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The Ground State Energy of a Two-Dimensional Bose Gas

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Abstract: We prove the following formula for the ground state energy density of a dilute Bose gas with density ρ in 2 dimensions in the thermodynamic limit

$$e^{2D}(\rho) = 4\pi\rho^2 Y \left(1 - Y|\log Y| + \left(2\Gamma + \frac{1}{2} + \log(\pi)\right)Y\right) + o(\rho^2 Y^2),$$

as $\rho a^2 \to 0$. Here $Y = |\log(\rho a^2)|^{-1}$ and a is the scattering length of the two-body potential. This result in 2 dimensions corresponds to the famous Lee–Huang–Yang formula in 3 dimensions. The proof is valid for essentially all positive potentials with finite scattering length, in particular, it covers the crucial case of the hard core potential.

Contents

1. Introduction

The calculation of the ground state energy of a dilute gas of bosons is of fundamental importance and has been the focus of much attention in recent years. This question can be posed in all dimensions of the ambient space, but of course, the most important case from the point of view of Physics is the 3-dimensional situation. However, also 1 and 2 dimensions are experimentally realizable. In this paper we study the 2-dimensional setting and prove an asymptotic formula analogous to the famous Lee–Huang–Yang formula in 3-dimensions.

Let us be more precise about the setting of the result. We consider positive, measurable potentials $v : \mathbb{R}^2 \to [0, +\infty]$ that are radial. Given such a potential, we will let $a = a(v)$

be its scattering length (for details on the scattering length see Sect. 3) and define the Hamiltonian

$$H(N, L) = \sum_{j=1}^{N} -\Delta_j + \sum_{j<k} v(x_j - x_k), \tag{1.1}$$

on $L^2(\Omega^N)$, with $\Omega = [-\frac{L}{2}, \frac{L}{2}]^2$. The ground state energy density in the thermodynamic limit $e^{2D}(\rho)$ is then defined by

$$e^{2D}(\rho) := \lim_{\substack{L\to\infty \\ N/L^2\to\rho}} L^{-2} \inf_{\Psi\in C_0^\infty(\Omega^N)} \frac{\langle \Psi, H(N, L)\Psi\rangle}{\|\Psi\|^2}. \tag{1.2}$$

It is a standard result that the limit exists, and actually our analysis of $e^{2D}(\rho)$ proceeds by giving upper bounds on the lim sup and lower bounds on the lim inf. It is also well-known that the limit is independent of the boundary conditions. The fact that we consider $\Psi \in C_0^\infty$ in the formula above, corresponds to the choice of Dirichlet boundary conditions for concreteness.

Theorem 1.1 (Main result). *For any constants $C_0, \eta_0 > 0$, there exist $C, \eta > 0$ (depending only on C_0 and η_0) such that the following holds. If the (measurable) potential $v : \mathbb{R}^2 \to [0, +\infty]$ is non-negative and radial with scattering length a and $\rho a^2 < C^{-1}$, and, furthermore,*

$$v(x) \le \frac{C_0}{|x|^2}\left(\frac{a}{|x|}\right)^{\eta_0}, \quad \text{for all } |x| \ge C_0 a. \tag{1.3}$$

Then

$$\left| e^{2D}(\rho) - 4\pi\rho^2\delta_0\left(1 + \left(2\Gamma + \frac{1}{2} + \log(\pi)\right)\delta_0\right)\right| \le C\rho^2\delta_0^{2+\eta}, \tag{1.4}$$

with

$$\delta_0 := |\log(\rho a^2|\log(\rho a^2)|^{-1})|^{-1}, \tag{1.5}$$

where $\Gamma = 0.577\ldots$ is the Euler–Mascheroni constant.

In terms of the simpler parameter $Y = |\log(\rho a^2)|^{-1}$, we get from (1.4), expanding δ_0 in terms of Y, the three-term asymptotics

$$e^{2D}(\rho) = 4\pi\rho^2 Y\left(1 - Y|\log Y| + \left(2\Gamma + \frac{1}{2} + \log(\pi)\right)Y\right) + \mathcal{O}(\rho^2 Y^{2+\eta}). \tag{1.6}$$

Here the third term in the asymptotics is analogous to the famous Lee–Huang–Yang term in the 3-dimensional situation.

Notice, in particular, that the decay assumption (1.3) is valid for potentials with compact support. So Theorem 1.1 applies to the very important special case of the hard core potential of radius a:

$$v_{hc}(x) = \begin{cases} 0, & |x| > a, \\ +\infty, & |x| \le a. \end{cases} \tag{1.7}$$

For this potential the radius of the support is equal to its scattering length.

The proof of Theorem 1.1 will proceed by establishing upper and lower bounds. In Theorems 2.1 and 2.3 below, we will state more precisely the estimates for the upper and lower bounds, respectively, and the assumptions necessary for each of these. In Sect. 2 below, we will give an outline of the paper as well as these precise statements.

The first term $4\pi\rho^2 Y$ in (1.6) was understood in [1] but a full proof was only given in 2001 in the paper [2]. Calculations beyond leading order were given in [3–6], but have so far not been rigorously proven. The recent papers [7,8] give an analogous expansion of the ground state energy in the setting of the Gross–Pitaevskii regime, giving furthermore information about the excitation spectrum. The constant in the second order term was also found in [9] by restricting to quasi-free states in a special scaling regime.

In the 3-dimensional case, the asymptotic formula for the energy density (with $e^{3D}(\rho)$ defined analogously to (1.2) and a being here the 3-dimensional scattering length) is

$$e^{3D}(\rho) = 4\pi a\rho^2 \left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3}\right) + o(a\rho^2\sqrt{\rho a^3}). \tag{1.8}$$

This is the famous Lee–Huang–Yang formula. The leading order term goes back to [10], and the second term—the Lee–Huang–Yang (LHY) term—were given in [11,12]. Mathematically rigorous proofs of the leading order term were given in [13] (upper bound) and [14] (matching lower bound). Upper bounds for sufficiently regular potentials to the precision of the LHY-term were given in [15] (correct order only), [16] (first upper bound with correct constant on the LHY-term) with recent improvements in [17]. Lower bounds of second order were given in [18] (potentials in L^1) and [19] (general case including the hard core potential). The upper bound in 3-dimensions in the case of potentials with large L^1-norm, in particular the key example of hard core potentials, is still open.

As can be understood from this overview of results from the analysis of the $3D$ case, it is difficult to prove precise results on the energy when $\int_{\mathbb{R}^3} v$ is much larger than the scattering length $a(v)$, i.e., the hard core case. In 2-dimensions the analogous comparison is between $\int_{\mathbb{R}^2} v$ and δ_0, which always satisfy $\int_{\mathbb{R}^2} v \gg \delta_0$. So in 2-dimensions we face similar challenges as in the 3D hard core case, even for regular potentials. This is one of the reasons why progress on the 2D problem has been slower. It is therefore remarkable that Theorem 1.1 can be established, including both upper and lower bounds, without any extra assumptions on the potentials. Also, the $2D$ case comes with its own challenges due to the logaritmic divergences and changes of the lengthscales. In particular, the small parameter in $3D$ is (ρa^3), i.e. it is a power of the density parameter, whereas in the present $2D$ case, our small parameter is $Y = |\log(\rho a^2)|^{-1}$ which is logarithmic in the density.

Throughout the paper we will use the standard convention that $C > 0$ will denote an arbitrarily large universal constant whose value can change from one line to the other. *Notation* We will use the following notation for Fourier transforms,

$$\widehat{f}(p) = \widehat{f}_p = \int e^{-ixp} f(x)\,\mathrm{d}x.$$

In the paper we will use the notation $A \ll B$ in a precise sense given by (H1).

2. Strategies of the Proofs

2.1. Upper bound. As upper bound we prove the following theorem.

Theorem 2.1. *For any constants C_0, $\eta_0 > 0$, there exists C (that depends only on C_0 and η_0) such that the following holds. Let $v : \mathbb{R}^2 \to [0, \infty]$ be a non-negative, measurable and radial potential with scattering length $a < \infty$, and satisfying the following decay property,*

$$v(x) \leq \frac{C_0}{|x|^2}\left(\frac{a}{|x|}\right)^{\eta_0} \quad for \quad |x| \geq C_0 a. \tag{2.1}$$

Then, if $\rho a^2 < C^{-1}$,

$$e^{2D}(\rho) \leq 4\pi \rho^2 \delta_0 \left(1 + \left(2\Gamma + \frac{1}{2} + \log(\pi)\right)\delta_0\right) + C\rho^2 \delta_0^3 |\log(\delta_0)|,$$

with δ_0 given by (1.5).

In order to prove Theorem 2.1, we will reduce the analysis to the case of compactly supported potentials on a smaller periodic box $\Lambda = \Lambda_\beta = [-\frac{L_\beta}{2}, \frac{L_\beta}{2}]^2$ with length

$$L_\beta = \rho^{-1/2} Y^{-\beta}, \qquad \beta > 0. \tag{2.2}$$

In this box, if the density is ρ, the number of particles is $N = \rho L_\beta^2 = Y^{-2\beta} \gg 1$. Throughout the paper we find conditions on β over which we will optimize. For a potential v with supp $v \subseteq B(0, \frac{L_\beta}{2})$, we consider the following Hamiltonian acting on the Fock space $\mathscr{F}_s(L^2(\Lambda_\beta))$,

$$\mathcal{H}_v = \bigoplus_{n \geq 0} \left(\sum_{i=1}^{n} -\Delta_{x_i}^{\text{per}} + \sum_{1 \leq i < j \leq n} v^{\text{per}}(x_i - x_j) \right). \tag{2.3}$$

Here Δ^{per} is the periodic Laplacian, and $v^{\text{per}}(x) = \sum_{m \in \mathbb{Z}^2} v(x + L_\beta m)$ is the periodic version of v. Note that for any $p \in \frac{2\pi}{L_\beta}\mathbb{Z}^2$, the Fourier coefficient of v^{per} is equal to the Fourier transform $\widehat{v}(p)$, because the radius of the support of v is smaller than L_β. In this setting we prove the following result.

Theorem 2.2. *For any $\beta \geq \frac{3}{2}$, there exists $C > 0$, depending only on β such that the following holds. Let $\rho > 0$ and $v : \mathbb{R}^2 \to [0, \infty]$ be a non-negative, measurable and radial potential with scattering length a and supp $v \subset B(0, R)$ for some $R > 0$. If $\rho R^2 \leq Y^{2\beta+2}$ and $\rho a^2 \leq C^{-1}$, then there exists a normalized trial state $\Psi \in \mathscr{F}_s(L^2(\Lambda_\beta))$, such that,*

$$\langle \mathcal{H}_v \rangle_\Psi \leq 4\pi L_\beta^2 \rho^2 \delta_0 \left(1 + \left(2\Gamma + \frac{1}{2} + \log(\pi)\right)\delta_0\right) + C L_\beta^2 \rho^2 \delta_0^3 |\log(\delta_0)|.$$

Moreover Ψ satisfies $\langle \mathcal{N} \rangle_\Psi \geq N(1 - CY^2)$, and $\langle \mathcal{N}^2 \rangle_\Psi \leq 9N^2$, where $\mathcal{N}$ is the number operator on $\mathscr{F}_s(L^2(\Lambda_\beta))$ and $N = \rho L_\beta^2 = Y^{-2\beta}$.

2.1.1. Strategy for the upper bound

1. We will show in "Appendix A" how Theorem 2.1 follows from Theorem 2.2. This corresponds to go from the result on the box Λ_β to the thermodynamic limit.
2. The rest of the proof, Sects. 4 and 5, is dedicated to the proof of Theorem 2.2. We first prove in Sect. 4 a weaker upper bound with the assumption that the potential is regular enough. We call it a *soft* potential. Under this assumption, we use a quasi-free trial state Φ built over a Weyl transform W_{N_0} to create the condensate and a unitary T_ν to deal with the excitations. We then minimize over the parameters of this state. This is an adaptation of the method of [15,20–22] to the 2D case. We show in Theorem 4.1 that, with a good choice of Φ to our level of precision, we have

$$\langle \mathcal{H}_v \rangle_\Phi \leq 4\pi L_\beta^2 \rho^2 \delta_0 \Big(1 + \Big(2\Gamma + \frac{1}{2} + \log(\pi) \Big) \delta_0 \Big)$$
$$+ C L_\beta^2 \rho^2 \delta_0 (\widehat{v_0} - \widehat{g_0}) + C L_\beta^2 \rho^2 \delta_0^2 \widehat{v_0}. \tag{2.4}$$

Here $g = \varphi v$ and φ is the scattering solution associated to v (see Sect. 3 for the precise definition of φ and with parameter δ_0). This provides a first upper bound, but it is not enough to prove Theorem 2.2, unless v admits a Fourier transform and $\widehat{v_0}$ is of order $\widehat{g_0}$.
3. In Sect. 5 we explain how to reduce from any v to a soft potential. To this end, we take care of the influence of the potential on a much shorter length scale by introducing φ_b as the scattering solution normalized at

$$b = \rho^{-\frac{1}{2}} Y^{\beta + \frac{1}{2}} \tag{2.5}$$

and use it to build a *Jastrow function* as follows

$$F_n(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} f(x_i - x_j), \tag{2.6}$$

with $f = \min(1, \varphi_b)$. Then our complete trial state will be the following product state

$$\Psi = \bigoplus_{n \geq 1} F_n \Phi_n, \quad F_n, \Phi_n \in L_s^2(\Lambda_\beta^n), \tag{2.7}$$

where $\Phi = \sum \Phi_n$ is a quasi-free state. When we compute the energy of such a state Ψ we get

$$\langle \mathcal{H}_v \rangle_\Psi \leq \langle \mathcal{H}_{\widetilde{v}} \rangle_\Phi + \langle \mathcal{R} \rangle_\Phi, \tag{2.8}$$

where $\mathcal{R}$ is an error term and $\widetilde{v}$ is the following soft potential,

$$\widetilde{v} = 2 f'(b) \delta_{\{|x|=b\}}. \tag{2.9}$$

The power of Y driven by the parameter β in b is chosen minimal such that $\|\Psi\|^2 = \|\Phi\|^2 + O(Y^2)$, see Lemma 5.3. For the result to apply for the widest range of potentials we will want to choose β as small as possible.

The potential in (2.9) is soft in the sense that it has a decaying Fourier transform and $\widehat{\widetilde{v}}_0 \simeq \widehat{\widetilde{g}}_0$ (see Lemma 3.10 for precise estimates). Then we can take for Φ the optimal quasi-free state satisfying (2.4) for $\widetilde{v}$ and this turns out to be enough to prove Theorem 2.2.

2.1.2. Remarks Since the Jastrow factor (2.6) encodes all 2-particle interactions—at least on short scales—it is a natural trial state for getting upper bounds on the energy. In particular, it has been used to get the correct first order upper bound, both in 3D [13] and 2D [2]. In the product state Ψ, the Jastrow factor deals with short distance correlations between particles (when $|x_i - x_j| \leq b$), while long range effects are dealt with by the quasi-free state Φ. In the case of hard core potentials, the Jastrow factor also imposes the necessary condition that our state vanishes whenever two particles are too close.

We emphasize the following major differences between 2D and 3D. To be able to reduce to the quasi-free state Φ, we need to bound $\mathcal{O}(N^2)$ terms of the form $f(x_i - x_j)$ by 1. The number of particles N in our box is not too large (powers of $|\log(\rho a^2)|$, since the relevant length-scale is $\rho^{-1/2}$ up to logarithmic factors) thus making this error controllable. This is not at all the case in dimension 3, because the number of particles in the box is of order $(\rho a^3)^{-2}$ (since the relevant length-scale in this case is $\frac{1}{\rho a^2}$). However, a similar state as ours was successfully used in the 3D Gross–Pitaevskii regime [22] (length-scale $\frac{1}{\sqrt{\rho a}}$). In this regime the number of particles is $(\rho a^3)^{-\frac{1}{2}}$, which allows the authors, with substantially more work, to get through to a good upper bound. More precisely, they use more accurate bounds on the Jastrow factor compared to our Sect. 5 and obtain the LHY order in the box. See Remark 5.1 for additional information.

Finally, one should notice that Φ is a quasi-free state, and does not include the soft pair interactions that were necessary in [16,17] to get the correct upper bound in 3D. Indeed, for a quasi-free state Φ the second order energy bounds are in terms of $\widehat{v}_0$ and to get the correct constant one needs to change $\widehat{v}_0$'s into $\widehat{g}_0$'s. This is the role of soft pairs. However, our potential $\widetilde{v}$ from (2.9) already satisfies $\widehat{\widetilde{v}}_0 - \widehat{\widetilde{g}}_0 = \mathcal{O}(Y^2|\log Y|)$ (see Lemma 3.10) and this replacement only gives errors of order $\rho^2 Y^3|\log Y|$. It is possible that we could add the soft pair interactions into Φ to reduce this error at the expense of a much longer and more technical proof.

We conclude this section by proving Theorem 2.1 using Theorem 2.2 and the classical theory of localization to smaller boxes which is added for convenience in "Appendix A".

2.2. Lower bound. In this section we provide the strategy of proof for the theorem below.

Theorem 2.3. *For any constant $\eta_1 > 0$ there exist $C, \eta > 0$ (depending only on η_1) such that the following holds. Let $\rho > 0$ and $v : \mathbb{R}^2 \to [0, +\infty]$ be a non-negative, measurable and radial potential with scattering length $a < \infty$. If $\rho a^2 < C^{-1}$ and*

$$\int_{\{|x|>\rho^{-1/2}\}} v(x) \log\left(\frac{|x|}{a}\right)^2 dx \leq Y^{\eta_1}, \tag{2.10}$$

then

$$e^{2D}(\rho) \geq 4\pi\rho^2\delta_0\left(1 + \left(2\Gamma + \frac{1}{2} + \log(\pi)\right)\delta_0\right) + C\rho^2\delta_0^{2+\eta}, \tag{2.11}$$

with δ_0 as defined in (1.5).

We introduce the lengths

$$\ell = \rho^{-1/2}Y^{-1/2-\alpha}, \qquad \ell_{\delta_0} = \frac{1}{2}e^\Gamma\rho^{-1/2}Y^{-1/2}, \tag{2.12}$$

for a certain $\alpha \in (0, 1)$, the second of which being called the *healing length*. The proof of Theorem 2.3 will depend on a precise choice of a number of parameters. For convenience these and the relations between them have been collected in "Appendix H".

We work at three different lengthscales:

- the thermodynamical scale, in the box $\Omega = [-L/2, L/2]^2$, where we state the main result in the limit $L \to +\infty$;
- the large box scale $\Lambda = [-\ell/2, \ell/2]^2$, where we prove most of the results and by the sliding localization techniques we integrate over all these boxes to prove the lower bound in the whole thermodynamical box;
- the small box scale $B = [-d\ell/2, d\ell/2]^2$, with $d \ll 1$, where we derive a bound for the number of particles excited out from the condensate, fundamental for the general strategy, obtaining the Bose–Einstein condensation (BEC).

The relations

$$d\ell \ll \ell_{\delta_0} \ll \ell \ll L, \tag{2.13}$$

guarantee that the boxes are in a chain of inclusions.

2.2.1. Strategy for the lower bound The overall strategy for the lower bound has the same structure as in the $3D$ hard core case analyzed in [19]. Therefore, many of the steps below are the same as in that case. We will only indicate when a step differs from its $3D$ counterpart. However, the $2D$ case comes with its own challenges due to the logaritmic divergences and changes of the lengthscales.

1. In Sect. 6.1 we reformulate the problem in a grand canonical setting, adding a chemical potential ρ_μ to the Hamiltonian, in order to control the distribution of particles in later localization steps. The resulting Hamiltonian $\mathcal{H}_{\rho_\mu}$ acts on the symmetric Fock space $\mathcal{F}_s(L^2(\Omega))$. We also reduce the analysis to compactly supported potentials with norm[1] $\|v\|_1 \leq Y^{-1/8}$, using the analysis of the scattering equation from Sect. 3 (the details of this part are different from the $3D$ case). Theorem 2.3 is shown to be a consequence of Theorem 6.1.

2. In order to prove Theorem 6.1, in Sect. 6.2 we use a sliding localization technique to reduce the problem from the thermodynamical box Ω to the large box Λ. The result of this procedure is an inequality of the form (in the quadratic form sense)

$$\mathcal{H}_{\rho_\mu} \geq \int_{\mathbb{R}^2} \mathcal{H}_{\Lambda_u}(\rho_\mu)\, du,$$

where $\mathcal{H}_{\Lambda_u}(\rho_\mu)$ is a Hamiltonian localized to a box Λ_u which is the translation of the fixed box Λ to be centered at u. The main result is then reduced to the proof of an analogous lower bound for $\mathcal{H}_\Lambda(\rho_\mu)$, namely Theorem 6.7. The next sections focus on this proof.

3. We split the potential energy on the large box in Sect. 7.1 by means of projectors P and Q onto and outside the condensate, respectively, or in other words onto the zero momentum sector and its complement. The splitting produces terms involving from 0 to 4 Q projectors. This is similar to the approach in [11,12].

   By an algebraic identity (see Lemma 7.1), we identify a positive term $\mathcal{Q}_4^{\text{ren}}$ that can be discarded for a lower bound. This procedure also changes the terms with 0 to 3

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[1] The power $\frac{1}{8}$ is not optimal but chosen for convenience, in particular to be in agreement with (H24).

Q's. By this procedure, all occurrences of the potential v are replaced by the function g related to the scattering equation and to the parameter δ_0. This idea has its roots in [23] and was a key step in [18,19]. Since $\widehat{g}(0) = 8\pi\delta_0 \ll \widehat{v}(0)$, this can be interpreted as a renormalization procedure.

4. In [16] it was understood how the interaction of the so-called *soft pairs* contributes significantly to the energy. These correspond to two interacting high-momenta producing one 0-momentum and one low-momentum. This is the main contribution of the $3Q$ term. The soft pairs appear after estimating the other parts of the $3Q$ term to be of lower order. This is done in two steps, the first (restriction to low outgoing momentum) is proved in Sect. 7.2 and the second one (high incoming momenta) in Lemma 8.2, the latter being easier treated in second quantization.

5. A key step in both 2 and 3 dimensions is to be able to focus on states where the operator counting the number of excitations satisfies a norm bound. To handle the 3 dimensional hard-core case, in [19] it was realized that such a bound is only possible when restricting to excitations with low momentum. In the 2 dimensional case, we face this difficulty even if the potential v has small integral (i.e., it is soft).

The reason for this difficulty is that the bound on the excitations involves the integral of v, i.e. $\widehat{v}(0)$. This has to be compared to the main term of the energy, where the relevant parameter is $\widehat{g}(0)$, and as previously noticed, in 2 dimensions $\widehat{g}(0) = 8\pi\delta_0 \ll \widehat{v}(0)$. The solution to this problem follows the same general approach as in [19], namely to not bound all excitations but only those with low momentum. This is the result of Theorem 7.7. The analysis for this bound is carried out in Sect. 7.3 and based on estimates on Bose–Einstein Condensation from Theorem 7.6 proven in "Appendix D". Some other important ingredients of the proof are delegated to "Appendix E". Theorem 7.7 and its proof are somewhat simpler and more along the lines of an IMS-localization estimate than the ones in [19].

6. Section 8 contains lower bounds that use a second quantization formalism in momenta space. We first write the Hamiltonian in this formalism in Sect. 8.2. Then we use the c-number substitution in Sect. 8.3, thus reducing to a problem of minimization for particles outside the condensate. The operators related to the condensate act as numbers over the class of coherent states over which we minimize. After this procedure we arrive at an operator containing terms of order up to 3 creation and annihilation operators of non-zero momenta.

7. In Sect. 9 we distinguish the two cases where the density of particles in the condensate ρ_z is far from or close to ρ_μ, the expected density. Since we have Bose–Einstein condensation, we expect on physical grounds to be in the second case, and indeed fairly rough bounds suffice in the first case. These are given in Sect. 9.1. In the second case, $\rho_z \approx \rho_\mu$, a more careful analysis is needed. We use standard techniques, collected in "Appendix B", to diagonalize the main quadratic part of the Hamiltonian the ground state energy of which appears as an integral.

This integral is calculated in "Appendix C", and we show how together with the constant term of the Hamiltonian, we get the energy to the desired precision.

What remains at this point is to show how the remainders, including the localized $3Q$ term, are error terms, and this is the content of the technical Sect. 9.3. There we show how the contribution of the soft pairs is compensated by the remaining quadratic part of the Hamiltonian. Here in particular, the logarithmic divergencies specific to the 2-dimensional situation makes many estimates delicate and require extra localizations in momentum space.

8. Finally, in Sect. 9.4 we use all the previous results to give a proof of Theorem 6.7, with the choices of the parameters in "Appendix H", where all the conditions used to prove the lower bound are collected.
9. In the proof we need two technical estimates, namely (8.12), (8.13) and (E1), which are taken from the 3D case and are independent of dimension. They are only stated and we refer to [19] for the proof.

3. The Scattering Solution in 2 Dimensions

3.1. Basic theory. In this section we establish the notation and results surrounding the two dimensional two body scattering problem. The standard properties of the scattering solutions stated below are well known and can be found in [24, Appendix A]. We will only consider radial and positive potentials $v : \mathbb{R}^2 \to [0, \infty]$, furthermore if v is compactly supported we denote by R the radius of the support of v, i.e., $v(x) = 0$ if $|x| \geq R$.

Definition 3.1. For a compactly supported v its scattering length $a = a(v)$ is defined as

$$\frac{2\pi}{\log(\frac{\widetilde{R}}{a})} = \inf \left\{ \int_{B(0,\widetilde{R})} \left(|\nabla u|^2 + \frac{1}{2} v |u|^2 \right) dx \ \middle| \ u \in H^1(B(0, \widetilde{R})), \quad u|_{\partial B(0,\widetilde{R})} = 1 \right\}, \tag{3.1}$$

where $\widetilde{R} > R$ is arbitrary.

By the positivity of the right hand side we find $a \leq R$. It is also easy to verify that a is an increasing function of v and is independent of $\widetilde{R} > R$. Furthermore for any $\widetilde{R}$ the above functional has a unique minimizer $\varphi_{v,\widetilde{R}} = \log(\frac{\widetilde{R}}{a(v)})^{-1} \varphi_v^{(0)}(x)$, where, for $v \in L^1(\mathbb{R}^2)$, we have

$$-\Delta \varphi_v^{(0)} + \frac{1}{2} v \varphi_v^{(0)} = 0 \quad \text{on } \mathbb{R}^2, \tag{3.2}$$

in the distributional sense. Furthermore,

$$\varphi_v^{(0)}(r) = \log\left(\frac{r}{a(v)}\right), \quad \text{for } r \geq R,$$

and $\varphi_v^{(0)}$ is a monotone, non-decreasing and non-negative, radial function. We will omit the v in the notation of the scattering length if the potential is clear from the context.

The logarithm in the $2D$-scattering solution is clearly unbounded for large values of r. This is a major difference to the $3D$ behaviour (where the scattering solution behaves as $1 - \frac{a}{r}$ at infinity). Therefore the scattering solution normalized to 1 at a certain length $\widetilde{R}$ is of much greater importance. Using the parameter

$$\delta = \frac{1}{2} \log\left(\frac{\widetilde{R}}{a}\right)^{-1}, \quad \text{i.e.} \quad \widetilde{R} = a e^{\frac{1}{2\delta}}, \tag{3.3}$$

we define on $\mathbb{R}^2$

$$\varphi = \varphi_{v,\delta} = 2\delta \varphi^{(0)}, \quad \omega = 1 - \varphi, \quad g = v\varphi = v(1 - \omega). \tag{3.4}$$

Clearly,

$$-\Delta\omega = \frac{1}{2}g, \tag{3.5}$$

and, using the divergence theorem,

$$\int g \, \mathrm{d}x = 8\pi\delta. \tag{3.6}$$

We remark here again a difference between the $2D$ and $3D$ case: in $3D$, φ would be normalized to 1 at infinity and (3.6) would have an a instead of δ.

Remark 3.2. (On the parameters δ and $\widetilde{R}$) We clearly have some freedom in the choice of δ, which amounts to determine a normalization lengthscale $\widetilde{R}$ for φ. Throughout the paper, we will need δ to be of the same order as $Y = |\log(\rho a^2)|^{-1}$, namely

$$\frac{Y}{2} \leq \delta \leq 2Y, \quad \text{or, equivalently,} \quad (\rho a^2)^{-1/4} \leq \frac{\widetilde{R}}{a} \leq (\rho a^2)^{-1}. \tag{3.7}$$

With this condition we can always exchange Y and δ when estimating errors. We thus get upper and lower bounds on the energy depending on the parameter δ. In both cases, it turns out that the optimal choice is given by (1.5), i.e.

$$\delta = \delta_0 = |\log(\rho a^2 |\log(\rho a^2)|^{-1})|^{-1}, \tag{3.8}$$

which corresponds to

$$\frac{\widetilde{R}}{a} = (\rho a^2 Y)^{-1/2}. \tag{3.9}$$

See also Remarks 4.9 and C.4.

3.2. Potentials without compact support.

Definition 3.3. For a potential v without compact support the scattering length is defined as

$$a(v) = \lim_{n\to\infty} a(v\mathbb{1}_{B(0,n)}).$$

Since a is an increasing function of v the limit exits if and only if $\{a(v\mathbb{1}_{B(0,n)})\}_n$ is bounded, which by [25, Lemma 1] is true if and only if there exists a $\widetilde{b} > 0$ such that

$$\int_{\{|x|>\widetilde{b}\}} v(x) \log\left(\frac{|x|}{\widetilde{b}}\right)^2 \mathrm{d}x < \infty.$$

We need to localize our potentials to have compact support. The next result estimates the change this localization induces in the scattering length.

Lemma 3.4. *For a potential v with finite scattering length a and $R > a$, let $v_R = \mathbb{1}_{B(0,R)}v$ and a_R be its associated scattering length. Then,*

$$0 \leq \frac{2\pi}{\log(\frac{R}{a})} - \frac{2\pi}{\log(\frac{R}{a_R})} \leq \frac{1}{2}\int_{\{|x|>R\}} v(x)\frac{\log(\frac{|x|}{a})^2}{\log(\frac{R}{a})^2} \mathrm{d}x. \tag{3.10}$$

Proof. Let φ_1 be the scattering solution for v_R normalized at R, and let

$$\varphi_n(x) := \begin{cases} \varphi_1(x)\frac{\log(\frac{R}{a})}{\log(n)}, & |x| \leq R, \\ \frac{\log(\frac{|x|}{a})}{\log(n)}, & |x| \geq R. \end{cases} \tag{3.11}$$

Notice that φ_n is normalized at $a \cdot n$ and continuous. We use it as a trial function in the variational problem of $v_n = \mathbb{1}_{B(0,a\cdot n)}v$, with $n \cdot a > R$, to get (with $a_n := a(v_n)$)

$$\frac{2\pi}{\log(\frac{a\cdot n}{a_n})} \leq \int_{\{|x|<R\}} \left(|\nabla\varphi_n|^2 + \frac{1}{2}v_n\varphi_n^2\right)dx + \int_{\{R<|x|<a\cdot n\}} \left(|\nabla\varphi_n|^2 + \frac{1}{2}v_n\varphi_n^2\right)dx. \tag{3.12}$$

Since φ_n is just a multiple of the scattering solution of φ_1 inside R the first integral gives

$$\int_{\{|x|<R\}} \left(|\nabla\varphi_n|^2 + \frac{1}{2}v_n\varphi_n^2\right)dx = \frac{2\pi}{\log(\frac{R}{a_R})}\frac{\log(\frac{R}{a})^2}{\log(n)^2}. \tag{3.13}$$

The second term is directly calculated using the explicit formula for φ_n,

$$\int_{\{R<|x|<a\cdot n\}} \left(|\nabla\varphi_n|^2 + \frac{1}{2}v_n\varphi_n^2\right)dx$$
$$\leq \frac{2\pi \log(\frac{a\cdot n}{R})}{\log(n)^2} + \frac{1}{2\log(n)^2}\int_{\{|x|>R\}} v\log\left(\frac{|x|}{a}\right)^2. \tag{3.14}$$

By (3.13), multiplying (3.12) through with $\log(n)^2$ and letting $n \to \infty$, whereby $a_n \to a$, yields

$$2\pi \log\left(\frac{R}{a}\right) \leq \frac{2\pi \log(\frac{R}{a})^2}{\log(\frac{R}{a_R})} + \frac{1}{2}\int_{\{|x|>R\}} v\log\left(\frac{|x|}{a}\right)^2 dx.$$

The result then follows by dividing through with $\log(\frac{R}{a})^2$. $\qquad\square$

3.3. Compactly supported potentials with large integrals.
We state and prove here in the $2D$ setting a similar approximation result as the one found in [19, Theorem 1.6] for the scattering length in $3D$.

Lemma 3.5. *For a radial, positive $v \in L^1(\mathbb{R}^2)$ with support contained in $B(0, R)$ there exists, for any $T > 0$, a $v_T : \mathbb{R}^2 \to [0, +\infty]$ satisfying*

$$0 \leq v_T(x) \leq v(x), \quad \text{for all } x \in \mathbb{R}^2, \text{ and} \quad \int v_T \leq 4\pi T, \tag{3.15}$$

and such that

$$\frac{2\pi}{\log(\frac{R}{a})} - \frac{2\pi}{\log(\frac{R}{a_T})} \leq \frac{2\pi}{\log(\frac{R}{a})^2 T}. \tag{3.16}$$

Proof. Due to the integrability assumption on v we may define

$$R_T = \inf\left\{ R' > 0 : \int_{\{|x| \geq R'\}} v \mathrm{d}x < 4\pi T \right\}$$

and

$$v_T := v \mathbb{1}_{\{|x| > R_T\}}. \tag{3.17}$$

Clearly,

$$\int v_T = 4\pi T. \tag{3.18}$$

Also, we may assume $R_T > 0$. Otherwise there is nothing to prove.

Let φ be the scattering solution of v and φ_T the scattering solution of v_T both normalized at $\widetilde{R} > R$. We have from (3.6), using that φ_T is a non-decreasing function,

$$\frac{4\pi}{\log(\frac{\widetilde{R}}{a_T})} = \int v_T \varphi_T \mathrm{d}x \geq \varphi_T(R_T) \int v_T = 4\pi \varphi(R_T) T,$$

and hence

$$\varphi(R_T) \leq \frac{1}{\log(\frac{\widetilde{R}}{a_T}) T}. \tag{3.19}$$

Next we define

$$u = \mathbb{1}_{\{|x| > R_T\}}(\varphi_T - \omega_T \varphi_T(R_T)) \quad \text{where} \quad \omega_T(x) = 1 - \frac{\log(\frac{|x|}{R_T})}{\log(\frac{\widetilde{R}}{R_T})}.$$

Observe that $u(\widetilde{R}) = 1$ and we may therefore apply it as a trial function in the functional for a to get

$$\frac{2\pi}{\log(\frac{\widetilde{R}}{a})} \leq \int_{\{|x| < \widetilde{R}\}} \left(|\nabla u|^2 + \frac{1}{2} vu^2 \right) \mathrm{d}x := E_1 + E_2 + E_3, \tag{3.20}$$

with

$$E_1 = \int_{\{R_T < |x| < \widetilde{R}\}} \left(|\nabla \varphi_T|^2 + \frac{1}{2} v_T \varphi_T^2 \right) \mathrm{d}x = \frac{2\pi}{\log(\frac{\widetilde{R}}{a_T})},$$

$$E_2 = -2\varphi_T(R_T) \int_{\{R_T < |x| < \widetilde{R}\}} \left(\nabla \varphi_T \nabla \omega_T + \frac{1}{2} v_T \varphi_T \omega_T \right) \mathrm{d}x = 0,$$

$$E_3 = \varphi_T(R_T)^2 \int_{\{R_T < |x| < \widetilde{R}\}} \left(|\nabla \omega_T|^2 + \frac{1}{2} v_T \omega_T^2 \right) \mathrm{d}x.$$

For E_2 we integrated by parts and used that φ_T is harmonic inside $B(0, R_T)$, thus constant, which makes the boundary term vanish. For E_3 we use that $\omega_T \leq 1$ on the given interval, so combining (3.18), (3.19) and (3.20) yields

$$\frac{2\pi}{\log(\frac{\widetilde{R}}{a})} - \frac{2\pi}{\log(\frac{\widetilde{R}}{a_T})} \leq E_3 \leq \frac{2\pi}{\log(\frac{\widetilde{R}}{a_T})^2} \left(\frac{1}{\log(\frac{\widetilde{R}}{R_T}) T^2} + \frac{1}{T} \right). \tag{3.21}$$

Using that $a \geq a_T$ we may replace a_T with a on the right hand side. Secondly, we observe that the function

$$\left(\frac{1}{\log(\frac{\widetilde{R}}{a})} - \frac{1}{\log(\frac{\widetilde{R}}{a_T})}\right) \log\left(\frac{\widetilde{R}}{a}\right)^2,$$

is increasing in $\widetilde{R}$ so we may replace $\widetilde{R}$ with R in the above expression and use (3.21) to get

$$\frac{2\pi}{\log(\frac{R}{a})} - \frac{2\pi}{\log(\frac{R}{a_T})} \leq \frac{2\pi}{\log(\frac{R}{a})^2}\left(\frac{1}{\log(\frac{\widetilde{R}}{R_T})T^2} + \frac{1}{T}\right).$$

Now the result follows by letting $\widetilde{R}$ go to infinity. $\square$

We are ready to prove the main theorem of this section which gives us the ability to deal with a wide range of potentials including, most notably, the hard core.

Theorem 3.6. *For a radial, positive potential $v : \mathbb{R}^2 \to [0, \infty]$ with finite scattering length a there exists, for any $R > a$ and $T, \epsilon > 0$, a potential $v_{T,R,\epsilon}$ such that*

$$\text{supp}(v_{T,R,\epsilon}) \subset B(0, R), \qquad 0 \leq v_{T,R,\epsilon}(x) \leq v(x), \qquad \int v_{T,R,\epsilon} \leq 4\pi T, \text{ (3.22)}$$

and its scattering length $a_{T,R,\epsilon}$ satisfies

$$\frac{2\pi}{\log(\frac{R}{a})} - \frac{2\pi}{\log(\frac{R}{a_{T,R,\epsilon}})} \leq \frac{1}{\log(\frac{R}{a})^2}\left(\frac{2\pi}{T}(1 + \epsilon) + \frac{1}{2}\int_{\{|x|>R\}} v(x) \log\left(\frac{|x|}{a}\right)^2 \right)dx.$$
$$\text{(3.23)}$$

Proof. Lemma 3.5 applied to $v_R^n = \mathbb{1}_{B(0,R)} \min(n, v)$ yields a $v_{R,T}^n$ satisfying all three conditions of (3.22) and

$$\frac{2\pi}{\log(\frac{R}{a_R^n})} - \frac{2\pi}{\log(\frac{R}{a_{T,R}^n})} \leq \frac{2\pi}{\log(\frac{R}{a})^2 T}, \tag{3.24}$$

for all $n \in \mathbb{N}$. In the above we used that $a_{T,R}^n \leq a_R^n \leq a_R \leq a$ (where $a_{T,R}^n, a_R^n, a_R$ are the scattering lengths of $v_{T,R}^n, v_R^n, v_R$, respectively). Choosing n_0 large enough such that $a_R^{n_0}$ is close enough to a_R gives an $a_{T,R,\epsilon} := a_{T,R}^{n_0}$ satisfying

$$\frac{2\pi}{\log(\frac{R}{a_R})} - \frac{2\pi}{\log(\frac{R}{a_{T,R,\epsilon}})} \leq \frac{2\pi}{\log(\frac{R}{a})^2 T}(1 + \epsilon). \tag{3.25}$$

We conclude using (3.10) which gives the integral term of (3.23). $\square$

3.4. Fourier analysis on the scattering equation. Due to Theorem 3.6 we may assume our potentials to be compactly supported and L^1, thus making the Fourier transform well defined. The scattering solution φ will be the one defined in (3.4) which is normalized to 1 outside the support of v. In order to discuss the Fourier transform of the scattering solution, we recall some standard results surrounding the Fourier transform of the logarithm. We denote by $\mathcal{S}$ and $\mathcal{S}'$ the Schwartz space and the space of tempered distribution on $\mathbb{R}^2$, respectively.

Lemma 3.7. *For $D > 0$, let L_D denote the tempered distribution given by the function* $\log(|x|/D)$ *in $\mathbb{R}^2$. The Fourier transform of L_D satisfies for any $h \in \mathcal{S}$*

$$\langle \widehat{L_D}, h \rangle_{\mathcal{S}',\mathcal{S}} = -(2\pi) \int_{\mathbb{R}^2} \frac{h(p) - h(0)\mathbb{1}_{\{|p| \le 2e^{-\Gamma} D^{-1}\}}}{p^2}\, dp, \tag{3.26}$$

where Γ denotes the Euler–Mascheroni constant,

$$\Gamma := -\int_0^\infty e^{-x} \log x\, dx \approx 0.5772. \tag{3.27}$$

The proof is an exercise in distribution theory, with details for instance given in the recent book [26, Theorem 4.73].

It follows from (3.26) that, for any $f \in \mathcal{S}$,

$$\iint_{\mathbb{R}^2 \times \mathbb{R}^2} \overline{f(x)} f(y) \log\left(\frac{|x - y|}{D}\right) dx dy$$
$$= -2\pi \int_{\mathbb{R}^2} \frac{|\widehat{f}(p)|^2 - |\widehat{f}(0)|^2 \mathbb{1}_{\{|p| \le 2e^{-\Gamma} D^{-1}\}}}{p^2}\, dp. \tag{3.28}$$

Using the notation from (3.4) and (3.5), we may compute the Fourier transform of ω. In the $3D$ case one gets that $\widehat{\omega}(p) = \frac{\widehat{g}(p)}{2p^2}$, but in $2D$ this formula has to be corrected by a distribution supported at the origin according to Lemma 3.7, see Lemma 3.8 below.

Lemma 3.8. *Let $\widehat{\omega}$ denote the Fourier transform of ω. Then $\widehat{\omega}$ is the tempered distribution given by*

$$\langle \widehat{\omega}, u \rangle_{\mathcal{S}',\mathcal{S}} = \int \frac{\widehat{g}(p)u(p) - \widehat{g}(0)u(0)\mathbb{1}_{\{|p| \le \ell_\delta^{-1}\}}}{2p^2}\, dp, \tag{3.29}$$

for any $u \in \mathcal{S}$ where, recalling the definition of $\widetilde{R}$ in (3.3),

$$\ell_\delta := \frac{ae^\Gamma}{2} e^{\frac{1}{2\delta}} = \frac{1}{2}e^\Gamma \widetilde{R}. \tag{3.30}$$

Notice that if $\delta = \delta_0$ from (1.5), then ℓ_δ coincides with ℓ_{δ_0} introduced in (2.12).

Proof. We first recall the definition (3.4) of ω and write

$$\omega = -\frac{\widehat{g}(0)}{4\pi} \log\left(\frac{r}{\widetilde{R}}\right) + \widetilde{\omega}, \tag{3.31}$$

where $\widetilde{\omega}$ is compactly supported, and we recall $\widehat{g}(0) = 8\pi \delta$. Hence, using the Fourier transform of the logarithm as recalled in Lemma 3.7,

$$\langle \widehat{\omega}, u \rangle_{\mathcal{S}',\mathcal{S}} = \widehat{g}(0) \int_{\mathbb{R}^2} \frac{u(p) - u(0)\mathbb{1}_{\{|p| \le 2e^{-\Gamma} \tilde{R}^{-1}\}}}{2p^2}\, dp + \int \widehat{\widetilde{\omega}}(p)u(p)\, dp. \tag{3.32}$$

Using the scattering equation (3.5) we find $\widehat{g}(p) = 2p^2\widehat{\omega}(p) = \widehat{g}(0) + 2p^2\widehat{\widetilde{\omega}}(p)$, where we used that the logarithm is the fundamental solution of the Laplacian. Since $\widetilde{\omega}$ is a smooth function we deduce

$$\widehat{\widetilde{\omega}}(p) = \frac{\widehat{g}(p) - \widehat{g}(0)}{2p^2},$$

and this concludes the proof.   □

Thanks to the previous lemma we are able to prove some important properties of $\widehat{g\omega}(0)$ which are going to be key through all the paper.

Lemma 3.9. *The following identity holds*

$$\widehat{g\omega}(0) = \int_{\mathbb{R}^2} \frac{\widehat{g}(k)^2 - \widehat{g}(0)^2 \mathbb{1}_{\{|k| \leq \ell_\delta^{-1}\}}}{2k^2} \, dk \tag{3.33}$$

and, furthermore, the following bounds hold

$$|\widehat{g\omega}(0)| \leq C\delta, \tag{3.34}$$

$$\left| \int_{\{|k| \leq \ell_\delta^{-1}\}} \frac{\widehat{g}(k)^2 - \widehat{g}(0)^2}{2k^2} \, dk \right| \leq C R^2 \delta^2 \ell_\delta^{-2}, \tag{3.35}$$

$$\left| \int_{\{|k| \geq \ell_\delta^{-1}\}} \frac{\widehat{g}(k)^2}{2k^2} \, dk \right| \leq C\delta. \tag{3.36}$$

Proof. Formula (3.33) is formally given by an application of Lemma 3.8 choosing $u = \widehat{g}$. Since $\widehat{g}$ is not a Schwartz function, we need to apply a regularization argument, by truncating in momentum space. This truncation can then be removed at the end and one arrives at (3.33).

The first bound (3.34) follows because in the support of g, $\omega \leq 1$ and $\widehat{g}_0 = 8\pi\delta$. The last bound (3.36) follows once we have proved the second one. In order to do that, we consider a Taylor expansion to the second order of $|\widehat{g}_k - \widehat{g}_0| \leq C|k|^2\|\widehat{g}_k''\|_\infty$ as a radial function, due to the symmetry of g. We use that v has a compact support R and the definition

$$\widehat{g}_k = \int_{\mathbb{R}^2} v\varphi e^{-ik.x} dx \tag{3.37}$$

to bound $\widehat{g}_k''$ by $R^2\widehat{g}_0$ to obtain

$$\left| \int_{\{|k| \leq \ell_\delta^{-1}\}} \frac{\widehat{g}_k^2 - \widehat{g}_0^2}{2k^2} \, dk \right| \leq C R^2 |\widehat{g}_0|^2 \int_{\{|k| \leq \ell_\delta^{-1}\}} dk = C R^2 \delta^2 \ell_\delta^{-2}. \tag{3.38}$$

□

3.5. Spherical measure potentials. For the upper bound, we will change the potential in order to ensure small L^1 norm. For a potential v supported in $B(0, R)$ and $b > R$, let

$$f(x) := \min\left(1, \varphi^{(0)}(x) \log\left(\frac{b}{a}\right)^{-1}\right).$$

Thus, f is the scattering solution in $B(0, b)$ normalized at b and extended by one. The new potential $\widetilde{v}$ will then be described by the deviation of f being the actual scattering solution, i.e.,

$$\widetilde{v} = 2\left(-\Delta f + \frac{1}{2}vf\right), \tag{3.39}$$

where the above equality is to be thought of in a distributional sense. The factor 2 is important and should be thought of as the number of particles involved in the scattering process. A quick calculation shows that

$$\widetilde{v} = 2f'(b)\delta_{\{|x|=b\}} = 2\frac{1}{b\log(\frac{b}{a})}\delta_{\{|x|=b\}}, \tag{3.40}$$

where $\delta_{\{|x|=b\}}$ is the uniform measure on the circle $\{|x| = b\}$ normalized so that $\int \delta_{\{|x|=b\}} = 2\pi b$, and where $f'(b)$ is to be understood as the radial derivative (from the left) of f at length b. We show in Sect. 5 how we reduce to this potential. The simple, but essential properties of $\widetilde{v}$ are stated in the lemma below.

Lemma 3.10. *Let v and $\widetilde{v}$ be given as above. We use the notation $\widetilde{a} = a(\widetilde{v})$ and $a = a(v)$. Furthermore, let $\widetilde{\varphi}$ be the scattering solution of $\widetilde{v}$ normalized at $\widetilde{R} > b$ and $\widetilde{g} = \widetilde{v}\widetilde{\varphi}$. Then*

1 The scattering lengths agree, i.e., $\widetilde{a} = a$.
2 $\widehat{\widetilde{v}}(p) = 2f'(b)bJ_0(b|p|)$, where J_0 is the zeroth spherical Bessel function. In particular there exists a universal constant $C > 0$ such that

$$|\widehat{\widetilde{v}}(p)| \leq C\frac{\widehat{\widetilde{v}}(0)}{\sqrt{b|p|}}. \tag{3.41}$$

3 $\widehat{\widetilde{v}}(0) := \langle v, 1 \rangle = \dfrac{4\pi}{\log(b/a)}$, and $\widehat{\widetilde{g}}(0) = \widehat{\widetilde{v}\widetilde{\varphi}}(0) = \dfrac{4\pi}{\log(\widetilde{R}/a)}.$

Proof. The potential $\widetilde{v}$ is a spherical measure on the sphere $\{|x| = b\}$ and thus $\widetilde{\varphi}$ is harmonic both inside and outside this sphere. We may therefore conclude from the continuity of $\widetilde{\varphi}$ that

$$\log(\widetilde{R}/\widetilde{a})\widetilde{\varphi}(r) = \begin{cases} \log(b/\widetilde{a}), & \text{if } r \leq b, \\ \log(r/\widetilde{a}), & \text{if } r > b. \end{cases} \tag{3.42}$$

From the scattering equation

$$-\Delta\widetilde{\varphi} + \frac{1}{2}\widetilde{v}\widetilde{\varphi} = 0,$$

applied to a $u \in C_c^\infty(\mathbb{R}^2)$ we obtain, using Green's formula,

$$-\int_{\{|x|=b\}} u\nabla\widetilde{\varphi} \cdot d\boldsymbol{n} = f'(b)\widetilde{\varphi}(b)\int_{\{|x|=b\}} u \tag{3.43}$$

and then deduce

$$\widetilde{\varphi}'(b) = f'(b)\widetilde{\varphi}(b) \tag{3.44}$$

where $\widetilde{\varphi}'(b)$ denotes the outgoing radial derivative of $\widetilde{\varphi}$ at length b. Combining (3.42) and (3.44) yields 1. Property 2. is a direct consequence of $\widetilde{v}$ being a uniform measure on the sphere $\{|x|\} = b$ and the behaviour of J_0 at infinity. Finally, the identities in 3.. follow immediately after realizing that

$$g = \widetilde{\varphi}(b)v.$$

$\square$

4. Upper Bound for a Soft Potential

We denote by $\mathscr{M}_c$ the set of potentials of the form $v = v_{\text{reg}} + v_{\text{m}}$, where $v_{\text{reg}} \in L^1(\mathbb{R}^2)$ is radial, positive and has compact support, and where $v_{\text{m}} = C\delta_{\{|x|=r\}}$ for some $C \geq 0$ and $r > 0$. If $v \in \mathscr{M}_c$, it admits a bounded and continuous Fourier transform $\widehat{v}$. The aim of this section is to prove an upper bound on the ground state energy of

$$\mathcal{H}_v = \bigoplus_{n\geq 0}\left(\sum_{i=1}^{n} -\Delta_{x_i}^{\text{per}} + \sum_{1\leq i<j\leq n} v^{\text{per}}(x_i - x_j)\right) \tag{4.1}$$

on the box $\Lambda_\beta = [-\frac{L_\beta}{2}, \frac{L_\beta}{2}]^2$ for potentials $v \in \mathscr{M}_c$, under some additional decay assumption on the Fourier transform of v. We recall that $L_\beta = \rho^{-\frac{1}{2}}Y^{-\beta}$.

In this section, we will denote by φ the scattering solution of the given v, normalized at length $\widetilde{R}$, and $g = \varphi v$, see (3.4). Notice here that the theory of Sect. 3 extends to potentials $v \in \mathscr{M}_c$; for this we use in particular, that if $u \in H^1(\mathbb{R}^2)$, then $u|_{\{|x|=r\}} \in L^2$ so the variational problem in Definition 3.1 is well posed. In particular, the scattering equation (3.2) is valid in the distributional sense. We recall that $0 \leq \widehat{g}_0 = 8\pi\delta \leq CY$ by (3.6) and (3.3). We prove the following upper bound, which is very similar in spirit to the upper bound of [15] in the $3D$ case.

Theorem 4.1. *For any given $c_0 > 0$ and $\beta \geq \frac{3}{2}$, there exists $C_\beta > 0$ (only depending on c_0 and β) such that the following holds. Let $\rho > 0$ and $v \in \mathscr{M}_c$ be a radial positive measure with scattering length a and $\text{supp}\, v \subset B(0, R)$, for some $R > 0$. Let $\mathcal{H}_v$ be as defined in (2.3). Assume that*

$$|\widehat{g}_p| \leq c_0 \frac{\widehat{g}_0}{\sqrt{R|p|}}, \qquad \forall |p| \geq a^{-1}. \tag{4.2}$$

Then, if $\rho R^2 \leq Y$ and $\rho a^2 \leq C_\beta^{-1}$, one can find a normalized trial state $\Phi \in \mathscr{F}_s(L^2(\Lambda_\beta))$ satisfying

$$\langle\mathcal{H}_v\rangle_\Phi \leq 4\pi L_\beta^2\rho^2\delta_0\left(1 + \left(2\Gamma + \frac{1}{2} + \log(\pi)\right)\delta_0\right) + CL_\beta^2\rho^2\delta_0(\widehat{v}_0 - \widehat{g}_0) + CL_\beta^2\rho^2\delta_0^2\widehat{v}_0$$

with $\langle\mathcal{N}\rangle_\Phi = N$, and $\langle\mathcal{N}^2\rangle_\Phi \leq 9N^2$, where $N = \rho L_\beta^2 = Y^{-2\beta}$.

Remark 4.2. Note that this result is much weaker than Theorem 2.2. Indeed, the remainders are only of order $\rho^2 L_\beta^2 \delta_0^2$ and $\rho^2 L_\beta^2 \delta_0$ and thus much larger than the 2D-LHY term, unless $\widehat{v}_0 = \widehat{g}_0 + o(\delta_0)$. Moreover, Theorem 4.1 only holds for potentials with finite integral and, in particular, it does not allow for a hard core. However, in the proof of Theorem 2.2 in Sect. 5 we will show how to reduce to such potentials. More precisely, we will apply Theorem 4.1 to a surface potential of the form (3.40) (with the choice of b given in (2.5)).

Remark 4.3. The specific $\delta = \delta_0$ defined in (1.5) is chosen to minimize the upper bound (4.22) up to the LHY precision. This corresponds to fixing the normalisation length of the soft potential $\widetilde{R} = a e^{\frac{1}{2\delta}}$. See also Remarks 3.2, 4.9 and C.4.

The rest of Sect. 4 is dedicated to the proof of Theorem 4.1. We will give an explicit trial state and state several technical calculations as lemmas. In the end we collect the pieces and finish the proof.

4.1. A quasi-free state.

We will define our trial state Φ in second quantization formalism. On the bosonic Fock space $\mathscr{F}(L^2(\Lambda_\beta))$, we will denote by $a_p^\dagger$ and a_p the creation and annihilation operators associated to the function $x \mapsto |\Lambda_\beta|^{-\frac{1}{2}} \exp(ipx)$, for $p \in \Lambda_\beta^* = (\frac{2\pi}{L_\beta}\mathbb{Z})^2$. Our quasi-free state is $\Phi = T_v W_{N_0} \Omega$ where Ω is the vacuum, W_{N_0} creates the condensate and T_v the excitations:

$$W_{N_0} = \exp\left(\sqrt{N_0}(a_0^\dagger - a_0)\right), \quad T_v = \exp\left(\frac{1}{2}\sum_{p \neq 0} v_p(a_p^\dagger a_{-p}^\dagger - a_p a_{-p})\right), \quad (4.3)$$

for a given $N_0 \leq N$ associated with $\rho_0 := N_0/L_\beta^2$. These operators have the nice properties that

$$W_{N_0}^* a_0 W_{N_0} = a_0 + \sqrt{N_0}, \quad \text{and} \quad T_v^* a_p T_v = \cosh(v_p)a_p + \sinh(v_p)a_{-p}^\dagger. \quad (4.4)$$

In particular, for any $p, q \in \Lambda_\beta^*$,

$$\langle a_q^\dagger a_p \rangle_\Phi = \begin{cases} N_0, & \text{if } p = q = 0, \\ 0, & \text{if } p \neq q, \qquad \text{and} \\ \gamma_q, & \text{if } p = q \neq 0, \end{cases}$$

$$\langle a_q a_p \rangle_\Phi = \langle a_q^\dagger a_p^\dagger \rangle_\Phi = \begin{cases} N_0, & \text{if } p = q = 0, \\ 0, & \text{if } p \neq -q, \\ \alpha_q, & \text{if } p = -q \neq 0, \end{cases} \quad (4.5)$$

where $\alpha_p = \cosh(v_p)\sinh(v_p)$ and $\gamma_p = \sinh(v_p)^2$. We choose the coefficient v_p such that

$$\alpha_p = \frac{-\rho_0 \widehat{g}_p}{2\sqrt{p^4 + 2\rho_0 \widehat{g}_p p^2}}, \quad \gamma_p = \frac{p^2 + \rho_0 \widehat{g}_p - \sqrt{p^4 + 2\rho_0 \widehat{g}_p p^2}}{2\sqrt{p^4 + 2\rho_0 \widehat{g}_p p^2}} \geq 0, \quad (4.6)$$

this specific choice coming from a minimization of the energy

$$(p^2 + \rho_0 \widehat{g}_p)\gamma_p + \rho_0 \widehat{g}_p \alpha_p$$

obtained in Lemma 4.6 up to changing $\widehat{v}$ into $\widehat{g}$. Note that by $(\cosh(x)^2 - \sinh(x)^2) = 1$ we have $\alpha_p^2 = \gamma_p(\gamma_p + 1)$, making it a possible choice. These coefficients satisfy the following estimates.

Lemma 4.4. *We estimate the sum (over Λ_β^*) of α_p and γ_p:*

$$\sum_{p \neq 0} |\alpha_p| \leq CN, \quad and \quad \sum_{p \neq 0} \gamma_p \leq CN\delta. \tag{4.7}$$

Proof. We start from the expression of α_p (4.6) and split the sum between $|p| \leq \sqrt{\rho_0 \widehat{g}_0}$ and $|p| \geq \sqrt{\rho_0 \widehat{g}_0}$:

$$\sum_{p \neq 0} |\alpha_p| \leq C\sqrt{\rho_0 \widehat{g}_0} \sum_{0 < |p| \leq \sqrt{\rho_0 \widehat{g}_0}} \frac{1}{|p|} + C\rho_0 \sum_{|p| \geq \sqrt{\rho_0 \widehat{g}_0}} \frac{|\widehat{g}_p|}{|p|^2}$$

$$\leq CL_\beta^2 \sqrt{\rho_0 \widehat{g}_0} \int_0^{\sqrt{\rho_0 \widehat{g}_0}} du + CL_\beta^2 \rho_0 \widehat{g}_0 \int_{\sqrt{\rho_0 \widehat{g}_0}}^{a^{-1}} \frac{du}{u} + CL_\beta^2 \rho_0 \int_{a^{-1}}^{+\infty} \frac{\widehat{g}_0}{R^{1/2} u^{3/2}} du$$

$$\leq CL_\beta^2 \rho_0 \widehat{g}_0 (1 + |\log(a^2 \rho_0 \widehat{g}_0)|) \leq CN,$$

where we used the decay of $\widehat{g}_p$ at infinity (4.2) and the bound $a \leq R$. For γ_p we also split the sum this way. For $p \leq \sqrt{\rho_0 \widehat{g}_0}$ we obtain that

$$\sum_{|p| \leq \sqrt{\rho_0 \widehat{g}_0}} |\gamma_p| \leq C \sum_{|p| \leq \sqrt{\rho_0 \widehat{g}_0}} \frac{\sqrt{\rho_0 \widehat{g}_0}}{|p|} \leq CL_\beta^2 \rho_0 \widehat{g}_0 \leq CN_0 \delta.$$

For $p \geq \sqrt{\rho_0 \widehat{g}_0}$ we expand the square root and find

$$\sum_{|p| \geq \sqrt{\rho_0 \widehat{g}_0}} |\gamma_p| \leq C \sum_{|p| \geq \sqrt{\rho_0 \widehat{g}_0}} \frac{(\rho_0 \widehat{g}_0)^2}{|p|^4} \leq CL_\beta^2 \rho_0 \widehat{g}_0 \leq CN_0 \delta,$$

which concludes the proof. $\qquad\square$

Finally choose N_0 such that

$$\rho L_\beta^2 = N = N_0 + \sum_{p \neq 0} \gamma_p. \tag{4.8}$$

Note that with this choice Φ has the expected average number of particles as stated in the next lemma.

Lemma 4.5. *The state $\Phi = T_v W_{N_0} \Omega$ satisfies*

$$\langle \mathcal{N} \rangle_\Phi = N, \quad \langle \mathcal{N}^2 \rangle_\Phi \leq CN^2,$$

where $\mathcal{N} = \sum_{p \in \Lambda_\beta^*} a_p^\dagger a_p$ is the number operator.

Proof. First we can use the property (4.5) to find

$$\langle \mathcal{N} \rangle_\Phi = N_0 + \sum_{p \neq 0} \gamma_p = N. \tag{4.9}$$

For $\mathcal{N}^2$ we split the sums according to zero and non-zero momenta, and then conjugate by W_{N_0},

$$\langle \mathcal{N}^2 \rangle_\Phi = \sum_{q,p} \langle a_p^\dagger a_p a_q^\dagger a_q \rangle_\Phi = N_0^2 + N_0 + \sum_{q \neq 0} \langle a_0^\dagger a_0 a_q^\dagger a_q + \text{h.c} \rangle_\Phi + \sum_{q \neq 0, p \neq 0} \langle a_p^\dagger a_p a_q^\dagger a_q \rangle_\Phi$$

$$= N_0^2 + N_0 \Big(1 + 2 \sum_{p \neq 0} \gamma_p \Big) + \sum_{q \neq 0, p \neq 0} \langle a_p^\dagger a_p a_q^\dagger a_q \rangle_\Phi.$$

Now we use Lemma 4.4 and apply Wick's Theorem [27, Theorem 10.2] to the state $T_\nu \Omega$ to find

$$\langle \mathcal{N}^2 \rangle_\Phi \leq 4N^2 + \sum_{q \neq 0, p \neq 0} \Big(\langle a_p^\dagger a_p \rangle_\Phi \langle a_q^\dagger a_q \rangle_\Phi + \langle a_p^\dagger a_q^\dagger \rangle_\Phi \langle a_p a_q \rangle_\Phi + \langle a_p^\dagger a_q \rangle_\Phi \langle a_p a_q^\dagger \rangle_\Phi \Big)$$

$$\leq 4N^2 + \Big(\sum_{p \neq 0} \gamma_p \Big)^2 + \sum_{p \neq 0} \alpha_p^2 + \sum_{p \neq 0} (\gamma_p^2 + \gamma_p) \leq CN^2$$

using Lemma 4.4. $\qquad\square$

4.2. Energy of Φ. In order to get an upper bound on the energy of Φ we first introduce the quantity

$$D(A, B) = \frac{1}{(4\pi^2)^2} \int \widehat{v} * A(p) B(p) \mathrm{d}p = \frac{1}{(4\pi^2)^2} \langle B, \widehat{v} * A \rangle, \tag{4.10}$$

and observe that it is symmetric in the entries. Then we prove the following result.

Lemma 4.6. *Under the assumptions of Theorem 4.1, there exists a constant $C > 0$, independent of v and ρ, such that*

$$|\Lambda_\beta|^{-1} \langle \mathcal{H}_v \rangle_\Phi \leq \frac{\rho^2}{2} \widehat{v}_0 + \int \Big((p^2 + \rho_0 \widehat{v}_p) \gamma_p + \rho_0 \widehat{v}_p \alpha_p \Big) \frac{\mathrm{d}p}{4\pi^2} + \frac{1}{2} D(\alpha, \alpha) + C \widehat{v}_0 \rho^2 Y^3.$$

Proof. One can write $\mathcal{H}_v$ in second quantization in momentum variable,

$$\mathcal{H}_v = \sum_{p \in \Lambda_\beta^*} p^2 a_p^\dagger a_p + \frac{1}{2|\Lambda_\beta|} \sum_{p,q,r} \widehat{v}_r a_{p+r}^\dagger a_q^\dagger a_{q+r} a_p,$$

and express the energy of Φ in terms of α_p and γ_p as follows. We conjugate by W_{N_0} using (4.4), which amounts to change the a_0's in $\sqrt{N_0}$. Since $\Phi = T_\nu W_{N_0} \Omega$ with no a_0

in T_ν (see (4.3)), when we apply Φ we find

$$
\langle \mathcal{H}_\nu \rangle_\Phi = \sum_{p \neq 0} p^2 \langle a_p^\dagger a_p \rangle_\Phi + \frac{N_0^2}{2|\Lambda_\beta|} \widehat{v}_0 + \frac{N_0}{|\Lambda_\beta|} \sum_{p \neq 0} (\widehat{v}_0 + \widehat{v}_p) \langle a_p^\dagger a_p \rangle_\Phi
$$

$$
+ \frac{N_0}{|\Lambda_\beta|} \sum_{r \neq 0} \widehat{v}_r \langle a_r a_{-r} \rangle_\Phi + \frac{\sqrt{N_0}}{|\Lambda_\beta|} \sum_{\substack{q,r \neq 0 \\ q+r \neq 0}} \widehat{v}_r \langle a_r^\dagger a_q^\dagger a_{q+r} \rangle_\Phi
$$

$$
+ \frac{\sqrt{N_0}}{|\Lambda_\beta|} \sum_{\substack{p,r \neq 0 \\ p+r \neq 0}} \widehat{v}_r \langle a_{p+r}^\dagger a_{-r}^\dagger a_p \rangle_\Phi + \frac{1}{2|\Lambda_\beta|} \sum_{\substack{p,q \neq 0 \\ p+r, q+r \neq 0}} \widehat{v}_r \langle a_{p+r}^\dagger a_q^\dagger a_{q+r} a_p \rangle_\Phi .
$$

$$(4.11)$$

We can use Wick's Theorem [27, Theorem 10.2] to the state $T_\nu \Omega$. By definition of α_p and γ_p in (4.5) together with $N^2 = (N_0 + \sum_{p \neq 0} \gamma_p)^2$ we deduce

$$
\langle \mathcal{H}_\nu \rangle_\Phi = \frac{N^2}{2|\Lambda_\beta|} \widehat{v}_0 + \sum_{p \neq 0} p^2 \gamma_p + \frac{N_0}{|\Lambda_\beta|} \sum_{p \neq 0} \left(\widehat{v}_p \gamma_p + \widehat{v}_p \alpha_p \right)
$$

$$
+ \frac{1}{2|\Lambda_\beta|} \sum_{\substack{q \neq 0 \\ q+r \neq 0}} \widehat{v}_r \alpha_q \alpha_{q+r} + \frac{1}{2|\Lambda_\beta|} \sum_{\substack{q \neq 0 \\ q+r \neq 0}} \widehat{v}_r \gamma_q \gamma_{q+r} .
$$

$$(4.12)$$

We bound the last term in the above using Lemma 4.4. With $\rho = N|\Lambda_\beta|^{-1}$ and $\rho_0 = N_0 |\Lambda_\beta|^{-1}$ we deduce

$$
|\Lambda_\beta|^{-1} \langle \mathcal{H}_\nu \rangle_\Phi \leq \frac{1}{2} \rho^2 \widehat{v}_0 + \frac{1}{|\Lambda_\beta|} \sum_{p \neq 0} \left((p^2 + \rho_0 \widehat{v}_p) \gamma_p + \rho_0 \widehat{v}_p \alpha_p \right)
$$

$$
+ \frac{1}{2|\Lambda_\beta|^2} \sum_{\substack{q \neq 0 \\ q+r \neq 0}} \widehat{v}_r \alpha_q \alpha_{q+r} + C \widehat{v}_0 \rho^2 Y^2 .
$$

$$(4.13)$$

Up to errors $\mathcal{E} \leq C \widehat{v}_0 \rho^2 Y^{\frac{1}{2}+\beta}$, we can approximate these Riemann sums by integrals (see Lemma G.1) and the lemma follows. In fact, the requirement $\beta \geq 3/2$ in Theorem 4.1 comes from here. $\qquad \square$

Lemma 4.7. *Under the assumptions of Theorem* 4.1*, there exists a constant* $C > 0$*, independent of* ν *and* ρ*, such that*

$$
|\Lambda_\beta|^{-1} \langle \mathcal{H}_\nu \rangle_\Phi
$$

$$
\leq \frac{\rho^2}{2} \widehat{g}_0 + \frac{1}{2} \int \left(\sqrt{p^4 + 2\rho_0 p^2 \widehat{g}_p} - p^2 - \rho_0 \widehat{g}_p + \rho_0^2 \frac{\widehat{g}_p^2 - \widehat{g}_0^2 \mathbb{1}_{\{p \leq 2e^{-\Gamma} \widetilde{R}^{-1}\}}}{2p^2} \right) \frac{dp}{4\pi^2}
$$

$$
+ \frac{1}{2} D(\alpha + \rho_0 \widehat{\omega}, \alpha + \rho_0 \widehat{\omega}) + C \rho^2 Y (\widehat{v}_0 - \widehat{g}_0) + C \widehat{v}_0 \rho^2 Y^2 .
$$

Proof. We recall the definition (3.4) of ω, and we insert $\rho_0\widehat{\omega}$ into $D(\alpha, \alpha)$,

$$D(\alpha, \alpha) = -\rho_0^2 D(\widehat{\omega}, \widehat{\omega}) - 2\rho_0 D(\alpha, \widehat{\omega}) + D(\alpha + \rho_0\widehat{\omega}, \alpha + \rho_0\widehat{\omega}).$$

Inserting this into Lemma 4.6 we find

$$\frac{\langle \mathcal{H}_v \rangle_\Phi}{|\Lambda_\beta|} \le \frac{\rho^2}{2}\widehat{v}_0 - \frac{\rho_0^2}{2}D(\widehat{\omega}, \widehat{\omega}) + \int \left((p^2 + \rho_0\widehat{v}_p)\gamma_p + \rho_0(\widehat{v}_p - \widehat{v\omega}_p)\alpha_p \right) \frac{\mathrm{d}p}{4\pi^2}$$
$$+ \frac{1}{2}D(\alpha + \rho_0\widehat{\omega}, \alpha + \rho_0\widehat{\omega}) + C\widehat{v}_0\rho^2 Y^2. \tag{4.14}$$

Now note that $\widehat{g}_p = \widehat{v}_p - (\widehat{v} * \widehat{\omega})_p$ and,

$$\frac{\rho^2}{2}\widehat{v}_0 = \frac{\rho^2}{2}\widehat{g}_0 + \frac{\rho_0^2}{2}(\widehat{v\omega})_0 + \frac{\rho^2 - \rho_0^2}{2}(\widehat{v\omega})_0$$
$$= \frac{\rho^2}{2}\widehat{g}_0 + \frac{\rho_0^2}{2}(\widehat{g\omega})_0 + \frac{\rho_0^2}{2}(\widehat{v\omega^2})_0 + \frac{\rho^2 - \rho_0^2}{2}(\widehat{v\omega})_0.$$

This equality inserted in (4.14), together with $D(\widehat{\omega}, \widehat{\omega}) = (\widehat{v\omega^2})_0$ implies

$$\frac{\langle \mathcal{H}_v \rangle_\Phi}{|\Lambda|} = \frac{\rho^2}{2}\widehat{g}_0 + \frac{\rho_0^2}{2}(\widehat{g\omega})_0 + \int \left((p^2 + \rho_0\widehat{v}_p)\gamma_p + \rho_0\widehat{g}_p\alpha_p \right) \frac{\mathrm{d}p}{4\pi^2}$$
$$+ \frac{1}{2}D(\alpha + \rho_0\widehat{\omega}, \alpha + \rho_0\widehat{\omega}) + \frac{\rho^2 - \rho_0^2}{2}(\widehat{v\omega})_0 + C\widehat{v}_0\rho^2 Y^2. \tag{4.15}$$

Our choice of γ and α minimizes the integral where we replaced $\widehat{v}_p$ by $\widehat{g}_p$, and by explicit computation using the definition (4.6) of α and γ we find

$$\int (p^2 + \rho_0\widehat{g}_p)\gamma_p + \rho_0\widehat{g}_p\alpha_p \frac{\mathrm{d}p}{2\pi^2} = \frac{1}{2}\int \left(\sqrt{p^4 + 2\rho_0 p^2 \widehat{g}_p} - p^2 - \rho_0\widehat{g}_p \right) \frac{\mathrm{d}p}{4\pi^2}. \tag{4.16}$$

Moreover the formula for $\widehat{g\omega}$ from Lemma 3.9 yields

$$(\widehat{g\omega})_0 = \langle \widehat{\omega}, \widehat{g} \rangle = \int \frac{\widehat{g}_p^2 - \widehat{g}_0^2 \mathbb{1}_{\{p \le \ell_\delta^{-1}\}}}{2p^2} \frac{\mathrm{d}p}{4\pi^2},$$

where $\ell_\delta = \frac{1}{2}e^\Gamma \widetilde{R}$. Inserting this and (4.16) into (4.15) we find

$$\frac{\langle \mathcal{H}_v \rangle_\Phi}{|\Lambda|} = \frac{\rho^2}{2}\widehat{g}_0 + \frac{1}{2}\int \left(\sqrt{p^4 + 2\rho_0 p^2 \widehat{g}_p} - p^2 - \rho_0\widehat{g}_p + \rho_0^2 \frac{\widehat{g}_p^2 - \widehat{g}_0^2 \mathbb{1}_{\{p \le \ell_\delta^{-1}\}}}{2p^2} \right) \frac{\mathrm{d}p}{4\pi^2}$$
$$+ \frac{1}{2}D(\alpha + \rho_0\widehat{\omega}, \alpha + \rho_0\widehat{\omega})$$
$$+ \frac{\rho^2 - \rho_0^2}{2}(\widehat{v\omega})_0 + \rho_0 \int (\widehat{v}_p - \widehat{g}_p)\gamma_p \frac{\mathrm{d}p}{4\pi^2} + C\widehat{v}_0\rho^2 Y^2,$$

where the last integral comes from the replacement of $\widehat{v}_p$ by $\widehat{g}_p$ in the first term of the integral in (4.15). Since $\rho - \rho_0 \le C\rho Y$ (Lemma 4.4 and Lemma G.1) and $|\widehat{v}_p - \widehat{g}_p| \le (\widehat{v\omega})_0 = \widehat{v}_0 - \widehat{g}_0$, we can bound

$$\frac{\rho^2 - \rho_0^2}{2}(\widehat{v\omega})_0 + \rho_0 \int (\widehat{v}_p - \widehat{g}_p)\gamma_p \frac{\mathrm{d}p}{4\pi^2} \le C\rho^2 Y(\widehat{v}_0 - \widehat{g}_0),$$

and the lemma follows. $\qquad\square$

In the following lemma we estimate the remainder term from Lemma 4.7.

Lemma 4.8. *There is a $C > 0$ independent of v and ρ such that:*

$$D(\alpha + \rho_0\widehat{\omega}, \alpha + \rho_0\widehat{\omega}) \leq \rho_0^2\delta^2\widehat{v}_0\left|\frac{1}{\delta} - \frac{1}{Y} + \log\delta\right| + \rho_0^2\delta^2\widehat{v}_0\left(\frac{1}{\delta} - \frac{1}{Y} + \log\delta\right)^2 + C\widehat{v}_0\rho_0^2\widehat{g}_0^2.$$

In particular, with $\delta = \delta_0$ defined in (1.5) *we deduce*

$$D(\alpha + \rho_0\widehat{\omega}, \alpha + \rho_0\widehat{\omega}) \leq C\widehat{v}_0\rho^2\delta_0^2.$$

Proof. We recall the definition of ℓ_δ in (3.30). We first estimate $h_p := \langle \alpha + \rho_0\widehat{\omega}, \widehat{v}_{p-.}\rangle$, using Lemma 3.8 as

$$h_p = \int \left(\frac{\rho_0\widehat{g}_q\widehat{v}_{p-q} - \rho_0\widehat{g}_0\widehat{v}_p\mathbb{1}_{\{|q|<\sqrt{\rho_0\widehat{g}_0}\}}}{2q^2} + \frac{\rho_0\widehat{g}_0\widehat{v}_p\mathbb{1}_{\{\ell_\delta^{-1}<|q|<\sqrt{\rho_0\widehat{g}_0}\}}}{2q^2}\right.$$
$$\left. - \frac{\rho_0\widehat{g}_q\widehat{v}_{p-q}}{2\sqrt{q^4 + 2\rho_0q^2\widehat{g}_q}}\right)\frac{\mathrm{d}q}{4\pi^2}$$
$$= h_p^{(1)} + h_p^{(2)} + h_p^{(3)}, \tag{4.17}$$

with

$$|h_p^{(1)}| = \left|\int_{\{|q|>\sqrt{\rho_0\widehat{g}_0}\}}\frac{\rho_0\widehat{g}_q\widehat{v}_{p-q}}{2q^2}\left(1 - \frac{1}{\sqrt{1 + \frac{2\rho_0\widehat{g}_q}{q^2}}}\right)\frac{\mathrm{d}q}{4\pi^2}\right|$$
$$\leq C\widehat{v}_0\int_{\{|q|>\sqrt{\rho_0\widehat{g}_0}\}}\frac{(\rho_0\widehat{g}_0)^2}{q^4}\mathrm{d}q \leq C\widehat{v}_0\rho_0\widehat{g}_0. \tag{4.18}$$

We also calculate

$$|h_p^{(2)}| = \left|\int_{\{\ell_\delta^{-1}<|q|<\sqrt{\rho_0\widehat{g}_0}\}}\frac{\rho_0\widehat{g}_0\widehat{v}_p}{2q^2}\frac{\mathrm{d}q}{4\pi^2}\right|$$
$$\leq C\rho_0\widehat{g}_0\widehat{v}_0\left|\log\left(\sqrt{\rho_0a^2}\sqrt{\widehat{g}_0}\frac{e^\Gamma}{2}e^{\frac{1}{2\delta}}\right)\right|$$
$$= \rho_0\delta\widehat{v}_0\left|\frac{1}{\delta} - \frac{1}{Y} + \log\delta + C\right|. \tag{4.19}$$

In the case where $\ell_\delta^{-1} \geq \sqrt{\rho_0\widehat{g}_0}$, the same estimates hold true. Only the inequalities inside the indicator function in (4.17) change. Using (3.30) we have,

$$|h_p^{(3)}| = \left|\int_{\{|q|<\sqrt{\rho_0\widehat{g}_0}\}}\frac{\rho_0(\widehat{g}_q\widehat{v}_{p-q} - \widehat{g}_0\widehat{v}_p)}{2q^2}\frac{\mathrm{d}q}{4\pi^2} - \int_{\{|q|<\sqrt{\rho_0\widehat{g}_0}\}}\frac{\rho_0\widehat{g}_q\widehat{v}_{p-q}}{2\sqrt{q^4 + 2\rho_0q^2\widehat{g}_q}}\frac{\mathrm{d}q}{4\pi^2}\right|$$
$$\leq \int_{\{|q|<\sqrt{\rho_0\widehat{g}_0}\}}\frac{\rho_0\widehat{g}_0|\widehat{v}_{p-q} - \widehat{v}_p| + \rho_0|\widehat{g}_q - \widehat{g}_0|\widehat{v}_0}{2q^2}\frac{\mathrm{d}q}{4\pi^2}$$
$$+ C\widehat{v}_0\sqrt{\rho_0\widehat{g}_0}\int_{\{|q|<\sqrt{\rho_0\widehat{g}_0}\}}\frac{1}{|q|}\mathrm{d}q$$
$$\leq C\|\nabla\widehat{v}\|_\infty\int_{\{|q|<\sqrt{\rho_0\widehat{g}_0}\}}\frac{\rho_0\widehat{g}_0}{|q|}\mathrm{d}q + C\widehat{v}_0\rho_0\widehat{g}_0 \leq C\widehat{v}_0\rho_0\widehat{g}_0,$$

where we used $\ell_\delta^{-1} \sim \sqrt{\rho \widehat{g_0}} \leq R$ and $\|\nabla \widehat{v_0}\|_\infty \leq R \widehat{v_0}$. In the end we obtain

$$|h_p| \leq C\widehat{v_0}\rho_0\widehat{g_0} + \rho_0\delta\widehat{v_0}\left(\frac{1}{\delta} - \frac{1}{Y} + \log\delta + C\right).$$

Similarly we have bounds on the gradient of h, namely

$$|\nabla h_p| \leq CR\widehat{v_0}\rho_0\widehat{g_0}. \tag{4.20}$$

Now we turn to

$$D(\alpha + \rho_0\widehat{\omega}, \alpha + \rho_0\widehat{\omega}) = \langle \alpha + \rho_0\widehat{\omega}, h \rangle$$

$$= \int \left(\frac{\rho_0\widehat{g_q}h_q - \rho_0\widehat{g_0}h_0 \mathbb{1}_{\{|q|>\ell_\delta^{-1}\}}}{2q^2} - \frac{\rho_0\widehat{g_q}h_q}{2\sqrt{q^4 + 2\rho_0 q^2\widehat{g_q}}}\right)\frac{dq}{4\pi^2},$$

which we in the same way write as $D_1 + D_2 + D_3$ with

$$|D_1| = \left|\int_{\{|q|>\sqrt{\rho_0\widehat{g_0}}\}} \frac{\rho_0\widehat{g_q}h_q}{2q^2}\left(1 - \frac{1}{\sqrt{1 + \frac{2\rho_0\widehat{g_q}}{q^2}}}\right)\frac{dq}{4\pi^2}\right|$$

$$\leq \frac{\|h\|_\infty}{8\pi^2}\int_{\{|q|>\sqrt{\rho_0\widehat{g_0}}\}} \frac{(\rho_0\widehat{g_0})^2}{q^4}dq$$

$$\leq C\widehat{v_0}\rho_0^2\widehat{g_0}^2 + \rho_0^2\delta^2\widehat{v_0}\left(\frac{1}{\delta} - \frac{1}{Y} + \log\delta + C\right),$$

and using the bounds on h, we find $|D_1| \leq C\widehat{v_0}\rho_0^2\widehat{g_0}^2$. The technique to bound D_2 is the same as for $h^{(2)}$ and its provides

$$|D_2| = \left|\int_{\{\ell_\delta^{-1}<|q|<\sqrt{\rho_0\widehat{g_0}}\}} \frac{\rho_0\widehat{g_0}h_0}{2q^2}\frac{dq}{4\pi^2}\right| \leq \widehat{v_0}\rho_0^2\delta^2\left(\frac{1}{\delta} - \frac{1}{Y} + \log\delta + C\right)^2.$$

Lastly D_3 is bounded just as $h^{(3)}$,

$$|D_3| = \left|\int_{\{|q|<\sqrt{\rho_0\widehat{g_0}}\}} \frac{\rho_0(\widehat{g_q}h_q - \widehat{g_0}h_0)}{2q^2}\frac{dq}{4\pi^2} - \int_{\{|q|<\sqrt{\rho_0\widehat{g_0}}\}} \frac{\rho_0\widehat{g_q}h_q}{2\sqrt{q^4 + 2\rho_0 q^2\widehat{g_q}}}\frac{dq}{4\pi^2}\right|$$

$$\leq C\widehat{v_0}\rho_0^2\widehat{g_0}^2,$$

from which the first result follows. The second comes from the fact that when $\delta = \delta_0$ we have

$$\delta_0^{-1} = Y^{-1} + |\log Y| + O(Y|\log Y|^2),$$
$$\log\delta_0 = \log Y + \log(1 - Y|\log Y|),$$

providing that

$$|\delta_0^{-1} - Y^{-1} + \log\delta_0| \leq C. \tag{4.21}$$

$\square$

Now we have all necessary ingredients to conclude the proof of Theorem 4.1.

Proof of Theorem 4.1. We take the trial state Φ defined in Sect. 4.1, which has the expected bounds on number of particles from Lemma 4.5. The energy of Φ is bounded by Lemma 4.7 together with Lemma 4.8, and using $\delta_0 \geq \frac{1}{2} Y$ we find

$$
\begin{aligned}
\frac{\langle \mathcal{H}_v \rangle_\Phi}{|\Lambda_\beta|} &\leq \frac{\rho^2}{2} \widehat{g}_0 + \frac{1}{2} \int \left(\sqrt{p^4 + 2\rho_0 p^2 \widehat{g}_p} - p^2 - \rho_0 \widehat{g}_p + \rho_0^2 \frac{\widehat{g}_p^2 - \widehat{g}_0^2 \mathbb{1}_{\{|p| \leq \ell_\delta^{-1}\}}}{2p^2} \right) \frac{dp}{4\pi^2} \\
&\quad + \rho_0^2 \delta^2 \widehat{v}_0 \left(\frac{1}{\delta} - \frac{1}{Y} + \log \delta \right) + \rho_0^2 \delta^2 \widehat{v}_0 \left(\frac{1}{\delta} - \frac{1}{Y} + \log \delta \right)^2 \\
&\quad + C\rho^2 \delta (\widehat{v}_0 - \widehat{g}_0) + C\widehat{v}_0 \rho_0^2 \widehat{g}_0^2.
\end{aligned}
$$

Now this integral can be estimated by Proposition C.3 and using $\rho - \rho_0 \leq C\rho Y$ we find

$$
\begin{aligned}
\frac{\langle \mathcal{H}_v \rangle_\Phi}{|\Lambda_\beta|} &\leq \frac{\rho^2}{2} \widehat{g}_0 + 4\pi \rho^2 \delta^2 \left(\frac{1}{\delta} - \frac{1}{Y} + \log \delta + \left(\frac{1}{2} + 2\Gamma + \log \pi \right) \right) \\
&\quad + \frac{1}{2} \rho^2 \delta^2 \widehat{v}_0 \left| \frac{1}{\delta} - \frac{1}{Y} + \log \delta \right| + \frac{1}{2} \rho^2 \delta^2 \widehat{v}_0 \left(\frac{1}{\delta} - \frac{1}{Y} + \log \delta \right)^2 \\
&\quad + C\rho^2 \delta (\widehat{v}_0 - \widehat{g}_0) + C\widehat{v}_0 \rho^2 \widehat{g}_0^2. \tag{4.22}
\end{aligned}
$$

Finally, with $\widehat{g}_0 = 8\pi \delta$ and the specific choice $\delta = \delta_0$ we deduce

$$
\frac{\langle \mathcal{H}_v \rangle_\Phi}{|\Lambda_\beta|} \leq 4\pi \rho^2 \delta_0 \left(1 + \left(2\Gamma + \frac{1}{2} + \log(\pi) \right) \delta_0 \right) + C\rho^2 \delta_0 (\widehat{v}_0 - \widehat{g}_0) + C\rho^2 \delta_0^2 \widehat{v}_0. \tag{4.23}
$$

□

Remark 4.9. In the case of the spherical measure potential (3.40) (with the choice of b given in (2.5)), one can see that the upper bound (4.22) is minimized (to the available energy precision) by the choice $\delta = \delta_0$. Indeed, even though the first three terms suggest to choose the smallest δ possible, including the remaining contributions yields a minimizer of the form

$$
\delta = Y(1 + cY|\log Y|). \tag{4.24}
$$

Notice that first line in (4.22) is independent of the choice of c to our precision. We pick for simplicity $c = -1$ to obtain our δ_0 providing useful cancelations, see (4.21). This also fixes the value of $\widetilde{R} = ae^{\frac{1}{2\delta}}$.

5. General Upper Bound

In this section we prove Theorem 2.2, using the results of Sect. 4. We let $\beta \geq \frac{3}{2}$ be given and we work on the box $\Lambda_\beta = [-\frac{L_\beta}{2}, \frac{L_\beta}{2}]^2$ of size $L_\beta = \rho^{-\frac{1}{2}} Y^{-\beta}$. Moreover, the number of particles at density ρ is $N = Y^{-2\beta}$.

5.1. Trial state.

Let v be a non-negative measurable and radial potential with scattering length a and supp$(v) \subset B(0, R)$, with $\rho R^2 \leq Y^{2\beta+2}$. We consider φ_b the associated scattering solution normalized at length $b = \rho^{-1/2} Y^{\beta+\frac{1}{2}}$. In other words $\varphi_b = 2\delta_\beta \varphi^{(0)}$ with $\delta_\beta = \frac{1}{2} \log(b/a)^{-1}$, see (3.3). Note that $R \ll b$. Let $f = \min(1, \varphi_b)$ be the truncated scattering solution. It satisfies

$$- \Delta f(x) + \frac{1}{2} v(x) f(x) = 0 \quad \text{on } B(0, b), \tag{5.1}$$

and is normalized such that $f(x) = 1$ for $|x| \geq b$. We define a grand canonical trial state as

$$\Psi = \sum_{n \geq 0} \Phi_n F_n \in \mathscr{F}_s(L^2(\Lambda_\beta)) \tag{5.2}$$

where $\Phi = \sum_n \Phi_n \in \mathscr{F}_s(L^2(\Lambda_\beta))$ is a quasi-free state defined in (4.3) and F_n is the Jastrow factor

$$F_n(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} f(x_i - x_j). \tag{5.3}$$

We will use the notation $f(x_i - x_j) = f_{ij}$ and $\nabla f(x_i - x_j) = \nabla f_{ij}$. Finally note that

$$\nabla_i F_n(x_1, \ldots, x_n) = \sum_{\substack{j=1 \\ j \neq i}}^{n} \nabla_i f_{ij} \frac{F_n}{f_{ij}}. \tag{5.4}$$

Remark 5.1. To estimate the energy of Ψ we use the bound

$$1 \geq \prod_{1 \leq i < j \leq n} f(x_i - x_j)^2 \geq 1 - \sum_{1 \leq i < j \leq n} (1 - f(x_i - x_j)^2). \tag{5.5}$$

A similar trial state is used in [22] in 3 dimensions but there it is necessary to expand the product (5.5) to one order higher to be able to reach the LHY precision. This substantially complicates the estimates in that case.

5.2. Reduction to a soft potential.

In this section we prove that the energy $\langle \Psi, \mathcal{H}_v \Psi \rangle$ can be bounded by $\langle \Phi, \mathcal{H}_{\tilde{v}} \Phi \rangle$ where $\tilde{v}$ is a nicer potential. This is the effect of the Jastrow factor F_n, and we are thus reduced to optimizing the choice of the quasi-free state Φ according to the potential $\tilde{v}$.

Lemma 5.2. *Consider the radial potential $\tilde{v}(x) = 2f'(b)\delta_{\{|x|=b\}}$ (with f' being understood as the radial derivative). Then the state Ψ defined in (5.2) satisfies*

$$\langle \Psi, \mathcal{H}_v \Psi \rangle \leq \langle \Phi, \mathcal{H}_{\tilde{v}} \Phi \rangle - \langle \Phi, \mathcal{R} \Phi \rangle,$$

where $\mathcal{R} = \oplus_n \mathcal{R}_n$ with

$$\mathcal{R}_n = \sum_{\{i,j,k\}} \frac{\nabla f_{ij}}{f_{ij}} \cdot \frac{\nabla f_{ik}}{f_{ik}} F_n^2,$$

where we introduced the notation

$$\{i, j, k\} = \{\text{set of pairwise distinct indices } i, j, k \text{ running from } 1 \text{ to } n\}.$$

Proof. The energy of the n-th sector state is

$$\langle \Psi_n, \mathcal{H}_n \Psi_n \rangle = \sum_{i=1}^{n} \int_{\Lambda^n} \left(F_n^2 |\nabla_i \Phi_n|^2 + |\nabla_i F_n|^2 \Phi_n^2 + 2 F_n \nabla_i F_n \cdot \Phi_n \nabla_i \Phi_n \right) dx$$

$$+ \sum_{1 \le i < j \le n} \int_{\Lambda^n} v(x_i - x_j) F_n^2 \Phi_n^2 dx. \tag{5.6}$$

The second term in (5.6) can be written via (5.4) as

$$\sum_{i=1}^{n} \int_{\Lambda^n} |\nabla_i F_n|^2 \Phi_n^2 dx = \sum_{i \ne j} \int_{\Lambda^n} |\nabla f_{ij}|^2 \frac{F_n^2}{f_{ij}^2} \Phi_n^2 dx + \sum_{\{i,j,k\}} \int_{\Lambda^n} \frac{\nabla f_{ij}}{f_{ij}} \cdot \frac{\nabla f_{ik}}{f_{ik}} F_n^2 \Phi_n^2 dx. \tag{5.7}$$

Note that, in the first part of (5.7) the integration in x_i is only supported on the ball $|x_i - x_j| \le b$, because $f_{ij} = 1$ outside this ball. We integrate by parts on this ball to find

$$\sum_{i=1}^{n} \int_{\Lambda^n} |\nabla_i F_n|^2 \Phi_n^2 dx \tag{5.8}$$

$$= -\sum_{i \ne j} \int_{\{|x_i - x_j| \le b\}} \Delta f_{ij} \frac{F_n^2}{f_{ij}} \Phi_n^2 dx - \sum_{i \ne j} \int_{\Lambda^n} \nabla f_{ij} \frac{F_n^2}{f_{ij}} \cdot \nabla_i (\Phi_n^2) dx$$

$$- \sum_{\{i,j,k\}} \int_{\Lambda^n} \frac{\nabla f_{ij}}{f_{ij}} \cdot \frac{\nabla f_{ik}}{f_{ik}} F_n^2 \Phi_n^2 dx + \sum_{i \ne j} \int_{\Lambda^{n-1}} \int_{\{|x_i - x_j| = b\}} \partial_r f(b) F_n^2 \Phi_n^2 dx_i d\hat{x}_i,$$

where $\hat{x}_i = (x_1, \ldots x_{i-1}, x_{i+1}, \ldots, x_n)$. The second term in the right hand side of (5.8) is precisely $-2 F_n \nabla_i F_n \cdot \Phi_n \nabla_i \Phi_n$ thanks to (5.4). We use the scattering equation (5.1) to transform

$$\sum_{i \ne j} \int_{\{|x_i - x_j| \le b\}} \Delta f_{ij} \frac{F_n^2}{f_{ij}} \Phi_n^2 dx = \sum_{1 \le i < j \le n} v(x_i - x_j) F_n^2 \Phi_n^2 dx, \tag{5.9}$$

in (5.8) (note that there is no half factor because the sum is on $i < j$). Using (5.8) and (5.9) in (5.6) we deduce

$$\langle \Psi_n, \mathcal{H}_n \Psi_n \rangle = \sum_{i=1}^{n} \int_{\Lambda^n} F_n^2 |\nabla_i \Phi_n|^2 + 2 \sum_{i<j} \int_{\Lambda^{n-1}} \int_{\{|x_i - x_j| = b\}} \partial_r f(b) F_n^2 \Phi_n^2 dx_i d\hat{x}_i$$

$$- \sum_{\{i,j,k\}} \int_{\Lambda^n} \frac{\nabla f_{ij}}{f_{ij}} \cdot \frac{\nabla f_{ik}}{f_{ik}} F_n^2 \Phi_n^2 dx. \tag{5.10}$$

In the first two terms we bound F_n by 1, and the last one we consider as a remainder. Thus,

$$\langle \Psi_n, \mathcal{H}_n \Psi_n \rangle \le \int_{\Lambda^n} |\nabla \Phi_n|^2 + \sum_{i<j} \int_{\Lambda^n} \Phi_n^2 \tilde{v}(x_i - x_j) dx - \mathcal{R}_n.$$

$$\square$$

We comment here how in the proof we used nowhere that Φ is a quasi-free state, therefore the lemma holds true for more general $\Phi \in \mathcal{F}_s(L^2(\Lambda_\beta))$.

5.3. Number of particles in our trial state. Now for Φ we choose the quasi-free state given by Theorem 4.1, applied to the potential $\widetilde{v}$. We recall that $\Phi = W_{N_0} T_v \Omega$ is defined in (4.3), and Ψ in (5.2). In this section we prove the following two lemmas, giving estimates on the norm of Ψ and the average number of particles in Ψ. The idea is to use the properties of F_n to derive the bounds on $\Psi_n = F_n \Phi_n$ from the bounds on the quasi-free state Φ.

Lemma 5.3. *There is a $C > 0$, independent of v and ρ, such that*

$$\|\Psi\|^2 \geq \|\Phi\|^2 \left(1 - CNY^{2\beta+2}\right).$$

Proof. The norm of our trial state is bounded from below by

$$\|\Psi\|^2 = \sum_{n \geq 0} \int_{\Lambda^n} F_n^2(x) \Phi_n^2(x) dx$$

$$\geq \sum_{n \geq 0} \left(\int_{\Lambda^n} \Phi_n^2 dx - \sum_{1 \leq i < j \leq n} \int_{\Lambda^n} (1 - f(x_i - x_j)^2) \Phi_n^2(x) dx\right), \qquad (5.11)$$

where we used the inequality

$$\prod_{1 \leq i < j \leq n} f(x_i - x_j)^2 \geq 1 - \sum_{1 \leq i < j \leq n} (1 - f(x_i - x_j)^2). \qquad (5.12)$$

The second term is the 2-body interaction potential energy of Φ, thus we can write it as

$$\sum_{n \geq 0} \sum_{1 \leq i < j \leq n} \int (1 - f(x_i - x_j)^2) \Phi_n^2 dx = \frac{1}{2|\Lambda_\beta|} \sum_{p,q,r} \widehat{(1 - f^2)}_r \langle a_q^* a_{p+r}^* a_{q+r} a_p \Phi, \Phi \rangle$$

$$\leq \frac{\widehat{(1 - f^2)}_0}{2|\Lambda_\beta|} \sum_{p,q,r} \langle a_q^* a_{p+r}^* a_{q+r} a_p \Phi, \Phi \rangle. \qquad (5.13)$$

Since $\Phi = W_{N_0} T_v \Omega$ is a quasi-free state we can estimate this term as already done in (4.11). We first conjugate by W_{N_0} which amounts to change the a_0's into $N_0 \leq N$. Together with Lemma 4.4 and Wick's theorem we deduce

$$\sum_{p,q,r} \langle a_q^* a_{p+r}^* a_{q+r} a_p \rangle_\Phi \leq CN^2 + \sum_{\substack{p \neq 0, q \neq 0, r \neq 0 \\ p+r \neq 0, q+r \neq 0}} \langle a_q^* a_{p+r}^* a_{q+r} a_p \rangle_\Phi. \qquad (5.14)$$

Then we use again Wick's Theorem to estimate the remaining sum, which is then bounded by CN^2 by Lemma 4.4. Thus Eq. (5.13) gives

$$\sum_{n \geq 0} \sum_{1 \leq i < j \leq n} \int (1 - f(x_i - x_j)^2) \Phi_n^2 dx \leq C \frac{N^2}{|\Lambda_\beta|} \int_\Lambda (1 - f(x)^2) dx \, \|\Phi\|^2. \qquad (5.15)$$

Using $\frac{d}{dr}[r^2 \log\left(\frac{r}{a}\right)^2 - r^2 \log\left(\frac{r}{a}\right) + \frac{r^2}{2}] = 2r \log\left(\frac{r}{a}\right)^2$ and $a \leq R \leq b$ we have that

$$\int_\Lambda (1 - f(x)^2)dx = 2\pi \int_R^b \left(1 - \frac{\log\left(\frac{r}{a}\right)^2}{\log\left(\frac{b}{a}\right)^2}\right)rdr + 2\pi \int_0^R (1 - f(r)^2)rdr$$

$$\leq C\frac{b^2}{\log\left(\frac{b}{a}\right)} + CR^2 \leq C\rho^{-1}Y^{2\beta+2}, \tag{5.16}$$

where we used $\rho R^2 \leq Y^{2\beta+2}$ and $b^2 = \rho^{-1}Y^{2\beta+1}$. We use this last bound in (5.15) and (5.11) to get

$$\|\Psi\|^2 \geq \|\Phi\|^2\left(1 - CNY^{2\beta+2}\right).$$

$\square$

Lemma 5.4. *There is a $C > 0$ independent of ρ and v such that,*

$$\langle\Psi, \mathcal{N}\Psi\rangle \geq N(1 - CY^2)\|\Psi\|^2, \qquad \langle\Psi, \mathcal{N}^2\Psi\rangle \leq CN^2\|\Psi\|^2.$$

Proof. First we have by Lemmas 4.5 and 5.3 that

$$\langle\Psi, \mathcal{N}^2\Psi\rangle = \sum_{n\geq0} n^2 \int_{\Lambda^n} F_n^2\Phi_n^2 dx \leq \sum_{n\geq0} n^2\|\Phi_n\|^2 = \langle\Phi, \mathcal{N}^2\Phi\rangle \leq CN^2\|\Psi\|^2.$$

For the bound on $\langle\mathcal{N}\rangle_\Psi$ we use the same idea as in the proof of Lemma 5.3. From inequality (5.12) we deduce

$$\langle\Psi, \mathcal{N}\Psi\rangle = \sum_{n\geq0} n \int_{\Lambda^n} F_n^2\Phi_n^2 dx$$

$$\geq \sum_{n\geq0} n\left(\int_{\Lambda^n} \Phi_n^2 dx - \sum_{i<j} \int_{\Lambda^n} (1 - f(x_i - x_j)^2)\Phi_n^2 dx\right). \tag{5.17}$$

In the second term we recognize a number operator and a 2-particles interaction energy, which can be rewritten as

$$\sum_{n\geq0} n \sum_{i<j} \int_{\Lambda^n} (1 - f(x_i - x_j)^2)\Phi_n^2 dx = \sum_{k,p,q,r\in\Lambda^*} \frac{\widehat{(1-f^2)}_r}{2L_\beta^2}\langle a_k^*a_k a_{p+r}^* a_q^* a_{q+r}a_p\Phi, \Phi\rangle.$$

We can compute this term using the same techniques as for (5.13), i.e., extract the a_0's and then apply Wick's Theorem yielding many terms of the form $A_1 A_2 A_3$ with $A_i \in \{\langle a_0^\dagger a_0\rangle_\Phi, \sum_{p\neq0}\alpha_p, \sum_{p\neq0}\gamma_p\}$ (see (4.5)). These terms are bounded by N^3 by Lemma 4.4. Thus

$$\sum_{n\geq0} n \sum_{i<j} \int_{\Lambda^n} (1 - f(x_i - x_j)^2)\Phi_n^2 dx \leq C\frac{N^3}{L_\beta^2}\int (1 - f(x)^2)dx\|\Phi\|^2.$$

Now we use the inequality (5.16) to bound the right hand side of the quantity above and plug it in (5.17) to obtain

$$\langle\Psi, \mathcal{N}\Psi\rangle \geq (N - CN^2Y^{2\beta+2})\|\Psi\|^2 = N(1 - CY^2)\|\Psi\|^2,$$

where in the equality used that $N = Y^{-2\beta}$. $\square$

5.4. Remainder term. Here we prove that the remainder term in Lemma 5.2 is indeed small.

Lemma 5.5. *There is a $C > 0$ independent of v and ρ such that*

$$|\langle \Phi, \mathcal{R}\Phi \rangle| \le C L_\beta^2 \rho^2 Y^{2\beta+2} \|\Phi\|^2.$$

Proof. The remainder term can be bounded by

$$|\langle \Phi, \mathcal{R}\Phi \rangle| \le \sum_{n \ge 3} \sum_{\{i,j,k\}} \int_{\Lambda^n} W(x_i - x_k) W(x_i - x_j) \Phi_n^2 dx,$$

where $W(x) = |f(x)\nabla f(x)|$. This is a three-body interaction potential, which can be rewritten in second quantization as

$$|\langle \Phi, \mathcal{R}\Phi \rangle| \le \frac{1}{|\Lambda_\beta|^2} \sum_{p,q,r,k,\ell \in \Lambda^*} \widehat{W}_k \widehat{W}_\ell \langle a_{p+\ell+k}^* a_{q-k}^* a_{r-\ell}^* a_r a_q a_p \rangle_\Phi \|\Phi\|^2.$$

We can again use Wick's Theorem to estimate this part, and since Lemma 4.4 provides

$$\max\left\{ \sum_{p \ne 0} \alpha_p, \sum_{p \ne 0} \gamma_p \right\} \le N,$$

we find

$$|\langle \Phi, \mathcal{R}\Phi \rangle| \le C \frac{N^3}{|\Lambda_\beta|^2} \widehat{W}_0^2 \|\Phi\|^2. \tag{5.18}$$

Now since $f(x) = \log\left(\frac{b}{a}\right)^{-1} \log\left(\frac{|x|}{a}\right)$ outside the support of v and is radially increasing, we have

$$\widehat{W}_0 \le 2\pi \int_R^b \frac{\log\left(\frac{r}{a}\right)}{\log\left(\frac{b}{a}\right)^2} dr + 2\pi \int_0^R f(r) f'(r) r \, dr$$

$$\le C \frac{b}{\log\left(\frac{b}{a}\right)} + CR\left(\frac{\log\left(\frac{R}{a}\right)}{\log\left(\frac{b}{a}\right)}\right)^2 \le C\rho^{-1/2} Y^{\beta+1},$$

where we used $|\log\left(\frac{b}{a}\right)|^{-1} \le Y$ and $|\log\left(\frac{R}{a}\right)| \le |\log\left(\frac{b}{a}\right)|$. Inserting this bound in (5.18) we get the result. $\qquad\square$

5.5. Conclusion: Proof of Theorem 2.2. Using Lemmas 5.2, 5.3 and 5.5 we know that our trial state Ψ satisfies

$$\langle \mathcal{H}_v \rangle_\Psi \le \left(\langle \mathcal{H}_{\widetilde{v}} \rangle_\Phi + C L_\beta^2 \rho^2 Y^{2\beta+2} \right)\left(1 + CNY^{2\beta+2} \right). \tag{5.19}$$

For Φ we choose the quasi-free state given by Theorem 4.1 applied to the soft potential $\widetilde{v}$. Recall the definition of $\widetilde{g}$ from Lemma 3.10. We deduce that

$$\frac{1}{|\Lambda_\beta|} \langle \mathcal{H}_{\widetilde{v}} \rangle_\Phi \le 4\pi\rho^2 \delta_0 \left(1 + \left(2\Gamma + \frac{1}{2} + \log(\pi)\right)\delta_0\right) + C\rho^2 \delta_0 \left(\widehat{\widetilde{v}}(0) - \widehat{\widetilde{g}}(0)\right) + C\rho^2 \delta_0^2 \widehat{\widetilde{v}}(0). \tag{5.20}$$

From Lemma 3.10 we have

$$\widehat{\widetilde{v}}(0) = \frac{4\pi}{\log b/a} \quad \text{and} \quad \widehat{\widetilde{g}}(0) = \frac{4\pi}{\log \widetilde{R}/a} = 8\pi \delta_0,$$

where we recall from (3.3) that $\widetilde{R} = a e^{\frac{1}{2\delta_0}}$. Therefore, remembering the choices $b = \rho^{-1/2} Y^{1/2+\beta}$ and $\beta \geq 3/2$, we can estimate $(\widehat{\widetilde{v}}(0) - \widehat{\widetilde{g}}(0)) \leq CY^2 \log Y$ and we get

$$\frac{1}{|\Lambda_\beta|} \langle \mathcal{H}_{\widetilde{v}} \rangle_\Phi \leq 4\pi \rho^2 \delta_0 \left(1 + \left(2\Gamma + \frac{1}{2} + \log(\pi) \right) \delta_0 \right) + \beta C \rho^2 \delta_0^3 |\log(\delta_0)| + C\rho^2 \delta_0^3. \tag{5.21}$$

We insert this into (5.19) together with $N = \rho L_\beta^2 = Y^{-2\beta}$ and $Y \leq 2\delta_0$, which concludes the proof of Theorem 2.2. $\qquad \square$

6. Localization to Large Boxes for the Lower Bound

In this section we reduce the proof of Theorem 2.3 to an analogous statement localized to a box of size ℓ defined in (6.6), namely Theorem 6.7.

6.1. Grand canonical ensemble.
We rewrite the Hamiltonian in a grand canonical setting to approach the problem in the Fock space description. To emphasize the fact that the density parameter appears through a chemical potential in this setting, we introduce the notation $\rho_\mu > 0$ as new parameter. The corresponding Y will be $Y = |\log(\rho_\mu a^2)|^{-1}$ and we fix δ to be

$$\delta = \delta_\mu, \qquad \delta_\mu := \frac{1}{|\log(\rho_\mu a^2 |\log(\rho_\mu a^2)|^{-1})|}. \tag{6.1}$$

This corresponds to normalizing the scattering solution at length $\widetilde{R} = (\rho_\mu Y)^{-1/2}$ in (3.3). With this choice we recall the definition (3.4) of g. This definition is analogous to the one of δ_0 (1.5) but with ρ_μ in place of ρ. We are going to choose, a posteriori, $\rho_\mu = \rho$ which implies $\delta_\mu = \delta_0$.

That this choice of δ is optimal follows by an evaluation of the relevant integral giving the constant in the correction term in (1.4). Please see Remark C.4 for the evaluation of this integral and the discussion of the optimal choice.

We consider the operator $\mathcal{H}_{\rho_\mu}$ acting on the symmetric Fock space $\mathscr{F}_s(L^2(\Omega))$ and commuting with the number operator, whose action on the N-bosons space is

$$\mathcal{H}_{\rho_\mu, N} = H(N, L) - 8\pi \delta \rho_\mu N = \sum_{j=1}^{N} -\Delta_j + \sum_{i<j} v(x_i - x_j) - 8\pi \delta \rho_\mu N$$

$$= \sum_{j=1}^{N} \left(-\Delta_j - \rho_\mu \int_{\mathbb{R}^2} g(x_j - y) \, dy \right) + \sum_{i<j} v(x_i - x_j). \tag{6.2}$$

We define the ground state energy density of $\mathcal{H}_{\rho_\mu}$:

$$e_0(\rho_\mu) := \lim_{|\Omega| \to +\infty} \frac{1}{|\Omega|} \inf_{\Psi \in \mathscr{F}_s(L^2(\Omega)) \setminus \{0\}} \frac{\langle \Psi \,|\, \mathcal{H}_{\rho_\mu} \,|\, \Psi \rangle}{\|\Psi\|^2}. \tag{6.3}$$

In the rest of the paper we prove the following lower bound on $e_0(\rho_\mu)$.

Theorem 6.1. *There exists C, $\eta > 0$ such that the following holds. Let $\rho_\mu > 0$ and $v \in L^1(\Omega)$ be a positive, spherically symmetric potential with scattering length a and* $\mathrm{supp}(v) \subset B(0, R)$ *such that $\|v\|_1 \leq Y^{-1/8}$ and $R \leq \rho_\mu^{-1/2}$. Then, if $\rho_\mu a^2 \leq C^{-1}$, we have, for any $\rho_\mu > 0$,*

$$e_0(\rho_\mu) \geq -4\pi \rho_\mu^2 \delta \left(1 - \left(2\Gamma + \frac{1}{2} + \log \pi\right)\delta\right) - C\rho_\mu^2 \delta^{2+\eta}. \tag{6.4}$$

We now show that Theorem 6.1 implies the main lower bound Theorem 2.3.

Proof of Theorem 2.3. We start by reducing the problem to a potential which is L^1 and compactly supported. For a given v satisfying the assumptions of Theorem 2.3, we apply Theorem 3.6 with $T = (4\pi Y)^{-1/8}$, $R = \rho^{-1/2}$ and $\varepsilon = 1$. This provides us with a potential $\widetilde{v} = v_{T,R,\epsilon}$ to which we can apply Theorem 6.1. Then for this new potential we use the ground state of $\mathcal{H}_N$ as a trial function for $\mathcal{H}_{\rho_\mu}$ and get

$$
\begin{aligned}
e^{2\mathrm{D}}(\rho, \widetilde{v}) &\geq e_0(\rho_\mu, \widetilde{v}) + 8\pi \widetilde{\delta} \rho \rho_\mu \\
&\geq -4\pi \rho_\mu^2 \widetilde{\delta} \left(1 - \left(2\Gamma + \frac{1}{2} + \log \pi\right)\widetilde{\delta}\right) - C\rho_\mu^2 \widetilde{\delta}^{2+\eta} + 8\pi \widetilde{\delta} \rho \rho_\mu,
\end{aligned}
$$

where $\widetilde{\delta} = \frac{1}{|\log(\rho \widetilde{a}^2 |\log(\rho \widetilde{a}^2)|^{-1})|}$ and $\widetilde{a}$ is the scattering length of $\widetilde{v}$. Since $\widetilde{v} \leq v$ we have $e^{2\mathrm{D}}(\rho, v) \geq e^{2\mathrm{D}}(\rho, \widetilde{v})$. Moreover, by Eq. (3.23) we can change $\widetilde{\delta}$ into δ up to an error of order

$$\frac{1}{\log\left(\frac{R}{a}\right)^2 T} + \frac{1}{\log\left(\frac{R}{a}\right)^2} \int_{\{|x|>R\}} v(x) \log\left(\frac{|x|}{a}\right)^2 \mathrm{d}x \leq C\delta^{2+\min\left(\frac{1}{8},\eta_1\right)}. \tag{6.5}$$

Choosing $\rho_\mu = \rho$ concludes the proof. $\square$

6.2. Reduction to large boxes.

We now make use of the sliding localization technique developed in [28] to reduce the proof of Theorem 2.3 to a localized problem in a large box $\Lambda \subset \Omega$. We introduce the length scale

$$\ell := K_\ell \, \rho_\mu^{-1/2} Y^{-\frac{1}{2}}, \tag{6.6}$$

where $K_\ell \gg 1$ is a parameter fixed in "Appendix H", and we carry out the analysis in the large box

$$\Lambda := \left[-\frac{\ell}{2}, \frac{\ell}{2}\right]^2. \tag{6.7}$$

For any $u \in \mathbb{R}^2$, we denote by

$$\Lambda_u := \ell u + \Lambda \tag{6.8}$$

the translated large box. Let us introduce the localization functions: the sharp characteristic function

$$\theta_u := \mathbb{1}_{\Lambda_u} \tag{6.9}$$

and the regular one: let $\chi \in C_0^M(\mathbb{R}^2)$, for $M \in \mathbb{N}$ with supp $\chi = [-\frac{1}{2}, \frac{1}{2}]^2$ be the spherically symmetric function defined in "Appendix F", and

$$\chi_\Lambda(x) := \chi\left(\frac{x}{\ell}\right), \qquad \chi_u(x) := \chi_\Lambda(x - \ell u). \tag{6.10}$$

The parameter M is fixed in "Appendix H". Define the following projections on $L^2(\Lambda)$,

$$P := \ell^{-2}|\mathbb{1}_\Lambda\rangle\langle\mathbb{1}_\Lambda|, \qquad Q := \mathbb{1} - P, \tag{6.11}$$

i.e. P is the orthogonal projection in $L^2(\Lambda)$ onto the constant functions and Q is the orthogonal projection to the complement. Using these definitions, we define the following operators on $\mathscr{F}_s(L^2(\Lambda))$ through their action on any N-particles sector:

$$n_0 := \sum_{j=1}^N P_j, \qquad n_+ := \sum_{j=1}^N Q_j = N - n_0. \tag{6.12}$$

The definition is based on the idea that low energy eigenstates of the system should concentrate in the constant function. Thus, n_0 counts the number of particles in the condensate and n_+ the number of particles excited out of the condensate.

We start by stating the result for the kinetic energy.

Lemma 6.2 (Kinetic energy localization). *Let $-\Delta_u^{\mathcal{N}}$ denote the Neumann Laplacian in Λ_u and $-\Delta$ the Laplacian on $\mathbb{R}^2$. If the regularity of χ is $M > 5$ and the positive parameters $\varepsilon_N, \varepsilon_T, d, s, b$ are smaller than some universal constant, then for all $\ell > 0$ we have*

$$-\Delta \geq \int_{\mathbb{R}^2} \mathcal{T}_u \, du, \tag{6.13}$$

in terms of quadratic forms in $H^1(\mathbb{R}^2)$, where

$$\mathcal{T}_u := \varepsilon_N(-\Delta_u^{\mathcal{N}}) + (1 - \varepsilon_N)(\mathcal{T}_u^{Neu,s} + \mathcal{T}_u^{Neu,l} + \mathcal{T}_u^{gap} + \mathcal{T}_u^{kin}), \tag{6.14}$$

with

$$\mathcal{T}_u^{Neu,s} := \frac{\varepsilon_T}{2(d\ell)^2} \frac{-\Delta_u^{\mathcal{N}}}{-\Delta_u^{\mathcal{N}} + (d\ell)^{-2}}, \tag{6.15}$$

$$\mathcal{T}_u^{Neu,l} := \frac{b}{\ell^2} Q_u, \tag{6.16}$$

$$\mathcal{T}_u^{gap} := b\frac{\varepsilon_T}{(d\ell)^2} Q_u \mathbb{1}_{(d^{-2}\ell^{-1},+\infty)}(\sqrt{-\Delta})Q_u, \tag{6.17}$$

$$\mathcal{T}_u^{kin} := Q_u \chi_u \left\{ (1 - \varepsilon_T)\left[\sqrt{-\Delta} - \frac{1}{2s\ell}\right]_+^2 + \varepsilon_T\left[\sqrt{-\Delta} - \frac{1}{2ds\ell}\right]_+^2 \right\} \chi_u Q_u. \tag{6.18}$$

Proof. The proof is identical to the one of [28, Lemma 3.7] and its adaptation to our context in [19, Lemma 6.4], which are independent of dimension. $\square$

Remark 6.3. The kinetic energy is composed of several terms which have to remedy some problems related to the main kinetic energy term and play the following roles:

- $\mathcal{T}_u^{kin}$ is the main kinetic energy term;

- $-\Delta^{\mathcal{N}}$ is the Neumann Laplacian and compensates the loss of ellipticity at the boundary caused by the localization function χ in $\mathcal{T}_u^{\mathrm{kin}}$;
- $\mathcal{T}_u^{\mathrm{Neu,s}}$ is the Neumann gap in the small box. Worth to remark is that, for large momenta, it behaves like a gap, while for small momenta its action is like a Neumann Laplacian;
- $\mathcal{T}_u^{\mathrm{Neu,l}}$ is a fraction b of the Neumann gap in the large box. We don't think of b as a parameter but as a fixed small constant. In the remaining we then choose and fix the value of b.
- $\mathcal{T}_u^{\mathrm{gap}}$ is another spectral gap which we need in order to control the number of excitations with large momenta.

The localization of the potential energy relies on a direct calculation of the integral which can be found in [28, Proposition 3.1]. Assuming that $R\ell^{-1}$ is sufficiently small, we can introduce the following localized potentials

$$W(x) := \frac{v(x)}{\chi * \chi(x/\ell)}, \qquad w(x, y) := \chi_\Lambda(x) W(x - y) \chi_\Lambda(y), \qquad (6.19)$$

$$W_1(x) := \frac{g(x)}{\chi * \chi(x/\ell)}, \qquad w_1(x, y) := \chi_\Lambda(x) W_1(x - y) \chi_\Lambda(y), \qquad (6.20)$$

$$W_2(x) := \frac{g(x) + g(x)\omega(x)}{\chi * \chi(x/\ell)}, \qquad w_2(x, y) := \chi_\Lambda(x) W_2(x - y) \chi_\Lambda(y), \qquad (6.21)$$

where we observe that W, W_1, W_2 and w, w_1, w_2 are localized versions of v, g, $(1+\omega)g$, respectively, defined in (3.4).

Furthermore, we introduce the translated versions for $u \in \Lambda$

$$w_{1,u}(x, y) = w_1(x - \ell u, y - \ell u) \qquad (6.22)$$

and similarly for $w_{2,u}$ and w_u. We are going to make use of the following approximation result. We recall the defintion of the lengthscale ℓ_δ from (3.30), which, with our choice $\delta = \delta_\mu$ from (6.1) becomes

$$\ell_\delta = \frac{e^\Gamma}{2} \rho_\mu^{-1/2} Y^{-1/2}, \qquad (6.23)$$

and corresponds to the so-called healing length.

Lemma 6.4. *There exists a universal constant $C > 0$ such that, if $R\ell^{-1} < C^{-1}$, we have*

- W_1 *can be approximated by g up to the following error*

$$0 \le W_1(x) - g(x) \le Cg(x) \frac{\min\{|x|^2, R^2\}}{\ell^2}, \qquad (6.24)$$

and in particular $\|W_1\|_{L^1} \le 8\pi\delta(1 + CR^2\ell^{-2})$ due to (3.6).
- *For any $h \in L^1(\mathbb{R}^2)$ such that $h(x) = h(-x)$ and $\mathrm{supp}\, h \subseteq B(0, R)$,*

$$\left| h * \chi_\Lambda(x) - \chi_\Lambda(x) \int_{\mathbb{R}^2} \mathrm{d}x\, h(x) \right| \le C \max_{i,j} \|\partial_i \partial_j \chi\|_\infty \frac{R^2}{\ell^2} \|h\|_{L^1}. \qquad (6.25)$$

- *It also holds*

$$\left| \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} dk \, \frac{\widehat{W}_1(k)^2 - \widehat{W}_1^2(0)\mathbb{1}_{\{|k|\leq\ell_\delta^{-1}\}}}{2k^2} - \widehat{g\omega}(0) \right| \leq C\frac{R^2}{\ell^2}\delta. \qquad (6.26)$$

- *It holds*

$$\left| \int_{\mathbb{R}^2} \frac{(\widehat{W}_1(k) - \widehat{g}(k))^2 - (\widehat{W}_1(0) - \widehat{g}(0))^2 \mathbb{1}_{\{|k|\leq\ell_\delta^{-1}\}}}{2k^2} dk \right| \leq C\frac{R^4}{\ell^4}\widehat{g\omega}(0). \quad (6.27)$$

Proof. For (6.24) we use that the support of g is contained in the set $\{|x| < R\}$, therefore it is enough to give here our estimate. Using the symmetries of χ, the normalization $\|\chi\|_2 = 1$ (Appendix F) and a Taylor expansion we see that

$$\left| 1 - \frac{1}{\chi * \chi(x/\ell)} \right| \leq \frac{1}{|\chi * \chi(x/\ell)|} \left| \int_{\mathbb{R}^2} \chi(y)[\chi(y) - \chi(x/\ell - y)] \right| dy$$
$$\leq C\frac{|x|^2}{\ell^2} \max_{i,j} \|\partial_i\partial_j\chi\|_\infty,$$

which implies the first bound. (6.25) is proved similarily. For the bound (6.26), by the Lemma 3.8 we know that

$$(\widehat{g\omega})_0 = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{\widehat{g}_k^2 - \widehat{g}_0^2 \mathbb{1}_{\{|k|\leq\ell_\delta^{-1}\}}}{2k^2} \, dk, \qquad (6.28)$$

and using (3.28) for both the expressions of W and g we get

$$\frac{1}{(2\pi)^2} \left| \int_{\mathbb{R}^2} \frac{\widehat{g}_k^2 - \widehat{g}_0^2 \mathbb{1}_{\{|k|\leq\ell_\delta^{-1}\}} - \widehat{W}_1^2(k) + \widehat{W}_1^2(0)\mathbb{1}_{\{|k|\leq\ell_\delta^{-1}\}}}{2k^2} dk \right|$$
$$\leq -C \iint |g(x)g(y) - W_1(x)W_1(y)| \log\left(\frac{x-y}{\ell_\delta}\right) dx dy$$
$$\leq -\frac{C}{\ell^2} \iint |x|^2 g(x)g(y) \log\left(\frac{x-y}{\ell_\delta}\right) dx dy$$
$$= \frac{C}{\ell^2} \int |x|^2 g(x)\omega(x) dx$$
$$\leq C\frac{R^2}{\ell^2}\delta,$$

where we used first (6.24), then the fact that in 2 dimension the log term produces a convolution with the Green's function of the Laplacian and finally formulas (3.5) and (3.6) (together with the bounds $\omega \leq 1$ in the support of g and $2R < \ell_\delta$). The last inequality has a similar proof and is omitted. $\qquad\square$

We now give a result of localization to large boxes for the potential part in the Hamiltonian (6.2).

Lemma 6.5 (Localization of the potential). *The following identity holds*

$$-\rho_\mu \sum_{j=1}^N \int_{\mathbb{R}^2} g(x_j - y)\mathrm{d}y + \sum_{i<j} v(x_i - x_j)$$

$$= \int_{\mathbb{R}^2} \left[-\rho_\mu \sum_{j=1}^N \int_{\mathbb{R}^2} w_{1,u}(x_j, y)\mathrm{d}y + \sum_{i<j} w_u(x_i, x_j) \right]\mathrm{d}u. \qquad (6.29)$$

Proof. It is proven by direct calculation following the same lines as [28, Proposition 3.1]. □

Therefore, joining the results from Lemmas 6.2, 6.5 and introducing the large box Hamiltonian acting on $\mathscr{F}_s(L^2(\Lambda_u))$ as

$$\mathcal{H}_{\Lambda_u}(\rho_\mu)_N := \sum_{j=1}^N \mathcal{T}_u^{(j)} - \rho_\mu \sum_{j=1}^N \int_{\mathbb{R}^2} w_{1,u}(x_j, y)\mathrm{d}y + \sum_{i<j} w_u(x_i, x_j), \quad (6.30)$$

where $\mathcal{T}_u^{(j)}$ is (6.14) for the x_j variable, and the ground state energy and its density

$$E_\Lambda(\rho_\mu) := \inf \mathrm{Spec}(\mathcal{H}_\Lambda(\rho_\mu)), \qquad e_\Lambda(\rho_\mu) := \frac{1}{\ell^2} E_\Lambda(\rho_\mu), \qquad (6.31)$$

we are able to prove the following. Recall that e_0 is defined in (6.3).

Lemma 6.6. *Under the assumptions of Lemma 6.2,*

$$e_0(\rho_\mu) \geq e_\Lambda(\rho_\mu). \qquad (6.32)$$

Proof. By direct application of Lemma 6.2 and Lemma 6.5 we have

$$\mathcal{H}_{\rho_\mu, N}(\rho_\mu) \geq \int_{\ell^{-1}(\Omega+B(0,\ell/2))} \mathcal{H}_{\Lambda_u}(\rho_\mu)_N \mathrm{d}u \geq \ell^{-2}|\Omega + B(0, \ell/2)|E_\Lambda(\rho_\mu), (6.33)$$

where the last inequality is guaranteed by the unitary equivalence $\mathcal{H}_{\Lambda_u} \cong \mathcal{H}_{\Lambda_{u'}}$ via the relation

$$w_{u'}(x, y) = w_u(x - \ell(u' - u), y - \ell(u' - u)). \qquad (6.34)$$

The proof is concluded taking the infimum of the spectrum of the left-hand side and dividing by $|\Omega|$ observing that $\frac{|\Omega+B(0,\ell/2)|}{|\Omega|} \xrightarrow[|\Omega|\to+\infty]{} 1$. □

Therefore, Lemma 6.6 shows that in order to prove our main result Theorem 6.1, it is enough to give an analogous estimate on the Hamiltonian on the large box, and it is the content of the next theorem.

Theorem 6.7. *There exist $C, \eta > 0$ such that the following holds. Let $\rho_\mu > 0$ and $v \in L^1(\Omega)$ be a positive, spherically symmetric potential with scattering length a and* $\mathrm{supp}(v) \subset B(0, R)$ *such that $\|v\|_1 \leq Y^{-1/8}$ and $R \leq \rho_\mu^{-1/2}$. Then, if $\rho_\mu a^2 \leq C^{-1}$, and the parameters are chosen as in "Appendix H", we have*

$$E_\Lambda(\rho_\mu) \geq -4\pi\ell^2\rho_\mu^2\delta\Big(1 - \Big(2\Gamma + \frac{1}{2} + \log\pi\Big)\delta\Big) - C\ell^2\rho_\mu^2\delta^{2+\eta}. \qquad (6.35)$$

The proof of Theorem 6.7 is given in the remaining sections of the article.

7. Lower Bounds in Position Space

7.1. Splitting of the potential. By the definitions (6.11) of the projectors P and Q, we see that we can split the potential in a way presented in the lemma below.

Lemma 7.1. *We have, recalling the definitions in* (6.19)*, that*

$$-\rho_\mu \sum_{j=1}^{N} \int_{\mathbb{R}^2} w_1(x_j, y) \mathrm{d}y + \frac{1}{2} \sum_{i \neq j} w(x_i, x_j) = \sum_{j=0}^{4} \mathcal{Q}_j^{ren} \tag{7.1}$$

with

$$0 \leq \mathcal{Q}_4^{ren} := \frac{1}{2} \sum_{i \neq j} \left[Q_i Q_j + \left(P_i P_j + P_i Q_j + Q_i P_j \right) \omega(x_i - x_j) \right] w(x_i, x_j)$$

$$\times \left[Q_j Q_i + \omega(x_i - x_j) \left(P_j P_i + P_j Q_i + Q_j P_i \right) \right], \tag{7.2}$$

$$\mathcal{Q}_3^{ren} := \sum_{i \neq j} P_i Q_j w_1(x_i, x_j) Q_i Q_j + h.c., \tag{7.3}$$

as well as

$$\mathcal{Q}_2^{ren} := \sum_{i \neq j} P_i Q_j w_2(x_i, x_j) Q_i P_j + \sum_{i \neq j} P_i Q_j w_2(x_i, x_j) P_i Q_j \tag{7.4}$$

$$+ \frac{1}{2} \sum_{i \neq j} P_i P_j w_1(x_i, x_j) Q_i Q_j + h.c. - \rho_\mu \sum_{j=1}^{N} Q_i \int_{\mathbb{R}^2} w_1(x_i, y) \mathrm{d}y \, Q_i, \tag{7.5}$$

$$\mathcal{Q}_1^{ren} := \sum_{i,j} Q_i P_j w_2(x_i, x_j) P_i P_j - \rho_\mu \sum_{i=1}^{N} Q_i \int_{\mathbb{R}^2} w_1(x_i, y) \mathrm{d}y \, P_i + h.c., \tag{7.6}$$

and

$$\mathcal{Q}_0^{ren} := \frac{1}{2} \sum_{i \neq j} P_i P_j w_2(x_i, x_j) P_i P_j - \rho_\mu \sum_{j=1}^{N} P_j \int_{\mathbb{R}^2} w_1(x_j, y) \mathrm{d}y \, P_j. \tag{7.7}$$

Proof. It follows from an elementary calculation, using that $P + Q = \mathbb{1}$ on $L^2(\Lambda)$ and, where needed, the identity

$$w_1 = w_2 - w\omega + w\omega^2. \tag{7.8}$$

□

We rewrite now some of the previous Q terms in the lemma below.

Lemma 7.2. *With the notation* $\rho_0 = \frac{n_0}{\ell^2}$ *we have*

$$\mathcal{Q}_0^{ren} = \frac{\rho_0(n_0 - 1)}{2}(\widehat{g}(0) + \widehat{g\omega}(0)) - \rho_\mu n_0 \widehat{g}(0), \tag{7.9}$$

$$\mathcal{Q}_1^{ren} = (\rho_0 - \rho_\mu)\sum_{i=1}^{N} Q_i \chi_\Lambda(x_i) W_1 * \chi_\Lambda(x_i) P_i + h.c.$$

$$+ \rho_0 \sum_{i=1}^{N} Q_i \chi_\Lambda(x_i)((W_1\omega) * \chi_\Lambda)(x_i) P_i + h.c., \tag{7.10}$$

and

$$\mathcal{Q}_2^{ren} \geq \sum_{i \neq j} P_i Q_j w_2(x_i, x_j) Q_i P_j + \frac{1}{2}\sum_{i \neq j}(P_i P_j w_1(x_i, x_j) Q_i Q_j + h.c.)$$

$$+ ((\rho_0 - \rho_\mu)\widehat{W}_1(0) + \rho_0 \widehat{W_1\omega}(0))\sum_{j=1}^{N} Q_j \chi_\Lambda(x_j)^2 Q_j - C(\rho_\mu + \rho_0)\delta\Big(\frac{R}{\ell}\Big)^2 n_+. \tag{7.11}$$

Proof. The first two identities are straightforward after having observed that

$$\sum_{j=1}^{N} P_j w_1(x_i, x_j) P_j = \frac{1}{\ell^2}\sum_{j=1}^{N} P_j \int_\Lambda w_1(x_i, y)\mathrm{d}y = \rho_0 \int_\Lambda w_1(x_i, y)\mathrm{d}y, \quad (7.12)$$

and

$$\int_\Lambda w_1(x_i, y)\mathrm{d}y = \chi_\Lambda(x_i)(W_1 * \chi_\Lambda)(x_i). \tag{7.13}$$

For the $\mathcal{Q}_2^{ren}$ term, the only parts which require a different approach are

$$(\rho_0 - \rho_\mu)\sum_{j=1}^{N} Q_j \chi_\Lambda(x_j) W_1 * \chi_\Lambda(x_j) Q_j + \rho_0 \sum_{j=1}^{N} Q_j \chi_\Lambda(x_j)((W_1\omega) * \chi_\Lambda)(x_j) Q_j. \tag{7.14}$$

Using Eq. (6.25) of Lemma 6.4 we can bound

$$\sum_{j=1}^{N} Q_j \chi_\Lambda(x_i) W_1 * \chi_\Lambda(x_i) Q_j \geq \|W_1\|_{L^1}\sum_{j=1}^{N} Q_j \chi_\Lambda(x_j)^2 Q_j$$

$$- C \max_{i,j}\|\partial_i \partial_j \chi\|_\infty \frac{R^2}{\ell^2}\|W_1\|_{L^1}\|\chi\|_\infty n_+. \tag{7.15}$$

Recalling that $\|W_1\|_{L^1} \leq C\delta$ (Lemma 6.4) and acting similarly for the other term, this concludes the proof. □

As a direct consequence of the lemma above, we can derive the following first lower bound for the large box Hamiltonian.

Corollary 7.3. *The following bound holds for the Hamiltonian in the large box*

$$\mathcal{H}_\Lambda(\rho_\mu)\big|_N \geq \sum_{j=1}^N \mathcal{T}^{(j)} + \frac{\rho_0(n_0 - 1)}{2}(\widehat{g}(0) + \widehat{g\omega}(0)) - \rho_\mu n_0 \widehat{g}_0 \tag{7.16}$$

$$+ \left(\rho_0 - \rho_\mu\right) \sum_{i=1}^N Q_i \chi_\Lambda(x_i) W_1 * \chi_\Lambda(x_i) P_i + h.c. \tag{7.17}$$

$$+ \rho_0 \sum_{i=1}^N Q_i \chi_\Lambda(x_i)((W_1\omega) * \chi_\Lambda)(x_i) P_i + h.c. \tag{7.18}$$

$$+ \sum_{i\neq j} P_i Q_j w_2(x_i, x_j) Q_i P_j + \frac{1}{2}\sum_{i\neq j}(P_i P_j w_1(x_i, x_j) Q_i Q_j + h.c.) \tag{7.19}$$

$$+ ((\rho_0 - \rho_\mu)\widehat{W}_1(0) + \rho_0 \widehat{W_1\omega}(0)) \sum_{j=1}^N Q_j \chi_\Lambda(x_j)^2 Q_j \tag{7.20}$$

$$- C(\rho_\mu + \rho_0)\delta\left(\frac{R}{\ell}\right)^2 n_+ + \mathcal{Q}_3^{ren} + \mathcal{Q}_4^{ren}. \tag{7.21}$$

In the lemma below we prove an estimate which is going to be useful in Sect. 7.2 to localize the $\mathcal{Q}_3^{ren}$ term.

Lemma 7.4. *Let Q' be a possibly non self-adjoint operator on $L^2(\Lambda)$ such that $QQ' = Q'$ and $\|Q'\| \leq 1$. Then for all $c \in (0, 1)$ there is a $C > 0$ such that, if $R \leq \ell$,*

$$\sum_{i\neq j}(P_i Q_j' w_1(x_i, x_j) Q_i Q_j + h.c.)$$

$$\geq -\frac{1}{4}\mathcal{Q}_4^{ren} - \sum_{i\neq j}(P_i Q_j' w_1\omega P_i P_j + h.c.) - \delta n_0\left(cK_\ell^{-2}\frac{n_+}{\ell^2} + C\frac{K_\ell^2}{\ell^2}\sum_{j=1}^N Q_j'(Q_j')^\dagger\right).$$

Proof. The idea is to reobtain the Q_4 term in the inequalities.

$$\sum_{i\neq j}(P_i Q_j' w_1 Q_i Q_j + h.c.) = \sum_{i\neq j} P_i Q_j' w_1\big[Q_i Q_j + \omega(P_i P_j + P_i Q_j + Q_i P_j)\big] + h.c.$$

$$- \sum_{i\neq j} P_i Q_j' w_1\omega(P_i P_j + P_i Q_j + Q_i P_j) + h.c. \tag{7.22}$$

We use Cauchy–Schwarz inequality on both the terms on the right-hand side. The first line of (7.22), using that $w_1 \leq w$, is controlled by

$$C\sum_{i\neq j} P_i Q_j' w_1(P_i Q_j')^\dagger + \frac{1}{4}\mathcal{Q}_4^{ren} = C\frac{n_0}{\ell^2}\sum_{j=1}^N Q_j'\chi_\Lambda(x_j)(W_1 * \chi_\Lambda)(x_j)(Q_j')^\dagger + \frac{1}{4}\mathcal{Q}_4^{ren}$$

$$\leq C\frac{n_0}{\ell^2}\|\chi_\Lambda\|_\infty^2 \delta \sum_{j=1}^N Q_j'(Q_j')^\dagger + \frac{1}{4}\mathcal{Q}_4^{ren},$$

where we used (7.12), (7.13), the bound $\|W_1\|_{L^1} \leq C\delta(1 + R^2\ell^{-2})$ and $R \leq \ell$. For the second line of (7.22) we keep the PP contribution and treat the other terms separately. They can be estimated as above. For instance,

$$\sum_{i \neq j} (P_i Q'_j w_1 \omega P_i Q_j + h.c.) \leq \varepsilon^{-1} \sum_{i \neq j} P_i Q'_j w_1 \omega (P_i Q'_j)^\dagger + \varepsilon \sum_{i \neq j} P_i Q_j w_1 \omega P_i Q_j$$

$$\leq C\delta \frac{n_0}{\ell^2} \Big(\varepsilon^{-1} \sum_{j=1}^N Q'_j (Q'_j)^\dagger + \varepsilon n_+ \Big), \tag{7.23}$$

where we used the Cauchy–Schwarz inequality with weight $\varepsilon > 0$. Choosing $\varepsilon = cC^{-1}K_\ell^{-2}$ with $c \in (0,1)$, we get

$$\sum_{i \neq j} P_i Q'_j w_1 \omega P_i Q_j \leq c^{-1} C^2 \delta \frac{n_0 d^2 K_\ell^2}{(d\ell)^2} \sum_{j=1}^N Q'_j (Q'_j)^\dagger + c\delta n_0 K_\ell^{-2} \frac{n_+}{\ell^2}, \tag{7.24}$$

and the lemma follows. $\qquad\square$

7.2. Localization of $3Q$ term.

In this section we show how we can restrict the action of one of the Q projectors in the $\mathcal{Q}_3^{\mathrm{ren}}$ term to low momenta. More precisely we define the following two sets of low and high momenta respectively,

$$\mathcal{P}_L := \{p \in \mathbb{R}^2 \mid |p| \leq d^{-2}\ell^{-1}\}, \qquad \mathcal{P}_H := \{p \in \mathbb{R}^2 \mid |p| \geq K_H \ell^{-1}\}. \tag{7.25}$$

We choose the parameters d and K_H satisfying (H6) so that the two sets are disjoint. We will localize the Q projector using the following cutoff function,

$$f_L(r) := f(d^2 \ell r), \qquad f(r) := \begin{cases} 1, & \text{if } r \leq 1, \\ 0, & \text{if } r \geq 2, \end{cases} \tag{7.26}$$

where $f \in C^\infty(\mathbb{R})$ is a non-increasing function. The localized projectors are

$$Q_L := Q f_L(\sqrt{-\Delta}), \qquad \overline{Q}_L := Q - Q_L, \tag{7.27}$$

and the localized version of $\mathcal{Q}_3^{\mathrm{ren}}$ is

$$\mathcal{Q}_3^{\mathrm{low}} := \sum_{i \neq j} (P_i Q_{L,j} w_1(x_i, x_j) Q_i Q_j + h.c.). \tag{7.28}$$

The number of high excitations, namely the number of bosons outside from the condensate and with momenta not in $\mathcal{P}_L$ is

$$n_+^H := \sum_{j=1}^N Q_j \mathbb{1}_{(d^{-2}\ell^{-1}, \infty)} (\sqrt{-\Delta_j}) Q_j. \tag{7.29}$$

It is easy to see that

$$\sum_{j=1}^N \overline{Q}_{L,j} \overline{Q}_{L,j}^\dagger \leq n_+^H. \tag{7.30}$$

The next lemma shows how the $\mathcal{Q}_3^{\mathrm{ren}}$ term added to a small contribution from $\mathcal{Q}_4^{\mathrm{ren}}$ and to the spectral gap from the kinetic energy (see (6.14)), can be bounded above by $\mathcal{Q}_3^{\mathrm{low}}$.

Lemma 7.5. *Assume $R \leq \ell$ and the relation* (H26) *between the parameters. Then there exists $C > 0$ such that, for any n-particles state $\Psi \in \mathscr{F}_s(L^2(\Lambda))$ with $n \leq 2\rho_\mu \ell^2$,*

$$\langle \mathcal{Q}_3^{ren} \rangle_\Psi + \frac{1}{4} \langle \mathcal{Q}_4^{ren} \rangle_\Psi + \frac{b}{100} \left(\frac{\langle n_+ \rangle_\Psi}{\ell^2} + \varepsilon_T \frac{\langle n_+^H \rangle_\Psi}{(d\ell)^2} \right) \geq \langle \mathcal{Q}_3^{low} \rangle_\Psi - C\delta \frac{n^2}{\ell^2} (d^{2M-2} + R^2 \ell^{-2})$$

where the fixed number b was introduced in Lemma 6.2.

Proof. By definition

$$\mathcal{Q}_3^{ren} - \mathcal{Q}_3^{low} = \sum_{i \neq j} (P_i \overline{Q}_{L,j} w_1(x_i, x_j) Q_i Q_j + h.c.). \tag{7.31}$$

We use now Lemma 7.4 with $Q' = \overline{Q}_L$ and the estimate (7.30) to get

$$\mathcal{Q}_3^{ren} - \mathcal{Q}_3^{low} \geq -\frac{1}{4} \mathcal{Q}_4^{ren} - \sum_{i \neq j} (P_i \overline{Q}_{L,j} w_1 \omega P_i P_j + h.c.)$$

$$- \delta n_0 \left(c K_\ell^{-2} \frac{n_+}{\ell^2} + C \frac{d^2 K_\ell^2}{(d\ell)^2} n_+^H \right). \tag{7.32}$$

By (7.12) we have

$$\sum_{i \neq j} (P_i \overline{Q}_{L,j} w_1 \omega P_i P_j + h.c.)$$

$$= \frac{n_0}{\ell^2} \left(\sum_{j=1}^{N} \overline{Q}_{L,j} \chi_\Lambda(x_j) \left(\|W_1\omega\|_{L^1} \chi_\Lambda(x_j) + \varepsilon(x_j) \right) P_j + h.c. \right), \tag{7.33}$$

with $\varepsilon(x_j) = W_1\omega * \chi_\Lambda(x_j) - \|W_1\omega\|_{L^1}\chi_\Lambda(x_j)$. The $\varepsilon(x_j)$-term can be bounded using a Cauchy–Schwarz inequality and (6.25),

$$\frac{n_0}{\ell^2} \left(\sum_{j=1}^{N} \overline{Q}_{L,j} \chi_\Lambda(x_j) \varepsilon(x_j) P_j + h.c. \right) \leq C \frac{n_0}{\ell^2} \sum_{j=1}^{N} (\overline{Q}_{L,j} \chi_\Lambda \varepsilon \overline{Q}_{L,j}^\dagger + P_j \chi_\Lambda \varepsilon P_j)$$

$$\leq C \frac{n_0 R^2}{\ell^4} \delta(n_+^H + n_0). \tag{7.34}$$

For the other term we take $M - 1 \leq 2\widetilde{M} \leq M$ and using the notation $D_M := (\ell^{-2} - \Delta_j)^{\widetilde{M}}$, we write

$$\overline{Q}_{L,j} \chi_\Lambda(x_j)^2 P_j + h.c. = \overline{Q}_{L,j} D_M^{-1} [D_M \chi_\Lambda(x_j)^2] P_j + h.c. \tag{7.35}$$

Therefore, by Cauchy–Schwarz inequality with weight $\varepsilon_0 > 0$,

$$\overline{Q}_{L,j} \chi_\Lambda(x_j)^2 P_j + h.c. \leq \varepsilon_0 P_j + \varepsilon_0^{-1} \|D_M \chi_\Lambda^2\|_\infty^2 \overline{Q}_{L,j} D_M^{-2} (\overline{Q}_{L,j})^\dagger.$$

Now using that $\|D_M \chi_\Lambda^2\| \leq C\ell^{-2\widetilde{M}}$ and that $\overline{Q}_L$ cut momenta lower than $d^{-2}\ell^{-1}$ we obtain

$$\overline{Q}_{L,j} \chi_\Lambda(x_j)^2 P_j + h.c. \leq \varepsilon_0 P_j + \varepsilon_0^{-1} C\ell^{-4\widetilde{M}} \overline{Q}_{L,j} (\ell^{-2} - \Delta_j)^{-2\widetilde{M}} (\overline{Q}_{L,j})^\dagger$$

$$\leq \varepsilon_0 P_j + \varepsilon_0^{-1} C d^{8\widetilde{M}} \overline{Q}_{L,j} (\overline{Q}_{L,j})^\dagger. \tag{7.36}$$

Therefore choosing $\varepsilon_0 = d^{4\widetilde{M}}$, we have

$$\frac{n_0}{\ell^2}\left(\sum_{j=1}^{N}\overline{Q}_{L,j}\chi_\Lambda(x_j)^2\|W_1\omega\|_{L^1}P_j + h.c.\right) \le C\delta d^{2M-2}\frac{n_0}{\ell^2}(n_+^H + n_0). \qquad (7.37)$$

Inserting (7.34) and (7.37) into (7.33) we find

$$\sum_{i\ne j}(P_i\overline{Q}_{L,j}w_1\omega P_iP_j + h.c.) \le C\delta\frac{n_0}{\ell^2}(n_+^H + n_0)(d^{2M-2} + R^2\ell^{-2}). \qquad (7.38)$$

We use this last bound in (7.32) and apply it to the state Ψ,

$$\begin{aligned}
\langle\mathcal{Q}_3^{\mathrm{ren}}\rangle_\Psi &- \langle\mathcal{Q}_3^{\mathrm{low}}\rangle_\Psi \\
&\ge -\frac{1}{4}\langle\mathcal{Q}_4^{\mathrm{ren}}\rangle_\Psi - C\delta\frac{n}{\ell^2}(\langle n_+^H\rangle_\Psi + n)(d^{2M-2} + R^2\ell^{-2}) \\
&\quad - c\delta n K_\ell^{-2}\frac{\langle n_+\rangle_\Psi}{\ell^2} - C\delta n d^2 K_\ell^2\frac{\langle n_+^H\rangle_\Psi}{(d\ell)^2} \\
&\ge -\frac{1}{4}\langle\mathcal{Q}_4^{\mathrm{ren}}\rangle_\Psi - C\delta\frac{n^2}{\ell^2}(d^{2M-2} + R^2\ell^{-2}) - c\frac{\langle n_+\rangle_\Psi}{\ell^2} - Cd^2 K_\ell^4\frac{\langle n_+^H\rangle_\Psi}{(d\ell)^2}, \qquad (7.39)
\end{aligned}$$

where we used $n \le 2\rho_\mu\ell^2$ and $\ell^2 = K_\ell^2\rho_\mu^{-1}Y^{-1}$. We conclude by choosing $c = \frac{b}{100}$ and using the relation (H26) between the parameters. $\square$

7.3. A priori bounds and localization of the number of excitations.

The purpose of this section is to get bounds on the number of excitations of the system. First of all, Theorem 7.6 gives a priori bounds on n_+.

Theorem 7.6. *There exists a universal constant $C > 0$ such that, if $\Psi \in \mathcal{F}_s(L^2(\Lambda))$ is a normalized n-bosons vector which satisfies*

$$\langle\mathcal{H}_\Lambda(\rho_\mu)\rangle_\Psi \le -4\pi\rho_\mu^2\ell^2 Y\left(1 - CK_B^2 Y|\log Y|\right), \qquad (7.40)$$

with K_B fixed in "Appendix H", then

$$\langle n_+\rangle_\Psi \le CK_B^2 K_\ell^2\rho_\mu\ell^2 Y|\log Y|, \qquad (7.41)$$

$$\langle\mathcal{Q}_4^{\mathrm{ren}}\rangle_\Psi \le CK_B^2 K_\ell^2\rho_\mu^2\ell^2 Y^2|\log Y|, \qquad (7.42)$$

$$\left|\rho_\mu - \frac{n}{\ell^2}\right| \le CK_B K_\ell\rho_\mu Y^{1/2}|\log Y|^{1/2}. \qquad (7.43)$$

Proof. It is proved in "Appendix D", using a second localization to "small boxes" of size $\ll \ell_\delta$. $\square$

We also need to bound the number of low excitations, defined in terms of our modified kinetic energy $\mathcal{T}$. More precisely we define, for a certain $\widetilde{K}_H \gg 1$ fixed in "Appendix H", the projectors

$$\overline{Q}_H = \mathbb{1}_{(0,\widetilde{K}_H^2\ell^{-2})}(\mathcal{T}), \qquad Q_H = Q - \overline{Q}_H, \qquad (7.44)$$

which satisfy

$$P + \overline{Q}_H + Q_H = 1_\Lambda. \tag{7.45}$$

We will consider the operators

$$n_+^L := \sum_j \overline{Q}_{H,j}, \qquad \tilde{n}_+^H := \sum_j Q_{H,j}, \tag{7.46}$$

for which we prove the following result.

Theorem 7.7 (Restriction on n_+^L). *There exist C, $\eta > 0$ such that the following holds. Let $\Psi \in \mathcal{F}_s(L^2(\Lambda))$ be a normalized n-particle vector which satisfies (7.40). Assume that the potential v is such that $\|v\|_1 \leq Y^{-1/8}$. Then, for $\mathcal{M} \gg 1$ satisfying condition (H24) there exists a sequence $\{\Psi^m\}_{m \in \mathbb{Z}} \subseteq \mathcal{F}_s(L^2(\Lambda))$ such that $\sum_m \|\Psi^m\|^2 = 1$ and*

$$\Psi^m = \mathbb{1}_{[0, \frac{\mathcal{M}}{2}+m]}(n_+^L)\Psi^m, \tag{7.47}$$

and such that the following lower bound holds true

$$\langle \Psi, \mathcal{H}_\Lambda(\rho_\mu)\Psi \rangle \geq \sum_{2|m| \leq \mathcal{M}} \langle \Psi^m, \mathcal{H}_\Lambda(\rho_\mu)\Psi^m \rangle - C\rho_\mu^2 \ell^2 Y^{2+\eta}$$

$$-4\pi\rho_\mu^2\ell^2 Y\left(1 - CK_B^2 Y|\log Y|\right) \sum_{2|m| > \mathcal{M}} \|\Psi^m\|^2.$$

Notice that, if such a state Ψ does not exist, then our lower bound is already proven (see when we apply Theorem 7.7 in (9.80)). From this result we see that, in order to prove Theorem 6.7, we only need to bound the energy of states satisfying the bound $n_+^L \leq \mathcal{M}$. In the remainder of this section, we prove Theorem 7.7. The following lemma states that for a lower bound we can restrict to states with finitely many excitations n_+, up to small enough errors. The proof of this lemma is inspired by the localization of large matrices result in [29]. It is also really similar to the bounds in [30, Proposition 21]. It could be interpreted as an analogue of the standard IMS localization formula. The error produced is written in terms of the following quantities d_1^L and d_2^L:

$$d_1^L := -\rho_\mu \sum_i (P_i + Q_{H,i}) \int w_1(x_i, y)\mathrm{d}y \, \overline{Q}_{H,i} + h.c.$$

$$+ \sum_{i \neq j} (P_i + Q_{H,i})\overline{Q}_{H,j} w(x_i, x_j)\overline{Q}_{H,i}\overline{Q}_{H,j} + h.c.$$

$$+ \sum_{i \neq j} \overline{Q}_{H,i}(P_j + Q_{H,j})w(x_i, x_j)(P_i + Q_{H,i})(P_j + Q_{H,j}) + h.c. \tag{7.48}$$

and

$$d_2^L := \sum_{i \neq j} (P_i + Q_{H,i})(P_j + Q_{H,j})w(x_i, x_j)\overline{Q}_{H,j}\overline{Q}_{H,i} + h.c. \tag{7.49}$$

Lemma 7.8. *Let* $\theta : \mathbb{R} \to [0, 1]$ *be any compactly supported Lipschitz function such that* $\theta(s) = 1$ *for* $|s| < \frac{1}{8}$ *and* $\theta(s) = 0$ *for* $|s| > \frac{1}{4}$. *For any* $\mathcal{M} > 0$, *define* $c_{\mathcal{M}} > 0$ *and* $\theta_{\mathcal{M}}$ *such that*

$$\theta_{\mathcal{M}}(s) = c_{\mathcal{M}}\theta\Big(\frac{s}{\mathcal{M}}\Big), \qquad \sum_{s \in \mathbb{Z}} \theta_{\mathcal{M}}(s)^2 = 1.$$

Then there exists a $C > 0$ *depending only on* θ *such that, for any normalized state* $\Psi \in \mathscr{F}_s(L^2(\Lambda))$,

$$\langle \Psi, \mathcal{H}_{\Lambda}(\rho_{\mu})\Psi \rangle \geq \sum_{m \in \mathbb{Z}} \langle \Psi^m, \mathcal{H}_{\Lambda}(\rho_{\mu})\Psi^m \rangle - \frac{C}{\mathcal{M}^2}\left(|\langle d_1^L \rangle_{\Psi}| + |\langle d_2^L \rangle_{\Psi}|\right), \quad (7.50)$$

where $\Psi^m = \theta_{\mathcal{M}}(n_+^L - m)\Psi$.

Proof. Notice that $\mathcal{H}_{\Lambda}$ only contains terms that change n_+^L by 0, ± 1 or ± 2. Therefore, we write our operator as $\mathcal{H}_{\Lambda}(\rho_{\mu}) = \sum_{|k| \leq 2} \mathcal{H}_k$, with $\mathcal{H}_k n_+^L = (n_+^L + k)\mathcal{H}_k$. Moreover, $\mathcal{H}_k + \mathcal{H}_{-k} = d_k^L$ for $k = 1, 2$. We use this decomposition to estimate the localized energy,

$$\sum_{m \in \mathbb{Z}} \langle \Psi^m, \mathcal{H}_{\Lambda}\Psi^m \rangle = \sum_{m \in \mathbb{Z}} \sum_{|k| \leq 2} \langle \theta_{\mathcal{M}}(n_+^L - m)\theta_{\mathcal{M}}(n_+^L - m + k)\Psi, \mathcal{H}_k\Psi \rangle$$

$$= \sum_{m,s \in \mathbb{Z}} \sum_{|k| \leq 2} \langle \theta_{\mathcal{M}}(s - m)\theta_{\mathcal{M}}(s - m + k)\mathbb{1}_{\{n_+^L = s\}}\Psi, \mathcal{H}_k\Psi \rangle$$

$$= \sum_{m,s \in \mathbb{Z}} \sum_{|k| \leq 2} \theta_{\mathcal{M}}(m)\theta_{\mathcal{M}}(m + k)\langle \mathbb{1}_{\{n_+^L = s\}}\Psi, \mathcal{H}_k\Psi \rangle,$$

where in the last line we changed the index m into $s - m$. We can sum on s to recognize

$$\sum_{m \in \mathbb{Z}} \langle \Psi^m, \mathcal{H}_{\Lambda}\Psi^m \rangle = \sum_{m \in \mathbb{Z}} \sum_{|k| \leq 2} \theta_{\mathcal{M}}(m)\theta_{\mathcal{M}}(m + k)\langle \Psi, \mathcal{H}_k\Psi \rangle. \qquad (7.51)$$

Furthermore the energy of Ψ can be rewritten as

$$\langle \Psi, \mathcal{H}_{\Lambda}\Psi \rangle = \sum_{|k| \leq 2} \langle \Psi, \mathcal{H}_k\Psi \rangle = \sum_{m \in \mathbb{Z}} \sum_{|k| \leq 2} \theta_{\mathcal{M}}(m)^2 \langle \Psi, \mathcal{H}_k\Psi \rangle, \qquad (7.52)$$

by definition of $\theta_{\mathcal{M}}$. Thus, the localization error is

$$\sum_{m \in \mathbb{Z}} \langle \Psi^m, \mathcal{H}_{\Lambda}\Psi^m \rangle - \langle \Psi, \mathcal{H}_{\Lambda}\Psi \rangle = \sum_{|k| \leq 2} \delta_k \langle \Psi, \mathcal{H}_k\Psi \rangle, \qquad (7.53)$$

with

$$\delta_k = \sum_{m \in \mathbb{Z}} \left(\theta_{\mathcal{M}}(m)\theta_{\mathcal{M}}(m + k) - \theta_{\mathcal{M}}(m)^2\right) = -\frac{1}{2}\sum_m \left(\theta_{\mathcal{M}}(m) - \theta_{\mathcal{M}}(m + k)\right)^2.$$

$$(7.54)$$

Since $\delta_0 = 0$, $\delta_k = \delta_{-k}$ and $d_k^L = \mathcal{H}_k + \mathcal{H}_{-k}$ we find

$$\sum_{m \in \mathbb{Z}} \langle \Psi^m, \mathcal{H}_{\Lambda}\Psi^m \rangle - \langle \Psi, \mathcal{H}_{\Lambda}\Psi \rangle = \delta_1 \langle d_1^L \rangle_{\Psi} + \delta_2 \langle d_2^L \rangle_{\Psi}, \qquad (7.55)$$

and only remains to prove that $|\delta_k| \le C\mathcal{M}^{-2}$. Using (7.54) and the definition of $\theta_{\mathcal{M}}$,

$$|\delta_k| = \frac{c_{\mathcal{M}}^2}{2} \sum_{m \in \mathbb{Z}} \left[\theta\left(\frac{m}{\mathcal{M}}\right) - \theta\left(\frac{m+k}{\mathcal{M}}\right) \right]^2. \tag{7.56}$$

We can restrict the sum to $m \in \left[-\frac{\mathcal{M}}{2}, \frac{\mathcal{M}}{2} \right]$, since the other terms vanish due to θ being a cutoff function. This sum contains $\mathcal{M} + 1$ terms which we can bound using the Lipschitz constant L of θ,

$$|\delta_k| \le c_{\mathcal{M}}^2 \frac{\mathcal{M}+1}{2} \frac{L^2 k^2}{\mathcal{M}^2} \le \frac{2L^2 k^2}{\mathcal{M}^2}, \tag{7.57}$$

where in the last inequality we used

$$c_{\mathcal{M}}^2 = \left(\sum_{s \in \mathbb{Z}} \theta\left(\frac{s}{\mathcal{M}}\right)^2 \right)^{-1} \le \frac{1}{\mathcal{M}/4 + 1}. \tag{7.58}$$

$\square$

To estimate the error in (7.50), we need the following bounds on d_1^L and d_2^L.

Lemma 7.9. *Let $\widetilde{\mathcal{M}} > 0$ and $\Psi \in \mathscr{F}_s(L^2(\Lambda))$ be a normalized n-bosons vector satisfying*

$$\Psi = \mathbb{1}_{[0,\widetilde{\mathcal{M}}]}(n_+^L)\Psi.$$

Then, assuming the choices of parameters in "Appendix H" we have

$$|\langle d_1^L \rangle_\Psi| + |\langle d_2^L \rangle_\Psi| \tag{7.59}$$

$$\le \rho_\mu^2 \ell^2 \|v\|_1 \left(\frac{\langle n_+ \rangle_\Psi^{1/2}}{n^{1/2}} + \frac{\widetilde{\mathcal{M}}^{1/2} \langle n_+ \rangle_\Psi^{1/2}}{n} \varepsilon_N^{-1/4} \widetilde{K}_H + \frac{\widetilde{\mathcal{M}} \langle n_+ \rangle_\Psi}{n^2} \varepsilon_N^{-1/2} \widetilde{K}_H^2 \right) + C \langle \mathcal{Q}_4^{ren} \rangle_\Psi.$$

Proof. We give the proof in "Appendix E". $\square$

Now we can combine Lemmas 7.8, 7.9 and Theorem 7.6 to prove Theorem 7.7.

Proof of Theorem 7.7. Given a n-sector state $\Psi \in L^2(\Lambda^n)$ satisfying (7.40), we can apply Lemma 7.8 and write $\Psi^m = \theta_{\mathcal{M}}(n_+^L - m)\Psi$. In (7.50) we split the sum into two. The first part, for $|m| < \frac{1}{2}\mathcal{M}$, we keep. For $|m| > \frac{1}{2}\mathcal{M}$, Ψ_m satisfies

$$\langle n_+ \rangle_{\Psi^m} \ge \langle n_+^L \rangle_{\Psi^m} \ge \frac{\mathcal{M}}{4} \|\Psi^m\|^2, \tag{7.60}$$

due to the cutoff $\theta_{\mathcal{M}}(n_+^L - m)$. Thanks to condition (H24) on $\mathcal{M}$, this is a larger bound than (7.41), and thus the assumption of Theorem 7.6 cannot be satisfied for Ψ^m and we must have the lower bound

$$\langle \Psi^m, \mathcal{H}_\Lambda(\rho_\mu)\Psi^m \rangle \ge -4\pi \rho_\mu^2 \ell^2 Y \left(1 - C K_B^2 Y |\log Y| \right) \|\Psi^m\|^2. \tag{7.61}$$

We finally bound the last term in (7.50), using Lemma 7.9 with $\widetilde{\mathcal{M}} = n$,

$$|\langle d_1^L \rangle_\Psi| + |\langle d_2^L \rangle_\Psi| \le \rho_\mu^2 \ell^2 \|v\|_1^2 \left(\frac{1 + \varepsilon_N^{-1/4} \widetilde{K}_H}{n^{1/2}} \langle n_+ \rangle_\Psi + \frac{\varepsilon_N^{-1/2} \widetilde{K}_H^2}{n} \langle n_+ \rangle_\Psi \right) + C \langle \mathcal{Q}_4^{ren} \rangle_\Psi.$$

Now we use the condensation estimate (7.41) and the bound (7.42) on Q_4^{ren} to obtain

$$|\langle d_1^L \rangle_\Psi| + |\langle d_2^L \rangle_\Psi| \tag{7.62}$$
$$\leq \rho_\mu^2 \ell^2 \|v\|_1 \left(Y^{1/2} |\log Y|^{1/2} K_\ell K_B \widetilde{K}_H \varepsilon_N^{-1/4} + Y |\log Y| K_\ell^2 K_B^2 \widetilde{K}_H^2 \varepsilon_N^{-1/2} \right).$$

The relation (H13) between the parameters implies that the largest term in (7.62) is the first one. Using the conditions (H11) and (H24) on ε_N and $\mathcal{M}$ respectively, and the assumptions on $\|v\|_1$ we find

$$\frac{|\langle d_1^L \rangle_\Psi| + |\langle d_2^L \rangle_\Psi|}{\mathcal{M}^2} \leq \rho_\mu^2 \ell^2 Y^{2+\eta}. \tag{7.63}$$

Using the estimates (7.61) for $m > \frac{1}{2}\mathcal{M}$ and (7.63) in formula (7.50) we conclude the proof. $\qquad\square$

8. Lower Bounds in Second Quantization

8.1. Second quantization formalism. We rewrite the Hamiltonian in the second quantization formalism. Let us introduce the operators, where # can be nothing or † for the annihilation or creation operators on the space $\mathscr{F}_s(L^2(\Lambda))$, respectively,

$$a_0^\# := \frac{1}{\ell} a^\#(\theta), \qquad \text{and} \qquad [a_0, a_0^\dagger] = 1, \tag{8.1}$$

being the creation and annihilation operators for bosons with zero momentum, where θ is the sharp localization function on Λ (see (6.9)). For $k \in \mathbb{R}^2 \setminus \{0\}$ we also define

$$\widetilde{a}_k^\# := \frac{1}{\ell} a^\#(Qe^{ikx}\theta), \tag{8.2}$$

the creation and annihilation operators for bosons with non-zero momentum with Q defined in (6.11), and their regular analogous

$$a_k^\# := \frac{1}{\ell} a^\#(Qe^{ikx}\chi_\Lambda), \tag{8.3}$$

where χ_Λ is the regular localization function defined in "Appendix F". We have the usual commutation relations, for $k, h \in \mathbb{R}^2 \setminus \{0\}$

$$[\widetilde{a}_k, \widetilde{a}_h] = [a_k, a_h] = 0, \qquad \text{and} \qquad [\widetilde{a}_k, \widetilde{a}_h^\dagger] = \frac{1}{\ell^2} \langle Qe^{ikx}, Qe^{ihx} \rangle. \tag{8.4}$$

Using that $P = \mathbb{1} - Q$ and $\widehat{\chi_\Lambda}(k) = \ell^2 \widehat{\chi}(k\ell)$,

$$[a_k, a_h^\dagger] = \frac{1}{\ell^2} \langle Qe^{ikx}\chi_\Lambda, Qe^{ihx}\chi_\Lambda \rangle = \widehat{\chi^2}((k-h)\ell) - \widehat{\chi}(k\ell)\overline{\widehat{\chi}}(h\ell), \tag{8.5}$$

and

$$[a_k, a_h^\dagger] \leq 1. \tag{8.6}$$

Let us observe, first of all, that

$$n_0 = a_0^\dagger a_0, \qquad n_+ = \frac{\ell^2}{(2\pi)^2} \int \widetilde{a}_k^\dagger \widetilde{a}_k \mathrm{d}k. \tag{8.7}$$

Let us introduce, for $k \in \mathbb{R}^2$, the kinetic Fourier multiplier

$$\tau(k) := (1 - \varepsilon_T)\Big[|k| - \frac{1}{2s\ell}\Big]_+^2 + \varepsilon_T\Big[|k| - \frac{1}{2ds\ell}\Big]_+^2. \tag{8.8}$$

We will need the following technical lemma to control the number operators.

Lemma 8.1. *Assume the relation* (H27) *between the parameters. Let* $\Psi \in \mathscr{F}_s(L^2(\Lambda))$ *be a normalized state satisfying*

$$\mathbb{1}_{[0,\mathcal{M}]}(n_+^L)\Psi = \Psi, \qquad \mathbb{1}_{[0,2\rho_\mu\ell^2]}(n_+)\Psi = \Psi, \tag{8.9}$$

then the following bounds hold

$$\Big\langle \ell^2 \int_{\{|k|\leq 2K_H\ell^{-1}\}} (a_k^\dagger a_k + \widetilde{a}_k^\dagger \widetilde{a}_k)\mathrm{d}k \Big\rangle_\Psi \leq C\mathcal{M}, \tag{8.10}$$

$$\Big\langle \ell^2 \int_{\mathbb{R}^2} (a_k^\dagger a_k + \widetilde{a}_k^\dagger \widetilde{a}_k)\mathrm{d}k \Big\rangle_\Psi \leq C\mathcal{M} + C\langle n_+^H \rangle_\Psi. \tag{8.11}$$

Proof. The proof is analogous for both the addends, therefore we give the proof only for the $a_k^\#$. We want to compare localization in terms of kinetic energy with localization in momenta. We use [19, Lemma 5.2] adapted to dimension 2:

$$Q\chi_\Lambda \mathbb{1}_{\{|p|\leq K_H\ell^{-1}\}}\chi_\Lambda Q \leq C\overline{Q}_H + C\Big(\Big(\frac{K_H}{\widetilde{K}_H}\Big)^M + \varepsilon_N^{3/2}\Big), \tag{8.12}$$

$$Q\mathbb{1}_{\{|p|\leq K_H\ell^{-1}\}}Q \leq C\overline{Q}_H + C\Big(\Big(\frac{K_H}{\widetilde{K}_H}\Big)^M + \varepsilon_N^{3/2}\Big), \tag{8.13}$$

where we recall the definition (7.44) of $\overline{Q}_H$. Using (8.12) we have the following inequality in the N-th Fock sector

$$\frac{\ell^2}{(2\pi)^2} \int_{\{|k|\leq 2K_H\ell^{-1}\}} a_k^\dagger a_k \mathrm{d}k \Big|_N = \sum_{j=1}^N Q_j\chi_\Lambda(x_j)\mathbb{1}_{(0,2K_H\ell^{-1}]}(\sqrt{-\Delta_j})\chi_\Lambda(x_j)Q_j$$

$$\leq Cn_+^L + \Big(\Big(\frac{K_H}{\widetilde{K}_H}\Big)^M + \varepsilon_N^{3/2}\Big)n_+.$$

Using the bounds from (8.9) and the relation (H27) we deduce

$$\Big\langle \ell^2 \int_{\{|k|\leq 2K_H\ell^{-1}\}} a_k^\dagger a_k \mathrm{d}k \Big\rangle_\Psi \leq C\mathcal{M} + C\Big(\Big(\frac{K_H}{\widetilde{K}_H}\Big)^M + \varepsilon_N^{3/2}\Big)\rho_\mu\ell^2 \leq C\mathcal{M}, \tag{8.14}$$

thus proving (8.10). In order to obtain (8.11) it is enough to estimate the integral on the complementary subset. We have, again on the N-th sector,

$$\ell^2 \int_{\{|k|\geq 2K_H\ell^{-1}\}} a_k^\dagger a_k \mathrm{d}k \Big|_N = \sum_{j=1}^N Q_j\chi_\Lambda(x_j)\mathbb{1}_{\{|k|\geq 2K_H\ell^{-1}\}}(\sqrt{-\Delta_j})\chi_\Lambda(x_j)Q_j.$$

$$\tag{8.15}$$

We insert $1 = \mathbb{1}_{\mathcal{P}_L} + \mathbb{1}_{\mathcal{P}_L^c}$ and use the Cauchy–Schwarz inequality to estimate the right-hand side,

$$
\begin{aligned}
Q\chi_\Lambda \mathbb{1}_{[2K_H\ell^{-1},+\infty)}(\sqrt{-\Delta})\chi_\Lambda Q \\
\leq 2Q\mathbb{1}_{\mathcal{P}_L^c}(\sqrt{-\Delta})\chi_\Lambda \mathbb{1}_{[2K_H\ell^{-1},+\infty)}(\sqrt{-\Delta})\chi_\Lambda \mathbb{1}_{\mathcal{P}_L^c}(\sqrt{-\Delta})Q. \\
+ 2Q\mathbb{1}_{\mathcal{P}_L}(\sqrt{-\Delta})\chi_\Lambda \mathbb{1}_{[2K_H\ell^{-1},+\infty)}(\sqrt{-\Delta})\chi_\Lambda \mathbb{1}_{\mathcal{P}_L}(\sqrt{-\Delta})Q.
\end{aligned}
$$

On $\mathcal{P}_L^c$ we can use the bound

$$
Q\mathbb{1}_{\mathcal{P}_L^c}(\sqrt{-\Delta})\chi_\Lambda \mathbb{1}_{[2K_H\ell^{-1},+\infty)}(\sqrt{-\Delta})\chi_\Lambda \mathbb{1}_{\mathcal{P}_L^c}(\sqrt{-\Delta})Q \leq \|\chi_\Lambda\|_\infty^2 Q\mathbb{1}_{\mathcal{P}_L^c}(\sqrt{-\Delta})Q.
$$

On $\mathcal{P}_L$ we bound the operator norm, multiplying and dividing by an M power of the Laplacian and using that χ has M bounded derivatives,

$$
\begin{aligned}
\|\mathbb{1}_{\mathcal{P}_L}(\sqrt{-\Delta})\chi_\Lambda \mathbb{1}_{[2K_H\ell^{-1},+\infty)}\| \\
\leq \|\mathbb{1}_{\mathcal{P}_L}(\sqrt{-\Delta})\chi_\Lambda(-\Delta)^{M/2}\|\|(-\Delta)^{-M/2}\mathbb{1}_{[2K_H\ell^{-1},+\infty)}\| \leq C(d^2K_H)^{-M}.
\end{aligned}
$$

We deduce

$$
\ell^2 \int_{\{|k|\geq 2K_H\ell^{-1}\}} a_k^\dagger a_k \mathrm{d}k \leq Cn_+^H + C(d^2K_H)^{-2M}n_+, \tag{8.16}
$$

and we conclude using (H27) and the assumptions on Ψ. $\square$

8.2. Second quantized Hamiltonian.

We can rewrite the $\mathcal{Q}_3^{\text{low}}$ term (7.28) in second quantized formalism

$$
\mathcal{Q}_3^{\text{low}} = \frac{\ell^2}{(2\pi)^4} \int_{\mathbb{R}^2 \times \mathbb{R}^2} f_L(p)\widehat{W}_1(k)a_0^\dagger \widetilde{a}_p^\dagger a_{p-k} a_k \mathrm{d}k\mathrm{d}p + h.c. \tag{8.17}
$$

An important consideration is that we can restrict the contributions in $\mathcal{Q}_3^{\text{low}}$ to the high momenta. This is the content of the next lemma.

Lemma 8.2 (Localization of $\mathcal{Q}_3^{\text{low}}$ to high momenta). *Assume $R \leq \ell$ and the relations (H16), (H17), (H27) between the parameters. If $\Psi \in \mathscr{F}_s(L^2(\Lambda))$ is a n-particle state satisfying (7.40) and $\mathbb{1}_{[0,\mathcal{M}]}(n_+^L)\Psi = \Psi$ then we have*

$$
\langle \Psi \mid \mathcal{Q}_3^{low}\Psi \rangle \geq \langle \Psi \mid \mathcal{Q}_3^{high}\Psi \rangle - \frac{b}{100\ell^2}\langle n_+\rangle_\Psi, \tag{8.18}
$$

where

$$
\mathcal{Q}_3^{high} = \frac{\ell^2}{(2\pi)^4} \int_{\mathcal{P}_H \times \mathbb{R}^2} f_L(p)\widehat{W}_1(k)a_0^\dagger \widetilde{a}_p^\dagger a_{p-k} a_k \mathrm{d}k\mathrm{d}p + h.c., \tag{8.19}
$$

with $\mathcal{P}_H$ defined in (7.25).

Proof. First note that

$$\langle \Psi \mid (\mathcal{Q}_3^{\text{low}} - \mathcal{Q}_3^{\text{high}})\Psi \rangle = \frac{\ell^2}{(2\pi)^4} \int_{\mathcal{P}_H^c \times \mathbb{R}^2} f_L(p) \widehat{W}_1(k) \langle \Psi \mid a_0^\dagger \widetilde{a}_p^\dagger a_{p-k} a_k \Psi \rangle \mathrm{d}k \mathrm{d}p + h.c.$$

$$(8.20)$$

For any $\varepsilon > 0$, using Cauchy–Schwarz on the creation and annihilation operators,

$$\langle \Psi \mid (\mathcal{Q}_3^{\text{low}} - \mathcal{Q}_3^{\text{high}})\Psi \rangle$$
$$\geq -C\delta\ell^2 \int_{\mathcal{P}_H^c \times \mathbb{R}^2} f_L(p) \Big(\varepsilon \langle \Psi \mid \widetilde{a}_p^\dagger a_0^\dagger a_0 \widetilde{a}_p \Psi \rangle + \varepsilon^{-1} \langle \Psi \mid a_k^\dagger a_{p-k}^\dagger a_{p-k} a_k \Psi \rangle \Big) \mathrm{d}k \mathrm{d}p,$$

$$(8.21)$$

where we used the fact that $\|\widehat{W}_1\|_\infty \leq \|W_1\|_1 \leq C\delta$ (from Lemma 6.4). We now use the following inequalities, obtained by Lemma 8.1 and bounding f_L by 1,

$$\ell^2 \int_{\mathcal{P}_H^c \times \mathbb{R}^2} f_L(p) \langle \Psi \mid \widetilde{a}_p^\dagger a_0^\dagger a_0 \widetilde{a}_p \Psi \rangle \mathrm{d}k \mathrm{d}p \leq n \langle n_+ \rangle_\Psi \int_{\mathcal{P}_H^c} \mathrm{d}k = \frac{n \langle n_+ \rangle_\Psi}{\ell^2} K_H^2, \qquad (8.22)$$

$$\ell^4 \int_{\mathcal{P}_H^c \times \mathbb{R}^2} f_L(p) \langle \Psi \mid a_k^\dagger a_{p-k}^\dagger a_{p-k} a_k \Psi \rangle \mathrm{d}k \mathrm{d}p \leq C\mathcal{M} \langle n_+ \rangle_\Psi. \qquad (8.23)$$

Therefore, applying to (8.21) we obtain

$$\langle \Psi \mid (\mathcal{Q}_3^{\text{low}} - \mathcal{Q}_3^{\text{high}})\Psi \rangle \geq -C\delta \frac{\langle n_+ \rangle_\Psi}{\ell^2} n \Big(\varepsilon K_H^2 + \varepsilon^{-1} \frac{\mathcal{M}}{n} \Big). \qquad (8.24)$$

Choosing $\varepsilon = K_H^{-1} \frac{\mathcal{M}^{1/2}}{n^{1/2}}$, we obtain

$$\langle \Psi \mid (\mathcal{Q}_3^{\text{low}} - \mathcal{Q}_3^{\text{high}})\Psi \rangle \geq -C\delta \frac{\langle n_+ \rangle_\Psi}{\ell^2} n \frac{K_H \mathcal{M}^{1/2}}{n^{1/2}}. \qquad (8.25)$$

We use Theorem 7.6 and (H17) to bound $n^{1/2}$ by $2\rho_\mu^{1/2}\ell$ and get

$$\langle \Psi \mid (\mathcal{Q}_3^{\text{low}} - \mathcal{Q}_3^{\text{high}})\Psi \rangle \geq -C\delta \rho_\mu^{1/2} \ell K_H \mathcal{M}^{1/2} \frac{\langle n_+ \rangle_\Psi}{\ell^2}.$$

By the assumption (H16) the error can be absorbed in a small fraction of the spectral gap.   □

We are ready to state a bound for the second quantized Hamiltonian.

Proposition 8.3. *Assume $R \ll (\rho_\mu \delta)^{-1/2}$ and the relations of "Appendix H" between the parameters. Let Ψ be a normalized n-particle state satisfying (7.40) and $\Psi = \mathbb{1}_{[0,\mathcal{M}]}(n_+^L)\Psi$. Then*

$$\langle \Psi \mid \mathcal{H}_\Lambda(\rho_\mu)\Psi \rangle \geq \langle \Psi \mid \mathcal{H}_\Lambda^{2nd}(\rho_\mu)\Psi \rangle - C\ell^2 \rho_\mu^2 \delta \Big(d^{2M-2} + R^2 \ell^{-2} \Big), \qquad (8.26)$$

where

$$\mathcal{H}_\Lambda^{2nd} := \frac{\ell^2}{(2\pi)^2} \int_{\mathbb{R}^2} (1 - \varepsilon_N)\tau(k)a_k^\dagger a_k \mathrm{d}k + \frac{b}{2\ell^2}n_+ + b\frac{\varepsilon_T}{8d^2\ell^2}n_+^H + b\frac{\varepsilon_T n_0 n_+^H}{16d^2\ell^2(\rho_\mu\ell^2)}$$

(8.27)

$$+ \frac{1}{2\ell^2}a_0^\dagger a_0^\dagger a_0 a_0 (\widehat{g}_0 + \widehat{g\omega}(0)) - \rho_\mu a_0^\dagger a_0 \widehat{g}_0 \tag{8.28}$$

$$+ \left(\left(\frac{1}{\ell^2}a_0^\dagger a_0 - \rho_\mu\right)\widehat{W}_1(0)\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\widehat{\chi}_\Lambda(k)a_k^\dagger a_0 \mathrm{d}k + h.c.\right) \tag{8.29}$$

$$+ \left(\frac{1}{\ell^2}a_0^\dagger a_0 \widehat{\omega W_1}(0)\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\widehat{\chi}_\Lambda(k)a_k^\dagger a_0 \mathrm{d}k + h.c.\right) \tag{8.30}$$

$$+ \mathcal{Q}_2^{rest} + \mathcal{Q}_3^{high} \tag{8.31}$$

$$+ \left(\left(\frac{1}{\ell^2}a_0^\dagger a_0 - \rho_\mu\right)\widehat{W}_1(0) + \frac{1}{\ell^2}a_0^\dagger a_0 \widehat{W_1\omega}(0)\right)\frac{\ell^2}{(2\pi)^2}\int_{\mathbb{R}^2}a_k^\dagger a_k \mathrm{d}k, \tag{8.32}$$

where $\tau(k)$ is defined in (8.8) *and with*

$$\mathcal{Q}_2^{rest} = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}(\widehat{W}_1(k) + \widehat{(W_1\omega)}(k))a_0^\dagger a_k^\dagger a_k a_0 \mathrm{d}k$$

$$+ \frac{1}{2}\int_{\mathbb{R}^2}\widehat{W}_1(k)\left(a_0^\dagger a_0^\dagger a_k a_{-k} + a_k^\dagger a_{-k}^\dagger a_0 a_0\right)\mathrm{d}k.$$

Proof. We use the lower bound for $\mathcal{H}_\Lambda(\rho_\mu)$ from Corollary 7.3. First of all, in the kinetic energy expression (6.14) we remove the positive parts depending on the Neumann Laplacian, namely $\varepsilon_N(-\Delta_\mathcal{N})$ and $\mathcal{T}^{Neu,s}$. Using the quantization, we obtain from (6.14) the expressions in (8.27) with the main kinetic energy term and the spectral gaps. We bounded part of the spectral gap to get the last term in (8.27) using $n_0 \leq 2\rho_\mu\ell^2$ (which follows from (7.43) and (H17)). This term will be useful later (in particular in the proof of Lemma 9.2).

The expressions (8.28), (8.29), (8.30), $\mathcal{Q}_2^{rest}$ and (8.32) are obtained from (7.16), (7.17), (7.18), (7.19) and (7.20) respectively, via a straightforward application of the quantization rules. Note that in (8.29) and (8.30) we have changed a $\widehat{W}_1(k)$ (resp. $\widehat{\omega W_1}(k)$) into $\widehat{W}_1(0)$ (resp. $\widehat{\omega W_1}(0)$). This can be justified by using (6.25) in (7.17) and (7.18), the error being of order $R^2\rho_\mu^2\delta$. We can reabsorb the term

$$-C(\rho_\mu + \rho_0)\delta R^2 \frac{n_+}{\ell^2},$$

in a fraction of the spectral gap because $R \ll (\rho_\mu\delta)^{-1/2}$. Let us observe that thanks to Lemma 7.5 we can replace $\mathcal{Q}_3^{ren} + \frac{1}{4}\mathcal{Q}_4^{ren}$ by $\mathcal{Q}_3^{low}$ in $\mathcal{H}_\Lambda(\rho_\mu)$. Part of the error is absorbed in the spectral gap, other part appears in (8.26). Finally we change $\mathcal{Q}_3^{low}$ into $\mathcal{Q}_3^{high}$ using Lemma 8.2, the error being absorbed in a fraction of the spectral gap again. □

8.3. c-number substitution. In this section we show how the energy can be bounded if we minimize over a specific class of coherent states, which are eigenvectors for the annihilation operator of the condensate. In this way we can turn the action of the condensate operators in the form of multiplication per complex numbers. Let us define

$$|z\rangle = e^{-\left(\frac{|z|^2}{2} + za_0^\dagger\right)}\Omega, \tag{8.33}$$

for any $z \in \mathbb{C}$. As anticipated, we have

$$a_0|z\rangle = z\,|z\rangle. \tag{8.34}$$

Given any state Ψ we define the z-dependent state

$$\Phi(z) := \langle z\,|\,\Psi\rangle, \tag{8.35}$$

obtained by the partial inner product in $\mathscr{F}_s(\mathrm{Ran}\,P)$. One can verify that these states generate the space $\mathscr{F}_s(\mathrm{Ran}\,Q)$. Moreover,

$$1 = \frac{1}{\pi} \int_{\mathbb{C}} |z\rangle\langle z|\,\mathrm{d}z. \tag{8.36}$$

We define the following z-dependent density,

$$\rho_z := \frac{|z|^2}{\ell^2}, \tag{8.37}$$

and z-dependent Hamiltonian,

$$\mathcal{K}(z) = \frac{1}{2}\rho_z^2\ell^2(\widehat{g}_0 + \widehat{g\omega}(0)) - \rho_\mu\rho_z\widehat{g}_0\ell^2 \tag{8.38}$$

$$+ \mathcal{K}^{\mathrm{Bog}} + \frac{b}{2\ell^2}n_+ + \frac{\varepsilon_T b}{8d^2\ell^2}n_+^H + b\frac{\varepsilon_T |z|^2 n_+^H}{16d^2\ell^2(\rho_\mu\ell^2)} + \varepsilon_R(\rho_\mu - \rho_z)^2\delta\ell^2 \tag{8.39}$$

$$+ (\rho_z - \rho_\mu)\widehat{W}_1(0)\frac{\ell^2}{(2\pi)^2}\int_{\mathbb{R}^2} a_k^\dagger a_k \mathrm{d}k + \mathcal{Q}_1^{\mathrm{ex}}(z) + \mathcal{Q}_2^{\mathrm{ex}}(z) + \mathcal{Q}_3(z), \tag{8.40}$$

where $\varepsilon_R \ll 1$ is fixed in "Appendix H", and

$$\mathcal{K}^{\mathrm{Bog}} := \frac{\ell^2}{2(2\pi)^2}\int_{\mathbb{R}^2}\Big(\mathcal{A}(k)(a_k^\dagger a_k + a_{-k}^\dagger a_{-k}) + \mathcal{B}(k)(a_k a_{-k} + a_k^\dagger a_{-k}^\dagger)$$

$$+ \mathcal{C}(k)(a_k^\dagger + a_{-k}^\dagger + a_k + a_{-k})\Big)\mathrm{d}k, \tag{8.41}$$

with

$$\mathcal{A}(k) := (1 - \varepsilon_N)\tau(k) + \mathcal{B}(k), \qquad \mathcal{B}(k) := \rho_z\widehat{W}_1(k),$$

$$\mathcal{C}(k) := \frac{(\rho_z - \rho_\mu)}{\ell^2}\widehat{W}_1(0)\widehat{\chi}_\Lambda(k)z,$$

$$\mathcal{Q}_1^{\mathrm{ex}}(z) := \rho_z\widehat{(\omega W_1)}(0)\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\widehat{\chi}_\Lambda(k)a_k^\dagger z\mathrm{d}k + h.c.,$$

$$\mathcal{Q}_2^{\mathrm{ex}}(z) := \frac{\ell^2}{(2\pi)^2}\rho_z\int_{\mathbb{R}^2}(\widehat{\omega W_1}(0) + \widehat{\omega W_1}(k))a_k^\dagger a_k\mathrm{d}k,$$

$$\mathcal{Q}_3(z) := \frac{\ell^2}{(2\pi)^4}\int_{\mathcal{P}_H\times\mathbb{R}^2} f_L(p)\widehat{W}_1(k)\Big(\bar{z}\tilde{a}_p^\dagger a_{p-k}a_k + h.c.\Big)\mathrm{d}k\mathrm{d}p \tag{8.42}$$

and $\tau(k)$ defined in (8.8). With these notations, the following theorem holds. Recall that $\mathcal{H}_\Lambda^{2\mathrm{nd}}$ is given by Proposition 8.3.

Theorem 8.4. *Assume $R \leq \ell$ and* (H17). *For any normalized n-particle state Ψ satisfying $\Psi = \mathbb{1}_{[0,\mathcal{M}]}(n_+^L)\Psi$ and* (7.40) *we have*

$$\langle \Psi \mid \mathcal{H}_\Lambda^{2nd}\Psi \rangle \geq \inf_{z \in \mathbb{R}_+} \inf_\Phi \langle \Phi \mid \mathcal{K}(z)\Phi \rangle - C\rho_\mu\delta(1 + \varepsilon_R K_\ell^4 K_B^2 |\log Y|), \quad (8.43)$$

where the second infimum is over all the normalized states in $\mathscr{F}(\mathrm{Ran}\,Q)$ such that

$$\Phi = \mathbb{1}_{[0,\mathcal{M}]}(n_+^L)\Phi, \quad and \quad \Phi = \mathbb{1}_{[0,2\rho_\mu\ell^2]}(n_+)\Phi. \quad (8.44)$$

Proof. The theorem is proven via a standard technique of calculating the actions of creation and annihilation operators for the condensate on the coherent state and using its eigenvector properties, for details see [19, Theorem 8.5]. Practically speaking it consists in the formal substitutions

$$a_0 \mapsto z, \quad a_0^\dagger \mapsto \bar{z}, \quad a_0^\dagger a_0 \mapsto |z|^2 - 1, \quad (8.45)$$

and getting rid of the lower order terms in $|z|$ because they produce errors of the form

$$\rho_\mu\delta = \rho_\mu^2\ell^2\delta^2 K_\ell^{-2}. \quad (8.46)$$

In order to make the last term in (8.39) appear, we add and subtract $\varepsilon_R(\rho_\mu - n_0\ell^{-2})^2\delta\ell^2$ to $\mathcal{H}_\Lambda^{2nd}$ and estimate the negative contribution, recalling the estimates in Theorem 7.7 and that $n_+^2 \leq nn_+$ we get

$$-\varepsilon_R\left(\rho_\mu - \frac{n_0}{\ell^2}\right)^2\delta\ell^2 \geq -2\varepsilon_R\delta\ell^{-2}((\rho_\mu\ell^2 - n)^2 + n_+n)$$

$$\geq -C\varepsilon_R\frac{\delta}{\ell^2}n^2K_B^2Y|\log Y|K_\ell^2 = -C\varepsilon_R\rho_\mu\delta K_B^2|\log Y|K_\ell^4,$$

which is coherent with the error terms. □

9. Lower Bounds for the Hamiltonian $\mathcal{K}$

9.1. Estimate of $\mathcal{K}$ for ρ_z far from ρ_μ. The purpose of this section is to show that for values of ρ_z far from the density ρ_μ it is possible to prove a rough estimate on the energy and eliminate these values from the analysis. This is the content of the proposition below. We recall that $\mathcal{K}(z)$ is defined in (8.38), and we use the notations $\varepsilon_{\mathcal{M}} = \frac{\mathcal{M}}{\rho_\mu\ell^2}$ and

$$\delta_1 = \frac{\varepsilon_T^2\varepsilon_{\mathcal{M}}}{d^8K_\ell^4}\left(1 + \frac{K_\ell^2}{K_H^2}\right), \quad \delta_2 = \varepsilon_{\mathcal{M}}^{1/2}, \quad \delta_3 = \delta|\log(dsK_\ell)| + \frac{(dK_\ell)^4}{\varepsilon_T^2}. \quad (9.1)$$

Proposition 9.1. *Assume the relations between the parameters in "Appendix H". There exists a $C > 0$, such that if we have $\rho_\mu a^2 \leq C^{-1}$ and*

$$|\rho_\mu - \rho_z| \geq C\rho_\mu \max\left((\delta_1 + \delta_2 + \delta_3)^{1/2}, \delta^{1/2}\right), \quad (9.2)$$

then for any state $\Phi \in \mathscr{F}(\mathrm{Ran}\,Q)$ satisfying (8.44), *we have*

$$\langle \Phi \mid \mathcal{K}(z)\Phi \rangle \geq -4\pi\rho_\mu^2\ell^2\delta + 8\pi\left(\frac{1}{2} + 2\Gamma + \log\pi\right)\rho_\mu^2\ell^2\delta^2. \quad (9.3)$$

Notice that the second order term in (9.3) is larger than the one aimed for in Theorem 6.7. So the statement of the proposition is that the energy is too large unless $|\rho_\mu - \rho_z|$ is small. The proof of the proposition relies on the technical estimate given by the following lemma.

Lemma 9.2. *Assume the relations between the parameters in "Appendix H". For any normalized $\Phi \in \mathcal{F}(\mathrm{Ran}\, Q)$ such that (8.44) holds,*

$$\langle \Phi \mid \mathcal{K}(z)\Phi \rangle \geq -4\pi\rho_\mu^2 \ell^2 \delta + 4\pi\ell^2(\rho_\mu - \rho_z)^2\delta - C\rho_z\rho_\mu\ell^2\delta\delta_1$$
$$- C\rho_\mu^{1/2}(\rho_\mu + \rho_z)^{3/2}\ell^2\delta\delta_2 - C\rho_z^2\ell^2\delta\delta_3 - C\rho_\mu\delta^2 K_\ell^{-2}(ds)^{-4}. \quad (9.4)$$

Proof of Lemma 9.2. We start by estimating the Q_1 terms. We have for any $\varepsilon > 0$

$$\int_{\mathbb{R}^2} \widehat{\chi}_\Lambda(k)(a_k^\dagger z + a_k \bar{z})\mathrm{d}k$$
$$\leq \int_{\mathbb{R}^2} |\widehat{\chi}_\Lambda(k)|(\varepsilon|z|^2 + \varepsilon^{-1}a_k^\dagger a_k)\mathrm{d}k$$
$$\leq C\Big(\varepsilon|z|^2 + \varepsilon^{-1}|\widehat{\chi}_\Lambda(0)| \int_{k\in\mathcal{P}_H^c} a_k^\dagger a_k \mathrm{d}k + \varepsilon^{-1}\int_{k\in\mathcal{P}_H} |\widehat{\chi}_\Lambda(k)|a_k^\dagger a_k \mathrm{d}k\Big).$$

Considering a Φ like in the assumption we have, using $|\widehat{\chi}_\Lambda(0)| = \ell^2\|\chi\|_1$ together with Lemma 8.1,

$$\Big\langle |\widehat{\chi}_\Lambda(0)| \int_{k\in\mathcal{P}_H^c} a_k^\dagger a_k \mathrm{d}k + \int_{k\in\mathcal{P}_H} |\widehat{\chi}_\Lambda(k)|a_k^\dagger a_k \mathrm{d}k \Big\rangle_\Phi$$
$$\leq C\Big(\mathcal{M} + \rho_\mu\ell^2 \sup_{k\in\mathcal{P}_H} (\ell^{-2}|\widehat{\chi}_\Lambda(k)|)\Big). \quad (9.5)$$

Now, using (F4) and optimizing with $\varepsilon = \sqrt{\mathcal{M}/|z|^2}$,

$$\langle -\frac{\ell^2}{(2\pi)^2} \int_{\mathbb{R}^2} \mathcal{C}(k)(a_k^\dagger + a_{-k}^\dagger + a_k + a_{-k})\mathrm{d}k + \mathcal{Q}_1^{ex}(z)\rangle_\Phi$$
$$\geq -C\delta\sqrt{\mathcal{M}}|z|(|\rho_z - \rho_\mu| + \rho_z)$$
$$\geq -C\Big(\frac{\mathcal{M}}{\rho_\mu\ell^2}\Big)^{1/2}\rho_\mu^{1/2}\ell^2\delta(\rho_\mu + \rho_z)^{3/2}. \quad (9.6)$$

For the terms that are quadratic in the field operators, we use the estimate

$$\Big|\Big\langle \ell^2 \int_{\mathbb{R}^2} \widehat{W}_1(k)a_k^\dagger a_k \mathrm{d}k \Big\rangle_\Phi\Big| \leq C\delta(\mathcal{M} + \langle n_+^H\rangle_\Phi), \quad (9.7)$$

from Lemma 8.1 to obtain that

$$\Big\langle \mathcal{Q}_2^{ex} + (\rho_z - \rho_\mu)\widehat{W}_1(0)\frac{\ell^2}{(2\pi)^2} \int_{\mathbb{R}^2} a_k^\dagger a_k \mathrm{d}k + \frac{\ell^2}{(2\pi)^2} \int_{\mathbb{R}^2} \mathcal{B}_k a_k^\dagger a_k \mathrm{d}k \Big\rangle_\Phi$$
$$\geq -C(\rho_z + \rho_\mu)\big(\rho_\mu\ell^2\delta\varepsilon_\mathcal{M} + \delta\langle n_+^H\rangle_\Phi\big), \quad (9.8)$$

where the $\mathcal{B}_k$ has been extracted from the expression of the $\mathcal{A}_k$. The first term is coherent with the error in the result and the last one can be reabsorbed in a fraction of the spectral gap because of relation (H8).

For the remaining part of $\mathcal{A}_k$ involving τ_k we add and subtract $-\rho_z\delta\varepsilon^{-1/2} + \varepsilon\tau_k$, with $\varepsilon \geq \varepsilon_N$ and estimate

$$(1 - \varepsilon_N)\tau_k \geq \widetilde{\mathcal{A}}_k - \rho_z\delta\varepsilon^{-1/2} + \varepsilon\tau_k, \tag{9.9}$$

with

$$\widetilde{\mathcal{A}}_k = (1 - 2\varepsilon)\Big[|k| - \frac{1}{2ds\ell}\Big]_+^2 + \rho_z\delta\varepsilon^{-1/2}. \tag{9.10}$$

We treat the terms in (9.9) separately, adding them to the remaining parts of the Hamiltonian. The simplest one is

$$-\frac{\ell^2}{(2\pi)^2}\rho_z\varepsilon^{-1/2}\delta\Big\langle \int_{\mathbb{R}^2} a_k^\dagger a_k\,\mathrm{d}k\Big\rangle_\Phi \geq -C\varepsilon^{-1/2}\rho_z\delta(\mathcal{M} + \langle n_+^H\rangle_\Phi), \tag{9.11}$$

where we used Lemma 8.1. We use this estimate to fix the choice of ε in order to absorb the last term in the fraction of the spectral gap represented by the second to last term in (8.39). This yields

$$\varepsilon = C^{-1}\varepsilon_T^{-2}(dK_\ell)^4, \tag{9.12}$$

for some sufficiently large constant C and the relations (H23), (H8) ensure that $\varepsilon_N \leq \varepsilon \ll 1$. For the $\widetilde{\mathcal{A}}$ term plus the B terms in the Hamiltonian we use the Bogoliubov diagonalization procedure stated in Theorem B.1 to obtain

$$\frac{\ell^2}{(2\pi)^2}\int_{\mathbb{R}^2} \widetilde{\mathcal{A}}_k a_k^\dagger a_k + \frac{\mathcal{B}_k}{2}(a_k^\dagger a_{-k}^\dagger + a_k a_{-k})\,\mathrm{d}k \geq -\frac{\ell^2}{2(2\pi)^2}\int_{\mathbb{R}^2} \widetilde{\mathcal{A}}_k - \sqrt{\widetilde{\mathcal{A}}_k^2 - \mathcal{B}_k^2}\,\mathrm{d}k, \tag{9.13}$$

and then we use Lemma C.5 and its proof choosing the parameters $K_1 = \rho_z\varepsilon^{-1/2}/2$, $K_2 = 2\rho_z$, $K = (2ds\ell)^{-1}$ and $\kappa = (1 - 2\varepsilon)$ to derive that

$$(9.13) \geq -\frac{\ell^2}{2(2\pi)^2}\Big(\rho_z^2\frac{1+\varepsilon}{1-2\varepsilon}\int_{\mathbb{R}^2}\mathrm{d}k\frac{\widehat{W}_1^2(k) - \widehat{W}_1^2(0)\mathbb{1}_{\{|k|\leq\ell_\delta^{-1}\}}}{2|k|^2} + C\rho_z\varepsilon^{1/2}\delta(ds\ell)^{-2}$$

$$+\frac{C}{(1-2\varepsilon)}\rho_z^2\delta^2(1 + R^2\ell_\delta^{-2}) + \frac{C\rho_z^2}{1-2\varepsilon}\delta^2|\log((ds\ell)^{-1}\ell_\delta)|\Big). \tag{9.14}$$

Using now Cauchy–Schwarz on the second term, Lemma 6.4, writing only the dominant terms due to the relations between the parameters and recalling the definition (6.23) of ℓ_δ we obtain

$$(9.13) \geq -\frac{1}{2}\rho_z^2\ell^2\widehat{g\omega}(0) - C\rho_z^2\ell^2\delta\big(\varepsilon + \delta^2\rho_\mu R^2 + \delta|\log(ds K_\ell)|\big) - C\delta\ell^2(ds\ell)^{-4}. \tag{9.15}$$

Due to relation (H3) the second term gives δ_3, while the third one gives the last term in (9.4).

We continue considering the third term in (9.9) and adding it to the $\mathcal{Q}_3$. The latter is an integral for $k \in \mathcal{P}_H$, and dropping the part of the τ_k for $k \in \mathcal{P}_H^c$ and using that for $k \in \mathcal{P}_H$ then $\tau_k \geq |k|^2/2$, we have to estimate

$$\frac{\ell^2}{(2\pi)^2} \int_{k \in \mathcal{P}_H} \left(\frac{\varepsilon}{2} k^2 a_k^\dagger a_k + \frac{1}{(2\pi)^2} \int f_L(p) \widehat{W}_1(k) (\bar{z} \tilde{a}_p^\dagger a_{p-k} a_k + a_k^\dagger a_{p-k}^\dagger \tilde{a}_p z) \right) \mathrm{d}p \mathrm{d}k. \tag{9.16}$$

We complete the square in the previous expression, introducing the operators

$$\sigma_k := a_k + \frac{2}{(2\pi)^2} \int f_L(p) \frac{\widehat{W}_1(k)}{\varepsilon |k|^2} z a_{p-k}^\dagger \tilde{a}_p \mathrm{d}p, \tag{9.17}$$

so that

$$\begin{aligned}
(9.16) &= \frac{\ell^2}{(2\pi)^2} \int_{k \in \mathcal{P}_H} \Big(\frac{\varepsilon}{2} k^2 \sigma_k^\dagger \sigma_k \\
&\quad - \frac{2|z|^2}{\varepsilon(2\pi)^4} \iint f_L(p) f_L(s) \frac{\widehat{W}_1(k)^2}{k^2} \tilde{a}_s^\dagger a_{s-k} a_{p-k}^\dagger \tilde{a}_p \Big) \mathrm{d}p \mathrm{d}s \mathrm{d}k \\
&\geq -\frac{2|z|^2 \ell^2}{\varepsilon(2\pi)^6} \int_{k \in \mathcal{P}_H} \frac{\widehat{W}_1(k)^2}{k^2} \\
&\quad \iint f_L(p) f_L(s) \tilde{a}_s^\dagger (a_{p-k}^\dagger a_{s-k} + [a_{s-k}, a_{p-k}^\dagger]) \tilde{a}_p \mathrm{d}p \mathrm{d}s \mathrm{d}k.
\end{aligned}$$

For the term without commutator, estimated on a state Φ which satisfies (8.44) and using Cauchy–Schwarz

$$\tilde{a}_s^\dagger a_{p-k}^\dagger a_{s-k} \tilde{a}_p \leq C(\tilde{a}_s^\dagger a_{p-k}^\dagger a_{p-k} \tilde{a}_s + \tilde{a}_p^\dagger a_{s-k}^\dagger a_{s-k} \tilde{a}_p), \tag{9.18}$$

we have

$$\begin{aligned}
\frac{2|z|^2 \ell^2}{\varepsilon(2\pi)^6} &\left\langle \int_{k \in \mathcal{P}_H} \mathrm{d}k \frac{\widehat{W}_1(k)^2}{k^2} \iint f_L(p) f_L(s) \tilde{a}_s^\dagger a_{p-k}^\dagger a_{p-k} \tilde{a}_s \mathrm{d}p \mathrm{d}s \right\rangle_\Phi \\
&\leq C|z|^2 \varepsilon^{-1} \frac{\ell^4 \delta^2}{K_H^2} \left\langle \int_{k \in \mathcal{P}_H} \int f_L(s) \tilde{a}_s^\dagger a_k^\dagger a_k \tilde{a}_s \mathrm{d}s \mathrm{d}k \right\rangle_\Phi \int_{p \in \mathcal{P}_L} \mathrm{d}p \\
&\leq C\varepsilon^{-1} \frac{\delta^2}{K_H^2} d^{-4} \mathcal{M} \rho_\mu \rho_z \ell^2, \tag{9.19}
\end{aligned}$$

where we used Lemma 8.1 since the support of f_L is included in the complement of $\mathcal{P}_H$, and the estimate, for $k \in \mathcal{P}_H$,

$$\frac{\widehat{W}_1(k)^2}{2k^2} \leq C K_H^{-2} \delta^2 \ell^2. \tag{9.20}$$

For the commutator part we use the estimate (8.6), the Cauchy–Schwarz inequality

$$\tilde{a}_s^\dagger [a_{s-k}, a_{p-k}^\dagger] \tilde{a}_p \leq C \tilde{a}_s^\dagger \tilde{a}_s + C \tilde{a}_p^\dagger \tilde{a}_p, \tag{9.21}$$

and Lemma 3.9 applied to $\widehat{W}_1$ instead of $\widehat{g}$ paying a small error, we get

$$-\frac{2|z|^2\ell^2}{\varepsilon(2\pi)^6}\left\langle \int_{k\in\mathcal{P}_H}\frac{\widehat{W}_1(k)^2}{k^2}\iint f_L(p)f_L(s)\widetilde{a}_s^\dagger[a_{s-k},a_{p-k}^\dagger]\widetilde{a}_p\,dp\,ds\,dk\right\rangle_\Phi$$

$$\geq -C\frac{|z|^2\ell^2}{\varepsilon}\delta\left\langle \iint f_L(p)f_L(s)\widetilde{a}_p^\dagger\widetilde{a}_p\,dp\,ds\right\rangle_\Phi \geq -C\varepsilon^{-1}\rho_z\delta\mathcal{M}d^{-4},\quad (9.22)$$

where in the last inequality we used Lemma 8.1.

Collecting formulas (9.6), (9.8), (9.15), (9.19) and (9.22) and observing that

$$\frac{1}{2}\rho_z^2\ell^2\widehat{g}_0 - \rho_z\rho_\mu\ell^2\widehat{g}_0 = \frac{1}{2}(\rho_z-\rho_\mu)^2\ell^2\widehat{g}_0 - \frac{1}{2}\rho_\mu^2\ell^2\widehat{g}_0,\qquad (9.23)$$

we obtain the result. $\square$

Proof of Proposition 9.1. We observe that, thanks to the relations (H6), (H8), (H22), we have $\delta_j \ll 1$ for $j=1,2,3$. Each coefficient of the δ_j in formula (9.4) can be bounded by

$$C\delta(\rho_\mu-\rho_z)^2\ell^2 + C\rho_\mu^2\ell^2\delta.\qquad (9.24)$$

Therefore, Lemma 9.2 and $\widehat{g}_0 = 8\pi\delta$ implies the bound

$$\langle\mathcal{K}(z)\rangle_\Phi \geq -\frac{1}{2}\rho_\mu^2\ell^2\widehat{g}_0 + \frac{1}{2}(\rho_\mu-\rho_z)^2\ell^2\widehat{g}_0(1-C(\delta_1+\delta_2+\delta_3))$$

$$-C\rho_\mu^2\ell^2\delta(\delta_1+\delta_2+\delta_3+\delta^2(K_\ell ds)^{-4})$$

$$\geq -\frac{1}{2}\rho_\mu^2\ell^2\widehat{g}_0 + \frac{1}{4}\ell^2\widehat{g}_0(\rho_\mu-\rho_z)^2 - C\rho_\mu^2\ell^2\delta(\delta_1+\delta_2+\delta_3+\delta^2(K_\ell ds)^{-4}).$$

Note that $\delta^2(K_\ell ds)^{-4} \ll \delta$ due to (H12) and (H17). By the assumption on $(\rho_\mu-\rho_z)^2$ the second term is of higher order both of the δ_j errors and of the desired quantity in the statement of the Proposition. $\square$

9.2. Estimate of $\mathcal{K}$ for $\rho_z \simeq \rho_\mu$. We study here the main case, that is when ρ_z is close to ρ_μ. More precisely, we consider the complementary situation to (9.2), when

$$|\rho_\mu-\rho_z| \leq K_\ell^{-2}\rho_\mu,\qquad (9.25)$$

where we used that, thanks to the choices of the parameters (H8), (H17) and (H22), we have

$$K_\ell^2\max\left((\delta_1+\delta_2+\delta_3)^{1/2},\delta^{1/2}\right) \leq C^{-1}.\qquad (9.26)$$

Using again (9.23) and reabsorbing the term $(\rho_z-\rho_\mu)\widehat{W}_1(0)\frac{\ell^2}{(2\pi)^2}\int a_k^\dagger a_k\,dk$ in part of the spectral gap of n_+, we have the estimate of $\mathcal{K}(z)$ from (8.38),

$$\mathcal{K}(z) \geq -\frac{1}{2}\rho_\mu^2\ell^2\widehat{g}_0 + \frac{1}{2}\rho_z^2\ell^2\widehat{g\omega}(0) + \frac{1}{2}(\rho_z-\rho_\mu)^2\ell^2\widehat{g}_0$$

$$+\mathcal{K}^{\text{Bog}} + \frac{b}{4\ell^2}n_+ + b\frac{\varepsilon_T}{8d^2\ell^2}n_+^H + b\frac{\varepsilon_T|z|^2n_+^H}{16d^2\ell^2(\rho_\mu\ell^2)} + \varepsilon_R(\rho_\mu-\rho_z)^2\delta\ell^2$$

$$+\mathcal{Q}_1^{\text{ex}}(z) + \mathcal{Q}_2^{\text{ex}}(z) + \mathcal{Q}_3(z),\qquad (9.27)$$

and in the following we want to give a lower bound for the expression above using a diagonalization method for the Bogoliubov Hamiltonian. In order to do that, let us introduce a couple of new creation and annihilation operators

$$b_k := \frac{1}{\sqrt{1 - \alpha_k^2}}(a_k + \alpha_k a^{\dagger}_{-k} + c_k), \tag{9.28}$$

where

$$\alpha_k := \mathcal{B}(k)^{-1}\left(\mathcal{A}(k) - \sqrt{\mathcal{A}(k)^2 - \mathcal{B}(k)^2}\right),$$

$$c_k := \frac{2\mathcal{C}(k)}{\mathcal{A}(k) + \mathcal{B}(k) + \sqrt{\mathcal{A}(k)^2 - \mathcal{B}(k)^2}}\mathbb{1}_{\{|k| \le \frac{1}{2}K_H\ell^{-1}\}},$$

with $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are defined in (8.41) and the diagonalized Bogoliubov Hamiltonian

$$\mathcal{K}_H^{\text{Diag}} := \frac{\ell^2}{(2\pi)^2}\int_{\{|k| \ge \frac{1}{2}K_H\ell^{-1}\}} \mathcal{D}(k)b_k^{\dagger}b_k\mathrm{d}k, \tag{9.29}$$

where

$$\mathcal{D}(k) := \frac{1}{2}\left(\mathcal{A}(k) + \sqrt{\mathcal{A}(k)^2 - \mathcal{B}(k)^2}\right). \tag{9.30}$$

Theorem 9.3. *Assume the relations between the parameters in "Appendix H". For any state* $\Phi \in \mathcal{F}_s(L^2(\Lambda))$ *such that* (8.44) *holds and* $\frac{9}{10}\rho_\mu \le \rho_z \le \frac{11}{10}\rho_\mu$ *we have*

$$\langle \mathcal{K}^{Bog}\rangle_\Phi + \frac{1}{2}\rho_z^2\ell^2(\widehat{g\omega})_0 + \frac{1}{2}(\rho_z - \rho_\mu)^2\ell^2\widehat{g}_0$$

$$\ge (1 - \varepsilon_K)\left\langle\mathcal{K}_H^{Diag}\right\rangle_\Phi + 4\pi\left(2\Gamma + \frac{1}{2} + \log\pi\right)\rho_z^2\ell^2\delta^2$$

$$- C(\rho_\mu - \rho_z)^2\ell^2\delta^2\rho_\mu R^2 - C\rho_\mu^2\ell^2\delta(K_H^{4-M}K_\ell\delta^{-1/2}) + Cr(\rho_\mu)\ell^2,$$

where the error term is given by

$$r(\rho_\mu) := \rho_\mu^2\delta^2\big(\delta|\log(\delta)|R^2\rho_\mu + \delta|\log(\delta)| + d + \varepsilon_T|\log\delta| + (sK_\ell)^{-1} + \varepsilon_N\delta^{-1}\big).$$

In the proof of Theorem 9.3 we are going to use the following formulas and estimates for the commutators of the operators, recalling that $\widehat{\chi}_\Lambda$ is even,

$$[b_k, b_h] = \frac{\alpha_k - \alpha_h}{\sqrt{1 - \alpha_k^2}\sqrt{1 - \alpha_h^2}}\left(\widehat{(\chi^2)}((k+h)\ell) - \widehat{\chi}(k\ell)\widehat{\chi}(h\ell)\right), \tag{9.31}$$

$$[b_k, b_h^{\dagger}] = \frac{1 - \alpha_k\alpha_h}{\sqrt{1 - \alpha_k^2}\sqrt{1 - \alpha_h^2}}\left(\widehat{(\chi^2)}((k-h)\ell) - \widehat{\chi}(k\ell)\widehat{\chi}(h\ell)\right), \tag{9.32}$$

$$[\widetilde{a}_p^{\dagger}, b_k^{\dagger}] = \frac{\alpha_k}{\sqrt{1 - \alpha_k^2}}[\widetilde{a}_p^{\dagger}, a_k] = \frac{\alpha_k}{\sqrt{1 - \alpha_k^2}}\ell^{-2}\langle e^{ipx}, Q\chi_\Lambda e^{ikx}\rangle, \tag{9.33}$$

$$[\widetilde{a}_p, b_{-k}^{\dagger}] = \frac{1}{\sqrt{1 - \alpha_k^2}}[\widetilde{a}_p, a_{-k}^{\dagger}] = \frac{1}{\sqrt{1 - \alpha_k^2}}(\widehat{\chi}((p+k)\ell) - \widehat{\theta}(p\ell)\widehat{\chi}(k\ell)). \tag{9.34}$$

Proof. Let us start by showing that the contribution coming from the $\mathcal{C}(k)$ gives an error term for $|k| > \frac{1}{2} K_H \ell^{-1}$.

By Cauchy–Schwarz we have $a_k^\dagger + a_k \le a_k^\dagger a_k + 1$ and then we recognize n_+ (8.7),

$$\frac{\ell^2}{2(2\pi)^2} \int_{\{|k| > \frac{1}{2} K_H \ell^{-1}\}} \mathcal{C}(k)(a_k^\dagger + a_{-k}^\dagger + a_k + a_{-k}) \, dk$$

$$\ge -C|\rho_\mu - \rho_z||\widehat{W}_1(0)||z| \int_{\{|k| > \frac{1}{2} K_H \ell^{-1}\}} |\widehat{\chi_\Lambda}(k)|(a_k^\dagger a_k + 1) \, dk$$

$$\ge -C\rho_\mu \delta |z|(n_+ + 1) K_H^{4-M},$$

where we use the assumption on ρ_z and that by Lemma F.1,

$$\ell^{-2} \sup_{|k| > \frac{1}{2} K_H \ell^{-1}} (1 + (k\ell)^2)^2 |\widehat{\chi_\Lambda}(k)| \le C K_H^{4-M}. \tag{9.35}$$

When we apply to Φ we have $n_+ \le 2\rho_\mu \ell^2$ and

$$\frac{\ell^2}{2(2\pi)^2} \int_{\{|k| > \frac{1}{2} K_H \ell^{-1}\}} \mathcal{C}(k) \langle a_k^\dagger + a_{-k}^\dagger + a_k + a_{-k} \rangle_\Phi \, dk \ge -C\rho_\mu^2 \ell^2 \delta (K_H^{4-M} \sqrt{\rho_\mu} \ell). \tag{9.36}$$

Therefore

$$\mathcal{K}^{\mathrm{Bog}} \ge \widetilde{\mathcal{K}}^{\mathrm{Bog}} - C\rho_\mu^2 \ell^2 \delta (K_H^{4-M} K_\ell \delta^{-1/2}), \tag{9.37}$$

where $\widetilde{\mathcal{K}}^{\mathrm{Bog}}$ is the same as $\mathcal{K}^{\mathrm{Bog}}$ but with $\mathcal{C}(k)$ substituted by

$$\widetilde{\mathcal{C}}(k) := \mathcal{C}(k) \mathbb{1}_{\{|k| \le \frac{1}{2} K_H \ell^{-1}\}}. \tag{9.38}$$

The bound on the commutator (8.6) allows us to use Theorem B.1 to diagonalize the Bogoliubov Hamiltonian

$$\widetilde{\mathcal{K}}^{\mathrm{Bog}} \ge \widetilde{\mathcal{K}}^{\mathrm{Diag}} - \frac{\ell^2}{2(2\pi)^2} \int_{\mathbb{R}^2} \left(\mathcal{A}(k) - \sqrt{\mathcal{A}(k)^2 - \mathcal{B}(k)^2} \right) dk$$

$$- (\rho_z - \rho_\mu)^2 \widehat{W}_1(0)^2 \frac{z^2}{(2\pi)^2 \ell^2} \int_{\{|k| \le \frac{1}{2} K_H \ell^{-1}\}} \frac{|\widehat{\chi_\Lambda}(k)|^2}{\mathcal{A}(k) + \mathcal{B}(k)} dk,$$

where

$$\widetilde{\mathcal{K}}^{\mathrm{Diag}} = \frac{\ell^2}{(2\pi)^2} \int (1 - \alpha_k^2) \mathcal{D}_k b_k^\dagger b_k dk \ge \frac{\ell^2}{(2\pi)^2} \int_{\{|k| > \frac{1}{2} K_H \ell^{-1}\}} (1 - \alpha_k^2) \mathcal{D}_k b_k^\dagger b_k dk. \tag{9.39}$$

Using the inequality $|\alpha_k| \le C\rho_z \delta k^{-2} \le C K_\ell^2 K_H^{-2}$ we find

$$\widetilde{\mathcal{K}}^{\mathrm{Diag}} \ge \mathcal{K}_H^{\mathrm{Diag}} (1 - C K_\ell^4 K_H^{-4}). \tag{9.40}$$

The calculation of the Bogoliubov integral is given in "Appendix C". Combining the results of Lemma C.1, Lemma C.2 and Proposition C.3 and multiplying everything by ℓ^2 we find

$$-\frac{\ell^2}{2(2\pi)^2}\int_{\mathbb{R}^2}\left(\mathcal{A}(k)-\sqrt{\mathcal{A}(k)^2-\mathcal{B}(k)^2}\right)dk+\frac{1}{2}\widehat{g\omega}(0)\rho_z^2\ell^2$$
$$\geq 4\pi\left(2\Gamma+\frac{1}{2}+\log\pi\right)\rho_z^2\ell^2\delta^2+r(\rho_\mu)\ell^2, \tag{9.41}$$

where $r(\rho_\mu)$ is defined in the statement of the theorem. For the remaining term we use the estimate

$$\mathcal{A}(k)+\mathcal{B}(k)\geq 2\rho_z\widehat{W}_1(k)\geq 2\rho_z\widehat{W}_1(0)(1-C\delta(kR)^2), \tag{9.42}$$

where we used a Taylor expansion and the fact that W_1 is even. By this last estimate, together with Lemma F.1 and (6.24) we obtain

$$-(\rho_z-\rho_\mu)^2\widehat{W}_1^2(0)\frac{z^2}{(2\pi)^2\ell^2}\int_{\{|k|\leq\frac{1}{2}K_H\ell^{-1}\}}\frac{|\widehat{\chi_\Lambda}(k)|^2}{\mathcal{A}(k)+\mathcal{B}(k)}$$
$$\geq -(\rho_z-\rho_\mu)^2\frac{\widehat{W}_1(0)}{2}\ell^2\left(1+C\rho_\mu\delta^2R^2K_H^2K_\ell^{-2}\right)$$
$$\geq -(\rho_z-\rho_\mu)^2\frac{\widehat{g}(0)}{2}\ell^2\left(1+C\rho_\mu\delta R^2\right),$$

where in the last line we used $K_H\ll\delta^{-1/2}$ from (H16). $\square$

9.3. Contribution of $\mathcal{Q}_3$.

The aim of this section is to bound the $3Q$ term from below, namely

$$\mathcal{Q}_3(z)=\frac{\bar{z}\ell^2}{(2\pi)^4}\int_{\mathcal{P}_H\times\mathbb{R}^2}\widehat{W}_1(k)f_L(p)(\widetilde{a}_p^\dagger a_{p-k}a_k+\text{h.c.})dkdp,$$

which turns out to be controlled by the quadratic Hamiltonian $\mathcal{K}_H^{\text{Diag}}$ defined in (9.29), absorbing $\mathcal{Q}_2^{\text{ex}}$ and $\mathcal{Q}_1^{\text{ex}}$. More precisely we prove

Theorem 9.4. *Assume the relations between the parameters in "Appendix H" to be satisfied. Then there exists a universal constant $C>0$ such that for any state Φ satisfying* (8.44) *we have*

$$\left\langle(1-\varepsilon_K)\mathcal{K}_H^{\text{Diag}}+\mathcal{Q}_3(z)+\mathcal{Q}_2^{\text{ex}}+\mathcal{Q}_1^{\text{ex}}+\frac{b}{100}\frac{n_+}{\ell^2}+\frac{\varepsilon_Tb}{100}\frac{n_+^H}{(d\ell)^2}\right\rangle_\Phi$$
$$\geq -C\rho_z^2\ell^2\delta^2\left(\delta K_H^{-8}K_\ell^{10}d^{-4}+\varepsilon_K^{-1}K_H^{-12}K_\ell^{10}d^{-8}+d^{-8}K_\ell^2K_H^{-4}\right.$$
$$+\varepsilon_K^{-1}K_H^{-2M-8}K_\ell^6d^{-8}+\delta K_\ell^2|\log\delta|^2+\delta^{-1}K_\ell^2d^{8M-2}\varepsilon_T^{-1}$$
$$\left.+\varepsilon_\mathcal{M}^{1/2}(K_\ell^4K_H^{-4}+\delta^{-1}K_H^{-M}d^{-2})\right).$$

Remark 9.5. Note that we control $\mathcal{Q}_3 + \mathcal{Q}_2^{\mathrm{ex}}$ using a large fraction of $\mathcal{K}_H^{\mathrm{Diag}}$. It is important to remember that $\mathcal{K}_H^{\mathrm{Diag}}$ is not the kinetic energy, but the Hamiltonian arising from the Bogoliubov diagonalization—sometimes $\mathcal{K}^{\mathrm{Diag}}$ is called the excitation Hamiltonian. The kinetic energy is already contributing to main order in the energy, and we use it to obtain the LHY term (Theorem 9.3). The operator $\mathcal{K}_H^{\mathrm{Diag}}$ is much smaller than the kinetic energy, and this is why we can use all of it to control $\mathcal{Q}_3 + \mathcal{Q}_2^{\mathrm{ex}}$.

In order to prove this theorem, we start by rewriting $\mathcal{Q}_3(z)$ in terms of the b_k's defined in (9.28). Notice that $c_k = c_{p-k} = 0$ if $k \in \mathcal{P}_H$ and $p \in \mathcal{P}_L$, and

$$a_k = \frac{b_k - \alpha_k b_{-k}^\dagger}{\sqrt{1 - \alpha_k^2}}, \qquad a_{p-k} = \frac{b_{p-k} - \alpha_{p-k} b_{k-p}^\dagger}{\sqrt{1 - \alpha_{p-k}^2}}. \tag{9.43}$$

Therefore,

$$a_{p-k} a_k = \frac{b_{p-k} b_k - \alpha_k b_{p-k} b_{-k}^\dagger - \alpha_{p-k} b_{k-p}^\dagger b_k + \alpha_{p-k} \alpha_k b_{k-p}^\dagger b_{-k}^\dagger}{\sqrt{1 - \alpha_k^2}\sqrt{1 - \alpha_{p-k}^2}},$$

and $\mathcal{Q}_3(z) = \mathcal{Q}_3^{(1)} + \mathcal{Q}_3^{(2)} + \mathcal{Q}_3^{(3)} + \mathcal{Q}_3^{(4)}$ where

$$\mathcal{Q}_3^{(1)} = \frac{z\ell^2}{(2\pi)^4} \int_{\mathcal{P}_H \times \mathbb{R}^2} \frac{f_L(p)\widehat{W}_1(k)}{\sqrt{1 - \alpha_k^2}\sqrt{1 - \alpha_{p-k}^2}} \left(\widetilde{a}_p^\dagger b_{p-k} b_k + \alpha_k \alpha_{p-k} \widetilde{a}_p^\dagger b_{k-p}^\dagger b_{-k}^\dagger + h.c.\right), \tag{9.44}$$

$$\mathcal{Q}_3^{(2)} = -\frac{z\ell^2}{(2\pi)^4} \int_{\mathcal{P}_H \times \mathbb{R}^2} \frac{f_L(p)\widehat{W}_1(k)\alpha_k}{\sqrt{1 - \alpha_k^2}\sqrt{1 - \alpha_{p-k}^2}} \left(\widetilde{a}_p^\dagger b_{-k}^\dagger b_{p-k} + b_{p-k}^\dagger b_{-k} \widetilde{a}_p\right), \tag{9.45}$$

$$\mathcal{Q}_3^{(3)} = -\frac{z\ell^2}{(2\pi)^4} \int_{\mathcal{P}_H \times \mathbb{R}^2} \frac{f_L(p)\widehat{W}_1(k)\alpha_{p-k}}{\sqrt{1 - \alpha_k^2}\sqrt{1 - \alpha_{p-k}^2}} \left(\widetilde{a}_p^\dagger b_{k-p}^\dagger b_k + b_k^\dagger b_{k-p} \widetilde{a}_p\right), \tag{9.46}$$

$$\mathcal{Q}_3^{(4)} = -\frac{z\ell^2}{(2\pi)^4} \int_{\mathcal{P}_H \times \mathbb{R}^2} \frac{f_L(p)\widehat{W}_1(k)}{\sqrt{1 - \alpha_k^2}\sqrt{1 - \alpha_{p-k}^2}} \alpha_k [b_{p-k}, b_{-k}^\dagger](\widetilde{a}_p^\dagger + \widetilde{a}_p). \tag{9.47}$$

In the remaining of this section, we get lower bounds on those four terms (Lemmas 9.7, 9.9 and 9.11 below) hence proving Theorem 9.4.

We collect here some important technical estimates which are going to be useful in the following.

Lemma 9.6. *The following bounds hold:*

$$|\alpha_k| \leq C\rho_z \delta |k|^{-2} \leq C K_\ell^2 K_H^{-2}, \quad \text{for } |k| \geq \frac{1}{2} K_H \ell^{-1}, \tag{9.48}$$

$$\mathcal{D}_k \geq \frac{1}{2}|k|^2 \geq \frac{1}{8} K_H^2 \ell^{-2}, \quad \text{for } |k| \geq \frac{1}{2} K_H \ell^{-1}, \tag{9.49}$$

$$\left| \rho_z \widehat{(\omega W_1)}(0) - \frac{1}{(2\pi)^2} \int_{\mathcal{P}_H} \widehat{W}_1(k)\alpha_k \mathrm{d}k \right| \leq C\rho_z \delta^2 |\log \delta|, \tag{9.50}$$

$$\left| \widehat{(\omega W_1)}(0) - \frac{1}{(2\pi)^2} \int_{\mathcal{P}_H} \frac{\widehat{W}_1(k)^2}{2\mathcal{D}_k} \mathrm{d}k \right| \leq C\delta^2 |\log \delta|, \tag{9.51}$$

and

$$\rho_z \frac{\ell^2}{(2\pi)^2} \int_{\mathbb{R}^2} \widehat{(W_1\omega)}(k) a_k^\dagger a_k dk$$

$$\geq \rho_z \widehat{(W_1\omega)}(0) \frac{\ell^2}{(2\pi)^2} \int_{\mathbb{R}^2} a_k^\dagger a_k dk - 4\rho_z \delta n_+^H - C\rho_z \delta d^{-2} \frac{R}{\ell} n_+. \tag{9.52}$$

Proof. The first two inequalities are straightforward from the definitions of the terms. For the third one we split the difference in the following way,

$$\left| \rho_z \widehat{(\omega W_1)}(0) - \frac{1}{(2\pi)^2} \int_{k\in\mathcal{P}_H} \widehat{W}_1(k)\alpha_k dk \right|$$

$$\leq C \left| \rho_z \int_{k\notin\mathcal{P}_H} \frac{\widehat{W}_1(k)\widehat{g}_k - \widehat{W}_1(0)\widehat{g}_0 \mathbb{1}_{\{|k|\leq\ell_\delta^{-1}\}}}{2k^2} dk \right| + C \left| \int_{k\in\mathcal{P}_H} \widehat{W}_1(k)\left(\alpha_k - \rho_z \frac{\widehat{g}_k}{2k^2}\right) dk \right|$$

$$=: (I) + (II). \tag{9.53}$$

For the first integral we do a further splitting of the domain of integration, considering $(I) \leq (I, <) + (I, >)$ for $|k| \leq \ell_\delta^{-1}$ or otherwise, respectively. For $(I, <)$ we consider a Taylor expansion of the numerator and we get, recalling the symmetry of g which in the integration drops the first order,

$$(I, <) \leq C\rho_z R^2 \delta^2 \int_{\{|k|\leq\ell_\delta^{-1}\}} \leq C\rho_z R^2 \delta^2 \ell_\delta^{-2}. \tag{9.54}$$

For the $(I, >)$ we proceed by a direct calculation and obtain

$$(I, >) \leq C\rho_z \delta^2 \log(K_H \ell^{-1} \ell_\delta). \tag{9.55}$$

Let us analyze the second integral. We have that $|\mathcal{B}_k/\mathcal{A}_k| \leq 1/2$ and therefore we can expand in the following way

$$\widehat{W}_1(k)\alpha_k = \rho_z^{-1}\mathcal{A}_k\left(1 - \sqrt{1 - \frac{\mathcal{B}_k^2}{\mathcal{A}_k^2}}\right) \simeq \rho_z \frac{\widehat{W}_1(k)^2}{2\mathcal{A}_k} + C\rho_z^3 \frac{\widehat{W}_1(k)^4}{\mathcal{A}_k^3}. \tag{9.56}$$

We deduce

$$(II) \leq C \left| \int_{k\in\mathcal{P}_H} \left(\widehat{W}_1(k)\alpha_k - \rho_z \frac{\widehat{W}_1(k)^2}{2\mathcal{A}_k}\right) dk \right| + C\rho_z \left| \int_{k\in\mathcal{P}_H} \widehat{W}_1(k)\left(\frac{\widehat{W}_1(k)}{2\mathcal{A}_k} - \frac{\widehat{g}_k}{2k^2}\right) dk \right|$$

$$\leq C\rho_z^3 \int_{k\in\mathcal{P}_H} \frac{\widehat{W}_1(k)^4}{\mathcal{A}_k^3} dk + C\rho_z \left| \int_{k\in\mathcal{P}_H} \widehat{W}_1(k)\left(\frac{\widehat{W}_1(k)}{2\mathcal{A}_k} - \frac{\widehat{g}_k}{2k^2}\right) dk \right|$$

$$\leq C\rho_z^3 \ell^4 \delta^4 K_H^{-4} + C\rho_z \left| \int_{k\in\mathcal{P}_H} \widehat{W}_1(k)^2 \left(\frac{1}{2\mathcal{A}_k} - \frac{1}{|k|^2}\right) dk \right|$$

$$+ C\rho_z \left| \int_{k\in\mathcal{P}_H} \left(\widehat{W}_1(k)\frac{\widehat{W}_1(k) - \widehat{g}_k}{2k^2}\right) dk \right|,$$

where we used that $\mathcal{A}_k \geq \frac{1}{2}|k|^2$ for $k \in \mathcal{P}_H$. For the remaining terms, we use that in $\mathcal{P}_H$ we have $0 < k^2 - \tau_k \leq 2|k|(ds\ell)^{-1}$,

$$C\rho_z \left| \int_{k \in \mathcal{P}_H} \frac{\widehat{W}_1(k)^2}{k^2} \left(\frac{k^2 - \mathcal{A}_k}{\mathcal{A}_k} \right) dk \right| \leq C\rho_z \int_{k \in \mathcal{P}_H} \frac{\widehat{W}_1(k)^2}{k^2} \left(\frac{2|k|(ds\ell)^{-1}}{k^2} + \rho_z \frac{\widehat{W}_1(k)}{k^2} \right)$$

$$\leq C\rho_z \delta^2 (ds)^{-1} K_H^{-1} + C\rho_z^2 \ell^2 \delta^3 K_H^{-2}.$$

By Cauchy–Schwarz inequality we get for the last term

$$\rho_z \left| \int_{k \in \mathcal{P}_H} \left(\widehat{W}_1(k) \frac{\widehat{W}_1(k) - \widehat{g}_k}{2k^2} \right) dk \right|$$

$$\leq C\rho_z \delta \int_{k \in \mathcal{P}_H} \frac{\widehat{W}_1(k)^2}{2k^2} dk + C\rho_z \delta^{-1} \int_{k \in \mathcal{P}_H} \frac{(\widehat{W}_1(k) - \widehat{g}_k)^2}{2k^2} dk.$$

We complete the domain of the integrals: by Lemma 6.4 we get

$$\rho_z \delta \int_{k \in \mathcal{P}_H} \frac{\widehat{W}_1(k)^2}{2k^2} dk \leq C\rho_z \delta \widehat{g\omega}(0) + C\rho_z \delta \int_{k \notin \mathcal{P}_H} \frac{\widehat{W}_1(k)^2 - \widehat{W}_1(0)^2 \mathbb{1}_{\{|k| \leq \ell_\delta^{-1}\}}}{2k^2} dk$$

$$\leq C\rho_z \delta^2 + C\rho_z \delta^3 (R^2 \ell_\delta^{-2} + \log(K_H \ell^{-1} \ell_\delta)),$$

and

$$\rho_z \delta^{-1} \int_{k \in \mathcal{P}_H} \frac{(\widehat{W}_1(k) - \widehat{g}_k)^2}{2k^2} dk$$

$$\leq C\rho_z \delta^{-1} \frac{R^4}{\ell^4} \widehat{g\omega}(0) + C\rho_z \delta^{-1} \left| \int_{k \notin \mathcal{P}_H} \frac{(\widehat{W}_1(k) - \widehat{g}_k)^2 - (\widehat{W}_1(0) - \widehat{g}_0)^2 \mathbb{1}_{\{|k| \leq \ell_\delta^{-1}\}}}{2k^2} dk \right|$$

$$\leq C\rho_z \frac{R^4}{\ell^4} + C\rho_z \delta \left(R^2 \ell_\delta^{-2} + \frac{R^2}{\ell^2} \log(K_H \ell^{-1} \ell_\delta) \right).$$

We conclude the proof of (9.50) by collecting all the previous estimates and exploiting the relations between the parameters so that $\rho_z \delta^2 |\log \delta|$ is the dominant term.

For the inequality (9.51), we can derive it from (9.50) and the control on the first term of (II) above using that, for $k \in \mathcal{P}_H$, $\left| 1 - \frac{\mathcal{A}_k}{\mathcal{D}_k} \right| \leq \frac{\mathcal{B}_k^2}{\mathcal{A}_k^2} \leq C\rho_z^2 \delta^2 |k|^{-4}$.

For the last inequality, we estimate the difference, splitting the integral for $|k| \leq \xi \ell^{-1}$ or otherwise,

$$\rho_z \frac{\ell^2}{(2\pi)^2} \int_{\mathbb{R}^2} ((\widehat{W_1 \omega})(k) - (\widehat{W_1 \omega})(0)) a_k^\dagger a_k dk$$

$$\geq -C\rho_z \xi^2 \delta \frac{R^2}{\ell^2} n_+ - \frac{2\ell^2}{(2\pi)^2} \rho_z \delta \int_{\mathbb{R}^2} a_k^\dagger \mathbb{1}_{\{|k| \geq \xi \ell^{-1}\}} a_k dk$$

where we used a Taylor expansion and estimated the integral for $|k| \leq \xi \ell^{-1}$. For the second term we exploit the second quantization in a N-bosons sector and we insert symmetrically the sum of projectors $1 = \mathbb{1}_{\{\sqrt{-\Delta} \in \mathcal{P}_L\}} + \mathbb{1}_{\{\sqrt{-\Delta} \in \mathcal{P}_L^c\}}$

$$\frac{\ell^2}{(2\pi)^2} \int_{\mathbb{R}^2} a_k^\dagger \mathbb{1}_{\{|k| \geq \xi \ell^{-1}\}} a_k dk \Big|_N = \sum_{j=1}^N Q_j \chi_\Lambda(x_j) \mathbb{1}_{\{\sqrt{-\Delta_j} \geq \xi \ell^{-1}\}} \chi_\Lambda(x_j) Q_j$$

$$\geq 2n_+^H + 2\mathcal{N} n_+$$

where we estimated by a Cauchy–Schwarz the cross terms $(\mathcal{P}_L, \mathcal{P}_L^c)$ to make them comparable to the diagonal terms and denoted by

$$\mathcal{N} := \|\mathbb{1}_{\{\sqrt{-\Delta}\in\mathcal{P}_L\}}\chi_\Lambda(x)\mathbb{1}_{\{\sqrt{-\Delta}\geq\xi\ell^{-1}\}}\|^2 \leq C\xi^{-2}d^{-4}. \tag{9.57}$$

Here we used the regularity properties of χ_Λ dividing and multiplying by $-\Delta$. We conclude optimizing ξ by the choice $\xi^2 = d^{-2}\frac{\ell}{R}$. $\qquad\square$

9.3.1. Estimates on $\mathcal{Q}_3^{(1)}$ The first part $\mathcal{Q}_3^{(1)}$ will absorb $\mathcal{Q}_2^{ex}$ using $\mathcal{K}_H^{Diag}$.

Lemma 9.7 (Estimates on $\mathcal{Q}_3^{(1)}$). *For any state* Φ *satisfying* (8.44) *we have*

$$\left\langle \mathcal{Q}_3^{(1)} + \mathcal{Q}_2^{ex} + \left(1 - 2\varepsilon_K\right)\mathcal{K}_H^{Diag} + \frac{b}{100}\frac{n_+}{\ell^2} + \frac{b}{100}\frac{\varepsilon_T n_+^H}{(d\ell)^2}\right\rangle_\Phi$$
$$\geq -C\rho_z^2\ell^2\delta^2\left(\delta K_H^{-8}K_\ell^{10}d^{-4} + \varepsilon_K^{-1}K_H^{-12}K_\ell^{10}d^{-8} + d^{-8}K_\ell^2 K_H^{-4}\right).$$

Proof. We first reorder the creation an annihilation operators, applying a change of variables $k \mapsto -k$, $p \mapsto -p$ in the α terms,

$$\mathcal{Q}_3^{(1)} = \frac{z\ell^2}{(2\pi)^4}\int_{\mathcal{P}_H\times\mathbb{R}^2} \frac{f_L(p)\widehat{W}_1(k)}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_{p-k}^2}}$$
$$\times \left(\tilde{a}_p^\dagger b_{p-k}b_k + \alpha_k\alpha_{p-k}\tilde{a}_{-p}^\dagger b_{p-k}^\dagger b_k^\dagger + b_k^\dagger b_{p-k}^\dagger \tilde{a}_p + \alpha_k\alpha_{p-k}b_k b_{p-k}\tilde{a}_{-p}\right)dkdp$$
$$= \frac{z\ell^2}{(2\pi)^4}\int_{\mathcal{P}_H\times\mathbb{R}^2} \frac{f_L(p)\widehat{W}_1(k)}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_{p-k}^2}}\left(\left(\tilde{a}_p^\dagger b_{p-k} + \alpha_k\alpha_{p-k}b_{p-k}\tilde{a}_{-p}\right)b_k\right.$$
$$+ b_k^\dagger\left(b_{p-k}^\dagger\tilde{a}_p + \alpha_k\alpha_{p-k}\tilde{a}_{-p}^\dagger b_{p-k}^\dagger\right)$$
$$\left.+ \alpha_k\alpha_{p-k}\left(\left[b_k, b_{p-k}\tilde{a}_{-p}\right] + \left[\tilde{a}_{-p}^\dagger b_{p-k}^\dagger, b_k^\dagger\right]\right)\right)dkdp.$$

We can complete the square to get, for $\varepsilon_K \ll 1$ fixed in "Appendix H",

$$\mathcal{Q}_3^{(1)} + (1 - 3\varepsilon_K)\mathcal{K}_H^{Diag} = (1 - 3\varepsilon_K)\frac{\ell^2}{(2\pi)^2}\int_{\mathcal{P}_H} \mathcal{D}_k\tilde{b}_k^\dagger\tilde{b}_k dk$$
$$+ \frac{\ell^2}{(2\pi)^2}\int_{\mathcal{P}_H} \left(\mathcal{T}_1(k) + \mathcal{T}_2(k)\right)dk, \tag{9.58}$$

where we mantained a small portion of $\mathcal{K}_H^{\text{Diag}}$ in order to bound other error terms and we defined

$$\tilde{b}_k := b_k + \frac{z}{\mathcal{D}_k(1 - 3\varepsilon_K)(2\pi)^2}$$

$$\int \frac{f_L(p)\widehat{W}_1(k)}{\sqrt{1 - \alpha_k^2}\sqrt{1 - \alpha_{p-k}^2}}\left(b_{p-k}^\dagger \tilde{a}_p + \alpha_k \alpha_{p-k}\tilde{a}_{-p}^\dagger b_{p-k}^\dagger\right)\mathrm{d}p, \tag{9.59}$$

$$\mathcal{T}_1(k) := \frac{z}{(2\pi)^2}\int_{\mathbb{R}^2}\frac{f_L(p)\widehat{W}_1(k)}{\sqrt{1 - \alpha_k^2}\sqrt{1 - \alpha_{p-k}^2}}\alpha_k \alpha_{p-k}\left([b_k^\dagger, \tilde{a}_{-p}^\dagger b_{p-k}^\dagger] + h.c.\right)\mathrm{d}p, \tag{9.60}$$

$$\mathcal{T}_2(k) := -\frac{|z|^2\widehat{W}_1(k)^2}{(1 - 3\varepsilon_K)\mathcal{D}_k(1 - \alpha_k^2)(2\pi)^4}\int\frac{f_L(p)f_L(s)}{\sqrt{1 - \alpha_{s-k}^2}\sqrt{1 - \alpha_{p-k}^2}}$$

$$\times \left(\tilde{a}_p^\dagger b_{p-k} + \alpha_k \alpha_{p-k}b_{p-k}\tilde{a}_{-p}\right)\left(b_{s-k}^\dagger \tilde{a}_s + \alpha_k \alpha_{s-k}\tilde{a}_{-s}^\dagger b_{s-k}^\dagger\right)\mathrm{d}p\mathrm{d}s. \tag{9.61}$$

• Let us estimate the error term $\mathcal{T}_1(k)$. We use $[b_k^\dagger, \tilde{a}_{-p}^\dagger b_{p-k}^\dagger] = \tilde{a}_{-p}^\dagger[b_k^\dagger, b_{p-k}^\dagger] + [b_k^\dagger, \tilde{a}_{-p}^\dagger]b_{p-k}^\dagger$ and the Cauchy–Schwarz inequality with weights $\varepsilon_1, \varepsilon_2 > 0$,

$$\mathcal{T}_1(k) \geq -Cz\int_{\mathbb{R}^2}\frac{f_L(p)\widehat{W}_1(k)}{\sqrt{1 - \alpha_k^2}\sqrt{1 - \alpha_{p-k}^2}}|\alpha_k \alpha_{p-k}|$$

$$\left(\left(\varepsilon_1\tilde{a}_{-p}^\dagger \tilde{a}_{-p} + \varepsilon_1^{-1}\right)|[b_k^\dagger, b_{p-k}^\dagger]| + |[b_k^\dagger, \tilde{a}_{-p}^\dagger]|\left(\varepsilon_2 b_{p-k}^\dagger b_{p-k} + \varepsilon_2^{-1}\right)\right)\mathrm{d}p.$$

By (9.33) and (9.31) we have $|[b_k^\dagger, \tilde{a}_{-p}^\dagger]| \leq C|\alpha_k|$ and $|[b_k^\dagger, b_{p-k}^\dagger]| \leq C|\alpha_k|$. Therefore using (9.48),

$$\frac{\ell^2}{(2\pi)^2}\int_{\mathcal{P}_H}\mathcal{T}_1(k)\mathrm{d}k \geq -C\frac{z\ell^2}{(2\pi)^4}\int_{\mathcal{P}_H \times \mathbb{R}^2}\frac{|f_L(p)|\rho_z^3\delta^4}{k^6}$$

$$\times \left(\left(\varepsilon_1\tilde{a}_{-p}^\dagger \tilde{a}_{-p} + \varepsilon_1^{-1}\right) + \left(\varepsilon_2 b_{p-k}^\dagger b_{p-k} + \varepsilon_2^{-1}\right)\right)\mathrm{d}k\mathrm{d}p.$$

Due to the presence of the cutoff f_L on low momenta and the bounds

$$\int_{\mathcal{P}_L}\left(\varepsilon_1\tilde{a}_{-p}^\dagger \tilde{a}_{-p} + \varepsilon_1^{-1}\right)\mathrm{d}p \leq C\frac{\varepsilon_1 n_+}{\ell^2} + \varepsilon_1^{-1}\frac{d^{-4}}{\ell^2}, \tag{9.62}$$

$$\int_{\mathcal{P}_L}\left(\varepsilon_2 b_k^\dagger b_k + \varepsilon_2^{-1}\right)\mathrm{d}p \leq C\frac{d^{-4}}{\ell^2}\left(\varepsilon_2 b_k^\dagger b_k + \varepsilon_2^{-1}\right), \tag{9.63}$$

where we changed the k variable, we find,

$$\frac{\ell^2}{(2\pi)^2}\int_{\mathcal{P}_H}\mathcal{T}_1(k)\mathrm{d}k \geq -Cz\rho_z^3\delta^4\int_{\mathcal{P}_H}\frac{1}{k^6}\left(\left(\varepsilon_1 n_+ + \varepsilon_1^{-1}d^{-4}\right) + d^{-4}\left(\varepsilon_2 b_k^\dagger b_k + \varepsilon_2^{-1}\right)\right)\mathrm{d}k.$$

We insert $\mathcal{D}_k \geq C^{-1}k^2$ in front of $b_k^{\dagger} b_k$ and get the bound

$$
\frac{\ell^2}{(2\pi)^2} \int_{\mathcal{P}_H} \mathcal{T}_1(k) \mathrm{d}k
$$
$$
\geq -C z \rho_z^3 \delta^4 \ell^6 K_H^{-4} \varepsilon_1 \frac{n_+}{\ell^2} - C z \varepsilon_1^{-1} d^{-4} \rho_z^3 \delta^4 \ell^4 K_H^{-4}
$$
$$
- C \varepsilon_2 \ell^6 z \rho_z^3 \delta^4 K_H^{-8} d^{-4} \ell^2 \int_{\mathcal{P}_H} \mathcal{D}_k b_k^{\dagger} b_k \mathrm{d}k - C \varepsilon_2^{-1} \ell^4 z \rho_z^3 \delta^4 d^{-4} K_H^{-4}.
$$

One can choose ε_1, ε_2 such that the first and third terms are absorbed in the positive $\frac{b}{100}\frac{n_+}{\ell^2}$ and $\varepsilon_K \mathcal{K}_H^{\mathrm{Diag}}$ respectively. With this choice the second and fourth terms are errors of respective sizes

$$
C\ell^2 \rho_z^2 \delta^2 (\delta K_\ell^{10} K_H^{-8} d^{-4}) \quad \text{and} \quad C\ell^2 \rho_z^2 \delta^2 (\delta K_\ell^{10} K_H^{-12} d^{-4} \varepsilon_K^{-1}).
$$

• Let us now focus on the square term $\mathcal{T}_2(k)$ in (9.61). One can write, in normal order,

$$
\widetilde{a}_p^{\dagger} b_{p-k} + \alpha_k \alpha_{p-k} b_{p-k} \widetilde{a}_{-p} = \widetilde{a}_p^{\dagger} b_{p-k} + \alpha_k \alpha_{p-k} \widetilde{a}_{-p} b_{p-k} + \alpha_k \alpha_{p-k} [b_{p-k}, \widetilde{a}_{-p}],
$$

and use the Cauchy–Schwarz inequality with weight ε_K on the cross terms to find

$$
\mathcal{T}_2(k) \geq (1 + \varepsilon_K) \mathcal{T}_2'(k) + (1 + \varepsilon_K^{-1}) \mathcal{T}_2''(k), \tag{9.64}
$$

with

$$
\mathcal{T}_2'(k) = -\frac{|z|^2 \widehat{W}_1(k)^2}{(1 - 3\varepsilon_K) \mathcal{D}_k (1 - \alpha_k^2)(2\pi)^4} \int \frac{f_L(p) f_L(s)}{\sqrt{1 - \alpha_{p-k}^2}\sqrt{1 - \alpha_{s-k}^2}}
$$
$$
\times (\widetilde{a}_p^{\dagger} + \alpha_k \alpha_{p-k} \widetilde{a}_{-p}) b_{p-k} b_{s-k}^{\dagger} (\widetilde{a}_s + \alpha_k \alpha_{s-k} \widetilde{a}_{-s}^{\dagger}) \mathrm{d}p \mathrm{d}s,
$$
$$
\mathcal{T}_2''(k) = -\frac{|z|^2 \widehat{W}_1(k)^2}{(1 - 3\varepsilon_K) \mathcal{D}_k (1 - \alpha_k^2)(2\pi)^4} \int \frac{f_L(p) f_L(s)}{\sqrt{1 - \alpha_{s-k}^2}\sqrt{1 - \alpha_{p-k}^2}}
$$
$$
\times \alpha_k^2 \alpha_{p-k} \alpha_{s-k} |[b_{p-k}, \widetilde{a}_{-p}]| |[\widetilde{a}_{-s}^{\dagger}, b_{s-k}^{\dagger}]| \mathrm{d}p \mathrm{d}s.
$$

$\mathcal{T}_2''$ we can estimate (for $k \in \mathcal{P}_H$),

$$
\frac{\ell^2}{(2\pi)^2} \int_{\mathcal{P}_H} \mathcal{T}_2''(k) \mathrm{d}k \geq -C\rho_z \ell^4 \left(\int_{\mathcal{P}_H} \frac{\widehat{W}_1(k)^2}{(1 - 3\varepsilon_K)\mathcal{D}_k} |\alpha_k|^4 \mathrm{d}k \right) d^{-8} \ell^{-4} \sup |[b_{p-k}, \widetilde{a}_{-s}]|^2,
$$

and by (9.51), (9.33) and (9.48) we get

$$
(1 + \varepsilon_K^{-1}) \frac{\ell^2}{(2\pi)^2} \int_{\mathcal{P}_H} \mathcal{T}_2''(k) \mathrm{d}k \geq -C\rho_z^2 \ell^2 \delta^2 (\varepsilon_K^{-1} K_H^{-12} K_\ell^{10} d^{-8}). \tag{9.65}
$$

Now we use a commutator to write $\mathcal{T}_2' = \mathcal{T}_{2,\mathrm{op}}' + \mathcal{T}_{2,\mathrm{com}}'$ in normal order for the b_k, with

$$\mathcal{T}_{2,\mathrm{op}}'(k) = -\frac{|z|^2 \widehat{W}_1(k)^2}{(2\pi)^4 (1 - 3\varepsilon_K)\mathcal{D}_k(1 - \alpha_k^2)} \int \frac{f_L(p) f_L(s)}{\sqrt{1 - \alpha_{p-k}^2}\sqrt{1 - \alpha_{s-k}^2}}$$

$$\times (\widetilde{a}_p^\dagger + \alpha_k \alpha_{p-k} \widetilde{a}_{-p}) b_{s-k}^\dagger b_{p-k} (\widetilde{a}_s + \alpha_k \alpha_{s-k} \widetilde{a}_{-s}^\dagger)\mathrm{d}p\mathrm{d}s,$$

$$\mathcal{T}_{2,\mathrm{com}}'(k) = -\frac{|z|^2 \widehat{W}_1(k)^2}{(2\pi)^4 (1 - 3\varepsilon_K)\mathcal{D}_k(1 - \alpha_k^2)} \int \frac{f_L(p) f_L(s)}{\sqrt{1 - \alpha_{p-k}^2}\sqrt{1 - \alpha_{s-k}^2}}$$

$$\times (\widetilde{a}_p^\dagger + \alpha_k \alpha_{p-k} \widetilde{a}_{-p})[b_{p-k}, b_{s-k}^\dagger](\widetilde{a}_s + \alpha_k \alpha_{s-k} \widetilde{a}_{-s}^\dagger)\mathrm{d}p\mathrm{d}s. \qquad (9.66)$$

- In order to estimate the error term $\mathcal{T}_{2,\mathrm{op}}'$, we introduce

$$\tau_s := \widetilde{a}_s + \alpha_k \alpha_{s-k} \widetilde{a}_{-s}^\dagger \quad \text{and} \quad \mathcal{C} := \sup_{p,s \in \mathcal{P}_L, k \in \mathcal{P}_H} |[b_{p-k}, \tau_s]|. \qquad (9.67)$$

In $\mathcal{T}_{2,\mathrm{op}}'$ we commute the b's trough the a's,

$$\tau_p^\dagger b_{s-k}^\dagger b_{p-k} \tau_s = b_{s-k}^\dagger \tau_p^\dagger \tau_s b_{p-k} + [\tau_p^\dagger, b_{s-k}^\dagger]\tau_s b_{p-k}$$

$$+ b_{s-k}^\dagger \tau_p^\dagger [b_{p-k}, \tau_s] + [\tau_p^\dagger, b_{s-k}^\dagger][b_{p-k}, \tau_s],$$

and use the Cauchy–Schwarz inequality

$$\tau_p^\dagger b_{s-k}^\dagger b_{p-k} \tau_s \leq C(b_{s-k}^\dagger \tau_p^\dagger \tau_p b_{s-k} + b_{p-k}^\dagger \tau_s^\dagger \tau_s b_{p-k} + \mathcal{C}^2).$$

Inserting it in $\mathcal{T}_{2,\mathrm{op}}'$, bounding $(1 - 3\varepsilon_K)(1 - \alpha_k) \geq 1/2$ and noticing that we can exchange s and p in the integral, we find

$$\mathcal{T}_{2,\mathrm{op}}'(k) \geq -C\frac{|z|^2 \widehat{W}_1(k)^2}{\mathcal{D}_k} \int \frac{f_L(p) f_L(s)}{\sqrt{1 - \alpha_{p-k}^2}\sqrt{1 - \alpha_{s-k}^2}}(b_{s-k}^\dagger \tau_p^\dagger \tau_p b_{s-k} + \mathcal{C}^2)\mathrm{d}p\mathrm{d}s.$$

When we apply this operator to the state Φ which satisfies $\mathbb{1}_{[0,\mathcal{M}]}(n_+^L)\Phi = \Phi$ we can apply Lemma 8.1 for the vector $b_{s-k}\Phi$ to get the estimate

$$\langle \mathcal{T}_{2,\mathrm{op}}'(k)\rangle_\Phi \geq -C\frac{|z|^2 \widehat{W}_1(k)^2}{\mathcal{D}_k}\left(\ell^{-2}\mathcal{M}\int f_L(s)\langle b_{s-k}^\dagger b_{s-k}\rangle_\Phi \mathrm{d}s + d^{-8}\ell^{-4}\mathcal{C}^2\right),$$

and finally, using again (9.51) and (9.49), and the fact that $\mathcal{C} \leq CK_\ell^2 K_H^{-2}$ by (9.33) and Lemma F.1,

$$\frac{\ell^2}{(2\pi)^2}\int_{\mathcal{P}_H}\langle \mathcal{T}_{2,\mathrm{op}}'(k)\rangle_\Phi \mathrm{d}k \geq -C\rho_z \ell^2 \delta^2 K_H^{-4} d^{-4}\mathcal{M}\langle \mathcal{K}_H^{\mathrm{Diag}}\rangle_\Phi - C\rho_z \delta d^{-8} K_\ell^4 K_H^{-4}$$

$$\geq -CK_\ell^4 K_H^{-4} d^{-4}\varepsilon_\mathcal{M} \mathcal{K}_H^{\mathrm{Diag}} - C\rho_z^2 \ell^2 \delta^2 d^{-8} K_\ell^2 K_H^{-4}. \qquad (9.68)$$

The first part can be absorbed in the positive $\varepsilon_K \mathcal{K}_H^{\mathrm{Diag}}$, as long as the relation (H21) holds, and the second part contributes to the error term.

• We now turn to $\mathcal{T}'_{2,\text{com}}$ given in (9.66). This term will absorb $\mathcal{Q}^{ex}_2$. We first use Lemma F.1, (9.32) and (9.48) to estimate the commutator,

$$
\begin{aligned}
\left|[\,b_{p-k}, b^\dagger_{s-k}] - \widehat{\chi^2}((p-s)\ell)\right| &= \left|\alpha_{p-k}\alpha_{s-k}\widehat{\chi^2}((p-s)\ell)\right| \\
&\quad + \left|(1 - \alpha_{p-k}\alpha_{s-k})\widehat{\chi}((p-k)\ell)\widehat{\chi}((s-k)\ell)\right| \\
&\leq CK^4_\ell K^{-4}_H,
\end{aligned}
$$

and bounding then by a Cauchy–Schwarz inequality

$$
\begin{aligned}
(\widetilde{a}^\dagger_p + \alpha_k\alpha_{p-k}\widetilde{a}_{-p})(\widehat{(\chi^2)}((p-s)\ell) + CK^4_\ell K^{-4}_H)(\widetilde{a}_s + \alpha_k\alpha_{s-k}\widetilde{a}^\dagger_{-s}) \\
\leq \widetilde{a}^\dagger_p\widehat{\chi^2}((p-s)\ell)\widetilde{a}_s + C(\widetilde{a}^\dagger_{-p}\widetilde{a}_{-p} + \widetilde{a}^\dagger_s\widetilde{a}_s)K^4_\ell K^{-4}_H.
\end{aligned}
$$

We get, by using Lemma 8.1

$$
\begin{aligned}
\frac{\ell^2}{(2\pi)^2}\int_{\mathcal{P}_H}\mathcal{T}'_{2,\text{com}}(k)\mathrm{d}k \geq &-\frac{\ell^2}{(2\pi)^2}\int_{\mathcal{P}_H}\frac{|z|^2\widehat{W}_1(k)^2}{(2\pi)^4(1-3\varepsilon_K)\mathcal{D}_k(1-\alpha^2_k)} \\
&\times \int\frac{f_L(p)f_L(s)}{\sqrt{1-\alpha^2_{p-k}}\sqrt{1-\alpha^2_{s-k}}}\widetilde{a}^\dagger_p\widehat{\chi^2}((p-s)\ell)\widetilde{a}_s\mathrm{d}p\mathrm{d}s\mathrm{d}k \\
&- C\left(\int_{\mathcal{P}_H}\frac{|z|^2\widehat{W}_1(k)^2}{\mathcal{D}_k}\mathrm{d}k\right)d^{-4}K^4_\ell K^{-4}_H\frac{n_+}{\ell^2}.
\end{aligned}
$$

Using (9.49) the last part is of order $K^6_\ell K^{-4}_H d^{-4}\frac{n_+}{\ell^2}$ and can be absorbed in a fraction of the positive $\frac{b}{100}\frac{n_+}{\ell^2}$ by (H8). For the first term we use the following formula valid in a Fock sector with N bosons

$$
\frac{\ell^4}{(2\pi)^4}\int f_L(p)f_L(s)\widetilde{a}^\dagger_p\widehat{\chi^2}((p-s)\ell)a_s\mathrm{d}s\mathrm{d}p|_N = \sum^N_{j=1}\mathcal{Q}^\dagger_{L,j}\chi^2_\Lambda(x_j)\mathcal{Q}_{L,j}, \quad (9.69)
$$

to rewrite, due to (H8) and by (9.49),

$$
\begin{aligned}
&\frac{\ell^2}{(2\pi)^2}\int_{\mathcal{P}_H}\mathcal{T}'_{2,\text{com}}(k)\mathrm{d}k \\
&\geq -\frac{(1+C\varepsilon_K)}{(2\pi)^2}\int\frac{\rho_z\widehat{W}_1(k)^2}{\mathcal{D}_k(1-\alpha^2_k)}\mathrm{d}k\sum^N_{j=1}\mathcal{Q}^\dagger_{L,j}\chi^2_\Lambda(x_j)\mathcal{Q}_{L,j} - \frac{b}{200}\frac{n_+}{\ell^2} \\
&\geq -(1+C\varepsilon_K + CK^4_\ell K^{-4}_H) \\
&\quad \times \left(2\rho_z\widehat{(\omega W_1)}(0) + C\rho_z\delta^2|\log\delta|\right)\sum^N_{j=1}\mathcal{Q}^\dagger_{L,j}\chi^2_\Lambda\mathcal{Q}_{L,j} - \frac{b}{200}\frac{n_+}{\ell^2}.
\end{aligned}
$$

In this last expression we want to replace $\mathcal{Q}_{L,j}$ by $\mathcal{Q}_j$. Using Cauchy–Schwarz with weight ε_0 we find

$$
\mathcal{Q}^\dagger_{L,j}\chi^2_\Lambda\mathcal{Q}_{L,j} \leq (1+\varepsilon_0)\mathcal{Q}_j\chi^2_\Lambda\mathcal{Q}_j + (1+\varepsilon^{-1}_0)\mathcal{Q}_j(f_L-1)\chi^2_j(f_L-1)\mathcal{Q}_j, \quad (9.70)
$$

and since f_L localizes on low momenta we can bound the second term by n_+^H, and the term $\varepsilon_0 Q_j \chi_j^2 Q_j$ by $C\varepsilon_0 n_+$,

$$\frac{\ell^2}{(2\pi)^2} \int_{\mathcal{P}_H} \mathcal{T}_{2,\text{com}}'(k) dk \geq -2\rho_z \widehat{(\omega W_1)}(0) \sum_{j=1}^{N} \left(Q_j \chi_j^2 Q_j + C\varepsilon_0^{-1} n_+^H + C\varepsilon_0 n_+ \right)$$

$$- C\left(\rho_z \ell^2 \delta^2 |\log \delta| + \rho_z \ell^2 \delta \varepsilon_K + \rho_z \ell^2 \delta K_\ell^4 K_H^{-4} + \frac{b}{200} \right) \frac{n_+}{\ell^2}.$$

The n_+^H-part can be absorbed by the positive $\frac{b}{100} \frac{\varepsilon_T n_+^H}{(d\ell)^2}$ if we choose $\varepsilon_0 \simeq \frac{\rho_z \delta d^2 \ell^2}{\varepsilon_T} \simeq \frac{d^2 K_\ell^2}{\varepsilon_T}$. With this choice the n_+ terms are of order

$$\left(\frac{d^2 K_\ell^4}{\varepsilon_T} + \delta K_\ell^2 |\log \delta| + K_\ell^2 \varepsilon_K + K_\ell^6 K_H^{-4} + \frac{b}{200} \right) \frac{n_+}{\ell^2}. \tag{9.71}$$

Those terms are absorbed in a fraction of the positive $\frac{b}{100} \frac{n_+}{\ell^2}$, as long as we have the relations (H8), (H10), (H17) and (H20). We deduce

$$\frac{\ell^2}{(2\pi)^2} \int_{\mathcal{P}_H} \mathcal{T}_{2,\text{com}}'(k) dk \geq -2\rho_z \widehat{(\omega W_1)}(0) \sum_{j=1}^{N} Q_j \chi_j^2 Q_j - \frac{b}{150} \frac{n_+}{\ell^2} - \frac{b}{100} \frac{\varepsilon_T n_+^H}{(d\ell)^2}.$$

To compare the remaining part with $\mathcal{Q}_2^{\text{ex}}$ we use (9.52) to find

$$\mathcal{Q}_2^{\text{ex}} = \rho_z \frac{\ell^2}{(2\pi)^2} \int \left(\widehat{W\omega}(k) + \widehat{W\omega}(0) \right) a_k^\dagger a_k dk$$

$$\geq 2\rho_z \frac{\ell^2}{(2\pi)^2} \widehat{W_1\omega}(0) \int a_k^\dagger a_k dk - C\rho_z \delta (d^{-2} R\ell^{-1} n_+ + C n_+^H)$$

$$= 2\rho_z \widehat{W_1\omega}(0) \sum_j Q_j \chi_\Lambda^2 Q_j - C\rho_z \delta d^{-2} R\ell^{-1} n_+ + C\rho_z \delta n_+^H.$$

Using that $\rho_z \simeq \rho_\mu$, the remaining parts are absorbed by the spectral gaps and then we get

$$\frac{\ell^2}{(2\pi)^2} \int_{\mathcal{P}_H} \mathcal{T}_{2,\text{com}}'(k) dk + \mathcal{Q}_2^{\text{ex}} \geq -\frac{b}{100} \frac{n_+}{\ell^2} - \frac{b}{100} \frac{\varepsilon_T n_+^H}{(d\ell)^2}.$$

This last estimate, together with (9.58), (9.64), (9.65) and (9.68) concludes the proof. □

Remark 9.8. It was necessary to replace a_k's by b_k's before estimating $\mathcal{Q}_3(z) + \mathcal{Q}_2^{ex}$, otherwise we would need a fraction of the kinetic energy instead of $\mathcal{K}_H^{\text{diag}}$ in Lemma 9.7, and this we cannot allow (see Remark 9.5). In other words, it is important that the positive term in (9.58) is given in terms of $\tilde{b}_k$ (Eq. (9.59)) whose main part is b_k.

9.3.2. Estimates on $\mathcal{Q}_3^{(2)}$ and $\mathcal{Q}_3^{(3)}$

Lemma 9.9 (Estimates on $\mathcal{Q}_3^{(2)}$ and $\mathcal{Q}_3^{(3)}$). *For any normalized state Φ satisfying* (8.44) *we have*

$$\left\langle \mathcal{Q}_3^{(2)} + \mathcal{Q}_3^{(3)} + \frac{\varepsilon_K}{100}\mathcal{K}_H^{Diag} \right\rangle_\Phi \geq -C\rho_z^2\ell^2\delta^2\varepsilon_K^{-1}K_H^{-2M-8}K_\ell^6 d^{-8}.$$

Proof. We focus on $\mathcal{Q}_3^{(3)}$ (the estimates on $\mathcal{Q}_3^{(2)}$ are similar), and decompose it into $\mathcal{Q}_3^{(3)} = I + II$, where

$$I = -\frac{z\ell^2}{(2\pi)^4}\int_{\mathcal{P}_H\times\mathbb{R}^2}\frac{f_L(p)\widehat{W}_1(k)\alpha_{p-k}}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_{p-k}^2}}\left(b_{k-p}^\dagger\widetilde{a}_p^\dagger b_k + b_k^\dagger\widetilde{a}_p b_{k-p}\right)\mathrm{d}k\mathrm{d}p, \quad (9.72)$$

and

$$II = -\frac{z\ell^2}{(2\pi)^4}\int_{\mathcal{P}_H\times\mathbb{R}^2}\frac{f_L(p)\widehat{W}_1(k)\alpha_{p-k}}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_{p-k}^2}}\left([\widetilde{a}_p^\dagger,b_{k-p}^\dagger]b_k + b_k^\dagger[b_{k-p},\widetilde{a}_p]\right)\mathrm{d}k\mathrm{d}p.$$
$$(9.73)$$

The first part we estimate using Cauchy–Schwarz with weight ε, and by (9.48)

$$I \geq -\frac{z\ell^2}{(2\pi)^4}\int_{\mathcal{P}_H\times\mathbb{R}^2}\frac{f_L(p)\widehat{W}_1(k)\alpha_{p-k}}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_{p-k}^2}}(\varepsilon b_{k-p}^\dagger\widetilde{a}_p^\dagger\widetilde{a}_p b_{k-p} + \varepsilon^{-1}b_k^\dagger b_k)\mathrm{d}k\mathrm{d}p$$

$$\geq -Cz\ell^2\delta K_\ell^2 K_H^{-2}\int_{\mathcal{P}_H\times\mathbb{R}^2}f_L(p)(\varepsilon b_{k-p}^\dagger\widetilde{a}_p^\dagger\widetilde{a}_p b_{k-p} + \varepsilon^{-1}b_k^\dagger b_k)\mathrm{d}k\mathrm{d}p,$$

and using Lemma 8.1,

$$\langle\Phi,\int f_L(p)b_{k-p}^\dagger\widetilde{a}_p^\dagger\widetilde{a}_p b_{k-p}\mathrm{d}p\Phi\rangle \leq C\ell^{-2}\mathcal{M}\langle\Phi,b_k^\dagger b_k\Phi\rangle. \quad (9.74)$$

We choose $\varepsilon = \sqrt{d^{-4}/\mathcal{M}}$, and insert $\mathcal{D}_k \geq K_H^2\ell^{-2}$,

$$\langle I\rangle_\Phi \geq -Cz\delta K_\ell^2 K_H^{-2}(\varepsilon\mathcal{M} + \varepsilon^{-1}d^{-4})\int_{k\in\mathcal{P}_H}\langle b_k^\dagger b_k\rangle_\Phi\mathrm{d}k$$

$$\geq -C(\rho_z^{1/2}\ell\delta K_\ell^2 K_H^{-4}\mathcal{M}^{1/2}d^{-2})\ell^2\int_{k\in\mathcal{P}_H}\mathcal{D}_k\langle b_k^\dagger b_k\rangle_\Phi\mathrm{d}k. \quad (9.75)$$

Thanks to condition (H21), I can be absorbed in the positive $\frac{\varepsilon_K}{100}\mathcal{K}_H^{Diag}$ term. Now we return to the commutator term, which can be estimated using a Cauchy–Schwarz inequality with new weight ε,

$$II \geq -2\frac{z\ell^2}{(2\pi)^4}\int_{\mathcal{P}_H\times\mathbb{R}^2}|[b_{k-p},\widetilde{a}_p]||f_L(p)|\widehat{W}_1(k)\alpha_k|(\varepsilon b_k^\dagger b_k + \varepsilon^{-1})\mathrm{d}k\mathrm{d}p.$$

We use the commutator bound $|[b_{k-p},\widetilde{a}_p]| \leq C\alpha_{k-p}\sup_{k\in\mathcal{P}_H}\widehat{\chi}(k\ell)$ from (9.33),

$$II \geq -Cz K_\ell^2 K_H^{-2}\left(\sup_{k\in\mathcal{P}_H}\widehat{\chi}(k\ell)\right)d^{-4}\left(\varepsilon K_\ell^2 K_H^{-4}\delta\ell^2\int_{\mathcal{P}_H}\mathcal{D}_k b_k^\dagger b_k\mathrm{d}k + \varepsilon^{-1}\int_{\mathcal{P}_H}\widehat{W}_1(k)\alpha_k\mathrm{d}k\right).$$

With $\varepsilon^{-1} \simeq \varepsilon_K^{-1} z d^{-4} K_\ell^4 K_H^{-6} \delta(\sup \widehat{\chi})$ and our choice of parameters, the first part is absorbed in the positive $\varepsilon_K \mathcal{K}^{\text{Diag}}$ term. We estimate the last part using (9.50) and Lemma F.1, and then II contributes with an error of order $\varepsilon_K^{-1} \rho_z^2 \ell^2 \delta^2 K_H^{-2M-8} K_\ell^6 d^{-8}$. $\square$

9.3.3. Estimates on $\mathcal{Q}_3^{(4)}$

First we rewrite $\mathcal{Q}_1^{\text{ex}}$ as a term appearing in $\mathcal{Q}_3^{(4)}$.

Lemma 9.10. *Assume that Assumptions of "Appendix H" are satisfied. Then there exists a universal constant $C > 0$ such that*

$$
\mathcal{Q}_1^{ex} + \frac{b}{100}\frac{n_+}{\ell^2} + \frac{b}{100}\frac{\varepsilon_T n_+^H}{(d\ell)^2} \geq \frac{z\ell^2}{(2\pi)^4} \int_{\mathcal{P}_H \times \mathbb{R}^2} \widehat{W}_1(k)\alpha_k \widehat{\chi^2}(p\ell)f_L(p)(\widetilde{a}_p^\dagger + \widetilde{a}_p)\mathrm{d}k\mathrm{d}p
$$
$$
- C\rho_z^2\ell^2\delta^3 K_\ell^2|\log\delta|^2 - \rho_z^2\ell^2\delta K_\ell^2 d^{8M-2}\varepsilon_T^{-1}.
$$

Proof. First we can rewrite $\mathcal{Q}_1^{\text{ex}}$ in terms of the $\widetilde{a}_p$'s,

$$
\mathcal{Q}_1^{\text{ex}} = z\rho_z\widehat{\omega W_1}(0)\frac{\ell^2}{(2\pi)^2}\int \widehat{\chi^2}(p\ell)(\widetilde{a}_p^\dagger + \widetilde{a}_p)\mathrm{d}p,
$$

and then we use (9.50) to compare $\widehat{\omega W_1}(0)$ with an integral in k, and using the bound $K_\ell^2 K_H^{-2} \ll 1$,

$$
\mathcal{Q}_1^{\text{ex}} \geq \frac{z\ell^2}{(2\pi)^4} \int_{\mathcal{P}_H \times \mathbb{R}^2} \widehat{W}_1(k)\alpha_k \widehat{\chi^2}(p\ell)(\widetilde{a}_p^\dagger + \widetilde{a}_p)\mathrm{d}k\mathrm{d}p
$$
$$
- C\rho_z\delta^2|\log\delta|z\ell^2 \int_{\mathbb{R}^2} \widehat{\chi^2}(p\ell)(\widetilde{a}_p^\dagger + \widetilde{a}_p)\mathrm{d}p. \tag{9.76}
$$

The second integral can be estimated using a Cauchy–Schwarz inequality with weight ε,

$$
\rho_z\ell^2\delta^2 z \int_{\mathbb{R}^2} \widehat{\chi^2}(p\ell)(\widetilde{a}_p^\dagger + \widetilde{a}_p)\mathrm{d}p
$$
$$
\leq \varepsilon\rho_z\ell^2\delta^2 z \int |\widehat{\chi^2}(p\ell)| + C\varepsilon^{-1}\rho_z\ell^2\delta^2 z \int |\widehat{\chi^2}(p\ell)|\widetilde{a}_p^\dagger\widetilde{a}_p\mathrm{d}p
$$
$$
\leq C\varepsilon\rho_z\delta^2 z + C\varepsilon^{-1}\rho_z\delta^2 zn_+. \tag{9.77}
$$

where we used Lemma (F.1). With $\varepsilon \simeq z\delta K_\ell^2|\log\delta|$, the second part is absorbed by the positive fraction of $\frac{n_+}{\ell^2}$, and the first term is of order $\rho_z^2\ell^2\delta^3 K_\ell^2|\log\delta|$. Hence,

$$
\mathcal{Q}_1^{\text{ex}} \geq \frac{z\ell^2}{(2\pi)^4} \int_{\mathcal{P}_H \times \mathbb{R}^2} \widehat{W}_1(k)\alpha_k \widehat{\chi^2}(p\ell)(\widetilde{a}_p^\dagger + \widetilde{a}_p)\mathrm{d}k\mathrm{d}p
$$
$$
- C\rho_z^2\ell^2\delta^3 K_\ell^2|\log\delta|^2 - \frac{b}{100}\frac{n_+}{\ell^2}. \tag{9.78}
$$

Finally we want to insert the cutoff $f_L(p)$ inside the integral. The error we make is estimated similarly,

$$\frac{z\ell^2}{(2\pi)^4} \int_{\mathcal{P}_H \times \mathbb{R}^2} \widehat{W}_1(k) \alpha_k \widehat{\chi^2}(p\ell)(1 - f_L(p))(\tilde{a}_p^\dagger + \tilde{a}_p)\mathrm{d}k\mathrm{d}p$$

$$\geq -Cz\ell^2 \rho_z \delta \int_{\mathcal{P}_L^c} \widehat{\chi^2}(p\ell)(\tilde{a}_p^\dagger + \tilde{a}_p)\mathrm{d}p$$

$$\geq -C\varepsilon z\rho_z\delta\ell^2 \int_{\mathcal{P}_L^c} |\widehat{\chi^2}(p\ell)|\mathrm{d}p - C\varepsilon^{-1}z\rho_z\delta\ell^2 \int_{\mathcal{P}_L^c} |\widehat{\chi^2}(p\ell)|\tilde{a}_p^\dagger \tilde{a}_p\mathrm{d}p$$

$$\geq -C\varepsilon z\rho_z\delta d^{4M-4} - C\varepsilon^{-1}z\rho_z\delta d^{4M}n_+^H,$$

where we used $\sup_{p\in\mathcal{P}_L^c} |\widehat{\chi^2}(p\ell)| \leq Cd^{4M}$. With $\varepsilon \simeq zK_\ell^2 d^{4M+2}\varepsilon_T^{-1}$ the first part is of order $\rho_z^2\ell^2\delta K_\ell^2 d^{8M-2}\varepsilon_T^{-1}$ and the second is absorbed in a fraction of $\frac{\varepsilon_T n_+^H}{(d\ell)^2}$. $\square$

Now we have all we need to estimate $\mathcal{Q}_3^{(4)}$.

Lemma 9.11 (Estimates on $\mathcal{Q}_3^{(4)}$). *For any state Φ satisfying (8.44) we have*

$$\left\langle \mathcal{Q}_3^{(4)} + \mathcal{Q}_1^{ex} + \frac{b}{100}\frac{n_+}{\ell^2} + \frac{b}{100}\frac{\varepsilon_T n_+^H}{(d\ell)^2}\right\rangle_\Phi \geq -C\rho_z^2\ell^2\delta^3 K_\ell^2|\log\delta|^2 - C\rho_z^2\ell^2\delta K_\ell^2 d^{8M-2}\varepsilon_T^{-1}$$

$$- C\rho_z^2\ell^2\delta^2\varepsilon_{\mathcal{M}}^{1/2}(K_\ell^4 K_H^{-4} + \delta^{-1}K_H^{-M}d^{-2}).$$

Proof. We use the commutator formula

$$[b_{p-k}, b_{-k}^\dagger] = (1 - \alpha_k\alpha_{p-k})\left(\widehat{\chi^2}(p\ell) - \widehat{\chi}(k\ell)\widehat{\chi}((p-k)\ell)\right),$$

and split into $\mathcal{Q}_3^{(4)} = I + II$, with

$$I = -\frac{z\ell^2}{(2\pi)^4} \int_{\mathcal{P}_H \times \mathbb{R}^2} \frac{f_L(p)\widehat{W}_1(k)}{\sqrt{1 - \alpha_k^2}\sqrt{1 - \alpha_{p-k}^2}} \alpha_k(1 - \alpha_k\alpha_{p-k})\widehat{\chi^2}(p\ell)(\tilde{a}_p^\dagger + \tilde{a}_p)\mathrm{d}k\mathrm{d}p,$$

and

$$II = -\frac{z\ell^2}{(2\pi)^4} \int_{\mathcal{P}_H \times \mathbb{R}^2} \frac{f_L(p)\widehat{W}_1(k)}{\sqrt{1 - \alpha_k^2}\sqrt{1 - \alpha_{p-k}^2}}$$

$$\times \alpha_k(1 - \alpha_k\alpha_{p-k})\widehat{\chi}(k\ell)\widehat{\chi}((p-k)\ell)(\tilde{a}_p^\dagger + \tilde{a}_p)\mathrm{d}k\mathrm{d}p.$$

In I we recognize the lower bound on $\mathcal{Q}_1^{ex}$ given by Lemma 9.10 with opposite sign, up to an error term:

$$I + \mathcal{Q}_1^{ex} + \frac{b}{100}\frac{n_+}{\ell^2} + \frac{b}{100}\frac{\varepsilon_T n_+^H}{(d\ell)^2} + C\rho_z^2\ell^2\delta^3 K_\ell^2|\log\delta|^2 + \rho_z^2\ell^2\delta K_\ell^2 d^{8M-2}\varepsilon_T^{-1}$$

$$\geq -Cz\ell^2 \int_{\mathcal{P}_H \times \mathbb{R}^2} f_L(p)\widehat{W}_1(k)\alpha_k^3\widehat{\chi^2}(p\ell)(\tilde{a}_p^\dagger + \tilde{a}_p)\mathrm{d}k\mathrm{d}p.$$

This remaining integral can be estimated, by (9.48), as

$$\left| z\ell^2 \int_{\mathcal{P}_H \times \mathbb{R}^2} f_L(p) \widehat{W}_1(k) \alpha_k^3 \widehat{\chi^2}(p\ell)(\widetilde{a}_p^\dagger + \widetilde{a}_p) \mathrm{d}k \mathrm{d}p \right|$$

$$\leq C|z|\ell^2 \rho_z^3 \delta^4 \int_{\mathcal{P}_H} k^{-6} \mathrm{d}k \int_{\mathcal{P}_L} |\widehat{\chi^2}(p\ell)|(\widetilde{a}_p^\dagger + \widetilde{a}_p) \mathrm{d}p,$$

and after applying to the state Φ we use a Cauchy–Schwarz inequality with weight $\sqrt{\mathcal{M}}$,

$$\left| z\ell^2 \int_{\mathcal{P}_H \times \mathbb{R}^2} f_L(p) \widehat{W}_1(k) \alpha_k^3 \widehat{\chi^2}(p\ell) \langle \widetilde{a}_p^\dagger + \widetilde{a}_p \rangle_\Phi \mathrm{d}k \mathrm{d}p \right|$$

$$\leq C z \rho_z^3 \ell^6 \delta^4 K_H^{-4} \left(\sqrt{\mathcal{M}} \int_{\mathcal{P}_L} |\widehat{\chi^2}(p\ell)| \mathrm{d}p + \frac{1}{\sqrt{\mathcal{M}}} \int_{\mathcal{P}_L} |\widehat{\chi^2}(p\ell)| \langle \widetilde{a}_p^\dagger \widetilde{a}_p \rangle_\Phi \mathrm{d}p \right)$$

$$\leq C z \rho_z^3 \ell^4 \delta^4 K_H^{-4} \sqrt{\mathcal{M}} \leq C \rho_z^2 \ell^2 \delta^2 \left(K_\ell^4 K_H^{-4} \sqrt{\varepsilon_\mathcal{M}} \right).$$

Finally we bound, by (F4) and (9.50),

$$|\langle II \rangle_\Phi| \leq z\ell^2 \sup_{h \in \mathcal{P}_H} |\widehat{\chi}(h\ell)| \int_{\mathcal{P}_H} |\widehat{W}_1(k)| \alpha_k |\widehat{\chi}(k\ell)| \mathrm{d}k \int_{\mathcal{P}_L} \langle \widetilde{a}_p^\dagger + \widetilde{a}_p \rangle_\Phi \mathrm{d}p$$

$$\leq C z \rho_z \ell^2 \delta K_H^{-M} \left(d^2 \mathcal{M}^{1/2} \int_{\mathcal{P}_L} \mathrm{d}p + d^{-2} \mathcal{M}^{-1/2} \int_{\mathcal{P}_L} \langle \widetilde{a}_p^\dagger \widetilde{a}_p \rangle_\Phi \mathrm{d}p \right),$$

where we used a Cauchy–Schwarz inequality with weight $d^2 \sqrt{\mathcal{M}}$. Thus,

$$|\langle II \rangle_\Phi| \leq C \rho_z^2 \ell^2 \delta K_H^{-M} d^{-2} \varepsilon_\mathcal{M}^{1/2}.$$

$\square$

9.4. Conclusion: Proof of Theorem 6.7. In Sect. 6 we showed how the proof of Theorem 2.3 is reduced to the proof of Theorem 6.7, which we give here.

Proof of Theorem 6.7. Recall the choices of the parameters in "Appendix H". Let us consider a normalized n-particle state $\Psi \in \mathscr{F}_s(L^2(\Lambda))$ which satisfies (7.40) for a certain large constant $C_0 > 0$,

$$\langle \mathcal{H}_\Lambda(\rho_\mu) \rangle_\Psi \leq -4\pi \rho_\mu^2 \ell^2 Y \left(1 - C_0 K_B^2 Y |\log Y| \right). \tag{9.79}$$

If such a state does not exists, our desired lower bound follows, because

$$-4\pi \rho_\mu^2 \ell^2 Y \left(1 - C_0 K_B^2 Y |\log Y| \right) \geq -4\pi \rho_\mu^2 \ell^2 \delta \left(1 - \left(2\Gamma + \frac{1}{2} + \log \pi \right) \delta \right). \tag{9.80}$$

So we can assume the existence of Ψ. By Theorem 7.7 there exists a sequence of n-particle states $\{\Psi^m\}_{m \in \mathbb{Z}} \subseteq \mathscr{F}_s(L^2(\Lambda))$ and $C_1, \eta_1 > 0$ such that

$$\langle \Psi, \mathcal{H}_\Lambda(\rho_\mu) \Psi \rangle \geq \sum_{2|m| \leq \mathcal{M}} \langle \Psi^{(m)}, \mathcal{H}_\Lambda(\rho_\mu) \Psi^m \rangle - C_1 \rho_\mu^2 \ell^2 \delta^{2+\eta_1}$$

$$-4\pi \rho_\mu^2 \ell^2 Y \left(1 - C_1 K_B^2 Y |\log Y| \right) \sum_{2|m| > \mathcal{M}} \|\Psi^m\|^2.$$

For $|m| \leq \frac{\mathcal{M}}{2}$, we have that $\Psi^m = \mathbb{1}_{[0,\mathcal{M}]}(n_+^L)\Psi^m$. If we prove the lower bound for all Ψ^m such that $|m| \leq \frac{\mathcal{M}}{2}$ then we would get (using (9.80) with C_0 replaced by C_1)

$$\langle \Psi, \mathcal{H}_\Lambda(\rho_\mu)\Psi \rangle \geq - 4\pi\rho_\mu^2 \ell^2 \delta \Big(1 - \Big(2\Gamma + \frac{1}{2} + \log \pi\Big)\delta\Big) \sum_m \|\Psi^{(m)}\|^2 - C_1 \rho_\mu^2 \ell^2 \delta^{2+\eta_1},$$

Therefore, the theorem is proven if we derive the corresponding lower bound for any n-particle, normalized state $\widetilde{\Psi} \in \mathscr{F}_s(L^2(\Lambda))$ such that

$$\widetilde{\Psi} = \mathbb{1}_{[0,\mathcal{M}]}(n_+^L)\widetilde{\Psi}. \tag{9.81}$$

By Proposition 8.3, for such a state there exists a constant $C_2 > 0$ such that

$$\langle \widetilde{\Psi}, \mathcal{H}_\Lambda(\rho_\mu)\widetilde{\Psi} \rangle \geq \langle \widetilde{\Psi}, \mathcal{H}_\Lambda^{2\mathrm{nd}}(\rho_\mu)\widetilde{\Psi} \rangle - C_2 \ell^2 \rho_\mu^2 \delta \big(d^{2M-2} + R^2 \ell^{-2}\big), \tag{9.82}$$

where the last term is an error term of order $\rho_\mu^2 \ell^2 \delta^{2+\eta_2}$, for some $\eta_2 > 0$, thanks to relations (H25) and (H3). Then, by Theorem 8.4, there exists a constant $C_3 > 0$ such that

$$\langle \widetilde{\Psi}, \mathcal{H}_\Lambda^{2\mathrm{nd}}\widetilde{\Psi} \rangle \geq \inf_{z \in \mathbb{R}_+} \inf_\Phi \langle \Phi, \mathcal{K}(z)\Phi \rangle - C_3 \rho_\mu \delta (1 + \varepsilon_R K_\ell^4 K_B^2 |\log Y|), \tag{9.83}$$

where the infimum is over the Φ's which satisfy (8.44). The last term is an error term of order $\rho_\mu^2 \ell^2 \delta^{2+\eta_3}$ for some $\eta_3 > 0$, thanks to relation (H19). The proof is reduced now to getting a lower bound for $\mathcal{K}(z)$. We have two cases, according to different values of z:

- If $|\rho_z - \rho_\mu| \geq C\rho_\mu \max((\delta_1 + \delta_2 + \delta_3)^{1/2}, \delta^{1/2})$ then Proposition 9.1 implies the bound

$$\langle \mathcal{K}(z) \rangle_\Phi \geq -\frac{1}{2}\rho_\mu^2 \ell^2 \widehat{g}_0 + 8\pi \Big(2\Gamma + \frac{1}{2} + \log \pi\Big)\rho_\mu^2 \ell^2 \delta^2, \tag{9.84}$$

and the second term is twice the LHY-term and positive, therefore there is nothing more to prove;

- Otherwise $|\rho_z - \rho_\mu| \leq \rho_\mu K_\ell^{-2}$ (see Sect. 9.2). In this case we can use (9.27) and Theorem 9.3 to obtain $C_4, \eta_4 > 0$, such that

$$\langle \mathcal{K}(z) \rangle_\Phi \geq -\frac{1}{2}\rho_\mu^2 \ell^2 \widehat{g}_0 + (1 - \varepsilon_K)\langle \mathcal{K}_H^{\mathrm{Diag}} \rangle_\Phi + 4\pi \Big(2\Gamma + \frac{1}{2} + \log \pi\Big)\rho_z^2 \ell^2 \delta^2$$

$$+ \Big\langle b\frac{n_+}{4\ell^2} + b\frac{\varepsilon_T n_+^H}{8d^2\ell^2} + \mathcal{Q}_1^{\mathrm{ex}}(z) + \mathcal{Q}_2^{\mathrm{ex}} + \mathcal{Q}_3(z)\Big\rangle_\Phi - C_4 \rho_\mu^2 \ell^2 \delta^{2+\eta_4}, \tag{9.85}$$

where we used that $C\rho_\mu^2 \ell^2 \delta (K_H^{4-M} K_\ell \delta^{-1/2}) + |r(\rho_\mu)|\ell^2 \leq C\rho_\mu^2 \ell^2 \delta^{2+\eta_4}$, thanks to the relations (H7), (H8) and that $M > 4$. We conclude observing that, thanks to Theorem 9.4, we have the existence of $C_5, \eta_5 > 0$ such that

$$\Big\langle (1 - \varepsilon_K)\mathcal{K}_H^{\mathrm{Diag}} + \mathcal{Q}_3(z) + \mathcal{Q}_2^{\mathrm{ex}} + \mathcal{Q}_1^{\mathrm{ex}} + \frac{b}{100}\frac{n_+}{\ell^2} + \frac{b\varepsilon_T}{100}\frac{n_+^H}{(d\ell)^2}\Big\rangle_\Phi \geq -C_5\rho_z^2 \ell^2 \delta^{2+\eta_5}, \tag{9.86}$$

where the error has been obtained using relations (H10), (H18), (H21), (H26) and (H27). Thanks to the assumptions on ρ_z and ρ_μ, there exist $C_6, \eta_6 > 0$ such that

$$|\rho_z^2 \ell^2 \delta^2 - \rho_\mu^2 \ell^2 \delta^2| \leq C_6 \rho_\mu^2 \ell^2 \delta^2 K_\ell^{-2} = C_6 \rho_\mu^2 \ell^2 \delta^{2+\eta_6}, \tag{9.87}$$

so that, plugging (9.86) into (9.85) and substituting the ρ_z by the ρ_μ using (9.87) gives the desired lower bound and the right order for the error terms.

We choose $C = \sum_{j=1}^{6} C_j$ and $\eta = \min_{j=1,\dots,6} \eta_j$. We conclude using that $\widehat{g}_0 = 8\pi\delta$ to get that

$$\inf_{z\in\mathbb{R}_+} \inf_{\Phi} \langle \Phi, \mathcal{K}(z)\Phi \rangle \geq -4\pi\ell^2 \rho_\mu^2 \delta \left(1 - \left(2\Gamma + \frac{1}{2} + \log\pi \right)\delta \right) - C\rho_\mu^2 \ell^2 \delta^{2+\eta}.$$

This finishes the proof of Theorem 6.7. $\hspace{2cm}\square$

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Appendix A: Reduction to Smaller Boxes for the Upper Bound

We provide here the necessary tools to go from a fixed box with compactly support potentials in the grand canonical setting, Theorem 2.2, to the thermodynamic limit with potentials allowing a tail, Theorem 2.1. The same techniques can be found in [17] with only minor deviations surrounding the non-compactness of the potential.

Given a potential v, we define

$$e(\rho) := \lim_{L\to\infty} e_L(\rho) = \lim_{L\to\infty} \inf_{\psi\in H_0^1(\Lambda_L^{\rho L^2})} \frac{\langle \psi, \mathcal{H}_v^{\rho L^2} \psi \rangle}{L^2},$$

where the limit is taken such that $\rho L^2 = N \in \mathbb{N}$ and

$$\mathcal{H}_v^N = \sum_{i=1}^{N} -\Delta_i + \sum_{i<j}^{N} v(x_i - x_j).$$

We write $v = v\mathbb{1}_{B(0,R)} + v\mathbb{1}_{B(0,R)^c} = v_R + v_{tail}$ where the v_{tail} will always be treated as an error term. Let $v_R^{per}(x) = \sum_{k\in\mathbb{Z}^2} v_R(x+kL)$. In order for this to be finite we understand

R to be smaller than L. We omit the N in the hamiltonian when it is operating on the Fock space.

The result below evaluates the error when going from periodic boundary conditions to Dirichlet boundary conditions.

Lemma A.1. *There exists a universal $C > 0$, such that given $R_0 > 0$ and a periodic, bosonic trial function $\Psi_L \in \mathscr{F}(\Lambda_L)$, there exists a Dirichlet trial function $\Psi_{L+2R_0}^D \in \mathscr{F}(L^2(\Lambda_{L+2R_0}))$ satisfying, for $j \in \mathbb{N}_0$,*

$$\langle \Psi_{L+2R_0}^D, \mathcal{N}^j \Psi_{L+2R_0}^D \rangle = \langle \Psi_L, \mathcal{N}^j \Psi_L \rangle, \tag{A1}$$

and

$$\langle \Psi_{L+2R_0}^D, \mathcal{H}_{v_R} \Psi_{L+2R_0}^D \rangle \leq \langle \Psi_L, \mathcal{H}_{v_R^{per}} \Psi_L \rangle + \frac{C}{L R_0} \langle \Psi_L, \mathcal{N} \Psi_L \rangle. \tag{A2}$$

Proof. The result is independent of dimension, see [31, Lemma 2.1.3] or [17, Lemma A.1] for a proof in the 3D case. □

Next step is to glue the Dirichlet boxes together in order to construct a trial function on a thermodynamic box.

Theorem A.2. *Let $\Psi_{L+2R_0}^D \in \mathscr{F}_s(L^2(\Lambda_{L+2R_0}))$ be a trial function with Diriclet boundary conditions and extend it to $\mathbb{R}^2$ by 0. Then for $L_k = k(L+2R_0+R)$, $k \in \mathbb{N}$, we define $\Psi_{L_k} \in \mathscr{F}_s(L^2(\Lambda_{L_k}))$ by*

$$\Psi_{L_k}^{(m)}(x_1, \ldots, x_m) = \frac{1}{\|(\Psi_{L+2R_0}^D)^{(n)}\|^{k^2-1}} \prod_{i=1}^{k^2} (\Psi_{L+2R_0}^D)^{(n)}(x_{1+n(i-1)} - c_i, \ldots, x_{in} - c_i), \tag{A3}$$

if $m = nk^2$, and $\Psi_{L_k}^{(m)} = 0$ otherwise. Here c_i defines an enumeration of the lattice points on $\mathbb{Z}^2(L + 2R_0)$. Then Ψ_{L_k} satisfies

$$\langle \Psi_{L_k}, \mathcal{N}^j \Psi_{L_k} \rangle = k^{2j} \langle \Psi_{L+2R_0}^D, \mathcal{N}^j \Psi_{L+2R_0}^D \rangle, \qquad j \in \mathbb{N}_0. \tag{A4}$$

Furthermore if v satisfies the decay condition (1.3) of Theorem 1.1, then there exits a constant C only depending on η_0 and C_0 such that

$$\langle \Psi_{L_k}, \mathcal{H}_v \Psi_{L_k} \rangle \leq k^2 \langle \Psi_{L+2R_0}^D, \mathcal{H}_{v_R} \Psi_{L+2R_0}^D \rangle + k^2 \langle \Psi_{L+2R_0}^D, \mathcal{N}^2 \Psi_{L+2R_0}^D \rangle \frac{C a^{\eta_0}}{R^{2+\eta_0}}. \tag{A5}$$

Proof. The expectation of $\mathcal{N}^j$ can be computed using that

$$\|\Psi_{L_k}^{(m)}\|^2 = \begin{cases} \|\Psi_{L+2R_0}^{(n)}\|^2 & \text{if } m = k^2 n, \\ 0 & \text{otherwise.} \end{cases}$$

However for the potential energy we need to estimate the interaction between the boxes and the long range interaction inside the box. We observe that

$$\langle \Psi_{L_k}, \mathcal{H}_v \Psi_{L_k} \rangle - k^2 \langle \Psi_{L+2R_0}^D, \mathcal{H}_{v_R} \Psi_{L+2R_0}^D \rangle = \sum_{n \geq 0} \sum_{i<j}^{k^2 n} \int |\Psi_{L_k}^{(k^2 n)}|^2 v_{tail}(x_i - x_j) \mathrm{d}x, \tag{A6}$$

where we used that the kinetic energy of the two terms are equal and only the tail of the potential survives due to the corridors between the boxes. We further estimate

$$\sum_{n\geq 0}\sum_{i<j}^{k^2 n}\int |\Psi_{L_k}^{(k^2 n)}|^2 v_{tail}(x_i - x_j)\mathrm{d}x \leq \sum_{n\geq 0}k^2 n\sum_{j=2}^{k^2 n}\int |\Psi_{L_k}^{(k^2 n)}|^2 v_{tail}(x_1 - x_j)\mathrm{d}x$$

$$\leq \sum_{n\geq 0}k^2 n\sum_{j=2}^{k^2 n}\int |\Psi_{L_k}^{(k^2 n)}|^2 \frac{Ca^{\eta_0}}{|x_1 - x_j|^{2+\eta_0}}\mathrm{d}x,$$

$$\text{(A7)}$$

where we used (1.3). If $s \in \mathbb{N}$ denotes the number of aligned boxes separating x_1 from x_j, then $|x_1 - x_j| \geq (s-1)L + R$ and there are $4(s+1) + 1$ of such possible boxes. Summing on s we get

$$(A7) \leq \sum_{n\geq 0}k^2 n\sum_{s=1}^{k}\frac{C_0 a^{\eta_0}n(4(s+1)+1)}{((s-1)L+R)^{2+\eta_0}}\|\Psi_{L_k}^{(k^2 n)}\|^2$$

$$\leq k^2\langle\Psi_{L+2R_0}^D, \mathcal{N}^2\Psi_{L+2R_0}^D\rangle C_0\left(\frac{9a^{\eta_0}}{R^{2+\eta_0}} + \frac{a^{\eta_0}}{L^{2+\eta_0}}\sum_{s=1}^{\infty}\frac{4}{s^{1+\eta_0}} + \frac{9}{s^{2+\eta_0}}\right).$$

In fact the largest term is the contribution of v_{tail} inside the box and its 8 neighbours which here is represented by the term $\frac{9a^{\eta_0}}{R^{2+\delta}}$. □

We have thus far constructed a sequence of grand canonical trial functions on larger and larger boxes, where we control the energy and the expected number of particles. The last part will be to relate this sequence to $e(\rho)$. For this we will use the continuity and convexity of $e(\rho)$ see [32, Thm. 3.5.8 and 3.5.11] together with the Legendre transformation being an involution on such functions.

Theorem A.3. *Let $\Psi_{L_k} \in \mathscr{F}(L^2(\Lambda_{L_k}))$ be a sequence with Dirichlet boundary conditions such that $L_k \to \infty$ as $k \to \infty$. Assume that there exist $C, c > 0$ such that, for all $k \in \mathbb{N}$,*

$$\langle\Psi_{L_k}, \mathcal{N}\Psi_{L_k}\rangle \geq \rho(1+c\rho)L_k^2, \qquad \langle\Psi_{L_k}, \mathcal{N}^2\Psi_{L_k}\rangle \leq C(\rho L_k^2)^2,$$

then

$$e(\rho) \leq \lim_{k\to\infty}\frac{\langle\Psi_{L_k}, \mathcal{H}_v\Psi_{L_k}\rangle}{L_k^2}.$$

Proof. We insert a chemical potential μ, and find that, using the positivity of $\mathcal{H}_v$ and $\mathcal{N}$, for any $\mu \geq 0$ and $M > 0$ we have

$$\frac{\langle\Psi_{L_k}, \mathcal{H}_v\Psi_{L_k}\rangle}{L_k^2}$$

$$\geq \frac{\langle\Psi_{L_k}, (\mathcal{H}_v - \mu\mathcal{N})\chi(\mathcal{N}\leq ML_k^2)\Psi_{L_k}\rangle}{L_k^2} + \frac{\mu}{L_k^2}\left(\langle\Psi_{L_k}, \mathcal{N}\Psi_{L_k}\rangle - \langle\Psi_{L_k}, \mathcal{N}\chi(\mathcal{N}\geq ML_k^2)\Psi_{L_k}\rangle\right)$$

$$\geq \sum_{m=0}^{ML_k^2}\left(e_{L_k}\left(\frac{m}{L_k^2}\right) - \mu\frac{m}{L_k^2}\right)\|\Psi_{L_k}^{(m)}\|^2 + \frac{\mu}{L_k^2}\left(\langle\Psi_{L_k}, \mathcal{N}\Psi_{L_k}\rangle - \frac{1}{ML_k^2}\langle\Psi_{L_k}, \mathcal{N}^2\Psi_{L_k}\rangle\right).$$

Fixing M large enough in terms of C and c then gives

$$\frac{\langle \Psi_{L_k}, \mathcal{H}\Psi_{L_k}\rangle}{L_k^2} \geq \sum_{m=0}^{ML^2}\left(e_{L_k}\left(\frac{m}{L_k^2}\right) - \mu\frac{m}{L_k^2}\right)\|\Psi_{L_k}^{(m)}\|^2 + \mu\rho. \tag{A8}$$

As in Theorem A.2, we glue several copies of a minimizer of $e_{L_k}(\rho)$, each copy living on a different box. We leave corridors of size $L_k^{1-\epsilon}$ between the boxes and this has the consequence of changing the density to $\rho(1 + L_k^{-\epsilon})^{-2}$. Assuming further that v satisfies the conditions of Theorem 1.1 we estimate the ignored interactions to find

$$e(\rho(1 + L_k^{-\epsilon})^{-2}) \leq e_{L_k}(\rho)(1 + L_k^{-\epsilon})^{-2} + \frac{C(\rho L_k^2)^2}{L_k^{4+\eta_0-(2+\eta_0)\epsilon}}. \tag{A9}$$

Using (A9) in (A8) yields

$$L_k^{-2}\langle \Psi_{L_k}, \mathcal{H}\Psi_{L_k}\rangle \geq \mu\rho + (1 + L_k^{-\epsilon})^2 \sum_{m=0}^{ML_k^2}\left(e\left(\frac{m}{L_k^2}(1 + L_k^{-\epsilon})^{-2}\right)\right.$$
$$\left. - (1 + L_k^{-\epsilon})^{-2}\mu\frac{m}{L_k^2} - \frac{Cm^2}{L_k^{4+\eta_0-(2+\eta_0)\epsilon}}\right)\|\Psi_{L_k}^{(m)}\|^2$$
$$\geq \mu\rho - (1 + L_k^{-\epsilon})^2 e^*(\mu) - C\rho^2 L_k^{-\eta_0+(2+\eta_0)\epsilon},$$

where * defines the Legendre transformation with respect to the interval $[0, M]$. Choosing $\epsilon > 0$ small enough and letting k go to infinity yields

$$\lim_{k\to\infty}\frac{\langle \Psi_{L_k}, \mathcal{H}\Psi_{L_k}\rangle}{L_k^2} \geq \sup_{\mu\in[0,\infty)}(\mu\rho - e^*(\mu)) = \sup_{\mu\in\mathbb{R}}(\mu\rho - e^*(\mu)) = e(\rho), \tag{A10}$$

where we used that $e^*(\mu) \geq 0$ for all $\mu \in \mathbb{R}$ and that the Legendre transformation is an involution. $\qquad\square$

We end the section by giving the proof of the final upper bound Theorem 2.1 using the result of Theorem 2.2.

Proof of Theorem 2.1. We first cut our potential in order to apply Theorem 2.2. We write

$$v = v\mathbb{1}_{B(0,R)} + v\mathbb{1}_{B(0,R)^c} = v_R + v_{tail},$$

where $R = \rho^{-\frac{1}{2}}Y^{\beta+2}$. We denote by a_R the scattering length of v_R. To get estimates on the energy density $e(\rho)$ we use the standard theory developed in "Appendix A". The idea is to extend L_β with R_0 and force the trial function to have Dirichlet boundary conditions on the box of sidelength $L_\beta + R_0$. Thereafter one glues together these small Dirichlet boxes, separated by corridors of size R. Since this process will slightly change the density, we choose for a given $\rho > 0$, the larger density $\widetilde{\rho}$ satisfying

$$\rho = \widetilde{\rho}(1 - 2C\widetilde{Y}^2)\left(1 + \frac{2R_0}{L_\beta} + \frac{R}{L_\beta}\right)^{-2},$$

where C is the same as in Theorem 2.2, and $R_0 = \rho^{-\frac{1}{2}}Y^{-\frac{1}{2}}$. This choice of R_0 is in fact optimal as one can see from the error term $C\frac{\rho}{L_\beta R_0}$ coming from the glueing process in

(A17). Here we use the notation $\widetilde{Y} = |\log(\widetilde{\rho}a_R^2)|^{-1}$ and $\widetilde{\delta}_0 = |\log(\widetilde{\rho}a_R^2\widetilde{Y})|^{-1}$. If ρa^2 is small enough then $\widetilde{\rho}a_R^2 \leq C^{-1}$, and we may use Theorem 2.2 to find a periodic trial state Ψ for the density $\widetilde{\rho}$ and potential v_R satisfying

$$\langle \mathcal{H}_{v_R} \rangle_\Psi \leq 4\pi L_\beta^2 \widetilde{\rho}^2 \widetilde{\delta}_0 \Big(1 + \Big(2\Gamma + \frac{1}{2} + \log(\pi) \Big) \widetilde{\delta}_0 \Big) + C L_\beta^2 \widetilde{\rho}^2 \widetilde{\delta}_0^{\,3} |\log(\widetilde{\delta}_0)|, \quad (A11)$$

with $\langle \mathcal{N} \rangle_\Psi \geq \widetilde{\rho} L_\beta^2 (1 - C\widetilde{Y}^2)$, and $\langle \mathcal{N}^2 \rangle_\Psi \leq C\widetilde{\rho}^2 L_\beta^4$.

Since $\frac{R}{L_\beta} \ll \frac{R_0}{L_\beta} = Y^{-\frac{1}{2}+\beta} \ll 1$, we have $|\rho - \widetilde{\rho}| \leq C\rho Y^{-\frac{1}{2}+\beta}$, and we can change $\widetilde{\rho}$ into ρ in (A11) up to smaller errors if

$$\beta \geq \frac{5}{2}. \tag{A12}$$

One can show that the C appearing in (A11) only increases in β (see (5.21)). Thus we find $\beta = 5/2$ to be optimal. We can also change a_R into a because the right-hand side of (A11) is an increasing function of the scattering length and $a_R \leq a$. Thus,

$$\langle \mathcal{H}_{v_R} \rangle_\Psi \leq 4\pi L_\beta^2 \rho^2 \delta_0 \Big(1 + \Big(2\Gamma + \frac{1}{2} + \log(\pi) \Big) \delta_0 \Big) + C L_\beta^2 \rho^2 \delta_0^3 |\log(\delta_0)|, \quad (A13)$$

and the bounds on the number of particles become

$$\langle \mathcal{N} \rangle_\Psi \geq (\rho + c\rho^2)(L_\beta + 2R_0 + R)^2, \quad \text{and} \quad \langle \mathcal{N}^2 \rangle_\Psi \leq C(\rho L_\beta^2)^2, \quad (A14)$$

for some $c > 0$.

Now we can use Lemma A.1 and Theorem A.2 to glue small boxes together. We get a sequence $\Psi_{k(L_\beta+2R_0+R)} \in \mathscr{F}_s\big(L^2\big(\Lambda_{k(L_\beta+2R_0+R)}\big)\big)$ with Dirichlet boundary conditions, for $k \in \mathbb{N}$, on the box

$$\Lambda_{k(L_\beta+2R_0+R)} = \Big[-\frac{1}{2}k(L_\beta + 2R_0 + R), \frac{1}{2}k(L_\beta + 2R_0 + R) \Big]^2,$$

satisfying

$$\langle \mathcal{N} \rangle_{\Psi_{k(L_\beta+2R_0+R)}} = k^2 \langle \mathcal{N} \rangle_\Psi, \qquad \langle \mathcal{N}^2 \rangle_{\Psi_{k(L_\beta+2R_0+R)}} = k^4 \langle \mathcal{N}^2 \rangle_\Psi, \quad (A15)$$

and

$$\langle \mathcal{H}_v \rangle_{\Psi_{k(L_\beta+2R_0+R)}} \leq k^2 \Big(\langle \mathcal{H}_{v_R} \rangle_\Psi + \langle \mathcal{N} \rangle_\Psi \frac{C}{L_\beta R_0} + \langle \mathcal{N}^2 \rangle_\Psi \frac{Ca^{\eta_0}}{R^{2+\eta_0}} \Big), \quad (A16)$$

where C only depends on η_0 and C_0. Note that we have the original potential in the left-hand side of (A16) because by (A5), v_{tail} only produces an error term. By construction this sequence satisfies the conditions on the number of particles for Theorem A.3, and we conclude

$$e^{2D}(\rho) \leq \lim_{k \to \infty} \frac{\langle \mathcal{H}_v \rangle_{\Psi_{k(L_\beta+2R_0+R)}}}{k^2 L_\beta^2} \leq \frac{\langle \mathcal{H}_{v_R} \rangle_\Psi}{L_\beta^2} + C \frac{\rho}{L_\beta R_0} + Ca^{\eta_0} \frac{\rho^2 L_\beta^2}{R^{2+\eta_0}}, \quad (A17)$$

where in the last inequality we used (A16), (A14) and that $\langle \mathcal{N} \rangle_\Psi^2 \leq \langle \mathcal{N}^2 \rangle_\Psi$. With our choice of parameters including (A12), the two last terms in (A17) are errors. Theorem 2.1 follows from (A17) and (A13). □

Appendix B: Bogoliubov Diagonalization

Theorem B.1. *Let $a_\pm$ be operators on a Hilbert space satisfying $[a_+, a_-] = 0$. For $\mathcal{A} > 0$, $\mathcal{B} \in \mathbb{R}$ satisfying either $|\mathcal{B}| < \mathcal{A}$ or $\mathcal{B} = \mathcal{A}$ and arbitrary $\kappa \in \mathbb{C}$, we have the operator identity*

$$\mathcal{A}(a_+^\dagger a_+ + a_-^\dagger a_-) + \mathcal{B}(a_+^\dagger a_-^\dagger + a_+ a_-) + \kappa(a_+^\dagger + a_-) + \bar{\kappa}(a_+ + a_-^\dagger)$$

$$= (1 - \alpha^2)\mathcal{D}(b_+^\dagger b_+ + b_-^\dagger b_-) - \frac{1}{2}(\mathcal{A} - \sqrt{\mathcal{A}^2 - \mathcal{B}^2})([a_+, a_+^\dagger] + [a_-, a_-^\dagger]) - \frac{2|\kappa|^2}{\mathcal{A} + \mathcal{B}},$$

where $\mathcal{D} = \frac{1}{2}(\mathcal{A} + \sqrt{\mathcal{A}^2 - \mathcal{B}^2})$ and

$$b_+ = \frac{1}{\sqrt{1 - \alpha^2}}(a_+ + \alpha a_-^\dagger + \bar{c}_0), \quad b_- = \frac{1}{\sqrt{1 - \alpha^2}}(a_- + \alpha a_+^\dagger + c_0), \tag{B1}$$

with

$$\alpha = \mathcal{B}^{-1}(\mathcal{A} - \sqrt{\mathcal{A}^2 - \mathcal{B}^2}), \quad c_0 = \frac{2\bar{\kappa}}{\mathcal{A} + \mathcal{B} + \sqrt{\mathcal{A}^2 - \mathcal{B}^2}}. \tag{B2}$$

Remark B.2. Note that the normalization of $b_\pm$ is chosen such that

$$[b_+, b_+^\dagger] = \frac{[a_+, a_+^\dagger] - \alpha^2[a_-, a_-^\dagger]}{1 - \alpha^2}, \tag{B3}$$

and we recover the canonical commutation relations $[b_+, b_+^\dagger] = 1$ when a_+ and a_- satisfies them as well.

Proof. This follows directly from algebraic computations.   □

Appendix C: Calculation of the Bogoliubov Integral

For functions α, β, and parameter $\varepsilon \geq 0$, we define

$$I_\varepsilon(\alpha, \beta) := \frac{1}{2(2\pi)^2} \int_{\mathbb{R}^2} \left(\sqrt{(1 - \varepsilon)^2 \alpha^2(k) + 2(1 - \varepsilon)\rho\alpha(k)\beta(k)} - (1 - \varepsilon)\alpha(k) - \rho\beta(k) \right.$$

$$\left. + \rho^2 \frac{\widehat{g}_k^2 - \widehat{g}_0^2 \mathbb{1}_{\{|k| \leq \ell_\delta^{-1}\}}}{k^2} \right) dk. \tag{C1}$$

We recall that $\widehat{g}_0 = 8\pi\delta$, where δ satisfies $\frac{1}{2}Y \leq \delta \leq 2Y$. We are mainly interested into two special cases, namely $I_0(k^2, \widehat{g})$ and $I_{\varepsilon_N}(\tau, \widehat{W}_1)$. In this section we estimate these integrals.

Lemma C.1. *We can replace τ_k by k^2 up to the following error,*

$$|I_{\varepsilon_N}(\tau, \widehat{W}_1) - I_{\varepsilon_N}(k^2, \widehat{W}_1)| \leq C\rho^2\delta^2 \big(d + \varepsilon_T|\log Y| + (sK_\ell)^{-1}\big).$$

Proof. We recall the definition (8.8) of τ_k,

$$\tau_k = (1 - \varepsilon_T)\left(|k| - \frac{1}{2s\ell}\right)_+^2 + \varepsilon_T\left(|k| - \frac{1}{2ds\ell}\right)_+^2,$$

from which we deduce the bounds

$$|\tau_k - k^2| \le \begin{cases} \frac{1}{2s\ell}|k| + \frac{1}{2(s\ell)^2}, & \text{if } |k| > \frac{1}{2ds\ell}, \\ \varepsilon_T|k|^2 + \frac{3}{2s\ell}|k|, & \text{if } \frac{1}{2s\ell} < |k| < \frac{1}{2ds\ell}. \end{cases} \qquad (C2)$$

We write the integral as

$$I_{\varepsilon_N}(\tau, \widehat{W}_1) = \frac{1}{2(2\pi)^2} \int_{\mathbb{R}^2} F_k(\tau_k, \widehat{W}_1(k)) dk, \qquad (C3)$$

with

$$F_k(\tau, w) = \sqrt{(1 - \varepsilon_N)^2\tau^2 + 2(1 - \varepsilon_N)\rho w\tau}$$

$$- (1 - \varepsilon_N)\tau - \rho w + \rho^2 \frac{\widehat{g}^2(k) - \widehat{g}^2(0)\mathbb{1}_{\{|k| \le \ell_\delta^{-1}\}}}{2k^2}.$$

We first consider separately the small k's. Indeed, $\tau_k = 0$ for $|k| \le \frac{1}{2s\ell}$ and thus

$$|F_k(\tau_k, \widehat{W}_1(k)) - F_k(k^2, \widehat{W}_1(k))| = \left|\sqrt{(1 - \varepsilon_N)^2 k^4 + 2(1 - \varepsilon_N)\rho\widehat{W}_1(k)k^2} - (1 - \varepsilon)k^2\right|$$

$$\le C\sqrt{\rho\delta}|k|,$$

(recall that $(sK_\ell)^{-1} \ll 1$) and

$$\frac{1}{2(2\pi)^2} \int_{|k| \le (2s\ell)^{-1}} |F_k(\tau_k, \widehat{W}_1(k)) - F_k(k^2, \widehat{W}_1(k))| dk \le C\rho^2\delta^2(sK_\ell)^{-3}. \quad (C4)$$

The part with larger k we bound using the derivatives of F and deduce

$$|I_{\varepsilon_N}(\tau, \widehat{W}_1) - I_{\varepsilon_N}(k^2, \widehat{W}_1)| \qquad (C5)$$

$$\le \frac{1}{2(2\pi)^2} \int_{\{|k| > (2s\ell)^{-1}\}} \sup_{\tau \in [\tau_k, k^2]} |\partial_\tau F_k(\tau, \widehat{W}_1(k))| \cdot |\tau_k - k^2| dk + C\rho^2\delta^2(sK_\ell)^{-3}.$$

The derivative of F is given by

$$\partial_\tau F(\tau, w) = \frac{(1 - \varepsilon_N)^2\tau + (1 - \varepsilon_N)\rho w}{\sqrt{(1 - \varepsilon_N)^2\tau^2 + 2(1 - \varepsilon_N)\rho w\tau}} - (1 - \varepsilon_N) \qquad (C6)$$

and can be estimated for $\tau \in [\tau_k, k^2]$ as

$$|\partial_\tau F_k(\tau, \widehat{W}_1(k))| \leq \begin{cases} C\dfrac{\sqrt{\rho\delta}}{|k| - (2s\ell)^{-1}}, & \text{if } (2s\ell)^{-1} < |k| < \sqrt{\rho\delta}, \\ C\dfrac{\rho^2\delta^2}{k^4}, & \text{if } |k| > \sqrt{\rho\delta}. \end{cases} \tag{C7}$$

Indeed, for $|k| < \sqrt{\rho\delta}$, we just need to bound individualy each term in (C6), whereas for $|k| > \sqrt{\rho\delta}$, we have $\tau_k > C\rho\widehat{W}_1(k)$ and we use a Taylor expansion of the square root to get

$$|\partial_\tau F_k(\tau, \widehat{W}_1(k))| \leq C\frac{\rho^2\widehat{W}_1(k)^2}{(1-\varepsilon_N)\tau^2} \leq C\frac{\rho^2\delta^2}{k^4}.$$

Now we split the integral in (C5) into 3 parts I_1, I_2, I_3, corresponding to the integration on the domains $\{(2s\ell)^{-1} < k < \sqrt{\rho\delta}\}$, $\{\sqrt{\rho\delta} < k < (2ds\ell)^{-1}\}$ and $\{k > (2ds\ell)^{-1}\}$, respectively, and we use (C7), (C2) to bound it and find:

$$I_1 \leq C\rho^2\delta^2\Big(\varepsilon_T + \frac{1}{sK_\ell}\Big),$$
$$I_2 \leq C\rho^2\delta^2\Big(\varepsilon_T|\log Y| + \frac{1}{sK_\ell}\Big),$$
$$I_3 \leq C\rho^2\delta^2 d.$$

$\square$

Lemma C.2. *We can replace $\widehat{W}_1(k)$ by $\widehat{g}_k$ up to the following error*

$$|I_{\varepsilon_N}(k^2, \widehat{W}_1) - I_{\varepsilon_N}(k^2, \widehat{g})| \leq C\rho^2\delta^2 K_\ell^{-1} + C\rho^2\delta\varepsilon_N.$$

Proof. Recall that $I_{\varepsilon_N}(k^2, \widehat{W}_1)$ is given by (C3). We first use (6.26) and (3.34) to replace the last part,

$$\rho^2\int_{\mathbb{R}^2}\frac{\widehat{g}_k^2 - \widehat{g}_0^2\mathbb{1}_{\{|k|\leq\ell_\delta^{-1}\}}}{2k^2}dk = \rho^2\int_{\mathbb{R}^2}\frac{\widehat{W}_1(k)^2 - \widehat{W}_1(0)^2\mathbb{1}_{\{|k|\leq\ell_\delta^{-1}\}}}{2(1-\varepsilon_N)k^2}dk$$
$$+ \mathcal{O}(\rho^2\delta\varepsilon_N + \rho^2\delta^2 K_\ell^{-2}),$$

so that

$$I_{\varepsilon_N}(k^2, \widehat{W}_1) = J(\widehat{W}_1) + \mathcal{O}(\rho^2\delta\varepsilon_N + \rho^2\delta^2 K_\ell^{-2}) \quad \text{and} \quad I_{\varepsilon_N}(k^2, \widehat{g}) = J(\widehat{g}), \tag{C8}$$

with

$$J(w) = \frac{1}{2(2\pi)^2}\int_{\mathbb{R}^2}G_k(w_k, w_0)dk, \tag{C9}$$

and

$$G_k(w, w_0) = \sqrt{(1 - \varepsilon_N)^2 k^4 + 2(1 - \varepsilon_N)\rho w k^2}$$
$$- (1 - \varepsilon_N)k^2 - \rho w + \rho^2 \frac{w^2 - w_0^2 \mathbb{1}_{\{|k| \le \ell_\delta^{-1}\}}}{2(1 - \varepsilon_N)k^2}.$$

Note that G_k is independent of w_0 for $|k| > \ell_\delta^{-1}$. Then we split $J(w)$ into two parts,

$$J(w) = \frac{1}{2(2\pi)^2} \int_{\{|k| < \ell_\delta^{-1}\}} G_k(w_k, w_0)\mathrm{d}k + \frac{1}{2(2\pi)^2} \int_{\{|k| > \ell_\delta^{-1}\}} G_k(w_k)\mathrm{d}k$$
$$=: J_<(w) + J_>(w). \tag{C10}$$

For $k > \ell_\delta^{-1}$ we use

$$|J_>(\widehat{W}_1) - J_>(\widehat{g})| \le \frac{1}{2(2\pi)^2} \int_{\{|k| > \ell_\delta^{-1}\}} \sup_{w \in [\widehat{g}_k, \widehat{W}_1(k)]} |\partial_w G_k(w)| \cdot |\widehat{W}_1(k) - \widehat{g}_k|\mathrm{d}k, \tag{C11}$$

with

$$\partial_w G = \frac{\rho}{\sqrt{1 + \frac{2\rho w}{(1 - \varepsilon_N)k^2}}} - \rho + \frac{\rho^2 w}{(1 - \varepsilon_N)k^2}. \tag{C12}$$

We use a Taylor expansion of the square root to get

$$|J_>(\widehat{W}_1) - J_>(\widehat{g})| \le C\rho^3 \int_{|k| > \ell_\delta^{-1}} \frac{\widehat{g}_k^2}{k^4} |\widehat{W}_1(k) - \widehat{g}_k|\mathrm{d}k.$$

Since $|\widehat{W}_1(k) - \widehat{g}_k| \le C\delta^2 K_\ell^{-1}$ (by (6.24)) and $\int \widehat{g}_k^2 k^{-2}\mathrm{d}k < C\delta$ (see (3.34)) we deduce

$$|J_>(\widehat{W}_1) - J_>(\widehat{g})| \le C\rho^2\delta^2 K_\ell^{-1}. \tag{C13}$$

For $k < \ell_\delta^{-1}$ we start by focusing on the first part of G_k,

$$F_k(w) = \sqrt{(1 - \varepsilon_N)^2 k^4 + 2(1 - \varepsilon_N)\rho w k^2} - (1 - \varepsilon_N)k^2 - \rho w. \tag{C14}$$

Since $|\partial_w F_k| \le C\rho$, we have

$$\left| \int_{\{|k| < \ell_\delta^{-1}\}} F_k(\widehat{W}_1(k)) - F_k(\widehat{g}_k)\mathrm{d}k \right| \le C\rho \int_{\{|k| < \ell_\delta^{-1}\}} |\widehat{W}_1(k) - \widehat{g}_k|\mathrm{d}k \le C\rho^2\delta^3 K_\ell^{-1}. \tag{C15}$$

Now

$$
\begin{aligned}
|J_<(\widehat{W}_1) - J_<(\widehat{g})| \leq\ & C \left| \int_{\{|k| < \ell_\delta^{-1}\}} F_k(\widehat{W}_1(k)) - F_k(\widehat{g}_k) \mathrm{d}k \right| \\
& + C \left| \int_{\{|k| < \ell_\delta^{-1}\}} \rho^2 \frac{\widehat{W}_1(k)^2 - \widehat{W}_1(0)^2}{2(1 - \varepsilon_T)k^2} \mathrm{d}k \right| \\
& + C \left| \int_{\{|k| < \ell_\delta^{-1}\}} \rho^2 \frac{\widehat{g}_k^2 - \widehat{g}_0^2}{2(1 - \varepsilon_T)k^2} \mathrm{d}k \right| \\
& \leq C\rho^2 \delta^3 K_\ell^{-1} + C\rho^2 R^2 \delta^2 \ell_\delta^{-2} \leq C\rho^2 \delta^2 K_\ell^{-1},
\end{aligned}
$$

where we used (3.35). Combining this with (C8) and (C13) the lemma is proved. $\qquad \square$

Proposition C.3. *There exists a universal constant $C > 0$ such that, for any $\varepsilon \in [0, 1)$,*

$$
\left| I_\varepsilon(k^2, \widehat{g}_k) - 4\pi\rho^2 \delta \left(1 - \frac{\delta}{Y} + \delta \log \delta + \left(\frac{1}{2} + 2\Gamma + \log(\pi) \right) \delta \right) \right| \\
\leq C\rho^2 \delta^3 \left(|\log(\delta)| R^2 \rho + 1 \right) + C\rho^2 \delta \varepsilon,
$$

where I_ε is defined in (C1). In particular when $\delta = \delta_0$ we deduce

$$
\left| I_\varepsilon(k^2, \widehat{g}_k) - 4\pi\rho^2 \delta_0^2 \left(\frac{1}{2} + 2\Gamma + \log(\pi) \right) \right| \leq C\rho^2 \delta^3 \left(|\log(\delta)| R^2 \rho + 1 \right) + C\rho^2 \delta \varepsilon.
$$

Proof. At first we want to replace $\widehat{g}_k$ by $\widehat{g}_0$ in the integral I:

$$
|I_\varepsilon(k^2, \widehat{g}_k) - I_\varepsilon(k^2, \widehat{g}_0)| \leq \int_{\mathbb{R}^2} |F(k^2, \widehat{g}_k) - F(k^2, \widehat{g}_0)| \mathrm{d}k \tag{C16}
$$

$$
\leq \int_{\mathbb{R}^2} \sup_{g \in [\widehat{g}_k, \widehat{g}_0]} \left| \partial_g F(k^2, g) \right| |\widehat{g}_k - \widehat{g}_0| \, \mathrm{d}k =: I_\leqslant + I_\geqslant, \tag{C17}
$$

where we split for values of $|k|$ under or above $(\rho\delta)^{1/2}$. Notice that

$$
\partial_g F(k^2, \widehat{g}_k) = \frac{\rho k^2}{\sqrt{k^4 + 2\rho\widehat{g}_k k^2}} - \rho + \frac{\rho^2 \widehat{g}_k}{k^2}. \tag{C18}
$$

By a Taylor expansion we can prove that

$$
I_\leqslant \leq C \int_{\{|k| \leq (\rho\delta)^{1/2}\}} \left(R^2 (\rho\widehat{g}_0)^{1/2} k^3 + \rho\widehat{g}_0 k^2 + R^2 (\rho\widehat{g}_0)^2 \right) \mathrm{d}k \leq C R^2 (\rho\delta)^3. \tag{C19}
$$

In the other case we have, by Taylor expansion of the square root in (C18),

$$
\begin{aligned}
I_\geqslant &\leq C\rho \int_{\{|k| \geq (\rho\delta)^{1/2}\}} \frac{(\rho\widehat{g}_0)^2}{k^4} |\widehat{g}_k - \widehat{g}_0| \mathrm{d}k \\
&\leq C (\rho\widehat{g}_0)^3 \left(\int_{\{(\rho\delta)^{1/2} \leq |k| \leq (\rho\delta)^{1/2} \widehat{g}_0^{-1/2}\}} \frac{R^2}{k^2} \mathrm{d}k + \int_{\{|k| \geq (\rho\delta)^{1/2} \widehat{g}_0^{-1/2}\}} \frac{\mathrm{d}k}{k^4} \right) \\
&\leq C (\rho\delta)^3 R^2 |\log \delta| + C (\rho\delta)^2 \delta.
\end{aligned}
$$

We deduce that $|I_\varepsilon(k^2, \widehat{g}_k) - I_\varepsilon(k^2, \widehat{g}_0)| \leq C\rho^2\delta^3(1 + R^2\rho\log(\delta))$. Now remains to compute $I_\varepsilon(k^2, \widehat{g}_0)$. In this integral we use the new variable $q = k(\rho\widehat{g}_0)^{-\frac{1}{2}}(1-\varepsilon)^{\frac{1}{2}}$,

$$I_\varepsilon(k^2, \widehat{g}_0) = \frac{(\rho\widehat{g}_0)^2}{2(2\pi)^2(1-\varepsilon)} \int_{\mathbb{R}^2} \left(\sqrt{q^4 + 2q^2} - q^2 - 1 + \frac{\mathbb{1}_{\{|q| > (1-\varepsilon)^{\frac{1}{2}}\ell_\delta^{-1}(\rho\widehat{g}_0)^{-\frac{1}{2}}\}}}{2q^2}\right)dq. \tag{C20}$$

In term of $c_0 = (1-\varepsilon)^{\frac{1}{2}}\ell_\delta^{-1}(\rho\widehat{g}_0)^{-\frac{1}{2}}$, this integral is explicitly computable and equal to

$$I_\varepsilon(k^2, \widehat{g}_0) = \frac{(\rho\widehat{g}_0)^2}{4\pi(1-\varepsilon)}\left(\frac{1}{8} - \frac{\log 2}{4} + \frac{1}{2}\log(c_0^{-1})\right). \tag{C21}$$

With $\widehat{g}_0 = 8\pi\delta$ and $c_0 = 2e^{-\Gamma}e^{-\frac{1}{2\delta}}(1-\varepsilon)^{\frac{1}{2}}\widehat{g}_0^{-\frac{1}{2}}(\rho a^2)^{-\frac{1}{2}}$ (see (3.30)), we find

$$I_\varepsilon(k^2, \widehat{g}_0) = 4\pi\rho^2\delta^2\left(\frac{1}{\delta} - \frac{1}{Y} + \log\delta + \frac{1}{2} + 2\Gamma + \log(\pi)\right)(1 + \mathcal{O}(\varepsilon)). \tag{C22}$$

$\square$

Remark C.4. With the arbitrary parameter δ (within the range $\frac{1}{2}Y \leq \delta \leq 2Y$), one can deduce from Proposition C.3 that our lower bound on the energy is

$$e^{\mathrm{2D}}(\rho) \geq 4\pi\rho^2\delta\left(2 - \frac{\delta}{Y} + \delta\log\delta + \left(\frac{1}{2} + 2\Gamma + \log(\pi)\right)\delta\right) - C\rho^2 Y^{2+\eta}. \tag{C23}$$

However, this lower bound is maximized by $\delta = Y(1 - Y|\log Y| + o(Y|\log Y|))$, thus leading to the optimal choice $\delta = \delta_0$.

We conclude this section by a general bound on Bogoliubov integrals that is used several times throughout the paper.

Lemma C.5. *For two functions* $\mathcal{A}, \mathcal{B} : \mathbb{R}^2 \to \mathbb{R}$ *such that*

$$\mathcal{A}(k) \geq \kappa[|k| - K]_+^2 + 2K_1\delta, \quad |\mathcal{B}(k)| \leq K_2\delta, \quad |\mathcal{B}(k) - \mathcal{B}(0)| \leq K_2 R^2\delta|k|^2, \tag{C24}$$

for constants $\kappa > 0, 0 < K_2 \leq K_1, \ell_\delta^{-1} < K$, *then there exists* $C > 0$ *such that*

$$\int_{\mathbb{R}^2} \left(\mathcal{A}(k) - \sqrt{\mathcal{A}(k)^2 - \mathcal{B}(k)^2}\right)dk$$

$$\leq \kappa^{-1}\int_{\mathbb{R}^2} \frac{\mathcal{B}^2(k) - \mathcal{B}^2(0)\mathbb{1}_{\{|k| \leq \ell_\delta^{-1}\}}}{2|k|^2}dk$$

$$+ C\frac{K_2^2}{K_1}\delta K^2 + C\kappa^{-1}K_2^2\delta^2(1 + R^2\ell_\delta^{-2}) + C\kappa^{-1}K_2^2\delta^2|\log(2K\ell_\delta)|$$

$$+ C\min\left(K_2^4\delta^4\kappa^{-3}K^{-4}, C\frac{K_2^2}{K_1^2}\kappa^{-1}\int_{\mathbb{R}^2} \frac{\mathcal{B}(k)^2 - \mathcal{B}(0)^2\mathbb{1}_{\{|k| < \ell_\delta^{-1}\}}}{|k|^2}dk\right). \tag{C25}$$

Proof. We show that the difference

$$\int_{\mathbb{R}^2}\left(\left(\mathcal{A}(k)-\sqrt{\mathcal{A}(k)^2-\mathcal{B}(k)^2}\right)-\kappa^{-1}\frac{\mathcal{B}^2(k)-\mathcal{B}^2(0)\mathbb{1}_{\{|k|\leq\ell_\delta^{-1}\}}}{2|k|^2}\right)dk, \quad \text{(C26)}$$

is bounded by the desired error terms.

For $|k|\leq 2K$ we have that

$$\int_{|k|\leq 2K}\left(\mathcal{A}(k)-\sqrt{\mathcal{A}(k)^2-\mathcal{B}(k)^2}\right)dk \leq C\int_{\{|k|\leq 2K\}}\frac{\mathcal{B}^2(k)}{\mathcal{A}(k)}dk$$

$$\leq C\frac{K_2^2}{K_1}\delta\int_{\{|k|\leq 2K\}}dk = C\frac{K_2^2}{K_1}\delta K^2,$$

while for the $\mathcal{B}$ part, using the assumption on $|\mathcal{B}(k)-\mathcal{B}(0)|$,

$$\kappa^{-1}\int_{\{|k|\leq\ell_\delta^{-1}\}}\frac{|\mathcal{B}^2(k)-\mathcal{B}^2(0)|}{2|k|^2}dk \leq C\kappa^{-1}K_2^2R^2\delta^2\int_{\{|k|\leq\ell_\delta^{-1}\}}dk = C\kappa^{-1}K_2^2R^2\delta^2\ell_\delta^{-2},$$
$$\text{(C27)}$$

and

$$\kappa^{-1}\int_{\{\ell_\delta^{-1}\leq|k|\leq 2K\}}d\frac{\mathcal{B}_k^2}{2|k|^2}dk \leq C\kappa^{-1}K_2^2\delta^2|\log(2K\ell_\delta)|. \quad \text{(C28)}$$

For $|k|\geq 2K$ we have, by a Taylor expansion,

$$\mathcal{A}(k)-\sqrt{\mathcal{A}(k)^2-\mathcal{B}(k)^2} \leq \frac{1}{2}\frac{\mathcal{B}(k)^2}{\mathcal{A}(k)}+C\frac{\mathcal{B}(k)^4}{\mathcal{A}(k)^3}. \quad \text{(C29)}$$

For the first term we observe that

$$\frac{\mathcal{B}(k)^2}{\mathcal{A}(k)} \leq \kappa^{-1}\frac{\mathcal{B}(k)^2}{(|k|-K)^2} \leq \kappa^{-1}\frac{\mathcal{B}(k)^2}{|k|^2}\left(1+\frac{K}{|k|}\right), \quad \text{(C30)}$$

giving

$$\int_{\{|k|\geq 2K\}}\left(\frac{\mathcal{B}(k)^2}{\mathcal{A}(k)}-\kappa^{-1}\frac{\mathcal{B}(k)^2}{2|k|^2}\right)dk \leq CK\kappa^{-1}\int_{\{|k|\geq 2K\}}\frac{\mathcal{B}(k)^2}{|k|^3}dk \leq C\kappa^{-1}K_2^2\delta^2,$$

while for the second one we can bound either

$$\int_{\{|k|\geq 2K\}}\frac{\mathcal{B}(k)^4}{\mathcal{A}(k)^3}dk \leq CK_2^4\delta^4\kappa^{-3}\int_{\{|k|\geq 2K\}}\frac{dk}{|k|^6} \leq CK_2^4\delta^4\kappa^{-3}K^{-4}, \quad \text{(C31)}$$

or as in the following,

$$\int_{\{|k|\geq 2K\}}\frac{\mathcal{B}(k)^4}{\mathcal{A}(k)^3}dk \leq \frac{K_2^2}{K_1^2}\kappa^{-1}\int_{\{|k|\geq 2K\}}\frac{\mathcal{B}(k)^2}{\mathcal{A}(k)}dk \leq C\frac{K_2^2}{K_1^2}\kappa^{-1}\int_{\{|k|\geq 2K\}}\frac{\mathcal{B}(k)^2}{|k|^2}dk,$$

adding and subtracting the term $C\frac{K_2^2}{K_1^2}\kappa^{-1}\int_{\{|k|<2K\}}\frac{\mathcal{B}(k)^2-\mathcal{B}(0)^2\mathbb{1}_{\{|k|<\ell_\delta^{-1}\}}}{|k|^2}$ we have

$$\int_{\{|k|\geq 2K\}}\frac{\mathcal{B}(k)^4}{\mathcal{A}(k)^3}dk \leq C\frac{K_2^2}{K_1^2}\kappa^{-1}\int_{\mathbb{R}^2}\frac{\mathcal{B}(k)^2-\mathcal{B}(0)^2\mathbb{1}_{\{|k|<\ell_\delta^{-1}\}}}{|k|^2}dk. \quad \text{(C32)}$$

This finishes the proof of Lemma C.5. $\square$

Appendix D: A Priori Bounds

In this section, we prove Theorem 7.6. We study a localized problem on a shorter length scale $d\ell$ such that

$$d\ell \ll \ell_\delta \ll \ell. \tag{D1}$$

where we recall that ℓ_δ is the healing length. We are able, in this section, to prove Bose–Einstein condensation in boxes with length scale smaller than the healing length. A key point is that, at this scale, we can use a larger Neumann gap to reabsorb the errors. We will show how the proof of Theorem 7.6 reduces to this localized problem.

We introduce the small box centered at $u \in \mathbb{R}^2$ to be

$$B_u = \Lambda \cap \left\{ d\ell u + \left[-\frac{d\ell}{2}, \frac{d\ell}{2} \right]^2 \right\}. \tag{D2}$$

The associated localization functions are

$$\chi_{B_u}(x) := \chi\left(\frac{x}{\ell}\right) \chi\left(\frac{x}{d\ell} - u\right), \tag{D3}$$

where we highlight that

$$\iint |\chi_{B_u}|^2 \mathrm{d}x\,\mathrm{d}u = \ell^2. \tag{D4}$$

In order to construct the small box Hamiltonian, we introduce the localized potentials

$$W^{\mathrm{s}}(x) := \frac{W(x)}{\chi * \chi(x/d\ell)}, \qquad w_{B_u}(x, y) := \chi_{B_u}(x) W^{\mathrm{s}}(x - y) \chi_{B_u}(y), \tag{D5}$$

$$W_1^{\mathrm{s}}(x) := \frac{W_1(x)}{\chi * \chi(x/d\ell)}, \qquad w_{1, B_u}(x, y) := \chi_{B_u}(x) W_1^{\mathrm{s}}(x - y) \chi_{B_u}(y), \tag{D6}$$

$$W_2^{\mathrm{s}}(x) := \frac{W_2(x)}{\chi * \chi(x/d\ell)}, \qquad w_{2, B_u}(x, y) := \chi_{B_u}(x) W_2^{\mathrm{s}}(x - y) \chi_{B_u}(y), \tag{D7}$$

where we recall that W, W_1, W_2 are localized versions of $v, g, (1 + \omega)g$, respectively (see formulas (6.19) and (3.4)). Since v has support in $B(0, R)$, we see that W^{s} is well-defined as $d\ell$ is required to be larger then R. Clearly W^{s} depends on $d\ell$ and thus ρ_μ, but we will not reflect this in our notation.

Similarly to Lemma 6.4, W_1^{s} satisfies the following inequalities which can be proven in analogous ways considering the length scale $d\ell$ in place of ℓ

$$\int W_2^{\mathrm{s}} \leq 2 \int W_1^{\mathrm{s}} \leq C\delta, \tag{D8}$$

$$0 \leq W_1^{\mathrm{s}}(x) - g(x) \leq C g(x) \frac{|x|^2}{(d\ell)^2}, \tag{D9}$$

$$\left| \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{\widehat{W}_1^{\mathrm{s}}(k)^2 - \widehat{W}_1^{\mathrm{s}}(0)^2 \mathbb{1}_{\{|k| \leq \ell_\delta^{-1}\}}}{2k^2} \mathrm{d}k - \widehat{g\omega}(0) \right| \leq C \frac{R^2\delta}{(d\ell)^2}. \tag{D10}$$

We define furthermore, as operators on $L^2(B_u)$,

$$P_{B_u} := \frac{1}{|B_u|} |\mathbb{1}_{B_u}\rangle\langle\mathbb{1}_{B_u}|, \qquad Q_{B_u} := \mathbb{1}_{B_u} - P_{B_u}, \tag{D11}$$

i.e., P_{B_u} is the orthogonal projection in $L^2(B_u)$ onto the constant functions and Q_{B_u} is the projection to the orthogonal complement. We can therefore introduce the number operators as well

$$n_{B_u} := \sum_{j=1}^{N} \mathbb{1}_{B_u,j}, \quad n_{B_u,0} := \sum_{j=1}^{N} P_{B_u,j}, \quad n_{B_u,+} := \sum_{j=1}^{N} Q_{B_u,j}, \qquad \text{(D12)}$$

and the small-box kinetic energy

$$\mathcal{T}_{B_u} := Q_{B_u}\left(\chi_{B_u}\left[\sqrt{-\Delta} - \frac{1}{ds\ell}\right]_+^2 \chi_{B_u} + \frac{\varepsilon_T}{2}(1 + \pi^{-2})\frac{1}{(d\ell)^2}\right)Q_{B_u}. \qquad \text{(D13)}$$

We are now ready to define the localized Hamiltonian $\mathcal{H}_{B_u}$ which acts on the symmetric Fock space $\mathcal{F}_s(L^2(B_u))$. It preserves particle number and is given as

$$\mathcal{H}_{B_u}(\rho_\mu)_N := \sum_{i=1}^{N}(1 - \varepsilon_N)\mathcal{T}_{B_u}^{(i)} - \rho_\mu \sum_{i=1}^{N}\int w_{1,B_u}(x_i, y)\,\mathrm{d}y + \frac{1}{2}\sum_{i\neq j} w_{B_u}(x_i, x_j), \qquad \text{(D14)}$$

on the N-particle sector where ε_N was introduced in Lemma 6.2.

An adaptation to dimension 2 of [28, Theorem 3.10] allows us relate $\mathcal{H}_B(\rho_\mu)$ to the original Hamiltonian in the large box, using the condition (H28). This gives the lower bound

$$\mathcal{H}_\Lambda(\rho_\mu) \geq (1 - \varepsilon_N)\frac{b}{\ell^2}\sum_{j=1}^{N} Q_{\Lambda,j} + \int_{\mathbb{R}^2} \mathcal{H}_{B_u}(\rho_\mu)\,\mathrm{d}u. \qquad \text{(D15)}$$

We would like to restrict the previous integral to boxes that are not too small. Therefore, we identify the following sets of integration, for $\xi \in [0, 1]$,

$$\Lambda_\xi := \left\{u \in \mathbb{R}^2 \,\middle|\, |\ell d u|_\infty - \frac{\ell}{2}(d + 1) \leq -\xi d\ell\right\}, \qquad \text{(D16)}$$

underlying the property

$$\Lambda_{\xi_1} \subseteq \Lambda_{\xi_2} \quad \text{if} \quad \xi_1 \geq \xi_2, \qquad \text{(D17)}$$

and we observe that integration outside Λ_0 is zero because there is no more intersection between the small box and Λ. The following Lemma guarantees that we can restrict the integration for the potential over set $\Lambda_{1/10}$ (where $1/10$ is chosen arbitrarily) and estimate the remaining part by a frame inside $\Lambda_{1/10}$.

Lemma D.1. *For all $x \in \Lambda$ we have the estimate*

$$-\rho_\mu \iint w_{1,B_u}(x, y)\mathrm{d}y\mathrm{d}u$$
$$\geq -\rho_\mu \int_{\Lambda_{\frac{1}{10}}} \int w_{1,B_u}(x, y)\mathrm{d}y\mathrm{d}u - 3\rho_\mu \int_{\Lambda_{\frac{1}{10}}\backslash\Lambda_{\frac{1}{5}}} \int w_{1,B_u}(x, y)\mathrm{d}y\mathrm{d}u. \qquad \text{(D18)}$$

Proof. The proof follows the same lines as in [19, Lemma E.1]. We split the domain of integration $\Lambda_{1/10}$ and $\Lambda_0 - \Lambda_{1/10}$ and we estimate the integral over the latter. By the definition of $w_{1,B}$ we have simply to estimate the quantity

$$\int_{\Lambda_0 \setminus \Lambda_{1/10}} \chi\left(\frac{x}{\ell d} - u\right) \chi\left(\frac{y}{\ell d} - u\right) \mathrm{d}u. \tag{D19}$$

We use that χ is a product of decreasing functions in the variables u_1, u_2 and observe that

$$\max_{\frac{1}{2}(d^{-1}+1) - \frac{(\ell d)-1}{10} \leq |u_1| \leq \frac{1}{2}(d^{-1}+1)} \chi\left(\frac{x}{\ell d} - u\right) \chi\left(\frac{y}{\ell d} - u\right)$$

$$\leq \min_{\frac{1}{2}(d^{-1}+1) - \frac{2(d\ell)-1}{10} \leq |u_1| \leq \frac{1}{2d}+\frac{1}{2} - \frac{(d\ell)-1}{10}} \chi\left(\frac{x}{\ell d} - u\right) \chi\left(\frac{y}{\ell d} - u\right), \tag{D20}$$

so that we can estimate the integral over the frame pointwise, getting a factor of 3 due to the presence of the corners.   □

Thanks to Lemma D.1 we can write

$$\mathcal{H}_\Lambda(\rho_\mu) \geq (1 - \varepsilon_N)\frac{b}{\ell^2}\sum_{j=1}^{N} Q_{\Lambda,j} + \int_{\Lambda_{\frac{1}{5}}} \mathcal{H}_{B_u}(\rho_\mu)\mathrm{d}u + \int_{\Lambda_{\frac{1}{10}} \setminus \Lambda_{\frac{1}{5}}} \mathcal{H}_{B_u}(4\rho_\mu)\mathrm{d}u, \tag{D21}$$

where we dropped the positive part of $\mathcal{H}_{B_u}$ in $\Lambda_0 \setminus \Lambda_{\frac{1}{10}}$. We are now ready to give lower bounds for kinetic and potential energies in terms of the number of particles. From this the lower bound for the small box Hamiltonian is going to follow in Corollary D.6 below.

In order to prove Theorem 7.6, we provide a lower bound on $\mathcal{H}_{B_u}(\rho_\mu)$. For notational simplicity we will remove the index u. Lemmas D.2 and D.3 below give first lower bounds on the potential and kinetic energy respectively.

Lemma D.2. *There exists a constant $C > 0$ depending only on χ such that*

$$-\rho_\mu \int \sum_{j=1}^{N} w_{1,B}(x, y)\mathrm{d}y + \frac{1}{2}\sum_{i \neq j} w_B(x_i, x_j)$$

$$\geq A_0 + A_2 + \frac{1}{2}\mathcal{Q}_4^{ren,\,s} - C\delta\left(\rho_\mu + \frac{n_{0,B}}{|B|}\right)n_{+,B},$$

with

$$A_0 := \frac{n_{0,B}(n_{0,B}+1)}{2|B|^2}\iint w_{2,B}(x, y)\mathrm{d}x\mathrm{d}y$$

$$- \left(\frac{\rho_\mu n_{0,B}}{|B|} + \frac{1}{4}\left(\rho_\mu - \frac{n_{0,B}-1}{|B|}\right)^2\right)\iint w_{1,B}(x, y)\mathrm{d}x\mathrm{d}y,$$

$$A_2 := \frac{1}{2}\sum_{i \neq j} P_i P_j w_{1,B} Q_i Q_j + h.c.,$$

and $\mathcal{Q}_4^{ren,s}$ is the analogue of (7.2), but for the small box B.

Proof. The proof follows from an analogous potential splitting like in Lemma 7.1 and Lemma 7.2 and by the same lines as [19, Lemma E.7]. □

Lemma D.3. *For the kinetic energy on the small box in the N'th sector we have*

$$Q_B \chi_B \Big[\sqrt{-\Delta} - \frac{1}{ds\ell}\Big]_+^2 \chi_B Q_B + A_2$$
$$\geq -\frac{1}{2}\widehat{g\omega}(0)\frac{N(N+1)}{|B|^2}\int \chi_B^2 + \mathcal{E}_2(N) + \mathcal{E}_4(N) - C\delta\frac{N+1}{|B|}n_{+,B},$$

where

$$\mathcal{E}_2(N) := -C\delta\Big(\frac{R^2}{(d\ell)^2} + \delta|\log(ds\,K_\ell)| + \delta^2\Big)\frac{N(N+1)}{|B|^2}\int \chi_B^2, \tag{D22}$$

$$\mathcal{E}_4(N) := -C\Big(\delta^4(ds\ell)^4\Big(\frac{N+1}{|B|}\Big)^3 + \delta(ds\ell)^{-2}\Big)\frac{N}{|B|}\int \chi_B^2. \tag{D23}$$

Proof. Let us introduce the operators

$$d_p^\dagger := \frac{1}{|B|^{1/2}}a^\dagger(Q_B \chi_B e^{-ipx})a_0, \tag{D24}$$

where $a_0 = \frac{1}{\ell}a(1)$ and $a, a^\dagger$ are the annihilation and creation operators on $\mathscr{F}_s(L^2(\Lambda))$. Further we introduce

$$A_1 := \frac{\widehat{W}_1^s(0)}{(2\pi)^2}\int_{\mathbb{R}^2}(d_p^\dagger d_p + d_{-p}^\dagger d_{-p})\mathrm{d}p. \tag{D25}$$

Now using that on the N'th sector we have

$$\int\Big[|p| - \frac{1}{ds\ell}\Big]_+^2 d_p^\dagger d_p \mathrm{d}p = \frac{(n_0+1)}{|B|}\sum_{j=1}^N Q_{B,j}\chi_B(x_j)\Big[\sqrt{-\Delta} - \frac{1}{ds\ell}\Big]_+^2\chi_B(x_j)Q_{B,j}, \tag{D26}$$

and that $n_0 \leq N$, we get, adding A_1 and A_2 to the kinetic energy

$$\sum_{j=1}^N Q_{B,j}\chi_{B,j}\Big[\sqrt{-\Delta} - \frac{1}{ds\ell}\Big]_+^2\chi_{B,j}Q_{B,j} + A_1 + A_2 \geq \frac{1}{2(2\pi)^2}\int h_p\mathrm{d}p, \tag{D27}$$

where

$$h_p := \mathcal{A}_p(d_p^\dagger d_p + d_{-p}^\dagger d_{-p}) + \mathcal{B}_p(d_p^\dagger d_{-p}^\dagger + d_{-p}d_p), \tag{D28}$$

with

$$\mathcal{A}_p := (1 - \varepsilon_N)\frac{|B|}{N+1}\Big[|p| - \frac{1}{ds\ell}\Big]_+^2 + 2\widehat{W}_1^s(0), \qquad \mathcal{B}_p := \widehat{W}_1^s(p). \tag{D29}$$

The additional A_1 term is estimated, thanks to (D8), by

$$A_1 \leq C\delta\frac{n_0+1}{|B|}n_{+,B}, \tag{D30}$$

which contributes to the last term in the result of the lemma. By an application of Theorem B.1 we get the bound

$$\frac{1}{2(2\pi)^2} \int h_p \mathrm{d}p \geq -\frac{1}{2(2\pi)^2} \frac{N}{|B|} \int \chi_B^2 \int \left(\mathcal{A}_p - \sqrt{\mathcal{A}_p^2 - \mathcal{B}_p^2}\right) \mathrm{d}p, \quad \text{(D31)}$$

and therefore we want to bound the latter. We observe that, thanks to (D9), $\widehat{W}_1^s(0) \geq C\delta$ for a certain $C < 8\pi$. Choosing the parameters

$$K = (ds\ell)^{-1}, \quad \kappa = (1 - \varepsilon_N) \frac{|B|}{N + 1}, \quad K_1 = 1, \quad K_2 = C, \quad \text{(D32)}$$

we can apply Lemma C.5 to obtain

$$-\frac{1}{2(2\pi)^2} \int (\mathcal{A}_p - \sqrt{\mathcal{A}_p^2 - \mathcal{B}_p^2}) \mathrm{d}p$$

$$\geq -\frac{N + 1}{|B|(1 - \varepsilon_N)} \widehat{g\omega}(0) - \frac{N + 1}{|B|} \frac{R^2}{(d\ell)^2} \delta + C\delta(ds\ell)^{-2}$$

$$- C\frac{N + 1}{|B|} \delta^2 \left(1 + \left(\frac{R}{\ell_\delta}\right)^2\right) - C\frac{N + 1}{|B|} \delta^2 |\log(2ds K_\ell)| - C\delta^4 \frac{(N + 1)^3}{|B|^3} (ds\ell)^4,$$

where we used that $\varepsilon_N \leq 1/2$ and (D10) to approximate the leading term by $\widehat{g\omega}(0)$ getting the second term as an error. Plugging the last estimate in (D31) we get the result with the error terms coherent with the definitions of $\mathcal{E}_2(n)$ and $\mathcal{E}_4(n)$.   □

We will also need the following estimates.

Lemma D.4. *Let ℓ_{min} denote the shortest length of the box B, then there exists a constant $C > 0$ such that*

$$\left| \iint w_{1,B}(x, y) \mathrm{d}x \mathrm{d}y - 8\pi\delta \int \chi_B^2 \right| \leq C\delta \left(\frac{R}{\ell_{min}}\right)^2 \int \chi_B^2, \quad \text{(D33)}$$

and

$$\iint w_{2,B}(x, y) \mathrm{d}x \mathrm{d}y \geq \iint w_{1,B}(x, y) \mathrm{d}x \mathrm{d}y + \widehat{g\omega}(0) \int \chi_B^2 - C\frac{R\delta^2}{\ell_{min}^2} \int \chi_B^2. \quad \text{(D34)}$$

Proof. Since $8\pi\delta = \int g$, and thanks to (D9), we can write the inequality

$$\left| \iint (W_1^s(x) - g(x)) \chi_B^2(y) \mathrm{d}x \mathrm{d}y \right| \leq C\delta \left(\frac{R}{\ell_{min}}\right)^2 \int \chi_B^2, \quad \text{(D35)}$$

where we used $\ell_{min} \leq d\ell$. By a Taylor expansion for the localization function and the fact that W is spherically symmetric and (D8), we have, on the other hand,

$$\left| \iint w_{1,B}(x, y) \mathrm{d}x \mathrm{d}y - \int W_1^s(x) \mathrm{d}x \int \chi_B^2 \right| \leq CR^2 \|\nabla^2 \chi_B\|_\infty \int W_1^s(x) \mathrm{d}x \int \chi_B$$

$$\leq C \left(\frac{R}{\ell_{min}}\right)^2 \delta \int \chi_B^2, \quad \text{(D36)}$$

and where we used that $|B|^{-1}(\int \chi_B)^2 \leq \int \chi_B^2$ and the bound (F6).

Then inequality (D33) follows by (D35) and (D36). The inequality (D34) follows from a very similar argument.   □

Combining the results of Lemma D.2 and D.3, we deduce that the Hamiltonian on the small box has the following lower bound, which is coherent with the main order of the energy expansion.

Theorem D.5. *Assume the conditions from "Appendix H", then for any box B we have the following bound on the N'th sector*

$$\mathcal{H}_B(\rho_\mu)|_N \geq \left(\frac{1}{4}\left(\rho_\mu - \frac{N}{|B|}\right)^2 - \frac{1}{2}\rho_\mu^2\right) \iint w_{1,B} + \frac{1}{2}\mathcal{Q}_4^{ren,s} + \mathcal{E}_2(N) + \mathcal{E}_4(N),$$

(D37)

with $\mathcal{E}_2$ and $\mathcal{E}_4$ defined in (D22).

Proof. The combination of Lemmas D.2, D.3 gives

$$\mathcal{H}_B(\rho_\mu) \geq \sum_{j=1}^{n} \mathcal{Q}_{B,j}\left(\frac{\varepsilon_T}{2}(1+\pi^{-2})\frac{1}{(d\ell)^2}\right)\mathcal{Q}_{B,j} + A_0 - \frac{1}{2}\widehat{g\omega}(0)\frac{N(N+1)}{|B|^2}\int \chi_B^2$$

$$+ \frac{1}{2}\mathcal{Q}_4^{ren,s} + \mathcal{E}_2(N) + \mathcal{E}_4(N) - C\delta\rho_\mu n_{+,B}.$$

We observe that we can choose a constant $C' > 0$ such that

$$\sum_{j=1}^{N} \mathcal{Q}_{B,j}\left(\frac{\varepsilon_T}{2}\frac{1}{(d\ell)^2}\right) \geq C'\rho_\mu\delta n_{+,B},$$

(D38)

where we used (H8) and, choosing the right C', we can cancel the last term with $n_{+,B}$ for a lower bound. The same can be said for the errors produced by replacing $n_0 = N - n_+$ by N. By Lemma D.4 we have

$$A_0 - \frac{1}{2}\widehat{g\omega}(0)\frac{N(N+1)}{|B|}\int \chi_B^2$$

$$\geq \left(-\frac{N^2}{2|B|^2} - \left(\rho_\mu\frac{N}{|B|} + \frac{1}{4}\left(\rho_\mu - \frac{N}{|B|}\right)^2\right)\right)\iint w_{1,B} - C\frac{N^2}{|B|^2}\frac{R\delta^2}{\ell_{min}^2}\int \chi_B^2$$

$$\geq \left(\frac{1}{4}\left(\rho_\mu - \frac{N}{|B|}\right)^2 - \frac{1}{2}\rho_\mu^2\right)\iint w_{1,B} - C\frac{N^2}{|B|^2}\frac{R\delta^2}{\ell_{min}^2}\int \chi_B^2,$$

and this gives the result since the last term can be reabsorbed in the $\mathcal{E}_2$ term. $\square$

We deduce the following corollary.

Corollary D.6. *Assume B is a small box with shortest side length $\ell_{min} \geq \frac{d\ell}{10}$ and that the conditions of "Appendix H" hold true. Then we have the following lower bound*

$$\mathcal{H}_B(\rho_\mu) \geq -\frac{1}{2}\rho_\mu^2\iint w_{1,B}(x,y)dxdy - C\rho_\mu^2\delta^2(ds K_\ell)^{-2}\int \chi_B^2 - C\rho_\mu\delta\frac{1}{|B|}\int \chi_B^2.$$

Proof. We split the particles in m subsets of n' particles and a remaining group of n'', with $n'' < n' < n$. If we ignore the positive interactions between the subsets, and denoting by $e_B(n, \rho_\mu)$ the ground state energy of $\mathcal{H}_B(\rho_\mu)$ restricted to states with n particles in the box B, then

$$e_B(n, \rho_\mu) \geq m e_B(n', \rho_\mu) + e_B(n'', \rho_\mu). \tag{D39}$$

From formula (D37) in Theorem D.5 applied for n' in place of n and, choosing $n' = 3\rho_\mu|B|$, we see that the first term becomes, thanks to Lemma D.4

$$\frac{1}{2}\rho_\mu^2 \iint w_{1,B} \geq 4\pi\rho_\mu^2 \delta\left(1 - C\left(\frac{R}{\ell_{\min}}\right)^2\right) \int \chi_B^2. \tag{D40}$$

From the following controls on the error terms

$$\mathcal{E}_2(n') \leq C\rho_\mu^2 \delta^2 (dK_\ell)^{-2} \int \chi_B^2, \tag{D41}$$

$$\mathcal{E}_4(n') \leq C\rho_\mu^2 \delta^2 (ds K_\ell)^{-2} \int \chi_B^2, \tag{D42}$$

$$C\rho_\mu^2 \delta\left(\frac{R}{\ell_{\min}}\right)^2 \int \chi_B^2 \leq C\rho_\mu^2 \delta \frac{R^2}{(d\ell)^2} \int \chi_B^2 \leq C\rho_\mu^2 \delta^2 (dK_\ell)^{-2} \int \chi_B^2, \tag{D43}$$

we see that the first term is the leading term of the energy. Since it is clearly positive, we obtain that with the aforementioned choice of n', we have $e_B(n', \rho_\mu) \geq 0$ and, then, using the previous equality, we can state that

$$e_B(n, \rho_\mu) \geq e_B(n'', \rho_\mu). \tag{D44}$$

The Corollary follows using again Theorem D.5 with n'' in place of n to obtain the lower bound and using (D41) and (D42) for n'' to control the errors, using that $s^{-1} \gg 1$ to obtain one of the error terms in the statement. A further error term of order

$$C\rho_\mu \delta \frac{1}{|B|} \int \chi_B^2,$$

is created by the substitutions of the terms $n'' \pm 1$ by n''. $\qquad \square$

We are finally ready to use the lower bound on the small box Hamiltonian to obtain a bound on the number of excited particles in the large box, result stated in the Theorem below. By an abuse notation, from now on, the operators n, n_+, n_0 start again to denote the number operators in the large box.

Theorem D.7. *We have the following lower bound for the large box Hamiltonian*

$$\mathcal{H}_\Lambda(\rho_\mu) \geq -4\pi\rho_\mu^2 \ell^2 Y\left(1 - \frac{1}{2}Y|\log Y|\right) + \frac{b}{2\ell^2}n_+, \tag{D45}$$

and if there exists a normalized $\Psi \in \mathscr{F}_s(L^2(\Lambda))$ with n particles in Λ such that (7.40) holds:

$$\langle\mathcal{H}_\Lambda(\rho_\mu)\rangle_\Psi \leq -4\pi\rho_\mu^2 \ell^2 Y(1 - CK_B^2 Y|\log Y|), \tag{D46}$$

then the bound (7.41) for the number of excitations holds:

$$\langle n_+\rangle_\Psi \leq Cn K_B^2 K_\ell^2 Y|\log Y|. \tag{D47}$$

Proof. We study the integration over $\Lambda_{1/10} \setminus \Lambda_{1/5}$ from formula (D21). By [18, (C.6)] we have $|\chi_{B_u}| \leq C(\ell_{\min}\ell^{-1})^M$ and then

$$\int_{\Lambda_{1/10}-\Lambda_{1/5}} \int \chi_{B_u}(x)^2 \mathrm{d}x\mathrm{d}u \leq C\Big(\frac{\ell_{\min}}{\ell}\Big)^{2M}(\ell d)^2 d^{-2} \leq C\Big(\frac{\ell_{\min}}{\ell}\Big)^{2M}\ell^2. \quad \text{(D48)}$$

By the joint action of Corollary D.6 and Lemma D.4 we get

$$\mathcal{H}_{B_u}(4\rho_\mu) \geq -C\rho_\mu^2\delta \int \chi_{B_u}^2 - C\rho_\mu\delta\Big(\rho_\mu\delta(ds\,K_\ell)^{-2} + \frac{1}{|B_u|}\Big)\int \chi_{B_u}^2, \quad \text{(D49)}$$

and therefore, using (D48) and that $|B| = d^2\ell^2$ we have

$$\int_{\Lambda_{1/10}\setminus\Lambda_{1/5}} \mathcal{H}_{B_u}(4\rho_\mu)\mathrm{d}u \geq -C\rho_\mu^2\ell^2\delta(1 + \delta(ds\,K_\ell)^{-2})\Big(\frac{\ell_{\min}}{\ell}\Big)^{2M} - C\rho_\mu\delta d^{-2}, \quad \text{(D50)}$$

Using the definition of $\ell_{\min} = d\ell/10$ and the relations between the parameters (H12) we get

$$\Big(\frac{\ell_{\min}}{\ell}\Big)^{2M} \leq d^{2M} \leq \delta, \qquad \rho_\mu\delta d^{-2} \leq \rho_\mu^2\ell^2\delta^2(K_\ell d)^{-2} \leq \rho_\mu^2\ell^2\delta^2 K_B^2, \quad \text{(D51)}$$

which makes the integral coherent with the statement of the Theorem using the expansion of δ,

$$\delta \simeq Y - Y^2|\log Y| + \mathcal{O}(Y^3|\log Y|^2). \quad \text{(D52)}$$

For the remaining integral in formula (D21) we use Corollary D.6 and Lemma D.4 to get

$$\int_{\Lambda_{1/10}} \mathcal{H}_{B_u}(\rho_\mu)\mathrm{d}u$$
$$\geq -\int_{\Lambda_{1/10}}\Big[\iint \mathrm{d}x\mathrm{d}y\,\frac{1}{2}\rho_\mu^2 w_{1,B_u}(x,y) + C\rho_\mu\delta\Big(\rho_\mu\delta(ds\,K_\ell)^{-2} + \frac{1}{|B_u|}\Big)\int \chi_{B_u}^2\Big]\mathrm{d}u$$
$$\geq -4\pi\rho_\mu^2\ell^2\delta - C\rho_\mu^2\ell^2\delta^2 K_B^2,$$

where we used (H3), (H12) and

$$\iiint w_{1,B_u}(x,y)\mathrm{d}x\mathrm{d}y\mathrm{d}u = 8\pi\delta\ell^2, \qquad \iint \chi_{B_u}(x)^2\mathrm{d}u\mathrm{d}x = \ell^2.$$

Collecting the previous estimates, together with (D21) and the fact that $\varepsilon_N \leq \frac{1}{2}$, we finally get (D45), using the expansion (D52) of δ.

The proof of the bound on n_+ is proven noting that, joining together the a priori bound (7.40) with the obtained lower bound we get

$$\frac{b}{2\ell^2}\langle n_+\rangle_\Psi \leq CK_B^2\rho_\mu\ell^2 Y^2|\log Y|, \quad \text{(D53)}$$

and conclude recalling that $\ell = \rho_\mu^{-1/2}Y^{-1/2}K_\ell$. $\qquad\qquad\square$

We follow now a similar strategy to obtain a lower bound for the large box Hamiltonian and get an a priori bound on the number of particles and a control on $\mathcal{Q}_4^{\text{ren}}$ in the large box.

Corollary D.8. *If there exists a n-particles state* $\Psi \in \mathscr{F}_s(L^2(\Lambda))$ *such that* (7.40) *holds, then the a priori bounds on n and* $\mathcal{Q}_4^{\text{ren}}$ *hold:*

$$\left| \rho_\mu - \frac{n}{\ell^2} \right| \le C K_B K_\ell \rho_\mu Y^{1/2} |\log Y|^{1/2}, \qquad \langle \mathcal{Q}_4^{ren} \rangle_\Psi \le C K_B^2 K_\ell^2 \rho_\mu^2 \ell^2 Y^2 |\log Y|. \tag{D54}$$

Proof. We observe that we have the following lower bound, reproducing analogous estimates for potential and kinetic energies from Lemmas D.2 and D.3 but adapted to the large box Λ (for details, see [19, Appendix E.2]), where we estimate the n_+ contributions thanks to Theorem D.7,

$$\langle \mathcal{H}_\Lambda(\rho_\mu) \rangle_\Psi \ge \frac{1}{2} \langle \mathcal{Q}_4^{\text{ren}} \rangle_\Psi - 4\pi \rho_\mu^2 \ell^2 \delta + 2\pi \left(\rho_\mu - \frac{n}{\ell^2} \right)^2 \ell^2 \delta - C K_B^2 K_\ell^2 \rho_\mu^2 \ell^2 \delta^2. \tag{D55}$$

By the assumption, the expansion of δ in terms of Y and (D55) we get

$$\left(\frac{n}{\ell^2} - \rho_\mu \right)^2 \ell^2 \delta + \langle \mathcal{Q}_4^{\text{ren}} \rangle_\Psi \le C K_B^2 K_\ell^2 \rho_\mu^2 \ell^2 Y^2 |\log Y|, \tag{D56}$$

which implies the desired bounds. $\qquad\square$

Appendix E: Technical Estimates for Off-Diagonal Excitation Terms

We give here a proof of Lemma 7.9, bounding the terms d_1^L and d_2^L defined in (7.48) and (7.49). We are going to use the following dimension independent estimates which are proven in [19, Corollary F.6] in order to prove the technical lemma below. There exists $C > 0$, such that, for any $\varphi \in \operatorname{Ran} \overline{Q}_H$,

$$\|\Delta(\chi_\Lambda \varphi)\| \le C \varepsilon_N^{-1/2} \frac{\widetilde{K}_H^2}{\ell^2}, \qquad \|\Delta^{\mathcal{N}} \varphi\| \le C \varepsilon_N^{-1} \frac{\widetilde{K}_H^2}{\ell^2}. \tag{E1}$$

Lemma E.1. *If we assume the relations between the parameters in "Appendix H", then there exists $C > 0$ such that*

$$\|\overline{Q}_{H,x} w(x, y) \overline{Q}_{H,x}\| \le C \varepsilon_N^{-1/2} \frac{\widetilde{K}_H^2}{\ell^2} \|v\|_1. \tag{E2}$$

Proof. The proof is an adaptation to 2 dimensions of [19, Lemma 5.3]. Let $\varphi \in \operatorname{Ran} \overline{Q}_{H,x}$ with $\|\varphi\|_2 = 1$, then

$$\|\overline{Q}_{H,x} w(x, y) \overline{Q}_{H,x} \varphi\| \le I_1 + I_2, \tag{E3}$$

where

$$I_1 = \int_{\mathbb{R}^2} \chi_\Lambda(x)^2 |\varphi(x)|^2 v(x - y) \mathrm{d}x,$$

$$I_2 = \int_{\mathbb{R}^2} \chi_\Lambda(x) |\chi_\Lambda(x) - \chi_\Lambda(y)| |\varphi(x)|^2 v(x - y) \mathrm{d}x.$$

We use the technical Lemma E.2 below to get

$$|I_1| \leq \|\chi_\Lambda \varphi\|_\infty^2 \|v\|_1 \leq C\|v\|_1 \|\chi_\Lambda \varphi\| \|\Delta \chi_\Lambda \varphi\| \leq C\frac{\widetilde{K}_H^2}{\ell^2} \varepsilon_N^{-1/2} \|v\|_1, \tag{E4}$$

by (E1) and (E6) and

$$|I_2| \leq C\frac{R}{\ell} \|\chi_\Lambda \varphi\|_\infty \|\varphi\|_\infty \|v\|_1 \leq C\frac{R}{\ell} \|\Delta(\chi_\Lambda \varphi)\|^{1/2} \|\Delta^{\mathcal{N}}\varphi\|_{L^2\left(\left[-\frac{L}{2}, \frac{L}{2}\right]^2\right)}^{1/2} \|v\|_1$$

$$\leq C\varepsilon_N^{-1/2}\left(\frac{R}{\ell} \varepsilon_N^{-1/4}\right)\frac{\widetilde{K}_H^2}{\ell^2} \|v\|_1 \leq C\varepsilon_N^{-1/2}\frac{\widetilde{K}_H^2}{\ell^2} \|v\|_1,$$

by a Taylor expansion for the localization function, (E5) and (E6) for φ and $\chi_\Lambda \varphi$, respectively, (E1) and the choice of the parameters in (H11), (H3), (H14) and this concludes the proof. □

In the proof of Lemma E.1 we used the following result.

Lemma E.2. *Let* $-\Delta^{\mathcal{N}}$ *denote the Neumann Laplacian on* $[-\frac{L}{2}, \frac{L}{2}]^2$. *There exists* $C > 0$ *such that, for all* $f \in \mathcal{D}(-\Delta^{\mathcal{N}})$ *such that* $\int_{[-\frac{L}{2}, \frac{L}{2}]^2} f(x)\mathrm{d}x = 0$, *we have*

$$\|f\|_\infty \leq C\|f\|_{L^2([-\frac{L}{2}, \frac{L}{2}]^2)}^{1/2} \|-\Delta^{\mathcal{N}} f\|_{L^2([-\frac{L}{2}, \frac{L}{2}]^2)}^{1/2}. \tag{E5}$$

Also, for all $f \in H^2(\mathbb{R}^2)$,

$$\|f\|_\infty \leq C\|f\|^{1/2}\|\Delta f\|^{1/2}. \tag{E6}$$

Proof. Let us prove the last inequality, the first one being proven by an adaptation for the box (essentially the only difference is to replace sums by integrals). We use a scaling argument defining $f_\lambda(x) := f(\lambda x)$, for $x \in \mathbb{R}^2$. Given an $f \in H^2(\mathbb{R}^2)$, it is clearly possible to choose λ such that $\|f_\lambda\| = \|\Delta f_\lambda\|$. Now, for the given λ, we have

$$\|f_\lambda\|_\infty^2 \leq \frac{1}{(2\pi)^2}\left(\int_{\mathbb{R}^2} |\hat{f}_\lambda(p)|\mathrm{d}p\right)^2 \leq C\int_{\mathbb{R}^2} (1+|p|^4)|\hat{f}_\lambda(p)|^2\mathrm{d}p = C\|\Delta f_\lambda\|^2,$$

where we multiplied and divided by $(1+|p|^4)^{1/2}$ and used the Cauchy–Schwarz inequality and the choice of λ. Applied to f_λ with the λ chosen above the previous inequality becomes

$$\|f\|_\infty^2 = \|f_\lambda\|_\infty^2 \leq C\|\Delta f_\lambda\|^2 = C\|f_\lambda\|\|\Delta f_\lambda\| = C\|f\|\|\Delta f\|, \tag{E7}$$

where in the last equality we used the scaling properties of the dilatation in λ w.r.t. the L^2 norm. □

Proof of Lemma 7.9. Let $\Psi \in \mathscr{F}_s(L^2(\Lambda))$ be satisfying the assumptions of Lemma 7.9. Our goal is to prove the following estimate

$$\langle (d_1^L + d_2^L) \rangle_{\widetilde{\Psi}}$$
$$\leq \rho_\mu \|v\|_1 \left(\langle n_+ \rangle_{\widetilde{\Psi}} + n^{1/2}\langle n_+^L \rangle_{\widetilde{\Psi}}^{1/2} + \langle (n_+^L)^2 \rangle_{\widetilde{\Psi}}^{1/2} \varepsilon_N^{-1/4}\widetilde{K}_H + \langle n_+^L \widetilde{n}_+^H \rangle_{\widetilde{\Psi}} \varepsilon_N^{-1/4}\widetilde{K}_H \right.$$
$$\left. + \left(\langle \widetilde{n}_+^H n_+^L \rangle_{\widetilde{\Psi}}^{1/2} \langle (n_+^L)^2 \rangle_{\widetilde{\Psi}}^{1/2} + \langle \widetilde{n}_+^H n_+^L \rangle_{\widetilde{\Psi}}\right)n^{-1}\varepsilon_N^{-1/2}\widetilde{K}_H^2\right) + C\langle Q_4^{\mathrm{ren}} \rangle_{\widetilde{\Psi}}. \tag{E8}$$

We split the d_j^L in several terms multiplying out the parentheses in (7.48) and (7.49). All these terms we treat individually using Cauchy–Schwarz inequalities. Similar bounds have been carried out in [19]. Here we just bound some representative examples to illustrate the procedure and the role played by Lemma E.1.

Let us start using the Cauchy–Schwarz inequality for any $\varepsilon > 0$ to get

$$\left| \left\langle -\rho_\mu \sum_i P_i \int dy\, w_1(x_i, y)\, \overline{Q}_{H,i} + h.c. \right\rangle_\Psi \right| \leq \frac{n}{\ell^2} \|w_1\|_1 \left(\varepsilon n + \varepsilon^{-1} \langle n_+^L \rangle_\Psi \right),$$

observing that $\|w_1\|_1 \leq C\delta$ and choosing $\varepsilon = \langle n_+^L \rangle_\Psi^{1/2} n^{-1/2}$, we obtain the desired quantity.

For the following term, for any $\varepsilon > 0$, we use the Cauchy–Schwarz inequality and Lemma E.1,

$$\left| \left\langle \sum_{i,j} P_i \overline{Q}_{H,j} w \overline{Q}_{H,i} \overline{Q}_{H,j} \right\rangle_\Psi \right| \leq \varepsilon \frac{n}{\ell^2} \|w\|_1 \langle n_+^L \rangle_\Psi + \varepsilon^{-1} \|\overline{Q}_H w \overline{Q}_H\| \sum_{i \neq j} \langle \overline{Q}_{H,i} \overline{Q}_{H,j} \rangle_\Psi$$

$$\leq \frac{\|w\|_1}{\ell^2} \left(\varepsilon n^2 + \varepsilon^{-1} \varepsilon_N^{-1/2} \widetilde{K}_H^2 \langle (n_+^L)^2 \rangle_\Psi \right)$$

where we used that $n_+^L \leq n_+$. Choosing $\varepsilon = \varepsilon_N^{-1/4} \widetilde{K}_H \langle (n_+^L)^2 \rangle_\Psi^{1/2} n^{-1}$, we obtain

$$\left| \left\langle \sum_{i,j} P_i \overline{Q}_{H,j} w \overline{Q}_{H,i} \overline{Q}_{H,j} \right\rangle_\Psi \right| \leq n \frac{\langle (n_+^L)^2 \rangle_\Psi^{1/2}}{\ell^2} \varepsilon_N^{-1/4} \widetilde{K}_H \|w\|_1. \tag{E9}$$

For the next term we want to apply a Cauchy–Schwarz inequality to reobtain a Q_4^{ren} term. In order to do that we are going to complete the Q_H to a $Q = Q_H + \overline{Q}_H$.

$$\left| \left\langle \sum_{i \neq j} \overline{Q}_{H,i} P_j w Q_{H,i} Q_{H,j} + h.c. \right\rangle_\Psi \right|$$

$$\leq \left| \left\langle \sum_{i \neq j} \overline{Q}_{H,i} P_j w Q_i Q_j \right\rangle_\Psi + h.c. \right|$$

$$+ \left| \left\langle \sum_{i \neq j} \overline{Q}_{H,i} P_j w (Q_{H,i} \overline{Q}_j + \overline{Q}_{H,i} Q_{H,j}) \right\rangle_\Psi + h.c. \right|$$

$$+ \left| \left\langle \sum_{i,j} P_i \overline{Q}_{H,j} w \overline{Q}_{H,i} \overline{Q}_{H,j} \right\rangle_\Psi \right|.$$

The second term and the third terms can be estimated in the same manner as above, so let us focus on completing the first term in order to obtain the Q_4.

$$\left| \left\langle \sum_{i \neq j} \overline{Q}_{H,i} P_j w Q_i Q_j \right\rangle_\Psi + h.c. \right| \tag{E10}$$

$$\leq \left| \left\langle \sum_{i \neq j} \overline{Q}_{H,i} P_j w (Q_i Q_j + \omega(P_i P_j + P_i Q_j + Q_i P_j)) \right\rangle_\Psi + h.c. \right| \tag{E11}$$

$$+ \left| \left\langle \sum_{i \neq j} \overline{Q}_{H,i} P_j w \omega (P_i Q_j + Q_i P_j)) \right\rangle_\Psi + h.c. \right| \tag{E12}$$

$$+ \left| \left\langle \sum_{i \neq j} \overline{Q}_{H,i} P_j w \omega P_i P_j) \right\rangle_\Psi + h.c. \right|. \tag{E13}$$

The second and the third terms are treated as above, using that $0 \leq \omega \leq 1$ on the support of w. By a Cauchy–Schwarz inequality on the first term we get

$$(E11) \leq \langle \mathcal{Q}_4^{\mathrm{ren}} \rangle_\Psi + C \frac{n}{\ell^2} \|w\|_1 \langle n_+ \rangle_\Psi. \tag{E14}$$

Collecting the previous estimates including the ones not explicitly treated, we obtain (E8).

Bounding $n_+^L \leq \widetilde{\mathcal{M}}$ in (E8) where it appears for higher moments than 1, using that $\widetilde{n}_+^H \leq n$ and that $\varepsilon_N^{-1/4} \widetilde{K}_H \geq 1$ by (H15) gives the result. This finishes the proof of Lemma 7.9. $\qquad \square$

Appendix F: Properties of the Localization Function

We collect here the definition and some important properties of the localization function that are used throughout the paper.

We define

$$\chi(x) := C_M (\zeta_1(x_1)\zeta_2(x_2))^{M+2}, \tag{F1}$$

where

$$\zeta(y) := \begin{cases} \cos(\pi y), & |y| \leq 1/2, \\ 0, & |y| > 1/2, \end{cases} \tag{F2}$$

where $M \in \mathbb{N}$ is chosen even and large enough. The normalization constant $C_M > 0$ is chosen in order to obtain $\|\chi\|_2 = 1$. We have $0 \leq \chi \in C^M(\mathbb{R}^2)$. We also define $\chi_\Lambda(x) = \chi(x/\ell)$.

Lemma F.1. *Let χ be the localization fuction defined above and let $M \in 2\mathbb{N}$. Then, for all $k \in \mathbb{R}^2$,*

$$|\widehat{\chi}(k)| \leq \frac{C_\chi}{(1 + |k|^2)^{M/2}}, \tag{F3}$$

where $C_\chi = \int |(1 - \Delta)^{M/2}\chi|$. If, furthermore, $|k| \geq \frac{1}{2}K_K\ell^{-1}$,

$$|\widehat{\chi_\Lambda}(k)| = \ell^2 |\widehat{\chi}(k\ell)| \leq C\ell^2 K_H^{-M}. \tag{F4}$$

An important property for the localization function χ_{B_u}, $u \in \mathbb{R}^2$, on the small boxes, namely

$$\chi_{B_u}(x) := \chi_\Lambda(x)\chi\left(\frac{x}{d\ell} - u\right), \tag{F5}$$

which is used in "Appendix D", is the following bound

$$\|\nabla^2 \chi_{B_u}\|_\infty \leq C_M \frac{1}{|B_u|\ell_{\min}^2} \int \chi_{B_u}, \tag{F6}$$

which is taken from [18, Appendix C]. Here it is key the fact that we do not consider a smooth function but we require χ to have a finite degree of regularity measured by the parameter M.

Appendix G: Comparing Riemann Sums and Integrals

We will show in this section that we could approximate integrals on $\mathbb{R}^2$ by Riemann sums when it was needed in (4.13) to prove the upper bound. Recall that the assumptions of Theorem 4.1 were

$$R \leq \rho^{-1/2} Y^{1/2}, \qquad L_\beta = \rho^{-1/2} Y^{-\beta}. \tag{G1}$$

We divide $\mathbb{R}^2$ into small squares $\square_p$ of size $\frac{2\pi}{L}$ centered at $p \in \Lambda_L^* = \frac{2\pi}{L}\mathbb{Z}^2$. Then, clearly

$$\left| \frac{4\pi^2}{L^2} \sum_{p \in \Lambda_L^*} f(p) - \int_{\mathbb{R}^2} f(k)\mathrm{d}k \right| \leq \frac{C}{L^3} \sum_{p \in \Lambda_L^*} \sup_{\square_p} |\nabla f|. \tag{G2}$$

We consider the functions present in the two sums of (4.13). With α_p and γ_p given in (4.6) the first term is

$$\begin{aligned}
f(p) &= p^2 + \rho_0 \widehat{g}_p - \sqrt{p^4 + 2\rho_0 \widehat{g}_p p^2} + \rho_0(\widehat{v}_p - \widehat{g}_p)(\gamma_p + \alpha_p) \\
&= p^2 + \rho_0 \widehat{g}_p - \sqrt{p^4 + 2\rho_0 \widehat{g}_p p^2} + \rho_0(\widehat{v}_p - \widehat{g}_p)\left(\frac{p^2}{2\sqrt{p^4 + 2\rho_0 \widehat{g}_p p^2}} - \frac{1}{2} \right),
\end{aligned} \tag{G3}$$

and the second term

$$d(p, r) = \widehat{v}_r \alpha_{p+r} \alpha_p. \tag{G4}$$

We then have the following estimates

Lemma G.1. *Let f, d be as in (G3) and (G4). Then,*

$$\left| \frac{1}{|\Lambda_\beta|} \sum_{p \in \Lambda_\beta^*} f(p) - \int_{\mathbb{R}^2} f(k) \frac{\mathrm{d}k}{4\pi^2} \right| \leq C\rho^2 Y^{1/2+\beta} \widehat{v}_0, \tag{G5}$$

and

$$\left| \frac{1}{|\Lambda_\beta|^2} \sum_{p,r \neq 0} d(p, r) - \int_{\mathbb{R}^4} d(p, r) \frac{\mathrm{d}p\mathrm{d}r}{(4\pi^2)^2} \right| \leq C\rho^2 Y^{1/2+\beta} \widehat{v}_0. \tag{G6}$$

Proof. In order to apply (G2), we start by calculating the gradient

$$\begin{aligned}
\partial_p f &= 2p + \rho_0 \partial_p \widehat{g}_p - 2p \frac{\left(1 + \frac{\rho_0 \widehat{g}_p}{p^2} + \frac{\rho_0 \partial_p \widehat{g}_p}{2p} \right)}{\sqrt{1 + \frac{2\rho_0 \widehat{g}_p}{p^2}}} \\
&\quad + \rho_0(\partial_p \widehat{v}_p - \partial_p \widehat{g}_p)\left(\frac{p^2}{2\sqrt{p^4 + 2\rho_0 \widehat{g}_p p^2}} - \frac{1}{2} \right) \\
&\quad + \rho_0^2 (\widehat{v}_p - \widehat{g}_p) \frac{\widehat{g}_p p^3 - \frac{1}{2}\partial_p \widehat{g}_p p^4}{\left(p^4 + 2\rho_0 \widehat{g}_p p^2 \right)^{\frac{3}{2}}} \\
&:= A_p + B_p + C_p.
\end{aligned} \tag{G7}$$

We will now systematically omit the constants and study separately the cases $p \leq \sqrt{2\rho_0\widehat{g}_0}$ (case 1 referring to A_p^1) and $p \geqslant \sqrt{2\rho_0\widehat{g}_0}$ (case 2 referring to A_p^2). We then get by elementary inequalities

$$\left|A_p^1\right| \leq (\rho\widehat{g}_0)^{1/2}, \qquad\qquad \left|A_p^2\right| \leq \frac{\rho^2 R\widehat{g}_0\widehat{g}_p}{p^2} + \frac{(\rho\widehat{g}_p)^2}{p^3},$$

$$\left|B_p^1\right| \leq \rho R(\widehat{v}_0 - \widehat{g}_0), \qquad\qquad \left|B_p^2\right| \leq \frac{R\rho^2(\widehat{v}_0 - \widehat{g}_0)\widehat{g}_p}{p^2},$$

$$\left|C_p^1\right| \leq \frac{\rho^{1/2}(\widehat{v}_0 - \widehat{g}_0)}{\widehat{g}_0^{1/2}}, \qquad\qquad \left|C_p^2\right| \leq (\widehat{v}_0 - \widehat{g}_0)\left(\frac{\rho^2\widehat{g}_0}{p^3} + \frac{R\rho^2\widehat{g}_p}{p^2}\right),$$

where we used $|\widehat{g}_0 - \widehat{g}_p| \leq |\widehat{g}_0^{3/2}|$ for $p \leq (\rho\widehat{g}_0)^{1/2}$. This way we can use inequality (G2) and the decay of $\widehat{g}_p$ (3.41) to get

$$\frac{1}{L_\beta^3} \sum_{p \in \Lambda^*} |\partial_p f| \, \mathrm{d}p \leq \frac{C}{L_\beta}\left((\rho\widehat{g}_0)^{3/2} + \rho^{3/2}\widehat{g}_0^{1/2}(\widehat{v}_0 - \widehat{g}_0) + \rho^2\widehat{g}_0 R + R\rho^2(\widehat{v}_0 - \widehat{g}_0)\right)$$

$$\tag{G8}$$

$$\leq C\widehat{v}_0\rho^2 Y^{1/2+\beta},$$

where we used (G1), and $\widehat{g}_0 \leq \widehat{v}_0$. We use the same method to prove (G6). We have

$$|\alpha_p| \leq \begin{cases} \frac{\sqrt{\rho\widehat{g}_0}}{p}, & \text{for } p \leq \sqrt{\rho\widehat{g}_0}, \\ \frac{\rho|\widehat{g}_p|}{p^2}, & \text{for } p \geq \sqrt{\rho\widehat{g}_0}. \end{cases} \tag{G9}$$

We have to calculate

$$\partial_p\alpha_p = -\frac{\rho\partial_p\widehat{g}_p}{2\sqrt{p^4 + 2\rho\widehat{g}_p p^2}} - \frac{\rho\widehat{g}_p\left(4p^3 + 4\rho\widehat{g}_p p + 2\rho\partial_p\widehat{g}_p p^2\right)}{2(p^4 + 2\rho\widehat{g}_p p^2)^{3/2}},$$

yielding

$$|\partial_p\alpha_p| \leq \begin{cases} \frac{\sqrt{\rho\widehat{g}_0}R}{p} + (\rho\widehat{g}_0)^{-1/2} + \frac{\sqrt{\rho\widehat{g}_0}}{p^2} + \frac{\sqrt{\rho\widehat{g}_0}R}{p}, & \text{for } p \leq \sqrt{\rho\widehat{g}_0}, \\ \frac{\rho\widehat{g}_p R}{p^2} + \frac{\rho\widehat{g}_p}{p^3} + \frac{(\rho\widehat{g}_p)^2}{p^5} + \frac{(\rho\widehat{g}_p)^2 R}{p^4}, & \text{for } p \geq \sqrt{\rho\widehat{g}_0}. \end{cases} \tag{G10}$$

The divergence in $p \to 0$ implies to remove a little box around the point 0

$$\left|\frac{16\pi^4}{L_\beta^4} \sum_{p,r \neq 0} d(p,r) - \int_{\mathbb{R}^4} d(p,r)\mathrm{d}p\mathrm{d}r\right|$$

$$\leq \left|\frac{16\pi^4}{L_\beta^4} \sum_{p,r \neq 0} d(p,r) - \int_{\left(\mathbb{R}^2\backslash[-\frac{1}{L},\frac{1}{L}]^2\right)^2} d(p,r)\mathrm{d}p\mathrm{d}r\right|$$

$$+ \left|\int_{\mathbb{R}^2 \times [-\frac{1}{L},\frac{1}{L}]^2} d(p,r)\mathrm{d}p\mathrm{d}r\right| + \left|\int_{[-\frac{1}{L},\frac{1}{L}]^2 \times \mathbb{R}^2} d(p,r)\mathrm{d}p\mathrm{d}r\right|.$$

where the last two terms in the above can be bounded by $\rho^2 Y^{1/2+\beta}\widehat{v}_0$. Finally a direct computation using the decay of $\widehat{g}_p$, the bounds (4.2), (G9), (G10), and (G2) yields

$$
\left| \frac{16\pi^4}{L_\beta^4} \sum_{p,r\neq 0} d(p,r) - \int_{\left(\mathbb{R}^2\setminus[-\frac{1}{L},\frac{1}{L}]^2\right)^2} d(p,r)\mathrm{d}p\mathrm{d}r \right|
$$
$$
\leq \frac{1}{L_\beta^5} \sum_{p,r\neq 0} \sup_{\square_p\times\square_r} \left|\nabla_{p,r} d(p,r)\right|
$$
$$
\leq C\frac{\widehat{v}_0}{L_\beta^5} \sum_{p\neq 0} |\partial_p\alpha_p| \sum_{r\neq 0} |\alpha_r| + C\frac{R\widehat{v}_0}{L_\beta^5}\left(\sum_r |\alpha_r|\right)^2
$$
$$
\leq C\widehat{v}_0\rho^2 Y^{1/2+\beta}, \tag{G11}
$$

where we used the estimates of Lemma 4.4. This concludes the proof.   □

Appendix H: Fixing Parameters for the Lower Bound

Here we collect all the relations and dependencies of the several parameters involved in the lower bound for the convenience of the reader. Furthermore, we end the section by making an explicit choice that satisfies all the relations. Recall that we have the small parameter

$$
Y = Y_\mu = |\log(\rho_\mu a^2)|^{-1}.
$$

We use the following notation throughout the article

$$
A \ll B \quad \text{if and only if there exist } C, \varepsilon > 0 \text{ s.t. } A \leq CY^\varepsilon B. \tag{H1}
$$

In the proof of the lower bound, a number of positive parameters are needed. These are the following

$$
d, s, \varepsilon_T, \varepsilon_K, \varepsilon_N, \varepsilon_{\mathcal{M}} \ll 1 \ll \mathcal{M}, K_\ell, K_H, \widetilde{K}_H, K_N, K_B. \tag{H2}
$$

These will be chosen below.

Furthermore, there are length scales ℓ_δ and R. These will be chosen to satisfy

$$
R \leq \rho_\mu^{-1/2}, \qquad\qquad \text{Condition on the radius of the support}, \tag{H3}
$$
$$
\ell_\delta = \frac{e^\Gamma}{2}\rho_\mu^{-1/2}Y^{-1/2}, \qquad\qquad \text{healing length condition.} \tag{H4}
$$

Some first relations between the parameters are

$$
d \ll 1 \ll K_\ell, \qquad\qquad \text{sep. of small and large boxes,} \tag{H5}
$$
$$
d^{-2} \ll K_H \ll \widetilde{K}_H, \qquad\qquad \text{sep. of low and high momenta,} \tag{H6}
$$
$$
d \ll (sK_\ell)^{-1} \ll 1, \qquad\qquad \text{condition for Bog. integral,} \tag{H7}
$$
$$
d^2 K_\ell^4 \ll \varepsilon_T \ll dsK_\ell, \qquad\qquad \text{spectral gap condition,} \tag{H8}
$$
$$
ds^{-1} \leq C, \qquad\qquad \text{localization to small boxes.} \tag{H9}
$$

The combination of (H6) and (H8) implies the following relations:

$$K_\ell \ll K_\ell^2 \ll sd^{-1} \ll d^{-1} \ll d^{-2} \ll K_H. \tag{H10}$$

Defining

$$\varepsilon_N := K_N^{-1}Y, \qquad \varepsilon_{\mathcal{M}} := \frac{\mathcal{M}}{\rho_\mu \ell^2}, \tag{H11}$$

we give the following conditions which control the magnitude of the large parameters in terms of Y:

$$(ds K_\ell)^{-1} \ll K_B, \qquad\qquad \text{condition errors in small box,} \tag{H12}$$

$$K_B K_\ell \widetilde{K}_H K_N^{1/4} \ll Y^{-1/4}, \qquad\qquad \text{small error in large matrices,} \tag{H13}$$

$$K_\ell^{-1} K_N^{1/4} \ll Y^{-1/2}, \qquad\qquad \text{technical estimate in large matrices,} \tag{H14}$$

$$\widetilde{K}_H K_N^{-1/4} \gg Y^{1/4}, \qquad\qquad \text{technical estimate in large matrices,} \tag{H15}$$

$$K_\ell^2 K_H^2 \mathcal{M} \ll Y^{-1}, \qquad\qquad \text{second localization of 3Q term,} \tag{H16}$$

$$K_B^2 K_\ell^2 \ll Y^{-1/4}, \qquad\qquad \text{number for high momenta,} \tag{H17}$$

$$K_\ell^{10} K_H^{-8} d^{-4} \ll Y^{-1}, \qquad\qquad \text{condition error in } \mathcal{T}_1. \tag{H18}$$

Here the magnitude of the small parameters:

$$\varepsilon_R \ll K_B^{-2} K_\ell^{-2} |\log Y|^{-1}, \qquad\qquad \text{Condition on } \varepsilon_R, \tag{H19}$$

$$\varepsilon_K \ll K_\ell^{-2}, \qquad\qquad \text{error in } \mathcal{T}'_{2,com}, \tag{H20}$$

$$\varepsilon_K \gg K_\ell^4 K_H^{-4} (d^{-2} \varepsilon_{\mathcal{M}}^{1/2} + d^{-4} \varepsilon_{\mathcal{M}}), \qquad \text{condition error in } \mathcal{T}_1 \text{ and } \mathcal{T}_2, \tag{H21}$$

$$\varepsilon_{\mathcal{M}} \ll d^8 K_\ell^4 \varepsilon_T^{-2}, \qquad\qquad \text{condition for error } \delta_1. \tag{H22}$$

$$\varepsilon_N \le \varepsilon_T^{-2} d^4 K_\ell^4, \qquad\qquad \text{bound from Lemma 9.2.} \tag{H23}$$

We use the fundamental property of the system that the number of excitations of our state is relatively small compared to the number of particles (expressed by the condition $\varepsilon_{\mathcal{M}} \ll 1$) but still larger that a certain threshold. This property is expressed by the following condition:

$$\mathcal{M} \gg Y^{-7/8} |\log Y|^{1/4} K_B^{1/2} K_\ell^{1/2} K_N^{1/8} \widetilde{K}_H^{1/2} \|v\|_1^{1/2}. \tag{H24}$$

The following are conditions that impose constraints on the size of M, the degree of regularity of the localization function χ:

$$d^{2M-2} \ll Y, \qquad\qquad \text{error in localization 3Q,} \tag{H25}$$

$$d^2 K_\ell^4 \ll \varepsilon_T, \qquad\qquad \text{error in localization 3Q,} \tag{H26}$$

$$\varepsilon_N^{3/2} + \left(\frac{K_H}{\widetilde{K}_H}\right)^M + (d^2 K_H)^{-2M} \le \varepsilon_{\mathcal{M}}, \qquad \text{number for high momenta,} \tag{H27}$$

$$(s^{-2} + d^{-2})(sd)^{-2} s^M \le C, \qquad\qquad \text{localization to small boxes.} \tag{H28}$$

A choice of parameters, non-optimal in the size of the error produced, fitting the previous conditions, is the following,

$$M = 258, \qquad \mathcal{M} = Y^{-\frac{31}{32}}, \qquad \varepsilon_T = Y^{\frac{1}{512} - \frac{1}{8192}},$$

$$K_\ell = Y^{-\frac{1}{2048}}, \qquad K_H = Y^{-\frac{1}{128}}, \qquad \widetilde{K}_H = Y^{-\frac{1}{64}},$$

$$d = Y^{\frac{1}{512}}, \qquad K_N = Y^{-\frac{1}{512}}, \qquad s = Y^{\frac{1}{4096}},$$

$$K_B = Y^{-\frac{1}{512}}, \qquad \varepsilon_K = Y^{\frac{1}{512}}, \qquad \varepsilon_\mathcal{M} = Y^{\frac{1}{32} + \frac{1}{1024}}. \qquad \text{(H29)}$$

This choice is not made with any particular view towards optimality.

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