



Algebraic Conditions for Conformal Superintegrability in Arbitrary Dimension

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Abstract: We consider second order (maximally) conformally superintegrable systems and explain how the definition of such a system on a (pseudo-)Riemannian manifold gives rise to a conformally invariant interpretation of superintegrability. Conformal equivalence in this context is a natural extension of the classical (linear) Stäckel transform, originating from the Maupertuis-Jacobi principle. We extend our recently developed algebraic geometric approach for the classification of second order superintegrable systems in arbitrarily high dimension to conformally superintegrable systems, which are presented via conformal scale choices of second order superintegrable systems defined within a conformal geometry. For superintegrable systems on constant curvature spaces, we find that the conformal scales of Stäckel equivalent systems arise from eigenfunctions of the Laplacian and that their equivalence is characterised by a conformal density of weight two. Our approach yields an algebraic equation that governs the classification under conformal equivalence for a prolific class of second order conformally superintegrable systems. This class contains all non-degenerate examples known to date, and is given by a simple algebraic constraint of degree two on a general harmonic cubic form. In this way the yet unsolved classification problem is put into the reach of algebraic geometry and geometric invariant theory. In particular, no obstruction exists in dimension three, and thus the known classification of conformally superintegrable systems is reobtained in the guise of an unrestricted univariate sextic. In higher dimensions, the obstruction is new and has never been revealed by traditional approaches.

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1. Introduction

Transformation groups play an important role in the natural sciences: the Poincaré group, for instance, and its subgroups, are pivotal in special relativity, for Maxwell’s field equations, in particle physics, and many other fields. Felix Klein’s Erlangen program has put the concept of transformations at the core of geometry, later generalised by Cartan [Kle00, SC00]. Inspired by this idea, the current paper reconsiders second order superintegrable Hamiltonian systems. These have been extensively studied as structures in (pseudo-)Riemannian geometry, but not yet as structures in conformal geometry.

Maximally superintegrable systems, traditionally, are Hamiltonian systems on a (pseudo-) Riemannian geometry that admit a maximal amount of (hidden) symmetry. Here *second order maximally* (conformally) superintegrable systems are exclusively considered. They are often seminal models in science. Historically, the theory of superintegrability arose from classical (and quantum) mechanics: While, in general, it is impossible, even for relatively simple models, to solve Hamilton’s or Schrödinger’s equation in exact, closed terms, for superintegrable systems the solution can be found by *quadrature*, i.e. using algebraic operations and the integration of known functions. Prominent examples of second order superintegrable systems are the Kepler-Coulomb and the Harmonic oscillator models. They have fundamental significance for the understanding of celestial mechanics, atomic orbitals, material science and many other disciplines.

1.1. What Geometry underpins superintegrability? Traditionally, second order superintegrable systems are defined on a (pseudo-)Riemannian manifold (M, g) . The suitable symmetry group for these systems is the semi-direct product $\mathfrak{S} = \text{Diff}(M) \times \text{Aff}(\mathbb{R})$ of diffeomorphisms (coordinate transformations on M) and the affine group $\text{Aff}(\mathbb{R}) = \mathbb{R}^* \ltimes \mathbb{R}$, where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, see Sect. 2. However, these are not the only possible transformations of superintegrable systems. Indeed, conformal geometry manifests itself in the theory of superintegrable systems through (classical) *Stäckel equivalence* or *coupling constant metamorphosis*; details and a comparison with the more recent concept of non-linear multi-parameter Stäckel transform are discussed in Sect. 2.1. Historically, the Stäckel transform can be traced back to the 18th century in the form of the Maupertuis principle [Tsi01, BKF95, Lag88, Jac84, dM50]. Classical Stäckel transformations are linked to very special conformal transformations, namely those that originate from superintegrable potentials.

Arbitrary conformal transformations, however, do not preserve superintegrability. Instead they lead to *conformally superintegrable systems*. Although these systems are well studied, their underlying conformal geometry is understood only superficially to date. *One purpose of the present paper is to remedy this, and to derive a suitable concept of conformal equivalence on conformally superintegrable systems from (a modification of) Stäckel equivalence.*

Given the significance of second order (conformally) superintegrable systems, it is natural to seek a classification. In [KSV23], the authors present an algebraic geometric framework for a classification of second order superintegrable systems in arbitrarily high dimension, and for arbitrary metrics. This framework put earlier attempts by various authors (see below) onto a firm base, yet it is not closed under conformal transformations of

superintegrable systems. *The current paper develops an algebraic geometric framework for conformally superintegrable systems that is closed under conformal transformations.* In particular we obtain: Non-degenerate second order (conformally) superintegrable systems are characterised by a (conformally invariant) *structure tensor*, more specifically a trace-free and totally symmetric tensor field S_{ijk} . This leads to a natural definition of *superintegrability on conformal manifolds*, whose symmetry group we identify as $\mathfrak{G} = \text{Conf}(M) \rtimes \mathbb{R}^*$. In this way we naturally incorporate conformal geometry into the theory of superintegrable systems. Somewhat surprisingly, this appears to never have been attempted before, although it sheds considerable light on the geometry underpinning superintegrability and opens the subject for subsequent studies using Cartan geometry, tractor calculus, algebraic geometry, representation theory and geometric invariant theory. In particular, superintegrable systems on (pseudo-) Riemannian geometries can be viewed as specific *conformal scale choices* of a conformally invariant superintegrable system.

1.2. State of the art. A vast literature exists both on second order conformally superintegrable systems and on Stäckel transformations. To date, second order conformally superintegrable systems are classified completely only in dimension 2 [KKM05c, KKM05a]. For conformally flat spaces in dimension 3, at least so-called non-degenerate systems are classified [Cap14, KKM06]. The conformal classes of non-degenerate systems are classified in dimensions 2 and 3 [Kre07, Cap14]. Existing classification results largely ignore the geometric structure of the classification space. However, an algebraic geometric classification exists for superintegrable systems in the Euclidean plane [KS18], and the algebraic varieties of superintegrable systems in dimensions 2 and 3 are addressed in [KKM07a, KKM07b]. While the classification of the conformal classes in dimension 2 is a mere list [Kre07], in dimension 3 a classification in terms of representations of the rotation group exists [Cap14, CK14, CKP15]. These latter references are one major inspiration for our work as they highlight the power of the geometric approach, revealing for example a natural algebraic hierarchy of systems related to an inclusion tree of certain algebraic ideals. Unfortunately, there is little hope of applying the methods from those references in higher dimensions, neither conceptually nor practically, as the equations become ever more extensive with increasing dimension. *The current paper develops a new approach, extending and generalising the framework from [KS18, KSV23]. We formulate the governing equations for second order conformally superintegrable systems in dimension $n \geq 3$ in a concise form, making the problem manageable in higher dimensions.*

Stäckel transformations can be traced back to *Maupertuis-Jacobi* transformations [Jac84], which take a Hamiltonian with potential to a potential-free one. The classical Stäckel transform is well understood as an equivalence relation on second order (conformally) superintegrable systems. It was first introduced, for integrable Hamiltonian systems with potential, in [HGDR84] under the name *coupling constant metamorphosis*. The name *Stäckel transformations* has been introduced in [BKM86] for transformations of integrable systems admitting separation of variables. In general, coupling constant metamorphosis and (classical) Stäckel transform are not identical, but they coincide for second order (conformal) integrals of the motion [Pos10]. Higher order integrals are discussed in [KMP09], showing that coupling constant metamorphosis in general does not even preserve the polynomiality in momenta of an integral. The *classical* Stäckel transform involves one parameter on which the Hamiltonian depends linearly. A multi-parameter generalisation of Stäckel transform was developed in [SB08, BM12, BM17],

allowing for several parameters on which the Hamiltonian may depend non-linearly. Here we only encounter Hamiltonians with *linear* parameter dependence, and therefore we do not need to exhaust the full scope of generalised multi-parameter Stäckel transformations. As we are interested in a *geometric approach*, we shall not require a specific parametrisation of the space of compatible potentials, but instead work with an $(n + 2)$ -dimensional *linear* space of potentials on an n -dimensional manifold, usually not specifying a particular basis.

For completeness we mention that conformal transformations are not the only possible transformations of superintegrable systems. For instance, Bôcher transformations of certain conformally superintegrable systems are studied in [KMS16, CKP15] and there is some indication that they can be understood as boundaries of orbit closures on the algebraic variety classifying the superintegrable systems [KS18]. Yet another transformation of superintegrable systems is possible if the underlying metrics share the same geodesics up to reparametrisation. Such metrics are called projectively (or geodesically) equivalent. For some examples of superintegrable systems defined on projectively equivalent geometries, see [Val16, KKMW03, Vol20].

To summarise, higher dimensions are out of the scope of traditional methods, which rely on the correspondence with properly superintegrable systems and on the extensive use of computer algebra. A particular challenge is the fast growth of the number of partial differential equations with increasing dimension. *In the current paper, we overcome this problem and outline how to approach the classification of second order conformally superintegrable systems in arbitrarily high dimension.* For the most prolific class of systems we find, somewhat surprisingly, that the underlying structure equations reduce to only a single, algebraic equation of degree 2.

1.3. Classifying second order superintegrability in arbitrarily high dimension. In reference [KSV23] the authors have developed an algebraic geometric framework for the classification of superintegrable systems. This framework generalises previous work in dimension two [KS18] to arbitrarily high dimensions. Older works in the field are [KKM07a] and [KKM07b] for dimensions two and three, respectively. While [KSV23] for the first time provides a framework to classify, in an algebraic geometric way, superintegrable systems in arbitrarily high dimensions, this framework in its original form is not (yet) closed under conformal transformations. *The present paper extends the existing algebraic geometric framework to conformally superintegrable systems on (pseudo-) Riemannian metrics. This new framework is closed under conformal transformations.*

Second order conformally superintegrable systems will be thoroughly introduced in Sects. 3.1 and 3.2. These systems are traditionally defined using a Hamiltonian

$$H = g^{ij}(\mathbf{q}) p_i p_j + V(\mathbf{q}) \quad (1.1)$$

where g_{ij} denotes the underlying metric and where \mathbf{q} and \mathbf{p} are the canonical position and momenta variables on the manifold. A second order conformally superintegrable system is a Hamiltonian system with a sufficiently high number of functions $F : T^*M \rightarrow \mathbb{R}$, $F = K^{ij}(\mathbf{q}) p_i p_j + W(\mathbf{q})$, satisfying

$$\{H, F\} = \omega_q(\mathbf{p}) H \quad (1.2)$$

for some 1-form $\omega = \omega_i dx^i$. The scalar function V is called a *potential* of H . Functions F satisfying (1.2) are called (*conformal*) *integrals*. The integrals F form a linear space \mathcal{F} . Likewise, the potentials V compatible with \mathcal{F} form a linear space \mathcal{V} . Following the

common convention, we take these spaces to be maximal, see Definitions 2.2 and 3.5 below.

1.4. First main result: conformal superintegrability in higher dimensions. The method carried out in [KSV23] facilitates the classification of second order properly superintegrable systems, in particular of so-called *abundant* systems. Abundantness is going to be introduced thoroughly later, and so here we limit ourselves to saying that these systems comprise all known non-degenerate second order conformally superintegrable systems. In the present paper, we extend the framework to conformally superintegrable systems. We find that it is closed under conformal transformations and leads to a well-defined concept of superintegrability on conformal geometries arising from conformal equivalence classes of conformally superintegrable systems. For the abundant case in dimensions $n \geq 3$, we show in Sects. 5 and 6 that such systems are in natural correspondence with harmonic cubic forms $\Psi_{ijk} p^i p^j p^k$ on \mathbb{R}^n that satisfy the simple algebraic equation

$$\left(g^{ab} \Psi_{ija} \Psi_{klb} - g^{ab} \Psi_{ika} \Psi_{jlb} \right)_o = 0 \quad (1.3)$$

where g^{ab} is an inner product on \mathbb{R}^n with the same signature as the metrics on the underlying manifold. The subscript “o” stands for projection onto the trace-free part.

We show that initial data in the form of a cubic form $\Psi_{ijk} p^i p^j p^k$ satisfying (1.3) can be extended, locally, to a *conformal structure tensor* S_{ijk} of an abundant second order superintegrable system. We make this precise by introducing the concept of c-superintegrable systems, i.e. conformal equivalence classes whose underlying geometry is a conformal manifold. In Sect. 6.4 we derive conformally invariant structural equations for abundant c-superintegrable systems. The equations governing abundant properly superintegrable systems [KSV23] naturally follow from the equations we present here. Condition (1.3) is conformally invariant, and therefore a suitable foundation for an algebraic geometric classification of second order systems on the level of conformal geometries. Condition (1.3) is also surprisingly simple, and in dimensions $n \geq 4$ it encodes new obstructions to conformal superintegrability. These obstructions do not exist in lower dimensions and have not been revealed by classical approaches.

We also show: *Abundant conformally superintegrable systems can only exist on conformally flat geometries. Such systems naturally correspond to solutions of Eq. (1.3). The task of classifying equivalence classes of n -dimensional conformally superintegrable systems is therefore equivalent to classifying harmonic cubics in n variables that satisfy (1.3).* Note that while a general classification of harmonic cubics under the rotation group is out of sight, a classification under the additional condition (1.3) may well be simple enough to admit a manageable solution.

In dimension $n = 3$, particularly, (1.3) is trivially satisfied. Thus abundant superintegrable systems in dimension 3 naturally correspond to harmonic ternary cubic forms, or, equivalently, to univariate sextic polynomials

$$p(x) = a_6 x^6 + a_5 x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0.$$

The details are discussed in Sect. 7. It is known that any conformally superintegrable system is Stäckel equivalent to a properly superintegrable systems [Cap14]. In dimension 3, every conformally superintegrable system is even Stäckel equivalent to a properly superintegrable system on a constant curvature space [KKM06, Theorem 4].

1.5. Second main result: superintegrable systems on constant curvature geometries. In our framework, second order conformally superintegrable systems are conformal scale choices of c-superintegrable systems. Thus the conformal structure tensor S_{ijk} determines a conformally superintegrable system up to the choice of a *conformal scale* expressed via a function σ that transforms as a weight-1 tensor density. For the prolific class of abundant second order conformally superintegrable systems we find that the conformal scale satisfies a Helmholtz like equation,

$$\left(R - 4 \frac{n-1}{n-2} \Delta \right) \sigma^{1-\frac{n}{2}} = S \sigma^{1-\frac{n}{2}}, \tag{1.4}$$

where $S = S_{abc} S^{abc}$ is a conformal density obtained from the structure tensor S_{ijk} mentioned earlier, and where R is the scalar curvature. The operator on the left hand side of (1.4) is the conformal Laplace operator.

If we restrict to properly superintegrable systems, the conformal structure tensor S_{ijk} determines a superintegrable system up to provision of a suitable conformal scale. In the present paper, we prove the following: *For abundant properly superintegrable systems on manifolds of constant sectional curvature, the conformal scale σ is (a power of) an eigenfunction of the Laplacian* for an eigenvalue determined by the curvature R ,

$$\Delta \sigma^{n+2} = -2 \frac{n+1}{n-1} R \sigma^{n+2}, \tag{1.5}$$

which holds in addition to (1.4), see Theorem 6.11. Note that the operator in (1.5) is not conformally invariant as Eq. (1.5) does not describe a property of the conformal class, but of an individual superintegrable system. In particular we find that on the n -sphere the conformal scale function satisfies a Laplace equation with quantum number $n + 1$. We show for any dimension (see Propositions 6.14 and 6.15): *The generic system on the n -sphere is never conformally equivalent to a superintegrable system on flat space, and a non-degenerate properly superintegrable system on the n -sphere is never conformally equivalent to the harmonic oscillator.*

2. Preliminaries

Before generalising to *conformally* superintegrable systems, it is instructive to briefly review *properly* superintegrable systems. We recall that, for clarity, the adjective “proper” is used to refer to superintegrable systems, whenever a distinction from *conformally* superintegrable systems is required. While self-contained, this review only highlights the aspects needed for a later comparison to conformally superintegrable systems. For a more in-depth review of proper superintegrability we refer the interested reader to the literature cited in the introduction and in particular to [KKM18].

Let (M, g) be a (pseudo-) Riemannian manifold. A Hamiltonian system on M is a dynamical system characterised by a function $H : T^*M \rightarrow \mathbb{R}$, $(\mathbf{p}, \mathbf{q}) \mapsto H(\mathbf{p}, \mathbf{q})$, referred to as *Hamiltonian*. We denote the position and momentum coordinates on the phase space T^*M by $\mathbf{q} = (q_1, \dots, q_n)$ and $\mathbf{p} = (p_1, \dots, p_n)$, respectively. The evolution of the system is determined by Hamilton’s equations

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}} \qquad \dot{\mathbf{q}} = +\frac{\partial H}{\partial \mathbf{p}}. \tag{2.1}$$

An *integral*, aka *first integral* or *constant of the motion*, for the Hamiltonian H is a function $F(\mathbf{p}, \mathbf{q})$ on phase space that commutes with H with respect to the canonical Poisson bracket. It is therefore constant along solutions of (2.1),

$$\dot{F} = \{F, H\} = \frac{\partial F}{\partial \mathbf{q}} \frac{\partial H}{\partial \mathbf{p}} - \frac{\partial F}{\partial \mathbf{p}} \frac{\partial H}{\partial \mathbf{q}} = 0. \quad (2.2)$$

In this equation, the partial derivatives may be replaced by covariant derivatives, i.e. using the Levi-Civita metric ∇^g , without changing the Poisson bracket. Note that for the ease of presentation we use Darboux coordinates (\mathbf{q}, \mathbf{p}) in the following, but that our results do not require the choice of specific coordinates on M .

An integral restricts the trajectory of the system to a hypersurface in phase space. A (properly) *superintegrable system* is a Hamiltonian system that possesses the maximal number of $2n - 1$ functionally independent constants of motion $F^{(0)}, \dots, F^{(2n-2)}$. Its trajectories in phase space are the (unparametrised) curves given as the intersections of the hypersurfaces $F^{(\alpha)}(\mathbf{p}, \mathbf{q}) = c^{(\alpha)}$, where the constants $c^{(\alpha)}$ are determined from the initial conditions. For convenience it is customary to choose $F^{(0)} = H$ without loss of generality. In particular, we assume the base manifold is endowed with a (pseudo-)Riemannian metric g and a natural Hamiltonian (1.1),

$$H = G(\mathbf{q}, \mathbf{p}) + V(\mathbf{q}),$$

where $G(\mathbf{q}, \mathbf{p}) = g_{\mathbf{q}}(\mathbf{p}, \mathbf{p})$ denotes the kinetic part and $V(\mathbf{q})$ is a smooth scalar function called *potential*. Note that, as we consider Hamiltonians on the manifold M , the appropriate transformation group is given by diffeomorphisms of M , which induce, via their pullback, fibre-preserving symplectomorphisms on T^*M .

Definition 2.1. A *second order* superintegrable system is a Hamiltonian together with a linear space \mathcal{F} of integrals of the form

$$F = K(\mathbf{q}, \mathbf{p}) + W(\mathbf{q}) := K^{ij}(\mathbf{q}) p_i p_j + W(\mathbf{q}), \quad (2.3)$$

satisfying (2.2). Moreover, \mathcal{F} must contain $2n - 1$ integrals that are functionally independent.

Note that $\dim(\mathcal{F}) \geq 2n - 1$. In case of the equality, it is common practice to only specify $2n - 1$ linearly independent generators $F^{(\alpha)}$. We also recall that, for (2.3), Eq. (2.2) is a polynomial condition in the momenta with homogeneous components of cubic and linear degree, respectively:

$$\{K, G\} = 0 \quad (2.4a)$$

$$\{K, V\} + \{W, G\} = 0 \quad (2.4b)$$

Condition (2.4a) is equivalent to the requirement that the (symmetric) components K_{ij} in $K(\mathbf{q}, \mathbf{p}) = K_{ij} p^i p^j$ are the components of a Killing tensor field, i.e.

$$K_{ij,k} + K_{jk,i} + K_{ki,j} = 0. \quad (2.5)$$

Here, the comma denotes a covariant derivative. Condition (2.4b) can be rewritten in the form

$$W_{,j} = K_j^k V_{,k} \quad \text{or even} \quad dW = K dV,$$

where by abuse of notation K denotes the endomorphism obtained from K_{ij} via the metric g . The integrability condition for W is known as the *Bertrand–Darboux condition* [Ber57, Dar01],

$$d(KdV) = 0. \tag{2.6}$$

The Bertrand–Darboux Equation (2.6) is the compatibility condition for the potential V and the space of Killing tensors K_{ij} . Let us denote the linear space of kinetic parts of the integrals $F \in \mathcal{F}$, viewed as endomorphisms, by

$$\mathcal{K} = \{K : K(\mathbf{p}, \mathbf{p}) + W \in \mathcal{F} \text{ for some } W\}.$$

Definition 2.2. For a second order superintegrable system with potential V , we introduce the spaces

$$\begin{aligned} \mathcal{V}^{\max} &= \{V : d(KdV) = 0 \text{ holds for every } K \in \mathcal{K}\} \\ \mathcal{K}^{\max} &= \{K : d(KdV) = 0 \text{ holds for every } V \in \mathcal{V}^{\max}\} \end{aligned}$$

Remark 2.3. A second order superintegrable system is said to be *irreducible* if the endomorphisms $K_i^j = g^{ja} K_{ia}$ obtained from its associated Killing tensors $K \in \mathcal{K}$ form an irreducible set. In reference [KSV23], it is shown that for such irreducible systems we can solve (2.6) for all second derivatives of the potential except its Laplacian. Thus the *Wilczynski equation* is obtained,

$$V_{,ij} = T_{ij}{}^k V_{,k} + \frac{1}{n} g_{ij} \Delta V, \tag{2.7}$$

where $T_{ij}{}^k$ is a tensor symmetric and trace-free in the first two indices, depending on the components of the Killing tensors $K^{(\alpha)}$ and their derivatives.

The properties of the partial differential Eq. (2.7) are discussed thoroughly in [KSV23], and similar equations appear in [KKM05b]. The most important fact is that in (2.7) the tensor $T_{ij}{}^k$ is determined by \mathcal{K} independently from the potential. More precisely, at a point $x_0 \in M$, $T_{ij}{}^k$ is determined by the values of the Killing tensors K_{ij} and their derivatives in x_0 .

In the classification theory of second order superintegrable systems, *non-degenerate systems* have received particular attention, e.g. [KKM18, KSV23, Cap14]. These are the systems satisfying (2.7) for which the dimension of \mathcal{V}^{\max} is maximal, i.e.

$$\dim(\mathcal{V}^{\max}) = n + 2.$$

The integrability conditions of (2.7) are then generically satisfied [KKM07b, KSV23]. Resubstituting (2.7) into (2.6) and considering the coefficients of ∇V and ΔV , one furthermore finds that for non-degenerate systems (see Sect. 2.3 for the notation of projectors via Young tableaux)

$$K_{ij,k} = \frac{1}{3} \begin{array}{|c|c|} \hline j & i \\ \hline k & \\ \hline \end{array} T^a{}_{ji} K_{ak}. \tag{2.8}$$

Now consider non-degenerate systems for which \mathcal{K}^{\max} has the maximal possible dimension, namely

$$\dim(\mathcal{K}^{\max}) = \frac{n(n+1)}{2}.$$

Such systems are called *abundant* [KSV23], implying that the integrability conditions of (2.8) are generically satisfied. Among second order superintegrable systems, abundant systems arguably are the most important ones: All non-degenerate examples known to date are abundant, and in dimensions two and three it is proven that non-degenerate systems are necessarily abundant [KKM05b]. In dimension two, all systems are restrictions of abundant ones [KKMP09].

2.1. Stäckel equivalence. The classical Stäckel transform is often introduced using a Hamiltonian with a coupling constant. Beginning along these lines, we then adopt an alternative, equivalent formulation better suited for our purposes. Consider a family of second order superintegrable systems on a (pseudo-) Riemannian manifold (M, g) given by the family of Hamiltonians

$$H_\beta = H_0 + \sum_{k=1}^N \beta_k U_k,$$

where $H_0 = g(\mathbf{p}, \mathbf{p})$ is the *free Hamiltonian*, and where U_k ($1 \leq k \leq N$) are functions $U_k : M \rightarrow \mathbb{R}$ with *coupling parameters* β_k . For a concise exposition, we use only one parameter and incorporate the remaining potentials into a background Hamiltonian. Classical Stäckel equivalence is based on the following fact, see for example [KMP09, Pos10, KKM05b, HGDR84, BKM86].

Lemma 2.4. *Let $H = H_0 + V + \beta U$ be a family of second order superintegrable Hamiltonians with integrals $F(\beta)$, for $V, U \in \mathcal{V}^{\max}$. Then the Hamiltonian $\tilde{H} = \frac{H+\eta}{U}$ admits the integral of motion $\tilde{F}(\eta) = F(\tilde{H})$, parametrised by η .*

The Hamiltonian \tilde{H} is called the *Stäckel transform* of H with conformal factor U . While either sign is permitted, it is often preferable to work with $U > 0$ in order to preserve the signature of the underlying metric (this is always possible locally by redefining β). Lemma 2.4 exploits the fact that any constant η can be added to H without changing the integrals. We could analogously write the Stäckel transform of $H = H_0 + V + \beta U + \eta$ with integrals $F^{(\alpha)}$ as [BKM86, Pos10]

$$\tilde{H} = U^{-1} H, \tag{2.9a}$$

$$\tilde{F}^{(\alpha)} = F^{(\alpha)} + \frac{1 - W^{(\alpha)}}{U} H. \tag{2.9b}$$

This transformation preserves the kinetic part up to a term proportional to $g(\mathbf{p}, \mathbf{p})$ with a coefficient that depends on the position only. For conciseness, we have restricted ourselves to one coupling parameter, but an analogous reasoning is valid whenever H depends *linearly* on the parameters (the multi-parameter generalisation of Stäckel transform in [SB08, BM12, BM17] exceeds this limitation, but is not needed here). Instead of working with a parametrised Hamiltonian, a parameter-free viewpoint is best suited for our purposes:

Definition 2.5. Two second order properly superintegrable systems are said to be Stäckel equivalent if their Hamiltonians and integrals satisfy (2.9).

For Stäckel equivalent Hamiltonians $H = H_0 + V$ and $\tilde{H} = \tilde{H}_0 + \tilde{V}$, the underlying metrics are conformally equivalent, i.e. $\tilde{g}_{ij} = \Omega^2 g_{ij}$, if $U = \Omega^2 > 0$. In case of negative sign, $U < 0$, the metric's signature is merely inverted. For the corresponding integrals $F = K^{ij} p_i p_j + W$ and $\tilde{F} = \tilde{K}^{ij} p_i p_j + \tilde{W}$, (2.9) implies

$$\begin{aligned}\tilde{K} &= K + \frac{1 - W}{U} g \\ \tilde{W} &= W + \frac{1 - W}{U} V.\end{aligned}$$

2.2. *Symmetry group.* Consider a non-degenerate second order properly superintegrable system with potential V . The symmetry group of such a system is the semi-direct product $\mathfrak{S} = \text{Diff}(M) \ltimes \text{Aff}(\mathbb{R})$ of the diffeomorphisms of the manifold M and the affine group $\text{Aff}(\mathbb{R}) \simeq \mathbb{R}^* \ltimes \mathbb{R}$. An element $\Phi = (\phi, (a, b)) \in \mathfrak{S}$ transforms a Hamiltonian according to

$$\Phi(g^{ij} p_i p_j + V) = \phi^*(g)^{ij} p_i p_j + a\phi^*(V) + b, \tag{2.10}$$

where ϕ^* is the pullback with ϕ . Indeed, the underlying geometry and the space of compatible Killing tensors does not change under \mathfrak{S} . Moreover, the structure tensor T_{ijk} in (2.7) remains unchanged under \mathfrak{S} . The structure tensor remains unchanged even¹ under $\mathfrak{S}' = \mathfrak{S} \times \mathbb{R}^*$, where $\Phi' = (\phi, (a, b), c) \in \mathfrak{S}'$ transforms a Hamiltonian according to

$$\Phi'(g^{ij} p_i p_j + V) = c\phi^*(g)^{ij} p_i p_j + a\phi^*(V) + b. \tag{2.11}$$

Note that two equivalent Hamiltonians remain equivalent after a Stäckel transformation.

2.3. *Young projectors.* In order to keep the notation concise, tensor symmetries are described by Young projectors in the following. In doing so, we adhere to the convention used for properly superintegrable systems in [KSV23], which we briefly review in the current section. A comprehensive introduction to representations of symmetric and linear groups is out of scope of the present paper, but can be found in [Fu197, FH00] for instance.

Let $n > 0$ be an integer. A partition of n into a sum of ordered, positive integers can be represented by a *Young frame*, i.e. by non-increasing rows of square boxes, which by convention are left-aligned. For instance, to denote the partition $5 = 3 + 1 + 1$ we may draw the Young frame



Irreducible representations of the permutation group S_n and the induced Weyl representations of $GL(n)$ can also be labelled by Young frames. A Young frame filled with tensor index names is called a *Young tableau*; it explicitly defines a projector onto an irreducible representation. Two simple examples are complete symmetrisation,

$$\boxed{i_1 \ i_2 \ \dots \ i_k},$$

and complete antisymmetrization,

$$\boxed{\begin{matrix} i_1 \\ i_2 \\ \vdots \\ i_k \end{matrix}}.$$

¹ Note that the action might not be proper.

For example, a 2-tensor τ_{ij} can be decomposed into its symmetric and antisymmetric parts,

$$\tau_{ij} = \frac{1}{2} (\tau_{ij} + \tau_{ji}) + \frac{1}{2} (\tau_{ij} - \tau_{ji}) = \frac{1}{2} \begin{bmatrix} i & j \\ & \end{bmatrix} \tau_{ij} + \frac{1}{2} \begin{bmatrix} i \\ j \end{bmatrix} \tau_{ij}.$$

The symmetric part can be decomposed further, according to irreducible representations of $SO(n)$, into a trace-free and a trace component. The projection onto the completely trace-free part of a tensor is denoted by a sub- or superscript “ \circ ”. For example,

$$\tau_{ij} = \frac{1}{2} \begin{bmatrix} i & j \\ & \end{bmatrix} \circ \tau_{ij} + \frac{1}{n} g_{ij} \tau^a_a + \frac{1}{2} \begin{bmatrix} i \\ j \end{bmatrix} \tau_{ij}.$$

A general Young tableau denotes the composition of its row symmetrisers and column antisymmetrisers. By convention the antisymmetrisers are applied first. For instance,

$$\begin{bmatrix} j & i \\ k \end{bmatrix} T_{ijk} = \begin{bmatrix} j & i \\ & \end{bmatrix} \begin{bmatrix} j \\ k \end{bmatrix} T_{ijk} = (T_{ijk} - T_{ikj}) + (T_{jik} - T_{jki}).$$

The adjoint of a Young tableau is given by applying the symmetrisers first,

$$\begin{bmatrix} j & i \\ k \end{bmatrix}^* T_{ijk} = \begin{bmatrix} j \\ k \end{bmatrix} \begin{bmatrix} j & i \\ & \end{bmatrix} T_{ijk} = (T_{ijk} + T_{jik}) - (T_{ikj} + T_{kij}).$$

For a 3-tensor T_{ijk} we have the decomposition

$$T_{ijk} = \frac{1}{6} \begin{bmatrix} i & j & k \\ & & \end{bmatrix} T_{ijk} + \frac{1}{3} \begin{bmatrix} i & j \\ k \end{bmatrix} T_{ijk} + \frac{1}{3} \begin{bmatrix} i & k \\ j \end{bmatrix} T_{ijk} + \frac{1}{6} \begin{bmatrix} i \\ j \\ k \end{bmatrix} T_{ijk}.$$

One particular 4-index Young tableaux that we make intensive use of is the projector

$$\begin{bmatrix} i & j \\ k & l \end{bmatrix}^* T_{ijkl} = \begin{bmatrix} i \\ k \end{bmatrix} \begin{bmatrix} j \\ l \end{bmatrix} \begin{bmatrix} i & j \\ & \end{bmatrix} \begin{bmatrix} k & l \\ & \end{bmatrix} T_{ijkl},$$

which projects onto algebraic curvature tensors. The well known Ricci decomposition can then be written as

$$R_{ijkl} = W_{ijkl} + \frac{1}{4(n-1)} \begin{bmatrix} i & k \\ j & l \end{bmatrix}^* \mathring{R}_{ik} g_{jl} + \frac{1}{8n(n-1)} \begin{bmatrix} i & k \\ j & l \end{bmatrix}^* g_{ik} g_{jl}.$$

where

$$W_{ijkl} = \frac{1}{12} \begin{bmatrix} i & k \\ j & l \end{bmatrix}^* \circ R_{ijkl}$$

is the Weyl curvature, \mathring{R}_{ij} the trace-free part of the Ricci tensor and R the scalar curvature. For later reference, we also introduce the *Schouten tensor*,

$$(n-2)P_{ij} = R_{ij} - \frac{1}{2(n-1)} R g_{ij} = \mathring{R}_{ij} + \frac{n-2}{2n(n-1)} R g_{ij}. \tag{2.12}$$

3. Conformal Structure Tensors

In the present chapter superintegrable systems on (pseudo-) Riemannian manifolds are generalised to *conformally superintegrable systems*. Before we begin, we need to introduce the conformal counterpart of Killing tensors.

Definition 3.1. A (second order) *conformal Killing tensor* is a symmetric tensor field C_{ij} on a (pseudo-) Riemannian manifold satisfying the conformal Killing equation

$$\boxed{i \mid j \mid k} C_{ij,k} = \boxed{i \mid j \mid k} g_{ij} \omega_k, \tag{3.1}$$

where ω is a 1-form.

The 1-form ω can be expressed in terms of the conformal Killing tensor. Indeed, contracting (3.1) in (i, j) , we find

$$\omega_k = \frac{1}{n+2} (2C^a_{k,a} + C^a_{a,k}). \tag{3.2}$$

Remark 3.2.

- (i) Any Killing tensor is also a conformal Killing tensor, with $\omega = 0$. In particular, the metric g is trivially a conformal Killing tensor.
- (ii) If K_{ij} is a conformal Killing tensor, any trace modification $C_{ij} = K_{ij} + \lambda g_{ij}$ is also a conformal Killing tensor. If K_{ij} is a proper Killing tensor, C_{ij} is a conformal Killing tensor with $\omega = d\lambda$.

We mention that while for a proper Killing tensor K_{ij} , the function $K(\mathbf{p}, \mathbf{p})$ is preserved along geodesics, for a conformal Killing tensor C_{ij} the function $C(\mathbf{p}, \mathbf{p})$ is preserved along null geodesics.

3.1. Conformally superintegrable systems. On (pseudo-) Riemannian manifolds, the concept of superintegrable systems is generalised as follows.

Definition 3.3. (i) By a *conformally (maximally) superintegrable system*, we mean a Hamiltonian system admitting $2n - 1$ functionally independent conformal integrals of the motion $F^{(\alpha)}$,

$$\{F^{(\alpha)}, H\} = \omega^{(\alpha)} H, \quad \alpha = 0, 1, \dots, 2n - 2, \tag{3.3}$$

with a function $\omega(\mathbf{p}, \mathbf{q})$ polynomial in momenta. The Hamiltonian can be assumed to be among the conformal integrals. Thus by convention we set

$$F^{(0)} = H, \quad \omega^{(0)} = 0.$$

- (ii) A conformal integral of the motion is *second order* if it is of the form

$$F^{(\alpha)} = C^{(\alpha)} + V^{(\alpha)}, \tag{3.4}$$

where

$$C^{(\alpha)}(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^n C_{ij}^{(\alpha)}(\mathbf{q}) p^i p^j$$

is quadratic in momenta and $V^{(\alpha)} = V^{(\alpha)}(\mathbf{q})$ a function depending only on positions. In this case, the function ω has to be linear in the momenta \mathbf{p} . A conformally superintegrable system is *second order* if its conformal integrals $F^{(\alpha)}$ are second order.

(iii) We call V a *conformal superintegrable potential* if the Hamiltonian (3.5) gives rise to a conformally superintegrable system.

A function $F(\mathbf{q}(t), \mathbf{p}(t))$ that satisfies (3.3) is constant on the null locus of the Hamiltonian, since

$$\dot{F} = \{F, H\} = \omega H.$$

By adding a constant c to the Hamiltonian, we achieve that F is constant on shells where the new Hamiltonian is constant and equal to c . Since we are concerned exclusively with second order maximally superintegrable systems, we omit the terms “second order” and “maximally” without further mentioning.

Assumption 3.4. From now on, unless otherwise stated, we assume that the quadratic parts correspond to *trace-free* conformal Killing tensors, except for the Hamiltonian $H = F^{(0)}$. This is no restriction, as the trace-free part of any such conformal Killing tensor is itself a conformal Killing tensor.

In view of Assumption 3.4 we distinguish the Hamiltonian by notation from the other (conformal) integrals,

$$H = G + V, \tag{3.5}$$

where

$$G(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^n g_{ij}(\mathbf{q}) p^i p^j$$

is given by the (pseudo-) Riemannian metric $g_{ij}(\mathbf{q})$ on the underlying manifold.

For (3.4), condition (3.3) splits into two homogeneous parts with respect to momenta. These parts are cubic respectively linear in \mathbf{p} :

$$\{C^{(\alpha)}, G\} = 2\omega^{(\alpha)} G \tag{3.6a}$$

$$\{C^{(\alpha)}, V\} + \{V^{(\alpha)}, G\} = 2\omega^{(\alpha)} V \tag{3.6b}$$

Condition (3.6a) for $C(\mathbf{p}, \mathbf{q}) = C_{ij}(\mathbf{q}) p^i p^j$ is equivalent to C_{ij} being a conformal Killing tensor. The components of $\omega(\mathbf{p}, \mathbf{q}) = \omega^i(\mathbf{q}) p_i$ are given by a 1-form, also denoted by ω in the following. Compare (3.6) to the analogous equations (2.4) for proper superintegrability.

The metric g allows us to identify symmetric forms and endomorphisms by abuse of notation. Interpreting a conformal Killing tensor in this way as an endomorphism on 1-forms, Eq. (3.6b) can be written in the form

$$dV^{(\alpha)} = C^{(\alpha)} dV + \omega^{(\alpha)} V. \tag{3.7}$$

Its integrability condition is the *Bertrand–Darboux condition*

$$d(C^{(\alpha)} dV) + V d\omega^{(\alpha)} + dV \wedge \omega^{(\alpha)} = 0 \tag{3.8a}$$

which in components reads (we suppress the superscript α)

$$\begin{bmatrix} i \\ j \end{bmatrix} \left(C^m{}_i V_{,jm} + C^m{}_{i,j} V_{,m} + \omega_i V_{,j} + \omega_{i,j} V \right) = 0. \tag{3.8b}$$

This is the counterpart to condition (2.6) from proper superintegrability.

By virtue of the Bertrand–Darboux equation (3.8), the potentials $V^{(\alpha)}$ for $\alpha \neq 0$ are eliminated from our equations. As we are going to see in the following, for non-degenerate systems the remaining potential $V = V^{(0)}$ can be eliminated as well, leaving equations on the conformal Killing tensors $C^{(\alpha)}$ alone. In analogy to proper superintegrability we denote by \mathcal{F} the linear space spanned by second order conformal integrals $F^{(\alpha)} = C^{(\alpha)}(\mathbf{p}, \mathbf{p}) + V^{(\alpha)}$ satisfying (3.3).

Definition 3.5. Let $H = g(\mathbf{p}, \mathbf{p}) + V$ with $\mathcal{F} = \langle F^{(\alpha)} \rangle$. Analogously to Definition 2.2 we introduce

$$\mathcal{C} = \{C : C(\mathbf{p}, \mathbf{p}) + W \in \mathcal{F} \text{ for some } W\}$$

and then

$$\begin{aligned} \mathcal{V}^{\max} &= \{V : (3.8) \text{ holds for every } C \in \mathcal{C}\} \\ \mathcal{C}^{\max} &= \{C : (3.8) \text{ holds for every } V \in \mathcal{V}^{\max}\}. \end{aligned}$$

From now on we (tacitly) work with these maximal spaces, facilitating a clean and concise exposition of the results, and following the most common convention in the relevant literature. Restricting to a subspace of \mathcal{F} with a basis of *functionally independent* integrals, we could instead work with a concrete Hamiltonian $H : T^*M \rightarrow \mathbb{R}$. Translating the results is straightforward, but omitted, as it does not add to a deeper understanding of the geometry underpinning superintegrability.

Assumption 3.6. Unless otherwise stated, a conformally superintegrable Hamiltonian will be considered together with all its conformal integrals $F = C_{ij} p^i p^j + W$ where $C \in \mathcal{C}^{\max}$ and where W satisfies Eq. (3.7), i.e.

$$dW = CdV + \omega V.$$

This assumption is no restriction, and ensures a tidy exposition as we do not need to specify the subspace $\mathcal{C} \subset \mathcal{C}^{\max}$ each time.

3.2. Conformal equivalence. In analogy to Sect. 2.2 we obtain a symmetry group of conformally superintegrable systems on (pseudo-) Riemannian manifolds. For the time being, we insist that the metric is preserved (up to coordinate transformations). In contrast to properly superintegrable systems, the affine group does not map potentials to potentials for conformally superintegrable systems, since (1.2) contains the potential V without derivatives. The symmetry group of a conformally superintegrable system on a fixed (pseudo-) Riemannian manifold therefore is $\mathfrak{S} = \text{Diff}(M) \rtimes \mathbb{R}^*$ where $\Phi = (\phi, a) \in \mathfrak{S}$ acts as

$$\Phi(g^{ij} p_i p_j + V) = \phi^*(g)^{ij} p_i p_j + a\phi^*(V).$$

This is the counterpart of the symmetry group (2.10) for properly superintegrable systems. Analogously to (2.11) in Sect. 2.2, the symmetry group of the structure tensor S_{ijk} of a conformally superintegrable system is $\mathfrak{S}' = \mathfrak{S} \times \mathbb{R}^*$.

We now introduce a concept of conformal equivalence for conformally superintegrable systems on (pseudo-) Riemannian geometries motivated by [Kre07, KKM05a, KKM06, BKM86]. While inspiration is drawn from the classical Stäckel transform, compare Lemma 2.4, we stress that no specific parametrisation is needed.

Definition 3.7. Consider two second order conformally superintegrable systems with Hamiltonians $H = g(\mathbf{p}, \mathbf{p}) + V$ respectively $\tilde{H} = \tilde{g}(\mathbf{p}, \mathbf{p}) + \tilde{V}$ and conformal integrals $F^{(\omega)}$ respectively $\tilde{F}^{(\tilde{\omega})}$ as in Definition 3.3.

We say that the two systems are *conformally equivalent* if the associated Hamiltonians H and \tilde{H} satisfy $\tilde{H} = \Omega^{-2}H$ for some function Ω , and if the trace-free parts of the associated conformal Killing tensors, viewed as $(2, 0)$ -tensors with upper indices, span the same linear space.

Later we reconsider this equivalence aiming at superintegrability as a concept on conformal geometries. We achieve this by Definition 3.9, for which the symmetry group of the structure tensor coincides with that of the superintegrable system. However, we first discuss Definition 3.7. Compare this to Definition 2.5. Classical Stäckel equivalence is a special case of conformal equivalence, insofar as in (2.9b) the conformal factor Ω is given by a potential $U \in \mathcal{V}^{\max}$. Regarding the classical Stäckel transformation of the integrals, Eq. (2.9b), we remark the following: In [Cap14, Theorem 4.1.8] it is proven that if $H = g(\mathbf{p}, \mathbf{p}) + V$ is a Hamiltonian with conformal integral $F = C(\mathbf{p}, \mathbf{p}) + W$, and $U \in \mathcal{V}^{\max}$, then there is a trace correction λ such that $\tilde{H} = \frac{H}{U}$ admits the proper integral $\tilde{F} = C(\mathbf{p}, \mathbf{p}) + \lambda g(\mathbf{p}, \mathbf{p}) - \frac{VW}{U}$. This fact provides us with a way to transform a second order conformally superintegrable system into a properly superintegrable system, possibly on a different (pseudo-) Riemannian manifold.

In dimensions 2 and 3 it is proven that any second order non-degenerate superintegrable system on a conformally flat manifold is Stäckel equivalent to a properly superintegrable system on a manifold of constant curvature; see [KKM05a, Theorem 3] and [KKM06, Theorem 4] respectively. Stäckel classes of 2-dimensional second order systems are studied in [Kre07] using properties of their associated quadratic algebras. As mentioned earlier, we impose trace-freeness on conformal Killing tensors, except for the metric, which thereby becomes a distinguished conformal Killing tensor. Trace-freeness is motivated, among other reasons, by the conformal transformation rules for conformal Killing tensors. Let us assume that H, \tilde{H} are two conformally equivalent Hamiltonians. Let $F(\mathbf{q}) = C_{\mathbf{q}}(\mathbf{p}, \mathbf{p}) + W(\mathbf{q})$ be a conformal integral for H , i.e.

$$\{H(\mathbf{p}, \mathbf{p}), C(\mathbf{p}, \mathbf{p}) + W\} = \omega(\mathbf{p})H(\mathbf{p}, \mathbf{p}).$$

Then we compute, with $\Upsilon = \ln \Omega$,

$$\{\tilde{H}, F\} = \left\{ \frac{H(\mathbf{p}, \mathbf{p})}{\Omega^2}, C(\mathbf{p}, \mathbf{p}) + W \right\} = (\omega(\mathbf{p}) - 2C(\mathbf{p}, d\Upsilon)) \tilde{H} =: \tilde{\omega}(\mathbf{p}) \tilde{H}. \quad (3.9)$$

We obtain the following statement, where the spaces $\mathcal{F}, \tilde{\mathcal{F}}$ are maximal due to Assumption 3.6.

Proposition 3.8. *Let g and $\tilde{g} = \Omega^2 g$ be a pair of conformally equivalent metrics, $\Omega > 0$.*

- (i) *If C_{ij} is a trace-free conformal Killing tensor for g then $\tilde{C}_{ij} = \Omega^4 C_{ij}$ is a trace-free conformal Killing tensor for \tilde{g} .*
- (ii) *If (g, V, \mathcal{F}) and $(\tilde{g}, \tilde{V}, \tilde{\mathcal{F}})$ are conformally equivalent second order conformally superintegrable systems, where for $F = C(\mathbf{p}, \mathbf{p}) + W \in \mathcal{F}$ with F non-proportional to H , the coefficients C_{ij} are the components of a trace-free conformal Killing tensor. Then $\tilde{\mathcal{F}} = \mathcal{F}$.*

Note that \tilde{C}_{ij} is trace-free with respect to \tilde{g} , but also g , since g and \tilde{g} are conformally equivalent.

Proof. For part (i), we have the equation $\boxed{i \ j \ k} (C_{ij,k} - g_{ij}\omega_k) = 0$ for the conformal Killing tensor C . A straightforward computation then shows that

$$\boxed{i \ j \ k} (\tilde{C}_{ij,k} - \tilde{g}_{ij}\tilde{\omega}_k) = 0$$

where

$$\tilde{C}_{ij} := \Omega^4 C_{ij}, \quad \tilde{\omega}_i := \Omega^2 (\omega_i - 2C_{ia}\Upsilon^a),$$

and $\Upsilon = \ln \Omega$.

Part (ii) then follows immediately from (3.9) in the light of Assumption 3.6. \square

Proposition 3.8 yields the transformation rules under conformal equivalence: The Hamiltonian is conformally modified, but the space of integrals is preserved. We can encode this in an efficient manner using weighted tensor densities. A conformal density δ of weight w is a section in $\mathcal{E}[w] := S^2 T^*M \otimes (\Lambda^n TM)^{-w/n}$ such that $\phi^*(\delta) = \Omega^w \delta$ if $\phi \in \text{Conf}(M)$ is a conformal transformation, i.e. $\phi^*(g) = \Omega^2 g$. Given the data (g, V, \mathcal{F}) , we observe that $g \in \mathcal{E}_{(0,2)}[2]$ and $V \in \mathcal{E}[-2]$ are weighted densities. We are thus able to define conformally invariant densities of weight 0, $\mathbf{g} \in \mathcal{E}_{(0,2)}[0]$ and $\mathbf{v} \in \mathcal{E}[0] = C^\infty(M)$, given in local components by

$$\mathbf{g}_{ij} = \frac{g_{ij}}{|\det(g)|^{\frac{1}{n}}},$$

compare [CG14], and

$$\mathbf{v} = |\det(g)|^{\frac{1}{n}} V,$$

respectively. One straightforwardly verifies that $\phi^*(\mathbf{g}) = \mathbf{g}$ and $\phi^*(\mathbf{v}) = \mathbf{v}$. This leads us to the following definition:

Definition 3.9. Let (M, \mathbf{g}) be a conformal manifold. A *second order c-superintegrable system* on M is given by a conformally invariant function \mathbf{v} and a maximal linear space \mathcal{F} of invariant scalar functions $F : T^*M \rightarrow \mathbb{R}$ on M such that

- (i) If $F \in \mathcal{F}$, then $F = C(\mathbf{q})^{ij} p_i p_j + W(\mathbf{q})$ where C^{ij} are the components of a trace-free conformal Killing tensor.
- (ii) There is a density $\Omega \in \mathcal{E}[1]$ of weight 1 such that $V = \Omega^2 \mathbf{v}$ satisfies the Bertrand-Darboux condition (3.8) for all $F \in \mathcal{F}$.

Note that the definition is conformally invariant, and for any $\Omega \in \mathcal{E}[1]$ we have that $(\Omega^{-2}\mathbf{g}, \Omega^2\mathbf{v}, \mathcal{F})$ is a conformally superintegrable system.

Remark 3.10. Let us consider the symmetry group of c-superintegrable systems. It is given by $\mathfrak{S} = \text{Conf}(M) \rtimes \mathbb{R}^*$, where \mathbb{R}^* acts as

$$(\mathbf{g}, \mathbf{v}, \mathcal{F}) \mapsto (\mathbf{g}, a\mathbf{v}, \mathcal{F})$$

It contains, in particular, all diffeomorphisms, and the transformations having a constant conformal factor. We remark that two different conformal factors do not necessarily result in different geometries. For instance, if g is a Euclidean metric in dimension two, then any conformally equivalent metric $\Omega^2 g$ is also Euclidean if it satisfies

$$\left(\frac{\partial \Omega}{\partial x}\right)^2 + \left(\frac{\partial \Omega}{\partial y}\right)^2 = \Omega \frac{\partial^2 \Omega}{\partial x^2} + \Omega \frac{\partial^2 \Omega}{\partial y^2}.$$

3.3. *Structure tensors of a conformally superintegrable system.* We consider the Bertrand-Darboux equation (3.8). It is a compatibility condition for the potential and the space of conformal Killing tensors associated to a second order superintegrable systems.

Our aim is to solve the overdetermined system of linear equations (3.8) for the highest derivatives of the potential V . Following the analogous discussion in [KSV23], we use a generalisation of Cramer’s rule.

Definition 3.11. On an inner product space, the *Gram Coefficients* $G_k(A)$ of a linear map A are the coefficients of the polynomial

$$\det(1 + tAA^*) = \sum_{k=0}^{\infty} G_k(A)t^k,$$

where A^* is the adjoint of A .

Up to sign and order, the $G_k(A)$ are the coefficients of the characteristic polynomial of AA^* . Consider a linear map A of rank r on an inner product space. The system of linear equations $Ax = b$ has a solution x if and only if the augmented matrix $(A|b)$ satisfies $G_{r+1}(A|b) = 0$. The minimal norm solution $x = A^\dagger b$ is obtained via the *Moore-Penrose inverse* [DTGVL05]

$$A^\dagger = \frac{1}{G_r(A)} \sum_{k=1}^r G_{r-k}(A)(-A^*A)^{k-1}A^*. \tag{3.10}$$

Concretely, consider the linear system, obtained from (3.8) written in local coordinates,

$$AX = b_1(dV) + b_0V, \tag{3.11}$$

where X is a vector that contains the unknown components of the trace-free Hessian $\overset{\circ}{\nabla}_{ij}^2 V$ of V and where the coefficient matrix A contains the components of the conformal Killing tensors. On the right hand side, b_1 and b_0 are the matrix and (column) vector containing the coefficients of components of dV and of V respectively.

The rank r for (3.11) does not need to be constant, but if the conformal Killing tensors are analytic, then so are the components of the matrix A . Consequently the Gram coefficients $G_k(A)$ are also analytic and then the rank of A is constant on an open and dense subset. Thus we can express the trace-free Hessian of V , using the Moore-Penrose inverse, as

$$X = A^\dagger b_1(dV) + A^\dagger b_0V. \tag{3.12}$$

Definition 3.12. A conformally superintegrable system on a (pseudo-) Riemannian manifold M has *rank* r , if the coefficient matrix A in (3.11) has rank r on an open and dense subset of M .

The rank of a conformally superintegrable system is at most

$$r_{\max} = \frac{n(n+1)}{2} - 1 = \frac{(n-1)(n+2)}{2}. \tag{3.13}$$

We characterise systems of maximal rank in terms of their trace-free conformal Killing tensors, tacitly identifying (trace-free) bilinear forms and (trace-free) endomorphisms on the tangent space. Due to Proposition 3.8, any conformally superintegrable system that is conformally equivalent to a maximal rank conformally superintegrable system is itself of maximal rank.

Definition 3.13.

- (i) A set of endomorphisms is *irreducible* if they do not have a non-trivial invariant subspace in common.
- (ii) A set of endomorphism fields on a (pseudo-) Riemannian manifold M is called *irreducible*, if they are pointwise irreducible on an open and dense subset of M .
- (iii) We call a conformally superintegrable system *irreducible*, if its conformal Killing tensors form an irreducible set.
- (iv) We call a c-superintegrable system *irreducible*, if its conformal Killing tensors form an irreducible set.

The next lemma follows analogously to the corresponding statement in the case of properly superintegrable systems [KSV23].

Lemma 3.14. *A conformally superintegrable system has maximal rank if and only if it is irreducible.*

Irreducibility thus ensures that we can solve (3.11) for X . In particular we find the minimal-norm solution (3.12), to which we may add any element from the kernel of A . For second order conformally superintegrable systems, the Bertrand–Darboux equation (3.8) can therefore be solved assuming irreducibility. We have the requirement that the trace of the Hessian of V is the Laplacian of V and thus the potential V of an irreducible conformally superintegrable system satisfies

$$V_{,ij} = T_{ij}{}^m V_{,m} + \frac{1}{n} g_{ij} \Delta V + \tau_{ij} V \tag{3.14}$$

with a (not necessarily unique) $(2, 1)$ -tensor T_{ijk} and a $(2, 0)$ -tensor τ_{ij} . We refer to (3.14) as *Wilczynski equation* because our methods are inspired by Wilczynski’s series of papers on the projective differential geometry of surfaces [Wil07, Wil09]. Equations similar to (3.14) appear in [KKM05b] in local coordinates and for dimension three. The tensors T and τ necessarily satisfy the following symmetries

$$\begin{matrix} i \\ | \\ j \end{matrix} (T_{ij}{}^m V_{,m} + \tau_{ij} V) = 0 \tag{3.15a}$$

$$g^{ij} (T_{ij}{}^m V_{,m} + \tau_{ij} V) = 0. \tag{3.15b}$$

We call T_{ijk} the *primary structure tensor* and τ_{ij} the *secondary structure tensor* of the conformally superintegrable system. Note that these tensors are not invariant under conformal transformations.

The analog of (3.14) in proper superintegrability is Eq. (2.7), which formally coincides with (3.14) for $\tau_{ij} = 0$. However, note that (2.7) was obtained from (2.6), where K is a proper Killing tensor. Instead, the Wilczynski Eq. (3.14) is obtained from the Bertrand–Darboux Eq. (3.8), where trace-free conformal Killing tensors appear. In spite of this difference, the following lemma shows that vanishing τ indeed follows from proper superintegrability. We are going to see below in Corollary 5.3 that, for non-degenerate second order conformally superintegrable systems, the converse holds as well.

Lemma 3.15. *Consider a second order superintegrable system with potential V and associated proper Killing tensors $K^{(\alpha)}$. Let $C^{(\alpha)} = \mathring{K}^{(\alpha)}$ and assume they satisfy the Wilczynski Eq. (3.14) with V . Then $\tau_{ij} = 0$.*

Proof. We choose a specific α and suppress the corresponding superscript for conciseness. By hypothesis, there is a function λ such that

$$K_{ij} = C_{ij} + \frac{1}{n}\lambda g_{ij} \quad (3.16)$$

satisfies (2.6).

The proper Killing tensor K_{ij} satisfies the Killing equation $\boxed{i \ j \ k} K_{ij,k} = 0$. Substituting (3.16) into the Killing equation, and then using the conformal Killing equation $\boxed{i \ j \ k} (C_{ij,k} - g_{ij}\omega_k) = 0$, we find $\lambda_{,k} = -n\omega_k$. We conclude that ω is exact, $d\omega = 0$. It follows that $\tau_{ij} = 0$, as (3.8) does not have a term involving V without derivative. \square

3.4. Non-degenerate and abundant systems. From now on we tacitly assume an analytic manifold and an analytic (pseudo-) Riemannian metric. In the previous section, it was shown that for irreducible second order superintegrable systems, the Bertrand–Darboux Eq. (3.8) can be solved for the second derivatives of V up to the Laplacian ΔV . The Wilczynski equation (3.14) then allows one to express all higher covariant derivatives of V linearly in V , ∇V and ΔV . Hence all higher derivatives of V at some point are determined by the value of $(V, \nabla V, \Delta V)$ at this point, i.e. by $n + 2$ constants. This motivates the following definition.

Definition 3.16. A conformally superintegrable system is called *non-degenerate* if it satisfies the Wilczynski condition (3.14), and if (3.14) admits an $(n + 2)$ -dimensional space of solutions V .²

Due to (3.15), the structure tensors satisfy the following symmetry conditions for a non-degenerate potential:

$$\begin{aligned} T_{ji}{}^m &= T_{ij}{}^m & g^{ij}T_{ij}{}^m &= 0 \\ \tau_{ij} &= \tau_{ji} & g^{ij}\tau_{ij} &= 0. \end{aligned}$$

Lemma 3.17. For a non-degenerate conformally superintegrable system the structure tensors T_{ijk} and τ_{ij} are unique.

Proof. Assume that the Wilczynski condition (3.14) were satisfied for two different pairs of structure tensors, say T_{ijk} , τ_{ij} and \tilde{T}_{ijk} , $\tilde{\tau}_{ij}$ respectively. Then, subtracting the corresponding copies of (3.14),

$$0 = (T_{ij}{}^k - \tilde{T}_{ij}{}^k) V_{,k} + (\tau_{ij} - \tilde{\tau}_{ij}) V.$$

By the hypothesis of non-degeneracy, the coefficients of $V_{,k}$ and V have to vanish independently, i.e. $T_{ij}{}^k = \tilde{T}_{ij}{}^k$ and $\tau_{ij} = \tilde{\tau}_{ij}$. \square

Example 3.18. The isotropic harmonic oscillator is an irreducible system in the sense of Definition 3.13 and has the potentials

$$V(\mathbf{x}) = \frac{\omega^2}{2}(\mathbf{x} - \mathbf{x}_0)^2 + V_0$$

² Note that for an analytic metric, the (trace-free) conformal Killing tensors are analytic, and thus the structure tensors and the potentials are also analytic.

with the $n + 2$ free parameters ω^2 , \mathbf{x}_0 and V_0 . $V(\mathbf{x})$ solves the Wilczynski equation (3.14). The *isotropic harmonic oscillator* on flat n -dimensional space has vanishing structure tensor T . It is properly superintegrable and therefore also the structure tensor τ_{ij} vanishes. Any conformally superintegrable system conformally equivalent to the isotropic harmonic oscillator is characterised by $\mathring{T}_{ijk} = 0$, and we obtain (3.14) in the form

$$V_{,ij} = \frac{n t_i V_{,j} + n t_j V_{,i} - 2 g_{ij} t^m V_{,m}}{(n - 1)(n + 2)} + \frac{1}{n} g_{ij} \Delta V + \tau_{ij} V. \tag{3.17}$$

We define *abundantness* in analogy to properly superintegrable systems.

Definition 3.19. We call a non-degenerate second order conformal superintegrable system in dimension n *abundant*, if the subspace

$$\mathring{C} = \{\mathring{C} : C \in \mathcal{C}^{\max} : \text{tr}(C) = 0\}$$

has dimension

$$\dim(\mathring{C}) = \frac{n(n + 1)}{2} - 1 = \frac{(n - 1)(n + 2)}{2} = r_{\max}.$$

Note that non-degeneracy and abundantness are conformally invariant, i.e. well-defined for c-superintegrable systems, by virtue of Proposition 3.8.

Definition 3.20.

- (i) We call a c-superintegrable system *non-degenerate* if one (and hence all) members of the corresponding equivalence class are non-degenerate in the sense of Definition 3.16.
- (ii) We call a c-superintegrable system *abundant*, if one (and hence all) members of the corresponding equivalence class are abundant in the sense of Definition 3.19.

It is manifest that for an abundant conformally superintegrable system, a properly superintegrable system conformally equivalent to it is also abundant in the sense of [KSV23]. In Sect. 5 we find that abundantness is tantamount with the generic integrability of the prolongation equations for the trace-free conformal Killing tensors arising from a second order conformally superintegrable system. Abundantness is trivial in dimension $n = 2$. For dimension $n = 3$, it follows from [KKM05b] that every non-degenerate second order c-superintegrable system on a conformally flat manifold is abundant (the so-called $(5 \Rightarrow 6)$ -Lemma).

From now on we restrict to non-degenerate systems that satisfy the Wilczynski Eq. (3.14). Our aim in the present section is to formulate and study the integrability conditions imposed onto the structure tensors by the non-degeneracy and (3.14).

Remark 3.21. For the sake of expositional simplicity, we require an abundant system to have $\frac{n(n+1)}{2} - 1$ linearly independent trace-free conformal Killing tensors. Functional independence of $2n - 1$ of the arising conformal integrals is not yet required in the definition, but in Lemma 6.7 we prove that systems which do admit $2n - 1$ functionally independent conformal integrals lie dense among abundant systems.

We devote the remainder of this paragraph to an alternative characterisation of abundantness.

Definition 3.22. We say that a collection of linearly independent Killing tensors $K_{ij}^{(\alpha)}$ is *conformally linearly independent* if

$$\sum_{\alpha} c_{\alpha} K^{(\alpha)} = f g, \quad (3.18)$$

with constants c_{α} and a function f , implies $f = 0$ and $c_{\alpha} = 0$ for all α .

The following lemma ensures that a conformally superintegrable system is abundant if the conformal Killing tensors $C^{(\alpha)}$ for $\alpha \in \{1, 2, \dots, \frac{n(n+1)}{2}\}$ in Definition 3.3 are conformally linearly independent.

Lemma 3.23. *Let $(K^{(1)}, \dots, K^{(m)})$ be a collection of Killing tensors and denote the corresponding trace-free conformal Killing tensors by $C^{(\alpha)} = K^{(\alpha)} - \frac{1}{n} \text{tr}(K^{(\alpha)}) g$. Then the tuple $(g, C^{(1)}, \dots, C^{(m)})$ is linearly independent if and only if $(K^{(1)}, \dots, K^{(m)})$ is conformally linearly independent.*

Proof. Assume first that $(g, C^{(1)}, \dots, C^{(m)})$ is linearly dependent. This means there is a combination

$$\sum c_{\alpha} C^{(\alpha)} = c_0 g$$

with constants $(c_0, c_1, \dots, c_m) \neq 0$. This means

$$\sum c_{\alpha} K^{(\alpha)} = f g$$

for a function f obtained from c_0 and the trace terms. This proves one implication.

For the other implication, we assume that $(K^{(1)}, \dots, K^{(m)})$ are conformally linearly dependent. Therefore we have

$$\sum_{\alpha=1}^m c_{(\alpha)} C^{(\alpha)} = \left(f + \frac{1}{n} \sum c_{(\alpha)} \text{tr}(K^{(\alpha)}) \right) g.$$

In this equation the left hand side and the right hand side have to vanish independently, as they are trace-free respectively pure trace. We conclude

$$\sum c_{\alpha} C^{(\alpha)} = 0, \quad f = -\frac{1}{n} \sum c_{\alpha} \text{tr}(K^{(\alpha)})$$

Because of the hypothesis of conformal linear dependence, we have $(c_1, \dots, c_m) \neq 0$. This implies that $(g, C^{(1)}, \dots, C^{(m)})$ are linearly dependent. \square

Moreover: if the integrals $F^{(\alpha)} = C^{(\alpha)ij} p_i p_j + W^{(\alpha)} \in \mathcal{F}$ are functionally independent, then their kinetic parts are associated to conformally linearly independent conformal Killing tensors. This follows from Theorem 1 of [KKM05b].

3.5. *Structure tensors and c-superintegrability.* We are now going to determine how the structure tensors behave under conformal changes of the superintegrable system.

Lemma 3.24. *Let $H = g(\mathbf{p}, \mathbf{p}) + V$ and $\tilde{H} = \Omega^{-2} H$ be a pair of conformally equivalent Hamiltonians, $\Omega > 0$. Assume H gives rise to an irreducible conformally superintegrable system such that the Wilczynski Eq. (3.14) is satisfied. Then \tilde{H} satisfies (3.14) as well. (We decorate the corresponding objects with a tilde). In particular, the structure tensors T_{ijk} and τ_{ij} are transformed, respectively, to $(\Upsilon = \ln \Omega)$*

$$\tilde{T}_{ij}{}^k = T_{ij}{}^k - 3 \boxed{i \mid j} \circ \Upsilon_{,i} g_j^k \tag{3.19a}$$

$$\tilde{\tau}_{ij} = \tau_{ij} + 2 T_{ij}{}^k \Upsilon_{,k} - \boxed{i \mid j} \circ (\Upsilon_{,ij} + 2 \Upsilon_{,i} \Upsilon_{,j}). \tag{3.19b}$$

Proof. By a straightforward computation, using the Wilczynski Eq. (3.14) and the product rule, we arrive at

$$\begin{aligned} \hat{\nabla}_{ij}^2 \hat{V} &= \frac{1}{n} \hat{g}_{ij} \hat{\Delta} \hat{V} + \left(T_{ij}{}^k - 3 \left(\Upsilon_i g_j^k + \Upsilon_j g_i^k \right) - \frac{6}{n} g_{ij} \Upsilon^{,k} \right) \hat{V}_k \\ &+ \left(\tau_{ij} + 2 T_{ij}{}^k \Upsilon_{,k} - 2 \frac{\nabla_{ij}^2 \Omega}{\Omega} + \frac{2}{n} g_{ij} \frac{\Delta \Omega}{\Omega} - 2 \Upsilon_{,i} \Upsilon_{,j} + \frac{2}{n} g_{ij} \Upsilon^{,a} \Upsilon_{,a} \right) \hat{V}. \end{aligned}$$

The result then simplifies further using $\Omega^{-1} \nabla_{ij}^2 \Omega = \Upsilon_{,i} \Upsilon_{,j} + \frac{1}{2} \boxed{i \mid j} \Upsilon_{,ij}$. \square

According to the Wilczynski Eq. (3.14), the structure tensor T_{ijk} of a conformally superintegrable system determines the conformally superintegrable potential up to the action of $\mathfrak{S} = \text{Diff}(M) \times \mathbb{R}^*$. The following corollary is straightforwardly obtained, but will be fundamental.

Corollary 3.25. *Let the hypothesis and notation be as in Lemma 3.24.*

(i) *Under conformal transformations, the trace $t_i = T_{ia}{}^a = T_{ai}{}^a$ undergoes a translation by $\Upsilon_{,i}$,*

$$\tilde{t}_i = t_i - \frac{3}{n} (n-1)(n+2) \Upsilon_{,i}. \tag{3.20}$$

(ii) *Under conformal transformations, the trace $T^a{}_{ai}$ remains unchanged,*

$$\tilde{T}^a{}_{ai} = T^a{}_{ai}.$$

(iii) *Under conformal transformations, the trace-free part of the primary structure tensor remains unchanged,*

$$\overset{\circ}{\tilde{T}}_{ij}{}^k = \overset{\circ}{T}_{ij}{}^k. \tag{3.21}$$

Remark 3.26. Although $\overset{\circ}{\tilde{T}}_{ij}{}^k$ is conformally invariant, it is often advantageous to work with the tensor $\overset{\circ}{\tilde{T}}_{ijk}$ instead, which in the context below turns out to be a totally symmetric tensor field. While $\overset{\circ}{\tilde{T}}_{ijk}$ is not actually invariant, it is still equivariant of conformal weight 2. According to its weight, under conformal transformations it is multiplied with a power of Ω ,

$$\overset{\circ}{\tilde{T}}_{ijk} = \Omega^2 g_{ka} \overset{\circ}{\tilde{T}}_{ij}{}^a = \Omega^2 \overset{\circ}{\tilde{T}}_{ijk}.$$

As shown in [KSV23], $\mathring{T}_{ij}{}^k$ is closely related to the Weyl curvature. Under mild assumptions we are going to find that $\mathring{T}_{ij}{}^k$ carries enough information to reconstruct the conformal equivalence class of a (conformally) superintegrable system. With properly superintegrable systems in mind, it might at first seem that the conformal case requires additional information, in the form of an additional structure tensor τ_{ij} . We find, however, that for abundant systems τ_{ij} is determined by T_{ijk} and the Ricci curvature. We summarise the discussion in this section with the following proposition.

Proposition 3.27. *On a (pseudo-) Riemannian manifold M , every non-degenerate irreducible conformally superintegrable system admits tensor fields T and τ with the following properties:*

- (i) T is well-defined and smooth on the open and dense subset $N = \{G_{r_{\max}}(A) \neq 0\} \subseteq M$. It is of valence 3, symmetric and trace-free in its first two indices:

$$T_{jik} = T_{ijk} \qquad g^{ij}T_{ijk} = 0 \qquad (3.22)$$

Moreover, τ is well-defined and smooth on N . It is of valence 2, symmetric and trace-free:

$$\tau_{ji} = \tau_{ij} \qquad g^{ij}\tau_{ij} = 0 \qquad (3.23)$$

- (ii) *The conformally superintegrable potential satisfies the Wilczynski Eq. (3.14).*
- (iii) *T and τ are uniquely determined by the metric and by the trace-free conformal Killing tensors $C^{(\alpha)}$ in the conformally superintegrable system, and depend only on the subspace \mathcal{C} spanned by these $C^{(\alpha)}$, i.e. it is invariant under basis changes on \mathcal{C} .*
- (iv) *The trace-free part $\mathring{T}_{ij}{}^k$ of the (2, 1)-tensor field $T_{ij}{}^k$ is invariant under conformal changes of the conformally superintegrable system.*

The components T_{ijk} of T are given explicitly in terms of the Killing tensors by the rank- r Moore-Penrose inverse, where $r = r_{\max}$ is the maximal rank (3.13).

Proof. The first three assertions follow analogous to the case of properly superintegrable systems, see [KSV23], such that the tensors T and τ are given by $A^\dagger b_1$ and $A^\dagger b_0$, respectively, using Eq. (3.11). To see (iv), take the trace-free part of (3.19a). \square

Let us reconsider the aforesaid in the light of Definition 3.9.

Corollary 3.28. (i) *Every non-degenerate irreducible c -superintegrable system on a conformal manifold (M, \mathbf{g}) admits a well-defined totally symmetric and trace-free tensor field $S = \mathring{T}$ that is invariant under conformal transformations.*
 (ii) *If this tensor coincides for two non-degenerate irreducible c -superintegrable systems on the same conformal manifold, then these systems are conformally equivalent if also*

$$\tau_{ij} - 2\mathring{P}_{ij} + \frac{2}{3}S_{ijk}\bar{i}^k + \frac{1}{3}\boxed{i \ j} \circ \bar{i} \bar{j}$$

coincides for both systems.

The corollary is easily obtained using (3.19) and recalling, see e.g. [CG14, Kul69, Kul70],

$$\mathring{R}_{ij} \rightarrow \mathring{R}_{ij} - (n - 2) (\Upsilon_{,ij} - \Upsilon_{,i} \Upsilon_{,j})_{\circ} \tag{3.24a}$$

$$R \rightarrow \Omega^{-2} \left(R + 2(n - 1) (\Delta \Upsilon - \frac{n-2}{2} \Upsilon^{,a} \Upsilon_{,a}) \right). \tag{3.24b}$$

The Schouten tensor transforms according to

$$P_{ij} \rightarrow P_{ij} - \Upsilon_{,ij} + \Upsilon_{,i} \Upsilon_{,j} - \frac{1}{2} g_{ij} \Upsilon_{,c} \Upsilon^{,c}. \tag{3.25}$$

4. Conformally Superintegrable Potentials

Written in local coordinates, the Wilczynski Eq. (3.14) is a second order partial differential equation (PDE) for the potential V . In Proposition 3.27 we have seen that for irreducible second order superintegrable systems, the coefficients in this PDE are determined by the space of trace-free conformal Killing tensors and the metric.

4.1. Prolongation of a superintegrable potential. The Wilczynski Eq. (3.14) expresses the trace-free part of the Hessian of the potential V linearly in the differential ∇V , the Laplacian ΔV , and the potential V itself. The coefficients in this expression are determined by the structure tensors T and τ . In the following Proposition, the Wilczynski Eq. (3.14) is extended by a second equation which expresses the derivatives of ΔV linearly in ΔV , ∇V and V . Again, the coefficients are determined by the structure tensors. The system (4.1) below is an extended system of PDEs for (3.14). Such an extended system is called a *prolongation* of the initial PDE, and it allows one to make use of the powerful theory of parallel linear connections. In particular, by virtue of (4.1) all higher derivatives of V , ∇V and ΔV are expressed in terms of these.

Proposition 4.1. *Equation (3.14) has the prolongation*

$$V_{,ij} = T_{ij}{}^m V_{,m} + \frac{1}{n} g_{ij} \Delta V + \tau_{ij} V \tag{4.1a}$$

$$\frac{n-1}{n} (\Delta V)_{,k} = q_k{}^m V_{,m} + \frac{1}{n} t_k \Delta V + \gamma_k V, \tag{4.1b}$$

with

$$q_j{}^m := Q_{aj}{}^{am} \tag{4.2a}$$

$$t_j := T_{aj}{}^a \tag{4.2b}$$

$$\gamma_k := \Gamma_{ak}{}^a \tag{4.2c}$$

where

$$Q_{ijk}{}^m := T_{ij}{}^m{}_{,k} + T_{ij}{}^l T_{lk}{}^m - R_{ijk}{}^m + \tau_{ij} g_k{}^m. \tag{4.3}$$

$$\Gamma_{ijk} := \tau_{ij,k} + T_{ij}{}^a \tau_{ak} \tag{4.4}$$

Proof. Equation (4.1a) is nothing but the Wilczynski Eq. (3.14). Substituting it into its covariant derivative, we obtain

$$V_{,ijk} = (T_{ij}{}^m{}_{,k} + T_{ij}{}^l T_{lk}{}^m + \tau_{ij} g_k^m) V_{,m} + \frac{1}{n} (T_{ijk} \Delta V + g_{ij}(\Delta V)_{,k}) + (\tau_{ij,k} + T_{ij}{}^m \tau_{mk}) V.$$

Antisymmetrising in j and k and applying the Ricci identity gives

$$R^m{}_{ijk} V_{,m} = \frac{\boxed{j}}{\boxed{k}} \left[(T_{ij}{}^m{}_{,k} + T_{ij}{}^l T_{lk}{}^m + \tau_{ij} g_k^m) V_{,m} + \frac{1}{n} (T_{ijk} \Delta V + g_{ij}(\Delta V)_{,k}) + \Gamma_{ijk} V \right].$$

Solving for the term involving $(\Delta V)_{,k}$ on the right hand side, we get

$$\frac{1}{n} \frac{\boxed{j}}{\boxed{k}} g_{ij}(\Delta V)_{,k} = - \frac{\boxed{j}}{\boxed{k}} (Q_{ijk}{}^m V_{,m} + \frac{1}{n} T_{ijk} \Delta V + \Gamma_{ijk} V).$$

The contraction of this equation in i and j now yields (4.1b), since T_{ijk} and $Q_{ijk}{}^m$ are trace-free in i and j by definition. \square

4.2. Integrability conditions for a non-degenerate potential. From the perspective of the Eq. (4.1), non-degeneracy implies that the corresponding integrability conditions be satisfied generically, independently of the potential. With this condition we finally eliminate the potential V (and therefore all $V^{(\alpha)}$) from the system, leaving a system of equations depending only on the structure tensors T_{ijk} and τ_{ij} , as well as on the underlying metric g and its curvature.

Proposition 4.2. *The Wilczynski Eq. (3.14) locally has a non-degenerate solution V if and only if the following integrability conditions hold:*

$$\frac{\boxed{j}}{\boxed{k}} \left(T_{ijk} + \frac{1}{n-1} g_{ij} t_k \right) = 0 \quad (4.5a)$$

$$\frac{\boxed{j}}{\boxed{k}} \left(Q_{ijkl} + \frac{1}{n-1} g_{ij} q_{kl} \right) = 0 \quad (4.5b)$$

$$\frac{\boxed{j}}{\boxed{k}} \left(\Gamma_{ijk} + \frac{1}{n-1} g_{ij} \gamma_k \right) = 0 \quad (4.5c)$$

$$\frac{\boxed{k}}{\boxed{l}} \left(q_{ki,l} + T_{mli} q_k{}^m + \frac{1}{n-1} t_k q_{li} + \gamma_k g_{li} \right) = 0 \quad (4.5d)$$

$$\frac{\boxed{i}}{\boxed{j}} \left(\gamma_{i,j} + q_i{}^m \tau_{mj} + \frac{1}{n-1} t_i \gamma_j \right) = 0. \quad (4.5e)$$

Proof. The system (4.1) allows one to write all higher derivatives of V , ∇V and ΔV as linear combinations of V , ∇V and ΔV . Necessary and sufficient integrability conditions are then given by the Ricci identities

$$\boxed{\frac{j}{k}} V_{,ijk} = R^m{}_{ijk} V_{,m} \qquad \boxed{\frac{k}{l}} (\Delta V)_{,kl} = 0,$$

where we substitute (3.14) in the left hand side of the equations. We obtain

$$\boxed{\frac{j}{k}} (Q_{ijk}{}^m + \frac{1}{n-1} g_{ij} q_k{}^m) V_{,m} + \frac{1}{n} \boxed{\frac{j}{k}} (T_{ijk} + \frac{1}{n-1} g_{ij} t_k) \Delta V + \boxed{\frac{j}{k}} (\Gamma_{ijk} + \frac{1}{n+1} g_{ij} \gamma_k) V = 0$$

and, respectively,

$$\begin{aligned} \boxed{\frac{k}{l}} (q_{k,l}{}^a + T_{ml}{}^a q_k{}^m + \frac{1}{n-1} t_k q_l{}^a + \gamma_k g_l{}^a) V_{,a} + \frac{1}{n} \boxed{\frac{k}{l}} (t_{k,l} + q_{kl}) \Delta V \\ + n \boxed{\frac{k}{l}} (\gamma_{k,l} + q_k{}^m \tau_{ml} + \frac{1}{n-1} t_k \gamma_l) V = 0. \end{aligned}$$

For a non-degenerate superintegrable potential the coefficients of ΔV , ∇V and V must vanish separately. In addition to the stated integrability conditions, this yields the condition

$$\boxed{\frac{k}{l}} (t_{k,l} + q_{kl}) = 0. \tag{4.6}$$

The latter is redundant, since it can be obtained from equation (4.5b) by a contraction over i and l . \square

Note that the Eq. (4.5) are not invariant under conformal transformations, as they emerge from coefficients of V , ∇V and ΔV , respectively. Still, after a conformal transformation as in Proposition 3.8, the form of (4.5) is the same, but the metric, the curvature and the structure tensors are replaced. We can solve Eq. (4.5a) right away, because it is linear and does not involve derivatives.

Proposition 4.3. *The first integrability condition for a superintegrable potential, Eq. (4.5a), has the solution*

$$T_{ijk} = S_{ijk} + \boxed{\frac{i}{j}} \left(\bar{t}_i g_{jk} - \frac{1}{n} g_{ij} \bar{t}_k \right), \tag{4.7}$$

where S is an arbitrary totally symmetric and trace-free tensor. The 1-form \bar{t}_i is given by

$$\bar{t}_i = \frac{n}{(n-1)(n+2)} t_i = \frac{n}{(n-1)(n+2)} T_{ia}{}^a. \tag{4.8}$$

Note that S_{ijk} and \bar{t}_i are uniquely determined by T .

Proof. Decompose T_{ijk} , which by definition is trace-free and symmetric in (i, j) , according to

$$\square\square \otimes \square \cong \square\square\square \oplus \square\square \cong \square\square\square \oplus \square \oplus \square \oplus \square \oplus \square.$$

Due to (4.5a), the penultimate component of this decomposition vanishes, and therefore we obtain

$$T_{ijk} = S_{ijk} + \frac{1}{6} \boxed{i \ j \ k} g_{ij} s_k + \frac{1}{4} \boxed{i \ j} \boxed{j} g_{ij} \xi_k,$$

where s_k and ξ_k are components of some 1-forms. Substituting (4.5a), and taking the trace in (j, k) ,

$$t_i = T_{ia}{}^a = \frac{n+2}{3} s_i - \frac{n-1}{4} \xi_i.$$

Taking the trace in (i, j) instead,

$$0 = T^a{}_{ai} = \frac{n+2}{3} s_i + \frac{n-1}{2} \xi_i.$$

Solving for s_i and ξ_i , we find

$$s_i = \frac{2t_i}{n+2}, \quad \xi_i = -\frac{4t_i}{3(n-1)}.$$

Resubstituting into the initial formula for T_{ijk} , we arrive at (4.7). \square

Corollary 4.4. (i) *The tensor q_{ij} is symmetric, i.e. $q_{ji} = q_{ij}$.*
(ii) *The 1-form \bar{t}_i is the derivative of a function \bar{t} , i.e. $\bar{t}_i = \bar{t}_{,i}$.*

Note that consequently also $t_i = t_{,i}$. Without loss of generality we impose $\bar{t} = \frac{n}{(n-1)(n+2)} t$ on the arbitrary integration constant.

Proof. The first statement follows from substituting (4.7) into the definition (4.2a) of q_{ij} . The second then follows from (4.6). \square

Proposition 4.5. *The third integrability condition for a superintegrable potential, Eq. (4.5c), has the solution*

$$\Gamma_{ijk} = \Sigma_{ijk} + \boxed{i \ j} (\bar{\gamma}_i g_{jk} - \frac{1}{n} g_{ij} \bar{\gamma}_k), \quad (4.9)$$

where Σ is an arbitrary totally symmetric and trace free tensor and

$$\bar{\gamma}_i = \frac{n}{(n+2)(n-1)} \gamma_i.$$

Note that Σ and γ are uniquely determined by Γ .

The proof is the same as that of Proposition 4.3.

4.3. *The conformal scale function.* As we have just seen, the trace t_i of the primary structure tensor is the differential of a function t . We thus obtain the following transformation rule under conformal changes of the Hamiltonian, which is an immediate consequence of (3.20).

Lemma 4.6. *Under a conformal change of the superintegrable system, say $H \mapsto \Omega^{-2}H$, $\Omega > 0$, the function t transforms as ($\Upsilon = \ln \Omega$)*

$$\bar{t} \mapsto \bar{t} - 3\Upsilon \text{ up to an irrelevant constant,} \tag{4.10}$$

Note that the function \bar{t} is determined by the structure tensor T only up to an irrelevant constant. The symmetry group of c-superintegrable systems is $\mathfrak{S} = \text{Conf}(M) \rtimes \mathbb{R}^*$, c.f. Remark 3.10. The second factor of \mathfrak{S} does not affect \bar{t} . Indeed, we see that if $\bar{t}^{\text{new}} - \bar{t}^{\text{old}} = c \in \mathbb{R}$, then $\Omega = e^{-\frac{c}{3}}$ and

$$H^{\text{new}} = e^{\frac{2c}{3}} H^{\text{old}}.$$

Lemma 4.6 above therefore confirms that \bar{t} behaves like a scale function.

Definition 4.7. The *conformal scale function* is the density of weight 1 defined by

$$\sigma = e^{-\frac{1}{3}\bar{t}}.$$

□

Lemma 4.6 allows us to change \bar{t} arbitrarily, resulting in a natural gauge freedom of a conformally superintegrable system. There are three natural scale choices, in particular, that are relevant here, each of which has specific features we can exploit to gain information or to simplify certain computations. Table 1 summarises some properties of these scale choices and the notation we use.

4.3.1. *Standard scale* This scale choice realises $\bar{t}_{,i} = 0$.

Definition 4.8. A conformally superintegrable system with $\bar{t}_i = 0$ is said to be in *standard scale*.

We shall use a specific notation for the metric and the secondary structure tensor when we work in standard scale. Given an arbitrary scale choice, we can apply a conformal transformation with conformal rescaling $\Omega = e^{\frac{1}{3}\bar{t}}$. Let $\Upsilon = \ln \Omega$. The transformed metric of the system in standard scale, then is

$$\tilde{g}_{ij} = e^{\frac{2}{3}\bar{t}} g_{ij} =: \mathfrak{g}_{ij}, \tag{4.11a}$$

and the new structure tensors become $\tilde{T}_{ij}{}^k = S_{ij}{}^k$ and

$$\tilde{\tau}_{ij} = \tau_{ij} + \frac{2}{3} S_{ij}{}^k \bar{t}_{,k} - \boxed{i \mid j} \circ \left(\frac{1}{3} \bar{t}_{,ij} - \frac{4}{9} \bar{t}_{,i} \bar{t}_{,j} \right) =: \mathfrak{s}_{ij}. \tag{4.11b}$$

For later reference, we mention the Schouten curvature \mathfrak{P}_{ij} of \mathfrak{g}_{ij} , which is given by

$$\mathfrak{P}_{ij} = P_{ij} - \Upsilon_{,ij} + \Upsilon_{,i} \Upsilon_{,j} - \frac{1}{2} g_{ij} \Upsilon_{,c} \Upsilon^{,c}, \tag{4.11c}$$

while the Weyl curvature remains unchanged under conformal transformations. Equations (4.11b) and (4.11c) are obtained, respectively, from (3.19b) and (3.25).

Note that the standard scale is not unique, as we may add any constant to $\bar{\tau}$. For simplicity we usually choose $\bar{\tau} = 0$. As discussed after Lemma 4.6, however, this only means that the Hamiltonian is multiplied by a constant, and moreover the structure tensor is not changed in the process. From the viewpoint of conformally superintegrable systems however, if we multiply the metric by a constant, this typically changes the underlying metric (unless the transformation is already in $\text{Diff}(M)$). Yet, the space \mathring{C} of conformal Killing tensors of a conformally superintegrable system remains unaffected by such a change.

The standard scale has two major advantages: on the one hand, it yields very compact equations, facilitating some otherwise tedious computations. On the other hand, the standard scale exposes the invariant data of the problem, which is going to be particularly helpful when we discuss conformal equivalence classes.

Example 4.9. The systems VII [5], O and A in Table 3 are in standard scale.

4.3.2. Flat scale This scale choice only exists for conformally flat metrics.

Definition 4.10. A conformally superintegrable system with flat curvature is said to be in *flat scale*.

We find, using (4.11a), that there is a function $\rho : \mathcal{C}^\infty(M) \rightarrow \mathbb{R}$ such that

$$g_{ij} = e^{\frac{2}{3}\rho} h_{ij}. \quad (4.12)$$

where h has vanishing curvature. A major advantage of flat scales is that covariant derivatives coincide with partial ones, facilitating concrete computations in local coordinates. Moreover, the existence of a flat scale permits us to express the Ricci curvature in terms of the scalar function ρ using (3.24).

Flat scales are not unique. For example, we can add any constant to ρ . According to [Kul70], any conformal change transforming a flat metric into a flat metric is given via $\rho \rightarrow \rho - \eta$ where η is a function satisfying

$$[Q(Y, Z) + g(Y, Z)r] X - [Q(X, Z) + g(X, Z)r] Y + g(Y, Z)Q(X) - g(X, Z)Q(Y) = 0$$

where

$$Q(X, Y) = (\nabla^2 \eta)(X, Y) - X(\eta)Y(\eta)$$

with $g(Q(X), Y) = Q(X, Y)$ and $r = g(d\eta, d\eta)$.

Example 4.11. All systems in Table 3 are in flat scale. In particular, note that the systems III [23] and V [32] are conformally equivalent.

4.3.3. Proper scale A third natural choice is the proper scale, in which the system is properly superintegrable (up to a trace correction of the trace-free conformal Killing tensors). As mentioned earlier, any conformally superintegrable system is conformally equivalent to a properly superintegrable system [Cap14, Theorem 4.1.8]. According to Lemma 3.15 it satisfies $\tau_{ij} = 0$.

Definition 4.12. A conformally superintegrable system with $\tau_{ij} = 0$ is said to be in *proper scale*.

Table 1. Notation and conventions for the three natural scale choices

Objects	Standard Scale	Flat scale	Proper scale
Function \bar{t}	0	ρ	\bar{t}
Schouten tensor P_{ij}	\mathfrak{P}_{ij}	0	P_{ij}
Secondary structure tensor τ_{ij}	\mathfrak{N}_{ij}	τ_{ij}	0
Metric g_{ij}	\mathfrak{g}_{ij}	h_{ij}	g_{ij}

Again, the proper scale choice is not unique. Its advantage is that all the known results about properly superintegrable systems can be invoked. Yet it is less useful for gaining insight into the underlying conformal geometry. Nevertheless, from the viewpoint of conformal geometry, proper scale choices have some interesting properties which we explore in Sect. 6.5 for constant curvature spaces.

Example 4.13. All systems in Table 3 are in proper scale. For an example that is in proper scale, but neither in flat nor standard scale, consider the metric $g = (z\bar{z} + 4)^{-1} dzd\bar{z}$ on the 2-sphere. It admits the superintegrable potential

$$V = a_0 \left(\frac{z\bar{z} + 4}{z + \bar{z}} \right)^2 + a_1 \left(\frac{z\bar{z} + 4}{z - \bar{z}} \right)^2 + a_2 \left(\frac{z\bar{z} + 4}{z\bar{z} - 4} \right)^2 + a_3$$

with parameters $a_i \in \mathbb{R}$, and satisfies

$$\bar{t} = \frac{3}{2} \ln \frac{i(z\bar{z} + 4)^3}{(z\bar{z} - 4)(z + \bar{z})(z - \bar{z})}$$

Example 4.14. (Generic system on the 3-sphere) Consider the 3-sphere with metric

$$g = d\phi^2 + \sin^2(\phi) \left(d\theta^2 + \sin^2(\theta) d\psi^2 \right).$$

The potential

$$V = \frac{a_1}{\cos^2(\phi)} + \frac{a_2}{\sin^2(\phi)\cos^2(\theta)} + \frac{a_3}{\sin^2(\phi)\sin^2(\theta)\cos^2(\psi)} + \frac{a_4}{\sin^2(\phi)\sin^2(\theta)\sin^2(\psi)} + a_0$$

is non-degenerate and defines the so-called generic system on the 3-sphere; in [KKM06] it is labelled VIII. It is in proper scale, but neither in flat nor in standard scale.

Note that the Harmonic Oscillator, see Example (3.18), is simultaneously in standard, flat and proper scale. In Sect. 6 we find that this is an immediate consequence of $T_{ijk} = 0$.

5. Conformal Killing Tensors in Conformally Superintegrable Systems

5.1. Prolongation equations for trace-free conformal Killing tensors. In Sect. 4.1 we have discussed a prolongation system for the potential V . Similarly, we can write down a prolongation for an arbitrary trace-free conformal Killing tensor C_{ij} . In general, this system can be rather complicated [Wei77], given also the explicit but complicated expressions well known for proper second order Killing tensors in [Wol98, GL19]. However, as is shown in [KSV23], the prolongation system for proper second order Killing tensors in non-degenerate superintegrable systems simplifies considerably. In fact, the prolongation system in this case closes after the first covariant derivative. We observe the same phenomenon with *trace-free* conformal Killing tensors, and trace-freeness is paramount. Indeed, for conformal Killing tensors with non-vanishing trace, the prolongation system would not be finite.

Theorem 5.1. *A trace-free conformal Killing tensor C_{ij} in a non-degenerate conformally superintegrable system satisfies*

$$C_{ij,k} = \frac{1}{3} \left(\begin{array}{c|c} j & i \\ \hline k & \end{array} T^m{}_{ji} g^n{}_k - \frac{2}{n} g_{ij} (t^m g^n{}_k - T_k{}^{mn}) \right) C_{mn}, \quad (5.1)$$

with the primary structure tensor $T_{ij}{}^k$ given by the Wilczynski Eq. (3.14).

The Bertrand–Darboux condition (3.8) in this situation is equivalent to (5.1) and

$$\begin{array}{c|c} j & \\ \hline k & \end{array} (\omega_{j,k} + C^m{}_j \tau_{km}) = 0. \quad (5.2)$$

Note that from (5.1) we obtain ω_i using Formula (3.2). We also remark that (5.1) does not contain the secondary structure tensor τ_{ij} . Indeed, we shall see that, under the hypothesis of the theorem, the tensor τ_{ij} is obtained from T_{ijk} and the Ricci curvature.

Remark 5.2. Equation (5.1) should be compared to the prolongation Eq. (2.8) for a Killing tensor in a properly superintegrable system. However, here $K_{ij,k}$ is not trace-free. Therefore, we need to subtract the trace, obtaining

$$K^a{}_{a,k} = \frac{2}{3} (t^m g^n{}_k - T_k{}^{mn}) K_{mn}. \quad (5.3)$$

Next, verify that

$$\left(\begin{array}{c|c} j & i \\ \hline k & \end{array} T^m{}_{ji} g^n{}_k - \frac{2}{n} g_{ij} (t^m g^n{}_k - T_k{}^{mn}) \right) g_{mn} = 0,$$

which, combined with (2.8) and (5.3), yields

$$C_{ij,k} = \frac{1}{3} \left(\begin{array}{c|c} j & i \\ \hline k & \end{array} T^m{}_{ji} g^n{}_k - \frac{2}{n} g_{ij} (t^m g^n{}_k - T_k{}^{mn}) \right) C_{mn}$$

where $C_{ij} = K_{ij} - \frac{1}{n} g_{ij} K^a{}_a$. Summarising, we have thus confirmed that the trace-free part C_{ij} of a properly superintegrable Killing tensor K_{ij} satisfies (5.1).

Proof of Theorem 5.1. We decompose $C_{ij,k}$ as

$$C_{ij,k} = \frac{1}{3} \begin{array}{c|c} j & i \\ \hline k & \end{array} C_{ij,k} + \frac{1}{6} \begin{array}{c|c|c} i & j & k \\ \hline & & \end{array} C_{ij,k} \quad (5.4)$$

The totally symmetric component is given by the conformal Killing equation,

$$\begin{array}{c|c|c} i & j & k \\ \hline & & \end{array} C_{ij,k} = \begin{array}{c|c|c} i & j & k \\ \hline & & \end{array} \omega_k g_{ij}$$

The hook symmetric component is obtained as follows: Substituting the Wilczynski Eq. (3.14) into the Bertrand–Darboux Eq. (3.8) gives

$$\begin{array}{c|c} j & \\ \hline k & \end{array} \left[(C^m{}_{j,k} - T_{ji}{}^m C^l{}_k + \omega_j g_k^m) V_{,m} + (C^m{}_j \tau_{km} + \omega_{j,k}) V \right] = 0. \quad (5.5)$$

From non-degeneracy it follows that the coefficients of $V_{,m}$ and V vanish independently. The coefficient of V yields (5.2). From the coefficients of $V_{,m}$ we obtain

$$\boxed{\begin{matrix} j \\ k \end{matrix}} C_{ij,k} = \boxed{\begin{matrix} j \\ k \end{matrix}} (T^l{}_{ji} C_{lk} + g_{ij} \omega_k).$$

Altogether, using (5.4),

$$C_{ij,k} = \frac{1}{3} \boxed{\begin{matrix} j & i \\ k \end{matrix}} (T^l{}_{ji} C_{lk} + \omega_k g_{ij}) + \frac{1}{6} \boxed{\begin{matrix} i & j & k \end{matrix}} \omega_k g_{ij}$$

The trace-freeness of C_{ij} now implies

$$\omega_k = -\frac{1}{3n} g^{ij} \boxed{\begin{matrix} j & i \\ k \end{matrix}} T^m{}_{ji} C_{mk} = \frac{2}{3n} (T_k{}^{ab} - t^a g^b{}_k) C_{ab}, \tag{5.6}$$

which completes the proof. \square

Equation (5.2) allows us to prove the converse of Lemma 3.15 for non-degenerate systems. Note that there is a natural mapping from the space \mathcal{K} of Killing tensors into the space $\mathring{\mathcal{C}}$ of trace-free conformal Killing tensors,

$$\mathcal{K} \rightarrow \mathring{\mathcal{C}}, \quad K_{ij} \mapsto C_{ij} = K_{ij} - \frac{1}{n} K^a{}_a g_{ij}.$$

This map is not surjective as not every conformal Killing tensor arises from a proper Killing tensor. Its range consists of trace-free conformal Killing tensors whose ω from (3.2) is exact, $\omega = d\lambda$, and thus

$$\left\{ C \in \mathring{\mathcal{C}} : 2C^a{}_{k,a} + C^a{}_{a,k} = \lambda_{,k} \text{ for some scalar } \lambda \right\} \rightarrow \mathcal{K}, \quad C \mapsto C - \frac{1}{n} \lambda g,$$

is surjective. It is not injective as we may add a constant multiple of the metric to any Killing tensor. From (5.2) we infer that ω is exact for trace-free conformal Killing tensors that commute with τ .

Corollary 5.3. *If $\tau_{ij} = 0$ for a non-degenerate second order conformally superintegrable system, then the system is properly superintegrable.*

Note that in the proof we do not take functional independence into account yet, but we will account for it in Lemma 6.7. This lemma ensures the existence of sufficiently many functionally independent conformal integrals for almost any potential of a non-degenerate system, which suffices here as we consider the space \mathcal{V}^{\max} .

Proof. Let C_{ij} be a trace-free conformal Killing tensor of the conformally superintegrable system. We need to find a function λ such that $K_{ij} = C_{ij} + \frac{1}{n} g_{ij} \lambda$ is a proper Killing tensor, i.e. it satisfies the Bertrand–Darboux condition (2.6).

We proceed in two steps. First we show that $d\omega = 0$. Then we prove that this leads to a properly superintegrable system.

For the first step, take the coefficient of V in (5.5). For a non-degenerate system (5.2) yields

$$2 d\omega_{ij} = \boxed{\begin{matrix} i \\ j \end{matrix}} \omega_{i,j} = \boxed{\begin{matrix} i \\ j \end{matrix}} C^a{}_j \tau_{ia} = \boxed{\begin{matrix} i \\ j \end{matrix}} K^a{}_j \tau_{ia}$$

and therefore ω is exact if $\tau_{ij} = 0$, i.e. $\omega = d\lambda$. Let $K_{ij} = C_{ij} + \frac{1}{n} g_{ij}\lambda$ with this specific function λ . We conclude, using the trace-freeness of C_{ij} ,

$$(d(KdV))_{ij} = \frac{1}{2} \begin{bmatrix} i \\ j \end{bmatrix} (K_{ia,j} V^{,a} + K_i^a V_{,aj}) = 0, \tag{5.7}$$

due to the conformal Bertrand–Darboux condition (3.8). So K_{ij} satisfies the proper Bertrand–Darboux Eq. (2.6). This proves the claim. \square

To allow for a concise notation, we introduce the shorthand

$$P_{ijk}{}^{mn} := \frac{1}{6} \begin{bmatrix} m & n \end{bmatrix} \left(\begin{bmatrix} j & i \\ k \end{bmatrix} T^m{}_{ji} g^n{}_k - \frac{2}{n} g_{ij} (t^m g^n{}_k - T^m{}_k{}^n) \right). \tag{5.8}$$

Consequently, we have (5.1) in the form

$$K_{ij,k} = P_{ijk}{}^{ab} K_{ab}.$$

Given T_{ijk} , we can compute $P_{ijk}{}^{mn}$. The following lemma shows that $P_{ijk}{}^{mn}$ contains all the information about τ_{ij} , i.e. for abundant systems the secondary structure tensor is redundant.

Lemma 5.4. *In an abundant conformally superintegrable system, the tensor τ_{ij} is given by*

$$\tau_{ij} = \frac{2}{n} \left(\Lambda_{ij}{}^a{}_{,a} - \Lambda^a{}_{ja,i} + \Lambda_i{}^{ab} P_{ab}{}^c{}_c{}_j - \Lambda^{cab} P_{abijc} \right) \tag{5.9}$$

where

$$\Lambda_k{}^{ab} = \frac{1}{3n} \begin{bmatrix} a & b \end{bmatrix} (T_k{}^{ab} - t^a g_k^b).$$

Moreover,

$$\begin{bmatrix} i \\ j \end{bmatrix} \begin{bmatrix} m & n \end{bmatrix} \left(\Lambda_{imn,j} + \Lambda_i{}^{ab} P_{abjmn} \right)_o = 0. \tag{5.10}$$

Note that τ_{ij} in (5.9) is symmetric and trace-free due to (5.10). Because of Eq. (5.9) the superintegrable potential is completely determined by the primary structure tensor $T_{ij}{}^k$, and this observation can be interpreted as follows: Any conformally superintegrable system corresponds to a properly superintegrable system [Cap14], for which the Wilczynski Eq. (3.14) holds with $\tau_{ij} = 0$. Applying a conformal transformation, due to (3.19b) the tensor τ_{ij} can only contain information from the properly superintegrable system (and the conformal factor). Indeed, this is the information appearing on the right hand side of Eq. (5.9).

Proof of Lemma 5.4. Equation (5.2) yields the antisymmetric part of $\omega_{i,j}$,

$$\begin{bmatrix} i \\ j \end{bmatrix} \omega_{i,j} = \begin{bmatrix} i \\ j \end{bmatrix} C^m{}_j \tau_{im}. \tag{5.11}$$

On the other hand we obtain from (5.6), after one differentiation,

$$\omega_{i,j} = \left(\Lambda_i{}^{mn}{}_{,j} + \Lambda_i{}^{ab} P_{abj}{}^{mn} \right) C_{mn}$$

Resubstituting into (5.11),

$$\boxed{\begin{matrix} i \\ j \end{matrix}} \left(\Lambda_i^{mn}{}_{,j} + \Lambda_i^{ab} P_{abj}{}^{mn} - \tau_i^m g^n{}_j \right) C_{mn} = 0.$$

Next, using the fact that there are $\frac{n(n+1)}{2} - 1 = r_{\max}$ linearly independent, trace-free and symmetric C_{ij} , we conclude that the symmetrisation of the coefficients of C_{mn} must vanish independently,

$$\boxed{\begin{matrix} i \\ j \end{matrix}} \boxed{\begin{matrix} m \\ n \end{matrix}} \left(\Lambda_{imn,j} + \Lambda_i^{ab} P_{abjmn} - \tau_{im} g_{nj} \right) = 0. \tag{5.12}$$

Contracting in (n, j) yields (5.9). Contracting (5.12) in (n, m) shows that (5.9) is the only independent trace of (5.12). The trace-free part of (5.12) is (5.10), and this completes the proof. \square

Lemma 5.4 ensures that $P_{ijk}{}^{mn}$ contains enough information to recompute the structure tensors T_{ijk} and τ_{ij} .

Corollary 5.5. *In an abundant system, the structure tensors can be recomputed from $P_{ijk}{}^{mn}$ defined in (5.8). We have*

$$t_k = -\frac{3}{n} P_{abi}{}^{ab}$$

$$S_{ijk} = \frac{3}{n} \left(P_{ijak}{}^a + \frac{n-1}{n} g_{ij} P_{abk}{}^{ab} + \boxed{\begin{matrix} i \\ j \end{matrix}} \frac{n-2}{n} g_{ik} P_{abj}{}^{ab} \right),$$

which yield T_{ijk} , and (5.9), which yields τ_{ij} .

Proof. This follows from $P_{abi}{}^{ab} = -\frac{n}{3} t_i$, and

$$P_{ijak}{}^a = \frac{n}{3} S_{ijk} + \frac{n-1}{3} g_{ij} t_k + \boxed{\begin{matrix} i \\ j \end{matrix}} \frac{n-2}{3} g_{ikt} j.$$

Together with Lemma 5.4 the claim follows. \square

5.2. Integrability conditions in an abundant system. A trace-free conformal Killing tensor in an abundant conformally superintegrable system satisfies the prolongation system (5.1). Due to the condition of abundantness, its integrability condition only depends on g, T and ∇T . We have already seen that non-degeneracy is the condition for the generic integrability of V . Along a similar line, abundantness is then the condition for the generic integrability of K_{ij} .

Proposition 5.6. *For the trace-free conformal Killing tensor fields in an abundant (conformally) superintegrable system, the integrability condition of (5.1) reads*

$$\boxed{\begin{matrix} k \\ l \end{matrix}} \left(P_{ijk}{}^{mn}{}_{,l} + P_{ijk}{}^{pq} P_{pql}{}^{mn} \right) = \frac{1}{2} \boxed{\begin{matrix} i \\ j \end{matrix}} \boxed{\begin{matrix} m \\ n \end{matrix}} R^m{}_{ikl} g^n{}_j. \tag{5.13}$$

Note that the integrability conditions (5.13) are not conformally invariant. This is entirely analogous to (4.5), which are not invariant either. However, our further analysis is going to show that we can distill invariant conditions out of (5.13) and, as we shall see, these already imply (4.5).

Proof. Writing

$$C_{ij,k} = P_{ijk}{}^{mn} C_{mn}, \tag{5.14}$$

and taking the covariant derivative, yields

$$C_{ij,kl} = P_{ijk}{}^{mn}{}_{,l} C_{mn} + P_{ijk}{}^{mn} C_{mn,l}.$$

After antisymmetrisation over (k, l) we can eliminate all derivatives of C by using the Ricci identity

$$\boxed{\begin{matrix} k \\ l \end{matrix}} C_{ij,kl} = \boxed{\begin{matrix} i & j \end{matrix}} R^m{}_{ikl} C_{mj}$$

on the left hand side, and substituting (5.14) for $C_{mn,l}$ on the right hand side. We obtain

$$\boxed{\begin{matrix} i & j \end{matrix}} R^m{}_{ikl} g^n{}_j C_{mn} = \boxed{\begin{matrix} k \\ l \end{matrix}} \left(P_{ijk}{}^{mn}{}_{,l} + P_{ijk}{}^{pq} P_{pql}{}^{mn} \right) C_{mn}. \tag{5.15}$$

An abundant conformally superintegrable system has $\frac{n(n+1)}{2} - 1 = r_{\max}$ linearly independent trace-free conformal Killing tensors C . Since this is exactly the number of independent components of the trace-free symmetric tensor C_{mn} , we can replace C_{mn} by a symmetrisation in m and n as the expression in parentheses in (5.15) is already trace-free in m and n . \square

Lemma 5.7. *For an abundant system, the curvature tensor $R^l{}_{ijk}$ satisfies*

$$R^l{}_{ijk} = \frac{2}{n+2} \boxed{\begin{matrix} j \\ k \end{matrix}} \left(P_{iaj}{}^{la}{}_{,k} + P_{iaj}{}^{pq} P_{pqk}{}^{la} \right). \tag{5.16}$$

Proof. Contracting (5.13) in n and j immediately yields the result. \square

Lemma 5.7 allows us to express the curvature in terms of the superintegrable structure tensor. Alternatively we can also view it as a curvature obstruction to the structure tensor. In any case, it enables us to (almost) eliminate the curvature from the integrability conditions.

Lemma 5.8. *An abundant conformally superintegrable system satisfies the curvature independent equation*

$$\boxed{\begin{matrix} k \\ l \end{matrix}} \left(P_{ijk}{}^{mn}{}_{,l} + P_{ijk}{}^{pq} P_{pql}{}^{mn} \right) = \boxed{\begin{matrix} i & j \end{matrix}} \boxed{\begin{matrix} m & n \end{matrix}} \boxed{\begin{matrix} k \\ l \end{matrix}} \left(P_{iak}{}^{ma}{}_{,l} + P_{iak}{}^{pq} P_{pql}{}^{ma} \right) \frac{g_j^n}{n+1}. \tag{5.17}$$

The proof is straightforward. By a tedious computation, the following corollary is confirmed.

Corollary 5.9. *For an abundant system, (5.9), (5.16) and (5.17) imply (4.5).*

Therefore, for abundant systems, the integrability conditions for the potential V , its trace-free conformal Killing tensors C_{ij} and their respective scalar parts W are equivalent to (5.9), (5.10), (5.16) and (5.17).

5.3. *Non-linear prolongation equations for the structure tensor.* We have found the prolongation (4.1) for the potential and the prolongation (5.1) for the trace-free conformal Killing tensors in a conformally superintegrable system. Now we show that these imply a third, non-linear prolongation for the structure tensor T_{ijk} , which expresses covariant derivatives of T_{ijk} polynomially in terms of T_{ijk} and the Ricci tensor R_{ij} .

Proposition 5.10. *For an abundant conformally superintegrable potential in dimension $n \geq 3$, the primary structure tensor T_{ijk} , decomposed according to (4.7) as*

$$T_{ijk} = S_{ijk} + \boxed{i \ j} \left(\bar{t}_i g_{jk} - \frac{1}{n} g_{ij} \bar{t}_k \right),$$

satisfies the following non-linear prolongation:

$$\nabla_i \bar{t}_j = \left(\frac{3}{(n-2)} R_{ij} + \frac{1}{3(n-2)} S_i{}^{ab} S_{jab} + \frac{1}{3} \bar{t}_i \bar{t}_j \right)_o + \frac{1}{n} g_{ij} \nabla^a \bar{t}_a \tag{5.18a}$$

$$\nabla^a \bar{t}_a = \frac{3}{2(n-1)} R + \frac{3n+2}{6(n-1)(n+2)} S^{abc} S_{abc} - \frac{(n-2)}{6} \bar{t}^a \bar{t}_a \tag{5.18b}$$

$$\nabla_l S_{ijk} = \frac{1}{18} \boxed{i \ j \ k} \left(S_{il}{}^a S_{jka} + 3 S_{ijl} \bar{t}_k + S_{ijk} \bar{t}_l + \left(\frac{4}{n-2} S_j{}^{ab} S_{kab} - 3 S_{jka} \bar{t}^a \right) g_{il} \right) \tag{5.18c}$$

Remarkably, the system (5.18) is the same as the one found for *properly* superintegrable systems in [KSV23], with the curvature term in (5.18a) replaced using (5.9). Here we leave the curvature term, in order to not re-introduce τ_{ij} into (5.18).

Proof of Proposition 5.10. The equations (5.18a) and (5.18b) are obtained from (5.16) by contraction. Now, using (5.17), define

$$E_{ijk}{}^{mn}{}_l = (n+1) \frac{\boxed{k}}{\boxed{l}} \left(P_{ijk}{}^{mn}{}_{,l} + P_{ijk}{}^{pq} P_{pql}{}^{mn} \right) - \boxed{i \ j} \boxed{m \ n} \frac{\boxed{k}}{\boxed{l}} \left(P_{iak}{}^{ma}{}_{,l} + P_{iak}{}^{pq} P_{pql}{}^{ma} \right) g^n{}_j,$$

which is pure trace.

Its trace components are given by $E_{kmnl}^{(1)} = E^a{}_{akmnl}$, $E_{ijkl}^{(2)} = E^a{}_{ja}{}^{mn}{}_l$, and $E_{ijnl}^{(3)} = E_{ija}{}^a{}_{nl}$, satisfying the relation $2(n-2) \left(E_{ijkl}^{(2)} - E_{kjil}^{(3)} \right) = (n^2 - 2n - 2) E_{lkji}^{(1)}$. This yields the trace-free part of (5.18c). The trace part is obtained from $E^{(1)}{}_{ija}{}^a$. \square

Notably, the system (5.18) is already conformally invariant, see Sect. 6.4 for details. Indeed, in (5.18a) and (5.18b) the terms involving \bar{t} absorb the transformation behaviour of R_{ij} under (3.24).

5.4. *The integrability conditions for abundant systems.* In addition to the integrability conditions for Killing tensors in abundant systems, we have two more equations, namely (5.9) and (5.10). Note that only (5.9) involves the secondary structure tensor τ_{ij} , as it allows us to express τ_{ij} in terms of the structure tensor T_{ijk} .

Lemma 5.11. (i) *The non-linear prolongation (5.18) implies Eq. (5.10).*

(ii) For abundant systems in dimension $n \geq 3$, the equations of the non-linear prolongation (5.18), together with

$$W_{ijkl} = \begin{matrix} \boxed{i} & \boxed{k} \\ \boxed{j} & \boxed{l} \end{matrix}^* S_{ik}^a S_{jla} = 0, \tag{5.19}$$

are equivalent to the integrability condition (5.13).

(iii) With (5.18), Eq. (5.9) becomes

$$3(n-2) \left(\overset{\circ}{P}_{ij} - \frac{1}{2} \tau_{ij} \right) = \left((n-2)(S_{ija} \bar{t}^a + \bar{t}_i \bar{t}_j) - S_i^{ab} S_{jab} \right)_{\circ}. \tag{5.20}$$

Proof. For the first part simply resubstitute (5.18) into (5.10). The proof of part (ii), namely of Equation (5.19), is analogous to that of Theorem 5.9 in [KSV23]. Finally, for part (iii), resubstitute (5.18) into Eq. (5.9). We obtain (5.20). \square

As an immediate consequence of (5.18) and (5.19), we obtain the following obstruction on the geometry underlying an abundant system.

Corollary 5.12. *Abundant conformally superintegrable systems can only exist on conformally flat manifolds.*

Proof. It follows immediately from (5.19) that an abundant conformally superintegrable system can only exist on a Weyl flat manifold. Therefore, for dimension $n \geq 4$, they can exist only on conformally flat manifolds. In dimension 2, any metric is conformally flat. We are therefore left with the case $n = 3$. Using standard scale, i.e. $\bar{t} = 0$, Eq. (5.18c) yields that

$$S_i^{ab} S_{jab} - \frac{3}{20} S^{abc} S_{abc} g_{ij}$$

is a Codazzi tensor. The claim then follows from the Weyl-Schouten Theorem. \square

6. Equivalence Classes of Abundant Superintegrable Systems

So far, we have considered conformally superintegrable systems whose underlying geometry is a (pseudo-) Riemannian manifold. We now turn towards conformal equivalence classes, i.e. towards c-superintegrable systems on conformal manifolds. For such systems, S_{ijk} is the conformally invariant structure tensor. According to (5.20), the secondary structure tensor τ_{ij} is determined by T_{ijk} and the Ricci curvature. Table 2 contrasts the setting of properly and conformally superintegrable systems as opposed to c-superintegrable systems.

6.1. Obstructions to the integrability of the non-linear prolongation. Consider the non-linear prolongation (5.18) of PDEs for \bar{t} and S_{ijk} . We now investigate the integrability conditions for this system. The prolongation Eq. (5.18) are non-linear in the components of S_{ijk} and \bar{t}_i . Therefore the Ricci conditions,

$$\begin{matrix} \boxed{l} \\ \boxed{m} \end{matrix} \nabla_m \nabla_l S_{ijk} = \begin{matrix} \boxed{i} & \boxed{j} & \boxed{k} \end{matrix} R_{ialm} S^a{}_{jk} \tag{6.1a}$$

Table 2. Synopsis of the main objects in proper, conformal and c-superintegrability

Comparison of abundant second order systems: proper versus conformal versus c-superintegrability			
Type	Proper superintegrability	Conformal superintegrability	c-superintegrability
Geometry metric g_{ij}	Pseudo-Riem. metric g_{ij}	Pseudo-Riem. metric \mathbf{g}_{ij}	Conformal
Constants of motion	(Proper) integrals	Conformal integrals	
Primary structure tensor	$T_{ijk} = S_{ijk} + \boxed{i \ j} \circ \bar{t}_i g_{jk}$		$S_{ijk} = \hat{T}_{ijk}$
Secondary structure tensor	$\tau_{ij} = 0$	Given by (5.20)	None (not conformally invariant)

$$\boxed{j \ k} \nabla_k \nabla_j \bar{t}_i = R_{iajk} \bar{t}^a, \tag{6.1b}$$

are necessary, but need not be sufficient for the integrability of (5.18). Sufficiency is guaranteed if not only the Ricci condition, but also all of its differential consequences are satisfied in a given point x_0 [Gol67]. We find that the integrability conditions, of which a priori there can be infinitely many, reduce to a single algebraic equation of the form (1.3).

Theorem 6.1. (i) *If Eq. (5.19) holds, then the integrability conditions (6.1) of (5.18) are satisfied.*

(ii) *Let M be a conformally flat, pseudo-Riemannian manifold of dimension $n \geq 3$ and let $x_0 \in M$ be a point on this manifold. Then any solution $\Psi_{ijk} = S_{ijk}(x_0)$ of (1.3) together with the arbitrary initial values $\bar{t}(x_0)$ and $\nabla \bar{t}(x_0)$ can be extended, in a neighborhood of x_0 , to solutions $S_{ijk}(x)$ and $\bar{t}(x)$ of the non-linear prolongation (5.18).*

Proof. (i) Since the integrability condition cannot depend on \bar{t} , we may w.l.o.g. perform a conformal transformation such that the transformed system is in standard scale, $\bar{t} = 0$. As a result, (5.18a) and (5.18b) determine the Ricci curvature tensor in this scale. We shall comment on this after finishing the proof. First, let us investigate the integrability condition for (5.18c), which by virtue of the aforementioned transformation has turned into

$$\nabla_l S_{ijk} = \frac{1}{18} \boxed{i \ j \ k} \circ \left(S_{il}^a S_{jka} + \frac{4}{n-2} S_j^{ab} S_{kab} g_{il} \right). \tag{6.2}$$

Using (5.18a), (5.18b) and Weyl-flatness (5.19), we replace the Riemann curvature in (6.1a) by a quadratic expression in S . Due to (5.19) in combination with the non-linear prolongation (5.18) and (5.20), Eq. (6.1a) is equivalent to the conformally invariant condition

$$\boxed{j \ k} \left(S^{abc} S_{ija} S_{kbc} \right) \circ = 0,$$

which is confirmed to be an algebraic consequence of (5.19) by way of contracting (5.19) with S . We have therefore established (5.19) as the only first order integrability condition of (5.18c). By an analogous computation we then confirm that all Ricci identities of (5.18) are satisfied if (5.19) holds. As explained earlier, however, the first order integrability conditions (6.1) need not be sufficient for the integrability of (5.18). We now proceed to show their sufficiency.

- (ii) In order to find sufficient, pointwise integrability criteria, higher order integrability conditions have to be taken into account. Concretely, all differential consequences of (5.19) need to be satisfied in a fixed point x_0 in order to allow us to extend S_{ijk} and \bar{t} such that the extensions satisfy (5.18) in a neighborhood of x_0 . Taking a covariant derivative of (5.19) and replacing derivatives of S_{ijk} by (6.2), we find an algebraic condition on S_{ijk} . Using (5.19) again, it is verified that this second order condition is an algebraic consequence of the first order one, i.e. (5.19). \square

If instead of a pseudo-Riemannian manifold we consider a conformal manifold, we obtain the following statement.

Corollary 6.2. *Let (M, \mathbf{g}) be a flat conformal manifold of dimension $n \geq 3$. A solution of (1.3) can be extended to $S_{ij}^k(x)$ in a neighborhood of a point $x_0 \in M$, such that for $g = \Omega^{-2}\mathbf{g}$ (where $\Omega > 0$ is a scalar function) the totally symmetric and trace-free tensor field $S_{ijk}(x)$ satisfies (6.2).*

Proof. Note that (1.3) is an invariant condition. The statement follows from statement (ii) in the theorem, making use of the fact that there exists a flat conformal scale choice, which removes the curvature from the non-linear prolongation system (5.18). If (1.3) holds in $x_0 \in M$ for this case, the system is integrable in a neighborhood of x_0 . Note that the specific scale choice is technical and irrelevant for the statement. \square

Corollary 6.3. *In dimension $n = 3$, (5.18) can be integrated for any $S_{ijk}(x_0)$.*

Proof. Note that (5.19) has Weyl symmetry and thus vanishes trivially in dimension $n = 3$. \square

6.2. The conformal factors between standard scale, proper scale and flat scale. In the standard scale $\bar{t} = 0$, Equations (5.18a) and (5.18b) become algebraic conditions on the curvature tensor. In contrast, if a flat scale exists, exactly the curvature terms disappear. In an arbitrary scale choice we have the following formula for the curvature, in terms of the conformally invariant tensor S_{ijk} and the conformal scale function σ .

Proposition 6.4. *The Ricci tensor satisfies*

$$R_{ij} = -\frac{1}{9} \left(S_i^{ab} S_{jab} + \frac{n}{3(n+2)} g_{ij} S^{abc} S_{abc} \right) + (n-2) \left(\frac{\hat{\nabla}_{ij}^2 \sigma}{\sigma} - \frac{4(n-1)}{n(n-2)^2} \frac{\Delta \sigma^{\frac{n-2}{2}}}{\sigma^{\frac{n-2}{2}}} g_{ij} \right) \tag{6.3}$$

where $\sigma = \exp(-\frac{1}{3}\bar{t})$ is the conformal scaling from Definition 4.7.

Proof. Solve (5.18a) and (5.18b) for the Ricci tensor. \square

On the right hand side of (6.3), the first term is invariant (up to a conformal factor), while the second term vanishes in standard scale. In a flat scale, $R_{ijkl} = 0$, and the left hand side vanishes. The function $\bar{t} = \rho$ arising from the primary structure tensor measures ‘how far’ the flat scale is from the standard scale. According to (4.12), the conformal scale factor between the two scales is

$$\theta := e^{-\frac{1}{3}\rho} \tag{6.4}$$

(note that we transform from the scale defined by h_{ij} back to the standard scale with metric g_{ij}). From (6.3) we see that the flat conformal scales θ are determined by S_{ijk} . The following lemma makes this explicit.

Lemma 6.5. *For any conformal class of abundant superintegrable systems there are functions $\theta > 0$ satisfying*

$$\overset{\circ}{\nabla}_{ij}^2 \theta = -\frac{(S_i^{ab} S_{jab})_{\circ}}{9(n-2)} \theta \tag{6.5a}$$

$$\Delta \theta^{1-\frac{n}{2}} = \frac{(n-2)(3n+2)}{36(n-1)(n+2)} S^{abc} S_{abc} \theta^{1-\frac{n}{2}}. \tag{6.5b}$$

where $\overset{\circ}{\nabla}^2$ is the flat, trace-free Hessian and Δ the flat Laplace operator, and where S_{ijk} is the conformal structure tensor in the flat scale.

Proof. Let the function $\bar{\tau} = \rho$ be the trace of the structure tensor after a conformal transformation to a flat scale. Then a conformal transformation with $\Upsilon = -\frac{\nabla \rho}{\rho}$ takes us back to the standard scale. Rewriting (5.18a) and (5.18b) in terms of θ , and decomposition of the result into its trace-free and trace parts, confirms the claim. \square

Solutions θ are not unique. Indeed, a positive constant scalar multiple of θ is again a solution, and, in general, more solutions can exist. For instance, in Table 3 the systems III and V are both in flat scale, and they are conformally equivalent, see [Cap14]. Two different solutions θ represent two different flat conformally superintegrable systems within the same conformal class. They share the same $\overset{\circ}{T}_{ij}{}^k = S_{ij}{}^k$ but the traces of T_{ijk} will be different unless the $\bar{\tau}$ differ only by an additive constant. In order to understand the space of solutions θ better, let us study (6.5a) further, ignoring the additional constraint (6.5b) for a moment. We find:

Lemma 6.6. *Equation (6.5a) has the linear prolongation*

$$\nabla_{ij}^2 \theta = -\frac{(S_i^{ab} S_{jab})_{\circ}}{9(n-2)} \theta + \frac{1}{n} g_{ij} \Delta \theta \tag{6.6a}$$

$$(\Delta \theta)_{,k} = \frac{n}{9(n-2)} S^{abc} S_{kab} \theta_{,c} + \frac{3n+2}{27(n-1)(n-2)} S_k{}^{ab} S_a{}^{cd} S_{bcd} \theta \tag{6.6b}$$

where ∇^2 is the flat Hessian and Δ the flat Laplace operator, and where S_{ijk} is the conformal structure tensor in the flat scale. The integrability conditions for (6.6) are equivalent to (5.19).

Proof. Equations (6.6) are obtained in formal analogy to (4.1), where formally $T \equiv 0$. Its integrability conditions are satisfied due to (5.19), as (6.6a) is a special case of (5.18a). \square

Equations (6.6) are a linear prolongation system for θ whose coefficients are determined by S_{ijk} . Its solutions θ lie in an (at most) $(n+2)$ -dimensional linear space, determined by the values of $\Delta \theta$, $\nabla \theta$ and θ in a fixed point. The additional constraint (6.5b) defines a quadric in this space.

6.3. Classifying the conformal classes of conformally superintegrable systems. In the previous section we have found algebraic integrability conditions whose form is the same for any conformally superintegrable system within a class (in the next section we reformulate them as equivariant conditions). As initial data we need to specify $\Psi_{ijk} = S_{ijk}(x_0)$. For a conformally flat geometry we may choose a flat metric, which

facilitates determining a solution for S_{ijk} . In order to reconstruct an abundant conformally superintegrable system from the initial data $\Psi_{ijk} = S_{ijk}(x_0)$, we recall that an abundant superintegrable system requires a conformally flat metric $g = \phi^2 h$ where h is the flat metric. Then we can use the following procedure.

- (i) Let $\Psi_{ijk} = S_{ijk}(x_0)$ be the initial data given in a point x_0 . Assume that Ψ_{ijk} solves the algebraic condition (1.3). The Ψ_{ijk} do not depend on ϕ , and if Ψ_{ijk} are solutions then so are $k\Psi_{ijk}$ for $k \neq 0$.
- (ii) We extend the initial data Ψ_{ijk} to a solution in a neighborhood of x_0 such that the non-linear prolongation (5.18) holds. This is possible by virtue of Corollary 6.2. For a concrete computation we should choose some conformal scale, and the flat scale is a reasonable choice. We then need to specify the initial data $\nabla\rho(x_0)$ and $\rho(x_0)$ in addition to Ψ_{ijk} .
- (iii) This yields T_{ijk} up to a conformal transformation. Integrating the Wilczynski Eq. (3.14) for V in the specific scale given by $\tilde{t} = \rho$, and computing $\mathbf{v} = e^{\frac{2\rho}{3}} V$, we find the conformally invariant potential as an $(n + 2)$ -parameter family of densities of weight -2 . This is the space \mathcal{V}^{\max} . The space \mathcal{C}^{\max} of conformal Killing tensors is similarly obtained by integration of (5.1).

Since all integrability conditions are satisfied generically, we find at least $\frac{1}{2} n(n+1)$ many linearly independent conformal integrals. We address their functional independence below in Lemma 6.7.

The procedure just outlined allows one to reconstruct an abundant c-superintegrable system from the given initial data and the knowledge of the underlying conformal metric up to the choice of the potential from \mathcal{V}^{\max} . We recall Assumption 3.6, but remark that with Lemma 6.7 below, we are able to restrict the space \mathcal{C}^{\max} in order to obtain $2n - 1$ functionally independent conformal integrals. Let us reinterpret the aforesaid in the light of classifying c-superintegrable systems. In [KSV23, Theorem 6.4] it is shown that the classification space for irreducible non-degenerate superintegrable systems on a (pseudo-) Riemannian manifold M with analytic metric is a quasi-projective subvariety $\mathcal{U} \subset G_{2n-1}(\mathcal{K}(M))$ in the Grassmannian of $(2n - 1)$ -dimensional subspaces in the space $\mathcal{K}(M)$ of Killing tensors on M . Since any c-superintegrable system admits at least one system in proper scale, it follows that the classification space of irreducible non-degenerate c-superintegrable systems with analytic metric is the quotient $\tilde{\mathcal{U}} = \mathcal{U}/\text{Conf}(M)$. For non-degenerate irreducible conformally superintegrable systems on analytic metrics, the classification space is a fibre bundle over $\tilde{\mathcal{U}}$.

The following lemma was proven for properly superintegrable systems in reference [KSV23]. We adapt it for conformal systems.

Lemma 6.7. *Let $\mathcal{C}^{(\alpha)}$ be $2n - 2$ linearly independent, trace-free conformal Killing tensors satisfying the integrability conditions (5.15) for (5.1), and (4.5) for (4.1). Then, in the linear space \mathcal{V}^{\max} of solutions V to Eq. (4.1), those V that give rise to functionally dependent integrals are confined to an affine subspace of \mathcal{V} with non-empty complement.*

Proof. Suppose the integrals (3.4) were functionally dependent. Then there is a function $\varphi : \mathbb{R}^{2n-2} \rightarrow \mathbb{R}$, non-zero in an open subset of its domain, such that

$$\varphi(F^{(1)}, \dots, F^{(2n-2)}) = 0 \tag{6.7}$$

This implies the infinitesimal condition

$$\sum_{\alpha=1}^{2n-2} \lambda_{(\alpha)} dF^{(\alpha)} = 0, \tag{6.8}$$

where

$$\lambda_{(\alpha)} = \frac{\partial \varphi}{\partial F^{(\alpha)}}(F^{(\alpha)}), \quad dF^{(\alpha)} = \frac{\partial F^{(\alpha)}}{\partial x^k} dx^k + \frac{\partial F^{(\alpha)}}{\partial p^k} dp^k.$$

By a direct computation we find

$$\frac{\partial F^{(\alpha)}}{\partial x^k} = C_{ij,k}^{(\alpha)} p^i p^j + V_{,k}^{(\alpha)}, \quad \frac{\partial F^{(\alpha)}}{\partial p^k} = 2C_{jk}^{(\alpha)} p^j.$$

Separating the components of (6.8) and substituting (3.7), we conclude

$$\sum_{\alpha} \lambda_{(\alpha)} \left(C_{ij,k}^{(\alpha)} p^i p^j + C_{jk}^{(\alpha)} V_{,j} + \omega_k^{(\alpha)} V \right) = 0 \tag{6.9a}$$

$$\sum_{\alpha} \lambda_{(\alpha)} C_{jk}^{(\alpha)} p^j = 0. \tag{6.9b}$$

Invoking (5.1), we obtain

$$C_{ij,k}^{(\alpha)} p^i p^j = \frac{2}{3} \left(T^a_{ij} C_{ka}^{(\alpha)} - T^a_{kj} C_{ia}^{(\alpha)} \right) p^i p^j - \frac{2}{3n} \left(T_k^{ab} C_{ab}^{(\alpha)} - t^a C_{ak}^{(\alpha)} \right) p^c p_c.$$

Multiplying with $\lambda_{(\alpha)}$ and summing over α , we find, using (6.9b) and the decomposition (4.7),

$$\sum_{\alpha} \lambda_{(\alpha)} C_{ij,k}^{(\alpha)} p^i p^j = \frac{2}{3} \sum_{\alpha} \lambda_{(\alpha)} \left(S^a_{ij} C_{ka}^{(\alpha)} - \frac{1}{n} t^a g_{ij} C_{ka}^{(\alpha)} - \frac{1}{n} S_k^{ab} C_{ab}^{(\alpha)} g_{ij} \right) p^i p^j.$$

Substituting this back into (6.9a), invoking (3.2), and using again the decomposition (4.7), we conclude

$$C_{ab} \eta^{kab} = 0, \tag{6.10}$$

where we use the abbreviations

$$C_{ab} := \sum_{\alpha} \lambda_{(\alpha)} C_{ab}^{(\alpha)}$$

and

$$\begin{aligned} \eta^{kab} &= g^{ka} \left(S^b_{ij} p^i p^j - \frac{1}{n} t^b p^c p_c + \frac{3}{2} V_{,b} - \frac{1}{n} \frac{(n+1)(n-2)}{(n-1)(n+2)} t^b V \right) \\ &\quad + \frac{1}{n} S^{kab} (V - p^c p_c). \end{aligned}$$

Note that $C(x_0) \neq 0$. Indeed, otherwise the $C_{ab}^{(\alpha)}(x_0)$ would be linearly dependent,

$$\sum_{\alpha} k_{(\alpha)} C_{ab}^{(\alpha)}(x_0) = 0, \quad k_{(\alpha)} = \lambda_{(\alpha)}(x_0).$$

Because of (5.1) the derivatives of $C_{ab}^{(\alpha)}$ are linearly dependent at x_0 , with the same constants $k_{(\alpha)}$. Iterated application of (5.1) to higher derivatives shows that the same is true for all higher derivatives. It readily follows that $\sum_{\alpha} k_{(\alpha)} C^{(\alpha)} = 0$ everywhere, which contradicts the linear independence of the conformal Killing tensors $C^{(\alpha)}$.

Now, for $x_0 \in M$, consider the mapping $\Xi : T_{x_0}^{\otimes 3} M \rightarrow T_{x_0} M$, given by contracting with C_{ab} ,

$$\Xi(\eta^{kab}) = C_{ab}\eta^{kab}.$$

By virtue of Eq. (6.10), we conclude that for any potential $V \in \mathcal{V}^{\max}$

$$\eta^{kab}(x_0) \in \ker \Xi.$$

Using the linearity of the kernel, we conclude further that

$$\left[\frac{3}{2} g^{kb} V_{,a} + \frac{1}{n} \left(S^{kab} - \frac{(n+1)(n-2)}{(n-1)(n+2)} g^{kb} t^a \right) V \right]_{x_0} \in \ker \Xi.$$

Choosing $V(x_0) = 0$, we obtain that

$$C_k^a V_{,a}(x_0) = 0$$

for any choice of $V_{,a}(x_0)$, contradicting that $C(x_0) \neq 0$. \square

Theorem 6.8. *Abundant conformally superintegrable Hamiltonians with their $(n+2)$ -parameter family of potentials, and identified under conformal transformations, are classified by (1.3).*

Proof. An abundant conformally superintegrable Hamiltonian with its $(n+2)$ -dimensional space \mathcal{V}^{\max} of all compatible potentials, can be recovered from S_{ijk} up to a conformal transformation of superintegrable systems and every abundant system satisfies (1.3). \square

6.4. Invariant formulation of the non-linear prolongation equations. In this section we express the non-linear prolongation equations (5.18) in a conformally invariant way.

Proposition 6.9. (i) *Equation (5.18a) is equivalent to*

$$\mathring{\mathbb{H}}_{ij}\sigma = -\frac{1}{9(n-2)} \left(S_i^{ab} S_{jab} \right)_\circ \sigma, \tag{6.11}$$

where $\sigma = e^{-\frac{1}{3}\bar{t}}$ and where $\mathring{\mathbb{H}}$ is the conformally invariant trace-free Hessian, defined by

$$\mathring{\mathbb{H}}_{ij} = \left(\nabla_{ij}^2 - P_{ij} \right)_\circ.$$

(ii) *Equation (5.18b) is equivalent to*

$$\mathbb{L}\sigma^{1-\frac{n}{2}} = -\frac{2}{9} \frac{3n+2}{n+2} S^{abc} S_{abc} \sigma^{1-\frac{n}{2}}, \tag{6.12}$$

where \mathbb{L} denotes the conformal Laplacian,

$$\mathbb{L} = -4 \frac{n-1}{n-2} \Delta + R.$$

(iii) Equation (5.18c) is equivalent to

$$\nabla_l^{\bar{\Gamma}} S_{ijk} = \frac{1}{3} \begin{bmatrix} i & j & k \end{bmatrix} \circ \left(S_{il}{}^a S_{jka} - \frac{4}{n-2} g_{kl} S_i{}^{ab} S_{jab} \right), \tag{6.13}$$

where $\nabla^{\bar{\Gamma}}$ is the conformally equivariant Weyl connection [Wey18] defined by

$$\nabla_i^{\bar{\Gamma}} \alpha_j = \nabla_i \alpha_j - \frac{w+1}{3} \bar{t}_i \alpha_j - \frac{1}{3} \bar{t}_j \alpha_i + \frac{1}{3} \bar{t}^a \alpha_a g_{ij}, \tag{6.14}$$

for α_j of conformal weight $w = -\frac{2}{3}$, i.e. $\alpha_j \mapsto \Omega^w \alpha_j$ under conformal transformations.

Proof. Parts (i) and (ii) are straightforward. For part (iii), apply (6.14) to S_{ijk} ,

$$\nabla_l^{\bar{\Gamma}} S_{ijk} = \nabla_l S_{ijk} - \frac{1}{18} \begin{bmatrix} i & j & k \end{bmatrix} \circ \left(3 S_{ijl} \bar{t}_k + S_{ijk} \bar{t}_l - 3 g_{kl} S_{ija} \bar{t}^a \right).$$

A direct computation indeed confirms that $\nabla^{\bar{\Gamma}}$ is invariant under conformal changes up to multiplication by a factor: The replacement rules are $g \rightarrow \Omega^2 g$ and $\bar{t} \rightarrow \bar{t} - 3 \ln |\Omega|$, as well as, respectively, $S_{ijk} \rightarrow \Omega^2 S_{ijk}$ or $\alpha_i \rightarrow \Omega^w \alpha_i$ with $w = -\frac{2}{3}$. \square

Two remarks are in place with regard to the above proposition. First, note that the conformal weights required for the trace-free conformal Hessian and the conformal Laplacian are different, leading to different powers of the conformal scale function. The second remark concerns the conformal invariance of the operators. Note that, under conformal transformations with rescale function Ω , we have

$$\sigma^{1-\frac{n}{2}} \rightarrow \Omega^{1-\frac{n}{2}} \sigma^{1-\frac{n}{2}},$$

and the conformal invariance of \mathbb{L} means

$$\mathbb{L} \circ \Omega^{1-\frac{n}{2}} = \Omega^{-1-\frac{n}{2}} \circ \mathbb{L},$$

which is consistent as $S^{abc} S_{abc} \rightarrow \Omega^{-2} S^{abc} S_{abc}$. Note that in the standard scale, i.e. for $\bar{t} = 0$ resp. $\sigma = 1$, Eq. (6.12) is an expression for the scalar curvature in terms of S_{ijk} , and (6.11) for the Schouten tensor.

Remark 6.10. To determine whether the non-linear system (5.18) is integrable, it is sufficient to know the invariants in (6.11), (6.12) and (6.13) as well as those in (6.6), which are constructed algebraically from S .

These invariants are

$$\begin{aligned} A_{ijkl} &= \begin{bmatrix} i & j & k \\ l \end{bmatrix} \circ (S_{il}{}^a S_{jka}) + \begin{bmatrix} i & j & k & l \end{bmatrix} \circ (S_{il}{}^a S_{jka}), \\ B_k &= S_k{}^{ab} S_a{}^{cd} S_{bcd}, \\ \Sigma_{ij} &= \begin{bmatrix} i & j \end{bmatrix} S_i{}^{ab} S_{jab}. \end{aligned}$$

The last of these invariants has the following geometric interpretation. Due to (5.18a),

$$\mathfrak{R}_{ij} = \frac{1}{9(n-2)} \mathring{\Sigma}_{ij},$$

where \mathfrak{P}_{ij} denotes the Schouten curvature for the standard scale. Its trace satisfies

$$\mathfrak{P}^a_a = \frac{3n + 2}{18(n - 1)(n + 2)} \Sigma^a_a.$$

The invariant B_k also has a geometric meaning. It is easy to show that

$$\mathfrak{P}^a_{a,k} = -\frac{(3n + 2) B_k}{27(n - 2)(n - 1)}, \tag{6.15}$$

and therefore $B_k = 0$ characterises the case when the standard scale system has constant scalar curvature. Note that B_k and Σ^a_a are not (differentially) independent.

6.5. Properly superintegrable systems on constant curvature manifolds. In the reference [KSV23] abundant properly superintegrable systems are studied. These systems satisfy $\tau_{ij} = 0$ due to Lemma 3.15 and thus (5.20) becomes

$$\left[(n - 2)(S_{ija}\bar{t}^a + \bar{t}_i\bar{t}_j) - S_i^{ab}S_{jab} \right]_{\circ} = 3 \mathring{R}_{ij}. \tag{6.16}$$

For a metric of constant sectional curvature, the right hand side of (6.16) vanishes. Moreover, [KSV23] shows that, for constant curvature, (6.16) and (5.18) imply that

$$S^{abc}S_{abc} - (n - 1)(n + 2)\bar{t}^a\bar{t}_a = 9R. \tag{6.17}$$

Theorem 6.11. *Let g be of constant sectional curvature. If $\tau_{ij} = 0$ and (6.17) hold, then*

$$\Delta\sigma^{n+2} = -2\frac{n + 1}{n - 1}R\sigma^{n+2} \tag{6.18}$$

where $\sigma = e^{-\frac{1}{3}\bar{t}}$ as in Definition 4.7.

Proof. If (6.17) holds, then $S^{abc}S_{abc}$ can be eliminated from Eq. (5.18b), yielding

$$\Delta\bar{t} = \frac{6(n + 1)}{(n - 1)(n + 2)}R + \frac{n + 2}{3}\bar{t}^a\bar{t}_{,a}.$$

In terms of $\sigma = e^{-\frac{1}{3}\bar{t}}$, this rewrites as (6.18). \square

For constant curvature spaces, Eq. (5.18b) thus becomes a Laplace eigenvalue problem, and a power of the scale function σ is an eigenfunction of Δ . For a flat manifold, (6.18) merely implies that σ^{n+2} is harmonic. On the round sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, we have spherical harmonics with the quantum number $\mu = n + 1$ satisfying

$$\mu(\mu + n - 1) := 2\frac{n + 1}{n - 1}Rr^2 = 2n(n + 1), \tag{6.19}$$

where the second equality follows from $R = \frac{n(n-1)}{r^2}$ with $r > 0$ denoting the radius of the sphere.

A close connection between the Helmholtz-Laplace equations and conformal superintegrability has been found in [KMS16, KKMP11]. Such links also appear in the present paper, although in different context: Earlier we have seen that on conformally flat spaces we find a scalar function $\theta^{1-\frac{n}{2}}$ satisfying the generalised Helmholtz Eq. (6.5b). Now we

have found (6.12), which is a conformally invariant generalised Helmholtz equation. In particular, in the case of proper superintegrability, the $(n + 2)$ -nd power of the conformal scale function satisfies the generalised Helmholtz Eq. (6.18). It is a proper Helmholtz equation in the case of constant scalar curvature.

We use Eq. (6.18) to study conformally equivalent properly superintegrable systems further.

Proposition 6.12. *Assume we are provided with an abundant second order properly superintegrable system on the sphere with the round metric g , which is conformally equivalent to a properly superintegrable system on flat space with the flat metric $h = \Omega^{-2}g$. Then the conformal factor σ on the sphere has to satisfy*

$$\Omega \left(\Delta \Omega - g(d \ln(\sigma^{n+2}), d\Omega) \right) + g(d\Omega, d\Omega) = 0. \tag{6.20}$$

Note that Ω is the conformal factor mediating between the standard and the spherical scale, while σ mediates between the spherical and the flat scale.

Proof. Due to (6.18), a properly superintegrable system on flat space must satisfy the condition $\Delta_{\text{flat}}(\Omega^{-(n+2)}\sigma^{n+2}) = 0$. A direct computation using (6.18) then shows

$$\Delta_{\text{flat}}(\Omega^{-(n+2)}\sigma^{n+2}) = (3n + 2) \sigma^{n+2} \Omega^{-(n+6)} \left[\Omega \Delta \Omega + \Omega \cdot^a \Omega_{,a} - \Omega \Omega \cdot^a (\ln \sigma^{n+2})_{,a} \right],$$

taking into account that

$$R = -2 \frac{n - 1}{\Omega^2} \left(\Omega \Delta \Omega - \frac{n}{2} \Omega \cdot^a \Omega_{,a} \right).$$

due to (3.24). \square

The following example generalises a result shown in [KKM06], which addresses the specific case of dimension 3, see also [Cap14]. The proof presented here is a relatively simple exercise, while with traditional methods the claim, if at all, cannot be obtained for arbitrary dimension in a straightforward fashion.

Example 6.13. (Generic system on the n -sphere) Consider the generic system on the n -sphere, with $n \geq 3$, introduced for dimension 3 in Example 4.14. In arbitrary dimension we have the metric

$$g = \sum_{m=1}^n \left(\prod_{k=2}^m \sin^2(\phi_{k-1}) \right) d\phi_m^2$$

with angular coordinates ϕ_1, \dots, ϕ_n . The superintegrable potential defining the generic system is

$$V = a_0 + \sum_{m=1}^n \left(\frac{a_m}{\cos^2(\phi_m) \prod_{k=2}^m \sin^2(\phi_{k-1})} \right) + \frac{a_{n+1}}{\prod_{k=1}^n \sin^2(\phi_k)}$$

For this system, σ^{n+2} satisfies the Laplace eigen-equation with quantum number $n + 1$,

$$\Delta \sigma^{n+2} = -2n(n + 1)\sigma^{n+2}.$$

Solutions of this equation span a vector space of dimension $(n + 2)^2$ whose basis is given by hyperspherical harmonics.

Concretely, for the generic system, we have

$$\sigma^{n+2} = \prod_{k=1}^n \cos(\phi_k) \sin^{n-k+1}(\phi_k),$$

which does *not* satisfy (6.20).

We have thus proven the following.

Proposition 6.14. *The generic system on the n -sphere is not conformally equivalent to a properly superintegrable system on flat space.*

We remark that the generic system on the n -sphere can be transformed into a proper superintegrable systems on flat space using Bôcher transformations or orbit degenerations [Cap14]. Opposed to conformal transformations, however, these are not equivalence relations on conformally superintegrable systems.

Proposition 6.15. *A non-degenerate properly superintegrable system on the n -sphere cannot be conformally equivalent to the harmonic oscillator.*

Proof. The harmonic oscillator has a vanishing structure tensor, $T_{ijk} = 0$. Therefore any system conformally equivalent to the harmonic oscillator must satisfy $S_{ijk} = 0$. Being proper, due to (6.11), (6.12) and (6.18), the system on the sphere satisfies

$$\overset{\circ}{\nabla}_{ij}^2 \sigma = 0, \quad \Delta \sigma^{1-\frac{n}{2}} = 0, \quad \Delta \sigma^{n+2} = -2 \frac{n+1}{n-1} R \sigma^{n+2} \neq 0,$$

where $\overset{\circ}{\nabla}^2$ and Δ are the trace-free Hessian and the Laplace-Beltrami operator on the sphere of constant scalar curvature $R \neq 0$. This system does not admit a solution. \square

Definition 6.16. We say that a c-superintegrable system is *basic* if it contains a member system that is an abundant properly superintegrable system on a manifold of constant curvature.

In reference [KSV23], it is proven that the structure tensor of an abundant second order properly superintegrable system on a constant curvature manifold of dimension $n \geq 3$ satisfies

$$T_{ijk} = \frac{1}{6} \boxed{i \mid j \mid k} \circ B_{,ijk} + \boxed{i \mid j} \circ \frac{1}{n+2} g_{ik} \left((\Delta B)_{,j} + \frac{2(n+1)}{n(n-1)} R B_{,j} \right). \quad (6.21)$$

where B is a scalar function, called its *structure function*. Obviously, the structure function is not unique, but according to [KSV23] it is determined up to addition of a solution δB of

$$\boxed{i \mid j \mid k} \circ \delta B_{,ijk} = 0, \quad (6.22a)$$

$$d \left(\Delta \delta B + \frac{2(n+1)}{n(n-1)} R \delta B \right) = 0. \quad (6.22b)$$

The next proposition allows one to extend structure functions to all basic c-superintegrable systems.

Proposition 6.17. *Consider two manifolds of constant curvature and with properly superintegrable systems that are conformally equivalent. Denote their metrics by g and $\tilde{g} = \Omega^2 g$. Assume the superintegrable systems with underlying metric g has the structure function B . Then the structure function \tilde{B} for the system of \tilde{g} can be chosen such that*

$$\tilde{B} = \Omega^2 B. \tag{6.23}$$

Proof. On a manifold with Hamiltonian $H = g^{ij} p_i p_j + V$, we infer from [KSV23] the following formula for B_{ijk} in an abundant constant-curvature system,

$$B_{ijk} = T_{ijk} + \frac{n+2}{n} g_{ij} \bar{t}_{,k} + \frac{1}{2(n-2)} \boxed{i \ j \ k} g_{ij} C_{,k}$$

where

$$C = \frac{n-2}{n+2} \Delta B + \frac{2(n+1)}{n(n-1)} RB - (n-2) \bar{t} \tag{6.24}$$

and

$$B_{ijk} = \frac{1}{6} \boxed{i \ j \ k} \left(B_{,ij} + \frac{4R}{n(n-1)} g_{ij} B \right)_{,k}$$

up to an irrelevant constant. By virtue of (3.19), we know the transformation rules

$$S_{ijk} \mapsto \tilde{S}_{ijk} = \Omega^2 S_{ijk} \quad \text{and} \quad \bar{t}_{,i} \mapsto \tilde{t}_{,i} - 3\Omega^{-1} \Omega_{,i},$$

and therefore that of B_{ijk} ,

$$\tilde{B}_{ijk} = \Omega^2 (B_{ijk} + \text{trace terms}). \tag{6.25}$$

Secondly, we also know by construction that $S_{ijk} = \boxed{i \ j \ k} \circ B_{,ijk}$, where on the right hand side we recall that comma denotes the covariant derivative. An analogous equation holds for \tilde{S}_{ijk} with a function \tilde{B} . Now, let us denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of g and \tilde{g} , respectively. Then, for the third derivatives,

$$\boxed{i \ j \ k} \circ \tilde{\nabla}_{ijk}^3 \tilde{B} = \Omega^2 \boxed{i \ j \ k} \circ \nabla_{ijk}^3 B \tag{6.26}$$

because of the invariance of S_{ijk} . A straightforward computation verifies that $\tilde{B} = \Omega^2 B$ satisfies (6.26). We have therefore confirmed that $\tilde{t} = \bar{t} - 3 \ln |\Omega|$ and $\tilde{B} = \Omega^2 B$ yield the correct structure tensors \tilde{S}_{ijk} and $\tilde{t}_{,i}$ for the conformally transformed manifold with Hamiltonian $\tilde{H} = \Omega^{-2} H$. The claim follows since structure functions are unique up to solutions of (6.22). \square

In [KSV23] the structure function B is defined for abundant properly superintegrable systems on constant curvature spaces only. We can now define it for any basic c-superintegrable system.

Corollary 6.18. *Abundant second order properly superintegrable systems on constant curvature spaces in dimension $n \geq 3$ are Stäckel equivalent if and only if their conformal densities $\mathbf{b} \in \mathcal{E}[-2]$,*

$$\mathbf{b} = B \det(g)^{\frac{1}{n}},$$

coincide up to a gauge transformation given by a solution of (6.22).

7. Application to Dimension Three

In the present section, we apply our framework to the 3-dimensional case. Non-degenerate second order conformally superintegrable systems in dimension 3 are classified in [KKM06,Cap14]. Also, it is known that all these systems are abundant [KKM05b]. In [KKM06] it has been established that any non-degenerate second order conformally superintegrable system is Stäckel equivalent to a non-degenerate second order abundant and proper system on a constant curvature geometry. We shall therefore restrict to the study of abundant systems for constant curvature metrics. All non-degenerate 3-dimensional systems are equivalent to these. Recall that in dimension 3, the non-linear condition (5.19) is void. Consequently, no further restriction exists on the tensor S_{ijk} . Hence any trace-free symmetric initial conditions $\Psi_{ijk} = S_{ijk}(x_0)$ in a point $x_0 \in M$ can be integrated for a structure tensor S_{ijk} of an abundant second order conformally superintegrable system, c.f. Corollary 6.3. Therefore the set of conformal equivalence classes of such systems is parametrised by the seven dimensional space of trace-free symmetric 3-tensors Ψ_{ijk} or, equivalently, harmonic ternary cubics $\Psi(\mathbf{p}) = \Psi_{ijk} p^i p^j p^k$. This parametrisation is equivariant with respect to the stabiliser subgroup of the point x_0 in the conformal group, which is isomorphic to $\text{SO}(3)$.

We comment that this agrees with the references [KKMP11,CK14,Cap14]. In [KKMP11] a 10-parameter classification space is mentioned, corresponding to a 10-dimensional representation of $\text{SO}(3)$ in [Cap14,CK14]. This 10-dimensional representation decomposes into two irreducible components of dimension 7 and 3, corresponding to S_{ijk} and \bar{t}_i in our framework. Note that the 3-dimensional component is restricted in the references, which corresponds to imposing proper superintegrability here. The 7-dimensional component is realised as the space of binary sextics and it is shown that no restrictions exist on this component. The relation to our framework is given by a known correspondence between harmonic ternary cubics and binary sextics as follows.

The adjoint action of $\text{SL}(2, \mathbb{C})$ on its Lie algebra $\mathfrak{sl}(2, \mathbb{C}) \cong \mathbb{C}^3$ preserves the Killing form. This defines a group morphism $\text{SL}(2, \mathbb{C}) \rightarrow \text{SO}(3, \mathbb{C})$ with kernel $\{\pm 1\}$ and hence an isomorphism $\text{SL}(2, \mathbb{C})/\mathbb{Z}_2 \rightarrow \text{SO}(3, \mathbb{C})$. The standard action of $\text{SL}(2, \mathbb{C})$ on \mathbb{C}^2 induces an $\text{SL}(2, \mathbb{C})$ -action on $S^2\mathbb{C}^2$ which descends to an $\text{SO}(3, \mathbb{C})$ -action, because the elements ± 1 act trivially. The latter induces an $\text{SO}(3, \mathbb{C})$ -action on $S^3 S^2\mathbb{C}^2$ which descends to $S^6\mathbb{C}^2$ under total symmetrisation $S^3 S^2\mathbb{C}^2 \rightarrow S^6\mathbb{C}^2$. Together with the isomorphism $S^2\mathbb{C}^2 \cong \mathbb{C}^3$, we obtain a morphism $S^3\mathbb{C}^3 \rightarrow S^6\mathbb{C}^2$ of $\text{SO}(3, \mathbb{C})$ -representations, giving an $\text{SO}(3, \mathbb{C})$ -equivariant morphism from the 10-dimensional space of ternary cubics to the 7-dimensional space of binary sextics. Its restriction to the 7-dimensional space of harmonic ternary cubics is non-trivial and hence an isomorphism by Schur's lemma. Explicitly, it is given by defining a sextic s from the cubic $\Psi(\mathbf{p})$ via

$$s(z, w) = \Psi(z^2 - w^2, 2zw, z^2 + w^2). \quad (7.1)$$

Note that the stabiliser subgroup contains only rotations. The action of translations is not linear and more involved.

Table 3 lists the established normal forms for 3-dimensional non-degenerate systems on flat space, see [KKM06,Cap14]. The functions B and \bar{t} are obtained as established in [KSV23], and due to Proposition 6.17 and Eq. (4.10), we may compute the corresponding functions for any 3-dimensional non-degenerate system conformally equivalent to one of the systems in Table 3. If in Table 3, we take the quotient under conformal equivalence for each example, then the systems III and V are identified (use e.g. Corollary 6.18) and

Table 3. The properly superintegrable systems in dimension $n = 3$ on flat space. The Systems III and V are conformally equivalent

Non-degenerate second order superintegrable systems in dimension $n = 3$, for Euclidean geometry $g = \sum_i dx_i^2$		B mod gauge terms	$-\frac{5}{3}i$ mod const.
Example	Potential mod constant		
<i>Regular systems</i> (linked with elliptic separation coordinates)			
“Generic system” I [2111]/ Smorodinsky-Winternitz I	$\sum_{i=1}^n \left(\frac{a_i}{x_i^2} + \omega x_i^2 \right)$	$-\frac{3}{2}(x^2 + y^2) \ln(x) + y^2 \ln(y) + z^2 \ln(z)$	$\sum_k \ln(x_k)$
System II [221]	$\omega(x^2 + y^2 + z^2) + a_1 \frac{x-iy}{(x+iy)^3} + a_2 \frac{1}{(x+iy)^2} + \frac{a_3}{z^2}$	$(x^2 + y^2) \left(\frac{1}{2} \ln(x^2 + y^2) - i \arctan\left(\frac{x}{y}\right) \right) + z^2 \ln(z)$	$\ln(z) + \ln(x^2 + y^2) - 2i \arctan\left(\frac{x}{y}\right)$
System III [23]	$\omega(x^2 + y^2 + z^2) + \frac{a_1}{(x+iy)^2} + \frac{a_2 z}{(x+iy)^3} + a_3 \frac{x^2 + y^2 - 3z^2}{(x+iy)^4}$	$(x^2 + y^2 + z^2) \left(\frac{1}{2} \ln(x^2 + y^2) - i \arctan\left(\frac{x}{y}\right) \right)$	$\frac{3}{2} \ln(x^2 + y^2) - 3i \arctan\left(\frac{x}{y}\right)$
System V [32]	$\omega(4x^2 + y^2 + z^2) + a_1 x + \frac{a_2}{(y+iz)^2} + a_3 \frac{y-iz}{(y+iz)^3}$	$(y^2 + z^2) \left(\frac{1}{2} \ln(y^2 + z^2) - i \arctan\left(\frac{y}{z}\right) \right)$	$\ln(y^2 + z^2) - 6i \arctan\left(\frac{y}{z}\right)$
System IV [311]/ Smorodinsky-Winternitz II	$\omega(4x^2 + y^2 + z^2) + a_1 x + \frac{a_2}{y^2} + \frac{a_3}{z^2}$	$y^2 \ln(y) + z^2 \ln(z)$	$\ln(y) + \ln(z)$
System VI [41]	$\omega(z^2 - 2(x-iy)^3 + 4x^2 + 4y^2) + a_1(2x + 2iy) - 3(x-iy)^2 + a_2(x-iy) + \frac{a_3}{z^2}$	$z^2 \ln(z) + \frac{1}{6}(x-iy)^3$	$\ln(z)$
System VII [5]	$\omega(x+iy) + a_1 \left(\frac{3}{4}(x+iy)^2 + \frac{a_2}{4} \right) + a_2((x+iy)^3 + \frac{1}{16}(x-iy) + \frac{3}{4}(x+iy)z) + \frac{a_3}{16}(5(x+iy)^4 + x^2 + y^2 + z^2) + 6(x+iy)^2 z$	$-\frac{1}{3}((x+iy)^2 + 6z)(x+iy)^2$	0
<i>Exceptional systems</i> (linked with degenerate separation coordinates)			
Isotropic oscillator O	$\omega^2 \sum_i x_i^2 + \sum_i a_i x_i$	0	0
System OO	$\frac{\omega}{2} \left(x^2 + y^2 + \frac{z^2}{4} \right) + a_1 x + a_2 y + \frac{a_3}{z^2}$	$z^2 \ln(z)$	$\ln(z)$
System A	$\omega((x-iy)^3 + 6(x^2 + y^2 + z^2)) + a_1((x-iy)^2 + 2(x+iy)) + a_2(x-iy) + a_3 z$	$-\frac{1}{18}(x-iy)^3$	0

we obtain a list of nine abundant c -superintegrable systems. Due to [Cap14] these are all abundant c -superintegrable systems in dimension 3, up to one exception. Indeed, from [KKM05b, KKM06, Cap14] it follows that there is one equivalence class of non-degenerate 3-dimensional superintegrable system that does not admit a representative *properly* superintegrable system on flat 3-space. This system is the generic system on the 3-sphere from Example 4.14, see also Proposition 6.14. Its conformally equivariant structure tensor S_{ijk} is generated by the structure function (up to gauge freedom)

$$B = -\frac{3}{2} \sum_k s_k^2 \ln(s_k).$$

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