_{X_{\text{EH}}} \underline{c}$, for some $\underline{c} \in \text{Ker}\, \delta_{f_* s(\sigma(y^*))} \subset \Omega^1(X_{\text{EH}}, f_* s(\sigma(y^*)))$. (Here, δ is the linearisation of the ASD equation as defined in Eq. (2.24).) Equation (4.112) then implies that $\underline{c} = 0$.
This contradicts Eq. (4.114) as follows: denote by $(z_i) \subset \mathbb{R}^3 \times X_{\text{EH}}$ the sequence corresponding to (x_i) under the map $s^{\sqrt{t},1/\sqrt{t}}_{1,t_i;\sigma(x_i)}$. Then (z_i) is a bounded sequence, as the $\mathbb{R}^3$-coordinate of all z_i is 0, and the X_{EH}-coordinates are bounded by the assumption that $\lim_{i\to\infty} t_i^{-1} r_{t_i}(x_i) < \infty$. Thus we can assume without loss of generality that $z_i \to z^* \in \mathbb{R}^3 \times X_{\text{EH}}$, and find that

$$w(z^*)^{1-\delta} \left|\underline{b}^*(z^*)\right| = \lim_{i\to\infty} w^{\nu}_{l,\delta;t}(z_i)^{1-\delta} \left|\underline{b}_i(z_i)\right| \geq \frac{1}{c}$$

by Proposition 4.91, which is a contradiction to $\underline{b}^* = 0$.

Case 3. "The sequence does not go to zero on the neck", i.e. there exists $y_i \in N_{t_i}$ such that $r_{t_i}(y_i) \to 0$, $t_i^{-1} r_{t_i}(y_i) \to \infty$, but $w_{-1,\delta;t}(y_i) \left|\underline{a}_i\right|(y_i) \nrightarrow 0$.
Assume without loss of generality that this is the sequence (x_i), i.e. $\lim_{i\to\infty} t_i^{-1} r_{t_i}(x_i) = \infty$ and $\lim_{i\to\infty} r_{t_i}(x_i) = 0$ (see Fig. 5).
Let

- $\epsilon_2^{(i)}$ such that $\epsilon_2^{(i)} \to 0$ and $\epsilon_2^{(i)} / r_{t_i}(x_i) \to \infty$,
- $\epsilon_3^{(i)}$ such that $\epsilon_3^{(i)} / r_{t_i}(x_i) \to 0$ and $\epsilon_3^{(i)} / t_i \to \infty$.

To ease notation, we write ϵ_2 instead of $\epsilon_2^{(i)}$ and ϵ_3 instead of $\epsilon_3^{(i)}$ in what follows. As before, write $\underline{a}_i = (\xi_i, a_i) \in (\Omega^0 \oplus \Omega^1)(N_t, \mathrm{Ad}\, E_t)$, and set

$$\underline{b}_i := (\zeta_i, b_i) := \left(s^{\nu,\sqrt{t_i},\epsilon_2,\epsilon_3}_{1,\sigma(x_i);t_i}(\xi_i), s^{\nu,\sqrt{t_i},\epsilon_2,\epsilon_3}_{1,\sigma(x_i);t_i}(a_i) \right)$$

and denote by (z_i) the sequence in $\mathbb{R}^3 \times \mathbb{C}^2/\{\pm 1\}$ corresponding to (x_i) under the map $s^{\nu,\sqrt{t_i},\epsilon_2,\epsilon_3}_{1,\sigma(x_i);t_i}$. Equation (4.114) implies

$$|\underline{b}_i(z_i)| \cdot w(z_i) > c, \tag{4.115}$$

Proposition 4.100 and Eq. (4.113) imply that

$$\left|\left| w^{\nu}_{l,\delta;t} s^{\nu,\epsilon_1,\epsilon_2,\epsilon_3}_{d,t;y} \underline{a} \right|\right|_{C^{1,\alpha}_0(U^{\nu}_{1/\sqrt{t},\epsilon_2/t,\epsilon_3/t;t})} \leq c, \tag{4.116}$$

Proposition 4.100 and Eq. (4.111) imply that

$$\left|\left| w^{\nu}_{l,\delta;t} L_{p^*_{X_{\mathrm{EH}}} A_0} s^{\nu,\epsilon_1,\epsilon_2,\epsilon_3}_{1,t;y} \underline{a} \right|\right|_{C^{1,\alpha}_0(U^{\nu}_{1/\sqrt{t},\epsilon_2/t,\epsilon_3/t;t})} \to 0 \text{ as } i \to \infty. \tag{4.117}$$

We will now conclude the argument as in case 2. The only difference is that, as it stands, the points z_i tend to infinity. Because of this, we cannot directly conclude that a limit of $\underline{b}_i$ would be non-zero. That is why we rescale by $|z_i|$ first. By passing to a subsequence we can assume without loss of generality to be in case 3.1 or 3.2 as below:

Case 3.1.: $|z_i| \leq 1/\sqrt{t_i}$. In this case let

$$\widetilde{\underline{b}}_i := (\widetilde{\zeta}_i, \widetilde{b}_i) := \left(|z_i|^{1-\delta}(\cdot|z_i|)^*\zeta_i, |z_i|^{-\delta}(\cdot|z_i|)^* b_i \right). \tag{4.118}$$

Equation (4.115) implies $|\widetilde{\underline{b}}_i(z_i/|z_i|)| \cdot r^{1-\delta}(z_i/|z_i|) = |\widetilde{\underline{b}}_i(z_i/|z_i|)| > c$, and Eq. (4.116) implies that on the sets $B^3(0, 1/\sqrt{t}) \times [B^4(0, \epsilon_2/|x_i|) \backslash B^4(0, \epsilon_3/|x_i|)]$, which exhaust $\mathbb{R}^3 \times (\mathbb{C}^2/\{\pm 1\} \backslash \{0\})$, we have:

$$\left|\left| \begin{cases} \widetilde{\underline{b}}_i r^{1-\delta} & \text{if } r \leq 1/(\sqrt{t} \cdot |z_i|) \\ \widetilde{\underline{b}}_i r^{1+\delta} t^{\delta} |z_i|^{2\delta} & \text{if } r > 1/(\sqrt{t} \cdot |z_i|). \end{cases} \right|\right|_{C^{1,\alpha}_0(B^3(0,1/\sqrt{t}) \times [B^4(0,\epsilon_2/|x_i|) \backslash B^4(0,\epsilon_3/|x_i|)])} \leq c. \tag{4.119}$$

Here is how to arrive at the exponents of the weight function for $\widetilde{\zeta}_i$ in the area $\{(u, v) \in \mathbb{R}^3 \times \mathbb{C}^2/\{\pm 1\} : r(v) > 1/(\sqrt{t} \cdot |z_i|)\}$:

$$\begin{aligned} \widetilde{\zeta}_i r^{1+\delta} t^{\delta} |z_i|^{2\delta} &= (\cdot|z_i|)^* \zeta_i |z_i|^{1+\delta} r^{1+\delta} t^{\delta} \\ &= (\cdot|z_i|)^* \left[\zeta_i r^{1+\delta} t^{\delta} \right], \end{aligned}$$

and $\zeta_i r^{1+\delta} t^{\delta}$ was bounded by Eq. (4.116). The exponents of the weight function on the area $\{(u, v) \in \mathbb{R}^3 \times \mathbb{C}^2/\{\pm 1\} : r(v) > 1/(\sqrt{t} \cdot |z_i|)\}$ and also for the 1-form part $\widetilde{b}_i$ are computed analogously and precisely give Eq. (4.119). Now, because of Eq. (4.119), we can use the Arzelà-Ascoli theorem and a diagonal sequence argument to extract a limit $\underline{b}^*$ on $\mathbb{R}^3 \times (\mathbb{C}^2/\{\pm 1\} \backslash \{0\})$. We denote the pullback under the quotient map $\mathbb{R}^3 \times (\mathbb{C}^2 \backslash \{0\}) \to \mathbb{R}^3 \times (\mathbb{C}^2/\{\pm 1\} \backslash \{0\})$ by the same symbol and end up with a tensor

$\underline{b}^*$ on $\mathbb{R}^3 \times (\mathbb{C}^2 \backslash \{0\})$. Again, by passing to a subsequence we can assume without loss of generality that we are in one of the following two cases:

Case 3.1.1: $\sqrt{t_i}|z_i| \to 0$ as $i \to \infty$.
In this case, the area $\{u \in \mathbb{R}^3 \times \mathbb{C}^2/\{\pm 1\} : r(u) > 1/(\sqrt{t} \cdot |z_i|)\}$ disappears as $i \to \infty$, and from Eq. (4.119) we get the estimate

$$\left|\left|\underline{b}^* r^{1-\delta}\right|\right|_{C_0^{1,\alpha/2}(\mathbb{R}^3 \times (\mathbb{R}^4 \backslash \{0\}))} \leq c. \tag{4.120}$$

The element $\underline{b}^*$ defines a distribution on all of $\mathbb{R}^3 \times \mathbb{C}^2$ and is smooth by elliptic regularity, e.g. [Fol95, Theorem 6.33]. We also get an L^∞-bound for $\underline{b}^*$ as in the proof of [Wal13a, Proposition 8.7]: away from $\mathbb{R}^3 \times \{0\}$, this is given by Eq. (4.120). To see that $\underline{b}^*$ does not blow up in the $\mathbb{R}^3$-direction near $\mathbb{R}^3 \times \{0\}$, consider any $y \in \mathbb{R}^3 \times \{0\}$. Let $1 < p < -4/(-1+\delta)$, then $\left|\left|\underline{b}^*\right|\right|_{L^p(B_1(y))} \leq c$, independent of y, by Eq. (4.120). So, by elliptic regularity $\left|\left|\underline{b}^*\right|\right|_{L^p_m(B_1(y))} \leq c$ for any $m \in \mathbb{N}$, and by the Sobolev embedding we have $\left|\left|\underline{b}^*\right|\right|_{L^\infty} \leq c$, where all of these estimates were independent of y.
Thus, by Proposition 4.88 applied to the case $\widetilde{X} = \mathbb{C}^2$, we get that $\underline{b}^*$ is independent of the $\mathbb{R}^3$-direction. Because of Eq. (4.86) we have that $\underline{b}^*$ is the pullback of a harmonic form of mixed degree (in degrees 0 and 1) on $\mathbb{C}^2$. So, $\underline{b}^*$ is harmonic and bounded on $\mathbb{C}^2$ by Eq. (4.120), therefore vanishes by Liouville's theorem. That contradicts Eq. (4.115).

Case 3.1.2: $\sqrt{t_i}|z_i| \to \kappa \in (0, \infty)$ as $i \to \infty$.
In this case, after passing to a subsequence, Eq. (4.119) gives the estimate

$$\left|\left|\begin{cases} \underline{b}^* r^{1-\delta} & \text{if } r \leq 1/\kappa \\ \underline{b}^* r^{1+\delta} & \text{if } r > 1/\kappa. \end{cases}\right|\right|_{C_0^{1,\alpha}(\mathbb{R}^3 \times (\mathbb{C}^2 \backslash \{0\})} \leq c. \tag{4.121}$$

Here is how to obtain this estimate: the assumption $\sqrt{t_i}|z_i| \to \kappa$ implies that $\sqrt{t_i}|z_i| > c$, at least up to a subsequence. Thus, we have $t^\delta \cdot |z_i|^{2\delta} < c$, and Eq. (4.119) becomes

$$\left|\left|\begin{cases} \widetilde{\underline{b}}_i r^{1-\delta} & \text{if } r \leq 1/(\sqrt{t} \cdot |z_i|) \\ \widetilde{\underline{b}}_i r^{1+\delta} & \text{if } r > 1/(\sqrt{t} \cdot |z_i|). \end{cases}\right|\right|_{C_0^{1,\alpha}(B^3(0,1/\sqrt{t}) \times [B^4(0,\epsilon_2/|x_i|) \backslash B^4(0,\epsilon_3/|x_i|)])} \leq c.$$

Here, taking the limit $i \to \infty$ gives Eq. (4.121). In this case, we arrive at a contradiction as in case 3.1.1.

Case 3.2.: $|z_i| > 1/\sqrt{t_i}$. In this case let

$$\widetilde{\underline{b}}_i := (\widetilde{\zeta}_i, \widetilde{b}_i) := \left(t^\delta |z_i|^{1+\delta} (\cdot |z_i|)^* \zeta_i, t^\delta |z_i|^\delta (\cdot |z_i|)^* b_i\right). \tag{4.122}$$

This gives us the following analogue of Eq. (4.119):

$$\left|\left|\begin{cases} \widetilde{\underline{b}}_i r^{1-\delta} t^{-\delta} |z_i|^{-2\delta} & \text{if } r \leq 1/(\sqrt{t} \cdot |z_i|) \\ \widetilde{\underline{b}}_i r^{1+\delta} & \text{if } r > 1/(\sqrt{t} \cdot |z_i|). \end{cases}\right|\right|_{C_0^{1,\alpha}(B^3(0,1/\sqrt{t}) \times [B^4(0,\epsilon_2/|x_i|) \backslash B^4(0,\epsilon_3/|x_i|)])} \leq c. \tag{4.123}$$

We can extract a limit $\underline{b}^*$ as in case 3.1 and are, without loss of generality, in one of the following two cases:

Case 3.2.1: $\sqrt{t_i} \cdot |z_i| \to \infty$ as $i \to \infty$. In this case we have the estimate

$$\left|\left|\underline{b}^* r^{1+\delta}\right|\right|_{C_0^{1,\alpha/2}(\mathbb{R}^3 \times (\mathbb{R}^4 \setminus \{0\}))} \leq c \tag{4.124}$$

and arrive at a contradiction as in case 3.1.1.

Case 3.2.2: $\sqrt{t_i} \cdot |z_i| \to \kappa \in (0, \infty)$ as $i \to \infty$. In this case we have exactly Eq. (4.121) and can conclude the proof as in case 3.1.2. □

4.3.6. Cross-term estimates In the beginning of Sect. 4.3 we explained the idea for the proof of the linear estimate. Namely, we want to separately consider two parts of the linearisation of the instanton equation: the first part near the resolution locus of the associative L, which is approximately equal to the linearisation of the Fueter equation. The second part is the linearised operator modulo deformations of the Fueter section. These parts were estimated in Sects. 4.3.2 and 4.3.5.
However, it is not true that the linearised instanton operator neatly decomposes as a sum of these two operators. We can take a deformation of the Fueter section, apply L_t to it, and then project it onto the part that does *not* come from a deformation of the Fueter section. In an ideal world, L_t near the resolution locus of the associative is exactly equal to the linearisation of the Fueter equation and the result is 0. In reality, we do not have that the result is 0, but we have that it is small. That is Eq. (4.126). There is also, roughly speaking, the converse of this, which is Eq. (4.127).
Like the results from Sect. 4.3.2, this proposition has been proved in a slightly different setting in [Wal17]. Again, the proof given therein carries over to our situation if we only have that $\widetilde{\psi}_t^N - \psi_t^P$ is small, which is true on resolutions of T^7/Γ by Proposition 4.38 and Theorem 3.2.

Proposition 4.125 (Proposition 8.29 in [Wal17]). *Let N_t be the resolution of T^7/Γ from Sect. 3.1. There exists a constant $c > 0$ such that for all $t \in (0, T)$ we have*

$$||\eta_t L_t \iota_t v||_{C^{0,\alpha}_{-2,0;t}} \leq ct^{2-\alpha} ||v||_{C^{1,\alpha}} \tag{4.126}$$

as well as

$$\left|\left|\pi_t L_t \eta_t \underline{a}\right|\right|_{C^{0,\alpha}} \leq ct^{-\alpha} \left|\left|\eta_t \underline{a}\right|\right|_{C^{1,\alpha}_{-1,0;t}}. \tag{4.127}$$

4.3.7. Proof of Proposition 4.78

Proof. Assume that the claim does not hold, and let $t_i \to 0$, $\underline{a}_i \in (\Omega^0 \oplus \Omega^1)(N_t, \mathrm{Ad}\, E_t)$ such that $\left|\left|\underline{a}_i\right|\right|_{\mathfrak{X}_t} = 1$, but $\left|\left|L_t \underline{a}_i\right|\right|_{\mathfrak{Y}_t} \to 0$.
We first prove that

$$t_i^{-\delta/2} \left|\left|\eta_{t_i} \underline{a}_i\right|\right|_{C^{1,\alpha}_{-1,\delta;t_i}} \to 0. \tag{4.128}$$

We have that

$$\begin{aligned} ||\eta_{t_i} \underline{a}_i||_{C^{1,\alpha}_{-1,\delta;t_i}} &\leq ||L_{t_i} \eta_{t_i} \underline{a}_i||_{C^{0,\alpha}_{-2,\delta;t_i}} \\ &\leq ||\eta_{t_i} L_{t_i} \eta_{t_i} \underline{a}_i||_{C^{0,\alpha}_{-2,\delta;t_i}} + ||\overline{\pi}_{t_i} L_{t_i} \eta_{t_i} \underline{a}_i||_{C^{0,\alpha}_{-2,\delta;t_i}} \end{aligned}$$

$$
\begin{aligned}
&\leq \left|\left|\eta_{t_i} L_t \underline{a}\right|\right|_{C^{0,\alpha}_{-2,\delta;t_i}} + \left|\left|\eta_{t_i} L_{t_i} \overline{\pi}_{t_i} \underline{a}_i\right|\right|_{C^{0,\alpha}_{-2,\delta;t_i}} + \left|\left|\overline{\pi}_{t_i} L_{t_i} \eta_{t_i} \underline{a}_i\right|\right|_{C^{0,\alpha}_{-2,\delta;t_i}} \\
&\leq \left|\left|\eta_{t_i} L_t \underline{a}\right|\right|_{C^{0,\alpha}_{-2,\delta;t_i}} + ||1||_{C^{0,\alpha}_{0,\delta;t_i}} \left|\left|\eta_{t_i} L_{t_i} \overline{\pi}_{t_i} \underline{a}_i\right|\right|_{C^{0,\alpha}_{-2,0;t_i}} + t^{1-\alpha} \left|\left|\pi_{t_i} L_{t_i} \eta_{t_i} \underline{a}_i\right|\right|_{C^{0,\alpha}} \\
&\leq c \left(\left|\left|\eta_{t_i} L_t \underline{a}\right|\right|_{C^{0,\alpha}_{-2,\delta;t_i}} + ct^{\delta/2} t^{2-\alpha} \left|\left|\pi_t \underline{a}_i\right|\right|_{C^{1,\alpha}} + t^{1-2\alpha} \left|\left|\eta_{t_i} \underline{a}_i\right|\right|_{C^{1,\alpha}_{-1,0;t}} \right) \\
&\leq c \left(\left|\left|\eta_{t_i} L_t \underline{a}\right|\right|_{C^{0,\alpha}_{-2,\delta;t_i}} + \mathcal{O}(t^{\delta/2+1-\alpha}) + \mathcal{O}(t^{1-2\alpha+\delta/2}) \right)
\end{aligned}
$$

where we used Proposition 4.108 in the first step; we used $\overline{\pi}_{t_i} + \eta_{t_i} = 1$ in the second and third steps; Propositions 4.22 and 4.66 in the fourth step; and Proposition 4.125 together with $||1||_{C^{0,\alpha}_{0,\delta;t_i}} \leq ct^{\delta/2}$ in the fifth step. Multiplying the last line with $t_i^{-\delta/2}$, the last two summands tend to zero as they are bounded by positive powers of t. The first summand tends to zero by the assumption $\left|\left|L_t \underline{a}_i\right|\right|_{\mathfrak{Y}_t} \to 0$.

It remains to prove that

$$
t_i \left|\left|\pi_{t_i} \underline{a}_i\right|\right|_{C^{1,\alpha}} \to 0. \tag{4.129}
$$

We have that

$$
\begin{aligned}
\lim_{i\to\infty} t_i \left|\left|\pi_{t_i} \underline{a}_i\right|\right|_{C^{1,\alpha}} &\leq \lim_{i\to\infty} t_i \left|\left|\pi_{t_i} L_{t_i} \iota_{t_i} \pi_{t_i} \underline{a}_i\right|\right|_{C^{0,\alpha}} \\
&\leq \lim_{i\to\infty} t_i \left(\left|\left|\pi_t L_t \underline{a}\right|\right|_{C^{0,\alpha}} + \left|\left|\pi_t L_t \eta_t \underline{a}\right|\right|_{C^{0,\alpha}} \right) \\
&\leq \lim_{i\to\infty} t_i \left(\left|\left|\pi_t L_t \underline{a}\right|\right|_{C^{0,\alpha}} + ct^{-\alpha} \left|\left|\eta_t \underline{a}\right|\right|_{C^{1,\alpha}_{-1,0;t}} \right).
\end{aligned}
$$

where we used Proposition 4.82 in the first step, $\overline{\pi}_{t_i} + \eta_{t_i} = 1$ in the second step, Proposition 4.125 in the third step. Here, the second summand tends to zero by Eq. (4.128), and the first summand tends to zero by the assumption $\left|\left|L_t \underline{a}_i\right|\right|_{\mathfrak{Y}_t} \to 0$. Altogether, $\left|\left|\underline{a}_i\right|\right|_{\mathfrak{X}_t} \to 0$, which is a contradiction. □

4.4. Quadratic estimate We state an estimate for the quadratic form Q_t from Eq. (4.62), where we denote its associated bilinear form by the same symbol. This statement is taken from [Wal17] and the proof can be directly adapted to our slightly different setting.

Proposition 4.130 (Proposition 9.1 in [Wal17]). *There exists a constant $c > 0$ such that for $t \in (0, 1)$ we have*

$$
\begin{aligned}
&\left|\left|\eta_t Q_t(\underline{a}_1, \underline{a}_2)\right|\right|_{C^{0,\alpha}_{-2,\delta;t}} \\
&\leq ct^{-\alpha} \left(\left|\left|\eta_t \underline{a}_1\right|\right|_{C^{0,\alpha}_{-1,\delta;t}} \cdot \left|\left|\eta_t \underline{a}_2\right|\right|_{C^{0,\alpha}_{-1,\delta;t}} + \left|\left|\eta_t \underline{a}_1\right|\right|_{C^{0,\alpha}_{-1,\delta;t}} \cdot \left|\left|\pi_t \underline{a}_2\right|\right|_{C^{0,\alpha}} \right. \\
&\quad \left. + \left|\left|\pi_t \underline{a}_1\right|\right|_{C^{0,\alpha}} \cdot \left|\left|\eta_t \underline{a}_2\right|\right|_{C^{0,\alpha}_{-1,\delta;t}} + \left|\left|\pi_t \underline{a}_1\right|\right|_{C^{0,\alpha}} \cdot \left|\left|\pi_t \underline{a}_2\right|\right|_{C^{0,\alpha}} \right)
\end{aligned} \tag{4.131}
$$

and

$$
\begin{aligned}
&t\,\big|\big|\pi_t Q_t(\underline{a}_1,\underline{a}_2)\big|\big|_{C^{0,\alpha}}\\
&\quad\le ct^{-\alpha}\Big(\big|\big|\eta_t\underline{a}_1\big|\big|_{C^{0,\alpha}_{-1,\delta;t}}\cdot\big|\big|\eta_t\underline{a}_2\big|\big|_{C^{0,\alpha}_{-1,\delta;t}}+\big|\big|\eta_t\underline{a}_1\big|\big|_{C^{0,\alpha}_{-1,\delta;t}}\cdot\big|\big|\pi_t\underline{a}_2\big|\big|_{C^{0,\alpha}}\\
&\qquad+\big|\big|\pi_t\underline{a}_1\big|\big|_{C^{0,\alpha}}\cdot\big|\big|\eta_t\underline{a}_2\big|\big|_{C^{0,\alpha}_{-1,\delta;t}}+t\,\big|\big|\pi_t\underline{a}_1\big|\big|_{C^{0,\alpha}}\cdot\big|\big|\pi_t\underline{a}_2\big|\big|_{C^{0,\alpha}}\Big).
\end{aligned}
\tag{4.132}
$$

4.5. Deforming to genuine solutions In this subsection we will complete the construction of G_2-instantons and show that in two favourable situations the G_2-instanton θ and the Fueter section s can be glued together to a G_2-instanton on N_t. The favourable situations are:

1. The Fueter section is a section of rigid ASD-instantons (cf. Theorem 4.133). This implies, in particular, that the Fueter section is infinitesimally rigid. In this case the map π_t from Definition 4.63 is just the zero map, which leads to better estimates of the quadratic part Q_t of the instanton equation.
2. We are in the special situation of Eq. (4.59), where we resolved the orbifold T^7/Γ.

The main reason we are confined to these two favourable scenarios is the following: in Corollaries 4.55 and 4.58 we proved a pregluing estimate with a good power of $t^{1/18}$ in the general case and a good power of t^2 in the case of T^7/Γ, roughly speaking. In Proposition 4.130 we stated an estimate for the quadratic part of the instanton operator which in particular implies

$$
\big|\big|Q_t(\underline{a}_1,\underline{a}_2)\big|\big|_{\mathfrak{Y}}\le t^{-2-\alpha-\delta/2}\,\big|\big|\underline{a}_1\big|\big|_{\mathfrak{X}}\,\big|\big|\underline{a}_2\big|\big|_{\mathfrak{X}}\,.
$$

To apply the inverse function theorem, we would need the bad power $t^{-2-\alpha-\delta/2}$ from this estimate to be absorbed by the good power from the pregluing estimate, but the pregluing estimate is only good enough to do this in the case of the orbifold T^7/Γ. If the Fueter section is actually the constant section of a rigid ASD-instanton, then we have a better estimate for the quadratic part of the instanton equation.

Theorem 4.133. *Assume that the section s is given by a rigid ASD-instanton in every point $x\in L$, and assume that the connection θ used to define the approximate G_2-instanton A_t from Proposition 4.28 is infinitesimally rigid.*
There exists $c>0$ such that for small t there exists $\underline{a}_t=(a_t,\xi_t)\in C^{1,\alpha}(\Omega^0\oplus\Omega^1(\operatorname{Ad}E_t))$ such that $\widetilde{A}_t:=A_t+a_t$ is a G_2-instanton. Furthermore, $\underline{a}_t$ satisfies $\big|\big|\underline{a}_t\big|\big|_{C^{1,\alpha}_{-1,\delta;t}}\le ct^{1/18}$.

Theorem 4.134. *Let $N\to Y'$ be the resolution of the orbifold $Y'=T^7/\Gamma$ from before. Assume that the connection θ used to define the approximate G_2-instanton A_t from Proposition 4.28 is infinitesimally rigid and that s is an infinitesimally rigid Fueter section.*
There exists $c>0$ such that for small t there exists an $\underline{a}_t=(a_t,\xi_t)\in C^{1,\alpha}(\Omega^0\oplus\Omega^1(\operatorname{Ad}E_t))$ such that $\widetilde{A}_t:=A_t+a_t$ is a G_2-instanton. Furthermore, $\underline{a}_t$ satisfies $\big|\big|\underline{a}_t\big|\big|_{\mathfrak{X}_t}\le ct^{2-2\alpha}$.

The proof of the theorems will use the following lemma:

Lemma 4.135 (Lemma 7.2.23 in [DK90]). *Let X be a Banach space and let $T : X \to X$ be a smooth map with $T(0) = 0$. Suppose there is a constant $c > 0$ such that*

$$||Tx - Ty|| \leq c(||x|| + ||y||)\,||x - y||\,.$$

Then if $y \in X$ satisfies $||y|| \leq \frac{1}{10c}$, there exists a unique $x \in X$ with $||x|| \leq \frac{1}{5c}$ solving

$$x + Tx = y.$$

The unique solution satisfies $||x|| \leq 2\,||y||$.

Proof of Theorem 4.133. In the case that s is a section of rigid ASD instantons, we have that the projection map π_t is zero. Therefore, Propositions 4.106 and 4.108 give

$$\left|\left|\underline{a}\right|\right|_{C^{1,\alpha}_{-1,\delta;t}} \leq c\left|\left|L_t\underline{a}\right|\right|_{C^{0,\alpha}_{-2,\delta;t}}. \tag{4.136}$$

This means that

$$L_t : C^{1,\alpha}((\Omega^0 \oplus \Omega^1)(N_t, \operatorname{Ad} E_t)) \to C^{0,\alpha}((\Omega^0 \oplus \Omega^1)(N_t, \operatorname{Ad} E_t))$$

is injective. Because L_t is formally self-adjoint, it is also bijective. Denote its inverse by L_t^{-1}. Furthermore, using $\pi_t = 0$, and therefore $\eta_t = \mathrm{Id}$, Proposition 4.130 gives

$$\left|\left|Q_t(\underline{a}_1, \underline{a}_2)\right|\right|_{C^{0,\alpha}_{-2,\delta;t}} \leq ct^{-\alpha}\left|\left|\underline{a}_1\right|\right|_{C^{0,\alpha}_{-1,\delta;t}} \cdot \left|\left|\underline{a}_2\right|\right|_{C^{0,\alpha}_{-1,\delta;t}}. \tag{4.137}$$

Set $T_t := Q_t \circ L_t^{-1}$. We then have

$$\begin{aligned}
\left|\left|T_t(\underline{b}_1) - T_t(\underline{b}_2)\right|\right|_{C^{0,\alpha}_{-2,\delta;t}} &= \left|\left|Q(L^{-1}\underline{b}_1 - L^{-1}\underline{b}_2, L^{-1}\underline{b}_1 + L^{-1}\underline{b}_2)\right|\right|_{C^{0,\alpha}_{-2,\delta;t}} \\
&\leq ct^{-\alpha}\left|\left|L^{-1}\underline{b}_1 - L^{-1}\underline{b}_2\right|\right|_{C^{0,\alpha}_{-1,\delta;t}}\left|\left|L^{-1}\underline{b}_1 + L^{-1}\underline{b}_2\right|\right|_{C^{0,\alpha}_{-1,\delta;t}} \\
&\leq ct^{-\alpha}\left|\left|L^{-1}\underline{b}_1 - L^{-1}\underline{b}_2\right|\right|_{C^{1,\alpha}_{-1,\delta;t}}\left|\left|L^{-1}\underline{b}_1 + L^{-1}\underline{b}_2\right|\right|_{C^{1,\alpha}_{-1,\delta;t}} \\
&\leq ct^{-\alpha}\left|\left|\underline{b}_1 - \underline{b}_2\right|\right|_{C^{0,\alpha}_{-2,\delta;t}}\left(\left|\left|\underline{b}_1\right|\right|_{C^{0,\alpha}_{-2,\delta;t}} + \left|\left|\underline{b}_1\right|\right|_{C^{0,\alpha}_{-2,\delta;t}}\right),
\end{aligned}$$

where we used Eq. (4.137) in the first inequality and Eq. (4.136) in the last inequality. For e_t we have

$$||e_t||_{C^{0,\alpha}_{-2,0;t}} \leq ct^{1/18}$$

by Corollary 4.55. For small t, we have that $t^{1/18} < \left(t^{-\alpha+\delta/2}\right)^{-1}$ due to our choices of α and δ in Definition 4.19. Thus, by applying Lemma 4.135 to the map T_t, we get a solution $\underline{b}_t$ to the equation $\underline{b}_t + T_t(\underline{b}_t) = -e_t$ for small t, satisfying the estimate $\left|\left|\underline{b}_t\right|\right|_{C^{0,\alpha}_{-2,0;t}} \leq ct^{1/18}$.

Letting $\underline{a}_t := L_t^{-1}(\underline{b}_t)$, this means precisely $L_t(\underline{a}_t) + Q_t(\underline{a}_t) = -e_t$, so $\widetilde{A}_t = A_t + a_t$ is a G_2-instanton, and $\underline{a}_t$ satisfies $\left|\left|\underline{a}_t\right|\right|_{C^{1,\alpha}_{-1,\delta;t}} \leq ct^{1/18}$ by Eq. (4.136), which proves the claim. □

Proof of Theorem 4.134. As in the proof of Theorem 4.133, set $T_t := Q_t \circ L_t^{-1}$. Then

$$\begin{aligned}
&\left|\left|T_t(\underline{b}_1) - T_t(\underline{b}_2)\right|\right|_{\mathfrak{Y}_t} \\
&= \left|\left|Q(L^{-1}\underline{b}_1 - L^{-1}\underline{b}_2, L^{-1}\underline{b}_1 + L^{-1}\underline{b}_2)\right|\right|_{\mathfrak{Y}_t} \\
&= t^{-\delta/2}\left|\left|\eta_t Q(L^{-1}\underline{b}_1 - L^{-1}\underline{b}_2, L^{-1}\underline{b}_1 + L^{-1}\underline{b}_2)\right|\right|_{C^{0,\alpha}_{-2,\delta;t}} \\
&\quad + t\left|\left|\pi_t Q(L^{-1}\underline{b}_1 - L^{-1}\underline{b}_2, L^{-1}\underline{b}_1 + L^{-1}\underline{b}_2)\right|\right|_{C^{0,\alpha}} \\
&\leq ct^{-\alpha-\delta/2}\Bigg(\left|\left|\eta_t L^{-1}(\underline{b}_1 - \underline{b}_2)\right|\right|_{C^{0,\alpha}_{-1,\delta;t}} \cdot \left|\left|\eta_t L^{-1}(\underline{b}_1 + \underline{b}_2)\right|\right|_{C^{0,\alpha}_{-1,\delta;t}} \\
&\quad + \left|\left|\eta_t L^{-1}(\underline{b}_1 - \underline{b}_2)\right|\right|_{C^{0,\alpha}_{-1,\delta;t}} \cdot \left|\left|\pi_t L^{-1}(\underline{b}_1 + \underline{b}_2)\right|\right|_{C^{0,\alpha}} \\
&\quad + \left|\left|\pi_t L^{-1}(\underline{b}_1 - \underline{b}_2)\right|\right|_{C^{0,\alpha}} \cdot \left|\left|\eta_t L^{-1}(\underline{b}_1 + \underline{b}_2)\right|\right|_{C^{0,\alpha}_{-1,\delta;t}} \\
&\quad + \left|\left|\pi_t L^{-1}(\underline{b}_1 - \underline{b}_2)\right|\right|_{C^{0,\alpha}} \cdot \left|\left|\pi_t L^{-1}(\underline{b}_1 + \underline{b}_2)\right|\right|_{C^{0,\alpha}}\Bigg) \\
&\quad + ct^{-\alpha}\Bigg(\left|\left|\eta_t L^{-1}(\underline{b}_1 - \underline{b}_2)\right|\right|_{C^{0,\alpha}_{-1,\delta;t}} \cdot \left|\left|\eta_t L^{-1}(\underline{b}_1 + \underline{b}_2)\right|\right|_{C^{0,\alpha}_{-1,\delta;t}} \\
&\quad + \left|\left|\eta_t L^{-1}(\underline{b}_1 - \underline{b}_2)\right|\right|_{C^{0,\alpha}_{-1,\delta;t}} \cdot \left|\left|\pi_t L^{-1}(\underline{b}_1 + \underline{b}_2)\right|\right|_{C^{0,\alpha}} \\
&\quad + \left|\left|\pi_t L^{-1}(\underline{b}_1 - \underline{b}_2)\right|\right|_{C^{0,\alpha}} \cdot \left|\left|\eta_t L^{-1}(\underline{b}_1 + \underline{b}_2)\right|\right|_{C^{0,\alpha}_{-1,\delta;t}} \\
&\quad + t\left|\left|\pi_t L^{-1}(\underline{b}_1 - \underline{b}_2)\right|\right|_{C^{0,\alpha}} \cdot \left|\left|\pi_t L^{-1}(\underline{b}_1 + \underline{b}_2)\right|\right|_{C^{0,\alpha}}\Bigg) \\
&\leq ct^{-\alpha-2-\delta/2}\Bigg(t^{-\delta}\left|\left|\eta_t L^{-1}(\underline{b}_1 - \underline{b}_2)\right|\right|_{C^{0,\alpha}_{-1,\delta;t}} \cdot \left|\left|\eta_t L^{-1}(\underline{b}_1 + \underline{b}_2)\right|\right|_{C^{0,\alpha}_{-1,\delta;t}} \\
&\quad + t^{1-\delta/2}\left|\left|\eta_t L^{-1}(\underline{b}_1 - \underline{b}_2)\right|\right|_{C^{0,\alpha}_{-1,\delta;t}} \cdot \left|\left|\pi_t L^{-1}(\underline{b}_1 + \underline{b}_2)\right|\right|_{C^{0,\alpha}} \\
&\quad + t^{1-\delta/2}\left|\left|\pi_t L^{-1}(\underline{b}_1 - \underline{b}_2)\right|\right|_{C^{0,\alpha}} \cdot \left|\left|\eta_t L^{-1}(\underline{b}_1 + \underline{b}_2)\right|\right|_{C^{0,\alpha}_{-1,\delta;t}} \\
&\quad + t^2\left|\left|\pi_t L^{-1}(\underline{b}_1 - \underline{b}_2)\right|\right|_{C^{0,\alpha}} \cdot \left|\left|\pi_t L^{-1}(\underline{b}_1 + \underline{b}_2)\right|\right|_{C^{0,\alpha}}\Bigg) \\
&\leq ct^{-\alpha-2-\delta/2}\left|\left|L^{-1}(b_1 - b_2)\right|\right|_{\mathfrak{X}_t}\left|\left|L^{-1}(b_1 + b_2)\right|\right|_{\mathfrak{X}_t} \\
&\leq ct^{-\alpha-2-\delta/2}||b_1 - b_2||_{\mathfrak{Y}_t}||b_1 + b_2||_{\mathfrak{Y}_t} \\
&\leq ct^{-\alpha-2-\delta/2}||b_1 - b_2||_{\mathfrak{Y}_t}\left(||b_1||_{\mathfrak{Y}_t} + ||b_2||_{\mathfrak{Y}_t}\right).
\end{aligned}$$

Here we used Proposition 4.130 in the third step, and Proposition 4.78 in the second to last step.
We have

$$||e_t||_{\mathfrak{Y}_t} \leq ct^{2-\alpha},$$

by Corollary 4.58. Applying Lemma 4.135 as in the proof of Theorem 4.133 shows the claim. □

5. Examples

5.1. Examples on the resolution of T^7/Γ In [Joy96], many examples of finite groups Γ acting on T^7 and resolutions of the resulting G_2-orbifolds T^7/Γ are explained. In [Wal13a], G_2-instantons on these resolutions were constructed. These examples can immediately be reproduced using Theorem 4.134 by choosing locally constant Fueter sections. However, Theorem 4.134 is more general in two ways.

1. It allows non-constant Fueter sections, as long as they are rigid. However, we found no example of such Fueter sections.
2. It allows θ to have non-trivial monodromy along L. Previously, no such examples have been constructed. Making use of Theorem 4.134, we now explain a large number of examples in the simplest case of the Generalised Kummer Construction.

Consider the group $\Gamma = \langle \alpha, \beta, \gamma \rangle$ acting on T^7 defined by

$$\begin{aligned} \alpha &: (x_1, \dots, x_7) \mapsto (x_1, x_2, x_3, -x_4, -x_5, -x_6, -x_7), \\ \beta &: (x_1, \dots, x_7) \mapsto \left(x_1, -x_2, -x_3, x_4, x_5, \frac{1}{2} - x_6, -x_7\right), \\ \gamma &: (x_1, \dots, x_7) \mapsto \left(-x_1, x_2, -x_3, x_4, \frac{1}{2} - x_5, x_6, \frac{1}{2} - x_7\right). \end{aligned} \tag{5.1}$$

The singular set $S \subset T^7/\Gamma$ consists of 12 copies of T^3. Let $p : \mathbb{R}^7 \to T^7/\Gamma$ be the quotient map. Then, $p^{-1}(T^7/\Gamma \setminus S) \subset \mathbb{R}^7$ is a universal cover of $T^7/\Gamma \setminus S$ and we can identify the orbifold fundamental group of T^7/Γ with the deck transformations of $p^{-1}(T^7/\Gamma \setminus S)$, namely

$$\pi_1^{\mathrm{orb}}(T^7/\Gamma) = \pi_1(T^7/\Gamma \setminus S) = \langle \alpha, \beta, \gamma, \tau_1, \dots, \tau_7 \rangle,$$

where α, β, γ are the maps from Eq. (5.1) but viewed as maps on $\mathbb{R}^7$ by abuse of notation, and τ_i is a translation by 1 in the ith coordinate of $\mathbb{R}^7$ for each $i \in \{1, \dots, 7\}$. These generators satisfy several relations (as observed in [Ma23, Example 6.2]), where the important ones for us are:

$$[\alpha, \beta] = \tau_6^{-1}, \quad [\alpha, \gamma] = \tau_5^{-1}\tau_7^{-1}, \quad [\beta, \gamma] = \tau_7^{-1}. \tag{5.2}$$

Let

$$\begin{aligned} a &= \mathrm{diag}(1, -1, -1), \quad b = \mathrm{diag}(-1, 1, -1), \\ c &= \mathrm{diag}(-1, -1, 1), \quad \mathbb{Z}_2^2 \cong \langle a, b, c \rangle \subset \mathrm{SO}(3). \end{aligned}$$

A representation $\rho : \pi_1^{\mathrm{orb}}(T^7/\Gamma) \to \langle a, b, c \rangle$ induces a flat connection θ on a bundle E_0 over T^7/Γ. By Eq. (5.2), a representation ρ is uniquely determined by specifying its value on $\alpha, \beta, \gamma, \tau_1, \tau_2, \tau_3, \tau_4$.

For any such choice we can carry out our pre-gluing construction as follows. Let A_0 be the product connection on the trivial SO(3)-bundle over Eguchi–Hanson space and M_0 its moduli space, which is just a single point. Then, for a $T^3 \subset T^7/\Gamma$ in the fixed point set of an element in $\Gamma \setminus \{\mathrm{Id}\}$ which is mapped to $\mathrm{Id} \in \mathrm{SO}(3)$, the bundle

$$\mathrm{Fr} \times (E_0|_{T^3}) \times_{\mathrm{U}(2)\times G} M_0$$

over T^3 has as its fibre a single point, so there exists a parallel section, which is in particular a Fueter section.

Likewise, let $A_{0,1}$ be the ASD instanton over X_{EH} from Proposition 2.32. This is defined on a U(1)-bundle and we view it as a reducible SO(3)-connection, and denote its moduli space by $M_{0,1}$. This connection has non-trivial holonomy $\rho_{0,1} : \mathbb{Z}_2 \to \mathrm{SO}(3)$ at infinity, thus $G_{\rho_{0,1}} \subsetneq G$. For each copy of T^3 fixed by an element in $\Gamma \setminus \{\mathrm{Id}\}$ which is mapped to a non-identity element in SO(3) we find that

$$\mathrm{Fr} \times (E_0|_{T^3}) \times_{\mathrm{U}(2)\times G_{\rho_{0,1}}} M_{0,1}$$

is again a bundle whose fibre is a single point, so we again have a Fueter section.
We chose a moduli bundle of connections over the singularities coming from α, β, and γ matching the monodromy of θ given by ρ. For example, if $\rho(\alpha) = \mathrm{Id}$, we chose the moduli bundle of trivial connections $\mathrm{Fr} \times (E_0|_{T^3}) \times_{\mathrm{U}(2)\times G} M_0$ over $\mathrm{fix}(\alpha)$. Because of this, the compatibility condition from Assumption 4.1 is satisfied.
If θ is irreducible and infinitesimally rigid, then Theorem 4.134 guarantees the existence of an irreducible G_2-instanton with structure group SO(3) on the resolution of T^7/Γ. We have the following criterion to check if θ is irreducible and/or rigid:

Proposition 5.3 (Proposition 9.2 in [Wal13a]). *A flat connection θ on a G-bundle E over a flat G_2-orbifold Y_0 corresponding to a representation $\rho : \pi_1^{\mathrm{orb}}(Y_0) \to G$ is irreducible (resp. unobstructed or, equivalently, infinitesimally rigid) if and only if the induced representation of $\pi_1^{\mathrm{orb}}(Y_0)$ on $\mathfrak{g}$ (resp. $\mathbb{R}^7 \otimes \mathfrak{g}$) has no non-zero fixed vectors. Here, the action on $\mathfrak{g}$ is ρ composed with the adjoint representation, and the action on $\mathbb{R}^7$ is given by identifying $\mathbb{R}^7 \cong T_x(T^7/\Gamma)$ for any fixed $x \in T^7/\Gamma$ and then acting by parallel transport with respect to the Levi-Civita connection of the flat metric.*

In our case, one checks that there are no non-zero vectors in $\mathfrak{so}(3)$ or $\mathbb{R}^7 \otimes \mathfrak{so}(3)$ fixed by $\pi_1^{\mathrm{orb}}(T^7/\Gamma)$, if at least two of the elements $\tau_1, \ldots, \tau_7$ are sent to different non-identity elements in $\langle a, b, c\rangle$. Thus, the connection θ is irreducible and infinitesimally rigid in this case by Proposition 5.3. The number of flat connections which do *not* satisfy this condition is

$$4^3 \cdot (2^4 \cdot 3 - 2).$$

Here, we got the factor 4^3 for the choice of different values for ρ on α, β, γ. The term $2^4 \cdot 3$ is the number of choices of values for ρ on $\tau_1, \ldots, \tau_7$ contained in $\{\mathrm{Id}, a\}$ or $\{\mathrm{Id}, b\}$ or $\{\mathrm{Id}, c\}$. However, we triple counted the choice $\rho(\tau_1) = \cdots = \rho(\tau_7) = \mathrm{Id}$, so overall we get the factor $(2^7 \cdot 3 - 2)$. Thus, we have $4^7 - 4^3 \cdot (2^4 \cdot 3 - 2) = 13440$ flat connections to which Theorem 4.134 can be applied, and we get this many G_2-instantons on the resolution of T^7/Γ.
Among these, there are 210 choices for ρ giving rise to a flat G_2-instanton on the resolved manifold, namely when the α, β, γ are all sent to the identity. Thus, we are left with $13440 - 210 = 13230$ novel, non-flat examples of G_2-instantons on the resolution of T^7/Γ.
This number contains gauge equivalent G_2-instantons, and we compute the number of different connections *up to gauge equivalence* as in [GPTW23, Remark 7.12]: as $\mathrm{Im}(\rho) \subseteq \langle a, b, c\rangle \subset \mathrm{SO}(3)$ contains at least two non-identity elements, the stabiliser of ρ in $\mathrm{Aut}(\langle a, b, c\rangle) \cong S_3$ is trivial. Hence, this number contains six gauge equivalence classes of each connection, and the number of new G_2-instantons up to gauge equivalence is $13230/6 = 2205$.

Corollary 5.4. *Let Γ act on T^7 as defined in Eq. (5.1) and let N'_t denote the one parameter family of resolutions of T^7/Γ from Sect. 3.1. Then, for t small enough, there exist* 2205 *non-flat, irreducible G_2-instantons with structure group* SO(3) *over N' which are pairwise not gauge equivalent.*

5.2. An example coming from a stable bundle

5.2.1. Review of the resolution of $(T^3 \times K3)/\Gamma$ Recall the G_2-manifold constructed in [JK21, Section 7.3]: consider the sextic

$$C = \{[z_0, z_1, z_2] \in \mathbb{CP}^2 : z_0^6 + z_1^6 + z_2^6 = 0\} \subset \mathbb{CP}^2$$

and let $\pi : X \to \mathbb{CP}^2$ be the double cover of $\mathbb{CP}^2$ branched over C. Then X is a complex K3 surface with a Hyperkähler triple of Kähler forms $\omega^I, \omega^J, \omega^K$, cf. [Huy16, Example 1.3]. On X we can define the following two maps: first, the map $\alpha : X \to X$ which swaps the two sheets of the branched cover. Second, there are two lifts $X \to X$ of the complex conjugation map $\sigma : \mathbb{CP}^2 \to \mathbb{CP}^2$. One of these two lifts acts freely on X, the other one does not. Denote the lift that does not act freely on X by $\beta : X \to X$, which has $\mathrm{fix}(\beta) = \pi^{-1}(\mathbb{RP}^2) \simeq S^2$. The Hyperkähler triple $\omega^I, \omega^J, \omega^K$ can be chosen to satisfy

$$\begin{aligned} \alpha^*\omega^I &= \omega^I, & \alpha^*\omega^J &= -\omega^J, & \alpha^*\omega^K &= -\omega^K, \\ \beta^*\omega^I &= -\omega^I, & \beta^*\omega^J &= \omega^J, & \beta^*\omega^K &= -\omega^K. \end{aligned}$$

Let α, β act on T^3 via

$$\alpha(x_1, x_2, x_3) = (x_1, -x_2, -x_3), \quad \beta(x_1, x_2, x_3) = \left(-x_1, x_2, \frac{1}{2} - x_3\right).$$

Denote $\Gamma = \langle \alpha, \beta \rangle$. Then $\alpha, \beta : T^3 \times X \to T^3 \times X$ preserve the product G_2-structure φ on $T^3 \times X$ defined by equation Eq. (2.11). Furthermore, $\mathrm{fix}(\alpha) = 4 \cdot (S^1 \times C)$, $\mathrm{fix}(\beta) = 4 \cdot (S^1 \times S^2)$, where the S^2-factors are the double cover of $\mathrm{fix}(\sigma) = \mathbb{RP}^2 \subset \mathbb{CP}^2$. Therefore, $L = \mathrm{fix}(\alpha) \cup \mathrm{fix}(\beta)$ admits a nowhere vanishing harmonic 1-form, namely the parallel 1-form in the S^1-direction of each component. Thus, this orbifold is of the type considered in Sect. 3 and its resolution $N_t \to (T^3 \times X)/\Gamma$ admits a 1-parameter family of G_2-structures with small torsion, inducing metrics g_t, which can be perturbed to torsion-free G_2-structures inducing metrics $\widetilde{g}_t$.

5.2.2. A connection on the orbifold $(T^3 \times K3)/\Gamma$ *coming from a stable bundle* The tangent bundle E of $\mathbb{CP}^2$ is a complex vector bundle of rank 2, which has an associated $\mathrm{SO}(3) = \mathrm{PU}(2)$-bundle F. The Levi-Civita connection on E is a Hermite-Einstein connection and induces an ASD instanton on F, denoted by A. We denote the standard Kähler structure on $\mathbb{CP}^2$ by $(J, g = g_{\mathrm{FS}}, \omega)$, where g_{FS} is the Fubini-Study metric. The pullback π^*A is then an ASD instanton on the bundle π^*F over (X, π^*g), but it need not be ASD with respect to the Calabi–Yau metric on X. We will show in Corollary 5.6 that π^*F also carries an instanton with respect to the Calabi–Yau metric.

Proposition 5.5 (Lemma 9.1.9 in [DK90]). *The bundle π^*E is stable with respect to ω.*

Corollary 5.6. *The bundle π^*E is stable with respect to the Calabi–Yau Kähler form ω^I.*

Proof of Corollary 5.6. Denote by $\hat{\omega} = \pi^*\omega$ the pullback of the Kähler form for the Fubini-Study metric on $\mathbb{CP}^2$ to X. By Yau's proof of the Calabi conjecture we have that $\omega^I = \hat{\omega} + i\partial\overline{\partial}\phi$ for some $\phi : X \to \mathbb{R}$. In particular, ω^I and $\hat{\omega}$ are in the same de Rham cohomology class.
By Proposition 5.5, π^*E is stable with respect to ω. The Kähler form enters into the definition of stability only through the definition of slope. But slopes do not change when switching between ω^I and $\hat{\omega}$ as they are in the same cohomology class. Thus π^*E is also stable with respect to ω^I. □

We also have the following:

Corollary 5.7 (p. 348 in [DK90]). *Denote by $\pi_F : F \to \mathbb{CP}^2$ the* SO(3)*-bundle over $\mathbb{CP}^2$ from Sect. 5.2.2. Let $\pi : X \to \mathbb{CP}^2$ be the branched double cover from Sect. 5.2.1 with Calabi–Yau metric $\hat{g}$. Then the bundle*

$$\hat{F} = \pi^*F = \{(x, u) \in X \times F : \pi_F(u) = \pi(x)\} \tag{5.8}$$

admits an infinitesimally rigid and unobstructed ASD instanton $\hat{A}$ with respect to $\hat{g}$.

Pulling back $(\hat{F}, \hat{A})$ under the projection onto the second factor, $p : T^3 \times X \to X$, gives a bundle with G_2-instanton by Example 2.41. Denote the bundle by E_0 and the connection by θ. The connection $\hat{A}$ was infinitesimally rigid, and the following proposition, which is proved like Proposition 4.88, implies that θ is infinitesimally rigid:

Proposition 5.9. *Let I be an ASD instanton on a bundle P over a compact* 4*-fold Y with deformation operator δ_I. Let $p : T^3 \times Y \to Y$ be the projection onto the second factor. Then the G_2-instanton p^*I is infinitesimally rigid if and only if I is infinitesimally rigid and unobstructed.*

The gluing theorems Theorems 4.133 and 4.134 require a connection on the orbifold, $(T^3 \times X)/\Gamma$. The following proposition states that θ can be viewed as such a connection:

Proposition 5.10. *There exist lifts $\alpha_0 : E_0 \to E_0$ of α and $\beta_0 : E_0 \to E_0$ of β such that $\alpha_0^2 = \beta_0^2 = \mathrm{Id}$, $\alpha_0^*\theta = \beta_0^*\theta = \theta$, α_0 being the identity over* $\mathrm{fix}(\alpha)$*, and β_0* not *being the identity over* $\mathrm{fix}(\beta)$.

This relies on the following construction on X:

Proposition 5.11. *There exists a lift $\hat{\beta} : \hat{F} \to \hat{F}$ of β such that $\hat{\beta}^2 = \mathrm{Id}$, $\hat{\beta}^*\hat{A} = \hat{A}$, and $\hat{\beta}$ not being the identity over* $\mathrm{fix}(\beta)$.

Proof. Denote by $\sigma : \mathbb{CP}^2 \to \mathbb{CP}^2$ the conjugation map and $E = T\mathbb{CP}^2$ as before. We can then view $\mathrm{d}\sigma$ as a complex linear map $E \to \overline{E}$ covering σ. Define

$$\begin{aligned} \hat{\sigma} : E \otimes \overline{E} &\to E \otimes \overline{E} \\ v \otimes w &\mapsto -\mathrm{d}\sigma w \otimes \mathrm{d}\sigma v, \end{aligned} \tag{5.12}$$

which is a complex linear map covering $\sigma : \mathbb{CP}^2 \to \mathbb{CP}^2$.
The manifold $\mathbb{CP}^2$ is Kähler-Einstein, so the Levi-Civita connection ∇^{LC} on E is a Hermite-Einstein connection. The connection ∇^{LC} on E induces the product connection $\nabla^{\otimes}$ on $E \otimes \overline{E}$, which is again a Hermite-Einstein connection. We have that σ is an isometry, so $\nabla^{\otimes}$ is preserved by $\hat{\sigma}$ in the sense of $\hat{\sigma} \circ \sigma^*\nabla^{\otimes} \circ \hat{\sigma} = \nabla^{\otimes}$.

Let $\hat{\beta}$ be the lift of $\hat{\sigma}$ to $\pi^*E \otimes \overline{\pi^*E}$, i.e. $\hat{\beta} : \pi^*E \otimes \overline{\pi^*E} \to \pi^*E \otimes \overline{\pi^*E}$ covering $\beta : X \to X$ and satisfying $p\hat{\beta} = \hat{\sigma} p$, where $p : \pi^*E \otimes \overline{\pi^*E} \to E \otimes \overline{E}$ is the obvious projection map. Then $\hat{\sigma}^*\nabla^{\otimes} = \nabla^{\otimes}$ implies $\hat{\beta}^*(\pi^*\nabla^{\otimes}) = \pi^*\nabla^{\otimes}$.
If $p \in \mathbb{CP}^2$ and (u_1, u_2) is a unitary basis of E_p, then $(\mathrm{d}\sigma(u_1), \mathrm{d}\sigma(u_2))$ is a unitary basis of $E_{\sigma(p)}$, and writing elements of the trace-free unitary endomorphism bundle $\mathfrak{u}_0(\pi^*E)$ in these bases, we see that $\hat{\beta}$ acts as

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \mapsto -\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \mapsto -\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Thus, $\hat{\beta}$ induces a map on $\hat{F} = \mathrm{SO}(\mathfrak{u}_0(\pi^*E))$ that is not the identity over $\mathrm{fix}(\beta)$ and preserves the ASD connection $\hat{A}$ on $\hat{F}$ induced by $\pi^*\nabla^{\otimes}$. □

Remark 5.13. This only works because we have a lift of complex conjugation $\sigma : \mathbb{CP}^2 \to \mathbb{CP}^2$ to F in Proposition 5.11. No lift of σ to E exists, because $c_1(\sigma^*E) = -c_1(E)$, so it is important to change from U(2)-bundles to SO(3)-bundles in this example.

Remark 5.14. Without the minus sign in Eq. (5.12), $\hat{\beta}$ would not descend to a map on $\mathrm{SO}(\mathfrak{u}_0(\pi^*E))$. That is because the map $-\mathrm{Id} : \mathfrak{u}_0(\pi^*E) \to \mathfrak{u}_0(\pi^*E)$ is orientation reversing, as $\mathfrak{u}_0(\pi^*E)$ has odd rank.

Proof of Proposition 5.10. The bundle $\hat{F}$ from Eq. (5.8) is the pullback of a bundle F from $\mathbb{CP}^2$ to X, thus we have the natural map

$$\begin{aligned} \hat{\alpha} : \hat{F} &\to \hat{F} \\ (x, u) &\mapsto (\alpha(x), u) \end{aligned}$$

covering $\alpha : X \to X$. The bundle E_0 is the pullback of $\hat{F}$ to $T^3 \times X$, and we can canonically extend the map $\hat{\alpha}$ and the map $\hat{\beta}$ from Proposition 5.11 to E_0 and find that they have the required properties. □

Because of Proposition 5.10, the connection θ defines a connection on the orbifold $(T^3 \times \mathrm{K3})/\Gamma$. The holonomy of θ around the four $S^1 \times C \subset (T^3 \times X)/\Gamma$ fixed by α is trivial, and the holonomy around the four $S^1 \times S^2$ fixed by β has order 2.

5.2.3. The resulting connection on the resolution of $(T^3 \times K3)/\Gamma$

Corollary 5.15. *Let N_t denote the one parameter family of resolutions of $(T^3 \times X)/\Gamma$ from Sect. 5.2.1. Then, for t small enough, there exists an irreducible G_2-instanton with structure group* SO(3) *over the resolution N_t.*

Proof. We make use of the α-invariant and β-invariant connection θ from Proposition 5.10 over $(T^3 \times X)/\Gamma$.

Next consider the product connection A_0 on the trivial SO(3)-bundle over Eguchi–Hanson space X_{EH}. Like in Sect. 5.1, we get a constant Fueter section on each connected component of $\mathrm{fix}(\alpha) = 4 \cdot (S^1 \times C)$, i.e.

$$S^1 \times C \to \mathrm{Fr} \times E_0|_{S^1 \times C} \times_{\mathrm{U}(2) \times G} M_0.$$

Likewise, let $A_{0,1}$ be the ASD instanton over X_{EH} from Proposition 2.32. As in Section 5.1, we get a constant Fueter section on each connected component of $\mathrm{fix}(\beta) = 4 \cdot (S^1 \times S^2)$, i.e.

$$S^1 \times S^2 \to \mathrm{Fr} \times E_0|_{S^1 \times S^2} \times_{\mathrm{U}(2) \times G_{\rho_{0,1}}} M_{0,1}.$$

By Proposition 5.10, the connection θ and the eight Fueter sections satisfy the necessary compatibility condition from Proposition 4.28. Thus, Theorem 4.133 applies and gives a G_2-instanton $\widetilde{A}_t$ on N_t. The connections $\widetilde{A}_t$ converge to θ on compact subsets of $(T^3 \times X)/\Gamma \setminus \mathrm{fix}(\Gamma)$ as $t \to 0$. The connection θ has full holonomy SO(3), as otherwise the Fubini-Study metric on $\mathbb{CP}^2$ would need to have reduced holonomy. Thus, $\widetilde{A}_t$ has full holonomy for small t and is therefore irreducible. □

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Declarations

Conflict of interest The author declares no competing interests. My manuscript has no associated data.

A. Appendix

A.1. The isometry group of Eguchi–Hanson space The following result is well known, but we were unable to locate a proof in the literature, so we provide it here.

Proposition A.1. *The group of holomorphic isometries of X_{EH} is isomorphic to* $\mathrm{U}(2)/\{\pm 1\}$.

Proof. We use the notation from the description of X_{EH} as a Hyperkähler reduction from before Definition 2.31. We view SU(2) embedded in $\mathbb{H}^{2\times 2}$ as quaternion valued matrices with no j or k components. Then SU(2) acts on $\mathcal{M}$ by right multiplication. This action restricts to $\mu^{-1}(\zeta)$ and commutes with the action of U(1). The action is not effective, as $-1 \in \mathrm{SU}(2)$ acts trivially, but the induced action of the quotient group $\mathrm{SU}(2)/\{\pm 1\} \simeq \mathrm{SO}(3)$ is effective. Next, let SO(2) act on $\mathcal{M}$ from the left via

$$q_a \mapsto e^{it} \cdot q_a, \quad t \in (0, 2\pi].$$

Again, the action restricts to $\mu^{-1}(\zeta)$ and commutes with the action of U(1), but is not effective as $-1 \in \mathrm{SO}(2)$ acts trivially. The actions of $\mathrm{SO}(2)/\{\pm 1\}$ and $\mathrm{SU}(2)/\{\pm 1\}$ commute, as the first group is acting from the left, the second is acting from the right. We thus get that the group $\mathrm{SO}(2)/\{\pm 1\} \times \mathrm{SU}(2)/\{\pm 1\}$ acts through isometries on X_{EH}. Last, one readily confirms that the map

$$\begin{aligned} U(1)/\{\pm 1\} \times \mathrm{SU}(2)/\{\pm 1\} &\to \mathrm{U}(2)/\{\pm 1\} \\ [\lambda], [A] &\mapsto [\lambda A] \end{aligned}$$

is a group isomorphism. Its inverse is given by $[B] \mapsto ([\sqrt{\det B}], [B/\sqrt{\det B}])$ which is not well-defined as a map $\mathrm{U}(1) \times \mathrm{SU}(2) \to \mathrm{U}(2)$ but is well-defined after dividing out $\{\pm 1\}$. □

Remark A.2. One may also recover the full isometry group of the Eguchi–Hanson space by noticing that there is an additional isometry induced by the map on $\mathcal{M}$ that swaps coordinates, i.e. $\mathcal{M} \to \mathcal{M}$, $(q_1, q_2) \mapsto (q_2, q_1)$. The group of *all* isometries (not necessarily holomorphic) of X_{EH} is isomorphic to $\mathrm{SO}(3) \times \mathrm{O}(2)$. The group of triholomorphic isometries of X_{EH} is isomorphic to $\mathrm{SO}(3)$.

A.2. Rigidity of finite subgroups Let G be a compact connected Lie group and Γ be a finite group. In Sect. 2.2 we took Γ to be a finite subgroup of $\mathrm{SU}(2)$, thereby acting on B^4. An orbifold G-bundle over B^4/Γ is a G-bundle P over B^4 together with a lift of the action of Γ to P. In Eq. (2.17) we extended elements of G to elements of the orbifold gauge group $\mathscr{G}(P)$. We could do this, because we assumed the lift of Γ to act in a standard way on P, see Eq. (2.14) for the precise statement. In other words: we used that up to gauge equivalence, orbifold bundles over B^4/Γ are determined by the homomorphism $\Gamma \to P_0 \simeq G$ induced by the lift of Γ to P. The proof of this fact was given in Proposition 2.13, but used that the homomorphism $\Gamma \to G$ is rigid, in some sense. We make this rigidity precise here and prove that every finite group in a compact Lie group is rigid. The proof is taken from [Bad21], where also the generalisation to non-compact G is explained.

Definition A.3. The set $\mathrm{Hom}(\Gamma, G) \subset G^{|\Gamma|}$ endowed with the restriction of the product topology on $G^{|\Gamma|}$ is called the *representation variety*.

Definition A.4. Let E be a Γ-module. A map $b \in \Gamma \to E$ is called *cocycle* if

$$b(\gamma\delta) = b(\gamma) + \gamma \cdot b(\delta) \text{ for all } \gamma, \delta \in \Gamma.$$

We denote the set of cocycles by $Z^1(\Gamma, E)$. A map $b \in \Gamma \to E$ is called *coboundary* if there exists $v \in E$ such that

$$b(\gamma) = v - \gamma \cdot v \text{ for all } \gamma \in \Gamma.$$

We denote the set of coboundaries by $B^1(\Gamma, E) \subset Z^1(\Gamma, E)$. The *first cohomology of* Γ *with coefficients in* E *is*

$$H^1(\Gamma, E) = Z^1(\Gamma, E)/B^1(\Gamma, E).$$

Theorem A.5 (Point 3 in [Wei64]). *Fix a group homomorphism $r : \Gamma \to G$. The group G is acting on $\mathfrak{g}$ through the adjoint representation, and together with r this gives $\mathfrak{g}$ the structure of a Γ-module. If $H^1(\Gamma, \mathfrak{g}) = 0$, then there exists a neighbourhood $U \subset \mathrm{Hom}(\Gamma, G)$ of r in which each element is conjugate to r, i.e. for all $s \in U$ there exists $g \in G$ such that*

$$s = l_g \circ r_{g^{-1}} \circ r.$$

Here, $l_g, r_{g^{-1}} : G \to G$ denote left translation and right translation on G, respectively.

Definition A.6. Fix $\pi : \Gamma \to \mathrm{Aut}(E)$. An *affine action* of Γ on E is a group homomorphism $\phi : \Gamma \to \mathrm{Aff}(E)$. We say that π is the *linear part* of the affine action ϕ if for all $\gamma \in \Gamma$ there exists $v_0 \in E$ such that

$$\phi(\gamma)(v) = \pi(\gamma)(v) + v_0 \text{ for all } v \in E.$$

Lemma A.7 (Lemma 2.1 in [DX16]). *The map* $\pi : \Gamma \to \mathrm{Aut}(E)$ *endows* E *with a* Γ*-module structure. We have* $H^1(\Gamma, E) = 0$ *with respect to this* Γ*-module structure if and only if every affine action with linear part* π *has a fixed point.*

Corollary A.8. *If* Γ *is finite, then* $H^1(\Gamma, E) = 0$ *for any* E.

Proof. Let $\phi : \Gamma \to \mathrm{Aff}(E)$ be an affine action. Then the element

$$X := \sum_{\delta \in \Gamma} \phi(\delta)(0) \in E$$

satisfies $\phi(\gamma)(X) = X$ for all $\gamma \in \Gamma$. By Lemma A.7 this implies that $H^1(\Gamma, E) = 0$. □

Corollary A.9. *The representation variety* $\mathrm{Hom}(\Gamma, G)$ *has finitely many connected components. For each connected component* C *there exists* $r \in \mathrm{Hom}(\Gamma, G)$ *such that*

$$C = U_r := \{l_g \circ r_{g^{-1}} \circ r : g \in G\}.$$

Proof. Because Γ is finite and G is compact we have that $\mathrm{Hom}(\Gamma, G)$ is compact and therefore has finitely many connected components. Fix some $r \in \mathrm{Hom}(\Gamma, G)$. Then U_r is compact because it is the image of G under the conjugation map. Thus, U_r is closed. On the other hand, U_r is open by Theorem A.5 together with Corollary A.8. Thus, each connected component of $\mathrm{Hom}(\Gamma, G)$ is of the form U_r for some $r \in \mathrm{Hom}(\Gamma, G)$. □

A.3. Removable singularities In Definition 2.21 we defined a map from the moduli space of ASD connections over the Eguchi–Hanson space X_{EH} into the moduli space of ASD connections over the one point compactification of X_{EH}. There, we used that every finite energy ASD connection that is defined over the complement of a point can be extended over this point. This statement was proved for Yang-Mills connections, not just ASD connections, in [Uhl82]. This is called the *Removable Singularities Theorem*. Because our map between moduli spaces should be a map between *framed* moduli spaces, we need a version of the Removable Singularities Theorem that respects framings. This is Proposition A.11 and we then apply it to our special case of connections over X_{EH} in Corollary A.14.

Theorem A.10 (Theorem 4.1 in [Uhl82], Theorem D.1 in [FU91]). *Let* G *be a compact Lie group and* A *be a connection on the trivial* G*-bundle over* $B^4\backslash\{0\}$, $A \in \mathscr{A}((B^4\backslash\{0\}) \times G)$, *which is in* $L^2_{1,loc}$ *and anti-self-dual with respect to a smooth metric on* B^4. *If*

$$\int_{B^4\backslash\{0\}} |F(A)|^2 < \infty,$$

then there exists an injective bundle homomorphism $\xi : (B^4\backslash\{0\}) \times G \to B^4 \times G$ *and a smooth connection* $A' \in \mathscr{A}(B^4 \times G)$ *such that* $\xi^* A' = A$ *over* $B^4 \setminus \{0\}$.

Theorem A.10 asserts existence of an extension over 0, and the following proposition asserts that this extension is essentially unique up to gauge:

Proposition A.11. *The data* ξ *and* A' *from Theorem A.10 are unique in the following sense: if* $\xi', \xi'' : (B^4\backslash\{0\}) \times G \to B^4 \times G$ *and* $A', A'' \in \mathscr{A}(B^4 \times G)$ *are such that* $(\xi')^* A' = (\xi'')^* A'' = A$, *then the map* $\xi'' \circ (\xi')^{-1} : (B^4\backslash\{0\}) \times G \to (B^4\backslash\{0\}) \times G$ *can be extended to a continuous map* $B^4 \times G \to B^4 \times G$.

Proof. We view the connections A', A'' on the trivial bundle $B^4 \times G$ as elements in $\Omega^1(B^4, \mathfrak{g})$, and view the gauge transformation $\xi'' \circ (\xi')^{-1}$ as a map $B^4 \backslash \{0\} \to G$, denoted by s. Without loss of generality assume that $A'(0) = A''(0) = 0$, which can be arranged by composing ξ', ξ'' with a suitable gauge transformation of $B^4 \times G$. Then $A'' = s^* A'$ on $B^4 \backslash \{0\}$, thus

$$0 = A''(0) = \lim_{x \to 0} s^{-1}(x) \, \mathrm{d}s(x)$$

and by taking norms we see that $\lim_{x \to 0} \mathrm{d}s(x) = 0$. This implies that $\lim_{x \to 0} s(x)$ exists: if the limit does not exist, then we have two sequences $x_i, x_i' \to 0$ such that $\lim_{i \to \infty} s(x_i) \neq \lim_{i \to \infty} s(x_i')$. Without loss of generality assume that x_i, x_i' can be joined by a line. The mean value theorem then gives a sequence $\theta_i \in B^4 \backslash \{0\}$ such that $|\mathrm{d}s(\theta_i)| \to \infty$, which is a contradiction.
Therefore $\lim_{x \to 0} s(x)$ exists and defines a continuous map $\bar{s} : B^4 \to G$, which in turn extends $\xi'' \circ (\xi')^{-1}$. □

Viewing the map ξ from Theorem A.10 as a map $\xi : B^4 \backslash \{0\} \to G$, the limit $\lim_{x \to 0} \xi(x)$ does not exist in general. But in important cases it does, according to the following proposition:

Proposition A.12. *Under the conditions of Theorem A.10, assume that A is bounded, viewed as an element in* $\Omega^1(B^4 \backslash \{0\}, \mathfrak{g})$*. Viewing* ξ *as a map* $\xi : B^4 \backslash \{0\} \to G$*, we have that the limit*

$$\lim_{x \to 0} \xi(x) \in G$$

exists.

Proof. Without loss of generality assume that $A'(0) = 0$. Then,

$$\xi^* A'(x) = A(x) \text{ for all } x \in B^4 \setminus \{0\}. \tag{A.13}$$

Taking norms in Eq. (A.13) and using $\xi^* A'(x) = \xi^{-1}(x) \, \mathrm{d}\xi(x) + A'(x)$ we see that $\mathrm{d}\xi$ is bounded on $B^4 \setminus \{0\}$, and we can conclude the proof as in the proof of Proposition A.11. □

This can be applied to the case of ASD instantons on ALE manifolds:

Corollary A.14. *Let* P *be a* G*-bundle over* X_{EH} *and denote by* $\mathscr{A}^{\mathrm{asd},-2}$ *the set of ASD-connections on* P *as in Eq. (2.17). Let* $A_0 + a \in \mathscr{A}^{\mathrm{asd},-2}$*, then there exists an orbifold* G*-bundle* P' *over* $\hat{X}_{EH}$ *together with a connection* $A' \in \mathscr{A}(P')$ *and an injective bundle homomorphism* $\xi : P \to P'$ *such that* $\xi^* A' = A_0 + a$*. Denote by* $f : B^4/\Gamma \to V$ *the chart of* $\hat{X}_{EH}$ *around* ∞ *from Proposition 2.12. Fixing a trivialisation of* P *over* $V \setminus \{\infty\}$ *induces a trivialisation of* P' *over* V *and we can view* ξ *as a map* $V \backslash \{\infty\} \to G$*. Then the limit* $\lim_{x \to \infty} \xi(x)$*, where* $\infty \in \hat{X}_{EH}$*, exists.*

Proof. The assumption $A_0 + a \in \mathscr{A}^{\mathrm{asd},-2}$ means that $a = \mathcal{O}(r^{-2})$, measured in the ALE metric. By inspecting how the inversion f acts on 1-forms, we find that $a = \mathcal{O}(1)$, measured in the orbifold metric, and Proposition A.12 gives the claim. □

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