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# Chiral Matter Multiplicities and Resolution-Independent Structure in 4D F-Theory Models

**Patrick Jefferson**[1], **Washington Taylor**[1], **Andrew P. Turner**[2]

[1] Department of Physics, Center for Theoretical Physics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139, USA. E-mail: pjeffers@mit.edu; wati@mit.edu

[2] Department of Physics and Astronomy, University of Pennsylvania, Philadelphia, PA 19104, USA. E-mail: turnerap@sas.upenn.edu

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**Abstract:** Motivated by questions related to the landscape of flux compactifications, we combine new and existing techniques into a systematic, streamlined approach for computing vertical fluxes and chiral matter multiplicities in 4D F-theory models. A central feature of our approach is the conjecturally resolution-independent intersection pairing of the vertical part of the integer middle cohomology of smooth elliptic Calabi–Yau fourfolds, relevant for computing chiral indices and related aspects of 4D F-theory flux vacua. We illustrate our approach by analyzing vertical flux backgrounds for F-theory models with simple, simply-laced gauge groups and generic matter content, as well as models with U(1) gauge factors. We explicitly analyze resolutions of these F-theory models in which the elliptic fiber is realized as a cubic in $\mathbb{P}^2$ over an arbitrary (e.g., not necessarily toric) smooth base, and confirm the independence of the intersection pairing of the vertical part of the middle cohomology for the resolutions we study. In each model, we find that vertical flux backgrounds can produce nonzero multiplicities for a spanning set of anomaly-free chiral matter field combinations, suggesting that F-theory geometry imposes no additional linear constraints on allowed matter representations beyond those implied by 4D anomaly cancellation.

## Contents

## 1. Introduction

F-theory [1–3] provides a powerful geometric framework for describing a large class of supersymmetric string theory vacua. In particular, F-theory can be used to describe a vast number of 4D $\mathcal{N} = 1$ supergravity theories with gauge symmetries.

Because F-theory provides a uniquely broad nonperturbative perspective on the set of supersymmetric string vacuum solutions, there are two rather fundamental questions about this set that can be explored fruitfully within this particular branch of string theory. First is the question of the extent to which F-theory, or string theory more generally, can provide a UV description of any low-energy field theory that has no known obstruction to coupling to quantum gravity; this question has been usefully framed as the problem of delineating the *swampland* [4,5] of apparently consistent low-energy effective theories of gravity not realized in string theory. Second is the question of how the gauge group, chiral matter content, and other physical features of the observed Standard Model of particle physics can be realized in string theory, and the extent to which this physics is typical or requires extensive fine tuning.

There has been a great deal of work on each of these questions in the context of F-theory over the last two decades (for some recent reviews see, e.g., [6,7]). However, neither question has been answered definitively.

In this paper, we investigate some aspects of F-theory flux backgrounds that are relevant for both of these questions. As part of our investigation, we bring together a variety of methods (some known and some new) to frame a systematic approach for characterizing chiral matter in broad classes of 4D F-theory models.

Many of the known methods we employ in our approach have been explored in different threads of the literature, as there has been extensive research on understanding how chiral matter arises from fluxes in 4D F-theory models. Chiral matter in F-theory GUT models was described locally in [8–10], and a more systematic description in terms of fluxes and global geometry was developed in [11–16], among others. Much of this work is reviewed in [17]; many of these papers compute the multiplicities of chiral matter by identifying geometric “matter surfaces” (i.e. specific holomorphic four-cycles in the elliptic Calabi–Yau fourfold) through which fluxes can be integrated to obtain the chiral indices, whereas by contrast [16] and some related works [18,19] indirectly compute the chiral indices by identifying fluxes through various holomorphic cycles with one-loop Chern–Simons couplings in 3D (which can be interpreted as linear combinations of the chiral indices). We follow the latter approach for explicit computations in this paper, though the resulting insights may shed light on some subtle aspects of the geometry of matter surfaces.

Our approach for studying 4D F-theory vacua offers computational and conceptual simplifications relevant for the two questions posed above. The computational simplification offered by our approach is that it combines the results of the previous work cited above with the techniques of [20] (used for computing intersection numbers) into a streamlined algorithm for analyzing chiral matter and vertical fluxes, which allows us to easily survey large families of F-theory flux vacua. We demonstrate the utility of our approach by analyzing numerous examples, some not previously studied in the literature, of flux vacua in models with fixed gauge group $\mathsf{G}$ over arbitrary smooth threefold base. Conceptually, our approach is simpler in that while previous work on chiral matter in 4D F-theory models has relied in an essential way upon specific choices of resolution of the singularities in the Weierstrass model defining the F-theory compactification, in this paper we take steps towards analyzing the chiral multiplicities, as well as the linear constraints they satisfy, in terms of (conjecturally) resolution-independent geometric structure intrinsic to the global elliptic Calabi–Yau fourfold.

The main resolution-independent structure that we make use of here is related to the intersection pairing on a particular subgroup of the middle cohomology $H^4(X, \mathbb{Z})$ of a smooth elliptic Calabi–Yau (CY) fourfold $X$ resolving the singular Weierstrass model. Specifically, we study the *nondegenerate* intersection pairing $M_{\text{red}}$ acting on the ("vertical") cohomology subgroup $H^{2,2}_{\text{vert}}(X, \mathbb{Z}) \subset H^4(X, \mathbb{Z})$ generated by products of divisors in $X$. The intersection pairing $M_{\text{red}}$ can be obtained by assembling the quadruple intersection numbers of $X$ into a matrix $M$ and removing its nullspace. While the quadruple intersection numbers of divisors are not generally independent of the choice of resolution $X$ of the singular Weierstrass model, we find evidence that for all models we study $M_{\text{red}}$ (and hence implicitly $M$ as well) is independent of the choice of $X$, up to an integral change of basis. Since $M_{\text{red}}$ encodes fluxes relevant for computing chiral matter multiplicities, we highlight the importance of $M_{\text{red}}$ as the primary geometric object of interest for analyzing chiral matter and vertical flux backgrounds in a manifestly resolution-independent manner. The apparent resolution-independence of $M_{\text{red}}$ and $M$ suggests that this intersection structure is in some sense an intrinsic mathematical feature of the singular elliptic CY fourfold that defines a 4D F-theory vacuum and may have a direct interpretation in this geometric language as well as in type IIB string theory, without any need for explicit resolution, although to our knowledge this statement has not been proven in the mathematical literature. There is perhaps a useful analogy to be made here: Just as the resolution-independent Dynkin diagram associated with a Kodaira singularity type encodes the resolution-invariant physics of the nonabelian gauge algebra of an F-theory compactification, this (conjecturally) resolution-independent part of the intersection structure encodes the resolution-invariant physics connecting vertical fluxes and chiral matter.[1]

The set of tools that this analysis provides for exploring the landscape of 4D F-theory flux vacua positions us to clarify aspects of the first question raised at the beginning of the paper. While 4D anomaly cancellation is satisfied by all F-theory constructions

[1] Note that other resolution-independent structures encoded in the intersection numbers of CY resolutions have been identified in the context of F-theory and M-theory compactifications. For example, the combined fiber diagrams (CFDs) of [21] appearing in non-flat resolutions of singular elliptic CY threefolds were shown to be manifestly flop-invariant. Furthermore, the intersection pairing between divisors and certain curve classes in smooth CY threefolds was shown to have invariant Smith normal form in [22]. We thank S. Schafer-Nameki for bringing these references to our attention.

that have been studied[2] and is expected to hold in all 4D $\mathcal{N} = 1$ supergravity theories that can be constructed in F-theory, it is unknown whether or not all anomaly-free families of chiral matter can be realized in F-theory. Interestingly, it turns out that in all cases we study this is indeed true, at least in the sense that for the most generic matter representations associated with a given gauge group, fluxes are available that produce combinations of massless chiral matter fields that span the linear space of anomaly-free matter representations. More specifically, we find in all cases that we study that the number of independent vertical flux backgrounds in $H^{\text{vert}}_{2,2}(X, \mathbb{Z})$ that lift to consistent F-theory flux backgrounds with unbroken gauge group—equivalently, the rank of $M_{\text{red}}$ minus the number of constraints required to preserve 4D local Lorentz and full gauge symmetry—is the same, and is in particular greater than or equal to the number of allowed independent families of anomaly-free chiral matter. Part of this result is to be expected: since the physics of any F-theory model is presumed resolution-invariant, given the relationship between chiral multiplicities and vertical flux backgrounds, it should follow that the number of independent vertical flux backgrounds corresponding to independent families of chiral matter multiplets is also a resolution-invariant property of the theory. Our results further suggest that this number is at least as large as the total number of linearly-independent families allowed by 4D anomaly cancellation. Since resolution-independence of the lattice pairing $M_{\text{red}}$ also implies that $M$ is resolution-independent, it may be possible to characterize part of the nullspace of $M$ in a canonical manner that is related to the 4D anomaly cancellation conditions. Since the nullspace of $M$ restricted to the subspace of 4D symmetry-preserving fluxes can be identified with the set of linear constraints (of which the 4D anomaly cancellation conditions must necessarily be a subset), this potentially points to a more systematic method for exploring possible swampland-like conditions obstructing the F-theory realization of certain families of chiral matter multiplets, or showing that no such additional linear conditions can exist, as we essentially conjecture here.

Regarding the second question raised at the beginning of this paper, one of the initial motivations was to analyze chiral matter in the family of $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$ models found in [27]. This model has three independent families of generic chiral matter fields that satisfy 4D anomaly cancellation, one of which corresponds to the matter content of the Minimal Supersymmetric Standard Model (MSSM). This seems to be the broadest class of F-theory models that have a tuned Standard Model–like gauge group,[3] and which naturally includes Standard Model–like matter. One subclass of these models arises naturally through a toric fiber ("$F_{11}$") construction [28], and has only the single family of chiral matter fields associated with the MSSM; chiral matter in some Standard Model–like $F_{11}$ constructions was recently intensively investigated in [29]. The approach developed here gives us a means to check whether F-theory models of the more general tuned $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$ type naturally contain chiral matter in the other two allowed families, or whether these are forbidden by string geometry for some reason and hence belong to the swampland. We find that indeed all three of the allowed chiral matter types are allowed; we briefly summarize these results here and report further on the details of this analysis in a forthcoming publication [30].

The structure of this paper is as follows: in Sect. 2, we give an overview of the main ideas, technical contributions, and results of the paper; in Sects. 3 and 4, we explore two complementary approaches to analyzing the set of cohomologically-distinct fluxes that

2 For example, analyses of such conditions were carried out in [23–26].

3 That is, a gauge group that is directly tuned in the Weierstrass model, as opposed to one that arises from breaking a larger GUT group or that is imposed as a generic feature of the F-theory base geometry.

preserve 4D local Lorentz and gauge symmetry, where the two approaches differ by the order in which the symmetry constraints and equivalence relations in (co)homology are imposed; Sect. 5 reviews the strategy we use for determining the precise relationship between chiral indices $\chi_{\mathsf{r}}$ and vertical fluxes, which exploits their relationship to one-loop Chern–Simons couplings appearing in the 3D low-energy effective action describing the F-theory Coulomb branch; in Sect. 6, we use our approach to study various F-theory models with simple gauge group $\mathsf{G}_{\mathrm{na}}$; in Sect. 7, we discuss the generalization of our analysis to models with gauge group $\mathsf{G} = (\mathsf{G}_{\mathrm{na}} \times \mathrm{U}(1))/\,\Gamma$, using the $(\mathrm{SU}(2)\times\mathrm{U}(1))/\mathbb{Z}_2$ and $(\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_6$ models to illustrate various aspects of the analysis; finally, Sect. 8 contains concluding remarks and future directions. A number of technical results related to e.g., anomaly cancellation, intersection theory, and resolutions of singular Weierstrass models, are collected in the appendices.

## 2. Overview

In this section, we give an overview of the steps needed to systematically describe a class of 4D F-theory models with a given gauge group and ultimately to compute the chiral matter content from vertical fluxes using our approach. In particular, we try to make clear how various techniques in the existing literature are integrated into our approach, and where this paper makes novel contributions. The current state of knowledge for many parts of this analysis is reviewed in more detail in a transcription of Weigand's excellent TASI lectures [17].

We are interested in finding a general formulation of the chiral matter multiplicities for a variety of F-theory constructions with different gauge groups, in a way that can be expressed succinctly in terms of the geometry of the base of the F-theory compactification and a choice of fluxes. In particular, for a given choice of gauge group and generic[4] matter representations, we are interested in identifying closed form expressions for the chiral matter multiplicities in a base-independent[5] (and resolution-independent) fashion. Expressions of this type have been found previously in the literature using related but distinct combinations of techniques for various gauge groups, such as SU(5) [14], $E_6$ [31], $\mathrm{U}(1) \times \mathrm{U}(1)$, $(\mathrm{SU}(5) \times \mathrm{U}(1) \times \mathrm{U}(1))/\mathbb{Z}_5$ [32], and $(\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)^2)/\mathbb{Z}_6$ [23] (see also [33]).

At a very heuristic level, the analysis can be described as follows: for any specific choice of gauge group, it should be possible to identify a multi-parameter family of Weierstrass models that describes F-theory models over an arbitrary base with that gauge group and generic matter. A resolution $X$ of any of the corresponding CY fourfolds gives rise to a well-defined set of intersection numbers, which can be organized into a matrix $M$ containing the intersection pairing $M_{\mathrm{red}}$ acting on the set of homology classes $H_{2,2}^{\mathrm{vert}}(X, \mathbb{Z})$. The intersection pairing is relevant for computing fluxes through certain homology classes dubbed "matter surfaces" that encode the multiplicities of chiral matter fields. The chiral matter multiplicities that are fixed by the choice of gauge flux and the intersection numbers can also be related directly to the 3D physics arising from a circle reduction of the F-theory model. While the choice of resolution and its associated intersection numbers are not unique, it should be possible in general to describe the multiplicities of chiral matter fields in the 4D limit in a resolution-independent fashion that depends only on the intersection structure of the compactification base and a choice

[4] See Sect. 2.3 for a precise definition of the notion of "generic matter" in F-theory compactifications.

[5] By "base-independent", we mean in a manner that does not rely on a specific choice of base. Clearly, the choice of base can change the physics of the F-theory vacuum.

of fluxes in an appropriate basis. One of the key ingredients in this paper is the identification of a conjecturally resolution-independent piece of the intersection structure of $X$, namely $M_{\text{red}}$, that is relevant for understanding the chiral multiplicities.

We now describe each of the steps in this procedure in a bit more detail, framing the analysis of the remainder of the paper.

*2.1. Selection of the base.* The first step in choosing an F-theory compactification is the choice of complex threefold base $B$. From the IIB string theory point of view, the 10D IIB theory is compactified on $B$, which we take here to be a compact Kähler threefold. Note that $B$ need not be a CY manifold, i.e., the canonical class $K$ of $B$ need not be trivial, though $-K$ must be an effective class. The F-theory model [1–3] is described by a Weierstrass model

$$y^2 = x^3 + fx + g, \tag{2.1}$$

defining an elliptic CY fourfold $X_0$ with base $B$, where $f$, $g$ are sections of the line bundles $\mathscr{O}(-4K)$, $\mathscr{O}(-6K)$, respectively. In general, the CY fourfold $X_0$ has singularities associated with loci in the base where the elliptic fiber degenerates. Degenerations over codimension-one loci in the base are associated with the gauge group of the F-theory model and degenerations over codimension-two loci are associated with matter. F-theory is frequently analyzed as a limit of M-theory on a smooth resolution $X$ of $X_0$, but the physics should in principle be independent of resolution as discussed further in Sect. 2.4.

The number of possible bases $B$ is quite large. The primary constraint is that $B$ cannot contain a divisor $\Sigma$ (codimension-one algebraic surface) that has a normal bundle that is so negative that $(f, g)$ need to vanish to orders $(4, 6)$ on $\Sigma$. When such a divisor exists, the singularity structure of the total space of the elliptic fibration goes beyond the classification of Kodaira [34] and Néron [35]; there is no smooth CY resolution and the resulting geometry lies at infinite distance in the moduli space of compactifications.[6] A large range of elliptic CY fourfolds have been studied in the literature, see, e.g., [36–38]. Restricting to the simple case of toric $B$, the number of possible bases has been shown by explicit construction to be at least $10^{755}$ [39] and is estimated through Monte Carlo analysis to be of order closer to $10^{3000}$ [40]. Many of these bases have codimension-two loci where $(f, g)$ vanish to orders $(4, 6)$. These codimension-two loci are generally associated with nonperturbative massless excitations in the low-energy 4D theory, see, e.g., [41–43]; in 6D, such excitations are generally associated with a superconformal sector in the theory [44,45], and while there are some parallel aspects of 4D F-theory models [46] the structure of these sectors in four space-time dimensions is less well understood.

Much of the detailed analysis of chiral matter in 4D F-theory models has been done in the context of toric geometry. One advantage of toric bases is that there are many powerful and simple tools for computing resolutions, intersection numbers, and other relevant features of toric varieties that extend to many elliptic CY fourfolds over toric bases that can be described as hypersurfaces in toric varieties. At least in the case of elliptic CY threefolds with relatively large Hodge numbers over complex surface bases, toric constructions seem to give a good representative sample of the set of possibilities [47], although for 4D F-theory models with chiral matter, some features such as

[6] Note that there can be higher-codimension $(4, 6)$ singularities without a crepant (CY) resolution (see, e.g., [36]); these geometries, however, lie at finite distance in moduli space and seem physically relevant as F-theory compactifications.

GUT breaking are not easily seen in purely toric contexts (see, e.g., [48,49]). Toric geometry has been used with great efficacy in many examples in the literature, e.g., [28,50–52]. By contrast, our analysis employs resolution techniques developed to study Weierstrass models defined over an arbitrary (toric or non-toric) base for certain gauge group and matter structures—see, e.g., [53,54]. Resolutions of general classes of elliptic fourfolds including non-toric constructions have also been considered in, e.g., [23,32], using somewhat different approaches.

*2.2. Non-degeneracy of the intersection pairing on the base.* For any threefold base $B$, there is a triple intersection form $D_\alpha \cdot D_\beta \cdot D_\gamma$ on the space $H_{2,2}(B, \mathbb{Z}) \cong H^{1,1}(B, \mathbb{Z})$ of divisors on $B$. One feature of a general F-theory threefold base that we will use in various places is the observation that for any such smooth $B$, the triple intersection form is nondegenerate, in the sense that for any divisor $A = A^\alpha D_\alpha$, there exists some $D'$, $D''$ for which $A \cdot D' \cdot D'' \neq 0$, so that there exists a curve $C$ whose class is of the form $C = D' \cdot D''$ with $C \cdot A \neq 0$. For a toric base, this follows from the standard result that the ring in intersection theory generated by the divisors (i.e., the Chow ring) generates the full linear space of homology classes $H_{i,i}(B, \mathbb{Z})$, combined with Poincaré duality, which states that the space of curves $H_{1,1}(B, \mathbb{Z})$ is dual to the space of divisors under the intersection product. More generally, the stated result follows from the hard Lefschetz theorem (see, e.g., [55]), which asserts that $J : H^{1,1}(B, \mathbb{Q}) \to H^{2,2}(B, \mathbb{Q})$ is an isomorphism over $\mathbb{Q}$ for any compact Kähler manifold $B$, where $J$ is a Kähler class (equivalently, a cohomology class Poincaré dual to the pullback of the hyperplane section in a projective realization of $B$ when $B$ is a smooth complex projective variety.) This nondegeneracy plays a useful role in our analysis of the structure of fluxes and the intersection numbers of CY fourfolds that can be realized as elliptic fibrations over $B$.

*2.3. Weierstrass model: gauge group and matter content.* A central feature of a 4D F-theory model is the gauge group $\mathsf{G}$ realized in the effective 4D theory constructed by compactifying F-theory on a Weierstrass model defined over a given threefold base $B$. In general, $\mathsf{G}$ is encoded in the Kodaira type of the singularities in the elliptic fibration over various divisors in $B$.

The gauge group $\mathsf{G}$ can arise either because it is forced from the geometry of $B$ or through explicit tuning of the Weierstrass model. In the first case, geometrically "non-Higgsable" gauge group factors can arise when certain divisors in $B$ have normal bundles that are sufficiently negative that $(f, g)$ are forced to vanish to orders at least $(1, 2)$ over those divisors [56,57]. Virtually all of the large number of threefold bases that support elliptic CY fourfolds have multiple non-Higgsable gauge group factors [39,40,58]. The gauge group can also be tuned by choosing a Weierstrass model where $f$, $g$, and the discriminant $\Delta = 4f^3 + 27g^2$ vanish to the appropriate orders over a given divisor in $B$ necessary to guarantee a desired nonabelian gauge factor. U(1) gauge factors can also be non-Higgsable [59–61] or tuned, and are subtler, as they rely on the global structure of the Mordell–Weil group of rational sections.

The allowed matter content in a given theory depends on the more detailed structure of singularities in the elliptic fibration over codimension-two loci in $B$. There is a natural distinction in F-theory between "generic" matter content for given $\mathsf{G}$, associated with the simplest codimension-two singularity types, and more exotic matter representations that can be realized through more complicated singularities. This notion of genericity can be made precise in 6D, where generic matter content is associated with the branch of moduli space of the largest dimension for a fixed $\mathsf{G}$ and anomaly coefficients that are not "too large" [62]. For given $\mathsf{G}$, in general we expect that there is a universal construction of a multi-parameter family of Weierstrass models that realize the full geometric moduli space of elliptic CY varieties over an arbitrary base that realize $\mathsf{G}$ and have generic matter content for that gauge group. Such "universal" $\mathsf{G}$ models were studied in [27],[7] where the universal $(\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_6$ Weierstrass model with generic matter was constructed, and a moduli-counting argument was introduced to check that a universal $\mathsf{G}$ model is fully parameterized; other universal $\mathsf{G}$ models with generic matter representations include the Tate-tuned models with various nonabelian gauge factors (see, e.g., [63]), and the Morrison–Park universal U(1) model [64]. In general, the parameters of the universal Weierstrass construction for a given $\mathsf{G}$ include discrete parameters associated with the divisor classes supporting the gauge factors and continuous parameters associated with complex structure moduli of the associated $X_0$. These discrete parameters, along with the canonical class of the base, form what for us will be the characteristic data of the F-theory model. While the definition of "generic" matter representations is most clear in 6D theories, the same representations are naturally generic for 4D F-theory constructions in terms of the dimension of the geometric moduli space and the complexity of the singularities; universal F-theory models with fixed $\mathsf{G}$ and these generic matter representations such as the Tate-tuned and Morrison–Park models take the same parameterized form in 6D and in 4D theories.

In this paper, we work with various universal $\mathsf{G}$ models with generic chiral matter content in 4D, meaning that we consider multi-parameter Weierstrass models with $\mathsf{G} = \mathrm{SU}(N)$, $\mathrm{SO}(4k+2)$, $\mathrm{E}_6$, $(\mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_2$ over arbitrary threefold bases $B$ that need not be toric. Note that even over toric $B$, only some universal $\mathsf{G}$ models have known toric constructions with fibers that can be constructed torically as elliptic curves within toric 2D fibers. For example, some $\mathrm{SU}(N)$ models can be constructed in this way torically, and a subset of the universal $(\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_6$ models can be so constructed torically, but not all.

In this paper we focus on the degrees of freedom of 4D F-theory models encoded in the Weierstrass model through the axiodilaton of type IIB theory and the fluxes that come from the 3-form field $C_3$ in the M-theory picture. There can be additional degrees of freedom such as "T-branes" [65] encoded in the world-volume dynamics of the 7-branes of the IIB theory; in the analysis here we do not consider the matter or other structures that these degrees of freedom may produce in the effective 4D $\mathcal{N} = 1$ supergravity theory.

*2.4. Resolution and intersection numbers.* As described above, we are interested in general families of Weierstrass models with particular structures of codimension-one and codimension-two singularities. Given a Weierstrass model in such a family, the standard approach taken for understanding F-theory models is to resolve the singular Weierstrass geometry into a smooth elliptic CY manifold and analyze the theory as a limit of M-theory, see, e.g., [17,66]. While this approach gives the best understood way of analyzing the physics of the resulting 4D F-theory model, the physics should be independent of the specific resolution; indeed, from the nonperturbative type IIB point of view, the physics should be well-defined directly in the context of the singular Weierstrass model. Note further that there can be terminal singularities at higher (i.e., $\geq 2$) codimension that do

[7] In that paper these universal Weierstrass model constructions were referred to as "generic"; here we change terminology to "universal" to avoid confusion with other uses of the term generic.

not admit a CY resolution at all; in many cases, these can be present without any apparent significant effects on the resulting physical model [67,68]. One of the broad motivations for the methods we explore in this paper is to find ways of characterizing the resolution-independent aspects of the physics of 4D chiral matter. Given a Weierstrass model defined by a singular elliptic CY fourfold, there are, in general, multiple distinct resolutions $X$ that preserve the CY structure of the singular fourfold. For most of the analysis here we do not concern ourselves with terminal singularities or higher-codimension (4, 6) loci where there is no "flat" resolution $X$ respecting the elliptic structure (although, see Sect. 2.9 for some further comments on codimension-three (4, 6) loci).[8] Since simple and manifestly resolution-independent methods are currently lacking for a complete analysis of physics like chiral matter, we use specific resolutions for explicit calculations and try to extract and identify the resolution-independent parts of the results.

One of the key features of a resolved CY fourfold $X$ is the set of quadruple intersection numbers of divisors $\hat{D}_I$.[9] Expanding an arbitrary set of divisors $\hat{C}$, $\hat{D}$, $\hat{E}$, $\hat{F}$ in an appropriate basis $\hat{D}_I$, we may write

$$\hat{C} \cdot \hat{D} \cdot \hat{E} \cdot \hat{F} = C^I D^J E^K F^L (\hat{D}_I \cdot \hat{D}_J \cdot \hat{D}_K \cdot \hat{D}_L). \tag{2.2}$$

These intersection numbers can also describe aspects of the dual pairing associated with Poincaré duality between divisors (codimension-one algebraic surfaces) and curves (codimension three) in the fourfold, e.g., when a curve is realized as an intersection of three divisors. As we discuss in further detail below, the quadruple intersection numbers of $X$ are not in general resolution-independent, although some are resolution-independent. A natural basis for the divisors in a fourfold with an elliptic fibration structure is suggested through the Shioda–Tate–Wazir [71] formula

$$h^{1,1}(X) = 1 + h^{1,1}(B) + \operatorname{rk} \mathsf{G}, \tag{2.3}$$

where the 1 comes from the zero section of the elliptic fibration, the second term comes from pullbacks of divisors in the base to the total space of the elliptic fibration, and the last term contains Cartan divisors of nonabelian gauge factors and additional sections for the free abelian part of $\mathsf{G}$. In view of this decomposition, and following standard notation in the literature (e.g., [18]), we use the following conventions for indices $I$:

- $I = 0$ denotes the zero section
- $I = a$ denotes a generating section associated to a non-Cartan U(1) gauge factor
- $I = \alpha$ denotes a divisor $\hat{D}_\alpha = \pi^*(D_\alpha)$ realized as a pullback of a divisor in the base

[8] In this paper, when we refer to a CY fourfold $X$ as a "resolution" of a singular Weierstrass model $X_0$ obtained by a sequence of blowups, we mean that $X$ is at least a partial resolution that is smooth through codimension-three loci in $B$ (for elliptic fibrations whose geometry does not force tunings leading to additional singular fibers beyond those suggested by the Weierstrass model), but may nonetheless contain singularities over special codimension-three loci (i.e., points) in $B$. These codimension-three singularities do not affect the results of our analysis, hence we ignore them and permit this abuse of terminology. Note that when $B$ is restricted to be a twofold $B^{(2)}$, these codimension-three singularities are absent; in the models we consider here there are also no codimension-two terminal singularities, hence the resulting CY spaces $X$ are in general genuine resolutions over $B^{(2)}$. A comprehensive analysis of the network of genuine CY fourfold resolutions (i.e., through codimension-three in $B$) using the physics of the low-energy effective 3D $\mathcal{N} = 2$ description of the F-theory Coulomb branch is presented in [69] (see also [70]); in [69], particular attention is given to the geometry of the singular elliptic fibers over codimension-two and codimension-three loci.

[9] We use hats to denote divisors in the fourfold $X$, as opposed to divisors in the base $B$; a glossary of notation commonly used throughout the paper is given in Appendix J.

- $I = i_s$ denotes a Cartan divisor of a nonabelian factor $\mathsf{G}_s \subset \mathsf{G}$.

To make contact with the low-energy gauge theoretic description of the Coulomb branch physics in the M-theory duality frame, we sometimes convert to the "physical" basis $\hat{D}_{\bar{I}} = \sigma^I_{\bar{I}} \hat{D}_I$ (see (B.5) for the definition of $\sigma^I_{\bar{I}}$), where

- $\bar{I} = \bar{0}$ denotes the $\mathrm{U}(1)_{\mathrm{KK}}$ divisor
- $\bar{I} = i = (\bar{a}, i_s)$ collectively denotes all other U(1) divisors.

Many of the intersection numbers are independent of resolution and are known for a general smooth elliptically-fibered CY. For example [18], when $\hat{D}_0$ is a holomorphic section and there are no abelian factors, one can use the fact that the quadruple intersection numbers of $X$ can be pushed down to the base $B$[10] to write

$$\pi_*(\hat{D}_0^4) = K^3, \quad \pi_*(\hat{D}_0 \cdot \hat{D}_\alpha \cdot \hat{D}_\beta \cdot \hat{D}_\gamma) = D_\alpha \cdot D_\beta \cdot D_\gamma. \tag{2.4}$$

The above intersection numbers are clearly independent of the geometric properties of the elliptic fibration, and only depend on the intersection numbers of the base. Other intersection numbers, particularly those with three or four indices of type $i_s$ or $a$, depend not only on the matter content of the theory but also on the particular resolution; see Appendix B for a more comprehensive discussion of the structure of these intersection numbers for rather general classes of elliptic fibrations.

One issue that arises in certain situations, for instance when $\mathsf{G} = (\mathsf{G}_{\mathrm{na}} \times \mathrm{U}(1)^k)/\Gamma$, is that there may not be a holomorphic section of the Weierstrass model; this occurs, in particular, when the section associated with the identity element of the Mordell–Weil group intersects with one or more sections associated with generators of U(1) factors over the discriminant locus. In some of these cases, the procedure of resolving the singular CY geometry and analyzing various physical properties in the dual M-theory frame on $X$ is more easily accomplished in models in which the elliptic fiber is realized as a general cubic in $\mathbb{P}^2$ rather than the usual Weierstrass model (e.g., the general cubic is used to define the resolutions studied in [28].)

In this paper, we adapt the mathematical techniques of [20] to compute intersection numbers of resolutions of models in which the elliptic fiber is presented as a general cubic in $\mathbb{P}^2$. These mathematical techniques enable us to evaluate the quadruple intersection numbers in terms of linear combinations of the triple intersection numbers of an arbitrary base $B$, much in the same manner as described above:

$$\pi_*(\hat{D}_I \cdot \hat{D}_J \cdot \hat{D}_K \cdot \hat{D}_L) = W_{IJKL} = W^{\alpha\beta\gamma}_{IJKL} D_\alpha \cdot D_\beta \cdot D_\gamma. \tag{2.5}$$

With the aid of a symbolic computing tool, the action of the above map $\pi_*$ can easily be used to compute intersection numbers (and other characteristic numbers[11]) of $X$ in terms of rational expressions involving divisor classes of the ambient fivefold in which

[10] Strictly speaking, the intersection products $\hat{D}_I \cdot \hat{D}_J \cdot \hat{D}_K \cdot \hat{D}_L$ live in the Chow ring of the variety $X$ (the Chow ring encodes the intersection structure in a smooth algebraic variety; see Sect. 3.1 for further discussion). However, since the pushforward is computed with respect to the canonical projection $\pi : X \to B$, the resulting intersection product $\pi_*(\hat{D}_I \cdot \hat{D}_J \cdot \hat{D}_K \cdot \hat{D}_L)$ lives in the Chow ring of $B$. For simplicity of notation we are often sloppy and omit explicit pushforward maps such as $\pi_*$ when the appropriate Chow ring is otherwise clear from the context.

[11] For example, the same methods have been used to compute the generating function of the Euler characteristics of smooth (up to codimension two) elliptic $n$-folds resolving singular Weierstrass models with gauge symmetry $\mathsf{G}$; see [20] for further details.

$X$ is realized as a hypersurface.[12] The techniques used to evaluate the pushforward map $\pi_*$ are described in detail in Appendix E. Note that the physical results obtained from a given resolutions $X$ should, and do in the cases we analyze, match with the expected physics from any other resolution, including the structure of the tensors $W^{\alpha\beta\gamma}_{IJKL}$ (see Appendix B) appearing on the right hand side of Eq. (2.5).

*2.5. Fluxes, consistency conditions, and linear algebra.* In order to obtain a chiral matter spectrum in 4D, it is necessary to turn on a nontrivial flux background, which in the M-theory duality frame corresponds to a nontrivial profile for the 4-form field strength $\mathrm{d}C_3$ whose key properties we now summarize.

*2.5.1. Flux conditions* It was argued in [72] that the cohomology class $G = \mathrm{d}C_3 \in H^4(X, \mathbb{R})$ satisfies a shifted quantization condition[13]

$$G_{\mathbb{Z}} = G - \frac{c_2(X)}{2} \in H^4(X, \mathbb{Z}). \tag{2.6}$$

To preserve supersymmetry, $G$ must satisfy

$$G \in H^{2,2}(X, \mathbb{R}) \cap H^4(X, \mathbb{Z}/2) \tag{2.7}$$

along with the primitivity condition

$$J \wedge G = 0, \tag{2.8}$$

where $J$ is the Kähler form of $X$ [73,74]. Finally, there is a tadpole condition [75–77] requiring that the net number of M2-branes (which are dual to D3-branes in the F-theory frame) is non-negative to ensure a stable vacuum,

$$N_{\mathrm{M2}} = \frac{\chi}{24} - \frac{1}{2}\int_X G \wedge G \in \mathbb{Z}_{\geq 0}, \tag{2.9}$$

where $\chi = \int_X c_4$ is the Euler characteristic of $X$; the integrality of $N_{\mathrm{M2}}$ follows from Eq. (2.6), as explained in [72].

Additional conditions must be imposed to ensure that $G$ dualizes to a suitable F-theory flux background. To preserve Poincaré symmetry, we require that the following fluxes vanish: [77],

$$\int_{S_{0\alpha}} G = 0, \quad \int_{S_{\alpha\beta}} G = 0. \tag{2.10}$$

Furthermore, to ensure that the flux background does not break the gauge symmetry $\mathsf{G}$ in the 4D limit, it is necessary to impose the conditions (see, e.g., [8])

$$\int_{S_{i\alpha}} G = 0, \tag{2.11}$$

[12] It should be possible to straightforwardly adapt these techniques to models in which the elliptic fiber is realized as a complete intersection in $\mathbb{P}^n$.

[13] It turns out that $c_2$ belongs to $H^{2,2}_{\mathrm{vert}}(X, \mathbb{Z})$ as defined in Eq. (2.13).

where we emphasize that $i$ collectively indexes all divisors dual to gauge U(1)s on the F-theory Coulomb branch. Note that since $G$ may be a half-integral cohomology class, in principle it seems there could be circumstances under which no flux satisfies these conditions; in all cases we have considered here, however, the integrals Eq. (2.10) and Eq. (2.11) take integer values and the constraints can be satisfied, and this is likely always true for the Poincaré symmetry constraints—in particular, the results of [78] show that the fluxes appearing in Eq. (2.10) are always integer-valued for any $G$ on a smooth elliptic fourfold. We describe these conditions explicitly in some families of models with gauge groups SU(2), SU(5) in Sect. 6.3 and Sect. 6.4.

*2.5.2. Vertical fluxes and intersection pairing* In addition to the usual Hodge decomposition, the cohomology group $H^4(X)$ admits the finer orthogonal decomposition [49,79]

$$H^4(X,\mathbb{C}) = H^{2,2}_{\text{vert}}(X,\mathbb{C}) \oplus H^{2,2}_{\text{rem}}(X,\mathbb{C}) \oplus H^4_{\text{hor}}(X,\mathbb{C}). \tag{2.12}$$

As in much of the previous literature, for the most part in this paper we focus on integral "vertical" fluxes, i.e., flux backgrounds $G$ belonging to the subgroup $H^{2,2}_{\text{vert}}(X,\mathbb{R}) \cap H^4(X,\mathbb{Z})$ spanned by wedge products of cohomology classes in $H^{1,1}(X)$, where $H^{1,1}(X)$ has a basis $\text{PD}(\hat{D}_I)$ of harmonic (1, 1)-forms on $X$ dual to divisors $\hat{D}_I$.[14] More precisely, for the purposes of this paper, we focus on fluxes belonging to the sublattice $H^{2,2}_{\text{vert}}(X,\mathbb{Z}) \subset H^{2,2}(X,\mathbb{R}) \cap H^4(X,\mathbb{Z})$, which we define as follows:

$$H^{2,2}_{\text{vert}}(X,\mathbb{Z}) := \text{span}_{\mathbb{Z}}(H^{1,1}(X,\mathbb{Z}) \wedge H^{1,1}(X,\mathbb{Z})), \tag{2.13}$$

which is to say that a "vertical" class for us means a class belonging to the integer linear span of forms $\text{PD}(\hat{D}_I) \wedge \text{PD}(\hat{D}_J)$. Note that it is in principle possible for there to exist integral vertical cohomology classes that do not lie in $H^{2,2}_{\text{vert}}(X,\mathbb{Z})$ as given by this definition and therefore it is possible that our definition excludes some consistent vertical flux backgrounds that could be included by permitting non-integer coefficients.

As reviewed in [17], vertical fluxes play a primary role in determining the chiral matter content of a 4D F-theory compactification, and for the most part we ignore components in $H^4_{\text{hor}}(X,\mathbb{C}) \oplus H^{2,2}_{\text{rem}}(X,\mathbb{C})$ for flux backgrounds.[15] Denoting by $\Lambda_S$ the $[h^{1,1}(X)(h^{1,1}(X)+1)/2]$-dimensional integral lattice spanned by the surfaces $S_{IJ} = \hat{D}_I \cap \hat{D}_J$ (here treated as formally independent objects for each $IJ$ pair), we can conveniently encode the flux integrals over vertical surfaces via the intersection pairing matrix

$$\begin{aligned} &M\colon \Lambda_S \times \Lambda_S \to \mathbb{Z}, \\ &M_{(IJ)(KL)} = S_{IJ} \cdot S_{KL} = \int_X \text{PD}(S_{IJ}) \wedge \text{PD}(S_{KL}). \end{aligned} \tag{2.14}$$

In the second line above, $\cdot$ indicates the intersection pairing on homology. As we explain in more detail in Sect. 3.1, $M$ can thus be viewed as an integral bilinear form on vectors

[14] Note that when $c_2(X)$ is not even, $G$ is a half-integer class; we neglect this refinement in our notation in various places, essentially restricting to the simplified cases where $c_2(x)$ is even, except when it is directly relevant to the discussion.

[15] The possibility that Poincaré duality and the inclusion of fluxes in $H^4_{\text{hor}}(X,\mathbb{C}) \oplus H^{2,2}_{\text{rem}}(X,\mathbb{C})$ may give a broader class of possible matter multiplicities is explored in Sect. 2.8, and more specifically in the case of the SU(5) model with generic matter in Sect. 6.4.

$\phi = \phi^{IJ} S_{IJ} \in \Lambda_S$, giving $\Lambda_S$ the structure of an integral lattice.[16] Following [16], we define the fluxes

$$\Theta_{IJ} = \int_{S_{IJ}} G = \int_X G \wedge \mathrm{PD}(S_{IJ}) = M_{(IJ)(KL)} \phi^{KL}, \tag{2.15}$$

where $\phi \in \Lambda_S$ represents the components of the Poincaré dual of the flux background $G$ expanded in a collection of classes $S_{IJ}$. Throughout the paper, we refer to $\phi$ as a "flux background". (Note that when $c_2(X)$ is not even, the possible values of $\phi$ are shifted appropriately by a half-integer lattice element $\mathrm{PD}(c_2(X)/2) \in \Lambda_S/2$.) In terms of the above notation, the symmetry constraints (2.10) and (2.11) can be expressed as

$$\Theta_{I\alpha} = 0, \tag{2.16}$$

where we note that for vertical fluxes the above conditions are both necessary and sufficient to preserve 4D gauge symmetry and local Lorentz symmetry. The fluxes $\Theta_{IJ}$ can be written as linear functions of the flux backgrounds $\phi^{IJ}$, with coefficients given by the pushforwards of the intersection numbers of divisors of $X$. In explicit computations we can for certain resolutions of $\mathsf{G}$ models (defined over arbitrary $B$) formally solve the equations Eq. (2.16) for a subset of the $\phi$s, so that the nonzero $\Theta$s that encode the chiral matter multiplicities are again linear functions of the remaining $\phi$s with coefficients that are polynomial functions of the intersection numbers.

Imposing the symmetry constraints is equivalent to restricting the flux background $\phi^{IJ}$ to lie in a sublattice $\Lambda_C \subset \Lambda_S$. For a given resolution $X$, the sublattice $\Lambda_C$ can be viewed as the lattice of $\phi^{IJ}$ whose image under $M$ is the sublattice of $\Theta_{IJ}$ satisfying (2.16), which in turn encodes the multiplicities of 4D chiral matter, as we review in more detail in the following subsection. We emphasize that while the intersection numbers entering the matrix $M$ are generically resolution-dependent, we expect that the allowed chiral multiplicities must be resolution-independent, consistent with the expectation that every set of M-theory vacua defined by a set of distinct resolutions $X$ lifts to a common set of F-theory vacua on a singular elliptic CY fourfold $X_0$.

What we have described above is essentially the standard perspective on analyzing chiral matter in 4D F-theory flux vacua. We now discuss a complementary perspective that illuminates additional aspects of the analysis. We begin with the observation that not all the cycles $S_{IJ}$ are independent in homology [23,24]. This implies that $M$ generically has a nontrivial nullspace, where the elements of the nullspace represent equivalence relations in homology, and hence the rank of the matrix $M$ is equal to the dimension $h^{\mathrm{vert}}_{2,2}(X)$ of $H^{\mathrm{vert}}_{2,2}(X, \mathbb{Z})$. We denote by $M_{\mathrm{red}}$ the *nondegenerate* intersection pairing

$$M_{\mathrm{red}} \colon H^{\mathrm{vert}}_{2,2}(X, \mathbb{Z}) \times H^{\mathrm{vert}}_{2,2}(X, \mathbb{Z}) \to \mathbb{Z}, \tag{2.17}$$

where we describe $M_{\mathrm{red}}$ explicitly as a matrix by restricting the action of the matrix $M$ to the quotient of $\Lambda_S$ by the nullspace, which projects $\Lambda_S$ to the quotient lattice $H^{\mathrm{vert}}_{2,2}(X, \mathbb{Z})$.

While the reduced matrix $M_{\mathrm{red}}$ produces the same results for the multiplicities of chiral matter as $M$ does in the procedure described above, $M_{\mathrm{red}}$ is a simple and useful tool for analyzing various aspects of fluxes and chiral matter (e.g. the number of independent

[16] While sometimes physicists refer to any subgroup of $\mathbb{R}^n$ that is isomorphic to $\mathbb{Z}^n$ as a lattice without reference to any associated bilinear form, throughout this paper we reserve the term lattice for a free abelian group of finite rank with a symmetric bilinear form. A standard reference for mathematical properties of lattices is [89].

families of chiral matter combinations can in principle be inferred from the rank of $M_{\mathrm{red}}$, without having to explicitly compute chiral indices.) Furthermore, we provide evidence suggesting that while the full set of quadruple intersection numbers are not in general resolution-independent, $M_{\mathrm{red}}$ is independent of the choice of $X$ up to a change of basis. Among other things, this implies that $M_{\mathrm{red}}$ makes the resolution-invariance of the chiral matter multiplicities manifest in terms of a canonical subspace of homology classes that parametrize the space of vertical fluxes lifting to consistent F-theory flux backgrounds. Since the dimension of the null space of $M$ is just $(h^{1,1}(X)(h^{1,1}(X)+1)/2) - h^{\mathrm{vert}}_{2,2}(X)$, resolution-independence of $M_{\mathrm{red}}$ also implies resolution-independence (up to an integral change of basis) of $M$; this argument is spelled out more explicitly in Sect. 4.2 and Appendix H. Previous work [23] has implemented the quotient by the nullspace taking $\Lambda_S$ to $H^{\mathrm{vert}}_{2,2}(X)$ in explicit resolutions by using methods related to the Stanley-Reisner ideal; here we carry out this quotient directly on the matrix $M$ computed for various $\mathsf{G}$ models defined over arbitrary $B$. To our knowledge, the observation that $M_{\mathrm{red}}$ and $M$ are resolution-invariant has not been made previously either in the mathematics or physics literature.

Both the symmetry constraints and the projection into nontrivial homology classes have a simple geometric interpretation, and it is clear that the composition of these two operations in either order leads to the same sublattice equipped with a nondegenerate bilinear form. Given the original lattice $\Lambda_S$ with bilinear form $M$, the constraints (2.10) and (2.11) restrict $\phi$ to the sublattice $\Lambda_C$. If $\dim(\Lambda_S) = m$ and there are $k$ (non-null) constraints, then $\dim(\Lambda_C) = m - k$. Imposing the homological equivalence relation $\phi \sim \psi \Leftrightarrow M(\phi - \psi) = 0$ (i.e., quotienting out $\Lambda_C$ by the nullspace $V$ of $M$, which satisfies $V \subset \Lambda_C$) gives us the lattice of independent vertical flux backgrounds $\Lambda_{\mathrm{phys}} = \Lambda_C/\sim$, with the nondegenerate bilinear form $M_{\mathrm{phys}}$ that is the restriction of $M_C$ to $\Lambda_C/\sim$. We can describe this procedure explicitly in a given basis for $\Lambda_S$. In particular, if we define an $m \times (m-k)$ matrix $C$ to have columns given by a set of generators of the lattice $\Lambda_C \subset \Lambda_S$, then $C : \mathbb{Z}^{m-k} \to \Lambda_S$ describes a lattice embedding of $\mathbb{Z}^{m-k}$ into $\Lambda_C \subset \Lambda_S$, and $M_C := C^{\mathrm{t}} M C$ is the restriction of $M$ to $\Lambda_C$ that results from imposing the symmetry constraints, expressed in a natural basis for $\Lambda_C$. The resulting form of $M_C$ plays an important role in our analysis, although with the simplest choice of coordinates there are some subtleties with integrality conditions that we discuss in more detail in Sect. 2.8 and Sect. 3.2. Alternatively, we can first impose the homological equivalence relation on $M$, leading to the reduced intersection pairing matrix $M_{\mathrm{red}}$, and then impose the symmetry constraints. The preferred order in which to perform these two operations depends on the circumstances. Nevertheless, these two operations lead to the same result when both are performed either over $\mathbb{Z}$ (more generally, over $\mathbb{R}$), so the analysis can be carried out in either order—see Fig. 1. Sections 4 and 3 essentially describe different perspectives on our analysis that arise from performing these two different orders of operation.

Each of these two approaches has value for understanding the structure of chiral matter multiplicities; explicit computation of $M_C$ in many cases provides an efficient method to compute the chiral indices as a function of the characteristic data, while the structure of $M_{\mathrm{red}}$ gives us insight into resolution-independence and the discretization structure of allowed chiral matter multiplicities.

To maintain clear control of the discrete quantization of allowed fluxes $\Theta$, some care is needed. While every integer quantized choice of flux background $\phi \in \Lambda_S$ must correspond to an integer vertical flux background $G$ by Poincaré duality, in some circumstances (i.e., when non-vertical fluxes are included) there may exist quantized flux

$$\begin{tikzcd} \Lambda_S \arrow[r] \arrow[d] & \Lambda_C \arrow[d] \\ H_{2,2}^{\text{vert}}(X,\mathbb{Z}) \arrow[r] & \Lambda_{\text{phys}} \end{tikzcd}$$

**Fig. 1.** Our approach to analyzing vertical fluxes and chiral matter involves the interplay of two commuting operations on the lattice of vertical flux backgrounds $\Lambda_S$ spanned by the vertical cycles $S_{IJ} = \hat{D}_I \cap \hat{D}_J$. One operation is the restriction of the $\Lambda_S$ to the sublattice $\Lambda_C$ of backgrounds satisfying the symmetry constraints (2.10) and (2.11) necessary to preserve 4D local Lorentz and gauge symmetry. The other operation is the restriction of $\Lambda_S$ to the vertical homology $H_{2,2}^{\text{vert}}(X,\mathbb{Z})$ by quotienting $\Lambda_S$ by homologically trivial cycles. Performed in either order, the composition of these two operations lead to the same sublattice $\Lambda_{\text{phys}}$ of consistent F-theory flux backgrounds that preserve gauge symmetries in the low-energy effective 4D $\mathcal{N}=1$ description of the F-theory compactification. We present evidence suggesting that $H_{2,2}^{\text{vert}}(X,\mathbb{Z})$ equipped with its symmetric bilinear form $M_{\text{red}}$ is resolution-independent up to an integer change of basis

backgrounds $G$ that give rise to more general fractional choices of $\phi$. We restrict attention in our analysis primarily to fluxes corresponding to integrally quantized $\phi \in \Lambda_S$ (except for the possible half-integer shift from $c_2(X)/2$), though we discuss in some places the possibility of more general fluxes, which may in turn lead to a larger set of possible chiral matter multiplicities. These issues are discussed further in Sect. 2.8.

*2.6. Chiral matter multiplicities.* The by now standard result in the F-theory literature [13–16,80] is that for any complex representation $\mathsf{r}$ of the gauge group $\mathsf{G}$, the chiral index is

$$\chi_{\mathsf{r}} = n_{\mathsf{r}} - n_{\mathsf{r}^*} = \int_{S_{\mathsf{r}}} G, \tag{2.18}$$

where the homology class $S_{\mathsf{r}} \in H_4(X,\mathbb{Z})$ is a "matter surface". For local F-theory matter, it is expected [14,81,82] that any cycle belonging to the class $S_{\mathsf{r}}$ is topologically the fibration of an irreducible component $C_w$ of the elliptic fibers (where $w \in \mathsf{r}$ is any weight) over an irreducible codimension-two component (i.e., a "matter curve", not to be confused with a matter surface) $C_{\mathsf{r}} \subset \{\Delta^{(2)} = 0\}$ of the discriminant locus $\{\Delta = 0\} \subset B$, associated to the local matter transforming in the quaternionic representation $\mathsf{r} = \mathsf{r} \oplus \mathsf{r}^*$. In practice the flux of $G$ through $S_{\mathsf{r}}$ is computed by way of Poincaré duality, i.e.

$$\int_{S_{\mathsf{r}}} G = \int_X G \wedge \text{PD}(S_{\mathsf{r}}), \tag{2.19}$$

and hence the analysis of vertical flux backgrounds we describe depends crucially on being able to identify an appropriate cohomology class $\text{PD}(S_{\mathsf{r}})$ dual to $S_{\mathsf{r}}$ (note that the choice of $\text{PD}(S_{\mathsf{r}})$ in general may depend on the choice of resolution $X$). With one exception [13], in all known examples $\text{PD}(S_{\mathsf{r}})$ can be characterized as an element of $H^{2,2}_{\text{vert}}(X)$ [17] (or equivalently $S_{\mathsf{r}} \in H_{2,2}^{\text{vert}}(X)$.) However, the precise definition is subtle and it is unclear that $S_{\mathsf{r}}$ always has non-trivial components in $H^{2,2}_{\text{vert}}(X)$; see Appendix G for a discussion about this subtlety in the context of certain resolutions of the SU(5) model. Our default assumption in this paper is that $S_{\mathsf{r}}$ always contains a non-trivial component in $H^{\text{vert}}_{2,2}(X, \mathbb{Z})$. Note that in cases we study where $c_2(X)$ is not an even class and hence (because of Eq. (2.6)) the flux $G$ is a half-integer class in cohomology, the chiral indices Eq. (2.18) nonetheless take integer values. This is presumably guaranteed for all physically allowed configurations though we do not know of a complete proof.

As discussed above, the fiber of $S_{\mathsf{r}}$ is a curve $C_w$ that an M2-brane wraps leading to 3D matter characterized by BPS central charges $C \cdot \hat{D}_i = w_i$ (here $\hat{D}_i$ are Cartan divisors associated to U(1) gauge factors characterizing the low-energy physics and $w_i$ are the Dynkin coefficients of the weight $w$). Thus by Poincaré duality, we can construct in homology the class $C$ associated with a particle labeled by any weight of any representation; note that to utilize Poincaré duality in this context one must project out, e.g., the fiber and zero section, as described in [83]. We describe an explicit example of a matter curve and some related quantization subtleties in the simple case of SU(2) in Sect. 6.3. Unfortunately, however, there is no universal approach known to explicitly construct $S_{\mathsf{r}}$ in homology simply from topological and representation theoretic considerations, without using a specific resolution. The issue is that, as reviewed in [17], the image of $S_{\mathsf{r}}$ in $\Lambda_C$ does not simply contain components of the form $S_{i\alpha} = \hat{D}_i \cap \hat{D}_\alpha$; indeed, these must be projected out to preserve gauge invariance. Rather, the image of $S_{\mathsf{r}}$ in $\Lambda_C$ must also include components in the $S_{ij}$ directions (as demonstrated explicitly by Eq. (4.8)), and since the intersection properties of the classes $S_{ij}$ in general may depend on the choice of $X$, it follows that the precise form of $S_{\mathsf{r}}$ is not known from first principles in a resolution-independent fashion. While the approach described in this paper does not rely on explicit computation of the matter surfaces, we remark that despite the apparent resolution dependence of $S_{\mathsf{r}}$, the resolution independence of $M_{\text{red}}$ suggests that there exists a natural description of $H^{\text{vert}}_{2,2}(X, \mathbb{Z})$ in terms of which the vertical components of the matter surfaces for any given anomaly-free combination of representations realized in F-theory can be characterized in a resolution-independent fashion.

Before addressing the explicit computation of chiral matter indices, we recall that, as described above, after both imposing the symmetry conditions and quotienting by the homology relations encoded in the nullspace of $M$, we are left with a set of independent flux backgrounds $\phi$, associated with a nontrivial (rk $M_{\text{phys}}$)-dimensional sublattice $M_{\text{phys}}\Lambda_{\text{phys}} \subset M\Lambda_S$ that for a given $X$ encodes the 4D chiral matter multiplicities $\chi_{\mathsf{r}}$. Thus, even without explicitly computing $S_{\mathsf{r}}$, we can expect in such cases for there to be an $m$-dimensional space of $\chi_{\mathsf{r}}$ (where $m \leq \text{rk}\, M_{\text{phys}}$) that can be realized by turning on different combinations of $\phi^{IJ}$. Since F-theory constructions are expected to always be consistent at low energies, these combinations of $\chi_{\mathsf{r}}$ should always satisfy 4D anomaly cancellation. Therefore, we expect that the rank of $M_{\text{phys}}$, or equivalently the rank of $M_{\text{red}}$ minus the number of independent constraints in (2.16), places an upper bound on the number of linearly independent combinations of chiral matter fields available in the theory.

As is evident from the above discussion, to explicitly compute $\chi_{\mathsf{r}}$ one must either identify $S_{\mathsf{r}}$, or proceed by more indirect means. Here we proceed in the latter fashion and follow a strategy similar to that of [19] (see also [32,84]), which exploits the following relationship between the set of $\Theta_{IJ}$ satisfying the symmetry constraints and linear combinations of $\chi_{\mathsf{r}}$ given by 3D one-loop Chern–Simons couplings appearing on the

Coulomb branch of the 4D F-theory vacuum compactified on a circle:

$$\Theta_{\bar{I}\bar{J}} = -\Theta^{3\mathrm{D}}_{\bar{I}\bar{J}}, \quad \bar{I} = \bar{0}, i \tag{2.20}$$

where

$$\Theta_{\bar{I}\bar{J}} := \sigma^{I}_{\bar{I}}\sigma^{J}_{\bar{J}}\Theta_{IJ}. \tag{2.21}$$

(Recall that the index $\bar{0}$ denotes the abelian Kaluza–Klein gauge field associated to the compact circle and we use the index $i$ to collectively denote all other $\mathrm{U}(1)$ gauge fields; see (B.5) for the precise definition of $\sigma^{I}_{\bar{I}}$.) In the above equation the couplings $\Theta^{3\mathrm{D}}_{\bar{I}\bar{J}}$, which receive contributions from integrating out all massive fermions on the Coulomb branch, can be expressed as linear combinations

$$\Theta^{3\mathrm{D}}_{ij} = x^{\mathsf{r}}_{ij}\chi_{\mathsf{r}}, \quad x^{\mathsf{r}}_{ij} \in \mathbb{Q}. \tag{2.22}$$

For every resolution that satisfies our default assumption that each matter surface has a vertical component, the above linear system can be inverted, which allows us to then write an explicit formula

$$\chi_{\mathsf{r}} = x^{ij}_{\mathsf{r}}\Theta^{3\mathrm{D}}_{ij}. \tag{2.23}$$

We expect the set of allowed chiral multiplicities $\chi_{\mathsf{r}}$ that can be realized for integer flux backgrounds $\phi^{ij}$ to be independent of the choice of resolution $X$, up to a choice of basis for $\phi^{ij}$.

*2.7. Linear constraints and anomaly cancellations.* The anomaly conditions for any 4D theory are linear relations on $\chi_{\mathsf{r}}$ (these conditions are reviewed in Appendix A). There are also linear relations that automatically hold on $\Theta_{IJ}$ by virtue of the nullspace of $M$. Connections between the anomaly relations and these geometric conditions were identified in [24] (see also [25,85]). Our finding here is that, in all cases we consider, the linear relations on $\Theta^{3\mathrm{D}}_{ij}$ imposed by the nullspace conditions and symmetry constraints are precisely the same as the anomaly conditions, so that not only does geometry encode the anomaly conditions, but there are also no further linear constraints coming from F-theory on the set of allowed chiral multiplicities, and thus fluxes exist that can turn on all anomaly-allowed combinations of chiral matter fields in all the cases we explore. (Note that this statement regards linear constraints on families of allowed chiral matter. Tadpole constraints will, of course, impose a bound on the magnitude of the number of families for any given linear combination of allowed chiral matter fields.)

In general, the linear constraints that hold on the fluxes $\Theta_{IJ}$ for an F-theory background where the 4D gauge group is unbroken (and hence equal to the geometric gauge group $\mathsf{G}$) are the union of those that come from the nullspace of $M$ and the constraints (2.16). It is helpful to consider how this set of constraints arises in the two approaches characterized in Fig. 1. When the nullspace of $M$ is quotiented out first, giving $M_{\mathrm{red}}$, and then the constraints are imposed, it is clear that the constraints listed above are precisely the constraints on the resulting $\Theta$s that can arise. The situation is slightly subtler, however, when the constraints are imposed first. In particular, the signature of the inner product matrix $M$ is not generally semi-definite, so in principle there can be vectors of vanishing norm that are not null vectors of $M$. If one of the constraints (2.16) can be described as $wM\phi$ for a vector $w$ of this type, then when the constraints are imposed first the matrix $M_C$ could have additional null vectors beyond those associated with homological equivalence in $\Lambda_S$; this would occur when the vector $w$ also lies in $\Lambda_C$. The subsequent quotient by homologically trivial cycles (i.e., null vectors of $M$) does not remove such null vectors from $M_C$. Nonetheless, any such vector would still correspond to a linear combination of the nullspace and symmetry constraints.[17] While this situation does not arise in practice in any of the models we analyze here, we do not have any way of strictly ruling it out, particularly for models with one or more U(1) factors, so this possibility must be kept in mind throughout the analysis.

*2.8. Quantization of fluxes and matter multiplicities.* One thorny issue, which has not been fully resolved to our knowledge anywhere in the literature, is the precise quantization condition on the fluxes and the consequent constraints on the multiplicities of chiral matter. Even in the most well-understood SU(5) F-theory GUT constructions, this question is left open in analyses of which we are aware. Note that this question arises whether or not there are issues related to the shifted quantization condition Eq. (2.6).

We do not fully resolve this quantization issue here but we do shed some light on the question and provide a set of sufficient conditions for matter with certain multiplicities to exist. In the basis for $\Lambda_S$ given by the surfaces $S_{IJ}$, the coefficients $\phi^{IJ}$ are always integers for purely vertical fluxes (or in some cases half-integers, when $c_2(X)$ is not even)[18], so that the lattice vectors $\phi$ live in the lattice $H_{2,2}^{\mathrm{vert}}(X, \mathbb{Z}) = \Lambda_S/\sim$ obtained by quotienting out homologically trivial $\phi$. However, a basic observation is that the matrix $M_{\mathrm{red}}$ that gives an inner product on this space (and which maps $\phi$ to some corresponding $\Theta$) does not in general have determinant equal to $\pm 1$, so that the possible values of $\Theta$ that can be realized generically imply a nontrivial quantization on possible chiral matter multiplicities induced by vertical fluxes. Furthermore, the symmetry constraints (2.10) and (2.11) impose further constraints on the allowed values of $\phi$ and hence the resulting nonzero $\Theta$ and associated chiral multiplicities may be subject to additional quantization constraints.

More explicitly, in many situations such as that of a purely nonabelian gauge group, the condition that certain $\Theta$s must vanish, needed to preserve local Lorentz and gauge symmetry of the 4D theory, can be written schematically in the form

$$\begin{pmatrix} 0 \\ \Theta'' \end{pmatrix} = \begin{pmatrix} M' & Q \\ Q^T & M'' \end{pmatrix} \begin{pmatrix} \phi' \\ \phi'' \end{pmatrix}, \tag{2.24}$$

so we have

$$M'\phi' + Q\phi'' = 0. \tag{2.25}$$

In the basis for $\Lambda_S$ given by the surfaces $S_{IJ}$, the coefficients $\phi^{IJ}$ comprising $\phi', \phi''$ are always integers. When the matrix $M'$ has a non-unit determinant, we can think of

[17] A simple proof of this statement can be made as follows: assume without loss of generality that $M$ has no nullspace, and the constraints are of the form $w\Theta = wM\phi = 0$ for $w \in W$, and $\Lambda_C$ is the orthocomplement $W^\perp$ of the set of constraints $W$. Then any null vector $u \in \Lambda_C$ of $M_C$ satisfies $uM_C\phi = uM\phi = 0$ for any $\phi \in W^\perp$. But then $u \in (W^\perp)^\perp = W$, so $u$ is a constraint vector. A similar proof follows when $M$ has nontrivial nullspace, though $u$ can also have a component then in this nullspace.

[18] For the most part we frame the discussion in terms of cases where $c_2(X)$ is an even class, so that the quantization issue of Eq. (2.6) leaves $G$ as an integer cohomology class; it should be kept in mind however that when $c_2(X)$ is not an even class, some of the flux background parameters must be half-integer, i.e. $\phi^{IJ} \in \mathbb{Z}+\frac{1}{2}$. We consider explicit examples of this in Sect. 6.4.

the image of $M'$ acting on vectors $\phi'$ in $\mathbb{Z}^k$ as a $k$-dimensional lattice $\Lambda'$. We can solve the equation (2.25) for integer values of $\phi'$ if and only if $Q\phi'' \in \Lambda'$. This gives a quantization condition on the flux coefficients $\phi''$ that is both necessary and sufficient to have an integer solution for $\phi'$. Thus, we can determine a condition on $\phi$, and hence on the nonzero $\Theta$s that parameterize the chiral matter, which is sufficient to guarantee the existence of an allowed flux background in $H_{2,2}^{\text{vert}}(X, \mathbb{Z})$. As we see in the more explicit analyses of Sect. 6, in cases of a simple gauge group like $\mathsf{G} = \mathrm{SU}(N)$, this kind of analysis leads to a natural understanding of the quantization condition on the fluxes from the appearance of the Cartan matrix of $\mathsf{G}$ in the role of at least a block of the matrix $M'$. The story is somewhat more complicated in the presence of U(1) factors.

The analysis just summarized focuses only on vertical fluxes. From Poincaré duality of $H_{2,2}(X, \mathbb{C})$, we know that there must exist a flux so that $\int_S G = 1$ for any primitive element $S$ in $H_{2,2}(X, \mathbb{Z})$. As mentioned above, however, the intersection form is not in general unimodular on $H_{\text{vert}}^{2,2}(X, \mathbb{Z})$. Thus, a complete analysis of the set of possible chiral matter multiplicities available may require including flux backgrounds with components in $H_{\text{rem}}^{2,2}(X, \mathbb{Z}) \oplus H_{\text{hor}}^{4}(X, \mathbb{Z})$ and/or fractional coefficients in terms of the basis $\mathrm{PD}(S_{IJ})$ for $H_{\text{vert}}^{2,2}(X)$—see Fig. 2 for an illustration of this point. This is discussed in more detail in the case of SU(5) in Sect. 6.4. Flux backgrounds with such fractional coefficients have been analyzed previously in, e.g., [29]. In that context, in the notation of this paper, fractional values of $\phi$ are considered and the necessary constraints that $M\phi$ gives integer values (i.e. that $G$ integrated over any surface in $H_{2,2}^{\text{vert}}(X)$ is integral) are imposed. However, not all such fractional values of $\phi$ necessarily correspond to allowed fluxes. As an example of this point, consider the self-dual lattice defined by the symmetric bilinear form diag(2, 2). This lattice consists of all vectors $(x, y)$ with $x, y \in \mathbb{Z}/2$, $x+y \in \mathbb{Z}$. The vector $(1/2, 0)$ has integer inner product with the elements of a non-unimodular basis $(1, 0)$, $(0, 1)$, but it is not an element of the given self-dual lattice. For similar reasons, the conditions that $M\phi$ is integral are not by themselves sufficient to guarantee that $\phi$ is an integer homology class. This question is further complicated by the lack of understanding of the components of $\phi$ in $H_{2,2}^{\text{rem}}(X, \mathbb{Z}) \oplus H_4^{\text{hor}}(X, \mathbb{Z})$. Thus, it is difficult to ascertain exactly which fractional values of $\phi$ correspond to vectors in the unimodular lattice $H_4(X, \mathbb{Z})$. We discuss this further in some specific examples in Sect. 6.3 and Sect. 6.4. When $H_{2,2}^{\text{vert}}(X, \mathbb{Z}) = H_4(X, \mathbb{Z}) \cap H_{2,2}(X, \mathbb{R})$, and $\phi$ is allowed to be a general element of $H_4(X, \mathbb{Z})$, then the unimodularity of $H_4(X, \mathbb{Z})$ implies that the proper conditions for the vertical component $\phi_{\text{vert}}$ are that it should lie in the dual lattice to $H_{2,2}^{\text{vert}}(X, \mathbb{Z})$ and also in the constrained lattice $\Lambda_C$, but is not subject to any further apparent constraints. This does not, however, imply that even in this case any chiral multiplicity is possible, without further information about whether the matter surface has components in $H_{\text{rem}}^{2,2}(X, \mathbb{Z}) \oplus H_{\text{hor}}^{4}(X, \mathbb{Z})$; we leave a more detailed investigation of these integrality conditions to future work.

*2.9. Codimension-three* (4, 6) *loci.* Many F-theory geometries contain (4, 6) (or higher) singularities in the elliptic fibration over codimension-three loci in the base [41,58]; these are often associated with non-flat fibers in the resolution [70]. In this paper, we focus on geometries without codimension-three (4, 6) singular loci in the elliptic fibration. We note that we have also analyzed a variety of situations, such as universal models with gauge groups $\mathsf{G} = \mathrm{SU}(N \geq 7)$, $\mathrm{SO}(N \geq 12)$, $\mathrm{E}_7$ and other cases that do have codimension-three (4, 6) loci, where we find that there is an additional allowed flux

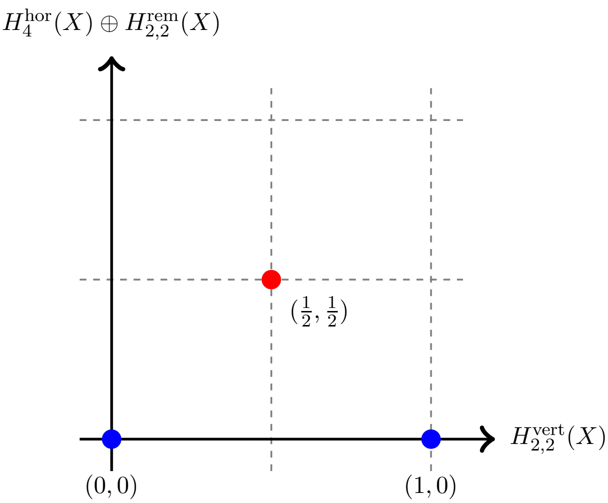


**Fig. 2.** Toy example of a possible realization of $H_4(X, \mathbb{Z})$ as an integral unimodular lattice (note that we implicitly include the shift by $c_2/2$), where we take the bilinear pairing to be $M_{\rm red} \oplus M_{\rm red}^{\perp} = \mathrm{diag}(2, 2)$. In this example we denote lattice vectors by $(\phi, \psi)$, with $\phi \in H_{2,2}^{\rm vert}(X)$, $\psi \in H_4^{\rm hor}(X) \oplus H_{2,2}^{\rm rem}(X)$. As can be seen by requiring the inner product $2\phi^2 + 2\psi^2$ to take integer values, the restriction of $(\phi, \psi)$ to $(\phi, 0)$ (represented by blue dots in the above graph) requires $\phi \in H_{2,2}^{\rm vert}(X, \mathbb{Z}) \cong \mathbb{Z}$ to preserve the integrality of the lattice, i.e. $\phi \in \frac{1}{2}\mathbb{Z}$ is forbidden. However, there exist lattice vectors with $\psi \neq 0$ for which $\phi \in \frac{1}{2}\mathbb{Z}$, for instance the vector $(\frac{1}{2}, \frac{1}{2})$ represented by the red dot in the above graph. This example shows that vertical flux backgrounds $\phi$ with rational coefficients $\phi^{IJ} \in \mathbb{Q}$, which preserve the integrality of both the lattice and chiral indices, could in principle exist

background parameter and the rank of $M_{\rm red}$ is larger than expected from the 4D anomaly cancellation conditions. A more detailed analysis of these models is left for future work.

*2.10. Summary of new results.* We summarize here the main results of the paper:

- We show, by way of example, that the pushforward technology of [20] can be used to easily compute the vertical fluxes of resolutions of singular Weierstrass models with any nonabelian gauge symmetry subgroup over an arbitrary smooth base. We also show that U(1) gauge factors can be incorporated into the analysis in a manner that depends explicitly on the intersections of the associated height pairing divisors with the curve classes of the base. We present an explicit expression for the vertical fluxes in terms of the pushforwards of the intersection numbers of the resolved elliptic CY fourfold to the base; in the special case of a purely nonabelian gauge group, these intersection numbers only depend on the intersections of the canonical class of the base and the classes of the gauge divisors wrapped by seven-branes whose gauge bundles correspond to the simple factors of the F-theory gauge group.
- We find that the reduced intersection pairing $M_{\rm red}$ on the vertical middle cohomology $H^{2,2}_{\rm vert}(X, \mathbb{Z})$ is independent of resolution (up to a change of basis) in all cases we consider explicitly. We furthermore show that this resolution-independence holds for all F-theory models with nonabelian gauge symmetry and generic matter, when the physically-relevant $M_{\rm phys}$ is resolution invariant and obeys certain compatibility conditions related to the weight lattice of the gauge algebra. We conjecture that the

resolution-independence of $M_{\rm red}$ (and hence also of the full matrix $M$ including the nullspace built from vertical cycles) holds more generally for F-theory models with arbitrary gauge groups, including those with U(1) factors, and give some explicit examples supporting this conjecture.

- Exploiting M-theory/F-theory duality, we match 3D Chern–Simons couplings with the vertical fluxes to obtain the chiral matter multiplicities associated to various examples of universal $\mathsf{G}$ models, some not previously studied in the literature. In particular we study low rank examples of models with simple, simply laced gauge group and generic matter (see Table 1), as well as the universal $(\mathrm{SU}(2)\times \mathrm{U}(1))/\mathbb{Z}_2$ model.
- We find that in all cases we study, the number of independent vertical fluxes remaining after imposing constraints necessary to preserve local Lorentz and gauge symmetry in 4D—equivalently, the rank of the nondegenerate intersection pairing of the vertical cohomology of the resolved elliptic CY fourfold with integer coefficients, minus the number of independent symmetry constraints—is greater than or equal to the number of allowed independent families of 4D chiral matter multiplicities plus the number of non-minimal codimension-three singularities in the F-theory base. In all these cases, allowed fluxes produce matter combinations that span the linear space of anomaly-free matter representations.[19]

## 3. Fluxes Preserving 4D Local Lorentz and Gauge Symmetry

Sections 3 and 4 present two different perspectives on the relation between flux backgrounds $\phi^{IJ}$ and fluxes $\Theta_{IJ}$ corresponding to the two paths from the upper left to the lower right of the commuting diagram in Fig. 1. In this section, we describe the sublattice of flux backgrounds $\Lambda_C \subset \Lambda_S$, which is the preimage of the lattice of fluxes $M\Lambda_S$ preserving 4D local Lorentz and gauge symmetry. The matrix elements of the inner product matrix $M_C$ on the constrained space depend on the pushforwards $W_{IJKL}$ of quadruple intersection numbers of a smooth elliptic CY fourfold $X$ resolving a Weierstrass model with gauge symmetry $\mathsf{G} = (\mathsf{G}_{\rm na} \times \mathrm{U}(1))/\Gamma$ (cases with additional U(1) factors are a straightforward generalization of the results presented here.) Note that in this section and the next, we do not concern ourselves with the shifted quantization condition (2.6) but simply treat $\Lambda_S$ as an integral lattice, with the understanding that sometimes this quantization condition gives an overall half-integer shift that must be incorporated in specific contexts.

In Sect. 3.1 we review the relationship between vertical fluxes $\Theta_{IJ}$ and intersection numbers of the types of smooth elliptic CY fourfolds $X$ with which we concern ourselves. Section 3.2 presents an explicit expression for $M_C$ that is valid in most of the cases we consider. In Sect. 3.3 we discuss further the relationship between the nullspace of $M_C$ and linear constraints on $\Theta_{IJ}$, along with the relationship of these constraints to 4D anomaly cancellation.

*3.1. Computing vertical fluxes with intersection theory.* Given a smooth CY fourfold $X$ and a basis of divisors $\hat{D}_I$ where $I = 0, 1, \alpha, i_s$ (we take $I = a = 1$ to be the only index, if there is one, denoting a U(1) section—see the discussion immediately below (2.3) for

[19] The resolutions we study of models with codimension-three (4, 6) loci are non-flat fibrations in which the fibers over the (4, 6) loci contain a Kähler surface as an irreducible component. See the comments in Sect. 2.9 for further discussion.

more details about the index structure), we may expand a vertical flux background $G \in H^{2,2}_{\text{vert}}(X, \mathbb{Z})$ in a basis of wedge products of (1, 1)-forms dual to divisors, $\text{PD}(\hat{D}_I)$, where 'PD' denotes the Poincaré dual.[20] In our analysis here we formally work in the Chow ring, which exhibits the intersection properties of elements of (co)homology that have a description in terms of algebraic subvarieties. The reason for this is that the pushforward technology that we use, which is described in more detail in Appendix B, is defined with respect to the Chow ring. For the purposes of the analysis here, however, the only elements of the Chow ring that concern us are the classes of divisors $\hat{D}_I$ and their intersections $S_{IJ} = \hat{D}_I \cap \hat{D}_J \in \Lambda_S$, which can be understood directly as elements of the homology groups $H_{3,3}(X, \mathbb{Z})$ and $H_{2,2}(X, \mathbb{Z})$ respectively.

As described in (2.15), the integrals of a flux background $G$ over the cycles of vertical surfaces can be evaluated in terms of intersection products of $\hat{D}_I$

$$\Theta_{IJ} = \int_{S_{IJ}} G = \int_X G \wedge \text{PD}(S_{IJ}) = \phi^{KL} S_{KL} \cdot S_{IJ} = \phi^{KL} \hat{D}_K \cdot \hat{D}_L \cdot \hat{D}_I \cdot \hat{D}_J, \tag{3.1}$$

and so we may parametrize a candidate vertical flux $G$ in terms of its Poincaré dual class $\phi$ in the Chow ring of $X$ as $\phi = \phi^{IJ} S_{IJ}$, leading to the more succinct expression

$$\Theta_{IJ} = \phi \cdot S_{IJ}. \tag{3.2}$$

This correspondence between integrals over cycles and intersection products implies that the intersection pairing matrix $M : \Lambda_S \times \Lambda_S \to \mathbb{Z}$ can be described as a matrix with indices given by pairs $IJ$, where the matrix elements are expressed in terms of quadruple intersection numbers,

$$M_{(IJ)(KL)} = S_{IJ} \cdot S_{KL}. \tag{3.3}$$

Thus, essentially every computation relevant for determining the multiplicities of chiral matter can be characterized in terms of linear algebra and performed using intersection theory.

In what follows, we assume that the smooth fourfold $X$ is a resolution of a singular Weierstrass model belonging to a family defined by the characteristic data $(K, \Sigma_s, W_{01})$, where $K$ is the canonical class of $B$, $\Sigma_s$ is the class of the gauge divisor in $B$ associated to the nonabelian gauge subalgebra $\mathfrak{g}_s$ and $W_{01}$ is the class of the pushforward $\pi_*(\hat{D}_0 \cdot \hat{D}_1)$ of the intersection of the zero and generating sections. We moreover assume that the resolved elliptic CY fourfold $\pi : X \to B$ can be realized as a hypersurface inside an ambient fivefold that is the blowup of a $\mathbb{P}^2$ bundle. These assumptions allow us to evaluate the quadruple intersection numbers explicitly by computing their pushforward to the Chow ring of the base $B$,

$$\pi_*(\hat{D}_I \cdot \hat{D}_J \cdot \hat{D}_K \cdot D_L) = W_{IJKL} = W^{\alpha\beta\gamma}_{IJKL} D_\alpha \cdot D_\beta \cdot D_\gamma, \tag{3.4}$$

where the right side of the above equation can be expressed as a linear combination of triple intersection products of the classes of the characteristic data $(K, \Sigma_s, W_{01})$.

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[20] Lefshetz's theorem on (1,1)-classes applied to projective varieties such as $X$ guarantees that given a basis of divisors $\hat{D}_I$ there always corresponds a Poincaré dual basis of harmonic (1, 1) forms $\text{PD}(\hat{D}_I)$—see e.g. [86] for a related discussion.

Furthermore, since $X$ is an elliptic fibration, for certain multi-indices $IJKL$ the push-forwards $W_{IJKL}$ have additional structure that remains applicable for all known crepant resolutions. For convenience we suppress the explicit pushforward map $\pi_*$ when the appropriate ring is otherwise clear from the context. See Appendix B for additional mathematical details about the pushforward map $\pi_*$ and the structure of the intersection numbers.

*3.2. Explicit solutions of the symmetry constraints.* The main result of this subsection is an explicit expression for the matrix $M_C$ and the resulting possible fluxes $\Theta_{IJ}$ that are the integrals of flux backgrounds $\phi^{IJ}$ restricted to live in the sublattice $\Lambda_C$ satisfying the symmetry constraints (2.10) and (2.11).

We now sketch the essential features of the computation; the details of this derivation can be found in Appendix C. As we have seen, the symmetry constraints (2.10) and (2.11) imply $\Theta_{I\alpha} = 0$. By ordering $\Theta_{IJ}$ so that those that $\Theta_{I\alpha}$ are listed first and likewise for $\phi^{I\alpha}$, as a matrix equation the symmetry constraints take the schematic form (2.24), which we reproduce here for convenience:

$$\begin{pmatrix} 0 \\ \Theta'' \end{pmatrix} = \begin{pmatrix} M' & Q \\ Q^T & M'' \end{pmatrix} \begin{pmatrix} \phi' \\ \phi'' \end{pmatrix}. \tag{3.5}$$

In solving the symmetry constraints (2.10) and (2.11) it is often convenient to eliminate (when possible) the parameters $\phi^{I\alpha}$, whose indices match those of the $\Theta_{I\alpha}$, which we require to vanish. We sometimes refer to objects carrying indices $\hat{I}\hat{J}$ (i.e., pairs $IJ$ for which $I, J \neq \alpha$) as "distinctive", and all others choices of index $I\alpha$ as "non-distinctive". For example, in the above matrix equation, $\phi''$ and $\Theta''$ have distinctive indices. However, even when we cannot solve for all non-distinctive $\phi$ parameters explicitly, we nevertheless sometimes denote by $\Theta''$ the set of fluxes obeying the symmetry constraints. Note also that even when we can solve for all the non-distinctive $\phi$ parameters it is sometimes useful to solve for a different set of $\phi$s; see, e.g., Eq. (7.14).

In cases for which the matrix block $M'$ is nondegenerate we can solve the equation (2.24) for the non-distinctive $\phi'$ parameters in terms of the distinctive $\phi''$ parameters, giving

$$\phi' = -(M')^{-1} Q \phi''. \tag{3.6}$$

When $|\det M'| = 1$, there is an integer solution in $\phi'$ for any $\phi''$. When $|\det M'| > 1$, however, the integrality condition imposed on $\phi''$ by requiring that the symmetry constraints be solved over $\mathbb{Z}$ is subtler, as discussed in Sect. 2.8. In some cases, such as imposing the constraints on $M_{\text{red}}$ after removing null vectors for a purely nonabelian group, the corresponding matrix $M'$ in the non-distinctive directions is non-degenerate and invertible, and this procedure of solving for the $\phi'$ flux backgrounds can be performed exactly as in Eq. (3.6). In other cases, in particular when we consider the constraints directly on $M$ and null vectors still are included among the $\phi$s, the matrix $M'$ is degenerate and cannot be inverted. In many such cases we can impose the constraints by simply using the pseudoinverse of $M'$ for $(M')^{-1}$, which for a symmetric matrix basically means taking the inverse on the orthocomplement of the null space and the zero matrix on the null space. This is equivalent to simply removing the null space and then taking the inverse. Note that this works when the null vectors of $M'$ are also null vectors of $M$.

We can use this more general sense of Eq. (3.6) to write an expression for the restriction of $M$ to $\Lambda_C \subset \Lambda_S$ when $M'$ is nondegenerate and thus invertible on the orthocomplement of the null space. In the following we denote by $(M')^{-1}$ the pseudoinverse of $M'$. In particular, as outlined in Sect. 2.5.2 we can define the $m \times (m-k)$ matrix

$$C = \begin{pmatrix} -(M')^{-1}Q \\ \mathrm{Id}_{m-k} \end{pmatrix}, \tag{3.7}$$

which defines an embedding of $\mathbb{Z}^{m-k}$ associated with the $m-k$ distinctive directions into a rational extension of the full lattice $\Lambda_S$, $C : \mathbb{Z}^{m-k} \to \Lambda_C(\mathbb{Q}) \subset \Lambda_S(\mathbb{Q})$. Note that null directions in $M'$ are associated with constraints that are automatically satisfied, so the corresponding combinations of $\phi'$ vanish, in accord with the definition of the pseudoinverse. As discussed in Sect. 2.8, when $\det M' = \pm 1$ (for the non-null part of $M'$) this map gives a one-to-one correspondence between $\mathbb{Z}^{m-k}$ and $\Lambda_C$; otherwise, the domain of $C$ must be taken to be the subset $\operatorname{dom} C = C^{-1}(\Lambda_C)$. In general, given such a mapping $C$, we can give an explicit description of the the inner product form

$$M_C = C^{\mathrm{t}} M C = M'' - Q^{\mathrm{t}}(M')^{-1}Q, \tag{3.8}$$

which gives the intersection pairing of flux backgrounds in the constrained space $\Lambda_C$ as parameterized by $\phi''$, recalling that in some situations there may be additional discrete constraints on the $\phi''$ values allowed for a valid flux background. We analyze these integrality conditions in more detail for nonabelian gauge groups in Sect. 6.2.2, and for specific examples in Sects. 6 and 7.1. In much of the discussion, however, we elide this subtlety.

Carrying this description slightly further, we can also define an $m \times m$ matrix

$$P = \begin{pmatrix} 0_{m\times k} & C \end{pmatrix}, \tag{3.9}$$

which is idempotent ($P^2 = P$) and gives by right multiplication of the matrix $M$

$$MP = P^{\mathrm{t}}MP = \begin{pmatrix} 0 & 0 \\ 0 & M_C \end{pmatrix}. \tag{3.10}$$

This extends the embedding map $C$ to be defined on all of $\Lambda_S$, where the extra (non-distinctive) parameters are essentially thrown out in the map, which becomes a projector; this form of $M_C$ will be useful in some places. In particular note that the $\Theta$s that result from the action of $MP$ on a given set of $\phi$s satisfying the constraint equations span the set of possible vertical fluxes. Recalling that $M_{\mathrm{phys}}$ can be defined as the inner product on $\Lambda_C/\sim$ after taking the quotient by homologically trivial cycles, we have

$$\operatorname{rk} M_{\mathrm{phys}} = \operatorname{rk} M_C. \tag{3.11}$$

This rank corresponds to the number of linearly independent families of allowed fluxes.

We present now a formal expression for the matrix elements of $M_C$ in the case of a gauge group $\mathsf{G} = (\mathsf{G}_{\mathrm{na}} \times \mathrm{U}(1))/\Gamma$ for generic characteristic data. This set of expressions is valid whenever $M'$ is nondegenerate and invertible (or pseudo-invertible, using null vectors of $M'$ that are also null vectors of $M$). This condition always holds when $\mathsf{G}$ is purely nonabelian and the F-theory geometry admits a holomorphic zero section, and is true in most situations we have considered with simple bases and/or generic characteristic data when the gauge group contains U(1) factors. As shown in the example in Sect. 7.1.3, however, there are some cases where $M'$ is degenerate even after removing null vectors in the non-distinctive directions; in such situations we can still analyze the spectrum by solving for a different set of $\phi$s, but the formulae given here do not apply in this form.

The fluxes satisfying the symmetry constraints take the form[21]

$$\Theta_{\hat{I}\hat{J}} = M_{C\,(\hat{I}\hat{J})(\hat{K}\hat{L})}\phi^{\hat{K}\hat{L}}. \tag{3.12}$$

In the above equation, the matrix elements of $M_C$ can be expressed as

$$M_{C\,(\hat{I}\hat{J})(\hat{K}\hat{L})} = M_{C_{\mathrm{na}}\,(\hat{I}\hat{J})(\hat{K}\hat{L})} - M_{C_{\mathrm{na}}\,(\hat{I}\hat{J})(1\alpha)} M^{+(1\alpha)(1\beta)}_{C_{\mathrm{na}}} M_{C_{\mathrm{na}}\,(1\beta)(\hat{K}\hat{L})}, \tag{3.13}$$

where $M_{C_{\mathrm{na}}} = C^{\mathrm{t}}_{\mathrm{na}} M C_{\mathrm{na}}$ is the restriction of $M$ to the sublattice $\Lambda_{C_{\mathrm{na}}}$ of backgrounds only satisfying the purely nonabelian constraints $\Theta_{i_s\alpha} = 0$. The components of $M_{C_{\mathrm{na}}}$ are

$$\begin{aligned} M_{C_{\mathrm{na}}\,(IJ)(KL)} = W_{IJKL} &- W_{IJ|i_s} \cdot W^{i_s|j_{s'}} W_{KLj_{s'}} \\ &- W_{0IJ} \cdot W_{KL} - W_{IJ} \cdot W_{0KL} \\ &+ W_{00} \cdot W_{IJ} \cdot W_{KL} \end{aligned} \tag{3.14}$$

where in particular

$$\begin{aligned} M_{C_{\mathrm{na}}\,(1\alpha)(KL)} &= D_\alpha \cdot W_{\bar{1}KL} \\ &= D_\alpha \cdot (-W_{1|k_{s''}} W^{k_{s''}|i_s} W_{i_sKL} + W_{1IJ} - W_{0KL} + (W_{00} - W_{01}) \cdot W_{KL}) \end{aligned} \tag{3.15}$$

$$\begin{aligned} M_{C_{\mathrm{na}}\,(1\alpha)(1\beta)} &= D_\alpha \cdot D_\beta \cdot W_{\bar{1}\bar{1}} \\ &= D_\alpha \cdot D_\beta \cdot (-W_{1|k_{s''}} W^{k_{s''}|i_s} W_{1i_s} + 2(W_{00} - W_{01})) \end{aligned} \tag{3.16}$$

and $M^{+(1\alpha)(1\beta)}_{C_{\mathrm{na}}}$ is the inverse of $M_{C_{\mathrm{na}}\,(1\alpha)(1\beta)}$.

The structure of the various pushforwards $W_{IJ}$ is explained in more detail in Appendix B; for example in Eq. (3.16), $W_{\bar{1}\bar{1}}$ is equal to (minus) the height pairing divisor associated to the U(1). For a purely nonabelian gauge group, there are no indices of the form $(1\alpha)$, the second term in Eq. (3.13) can be dropped, and $M_C = M_{C_{\mathrm{na}}}$ from Eq. (3.14). The fact that the restriction of $M'$ (see Eq. (2.24)) to the nonabelian part of the theory (i.e., taking all indices $I\alpha$ except $1\alpha$) contains a non-trivial invertible submatrix for generic characteristic data over arbitrary $B$ can be deduced from the explicit form of the components of $M'$, which are all resolution-independent, as discussed in more detail in Sect. 4.3.

The presence of a U(1) factor introduces additional complications, as we now describe in more detail. The submatrix $M^{+(1\alpha)(1\beta)}_{C_{\mathrm{na}}}$ is generically the inverse of the matrix $M_{C_{\mathrm{na}}\,(1\alpha)(1\beta)} = [[W_{\bar{1}\bar{1}} \cdot D_\alpha \cdot D_\beta]]$.[22] For bases with $h^{1,1}(B)$ not too large relative to $h^{1,1}(W_{\bar{1}\bar{1}}$[23] and generic characteristic data, this matrix is invertible. When $M_{C_{\mathrm{na}}(1\alpha)(1\beta)}$

[21] Hatted indices are of type $\hat{I} = 0, 1, i_s$, i.e., a restriction of the usual indices to the case $I \neq \alpha$.

[22] Barred ("physical") indices are of type $\bar{I} = \bar{0}, \bar{1}, \alpha, i_s$. In the basis $\hat{D}_{\bar{I}}$, $\hat{D}_{\bar{0}}$ is the KK U(1) divisor and $\hat{D}_{\bar{1}}$ is the abelian U(1) divisor (i.e., the image of the generating section under the Shioda map), whereas in the basis $\hat{D}_I$, $\hat{D}_0$ is simply the zero section and $\hat{D}_1$ is the generating section. The matrices $\sigma^I_{\bar{I}}$ in (B.5) and their inverses can be used to convert between these two bases.

[23] Since $W_{\bar{1}\bar{1}}$) is the class of a surface in $B$, whenever $h^{1,1}(W_{\bar{1}\bar{1}}) < h^{1,1}(B)$ the matrix $[[W_{\bar{1}\bar{1}} \cdot D_\alpha \cdot D_\beta]]$ will be singular.

is not invertible, however, the expression (3.13) is no longer valid; in such cases a further analysis must be done, which often involves solving for a different set of $\phi$ components, though the essentially the same procedure (i.e. solving the symmetry constraints $\Theta_{I\alpha} = 0$ by eliminating certain components of $\phi$) still works. An explicit example of this type of situation is illustrated in Sect. 7.1.3. We expect that while some null vectors of $M_{C_{\rm na}(1\alpha)(1\beta)}$ can be dealt with by solving some of the constraints $\Theta_{1\alpha} = 0$ for other $\phi$s, this can be done for at most the total number of parameters $\phi^{\hat{I}\hat{J}}$, $\frac{1}{2}(\mathrm{rk}\,\mathsf{G}{+}2)(\mathrm{rk}\,\mathsf{G}{+}1)$. Null vectors of $M_{C_{\rm na}(1\alpha)(1\beta)}$ that cannot be treated in this manner, i.e., by solving for other $\phi^{\hat{I}\hat{J}}$, should correspond to extra null vectors of $M$. It is also possible that even in the cases where null vectors of $M_{C_{\rm na}(1\alpha)(1\beta)}$ can be treated by solving for parameters $\phi^{\hat{I}\hat{J}}$, this may increase the number of null vectors of $M$ and decrease the number of independent possible fluxes $\Theta$ (since rk $M$ = dim $M$ − nullity $M$ is equal to the number of independent flux backgrounds plus the number of independent constraints, so that an increase in the number of null vectors corresponds either to a decrease in the number of independent constraints or a decrease in the number of independent fluxes); we have not encountered any explicit examples where this behavior occurs, though we have not attempted to systematically construct such examples. We explore the detailed structure of null vectors of $M$ in more detail in Sect. 4.

Note that the analysis here can in general lose information about the integer quantization on the $\phi$s, since in principle the inverse matrices $W^{i_s|j_{s'}}$ and $M^{+(1\alpha)(1\beta)}_{C_{\rm na}}$ may be rational and not integer valued. We address these issues more explicitly in Sect. 4 in the context of the analysis where the nullspace is removed first to give the reduced matrix $M_{\rm red}$.

*3.3. Homology relations and anomaly cancellation.* As discussed in Sect. 2.7, the null vectors of $M_C$, considered as elements of $\Lambda_C \subset \Lambda_S$, encode the full set of F-theory constraints on the possible vertical fluxes $\Theta_{IJ}$, which must include at least the anomaly cancellation conditions but in principle may impose stronger constraints. (See [24] for a closely related discussion about anomalies in F-theory.) When we can explicitly solve for a subset of the $\phi$ variables and write an expression for $M_C$ in terms of the remaining variables, such as is done in terms of the distinctive parameters $\phi''$ in the preceding section, we can gain explicit information that is relevant for understanding 4D chiral matter multiplicities—in particular, the nullspace of such an $M_C$ contains complete information about the linear constraints satisfied by the F-theory fluxes, as we now explain in more detail. This approach to understanding the number of independent families of chiral matter available in universal F-theory models for a given $\mathsf{G}$ complements the related analysis of this question using $M_{\rm red}$ as discussed in the following section. In the remainder of this discussion we assume that we have an explicit description of $M_C$ in terms of a subset of the flux degrees of freedom, as realized concretely in the preceding subsection in cases where $M'$ is (pseudo-)invertible, so that in this subspace $\Theta'' = M_C\phi''$ and the remaining $\Theta$s vanish.

Notice that since $M_C$ is symmetric, any null vector $\nu$ satisfying $M_C\nu = \nu^{\rm t}M_C = 0$ must also satisfy $\nu^{\rm t}\Theta'' = \nu^{\rm t}M_C\phi'' = 0$. Thus, identifying the nullspace of $M_C$ is equivalent to identifying the linear constraints that must be satisfied by the fluxes $\Theta''$. This can be accomplished in all purely nonabelian models admitting a resolution with a holomorphic zero section by using the explicit expression for the nontrivial matrix elements of $M_C$ given in (3.14).

The physical significance of the nullspace equations $\nu^{\mathrm{t}}\Theta'' = 0$ is that they are the complete set of linear conditions that must be obeyed by the symmetry constrained fluxes; provided it is possible to express the chiral multiplicities as rational linear combinations of fluxes as in (2.23), this further implies that the nullspace equations lead to the full set of linear constraints that must be obeyed by the chiral matter multiplicities. Since all allowed F-theory models are by assumption consistent with 4D anomaly cancellation, the nullspace equations include as a subset the linear 4D anomaly constraints.

This observation has immediate applications to the question of whether or not F-theory geometry imposes additional linear constraints on chiral matter multiplicities beyond those associated with 4D anomaly cancellation, as the nullspace equations can easily be recovered from (3.14). When $\mathsf{G}$ is purely nonabelian and the corresponding resolution admits a holomorphic zero section, the fact that (3.14) is true for arbitrary base $B$ implies that the linear constraints on the chiral multiplicities can in principle be read off for all $\mathsf{G}$ models in full generality, provided a resolution $X$ can be identified such that the chiral indices can be expressed in terms of the vertical fluxes $\Theta''$. In Sect. 6 we make extensive use of this structure to confirm that for all universal $\mathsf{G}$ models of this type that we study, F-theory geometry imposes no additional linear constraints on the chiral multiplicities of matter charged under $\mathsf{G}$ beyond the 4D anomaly cancellation constraints; we also find this to be true for all models we study with U(1) gauge factors, as discussed in Sect. 7.

For models with a U(1) gauge factor, some additional care is needed since, as explained towards the end of Sect. 3.2, it does not seem possible to easily compute a fully general form for the nullspace of $M_C$ for a model with U(1) gauge factors over an arbitrary base. Nevertheless, in many circumstances it does appear possible to first solve for $\Lambda_{C_{\mathrm{na}}}$, then further restrict $\Lambda_{C_{\mathrm{na}}}$ to the sublattice $\Lambda_{C_{\mathrm{na}}} \cap \{\phi^{1\alpha} = 0\}$, for which the remaining symmetry constraints $\Theta_{1\alpha} = 0$ can be solved over arbitrary $B$ without modifying the nullspace equations $\nu^{\mathrm{t}}\Theta'' = 0$. The basic idea here is that as long as there exists a linearly independent subset of null vectors of $M$ that span the $S_{1\alpha}$ directions, setting $\phi^{1\alpha} = 0$ for all $\alpha$ will not reduce the rank of the set of $\Theta$s that are realized by acting with $M$ on $\Lambda_C$, and hence will not change the nullspace equations $\nu^{\mathrm{t}}\Theta'' = 0$ that encode linear constraints on the matter multiplicities. We expect that generically the null vectors should have this property, and while we cannot prove that this is always the case we have not encountered any instances where this does not hold. Thus, we can often simplify the analysis of the linear constraints from null vectors by restricting to background fluxes satisfying $\phi^{1\alpha} = 0$. (Note, however, that even though we do not expect this to modify the number of linear constraints, this strategy will not keep track of the precise lattice of allowed fluxes, for reasons similar to the analysis following Eq. (H.6).) With the restriction to $\Lambda_{C_{\mathrm{na}}} \cap \{\phi^{1\alpha} = 0\}$, the U(1) symmetry constraints take the form

$$M_{C_{\mathrm{na}}(1\alpha)(\hat{I}\hat{J})}\phi^{\hat{I}\hat{J}} = 0\,. \tag{3.17}$$

In this case, the expressions for the symmetry constrained fluxes induced by flux backgrounds restricted to the sublattice $\phi^{1\alpha} = 0$ only depend polynomially on triple intersections of the characteristic data since setting $\phi^{1\alpha} = 0$ eliminates dependence on the matrix $W_{\bar{1}\bar{1}} \cdot D_\alpha \cdot D_\beta$; therefore we can again compute the symmetry-preserving fluxes in terms of the characteristic data without committing to a specific choice of $B$. Provided there are null vectors with components spanning the $S_{1\alpha}$ directions as described above, we can then easily determine the linear constraints in this simpler setting with the understanding that the same constraints apply to the unrestricted fluxes as well, at least for generic characteristic data. We do not attempt to specify the precise conditions under which this is true; rather, we simply note that we have yet to identify any counterexamples, i.e., any specific models with more restrictive linear constraints among the fluxes (when $\phi^{1\alpha} \neq 0$) than those implied by anomaly cancellation. We give an explicit example of this type of analysis in Sect. 7.1.

## 4. Reduced Intersection Pairing

In the previous section we explained how to restrict the lattice of vertical flux backgrounds $\Lambda_S$ to the sublattice $\Lambda_C$ of vertical M-theory flux backgrounds that lift to consistent F-theory flux backgrounds compatible with unbroken 4D local Lorentz and gauge symmetry $\mathsf{G}$. Specifically, we showed how to compute the symmetric bilinear form matrix $M_C$ on $\Lambda_C$ so that the symmetry constrained fluxes $\Theta''$ can be realized explicitly as elements of the lattice $M_C\Lambda_C$.

In this section, we present a complementary approach, namely first quotienting out the nullspace of $M$ to get the reduced inner product matrix $M_{\text{red}}$, and then imposing $\Theta_{I\alpha} = 0$. The conceptual advantage of this approach centers on the observation that $M_{\text{red}}$ (equivalently, the lattice $H_{2,2}^{\text{vert}}(X, \mathbb{Z})$, equipped with the intersection pairing $M_{\text{red}}$) appears to be independent of the choice of resolution $X$. It is also slightly easier in this approach to keep track of the integer quantization on the fluxes. Furthermore, $M_{\text{red}}$ may be used to consider F-theory models with flux backgrounds that break part of the gauge symmetry, though we do not explore such configurations here.

In Sect. 4.1, we briefly describe how to obtain the vertical cohomology as a lattice quotient, $H_{2,2}^{\text{vert}}(X, \mathbb{Z}) = \Lambda_S/\sim$, with some details of this analysis relegated to Appendix H. In Sect. 4.2, we summarize the evidence suggesting that $M_{\text{red}}$ is independent of the choice of resolution up to an integral change of basis. Although we are unable to produce a completely general expression for $M_{\text{red}}$, in Sects. 4.3 and 4.4 we describe the nullspace of the intersection pairing $M$ in as much detail as we are able for various $\mathsf{G}$ models, and we defer specific examples to Sects. 6 and 7. Section 4.5 presents an immediate physical application of the invariance of $M_{\text{red}}$.

*4.1. Nullspace quotient and integrality structure.* Considered as an abstract lattice quotient, the integrality structure of $\Lambda_{\text{red}} := \Lambda_S/\sim$ is automatically respected and the quantization condition on flux backgrounds $\phi \in \Lambda_{\text{red}}$ is clear—it is simply the condition that $\Lambda_{\text{red}}$ only contains integral elements. It is not always completely straightforward, however, starting from a given matrix $M$ and associated nullspace, to compute an integer basis for $\Lambda_{\text{red}} = H_{2,2}^{\text{vert}}(X, \mathbb{Z})$ and the associated symmetric bilinear form $M_{\text{red}}$ explicitly. For example, when the nullspace of an integer matrix describing the bilinear form on a lattice is determined, any null vector that contains a unit entry in some coordinate $IJ$ can be modded out by simply removing that vector. If there are no obvious unit entries, however, the projection to integer homology is less transparent, and typically one must identify an appropriate basis for the quotient lattice. A general methodology for performing this quotient and determining the resulting inner product matrix $M_{\text{red}}$ is described in Appendix H. In general, this will require a choice of basis vectors for $\Lambda_{\text{red}}$ that have multiple nonzero components in the original basis for $\Lambda_S$. In all the cases we have studied explicitly, it is possible to identify a subset of the basis vectors of $\Lambda_S$ that form a good basis for $\Lambda_{\text{red}}$; while we have not tried to prove that this is always possible it simplifies the analysis in the cases where this works.

*4.2. Resolution independence.* The quotient of the lattice $\Lambda_S$ by the nullspace of $M$ gives the lattice of vertical classes $H_{2,2}^{\text{vert}}(X, \mathbb{Z})$. The restriction of the intersection pairing on $M$ gives a nondegenerate symmetric bilinear form $M_{\text{red}}$ that maps pairs of elements of $H_{2,2}^{\text{vert}}(X, \mathbb{Z})$ to $\mathbb{Z}$. An intriguing feature of $M_{\text{red}}$ is that in all examples we study, $M_{\text{red}}$ appears to be independent of the choice of resolution $X$ up to an integer change of basis. It is thus tempting to conjecture that given any two resolutions $X, \tilde{X}$ of a singular Weierstrass model with corresponding nondegenerate intersection pairing matrices (resp.) $M_{\text{red}}, \tilde{M}_{\text{red}}$, there exists a matrix $U$ such that

$$\tilde{M}_{\text{red}} = U^{\text{t}} M_{\text{red}} U \,, \quad U \in \text{GL}(h_{\text{vert}}^{2,2}(X), \mathbb{Z}) \,, \tag{4.1}$$

where $h_{\text{vert}}^{2,2}(X) = h_{\text{vert}}^{2,2}(\tilde{X})$. Note that since $\text{GL}(n, \mathbb{Z})$ is a group, every $U$ must be invertible and therefore $\det U = \pm 1$. While we have not checked the resolution-independence of $M_{\text{red}}$ for every possible resolution of every $\mathsf{G}$ model we study, nor for all choices of characteristic data $(K, \Sigma_s, W_{01})$, for all cases in which we compute the matrices $M_{\text{red}}, \tilde{M}_{\text{red}}$ explicitly, we find that there is indeed an invertible integer matrix $U$ satisfying Eq. (4.1).

We remark that it would be desirable to be able to determine whether or not two lattices $H_{2,2}^{\text{vert}}(X, \mathbb{Z})$, $H_{2,2}^{\text{vert}}(\tilde{X}, \mathbb{Z})$ are equivalent up to a change of basis as in Eq. (4.1) by comparing invariants such as the determinant, signature, and rank, which must clearly be equal whenever the two are in fact equivalent. However, to the authors' knowledge, there does not exist a general classification of (non-degenerate) lattices of arbitrary signature in terms of easily-computable invariants such as these.[24] It is also tempting to attempt to check equivalence over the reals—since $M_{\text{red}}$ is symmetric, Sylvester's law of inertia implies that any two intersection pairing matrices with these common features are congruent to one another via an invertible real (not necessarily integer) matrix $U$. However, this is not enough to show that $U$ is an integral matrix, so to show that $M_{\text{red}}$ and $\tilde{M}_{\text{red}}$ are equivalent it appears necessary to explicitly compute a matrix $U$ satisfying $\tilde{M}_{\text{red}} = U^{\text{t}} M_{\text{red}} U$ and confirm that it is a unimodular matrix, which can quickly become a cumbersome task for lattices of large rank. It is for these reasons that we content ourselves to provide evidence that various pairing matrices are equivalent, rather than attempting a conclusive proof.

When $M_{\text{red}}, \tilde{M}_{\text{red}}$ are related by an integer change of basis $U$, it furthermore follows that the associated degenerate matrices $M, \tilde{M}$ are also related by an integer change of

[24] Likewise, one might hope that lattices can be classified and compared in terms of their automorphism groups. Abstractly, the automorphism group of a lattice (see, e.g., [87]) is defined to be the set of linear isometries that map the lattice to itself and preserve the inner product. There are multiple, equivalent ways to explicitly identify matrix representatives of automorphisms of the lattice $H_{2,2}^{\text{vert}}(X, \mathbb{Z})$. One method is to consider all elements $U \in \text{GL}(h_{\text{vert}}^{2,2}(X), \mathbb{Z})$ that act trivially by congruence, i.e., the group of $U$ such that $U^{\text{t}} M_{\text{red}} U = M_{\text{red}}$. Equivalently, giving an embedding of $H_{2,2}^{\text{vert}}(X, \mathbb{Z})$ into $\mathbb{R}^{n,m}$ (where $n + m = h_{\text{vert}}^{2,2}(X)$), it is possible to represent the elements of the automorphism group using orthogonal matrices $O \in \text{O}(n, m; \mathbb{Z})$, as follows: Suppose that $V$ is a matrix of generators of $H_{2,2}^{\text{vert}}(X, \mathbb{Z})$, i.e., the matrix satisfying $V^{\text{t}} I_{n,m} V = M_{\text{red}}$, where $I_{n,m}$ is a diagonal matrix comprising $n$ 1 s and $m$ $-1$s. Then, a matrix $O$ satisfying $O^{\text{t}} I_{n,m} O$ is additionally an element of the automorphism group iff $OV = VU$, for some $U \in \text{GL}(n+m, \mathbb{Z})$. It is straightforward to verify that both representations are equivalent, since by assumption $U^{\text{t}} M_{\text{red}} U = (VU)^{\text{t}} I_{n,m} VU = V^{\text{t}} O^{\text{t}} I_{n,m} OV = M_{\text{red}}$. While it is in principle possible to determine the automorphism group using either construction, to the authors' knowledge, there is no known algorithm for computing the automorphism group of a general lattice of arbitrary signature (although algorithms have been proposed for special cases—see, e.g., [88]), and hence we do not attempt to compute lattice automorphism groups in this paper.

basis. This can be seen by first putting each of the $M$ matrices in the canonical form (H.1) with $M_{\text{red}}$ in the upper left block, as described in Appendix H, and then using a linear transformation with $U$ in the upper left block and the identity in the remaining part of the matrix to relate the two canonical forms of $M$, $\tilde{M}$.

In purely nonabelian cases $\mathsf{G} = \mathsf{G}_{\text{na}} = \prod_s \mathsf{G}_s$, a general form for a matrix $U$ relating two different versions of $M_{\text{red}}$ can be constructed explicitly provided that we make the physically natural assumption that $M_{\text{phys}}$ is the same for both resolutions; we carry out this analysis in Appendix D. As discussed in more detail in Sect. 4.3.3, the resulting $U$ is only constrained to be rational and not integral from these considerations, and a certain compatibility condition is required for $U$ to be integral. In all cases we have considered, however, we have found an integral $U$ of this form. In more general models with U(1) gauge factors and rational zero sections, we do not know of such an explicit construction of $U$; nevertheless, a similar structure should hold in those cases, and for specific choices of characteristic data it still appears to be true that $M_{\text{red}}$ is independent of the specific choice of $X$ as we illustrate in the context of the $(\text{SU}(2) \times \text{U}(1))/\mathbb{Z}_2$ model in Sect. 7.

If $M_{\text{red}}$ is indeed resolution independent, this further suggests that the vertical cohomology $H^{2,2}_{\text{vert}}(X, \mathbb{Z})$ of any elliptic CY fourfold $X$ resolving a singular Weierstrass model with gauge symmetry $\mathsf{G}$ is in some sense a mathematical invariant characterizing properties of the singular locus of $X_0$.

*4.3. Purely nonabelian gauge groups.* Before discussing the more general case including a U(1) gauge factor, we study some properties of the nullspace of the intersection pairing $M$ that hold in the situation that the gauge group is purely nonabelian,

$$\mathsf{G} = \mathsf{G}_{\text{na}} = \prod_s \mathsf{G}_s, \tag{4.2}$$

and the zero section is holomorphic.

*4.3.1. Null space structure of M with purely nonabelian gauge group* When the group $\mathsf{G}$ is purely nonabelian, the intersection pairing $M$ between pairs of vertical cycles $S_{00}$, $S_{0\alpha}$, $S_{0i_s}$, $S_{\alpha\beta}$, $S_{\alpha i_s}$, $S_{i_s j_t}$ can be expressed as

$$M = \begin{pmatrix} K^3 & [K^2 \cdot D_\alpha] & 0 & [K \cdot D_\alpha \cdot D_\beta] & 0 & 0 \\ [K^2 \cdot D_{\alpha'}] & [[K \cdot D_\alpha \cdot D_{\alpha'}]] & 0 & [[D_\alpha \cdot D_{\alpha'} \cdot D_\beta]] & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ [K \cdot D_{\alpha'} \cdot D_{\beta'}] & [[D_\alpha \cdot D_{\alpha'} \cdot D_{\beta'}]] & 0 & 0 & 0 & [[W_{\alpha'\beta' i_s j_t}]] \\ 0 & 0 & 0 & 0 & [[W_{\alpha' \alpha i'_{s'} i_s}]] & [[W_{\alpha' i'_{s'} i_s j_t}]] \\ 0 & 0 & 0 & [[W_{\alpha\beta i'_{s'} j'_{t'}}]] & [[W_{\alpha i_s i'_{s'} j'_{t'}}]] & [[W_{i'_{s'} j'_{t'} i_s j_t}]] \end{pmatrix}. \tag{4.3}$$

(Above, single brackets $[\cdot]$ denote a sub-vector and double brackets $[[\cdot]]$ denote a submatrix; moreover, unprimed free indices correspond to rows while primed free indices correspond to columns.) Note that, as described in more detail in Appendix B, the only intersection numbers in Eq. (4.3) that are resolution-dependent are those that contain at least three indices of type $I = i_s$; the values in the upper left are all included explicitly, and we have

$$W_{\alpha\beta i_s j_t} = D_\alpha \cdot D_\beta \cdot W_{i_s j_t} = W_{i_s|j_t} D_\alpha \cdot D_\beta \cdot \Sigma_s = -\delta_{st} D_\alpha \cdot D_\beta \cdot \Sigma_s \kappa^{(s)}_{ij}, \tag{4.4}$$

where

$$\kappa_{ij}^{(s)} = -W_{i_s|j_s} \tag{4.5}$$

is the inverse Killing metric of the gauge factor $s = t$ (which is equal to the Cartan matrix for ADE groups) and $\Sigma_s$ is the divisor supporting that gauge factor.

The nullspace of $M$ is the set of solutions to the equation

$$M_{(IJ)(KL)}\nu^{KL} = 0\,. \tag{4.6}$$

Some elements of the nullspace correspond to linear combinations of intersections $S_{\alpha\beta}$ that are trivial in the base homology.[25] From the conditions of Poincaré duality on the base and nondegeneracy of the triple intersection product as discussed in Sect. 2.2, the number of independent homology classes represented by $S_{\alpha\beta}$ and $S_{0\alpha}$ are both equal to $h^{2,2}(B) = h^{1,1}(B)$; null directions associated with trivial homology classes in the linear space of $S_{\alpha\beta}$ can thus be removed, though for notational simplicity we continue to use the same symbol $S_{\alpha\beta}$ for the reduced basis. Similarly, there are (linear combinations of) intersections $S_{\alpha i_s}$ that correspond to trivial classes $D_\alpha \cdot \Sigma_s$ in the base. In general, the number of independent nontrivial classes $S_{\alpha i_s}$ is at most $h^{1,1}(\Sigma_s)$, but may be smaller. All these null vectors depend only on the geometry of the base.

After removing the nullspace elements associated with the base geometry, which are independent of resolution, we can proceed further by solving explicitly for more general nullspace elements; we find that additional elements of the nullspace are generated by the vectors

$$\begin{aligned}
\nu_{\langle 0\rangle} &= (1, -[K^\alpha], 0, 0, 0, 0)\\
\nu_{\langle i'_{s'}\rangle} &= (0, 0, [\delta^{j_t}_{i'_{s'}}], 0, 0, 0)\\
\nu_{C\langle a\rangle} &= \nu^{i'_{s'}j'_{t'}}_{C\langle a\rangle}(0, [W^\alpha_{i'_{s'}j'_{t'}}], 0, -[W^\alpha_{i'_{s'}j'_{t'}}K^\beta], -[W^{k_u|k'_{u'}}W^\alpha_{i'_{s'}j'_{t'}|k'_{u'}}], [\delta^{k_u l_v}_{i'_{s'}j'_{t'}}])\,,
\end{aligned} \tag{4.7}$$

where the expression in parentheses in the third line above may be viewed as the components of a basis of symmetry-constrained 4-cycles $S_{Ci_sj_t} = CS_{i_sj_t} \in \Lambda_C \subset \Lambda_S$ given by

$$S_{Ci_sj_t} = W^\alpha_{i_sj_t}(S_{0\alpha} - K^\beta S_{\alpha\beta}) - W^{k_v|l_u}W^\alpha_{i_sj_t|l_u}S_{\alpha k_v} + S_{i_sj_t}\,, \tag{4.8}$$

which satisfy

$$S_{Ci_sj_t}\cdot S_{Ck_ul_v} = S_{i_sj_t}\cdot S_{Ck_ul_v} = M_{C(i_sj_t)(k_ul_v)}\,, \tag{4.9}$$

and $\nu^{i_sj_t}_{C\langle a\rangle}$ are the coefficients of null vectors $M_C$, i.e.

$$0 = \nu^{i'_{s'}j'_{t'}}_{C\langle a\rangle}(W^\alpha_{k_ul_v}K^\beta W_{i'_{s'}|j'_{t'}} - W^{m_w|k'_{u'}}W^\alpha_{k_ul_v|k'_{u'}}W^\beta_{m_wi'_{s'}|j'_{t'}} + W^{\alpha\beta}_{k_ul_vi'_{s'}|j'_{t'}}) \tag{4.10}$$

or equivalently

$$0 = \nu^{i'_{s'}j'_{t'}}_{C\langle a\rangle}M_{C(k_ul_v)(i'_{s'}j'_{t'})}\,. \tag{4.11}$$

[25] For example, when $B = (\mathbb{P}^1)^{\times 3}$ with classes $H_i$, $i = 1, 2, 3$ corresponding to points in the three factors crossed with $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ from the other two factors, the only nontrivial intersection is $H_1 \cdot H_2 \cdot H_3 = 1$, and the curves $H_i \cap H_i$ are trivial in homology.

In the above expression we have used the fact that the expression in parentheses in Eq. (4.10) is $M_C$ in the special case of a purely nonabelian gauge group and holomorphic zero section—see (3.14) and note here $W_{00} = K$. The above computation shows the null vectors of $M_C$ are, in these cases, in one-to-one correspondence with the null vectors of $M$ (note that in this situation the subtlety of zero-norm null vectors of $M_C$ described in the second paragraph of Sect. 2.7 does not arise since in the notation of Eq. (3.7), the submatrix $M'$ restricted to the non-distinctive parameters $\phi^{I\alpha}$ is invertible after removing the null vectors that depend on the base geometry, as well as those from the first two rows of Eq. (4.7)). In principle the appearance of the inverse matrix $W^{k|k'}$ in the third set of vectors (4.7) may mean that even when (4.10) is satisfied for all $\nu^{ij}_{C\langle a\rangle}$, these null vectors are rational with integer $\nu^{ij}_{C\langle a\rangle}$, so that the $ij$ fluxes cannot simply be projected out, as discussed in Sect. 4.1. In all cases we have examined explicitly, however, the entries are integer despite the presence of the inverse matrix; we suspect that this occurs generally, though we have not tried to prove this statement.

As discussed in Sect. 3.3, the structure of the nullspace elements corresponds with constraints on the fluxes $\Theta_{IJ}$. In particular, the property $M = M^{\mathrm{t}}$ implies that the above nullspace equations must also be satisfied by the fluxes:

$$S_{KL} \cdot S_{IJ}\nu^{IJ} = 0 \quad \Longrightarrow \quad \Theta_{IJ}\nu^{IJ} = \nu^{IJ} S_{IJ} \cdot S_{KL}\phi^{KL} = 0\,. \tag{4.12}$$

The linear relations on fluxes coming from the first two classes of null vectors in Eq. (4.7), namely

$$\Theta_{IJ}\nu^{IJ}_{\langle 0\rangle} = \Theta_{00} - K^{\alpha}\Theta_{0\alpha} = 0, \quad \Theta_{IJ}\nu^{IJ}_{\langle i'_{s'}\rangle} = \Theta_{0i'_{s'}} = 0 \tag{4.13}$$

are true in the special case of a holomorphic zero section; see, e.g., [18]. The possible coefficients $\nu^{i'_{s'}j'_{t'}}_{C\langle a\rangle}$ appearing in the third linear condition

$$\Theta_{i'_{s'}j'_{t'}}\nu^{i'_{s'}j'_{t'}}_{C\langle a\rangle} = \phi^{k_u l_v} M_{C(k_u l_v)(i'_{s'}j'_{t'})}\nu^{i'_{s'}j'_{t'}}_{C\langle a\rangle} = 0 \tag{4.14}$$

can be determined in any given situation by explicitly identifying the nullspace vectors of the form in the last line in Eq. (4.7). In cases where there are no constraints on these coefficients, these conditions force all fluxes $\Theta_{ij}$ to vanish and there is no chiral matter.

While in principle for any base and characteristic data the nullspace of the intersection matrix $M$ is straightforward to compute directly, because of the relation (4.11), the structure of the constrained matrix $M_C$ studied in the previous section can be used to analyze the nullspace of $M$ in many cases. In some cases, for the purposes of practical computation, this analysis can be simplified when rows/columns of $M_C$ vanish identically. The subset of indices $i'_{s'}j'_{t'}$ for which $M_{C(k_u l_v)(i'_{s'}j'_{t'})} = 0$ vanishes for all $k_u l_v$ are indices for which the coefficients $\nu^{i'_{s'}j'_{t'}}_{C\langle a\rangle}$ can be set equal to unity. In these cases, the appearance of the Kronecker delta function in explicit coefficients of $S_{Ci'_{s'}j'_{t'}}$ given in Eq. (4.7) indicates that the basis elements $S_{i'_{s'}j'_{t'}}$ are redundant and may be removed from the generating set. On the other hand, the subset of indices $i'_{s'}j'_{t'}$ for which $M_{C(k_u l_v)(i'_{s'}j'_{t'})}$ does not vanish for all $k_u l_v$ are those for which nontrivial elements can be found spanning the nullspace of $M_C$ by taking appropriate linear combinations of $S_{Ci'_{s'}j'_{t'}}$. Removing these primitive directions from the lattice $\Lambda_C$ completes the nullspace quotient and leaves behind a basis of homologically nontrivial cycles spanning $\Lambda_C/\sim$. Since $M_C$ can be used to indirectly define these remaining elements via Eq. (4.11), it follows that explicitly computing $M_C$ automatically determines $M_{\text{red}}$; we elaborate on this point in Sect. 4.3.3.

We next examine the structure of $M_{\text{red}}$ in the context of specific models with purely nonabelian gauge symmetry.

*4.3.2. Nonabelian groups without chiral matter* Let us first assume that all of the vectors $S_{Ci'_{s'}j'_{t'}}$ appearing in the third class of vectors listed in Eq. (4.7) are null vectors; i.e., the expression in parentheses in (4.10) vanishes identically for all coefficients $\nu_C^{i'_{s'}j'_{t'}}$, so that $S_{Ci'_{s'}j'_{t'}} = 0$ in homology. We can then use the null vectors to eliminate the $(\text{rk}\,\mathsf{G}_{\text{na}})^2 + \text{rk}\,\mathsf{G}_{\text{na}} + 1$ redundant elements $S_{00}, S_{0i_s}, S_{i_s j_t}$ to form a basis consisting of the at most $h^{1,1}(B)(h^{1,1}(B) + \text{rk}\,\mathsf{G}_{\text{na}} + 1)$ classes $S_{0\alpha}, S_{\alpha\beta}, S_{\alpha i_s}$. (Since there may exist additional null vectors, in addition to $S_{0\alpha}$ we keep only homologically nontrivial basis elements among $S_{\alpha\beta}, S_{\alpha i_s}$, and hence the total number basis elements may be less than $h^{1,1}(B)(h^{1,1}(B)+\text{rk}\,\mathsf{G}_{\text{na}}+1)$.) This reduces the intersection matrix $M$ to the intersection pairing

$$M_{\text{red}} = [[D_\alpha \cdot D_{\alpha'}]] \otimes \begin{pmatrix} K & [D_\beta] & 0 \\ [D_{\beta'}] & 0 & 0 \\ 0 & 0 & [[W_{i'_{s'} i_s}]] \end{pmatrix}, \qquad (4.15)$$

where the Kronecker product $\otimes$ in the above expression is understood to imply the intersection product $\cdot$ component-wise. Note that the integrality condition on flux backgrounds in $H^{2,2}_{\text{vert}}(X, \mathbb{Z})$ is preserved through the projection to this basis when all the components in the vectors appearing in Eq. (4.7) are integer, which as discussed above occurs in all the cases we have studied. The pairing $M_{\text{red}}$ is manifestly independent of any choice of resolution $X$ with a holomorphic zero section, since in such cases $W_{i'_{s'} i_s} = \delta_{s's}\kappa^{(s)}_{i'i}\Sigma_s$ and the characteristic data $K, \Sigma_s$ remains unchanged.

Since the intersection pairing on $B$ is nondegenerate, as discussed in Sect. 2.2, and we have explicitly removed null combinations of $S_{\alpha i_s}$, $M_{\text{red}}$ is manifestly nondegenerate and resolution-independent. It follows immediately that the symmetry constraints (2.10) and (2.11) $\Theta_{0\alpha} = \Theta_{\alpha\beta} = \Theta_{\alpha i_s} = 0$ force all independent fluxes to vanish. Stated different, the symmetry constraints together with Eq. (4.14) imply $\Theta_{i_s j_t} = 0$. Hence there are no nontrivial fluxes in these cases, and consequently no chiral matter in the resulting 4D F-theory models, as assumed. We give explicit examples of systems of this kind in Sect. 6, in particular for the groups $\mathsf{G} = \text{SU}(N < 5)$.

*4.3.3. Nonabelian groups with chiral matter* Next we consider the case that the expression in parentheses in Eq. (4.10) is non-vanishing for some non-empty subset of indices $i'_{s'} j'_{t'}$. This implies that there are allowed nontrivial flux backgrounds $\phi$ and corresponding fluxes $\Theta$; however, not all of the nonzero fluxes $\Theta_{i_s j_t}$ are independent in $M_C\Lambda_C$. We can remove all null vectors and project $\phi = \phi^{IJ}S_{IJ}$ onto a basis of surfaces $S_{0\alpha}, S_{\alpha\beta}, S_{\alpha i_s}, S_{j_t k_u}$ (again keeping only homologically non-trivial $S_{\alpha\beta}, S_{\alpha i_s}$), leading to the intersection pairing

$$M_{\text{red}} = \begin{pmatrix} [[D_{\alpha'} \cdot K \cdot D_\alpha]] & [[D_{\alpha'} \cdot D_\alpha \cdot D_\beta]] & 0 & 0 \\ [[D_{\alpha'} \cdot D_{\beta'} \cdot D_\alpha]] & 0 & 0 & [[W_{\alpha'\beta' j_t k_u}]] \\ 0 & 0 & [[W_{\alpha' i'_{s'} \alpha i_s}]] & [[W_{\alpha' i'_{s'} j_t k_u}]] \\ 0 & [[W_{j'_{t'} k'_{u'} \alpha\beta}]] & [[W_{j'_{t'} k'_{u'} \alpha i_s}]] & [[W_{j'_{t'} k'_{u'} j_t k_u}]] \end{pmatrix}, \qquad (4.16)$$

where we keep in mind that only a linearly independent subset of the $(\text{rk}\,\mathsf{G}_{\text{na}})(\text{rk}\,\mathsf{G}_{\text{na}}+1)/2$ possible 4-cycles $S_{j_t k_u}$ is represented in the above expression for $M_{\text{red}}$. In principle this choice of a subset of the basis elements may not be compatible with the integral lattice structure through the projection, but as mentioned above this kind of issue does not occur for any of the cases we have considered explicitly and we can always choose such a basis in these cases. The specific set of independent fluxes of the form $\Theta_{i_s j_t}$ (equivalently, the set of independent 4-cycles of the form $S_{i_s j_t}$) depends on the characteristic data of the resolution $X$, and hence we cannot be more precise at this point without specifying the characteristic data of the elliptic fibration, although we expect that for every F-theory model the number of independent fluxes is independent of resolution. Nevertheless, we can see clearly that imposing the symmetry conditions on the reduced intersection pairing $M_{\text{red}}$ leaves behind a subset of independent fluxes in the "pure Cartan" (i.e. $S_{i_s j_t}$) directions that parametrize the combinations of 4D chiral indices realized by the F-theory compactification. We give more explicit examples of systems of this kind in Sect. 6, see in particular Table 1.

While it is not obvious that (4.16) is resolution independent, as shown in Appendix D, with some natural physical assumptions (essentially that $M_{\text{phys}}$ is the same for the two resolutions) we can determine an explicit form for a change of basis matrix $U$ that converts between two different presentations of $M_{\text{red}}$ associated to any pair of resolutions $X, \tilde{X}$ for which $M_{\text{phys}}, \tilde{M}_{\text{phys}}$ are related by an integral linear transformation $\tilde{M}_{\text{phys}} = U_p^{\text{t}} M_{\text{phys}} U_p$. The transformation $U$ has the schematic form

$$U = \begin{pmatrix} 1 & u \\ 0 & U_p \end{pmatrix}, \tag{4.17}$$

where $u$ may contain rational parts with a denominator of $\det\kappa$. As discussed in Appendix D, $U$ is an integral matrix when a certain compatibility condition is satisfied between the off-diagonal blocks on the lowest row and rightmost column of the two presenations of $M_{\text{red}}$, for an allowed choice of equivalence $U_p$ (which has an ambiguity up to automorphisms of $M_{\text{phys}}$). In all cases we have analyzed this compatibility condition is satisfied for some $U_p$, and the resulting $U$ is an integer change of basis, but we do not have a complete proof that this is generally the case.

In a related fashion, there is a transformation of the form (4.17) that takes $M_{\text{red}}$ to a canonical product form

$$M_{\text{red}}^{\text{cp}} = U^{\text{t}} M_{\text{red}} U = \begin{pmatrix} [[D_{\alpha'} \cdot K \cdot D_\alpha]] & [[D_{\alpha'} \cdot D_\alpha \cdot D_\beta]] & 0 & 0 \\ [[D_{\alpha'} \cdot D_{\beta'} \cdot D_\alpha]] & 0 & 0 & 0 \\ 0 & 0 & [[W_{\alpha' i'_{s'}, \alpha i_s}]] & 0 \\ 0 & 0 & 0 & M_{\text{phys}}/(\det\kappa)^2 \end{pmatrix}, \tag{4.18}$$

where we simply use the upper right components of $U$ to transform away the off-diagonal bottom row and right column of Eq. (4.16). This inner product matrix must be treated with respect to the lattice $\Lambda^{\text{cp}} = U^{-1}\mathbb{Z}^n$, which is not in general an integer lattice in this case. This form is, however, useful since the symmetry constraints can be solved trivially by setting all components except the last of the flux background $\phi \in \Lambda_{\text{cp}}$ to vanish; an explicit example of this is illustrated in Sect. 6.4.3. The appearance of $\det\kappa$ in the bottom right component comes from the fact that in general the off-diagonal values of $U$, associated with this transformation to the canonical product form in Eq. (4.18), are rational with denominator $\det\kappa$, and $\Lambda_{\text{phys}} = ((\det\kappa)\mathbb{Z})^m$, as discussed further in Appendix D.

*4.4. Gauge groups with a* $U(1)$ *factor.* For the more general case of Weierstrass models with gauge group $\mathsf{G} = (\mathsf{G}_{\text{na}} \times \mathrm{U}(1))/\Gamma$, we find in practice it is typically easier to compute resolutions of physically-equivalent singular models in which the elliptic fiber is realized as a general cubic in $\mathbb{P}^2$, see, e.g., [28]. These models generically admit rational (as opposed to holomorphic) sections associated to $\mathrm{U}(1)_{\text{KK}}$, $\mathrm{U}(1)$ and consequently the structure of the pushforwards of quadruple intersection numbers involving the divisors $\hat{D}_0$, $\hat{D}_1$ are not known in full generality as is the case in models with a single holomorphic zero section. For example, in these cases

$$W_{000\gamma} \neq K^2 \cdot D_\gamma, \quad W_{0000} \neq K^3 \tag{4.19}$$

and so on. This unfortunately complicates the computation of $M_{\text{red}}$ as our incomplete understanding of intersection products involving $\hat{D}_0$, $\hat{D}_1$ makes the solutions to the nullspace equations unclear, and thus at present we are unable to present even a formal general expression for $M_{\text{red}}$ for models with U(1) factors over arbitrary $B$ with arbitrary characteristic data. Nevertheless, we can follow the procedure to construct $M_{\text{red}}$ outlined in Sect. 4.1 for any specific $B$ and $\mathsf{G}$, and we find in all examples we have considered that $M_{\text{red}}$ is also resolution independent for models with a U(1) gauge factor—see Sect. 7 for some examples. It is natural to conjecture that this is generally the case although a more complete proof is clearly desirable.

*4.5. Dimension of* $\Lambda_{phys}$. One immediate application of the conjectural resolution invariance of $M_{\text{red}}$ is for understanding the number of independent F-theory vertical flux backgrounds and fluxes that can arise in a given model. After computing $M_{\text{red}}$, one can impose the symmetry constraints in order to further restrict the lattice $H^{\text{vert}}_{2,2}(X, \mathbb{Z})$ to the sublattice $\Lambda_{\text{phys}}$ of independent F-theory flux backgrounds; the restriction of the action of $M_{\text{red}}$ to $\Lambda_{\text{phys}}$ can be expressed as a matrix $M_{\text{phys}}$.

The number of independent fluxes $\Theta$ subject to the symmetry constraints is equal to $\text{rk}\, M_{\text{phys}}$. While in principle, as discussed in Sect. 2.7, $M_{\text{phys}}$ can have null vectors associated with constraints characterized by zero-norm non-null elements of $\Lambda_S$, we have not encountered any situations where this occurs. Indeed, this is impossible in purely nonabelian theories since all non-distinctive flux background parameters $\phi'$ are determined by the constraints as linear functions of the $\phi''$ as in Eq. (3.6). We suspect, but have not proven, that this also does not happen in theories with U(1) factors. When there are no such null vectors of $M_{\text{phys}}$, then we have

$$\begin{aligned} \text{rk}\, M_{\text{phys}} &= \dim \Lambda_{\text{phys}} = \#\text{ independent fluxes} \\ &= \text{rk}\, M_{\text{red}} - \#\text{ independent constraints}\,. \end{aligned} \tag{4.20}$$

The number of independent constraints is at most the number of basis elements $S_{0\alpha}$, $S_{\alpha\beta}$, $S_{\alpha i}$, i.e., $h^{1,1}(B) + \frac{1}{2}h^{1,1}(B)(h^{1,1}(B) + 1 + 2\text{rk}\,\mathsf{G})$, but in general can be smaller when there are homologically trivial cycles $S_{\alpha\beta}$, $S_{\alpha i}$ as discussed in Sect. 4.3.1. This number must be resolution-independent and can be identified directly from the structure of $M_{\text{red}}$ and the geometry of $B$.

All 4D F-theory models with generic matter that we have studied have the property that the rank of $M_{\text{phys}}$ is greater than or equal to the number of independent realized families of chiral matter multiplicities:

$$\text{rk}\, M_{\text{phys}} \geq \#\text{ of families of 4D chiral matter multiplets realized in F-theory.} \tag{4.21}$$

The fact that rk $M_{\text{phys}}$ is at least equal to the number of independent chiral matter multiplicities seems to follow from the assumption that all matter surfaces $S_{\mathsf{r}}$ have a non-trivial vertical component. We have furthermore found in all of these cases that the number of independent families of chiral matter multiplicities realized in F-theory matches the number of independent families satisfying 4D anomaly cancellation. We know of no natural geometric reason why this should always be true; the observation that in all cases considered this holds can be thought of as a statement regarding the absence of swampland type models in which entire families of anomaly-free 4D supergravity theories would lack an F-theory realization.

If it is indeed true that $M_{\text{red}}$ is resolution-independent and moreover that the number of independent families of chiral matter is bounded above by the rank of $M_{\text{phys}}$, computing $M_{\text{red}}$ may serve as an efficient strategy for scanning the F-theory landscape for vacua that impose stronger constraints than 4D anomaly cancellation without requiring the additional step of identifying the matter surfaces $S_{\mathsf{r}}$.

## 5. Computing Chiral Indices

In Sect. 3 and Sect. 4 we gave a prescription for computing the lattice of vertical F-theory flux backgrounds for $\mathsf{G}$ models with gauge group $\mathsf{G} = (\mathsf{G}_{\text{na}} \times \mathrm{U}(1))/\Gamma$ and chiral matter transforming in representation $\oplus \mathsf{r}^{\oplus n_{\mathsf{r}}}$. Here, we review a method to compute the multiplicities

$$\chi_{\mathsf{r}} = \int_{S_{\mathsf{r}}} G = n_{\mathsf{r}} - n_{\mathsf{r}^*} \tag{5.1}$$

of the 4D chiral matter representations $\mathsf{r}$ in terms of the fluxes $\Theta''$, without explicit knowledge of the matter surface $S_{\mathsf{r}}$.

Section 5.1 reviews the relationship [16] (see also [18,19]) between Chern–Simons couplings appearing in the low-energy effective 3D $\mathcal{N} = 2$ supergravity action describing M-theory compactified on a CY fourfold in a nontrivial flux background $G$, and the vertical fluxes $\int_{S_{IJ}} G = \Theta_{IJ}$. In Sect. 5.2, we explain how to compute the chiral indices by solving the linear system obtained by matching the vertical fluxes to one loop exact field theoretic expressions for CS couplings appearing in the 3D $\mathcal{N} = 2$ supergravity action, using a similar strategy to that used in [32].

*5.1. 3D Chern–Simons terms and M-theory fluxes.* The key step in our analysis that enables us to determine the chiral indices $\chi_{\mathsf{r}}$ in terms of vertical fluxes without explicit knowledge of the matter surfaces $S_{\mathsf{r}}$ is the identification [16,18]

$$\Theta_{\bar{I}\bar{J}} = -\Theta^{3\text{D}}_{\bar{I}\bar{J}}\,, \quad \bar{I} = \bar{0}, i\,. \tag{5.2}$$

On the right hand side of the above equation, $\Theta^{3\text{D}}_{\bar{I}\bar{J}}$ are Chern–Simons (CS) couplings that characterize the 3D effective action describing M-theory compactified on $X$ at low energies (recall that the index $\bar{I} = \bar{0}$ denotes the KK $U(1)$, see Eq. (B.5)).

The identification (5.2) holds for all M-theory compactifications on CY fourfolds $X$ with nontrivial flux backgrounds $G$, and follows from the dimensional reduction of 11D supergravity on $X$. In the special case that $X$ is a resolution of a singular elliptic CY fourfold, M-theory/F-theory duality implies that the low-energy effective 3D theory is a Kaluza–Klein (KK) theory equivalent to a circle compactification of the 4D $\mathcal{N} = 1$ supergravity theory describing a flux compactification of F-theory on the singular fourfold. Because of this duality, the one-loop exact quantum dynamics on the F-theory Coulomb branch gets mapped to the classical dynamics of M-theory; in particular, this means that the contributions of massive fermions on the F-theory Coulomb branch are captured by classical CS couplings $\Theta^{3\mathrm{D}}_{\bar{I}\bar{J}}$.

Concretely, given a collection of real Coulomb branch moduli $\varphi$ corresponding to the holonomies of Cartan U(1) gauge fields around the KK circle, the F-theory Coulomb branch is characterized by a collection of massive BPS hyperinos, with masses given by

$$m_{\text{hyp}} = n m_{\text{KK}} + \varphi \cdot w, \quad n \in \mathbb{Z}, \quad \varphi \cdot w = \varphi^i w_i, \quad i = 1, \ldots, \operatorname{rk} \mathsf{G}, \tag{5.3}$$

where $w_i$ may be regarded as the Dynkin coefficients of a weight in a basis of fundamental weights, associated with the charges (under $\mathrm{U}(1)^{\operatorname{rk}\mathsf{G}}$) of each hyperino on the Coulomb branch. In terms of the Cartan charges $(n, w_i)$ above, the one-loop exact CS couplings are given by [19]

$$\begin{aligned}
\Theta^{3\mathrm{D}}_{ij} &= \sum_w \left(\tfrac{1}{2} + \lfloor |r_{\text{KK}}\varphi \cdot w| \rfloor\right) \operatorname{sign}(\varphi \cdot w) w_i w_j, \\
\Theta^{3\mathrm{D}}_{\bar{0}i} &= \sum_w \left(\tfrac{1}{12} + \tfrac{1}{2} \lfloor |r_{\text{KK}}\varphi \cdot w| \rfloor (\lfloor |r_{\text{KK}}\varphi \cdot w| \rfloor + 1)\right) w_i, \\
\Theta^{3\mathrm{D}}_{\bar{0}\bar{0}} &= \sum_w \tfrac{1}{6} \lfloor |r_{\text{KK}}\varphi \cdot w| \rfloor (\lfloor |r_{\text{KK}}\varphi \cdot w| \rfloor + 1)(2\lfloor |r_{\text{KK}}\varphi \cdot w| \rfloor + 1),
\end{aligned} \tag{5.4}$$

where $r_{\text{KK}} := 1/m_{\text{KK}}$ is the KK radius.

The sign and floor functions in the above expressions encode the dependence of the CS couplings on the phase of the vector multiplet moduli space parametrized by the Coulomb branch moduli $\varphi_i$ and KK modulus $m_{\text{KK}}$; we return to the issue of explicitly evaluating these functions shortly. For now, we point out that the CS couplings can be expressed as linear combinations of the chiral indices $\chi_{\mathsf{r}}$ by making the replacement $\Sigma_w \to \sum_{\mathsf{r}} n_{\mathsf{r}} \sum_{w \in \mathsf{r}}$ (where $n_{\mathsf{r}}$ is the multiplicity of each type of representation $\mathsf{r}$ appearing in the 4D spectrum and we only sum over each distinct representation $\mathsf{r}$ once) and using the fact that the summands are odd under $\mathsf{r} \to \mathsf{r}^*$. Combining Eq. (5.2) and Eq. (5.4), we may thus write

$$\Theta_{\bar{0}\bar{0}} = x^{\mathsf{r}}_{\bar{0}\bar{0}} \chi_{\mathsf{r}}, \quad \Theta_{\bar{0}i} = x^{\mathsf{r}}_{\bar{0}i} \chi_{\mathsf{r}}, \quad \Theta_{ij} = x^{\mathsf{r}}_{ij} \chi_{\mathsf{r}}, \tag{5.5}$$

and under our assumption that all matter surfaces have components in $S_{IJ}$[26] it is possible to invert the coefficients $x^{\mathsf{r}}_{ij}$ so that

$$\chi_{\mathsf{r}} = x^{ij}_{\mathsf{r}} \Theta_{ij} = x^{ij}_{\mathsf{r}} S_{Cij} \cdot S_{kl} \phi^{kl} = S_{\mathsf{r}} \cdot \phi, \tag{5.6}$$

where on the right hand side of the above equation we have used the fact that the matter surfaces are given by

$$S_{\mathsf{r}} = x^{ij}_{\mathsf{r}} S_{Cij} \tag{5.7}$$

and $S_{Cij}$ are defined in Eq. (4.8).

[26] See Appendix G for a possible counterexample to this assumption.

*5.2. Computing 3D Chern–Simons terms using triple intersection numbers.* The explicit expressions for $\Theta^{3D}_{\bar{I}\bar{J}}$ given in the previous subsection depend on the values of the moduli-dependent functions $\mathrm{sign}(\varphi \cdot w)$ and $\lfloor |r_{\mathrm{KK}}\varphi \cdot w| \rfloor$ as input. These functions partially characterize the field theoretic regime (i.e., the "phase") of the F-theory Coulomb branch described by the 3D KK theory, or in geometric terms the regime of the Kähler moduli space to which the resolution $X$ corresponds. The Coulomb branch phase can in principle be computed geometrically by studying the fibers of $X$ over the codimension-two components of the discriminant locus in the base $B$, which carry local matter transforming in the representation $\mathsf{r} = \mathsf{r} \oplus \mathsf{r}^*$.

Unfortunately, this procedure is often delicate and sometimes difficult to carry out systematically, so we instead use an alternative approach that relies on the assumption that the hypermultiplet representations characterizing the gauge sector of a 6D supergravity theory can be recovered from a 5D KK theory, at least for representations $\mathsf{r}$ that correspond to local matter in the F-theory geometry. In particular, we exploit the fact that the matter representations are encoded in codimension-two components of the discriminant locus in the base $B^{(2)}$ of an elliptic CY threefold to extract the phase of the Coulomb branch from the triple intersections of Cartan divisors $\hat{D}_i$. Closely following the strategy described in [32], we now explain in detail how to use this trick to compute the sign and floor functions appearing in the field theoretic expressions for the 3D CS couplings in the previous subsection.

Recall that in the case of M-theory compactified on an elliptic CY threefold $X^{(3)}$, M-theory/F-theory duality (similar to the case of a CY fourfold) identifies the triple intersection numbers with one-loop quantum corrected CS couplings in 5D,

$$\hat{D}_{\bar{I}} \cdot \hat{D}_{\bar{J}} \cdot \hat{D}_{\bar{K}} = k^{5D}_{\bar{I}\bar{J}\bar{K}}, \quad \bar{I} = \bar{0}, i\,, \tag{5.8}$$

where field theoretic expressions analogous to Eq. (5.4) have also been worked out for the 5D one-loop CS couplings [90] (see also [91–93]):

$$\begin{aligned}
k^{5D}_{ijk} &= -\sum_w (\lfloor |r_{\mathrm{KK}}\varphi \cdot w| \rfloor + \tfrac{1}{2})\,\mathrm{sign}(\varphi \cdot w) w_i w_j w_k \\
k^{5D}_{\bar{0}ij} &= -\sum_w (\tfrac{1}{12} + \tfrac{1}{2} \lfloor |r_{\mathrm{KK}}\varphi \cdot w| \rfloor (\lfloor |r\varphi \cdot w| \rfloor + 1)) w_i w_j \\
k^{5D}_{\bar{0}\bar{0}i} &= -\sum_w \tfrac{1}{6} \lfloor |r_{\mathrm{KK}}\varphi \cdot w| \rfloor (\lfloor |r_{\mathrm{KK}}\varphi \cdot w| \rfloor + 1)(2\lfloor |r_{\mathrm{KK}}\varphi \cdot w| \rfloor + 1)\,\mathrm{sign}(\varphi \cdot w) w_i\,.
\end{aligned} \tag{5.9}$$

Importantly, the sign and floor functions appearing in (5.9) are the same as those in (5.4), which means they can equally well be determined from the 5D CS couplings provided the 5D and 3D CS couplings correspond to the same Coulomb branch phase in an appropriate sense.

It turns out to be possible to determine the 5D CS terms from the types of (partial) resolutions $X$ we consider, as the sequences of blowups we use to obtain $X$ for a given $\mathsf{G}$ model defined over a threefold base $B$ can also be used to obtain resolutions $X^{(3)}$ of the same $\mathsf{G}$ model defined over a twofold base $B^{(2)}$.[27] Consequently, for a given $\mathsf{G}$ model

[27] Recall that in our case we only consider resolutions of singularities through codimension-two sub-loci of the discriminant locus in $B$. This is actually true for bases of arbitrary dimension, $B^{(d)}$, so long as the sequence of blowups used to obtain a (partial) resolution is formally identical through codimension two. In the case of a twofold (i.e. $d = 2$), this implies that the resulting threefold $X^{(3)}$ is a genuine resolution.

and a common sequence of blowups resolving singularities through codimension two, the triple intersection numbers of $X^{(3)}$ are closely related to the quadruple intersection numbers $W_{\bar{I}\bar{J}\bar{K}\alpha}$ of $X$. More precisely, the pushforwards $W_{\bar{I}\bar{J}\bar{K}}$ are formally identical to the pushforwards of the triple intersection numbers of $X^{(3)}$ to the base, with the key difference that the pushforwards $W_{\bar{I}\bar{J}\bar{K}}$ are "promoted" from numbers to classes of curves in the threefold base $B$. In the fourfold case, one then computes quadruple intersection numbers involving three divisors carrying nonabelian Cartan indices by computing the intersections of these classes with other divisors in the base, i.e., $W_{\bar{I}\bar{J}\bar{K}} \cdot D_\alpha$. One can use this fact to make the formal identification

$$k^{5\mathrm{D}}_{\bar{I}\bar{J}\bar{K}} \to W_{\bar{I}\bar{J}\bar{K}} \tag{5.10}$$

provided we replace specific coefficients in the sums (5.9) with the intersection products of classes of certain curves in $B$. If, as in the 4D case, we organize the expressions for the CS couplings in (5.9) into sums over representations by making the replacement $\sum_w \to \sum_{\mathsf{R}} n_{\mathsf{R}} \sum_{w\in\mathsf{R}}$ (where $n_{\mathsf{R}}$ is the multiplicity of hypermultiplets in the 6D spectrum and we only sum over each distinct quaternionic representation $\mathsf{r}$ once), then we simply need to promote $n_{\mathsf{r}}$ to the classes of matter curves $C_{\mathsf{r}}$ (matter curves are discussed in Sect. 2.6; see also Appendix B for an explicit description of how $C_{\mathsf{r}}$[28] appear in the expressions for $W_{i_s j_t k_u}$.)[29]

Alternatively, we could rephrase this discussion as indicating that the formal expressions $W_{\bar{I}\bar{J}\bar{K}}$ should match the triple intersection numbers that arise when the threefold base $B$ is instead "demoted" to a twofold $B^{(2)}$. Either way, the upshot is that the sign and floor functions are captured by the terms $W_{\bar{I}\bar{J}\bar{K}}$, as is made clear by the matching (5.10). Since the linear system (5.10) does not involve any undetermined parameters, the system can be solved explicitly for the values of $\mathrm{sign}(\varphi \cdot w)$ and $\lfloor |r_{\mathrm{KK}}\varphi \cdot w| \rfloor$. Thus, we find that matching triple intersections with the low energy effective 5D physics of M-theory compactified on an elliptic CY threefold $X^{(3)}$ allows us to circumvent the task of computing the sign and floor functions directly from geometry, and we may subsequently use these values as input for the 3D case.

We illustrate this procedure for the SU(2) model in Sect. 6.3.

## 6. Models with Simple Gauge Group

We apply our systematic approach for analyzing flux backgrounds described in the previous sections in several examples of models with simple nonabelian gauge groups, $\mathsf{G} = \mathsf{G}_{\mathrm{na}}$. In Sect. 6.1, we explain why the only simple $\mathsf{G}$ models with generic matter admitting nontrivial chiral multiplicities are the simply-laced groups $\mathsf{G}_{\mathrm{na}} = \mathrm{SU}(N)$, $\mathrm{SO}(4k+2)$, $\mathrm{E}_6$, with $N \geq 5, k \geq 2$. Section 6.2 describes the common features of the F-theory fluxes $\Theta''$ for these models; the full set of results can be found in Table 1. We turn our attention to specific examples in Sects. 6.3 to 6.7.

*6.1. Chiral matter for simply-laced gauge groups.* The groups $\mathrm{SU}(N)$, $N \geq 5$, $\mathrm{SO}(4k+2)$, $k \geq 2$, and $\mathrm{E}_6$ are precisely the compact simple Lie groups for which we expect a

[28] Note that $C_{\mathsf{r}}$ are known for large classes of singular F-theory models [94] and (in contrast to $S_{\mathsf{r}}$) are manifestly resolution-independent.

[29] The fact that the pushforward technology used to evaluate the intersection numbers does not rely explicitly on the dimension of $B$ is a key part of what makes it such an efficient computational tool for this purpose.

one-dimensional family of anomaly-consistent chiral matter spectra with generic matter in 4D; all other compact simple Lie groups have no chiral solutions to the anomaly cancellation conditions with only generic matter representations. For reference, generic matter in these models includes the following complex representations:

- SU($N$): fundamental and two-index antisymmetric;
- SO($4k+2$): spinor;
- $E_6$: fundamental.

To see that these are the only gauge groups admitting chiral generic matter, note first that the set of generic matter for a simple gauge group comprises three representations if the group has an independent quartic Casimir and two representations otherwise (this can be seen in the 6D context as coming from the fact that the anomaly cancellation conditions depend on quadratic and quartic invariants of the gauge group). One of these representations is always the adjoint, which is self-conjugate, and thus there are at most two representations that can contribute chirally to the spectrum in any case. For groups with an independent cubic Casimir, the number of independent chiralities is further reduced by one by the 4D anomaly cancellation equations. One can then carry out a case-by-case analysis of the compact simple Lie groups to determine the number of independent chiralities in each case.

For SU($N$), $N \geq 5$, there is an independent quartic Casimir, giving two complex generic representations, and an independent cubic Casimir, reducing the number of independent chiral families to one. For SU(2), every representation is self-conjugate; for SU(3), there is an independent cubic Casimir but no independent quartic Casimir; and for SU(4), there is an independent cubic Casimir and the two-index antisymmetric representation is self-conjugate; thus, in all these cases, there are no chiral solutions. The group SO($N$) only has complex representations for $N = 4k+2$, with only the spinor being complex among generic matter representations, and has no independent cubic Casimir, thus having a one-dimensional family of generic chiral spectra for these $N$. None of the exceptional groups has an independent quartic Casimir, giving only one generic representation other than the adjoint, and of these, only $E_6$ has complex representations; the $E_6$ fundamental is complex, and $E_6$ has no independent cubic Casimir, leaving a one-dimensional family of generic chiral spectra.

Thus, we expect a one-dimensional family of chiral solutions for the simple gauge groups SU($N$), $N \geq 5$, SO($4k+2$), $k \geq 2$, and $E_6$, and no chiral solutions for all other compact simple Lie groups.

### *6.2. Summary of F-theory fluxes for simple nonabelian models.*

*6.2.1. Fluxes in universal (simple)* G *models* Universal G models with simple nonabelian gauge symmetry can be described in F-theory using Tate models, i.e., Weierstrass models presented in Tate form [63]

$$y^2 z + a_1 xyz + a_3 yz^2 - (x^3 + a_2 x^2 z + a_4 xz^2 + a_6 z^3) = 0 \tag{6.1}$$

with a choice of tuning

$$a_n = a_{n,m_n} \sigma^{m_n} , \tag{6.2}$$

which characterizes the sections $a_n$ of the $n$th tensor power of the anticanonical bundle of the base $B$ in the vicinity of the gauge divisor $\sigma = 0$. Note that the divisor class of $\sigma = 0$ is $[\sigma] = \Sigma$ and hence the divisor classes of the tuned sections are

$$[a_{n,m_n}] = n(-K) - m_n \Sigma\,. \tag{6.3}$$

In all nontrivial cases that we study without codimension-three (4, 6) singularities, and excluding the case $\mathsf{G} = \mathrm{SO}(11)$, we use the methods of Appendices E and F to show there is a one-dimensional family of independent F-theory fluxes preserving the 4D gauge group $\mathsf{G}_{\mathrm{na}}$ that take the form

$$\Theta_{\mathrm{phys}} = \phi \Sigma \cdot [p(a_{n,s_n})] \cdot [p'(a_{n,s_n})], \quad \phi \in \mathbb{Z}\,, \tag{6.4}$$

where in the above formula the bracketed expressions are the classes of polynomials $p$ of the sections $a_{n,s_n}$ that depend on the choice of gauge group. See Table 1 for results for various groups $\mathsf{G}_{\mathrm{na}}$.

The physical significance and interpretation of the above results is perhaps more transparent in the lattice $M_C \Lambda_C$ of symmetry-constrained fluxes. For example, consistent with the argument in the previous subsection, that each model admits at most a one-parameter family of chiral multiplicities, we find in each case we study that the non-trivial fluxes $\Theta_{ij} \in M_C \Lambda_C$ can be expressed as

$$\Theta_{ij} = M_{C(ij)(kl)} \phi^{kl} \propto \frac{\ell(\phi^{kl})}{\det \kappa} \Sigma \cdot [p(a_{n,s_n})] \cdot [p'(a_{n,s_n})] = \chi_{\mathsf{r}}, \tag{6.5}$$

for some complex representation $\mathsf{r}$ and where $\ell(\phi^{kl})$ is a linear combination of the parameters $\phi^{kl}$ whose precise form depends on $\mathsf{G}$; here, since $\mathsf{G}$ is simply-laced, $\kappa_{ij} = -W_{i|j}$ is the Cartan matrix for $\mathsf{G}$. Moreover, since rk $M_C = 1$ and $M_C^{\mathrm{t}} = M_C$, the coefficients of the parameters $\phi^{ij}$ in the linear expressions $\ell(\phi^{ij})$ are identical to the proportionality constants relating different nontrivial $\Theta_{ij}$, it follows that straightforwardly that

$$\Theta_{kl} \phi^{kl} = \frac{\ell(\phi^{ij})^2}{\det \kappa} \Sigma \cdot [p(a_{n,s_n})] \cdot [p'(a_{n,s_n})] = \int_X G \wedge G, \tag{6.6}$$

which is non-negative provided $\Sigma \cdot [p] \cdot [p'] \geq 0$ [30]; this is consistent with the assertion that $G$ is self-dual [74], which in turn implies $\int G \wedge G = \int G \wedge *G = \int |G|^2 \geq 0$; see e.g. Equation (6.18) for a specific example of Eq. (6.6) in the context of the SU(5) model. As we explain in Sect. 6.2.2, $\ell(\phi^{ij})/\det \kappa$ is an integer and hence $\Theta_{ij}$ are manifestly integer-valued.

*6.2.2. Integrality conditions for symmetry preserving flux backgrounds* Before proceeding to examples, let us justify the integrality condition $\ell(\phi^{ij})/\det \kappa \in \mathbb{Z}$, which ensures that our expression (3.14) for $M_{C(IJ)(KL)}$ leads to integer fluxes $\Theta_{ij}$ in (6.5). This integrality condition is of course guaranteed, provided that the symmetry constraints $\Theta_{I\alpha} = 0$ are solved over $\mathbb{Z}$ (assuming $\phi^{IJ} \in \mathbb{Z}$), since $M$ is an integer matrix, but nevertheless for the sake of clarity we spell out explicitly how the integrality condition propagates through to the final expressions in the case where the constraints are imposed

[30] In order for Tate models to be consistent and not give rise to a larger gauge algebra, it is necessary that the line bundles to which the $a_{n,s_n}$ appearing in the table are associated have non-empty spaces of sections. Mathematically, this can be expressed as the requirement that the divisor classes $[a_{n,s_n}]$ are effective; the resulting intersection products are negative as long as these divisors are non-rigid.

first and there is an additional quantization condition on the domain of the mapping $C$ defined in Eq. (3.7). This integrality condition is an explicit example of the type of quantization constraint discussed earlier in Sect. 2.8 and Sect. 3.2.

To see how this works, we use the symmetry constraints to derive a condition on the combination of intersection numbers and parameters in the numerator of (6.5). First notice that the local Lorentz symmetry constraint $\Theta_{\alpha\beta} = 0$ implies (see (C.6) and (C.7)) $\phi^{0\gamma} = -\phi^{00} K^\gamma + \phi^{ij}\kappa_{ij}\Sigma^\gamma$ and $\phi^{\beta\gamma} = -\phi^{ij}\kappa_{ij} K^\beta \Sigma^\gamma$. Since these expressions are polynomial in the distinctive parameters, it is evident that solving the constraints does not impose any conditions on $\phi^{ij}$. We thus turn our attention to the gauge symmetry constraints $\Theta_{\alpha i} = 0$, which imply $\phi^{\beta j}\kappa_{ij}\Sigma^\gamma = -\phi^{jk}\Delta^\beta_{\mathsf{R}}\rho^{\mathsf{R}}_{ijk}\Sigma^\gamma$ and for nonzero $\Sigma^\gamma$ further imply

$$\phi^{\beta j}\kappa_{ij} = -\phi^{jk}\Delta^\beta_{\mathsf{R}}\rho^{\mathsf{R}}_{ijk}\,, \tag{6.7}$$

where we note that $\rho^{\mathsf{r}}_{ijk}$ in the above equation is defined by the intersection numbers $W_{ijk} \cdot D_\alpha = \rho^{\mathsf{r}}_{ijk} C_{\mathsf{r}} \cdot D_\alpha$; see Eq. (B.9). Since $\phi^{\beta j}$ is assumed to be an integral lattice vector for every $\beta$, the right-hand side of the above equation must lie in the root lattice of the group $\mathsf{G}$. Comparing the above equation to the list of necessary and sufficient conditions in Table 7 of Appendix I for a lattice vector to lie in the root lattice of a simple Lie group, we obtain for each universal (simple and simply-laced) $\mathsf{G}$ model an integrality condition of the form

$$\frac{\Delta^\beta_{\mathsf{R}}\rho^{\mathsf{R}}_{ijk} c^i \phi^{jk}}{\det\kappa} \in \mathbb{Z}\,, \tag{6.8}$$

where the choice of coefficients $c^i$ depends on $\mathsf{G}$. In all cases we study, we find that $\Delta^\beta_{\mathsf{R}}\rho^{\mathsf{R}}_{ijk} c^i \phi^{jk} \equiv D^\beta \ell(\phi^{ij}) \mod \det\kappa$, and hence for generic coefficients $\Delta^\beta_{\mathsf{R}}\rho^{\mathsf{R}}_{ijk}$ the above condition reduces to

$$\frac{\ell(\phi^{ij})}{\det\kappa} \in \mathbb{Z}\,. \tag{6.9}$$

The above integrality condition must be satisfied for any allowed set of integer fluxes $\phi^{ij}$ that preserve 4D local Lorentz and gauge symmetry, guaranteeing that all chiral matter indices (6.5) automatically take integer values. This is demonstrated explicitly in the case of $\mathsf{G} = \mathrm{SU}(5)$ in Sect. 6.4.

This analysis in general gives a sufficient condition, for each of the $\mathsf{G}$ models studied here, for the chiral matter spectrum to have certain multiplicities. As discussed in Sect. 2.8, however, inclusion of fluxes in $H^{\mathrm{hor}}_4(X,\mathbb{Z}) \oplus H^{\mathrm{rem}}_{2,2}(X,\mathbb{Z})$ may permit a broader set of possible chiral multiplicities.

*6.3. SU(2) model.* We now discuss explicit examples. We begin with a very simple example that has been well studied in the literature, but which nevertheless illustrates the issue of unimodularity of the intersection pairing $M_{\mathrm{red}}$, namely the universal SU(2) model. (For additional background about SU($N$) models and their resolutions, see Appendix F.1.) For $\mathsf{G} = \mathrm{SU}(2)$, we find that the reduced intersection pairing $M_{\mathrm{red}}$ is resolution-invariant and the constraints that local Lorentz and SU(2) symmetry are unbroken in 4D forces all the flux backgrounds $\phi$ to vanish, so there are no nontrivial vertical fluxes and no chiral matter. Identical conclusions follow for $\mathsf{G} = \mathrm{SU}(3)$ and SU(4).

**Table 1.** F-theory fluxes for universal G models with arbitrary characteristic data

| Kodaira fiber | G | $\Delta^{(2)}$ | $\Theta_{\text{phys}}$ ($\Theta_{(4,6)}$) | geometric constraints |
|---|---|---|---|---|
| $I_5^s$ | SU(5) | $a_1^4(a_{6,5}a_1^2 - a_{3,2}a_{4,3}a_1 + a_{2,1}a_{3,2}^2)$ | $\phi\Sigma \cdot [a_1] \cdot [a_{6,5}]$ | $\chi_{\mathbf{5}} + \chi_{\mathbf{10}} = 0$ |
| $I_6^s$ | SU(6) | $a_1^4(a_1^2 a_{6,6} - a_1 a_{3,3} a_{4,3} - a_{4,3}^2)$ | $\phi\Sigma \cdot [a_1] \cdot [a_{4,3}]$ | $\chi_{\mathbf{6}} + 2\chi_{\mathbf{15}} = 0$ |
| $I_6^s$ | SU(6)° | $a_1^3(a_{6,6}a_1^3 - a_{3,2}a_{4,4}a_1^2 + a_{2,2}a_{3,2}^2 a_1 - a_{3,2}^3)$ | $(\phi\Sigma \cdot [a_1] \cdot [a_{3,2}^3])$ | $\chi_{\mathbf{6}} = 0$ |
| $I_7^s$ | SU(7) | $a_1^4(a_1^2 a_{6,7} - a_1 a_{3,3} a_{4,4} + a_{2,1} a_{3,3}^2)$ | $\phi\Sigma \cdot [a_1] \cdot [a_{6,7}]$ $(\phi'\Sigma \cdot [a_1] \cdot [a_{2,1}])$ | $\chi_{\mathbf{7}} + 3\chi_{\mathbf{21}} = 0$ |
| $I_6^{ns}$ | Sp(6) | $a_2^2(a_2 a_{3,3}^2 - a_{4,3}^2 + 4a_2 a_{6,6})$ | (?) | — |
| $I_1^{*s}$ | SO(10) | $a_{2,1}^3 a_{3,2}^2$ | $\phi\Sigma \cdot [a_{2,1}] \cdot [a_{6,5}]$ | any $\chi_{\mathbf{16}}$ |
| $I_2^{*ns}$ | SO(11) | $a_{2,1}^2(4a_{2,1}a_{6,5} - a_{4,3}^2)$ | $\phi\Sigma \cdot [a_{2,1}] \cdot [a_{6,5}]$ | — |
| $I_2^{*s}$ | SO(12) | $a_{2,1}^2(4a_{2,1}a_{6,5} - a_{4,3}^2)$ | $(\phi\Sigma \cdot [a_{2,1}] \cdot [a_{4,3}^2])$ | — |
| $IV^{*s}$ | $E_6$ | $a_{3,2}^4$ | $\phi\Sigma \cdot [a_{3,2}] \cdot [a_{6,5}]$ | any $\chi_{\mathbf{27}}$ |
| $III^*$ | $E_7$ | $a_{3,3}^4$ | $(\phi\Sigma \cdot [a_{4,3}] \cdot [a_{6,5}])$ | — |
| $IV^{*ns}$ | $F_4$ | $a_{6,4}^2$ | (?) | — |
| $I_0^{*ns}$ | $G_2$ | $4a_{4,2}^3 + 27a_{6,3}^2$ | 0 | — |

(Partial) resolutions of these models are taken from [20]. The final column matches the linear 4D anomaly conditions in all known examples. $\Delta^{(2)}$ is the codimension-two component of the discriminant locus restricted to the gauge divisor $\sigma = 0$. Models that admit 4D chiral matter are indicated in blue and satisfy $\Theta_{\text{phys}} = \chi_{\mathsf{r}_*}$ (where $\chi_{\mathsf{r}_*}$ is the minimal chiral index), while models whose fluxes are proportional to some number of codimension-three (4, 6) loci are indicated in red (note that the $F_4$ and $Sp(n)$ model fluxes appear to correspond to (4, 6) points that have not been resolved, and hence for which the explicit form of the flux is still unknown.) The SO-type groups listed above range from those of smallest rank that admit nontrivial flux to those of largest rank for which the corresponding model does not have (4, 6) loci in codimension two; the same is true of the Sp-type groups, with the caveat that none have been identified that admit nontrivial flux (note that the bases of divisors and resolutions for the $I_{2n}^{ns}$ and $I_{2n+1}^{ns}$ models appear to be identical.) The $E_8$ model is suppressed because it has codimension-two (4, 6) singularities for generic characteristic data. SU($N$) models with $N > 6$ contain codimension-three (4, 6) points; however for SU(6)°, SU(7) the geometric constraints are stated under the restriction that the characteristic data are chosen to ensure that these points are absent, noting that under analogous conditions a similar pattern of fluxes may persist for SU($N > 7$). Note that the SO(11) model flux does not correspond to chiral matter

*6.3.1. Absence of chiral matter* The SU(2) model is characterized by an $\mathrm{I}_2$ Kodaira singularity over the divisor $\Sigma$ and has matter in the representations **2**, **3** with weights

$$\begin{aligned} &w_+^{\mathbf{2}} = (1), \quad w_-^{\mathbf{2}} = (-1) \\ &w_+^{\mathbf{3}} = (2), \quad w_0^{\mathbf{3}} = (0), \quad w_-^{\mathbf{3}} = (-2)\,. \end{aligned} \tag{6.10}$$

The unique resolution $X_1 \to X_0$ admitting a holomorphic zero section consists of a single blowup. In this case, there is a single SU(2) Cartan divisor $\hat{D}_i$ whose nonzero quadruple intersection numbers are (see Eq. (E.23) and above for details on how to evaluate the pushforwards explicitly)

$$\begin{aligned} W_{\alpha\beta ii} &= W_{ii} \cdot D_\alpha \cdot D_\beta = -2\Sigma \cdot D_\alpha \cdot D_\beta \\ W_{\alpha iii} &= W_{iii} \cdot D_\alpha = 2\Sigma \cdot (2K - \Sigma) \cdot D_\alpha \\ W_{iiii} &= W_{iiii} = 2\Sigma \cdot (-4K^2 + 2K \cdot \Sigma - \Sigma^2)\,, \end{aligned} \tag{6.11}$$

with the remaining intersection numbers involving $\hat{D}_i$ vanishing.

The SU(2) model provides a simple illustration of the procedure, discussed towards the end of Sect. 5, for using low-energy effective 5D physics as a shortcut to determine the values of the sign and floor functions appearing in the field theoretic expressions for the 3D CS couplings. In this case, the floor functions vanish because the zero section is holomorphic. Furthermore, matching the pushforwards of the above intersection numbers with the 5D CS couplings (where the multiplicities $n_{\mathsf{R}}$ are replaced by the matter curves $C_{\mathsf{R}}$) fixes the values of the sign functions to be $\mathrm{sign}(\varphi \cdot w_\pm^{\mathsf{R}}) = \pm 1$. In detail,

$$\begin{aligned} k_{\bar{0}ii} &= -\frac{1}{12}\sum_{w\in\mathsf{r}} C_{\mathsf{r}} \sum_{i=\pm} (w_i^{\mathsf{r}})^2 \\ &= -\frac{1}{12}\left[\frac{1}{2}\Sigma \cdot (\Sigma + K)\left(2^2 + (-2)^2\right) + \Sigma \cdot (-8K - 2\Sigma)\left(1^2 + (-1)^2\right)\right] \\ &= (-2\Sigma) \cdot (-\frac{1}{2}K) \\ &= W_{\bar{0}ii} \end{aligned} \tag{6.12}$$

$$\begin{aligned} k_{iii} &= -\frac{1}{2}\sum_{w\in\mathsf{R}} C_{\mathsf{R}} \sum_{i=\pm} (w_i^{\mathsf{R}})^3 \mathrm{sign}(\varphi \cdot w_i) \\ &= -\frac{1}{2}\left[\frac{1}{2}\Sigma \cdot (\Sigma + K)\left(2^3 - (-2)^3\right) + \Sigma \cdot (-8K - 2\Sigma)\left(1^3 - (-1)^3\right)\right] \\ &= 2\Sigma \cdot (2K - \Sigma) \\ &= W_{iii}\,. \end{aligned} \tag{6.13}$$

The matrix $M$ from this resolution takes the form Eq. (4.3), where the $W$ entries with two or more $i$ indices in the bottom right blocks are all even. Note also that the second Chern class in the basis of Eq. (4.3) is given by $c_2(X) = (27, [-39K^\alpha], 0, [(c_2(B))^{\alpha\beta} + 11K^\alpha K^\beta], [7K^\alpha], 0)$; since for any any F-theory base the class $c_2(B) + K^2$ is even [78], it follows that the constrained fluxes $\Theta_{I\alpha}$ are integral even when $c_2(X)$ is not even. Thus, we can always remove the null vectors and impose the constraints by setting $\Theta_{I\alpha} = 0$ without worrying about half-integer shifts.

It is straightforward to verify that there is one null vector of the form $\nu_{C\langle a\rangle}$, so $M_{\text{red}}$ takes the resolution-independent form Eq. (4.15) and $M_C = M_{\text{phys}} = 0$. It follows that 4D SU(2) models exist but do not admit chiral matter. This conclusion is well known for F-theory models with a tuned SU(2) gauge invariance and is reviewed in [17]. This result is consistent with expectations from 4D anomaly cancellation since the **2** of SU(2) is a self-conjugate representation.

*6.3.2. Unimodularity and integrality of the intersection pairing* We pause briefly to illustrate issues of unimodularity and integrality of the reduced intersection pairing $M_{\text{red}}$ in the context of the SU(2) model. Although the lattice $H^4(X, \mathbb{Z})$ is unimodular, the simple example of SU(2) illustrates that fact the intersection pairing acting on $H^{2,2}_{\text{vert}}(X_1, \mathbb{Z})$ is generically not unimodular since the absolute value of the determinant of $M_{\text{red}}$ is generically greater than one. As an example, for a gauge divisor $\Sigma = H \subset B = \mathbb{P}^3$ where $H$ is the hyperplane class, we find

$$M_{\text{red}} = \begin{pmatrix} -4 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \qquad \det M_{\text{red}} = 2\,. \tag{6.14}$$

While one might imagine that we have simply chosen the wrong basis for $H^{2,2}_{\text{vert}}(X_1, \mathbb{Z})$, the story is slightly subtler.

To further analyze the situation, we briefly digress to a related situation in the case of 6D F-theory compactifications, focusing in particular on the parallel case where we have a tuned SU(2) F-theory model over the base $B = \mathbb{P}^2$. In this case, the triple intersection numbers of the Cartan divisors $\hat{D}_i$ have the related value $W_{iii} = 2$. Thus, the intersection number of the Cartan divisor $\hat{D}_i$ with the curve $C_{ii} = \hat{D}_i \cap \hat{D}_i$ is 2. Unlike in the 4D case, the dimension of $H^4(X^{(3)})$ is equal to that of $H^2(X^{(3)})$ by Poincaré duality, and again by Poincaré duality we know that there must be a curve $C$ satisfying $C \cdot \hat{D}_i = 1$, $C \cdot \hat{D}_{I\neq i} = 0$ corresponding to a (possibly massive) state in the fundamental representation of SU(2) (see [83] for a related discussion). Thus, in this situation the curve $C_{ii}$ is not a primitive curve in $H_2(X^{(3)}, \mathbb{Z})$, but rather $C = C_{ii}/2$ is such a primitive curve and is Poincaré dual to $\hat{D}_i$.

This same story cannot hold, however, in the 4D SU(2) model. We do expect that there is a matter surface $S$ associated with matter in the fundamental representation of SU(2). This surface cannot simply be identified with $S_{ii}/2$, however, since the intersection of that surface with itself under the matrix (6.14) is $(1/2) \times (-2) \times (1/2) = -1/2$. Thus, the Poincaré dual of the surface $S_{ii} \in H^{\text{vert}}_{2,2}(X_1, \mathbb{Z})$ is not itself contained entirely in $H^{\text{vert}}_{2,2}(X_1, \mathbb{Z})$ and we see that the orthogonal decomposition (2.12) of $H_4(X_1, \mathbb{Z})$ with respect to the intersection pairing does not hold over $\mathbb{Z}$.

As we discuss below in the context of SU(5) models with chiral matter, this point indicates that the assumption $\phi^{IJ} \in \mathbb{Z}$ may be too restrictive for our analysis to explore all possible chiralities. Rather, it appears to be necessary to extend the analysis to account for contributions from the orthogonal complement of $H^{\text{vert}}_{2,2}(X)$ in $H_4(X)$, which to our knowledge has yet to be completely understood.

*6.4. SU(5) model.* The SU(5) model (see Appendix F.1) is the simplest example of a universal SU($N$) model with chiral matter. The full set of resolutions of the SU(5) model admitting a holomorphic zero section were worked out in [54] (see also [53,95]) and the chiral indices were computed for a subset of these resolutions in [16] using similar methods to those described in this paper, as well as by other methods in e.g. [96,97].

The SU(5) model describes chiral matter in the fundamental (**5**) and two-index antisymmetric (**10**) representations. The 4D chiral anomaly cancellation condition requires that

$$\chi_{\mathbf{5}} + \chi_{\mathbf{10}} = 0\,. \tag{6.15}$$

We begin by analyzing the model with a specific resolution, $X_4 \to X_0$, which was described as the toric 'phase I' resolution in [16] and as the resolution '$\mathscr{B}_{1,3}$' in [54]. The signs associated to the central charges of BPS particles transforming in the (complex) representations **5**, **10** can be found in Table 3. In Sect. 6.4.3 we consider several other resolutions and show explicitly that $M_{\text{red}}$ is the same up to an integral choice of basis for each of these resolutions.

*6.4.1. Chiral matter multiplicities* Plugging intersection numbers into (3.14) for the specific resolution just mentioned, we learn that there are four nontrivial constrained fluxes $\Theta_{ij}$ (Cartan divisors take indices $i = 2, \ldots, 5$) that satisfy three linear relations, in agreement with the solution described in [16]:

$$\Theta_{33} = -\Theta_{35} = -\Theta_{44} = \Theta_{45}\,. \tag{6.16}$$

In particular (compare also to [14]),

$$\Theta_{33} = \frac{\ell(\phi^{ij})}{5}\Sigma \cdot [a_1] \cdot [a_{6,5}] = \frac{1}{5}(\phi^{33} - \phi^{35} - \phi^{44} + \phi^{45})\Sigma \cdot K \cdot (6K + 5\Sigma)\,. \tag{6.17}$$

Note that the proportionalities in Eq. (6.16) imply

$$\phi^{33}\Theta_{33} + \phi^{35}\Theta_{35} + \phi^{44}\Theta_{44} + \phi^{45}\Theta_{45} = (\phi^{33} - \phi^{35} - \phi^{44} + \phi^{45})\Theta_{33} = \ell(\phi^{ij})\Theta_{33}\,. \tag{6.18}$$

Comparing Eq. (6.16) to the one-loop 3D CS couplings, we learn that

$$\chi_{\mathbf{5}} = -\Theta_{33}, \quad \chi_{\mathbf{10}} = -\Theta_{44}, \tag{6.19}$$

and thus we recover the 4D anomaly cancellation equation (6.15). As explained in Sect. 6.2.2, solving the the gauge symmetry constraints $\Theta_{i\alpha} = 0$ over $\mathbb{Z}$ ensures that the flux (6.17) is integer-valued despite the factor of 5 in the denominator; we go through this analysis in some detail here to illustrate this point. First note that a necessary and sufficient condition for an integral vector $v_i$ to lie in the $\mathfrak{su}(5)$ root lattice is (see Appendix I) $v_2 + 2v_3 + 3v_4 + 4v_5 \equiv 0 \pmod 5$, which is equivalent to the condition

$$-2v_2 + v_3 + 4v_4 + 2v_5 \in 5\mathbb{Z}. \tag{6.20}$$

We can use this condition and the logic following (2.25) to determine a further constraint on the parameters $\phi^{ij}$ by noting that from (C.5) and (B.11), the gauge symmetry constraints imply

$$\Theta^{\text{d}}_{\alpha i} = \phi^{\beta j}(D_\beta \cdot D_\alpha \cdot \Sigma)\kappa_{ij}, \tag{6.21}$$

where the superscript 'd' indicates that $\Theta^{\mathrm{d}}_{\alpha i}$ is the part of $\Theta_{\alpha i}$ that only depends explicitly on the distinctive parameters $\phi^{\hat{I}\hat{J}}$. It follows that $\Theta^{\mathrm{d}}_{\alpha i}$ lies in the tensor product of the $\mathfrak{su}(5)$ root lattice (for the $i$ index) and the sublattice of $H_{1,1}$ spanned by $\Sigma \cap D_\alpha$ (for the $\alpha$ index). Applying the linear combination from the condition (6.20) to the fluxes $\Theta^{\mathrm{d}}_{\alpha i}$ we find

$$-2\Theta^{\mathrm{d}}_{\alpha 2} + \Theta^{\mathrm{d}}_{\alpha 3} + 4\Theta^{\mathrm{d}}_{\alpha 4} + 2\Theta^{\mathrm{d}}_{\alpha 5} = -(\Sigma \cdot D_\alpha) \cdot K(\phi^{33} - \phi^{35} - \phi^{44} + \phi^{45}) \,. \tag{6.22}$$

and thus $K(\phi^{33} - \phi^{35} - \phi^{44} + \phi^{45})$ must lie in $5H^{1,1}(B, \mathbb{Z})$. This condition, which is necessary to ensure that the full SU(5) gauge symmetry is preserved, is sufficient to guarantee that the chiral matter multiplicities determined by the flux (6.17) are integer-valued. Note that for a generic base $B$, $K$ is not 5 times an integral divisor, so the parameters must typically satisfy the condition

$$(\phi^{33} - \phi^{35} - \phi^{44} + \phi^{45}) \in 5\mathbb{Z} \,. \tag{6.23}$$

*6.4.2. Flux quantization* We illustrate the non-unimodularity of $M_{\mathrm{red}}$ and the quantization of the parameters $\phi^{ij}$ in some further detail with a concrete one-parameter family of SU(5) examples. Consider the case $B = \mathbb{P}^3$, $K = -4H$, $\Sigma = nH$ where $H$ is the hyperplane class of $\mathbb{P}^3$. Using the parametrization of the nullspace of $M_C$ given in (4.7) along with the explicit results (6.16), we find that a suitable basis for $H^{\mathrm{vert}}_{2,2}(X, \mathbb{Z})$ is $S_{0\alpha}, S_{\alpha\alpha}, S_{\alpha i}, S_{35}$ (with the index $\alpha$ for the only base divisor) in terms of which the intersection pairing is given by

$$M_{\mathrm{red}} = \begin{pmatrix} -4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2n & n & 0 & 0 & 4n \\ 0 & 0 & n & -2n & n & 0 & -4n \\ 0 & 0 & 0 & n & -2n & n & 4n \\ 0 & 0 & 0 & 0 & n & -2n & -4n \\ 0 & 0 & 4n & -4n & 4n & -4n & -4n^2 \end{pmatrix} \tag{6.24}$$

In this example, $K$ is not 5 times an integer divisor, and the integrality condition (6.23) follows from assuming that $\phi^{IJ} \in \mathbb{Z}$. This can be seen explicitly as follows: imposing the local Lorentz and gauge symmetry constraints, one finds that the flux backgrounds $\phi^{\alpha i}$ can be solved for in terms of the flux backgrounds $\phi^{ij}$

$$\phi^{\alpha 2} = \frac{8}{5}\phi^{35}, \quad \phi^{\alpha 3} = -\frac{4}{5}\phi^{35}, \quad \phi^{\alpha 4} = \frac{4}{5}\phi^{35}, \quad \phi^{\alpha 5} = -\frac{8}{5}\phi^{35} \,. \tag{6.25}$$

The second Chern class for this class of models over $\mathbb{P}^3$ in the reduced basis for the resolution $\mathscr{B}_{1,3}$ is

$$\begin{aligned} c_2(X_4) = {} & 2(n-22)S_{H3} + 2(3n-34)S_{H4} + 2(n-20)S_{H5} \\ & +48S_{0H} + 182S_{HH} - 16S_{H2} + S_{35}. \end{aligned} \tag{6.26}$$

Since all coefficients except that of $S_{35}$ are even integers, we see that the proper shifted lattice for allowed values of $\phi$ keeps all $\phi^{IJ}$ integral except $\phi^{35}$, which must take a half-integral value.

The above relations imply that for any nontrivial integer solution to these conditions we have[31]

$$\phi^{35} = \frac{5}{2}(2k+1) \in \frac{5}{2}(2\mathbb{Z}+1)\,. \tag{6.27}$$

The single flux spanning $M\Lambda_{\text{phys}}$ is then given by

$$\Theta_{35} = \chi_{\mathbf{5}} = \frac{4}{5}n(24-5n)\phi^{35} = 2n(24-5n)(2k+1)\,, \tag{6.28}$$

so the chiral matter multiplicities are necessarily integral.

On the other hand, clearly unimodularity of $H_{2,2}^{\text{vert}}(X_4, \mathbb{Z})$ is not satisfied as for any $n \geq 1$

$$\det M_{\text{red}} = n^5(20n-96) \neq \pm 1\,. \tag{6.29}$$

For example, for $n = 1$ the determinant is $-76$. This means that Poincaré duality guarantees that there are flux backgrounds that are not of the simple form characterized by (half-)integer $\phi^{ij}$. This is parallel to the situation discussed for SU(2) models above in Sect. 6.3.2. Taking for example the $n = 1$ case, as for SU(2) the elements of the dual lattice to the lattice $H_{2,2}^{\text{vert}}(X_4, \mathbb{Z})$ with inner product (6.24) do not have integer norms. Thus, the Poincaré dual to e.g. the surface $S_{33}$ must project to a fractional vector in $H_{2,2}^{\text{vert}}(X_4)$ and therefore must also contain a component of $H_{2,2}^{\text{rem}}(X_4) \oplus H_4^{\text{hor}}(X_4)$.

An interesting question, which to our knowledge is not addressed anywhere in the literature and to which the answer seems unknown, is whether or not including such flux backgrounds can produce chiral matter multiplicities that are more general than those given by, e.g., Eq. (6.28).[32] For example, for $n = 1$ the allowed chiral multiplicities from this analysis should be 38, 114, .... Naively it might seem that Poincaré duality would suggest that arbitrary integer matter multiplicities should be possible since there is always an integral flux background in $H_4(X_4, \mathbb{Z})$ that gives $\Theta_{33} = 1, \Theta_{\alpha i} = 0$. It may be, however, that the components of $H_4(X_4, \mathbb{Z})$ that must be turned on for this flux background Poincaré dual to $S_{33}$ (recall the discussion of Poincaré duality and its relation to flux quantization at the of Sect. 2.8) would break gauge invariance (as discussed, e.g., in [49]) or some other necessary feature of the F-theory vacuum so that such further chiral multiplicities would be ruled out. Appealing to a heterotic dual description also does not immediately clarify this question, since (as demonstrated in e.g. [16]) the chiral multiplicities achieved through the spectral cover construction match the F-theory chiral multiplicities coming from purely vertical flux backgrounds, though it is possible that additional chiral multiplicities could be realized through more general bundle constructions. We leave further investigation of these questions for future work.

[31] Note that the quantization condition is insensitive to the fact that we eliminated the redundant homology classes before imposing the Poincaré and gauge symmetry conditions. In particular, the homology relations can be used to show that $\phi^{35}$, $\phi^{44}$, $\phi^{45}$ are each proportional to $\phi^{33}$, and these proportionality factors can be used to convert (6.23) into a relation of the form (6.27).

[32] As discussed in Sect. 2.8, some necessary conditions on fractional coefficients for flux backgrounds in $H_{2,2}^{\text{vert}}(X, \mathbb{Z})$ have been considered in [29], but not all flux backgrounds satisfying these conditions need be permissible.

*6.4.3. Resolution-independence of the reduced intersection pairing* In this section we demonstrate the resolution independence of $M_{\mathrm{red}}$ for the three resolutions $\mathscr{B}_{1,3}$, $\mathscr{B}_{1,2}$, $\mathscr{B}_{2,1}$ described in [54]. In order to compute $M_{\mathrm{red}}$ for these three cases, we first write down the symmetry constrained fluxes:

$$\begin{aligned} \mathscr{B}_{1,3} &: \Theta_{33} = -\Theta_{44} = -\Theta_{35} = \Theta_{45} \\ \mathscr{B}_{1,2} &: \Theta_{34} = -2\Theta_{44} = -\Theta_{35} = \Theta_{45} \\ \mathscr{B}_{2,1} &: \Theta_{23} = -2\Theta_{33} = -\Theta_{24} = \Theta_{34} \,. \end{aligned} \tag{6.30}$$

The indices $jk$ of the above fluxes determine the basis elements $S_{0\alpha}$, $S_{\alpha\beta}$, $S_{\alpha i}$, $S_{jk}$ spanning $H_{2,2}^{\mathrm{vert}}(X,\mathbb{Z})$ in each of these three resolutions. We illustrate this specifically in the case $B = \mathbb{P}^3$, $\Sigma = nH$, where $H = D_\alpha$ is the hyperplane class of $\mathbb{P}^3$. First, we compare the resolutions $\mathscr{B}_{1,3}$ and $\mathscr{B}_{1,2}$, for which a common basis is $S_{0\alpha}$, $S_{\alpha\beta}$, $S_{\alpha i}$, $S_{35}$. We find (see (6.24))

$$M_{\mathrm{red}}(\mathscr{B}_{1,3}) = M_{\mathrm{red}}(\mathscr{B}_{1,2}) = \begin{pmatrix} -4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2n & n & 0 & 0 & 4n \\ 0 & 0 & n & -2n & n & 0 & -4n \\ 0 & 0 & 0 & n & -2n & n & 4n \\ 0 & 0 & 0 & 0 & n & -2n & -4n \\ 0 & 0 & 4n & -4n & 4n & -4n & -4n^2 \end{pmatrix}, \tag{6.31}$$

i.e., the two intersection pairing matrices are identical for these two resolutions. On the other hand, to compare the two resolutions $\mathscr{B}_{1,2}$ and $\mathscr{B}_{2,1}$, we must use a different basis. A suitable basis in which to compare $M_{\mathrm{red}}(\mathscr{B}_{1,2})$, $M_{\mathrm{red}}(\mathscr{B}_{2,1})$ is $S_{0\alpha}$, $S_{\alpha\beta}$, $S_{\alpha i}$, $S_{34}$, for which we find

$$M_{\mathrm{red}}(\mathscr{B}_{1,2}) = M_{\mathrm{red}}(\mathscr{B}_{2,1}) = \begin{pmatrix} -4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & n \\ 0 & 0 & -2n & n & 0 & 0 & 0 \\ 0 & 0 & n & -2n & n & 0 & 16n-2n^2 \\ 0 & 0 & 0 & n & -2n & n & 3n^2-20n \\ 0 & 0 & 0 & 0 & n & -2n & 4n \\ 0 & n & 0 & 16n-2n^2 & 3n^2-20n & 4n & -6n^3+72n^2-224n \end{pmatrix}. \tag{6.32}$$

Again, we find that for an appropriate choice of basis the intersection pairing matrices are identical. This implies that were we to identify $M_{\mathrm{red}}$ for the resolutions $\mathscr{B}_{1,3}$ and $\mathscr{B}_{2,1}$, we would be forced to identify a change of basis, from (6.31) to (6.32); the explicit matrix $U$ presented in (4.17) does the job for a particular choice of sign in $U_p = (\pm 1)$: solving for the undetermined coefficients in $U$ we find that they take integer values compatible with the congruence

$$M_{\mathrm{red}}(\mathscr{B}_{2,1}) = U^{\mathrm{t}} M_{\mathrm{red}}(\mathscr{B}_{1,3}) U \,. \tag{6.33}$$

A related change of basis illuminates further the question of flux quantization discussed in Sect. 6.4.2. As discussed in general in Sect. 4.3.3 and Appendix D, there is a (non-integral) change of basis $U$ of the form (4.17) from both of the forms (6.31) and

(6.32) to a canonical product form (4.18), given here by

$$M_{\mathrm{red}}^{\mathrm{cp}} = U^{\mathrm{t}} M_{\mathrm{red}} U = \begin{pmatrix} -4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2n & n & 0 & 0 & 0 \\ 0 & 0 & n & -2n & n & 0 & 0 \\ 0 & 0 & 0 & n & -2n & n & 0 \\ 0 & 0 & 0 & 0 & n & -2n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & n(96-20n)/5 \end{pmatrix}. \tag{6.34}$$

Because $U$ is non-integral in these cases, the lattice $\Lambda^{\mathrm{cp}}$ of flux backgrounds on which Eq. (6.34) acts is not $\mathbb{Z}^n$. On the other hand, the constraint equations simply set all but the last of the flux background parameters to vanish. The non-integer elements of $U$ involve terms of the form $m/5, m \in \mathbb{Z}$ in the lowest row, and the set of physical flux background parameters $(0, \ldots, 0, \phi) \in \Lambda_{\mathrm{cp}}$ are thus constrained so $\phi \in 5(2\mathbb{Z}+1)/2$, analogous to the constraint (6.23).

From this analysis we see that the chiral multiplicity given by, e.g., $\chi_{\mathbf{5}} = \Theta_{35}$ is $\chi_{\mathbf{5}} = 2n(24-5n)(2k+1)$, with $k \in \mathbb{Z}$, in agreement with Eq. (6.28). We can relate the canonical form (6.34) to the lattice $\Lambda_{\mathrm{phys}} = \mathbb{Z}$, under which the intersection form becomes $M_{\mathrm{phys}} = (5n(96-20n))$. We expect on physical grounds that any valid F-theory resolutions should give rise to the same physics and the same $M_{\mathrm{phys}}$. Note that while any two such resolutions would admit non-integer transformations $U, V$ taking $M_{\mathrm{red}}$ to the canonical form Eq. (6.34), this is not quite sufficient to prove that $M_{\mathrm{red}}$ are the same in those two resolutions since there is no guarantee from what we have said here that $UV^{-1}$ is an integer matrix, as discussed further in Sect. 4.3.3 and Appendix D.

It is also worth noting that Eq. (6.34) gives an example of the self-duality condition as discussed in more general terms in Sect. 6.2.1. Namely, the SU(5) Weierstrass model on $\mathbb{P}^3$ is only consistently defined without enhancement when $n \leq 4$, in which case $M_{\mathrm{phys}}$ has a positive matrix entry and the flux background $\phi$ is self-dual.

*6.5. SU(6) model.* The SU(6) model describes chiral matter in the fundamental (**6**) and two-index antisymmetric (**15**) representations. The 4D anomaly conditions give

$$\chi_{\mathbf{6}} + 2\chi_{\mathbf{15}} = 0. \tag{6.35}$$

The signs of the BPS central charges associated to the fundamental and two-index antisymmetric representations can be found in Table 3. Plugging the intersection numbers into (3.14), we find that the only nonzero fluxes $\Theta_{ij}$ (Cartan indices are $i = 2, \ldots, 6$) satisfy the linear relations

$$\Theta_{33} = -\Theta_{34} = \Theta_{35} = \Theta_{45} = -\frac{1}{3}\Theta_{55} = -\Theta_{36} = \Theta_{56}\,, \tag{6.36}$$

where

$$\begin{aligned} \Theta_{33} &= \frac{\ell(\phi^{ij})}{6}\Sigma \cdot [a_1] \cdot [a_{4,3}^2] \\ &= \frac{1}{6}(\phi^{33} - \phi^{34} + \phi^{35} - \phi^{36} + \phi^{45} - 3\phi^{55} + \phi^{56})\Sigma \cdot K \cdot (8K + 6\Sigma)\,. \end{aligned} \tag{6.37}$$

Matching fluxes with the corresponding one-loop CS couplings implies $0 = \Theta_{22} = -\chi_{\mathbf{6}} - 2\chi_{\mathbf{15}}$, which, using Eq. (6.36), reproduces the 4D anomaly cancellation condition

(6.35). Inverting the linear system arising from matching the fluxes with 3D one-loop CS couplings, we find that the chiral indices can be expressed geometrically as

$$-\frac{\chi_{\mathbf{6}}}{2} = \chi_{\mathbf{15}} = \Theta_{33}\,. \tag{6.38}$$

Note that the gauge symmetry condition (2.11) implies, in an analogous fashion to the $\mathsf{G} = \mathrm{SU}(5)$ case,

$$2K^\alpha(\phi^{33} - \phi^{34} + \phi^{35} - \phi^{36} + \phi^{45} - \phi^{56}) \in 6\mathbb{Z}\,, \tag{6.39}$$

ensuring that $\Theta_{33} \in \mathbb{Z}$.

For comparison, we also comment on the flux for an alternative Tate tuning [98] (see also [99,100]) of the SU(6) model, denoted the SU(6)° model, that has matter in the exotic three-index antisymmetric representation rather than the usual two-index antisymmetric representation. Because the three-index antisymmetric representation is self-conjugate, one would naively expect the space of vertical F-theory fluxes to be empty. However, it turns out the SU(6)° model contains codimension-three (4, 6) singularities, leading to a nontrivial flux presumably given by the integral of the flux background over the surface component of the non-flat fiber visible at the (4, 6) point in the resolution $X_5$. See Table 1 for additional details.

*6.6. SO*(10) *model.* The SO(10) Tate model is characterized by a $\mathrm{I}_1^{*\mathrm{split}}$ singularity over a gauge divisor $\Sigma$ and contains chiral matter in the spinor representation (**16**); the multiplicity of matter in this representation is unconstrained by anomalies. We label Cartan divisors with indices $i = 2, \ldots, 6$. The signs of the BPS central charges associated to the spinor can be found in Table 5.

Using (3.14), we find that the F-theory fluxes satisfy the homology relations

$$\Theta_{22} = -\Theta_{24} = \Theta_{25} = \Theta_{44} = -\Theta_{46} = -\Theta_{55} = \frac{1}{2}\Theta_{66} \tag{6.40}$$

where

$$\begin{aligned}\Theta_{22} &= \frac{\ell(\phi^{ij})}{4} \cdot [a_{2,1}] \cdot [a_{6,5}]\\ &= \frac{1}{4}(\phi^{22} - \phi^{24} + \phi^{25} + \phi^{44} - \phi^{46} - \phi^{55} + 2\phi^{66})\Sigma \cdot (2K + \Sigma) \cdot (6K + 5\Sigma)\,.\end{aligned} \tag{6.41}$$

Matching with 3D one-loop CS terms, we find

$$\chi_{\mathbf{16}} = -\Theta_{22}\,. \tag{6.42}$$

The gauge symmetry condition (2.11) implies

$$(2K + \Sigma)^\alpha(\phi^{22} - \phi^{24} + \phi^{25} + \phi^{44} - \phi^{46} - \phi^{55} + 2\phi^{66}) \in 4\mathbb{Z}\,, \tag{6.43}$$

hence $\Theta_{22}$ is integer-valued. We again find no linear constraints on the SO(10) chiral spectrum other than those implied by anomaly cancellation.

*6.7. $E_6$ model.* Our final purely nonabelian example is $\mathsf{G} = E_6$, which is the only exceptional group with complex representations and hence the only exceptional group admitting chiral matter preserving the full gauge symmetry. The $E_6$ Tate model is characterized by a $IV^{*\text{split}}$ singularity over gauge divisor $\Sigma$ and contains chiral matter in the fundamental (**27**) representation, with a multiplicity unconstrained by local anomalies. Additional details about the resolution and corresponding signs of BPS central charges can be found in Appendix F.4.

Computing intersection numbers and substituting their values into the expression in (3.14), we find that the nontrivial constrained fluxes satisfy the homology relations

$$\frac{1}{2}\Theta_{22} = -\Theta_{25} = -\Theta_{33} = \Theta_{35} = -\Theta_{55} = \Theta_{56} = -\frac{1}{2}\Theta_{66} \tag{6.44}$$

where

$$\begin{aligned}\Theta_{35} &= \frac{\ell(\phi^{ij})}{3}\Sigma\cdot[a_{3,2}]\cdot[a_{6,5}]\\ &= \frac{1}{3}(2\phi^{22}-\phi^{25}-\phi^{33}+\phi^{35}-\phi^{55}+\phi^{56}-2\phi^{66})\Sigma\cdot(3K+2\Sigma)\cdot(6K+5\Sigma)\,.\end{aligned} \tag{6.45}$$

Comparing with the corresponding 3D one-loop CS couplings, we find

$$\chi_{\mathbf{27}} = \Theta_{35}\,, \tag{6.46}$$

in agreement with, e.g., Eq. (4.12) in [31]. Note that the gauge symmetry conditions (2.11) imply

$$\Sigma^{\alpha}(2\phi^{22}-\phi^{25}-\phi^{33}+\phi^{35}-\phi^{55}+\phi^{56}-2\phi^{66}) \in 3\mathbb{Z}\,, \tag{6.47}$$

which ensures that $\Theta_{35}$ is integer-valued.[33]

## 7. Models with a U(1) Gauge Factor

We now turn to the more general case of models with gauge symmetry $\mathsf{G} = (\mathsf{G}_{\text{na}} \times U(1))/\Gamma$. As discussed in previous sections, these models are complicated by the fact that the fluxes do not simply depend on the mutual triple intersections of the characteristic data $(K, \Sigma_s, W_{01})$, but rather also depend on the intersection products of all divisors $D_\alpha \in B$ with the height pairing divisor $W_{\bar{1}\bar{1}}$ associated to the U(1) factor—this is a reflection of the global geometric nature of U(1) gauge factors in F-theory, in contrast to the local nature of nonabelian gauge factors $\mathsf{G}_s \subset \mathsf{G}_{\text{na}}$. One notable consequence is that the nullspace of $M_C$ is not obviously computable for such models in a very general way without explicitly specifying $B$. A possible workaround to this complication, as discussed at the end of Sect. 3.3, is to further restrict to the sublattice $\Lambda_S \cap \{\phi^{1\alpha} = 0\}$; we describe an example of this analysis in Sect. 7.1.1. In the rest of this section we focus attention on specific bases $B$, where we can explicitly carry out the full flux analysis.

In Sect. 7.1 we analyze the $(SU(2) \times U(1))/\mathbb{Z}_2$ model in detail. In Sect. 7.2, we briefly summarize the results of the forthcoming paper [30] in which we use the methods of this paper to analyze the universal $(SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$ model from [27].

33 Curiously, in the special case that $\Sigma^\alpha \in 3\mathbb{Z}$ the gauge symmetry condition does not appear to place any special conditions on the parameters $\phi^{ij}$.

*7.1.* $(SU(2) \times U(1))/\mathbb{Z}_2$ *model.* Perhaps the simplest example of a model with gauge group $\mathsf{G}$ containing a U(1) gauge factor is the $F_6$ model studied in [28], with $\mathsf{G} = (\mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_2$ and matter transforming in the representations $\mathbf{1}_1, \mathbf{1}_2, \mathbf{2}_{\frac{1}{2}}, \mathbf{2}_{-\frac{3}{2}}$. A related F-theory model was recently analyzed in [101].

The 4D anomaly cancellation conditions impose the constraints that the chiral matter multiplicities must correspond to an integer multiple of the family

$$(\chi_{\mathbf{1}_1}, \chi_{\mathbf{1}_2}, \chi_{\mathbf{2}_{\frac{1}{2}}}, \chi_{\mathbf{2}_{-\frac{3}{2}}}) = (2, -1, -3, -1). \tag{7.1}$$

The characteristic data of this class of F-theory models consists of the canonical class $K$ and the two divisor classes $S_7, S_9$, in terms of which the SU(2) gauge divisor is given by $S_8 = -K + S_9 - S_7$. Explicitly, the singular $F_6$ model $X_0$ is realized as a hypersurface in an ambient $\mathbb{P}^2$ bundle over arbitrary smooth base $B$, given by

$$s_1u^3 + s_2u^2v + s_3uv^2 + s_4v^3 + s_5u^2w + s_6uvw + s_7v^2w + s_8uw^2 = 0 \tag{7.2}$$

where $[u : v : w]$ are homogeneous coordinates of the ambient space fibers defined by the hyperplane classes

$$[u] = \boldsymbol{H} + \boldsymbol{K} + \boldsymbol{S}_9, \quad [v] = \boldsymbol{H} - \boldsymbol{S}_7 + \boldsymbol{S}_9, \quad [w] = \boldsymbol{H} \tag{7.3}$$

(note $\boldsymbol{H}$ is the hyperplane class of the fibers and $\boldsymbol{K}, \boldsymbol{S}_7, \boldsymbol{S}_9$ are the pullbacks of the classes $K, S_7, S_9$ in the base to the Chow ring of the ambient space) and the divisor classes of the sections $s_m$ appearing the above hypersurface equation, namely $S_m = [s_m]$, are given by

$$S_1 = -3K - S_7 - S_9, \quad S_4 = 2S_7 - S_9 \tag{7.4}$$

along with

$$S_2 = \frac{1}{3}(2S_1 + S_4), \quad S_3 = \frac{1}{3}(S_1 + 2S_4), \quad S_5 = \frac{1}{2}(S_1 + S_8), \quad S_6 = \frac{1}{2}(S_7 + S_8). \tag{7.5}$$

For a good model with gauge group $\mathsf{G} = (\mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_2$, the characteristic data is constrained so that the divisor classes $S_1, S_4, S_7, S_8$ are effective.[34]

*7.1.1. Resolution, Chern–Simons terms, chiral index* The resolution $X_2 \to X_0$ described in [28] entails a sequence of two blowups acting on the $\mathbb{P}^2$ fibers so that the resulting smooth model $X_2$ may be viewed as a hypersurface in an ambient projective bundle with fibers isomorphic to $\mathbb{P}_{F_6}$, i.e., the blowup of $\mathbb{P}^2$ at two points. The hypersurface equation defining $X_2$ can be computed systematically by exploiting the fact that to every two-dimensional toric variety $\mathbb{P}_{F_i}$ is associated a canonically defined genus one curve in $\mathbb{P}_{F_i}$ that can be realized as a a zero section of the anticanonical bundle.

In order to use the pushforward technology to compute intersection numbers and other relevant characteristic numbers associated to $X_2$, we regard the singular model $X_0$ as a hypersurface of the ambient projective bundle $Y_0 = \mathbb{P}(\mathscr{V}) \to B$, with $\mathbb{P}^2$ fibers described by Eq. (7.3). Combining this data with the classes of the generators of the

[34] These divisor classes are associated with the vertices of the dual polytope of the toric fiber defining the $F_6$ model in [28].

centers of the blowups comprising the resolution $X_2 \to X_0$, it is straightforward to explicitly compute the pushforwards of the intersection numbers in a suitable basis and evaluate the geometric expressions for the fluxes using (3.14)–(3.16).

The computation of the 3D Chern–Simons terms following the strategy of matching intersection numbers of the form $W_{\bar{I}\bar{J}\bar{K}\alpha}$ to 5D Chern–Simons terms $k^{5D}_{\bar{I}\bar{J}\bar{K}}$ (where $\bar{I}, \bar{J}, \bar{K} = \bar{0}, \bar{1}, i$) in this case is more involved due to the fact that the resolved model $X_2$ has a rational, as opposed to holomorphic, zero section. This is because one needs to determine, in addition to the signs of the BPS central charges, the ratio of their magnitudes to the KK modulus $m_{\text{KK}}$. Fortunately there is a simple geometric computation one can do to determine which particles (descending from M2 branes wrapping irreducible holomorphic curves in the M-theory background) have nontrivial KK charge. Notice that the pushforward of the intersection of the zero section and generating section is given by

$$\pi_*(\hat{D}_0 \cdot \hat{D}_1) = W_{01} = S_7 \,. \tag{7.6}$$

From the above expression we can infer that the primitive BPS particles in the representation $\mathsf{R}'$ carrying nontrivial KK charge must be associated to matter loci of the schematic form

$$C_{\mathsf{R}'} = S_7 \cdot (\cdots) \,. \tag{7.7}$$

Exploiting the fact that the spectrum of the $F_6$ model is known, we see that the classes of the relevant matter loci fitting this criterion are

$$C_{\mathbf{2}_{-\frac{3}{2}}} = S_7 \cdot (-K - S_7 + S_9), \quad C_{\mathbf{1}_2} = S_7 \cdot (2S_7 - S_9) \,. \tag{7.8}$$

The above analysis implies that the BPS particles transforming in the representations $\mathsf{R}' = \mathbf{2}_{-\frac{3}{2}}, \mathbf{1}_2$ have nontrivial KK charge. It follows that these are the only particles for which the KK mass is not larger than the Coulomb branch mass; we may therefore set $\lfloor |r_{\text{KK}}\varphi \cdot w_i^{\mathsf{R}}| \rfloor = 0$ for all other representations $\mathsf{R}$. Utilizing this simplifying assumption, we find a perfect match between the 5D Chern–Simons terms and the triple intersection numbers involving precisely three Cartan divisors, provided we use the signs in Table 2 and set

$$\lfloor |r_{\text{KK}}\varphi \cdot w_+^{\mathbf{2}_{-\frac{3}{2}}}| \rfloor = 1 \tag{7.9}$$

where $w_+^{\mathbf{2}_{-\frac{3}{2}}} = (-\frac{3}{2}, 1)$ is the highest weight.

While we have not found a way to determine a general form for the the nullspace of $M_C$ for arbitrary characteristic data without specifying $B$, as discussed in Sect. 3.3 we can attempt to get a general picture of the nullspace by restricting to the sublattice $\Lambda_S \cap \{\phi^{1\alpha} = 0\}$. Combining the pushforwards of the intersection numbers with formulae for the constrained fluxes in (3.14) to (3.16) reveals that after imposing the additional restriction $\phi^{1\alpha} = 0$, the fluxes satisfy (the SU(2) Cartan index is $i = 2$)

$$2\Theta_{00} = -2\Theta_{01} = 2\Theta_{11} = 2\Theta_{02} = \Theta_{22} \,. \tag{7.10}$$

Continuing to work in the restricted case $\phi^{1\alpha} = 0$, comparing the fluxes with the corresponding 3D Chern–Simons theory shows that the multiplicity of chiral matter in $\mathbf{2}_{-\frac{3}{2}}$ is

$$2\chi_{\mathbf{2}_{-\frac{3}{2}}} = \Theta_{22}\,. \tag{7.11}$$

We thus expect in general that F-theory models can realize the one-parameter family Eq. (7.1) of anomaly-free chiral matter fields for this gauge group with generic matter. Since in the unrestricted case $\phi^{1\alpha} \neq 0$ we cannot write down a completely general expression for the fluxes without specifying $B$, we next turn our attention to specific examples.

*7.1.2. Example:* $B = \mathbb{P}^3$ As a first simple example, we study the case $B = \mathbb{P}^3$, $K = -4\,H$, $S_7 = s_7 H$, $S_9 = s_9 H$. We define $n = s_9+4-s_7$ for convenience, and parameterize the results in terms of $n, s_7$ (the parameter $n$ corresponds to the parameter $s_8$ in the $F_6$ Weierstrass model of [28], which as discussed above is the degree of the divisor class $S_8 = nH$ associated with the SU(2) factor; the U(1) factor is associated with the height pairing parameter $h := -2W_{\bar{1}\bar{1}} \cdot H^2 = 16+4s_7-n$). Such a model is defined for integer values of $n, s_7$ satisfying the conditions $n, s_7, 16-n-2s_7 = 24-3n/2-h/2, 4+s_7-n = (h-3n)/4 > 0$; from these conditions we see that the height pairing also satisfies $h = 16+4s_7-n > 0$. In this set of cases, the matrix $M_{C_{\text{na}}(1\,H)(1\,H)} = (n-4s_7-16) = -h/2$ would fail to have an inverse when $n = 4s_7+16$, but this does not happen in the parameter range of interest as the height pairing divisor is always positive/effective, so for these models the matrix $M_{C_{\text{na}}(1\alpha)(1\beta)}$ is always invertible, making it possible in all cases to solve for $\phi^{1H}$ with this expression as a denominator. However, as we demonstrate below, in this case the formal rational expression for the flux (which must take integer values) is not an invariant property of the solution, but rather a feature of our choice of solution. As an alternative, we can solve the equation $\Theta_{1\,H} = 0$ by eliminating a different flux background parameter so as to produce a polynomial expression for $\Theta_{22}$ that is manifestly integer-valued.

As discussed in Sect. 2.5, we can analyze this model by following one of two approaches: either we first impose the symmetry constraints and then study the nullspace of $M_C$ in order to determine linear constraints on the fluxes, or we first quotient out the nullspace of $M$ and then impose the symmetry constraints. Despite producing identical results, both approaches have their respective unique advantages. In the following discussion we briefly describe three versions of the calculation (two of which follow the former approach, with the third version following the latter approach) in order to illustrate the different aspects of the problem. In particular, the full set of constraints determining the quantization of the number of chiral matter fields is clearest in the analysis beginning with $M_{\text{red}}$ in this class of examples, though this may not be the case for other choices of $\mathsf{G}$ or $B$.

As stated more generally in Sect. 2.5.2, throughout the analysis of this section we disregard the possible half-integer shift in $\phi$ that may be required when $c_2(X)$ is not even; this can easily be incorporated, as described for the SU(5) model in Sect. 6.4.

**Rational solution.** Solving directly for $\phi^{1H}$ gives

$$\begin{aligned}\Theta_{22} = \chi_{\mathbf{2}_{-\frac{3}{2}}} &= \frac{2ns_7(16-n-2s_7)(4+s_7-n)}{h}\ell(\phi^{\hat{I}\hat{J}})\\ \ell(\phi^{\hat{I}\hat{J}}) &= \phi^{00} - \phi^{01} + \phi^{02} + \phi^{11} + 2\phi^{22}\,,\end{aligned} \tag{7.12}$$

As in the discussion above, this is well-defined since $h > 0$.

As for the purely nonabelian cases described previously, the symmetry constraints and integer conditions on the flux backgrounds $\phi^{IJ}$ are sufficient to guarantee that this expression is integer- (and even-)valued, although we have not identified as simple a structure underlying this integrality as the group theoretic structure underlying e.g. the combination of flux background parameters appearing in Eq. (6.23). In this case, the symmetry constraints $\Theta_{H2} = 0$ imply additional conditions, including

$$2s_7(4 + s_7 - n)\ell(\phi^{\hat{I}\hat{J}}) \in h\mathbb{Z}\,, \tag{7.13}$$

which in turn implies that $\Theta_{22}$ is integer-valued, and is in fact an integer multiple of $n(16 - n - 2s_7)$. The full set of constraints is seen most easily after imposing the homology equivalence conditions on $S_{IJ}$, as discussed further below.

**Polynomial solution.** Alternatively, we can solve $\Theta_{1H} = 0$ in this case for the flux $\phi^{22}$, which gives us the result

$$\begin{aligned}\Theta_{22} = \chi_{\mathbf{2}_{-\frac{3}{2}}} &= \frac{2}{3}(16 - n - 2s_7)(4 - n + s_7)\ell'(\phi^{IJ})\\ \ell'(\phi^{IJ}) &= \phi^{1H} + (s_7 - 4)\phi^{11} + n\phi^{12}\,.\end{aligned} \tag{7.14}$$

In this analysis, the condition $\Theta_{1H} = 0$ implies that $(16 - n + 4s_7)\ell'(\phi^{IJ}) = 3ns_7k$ where $k$ is an integer combination of fluxes, so generically we expect $\Theta_{22}$ to be an integer multiple of $2ns_7(16 - n - 2s_7)(4 - n + s_7)$.

The two presentations of the chiral multiplicity (7.12) and (7.14) must give equivalent answers after all quantization conditions are properly taken account of. The second expression is simpler since only a factor of 3, and not the U(1) height pairing $h$, appears in the denominator. On the other hand, the expression for $\ell$ is simpler than that of $\ell'$ as it does not depend on the characteristic data. For the full analysis of the quantization conditions we now turn to the analysis using $M_{\text{red}}$.

**Vertical homology and flux quantization.** An important difference in resolutions only admitting a rational zero section from those with a holomorphic zero section such as those associated with the purely nonabelian groups studied in previous sections is that solving the symmetry conditions (2.10) and (2.11) over the integers generically imposes additional constraints on the parameters beyond those necessary to ensure integrality of the solutions.

To complete the discussion of this simple example we describe the complete quantization condition following from the integrality of the fluxes $\phi^{IJ}$. We can first explicitly remove the nullspace of $M$ by dropping the fluxes $\phi^{01}, \phi^{02}, \phi^{11}, \phi^{12}, \phi^{22}$, which each appear with a coefficient of 1 in a nullspace vector with all other entries integer. With this simplification, the matrix $M_{\text{red}}$ in the basis $S_{0H}, S_{HH}, S_{H2}, S_{1H}, S_{00}$ becomes

$$M_{\text{red}} = \begin{pmatrix} -4 & 1 & 0 & s_7 & 16 - ns_7 \\ 1 & 0 & 0 & 1 & -4 \\ 0 & 0 & -2n & n & -ns_7 \\ s_7 & 1 & n & -4 & s_7(n-4) \\ 16 - ns_7 & -4 & -ns_7 & s_7(n-4) & -64 + 12ns_7 - n^2s_7 - ns_7^2 \end{pmatrix}, \tag{7.15}$$

and

$$\det M_{\text{red}} = -n^2 s_7(16 - n - 2s_7)(4 - n + s_7). \tag{7.16}$$

**Table 2.** Signs and Cartan charges associated to the BPS spectrum of the resolved $F_6$ model with gauge group $\mathsf{G} = (\mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_2$ analyzed in [28]

| **1** | **2** |
|---|---|
| $\left(\begin{array}{c\|cc} \frac{\varphi\cdot w}{\|\varphi\cdot w\|} & w_1 & w_2 \\ \hline + & 1 & 0 \\ \hline + & 2 & 0 \end{array}\right)$ | $\left(\begin{array}{c\|cc} \frac{\varphi\cdot w}{\|\varphi\cdot w\|} & w_1 & w_2 \\ \hline + & \frac{1}{2} & 1 \\ - & \frac{1}{2} & -1 \\ \hline + & -\frac{3}{2} & 1 \\ + & -\frac{3}{2} & -1 \end{array}\right)$ |

The Cartan charges are the Dynkin coefficients of the weights in the representations $\mathbf{1}_1, \mathbf{1}_2, \mathbf{2}_{\frac{1}{2}}, \mathbf{2}_{-\frac{3}{2}}$ of SU(2) and the signs correspond to the signs of the BPS central charges $\varphi \cdot w$ for a given choice of Coulomb branch moduli $\varphi^i$

Note that the top left $4 \times 4$ block is resolution-invariant and the top left $3 \times 3$ block corresponds to the generalization of (6.14) to arbitrary $n$. Given this form of the reduced matrix, we can directly solve the constraint equations $\Theta_{0H}, \Theta_{HH}, \Theta_{H2} = 0$ for $\phi^{0H}, \phi^{HH}, \phi^{H2}$. The first two of these each can be described as an integer linear combination of remaining fluxes, and the third can be solved as an integer whenever the flux combination $\phi^{1H} - s_7\phi^{00}$ is even. The remaining equation $\Theta_{1H} = 0$ becomes

$$3ns_7\phi^{00} = h\phi^{1H}. \tag{7.17}$$

These are therefore the only nontrivial constraints on these fluxes. With this simplification for removing the nullspace, the parameters $\ell, \ell'$ become $\phi^{00}, \phi^{1H}$ respectively.

From these constraints for any fixed values of $n, s_7$ we can explicitly determine the quantization of the chiral multiplicity encoded by $\Theta_{22}$. For example, when $n$ is odd, $h$ is odd as well and the even parity constraint on $\phi^{1H} - s_7\phi^{00}$ is automatically satisfied, so when $h$ and $3ns_7$ furthermore have no common divisors, it follows that $\Theta_{22}$ can be an arbitrary integer multiple of $2ns_7(16 - n - 2s_7)(4 - n + s_7)$, up to bounds determined by the tadpole condition (and where, as noted above, to simplify the discussion we have ignored possible half-integer shifts for non-even $c_2(X)$).

As explicit examples, if $n = s_7 = 1$ ($h = 19$), the chiral index will be a multiple of $2 \times 13 \times 4 = 104$, and if $n = 1, s_7 = 4$ ($h = 43$), the chiral index will be a multiple of 392. If, however, e.g., $n = 3, s_7 = 2$ ($h = 21$), then $h$ has a common factor with $3ns_7$, in particular, $4 - n + s_7 = 3$, $16 - n - 2s_7 = 9$, and $\Theta_{22}$ can be any multiple of 81 (instead of 243), up to tadpole constraints. And when $n = 4, s_7 = 1$ ($h = 16$), the even parity constraint imposes the additional condition that $\phi^{1H}$ must be even, so $\Theta_{22}$ is a multiple of $4ns_7(16 - n - 2s_7)(4 - n + s_7) = 160$.

*7.1.3. Example: $B = \tilde{\mathbb{F}}_n$* We next consider a one-parameter family of examples where the F-theory base is taken to be a Hirzebruch threefold, $B = \tilde{\mathbb{F}}_n$, in order to illustrate how the submatrix $M_{C_{\mathrm{na}}(1\alpha)(1\beta)}$ can fail to be invertible for certain choices of characteristic data $K, S_7, S_9$. A Hirzebruch threefold is a generalization of a Hirzebruch surface $\mathbb{F}_n$ (i.e., a $\mathbb{P}^1$ fibration over a $\mathbb{P}^1$ base) in which the base of the $\mathbb{P}^1$ fibration is taken to be $\mathbb{P}^2$ instead of $\mathbb{P}^1$. For our purposes, we simply need to know the intersection theory of $\tilde{\mathbb{F}}_n$. To make an analogy, note that Hirzebruch $\mathbb{F}_n$ has two independent classes $F, E$, where $F$ is the class of the $\mathbb{P}^1$ fiber (meaning that $F$ is the divisor class of a point in the $\mathbb{P}^1$ base) and $E$ is the class of the $\mathbb{P}^1$ base (meaning that $E$ is the divisor class of a point in a $\mathbb{P}^1$ fiber). These two classes have the following intersection properties:

$$F^2 = 0, \quad F \cdot E = 1, \quad E^2 = -n, \quad n \in \mathbb{Z}_{\geq 0}. \tag{7.18}$$

The threefold $\tilde{\mathbb{F}}_n$ similarly has two independent divisor classes $D_2 := F, D_1 := E$ satisfying

$$D_2^3 = 0, \quad D_2^2 \cdot D_1 = 1, \quad D_2 \cdot D_1^2 = -n, \quad D_1^3 = n^2, \quad n \in \mathbb{Z}_{\geq 0}. \tag{7.19}$$

Since $\tilde{\mathbb{F}}_n$ is a toric variety, the canonical class $K$ of $\tilde{\mathbb{F}}_n$ is as usual given by minus the sum of all divisors corresponding to one-dimensional cones of the toric fan:

$$K = -\sum D_\alpha = -(D_2 + D_2 + D_2 + D_1 + (D_1 + nD_2)) = -(3+n)D_2 - 2D_1. \tag{7.20}$$

(Note that the above results can easily be derived by adapting the pushforward technology described in Appendix E to the projectivization $\mathbb{P}(\mathscr{V}) \to B^{(2)}$ of a rank one vector bundle $\mathscr{V} = \mathscr{L} \oplus \mathscr{O}_{B^{(2)}}$.) We expand the divisors $S_m$ in the basis $D_\alpha$, $S_m = s_{m\alpha} D_\alpha$. In terms of this basis of divisors, the constraints on the characteristic data for a good $(\mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_2$ model are then

$$\begin{aligned} (s_{71}, s_{72}) &> (0,0) \\ (s_{81}, s_{82}) &> (0,0) \\ (8 - 2s_{71} - s_{81}, 12 + 4n - 2s_{72} - s_{82}) &> (0,0) \\ (2 + s_{71} - s_{81}, 3 + n + s_{72} - s_{82}) &> (0,0)\,, \end{aligned} \tag{7.21}$$

where a (Weil) divisor $S_m$ is effective if $s_{m\alpha} \geq 0$ and either $s_{m1} > 0$ or $s_{m2} > 0$. Note that when $n > 3$ there is a non-Higgsable gauge factor on the divisor $D_2$, which may lead to an enhancement of the gauge symmetry in the class of universal $(\mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_2$ models.

In this family of examples, (minus) the height pairing divisor is then given by

$$W_{\bar{1}\bar{1}} = \frac{1}{2} S_8 + 2(K - S_7) = \left(-4 - 2s_{71} + \frac{s_{81}}{2}\right) D_1 + \left(-2(3+n) - 2s_{72} + \frac{s_{82}}{2}\right) D_2. \tag{7.22}$$

Combining the above expression for $W_{\bar{1}\bar{1}}$ with the $\tilde{\mathbb{F}}_n$ intersection numbers in Eq. (7.19) we find

$$\begin{aligned} &[[M_{C_{\mathrm{na}}(1\alpha)(1\beta)}]] \\ &= \begin{pmatrix} n^2(4s_{71} - s_{81} + 4) + n(-4s_{72} + s_{82} - 12) & n(-4s_{71} + s_{81} - 4) + 4s_{72} - s_{82} + 12 \\ n(-4s_{71} + s_{81} - 4) + 4s_{72} - s_{82} + 12 & 4s_{71} - s_{81} + 8 \end{pmatrix} \end{aligned} \tag{7.23}$$

from which it follows

$$\begin{aligned} \det[[M_{C_{\mathrm{na}}(1\alpha)(1\beta)}]] = &-\frac{1}{2}(4(n + s_{72} + 3) - s_{82})(n(4s_{71} - s_{81} + 4) \\ &-4(s_{72} + 3) + s_{82}). \end{aligned} \tag{7.24}$$

Hence we see that if we choose the characteristic data such that

$$s_{82} = \begin{cases} 12 + 4n + 4s_{72} \\ 12 - 4n - 4ns_{71} + 4s_{72} + ns_{81} \end{cases} \tag{7.25}$$

the matrix $M_{C_{\text{na}}(1\alpha)(1\beta)}$ will be singular.

We now turn our attention to some specific choices of $n \leq 3$ and look in particular for flux compactifications on $(\text{SU}(2) \times \text{U}(1))/\mathbb{Z}_2$ models with characteristic data satisfying the special conditions Eq. (7.25) that lead to non-invertibility of $M_{C_{\text{na}}(1\alpha)(1\beta)}$.

**Example:** $B = \tilde{\mathbb{F}}_0 \cong \mathbb{P}^2 \times \mathbb{P}^1$. As a specific example, consider the case $n = 0$. The matrix of intersection pairings with $W_{\bar{1}\bar{1}}$ takes the form

$$[[M_{C_{\text{na}}(1\alpha)(1\beta)}]] = -\frac{1}{2}\begin{pmatrix} 0 & 4s_{72} - s_{82} + 12 \\ 4s_{72} - s_{82} + 12 & 4s_{71} - s_{81} + 8 \end{pmatrix} \tag{7.26}$$

with $\det[[M_{C_{\text{na}}(1\alpha)(1\beta)}]] = -(4s_{72} - s_{82} + 12)^2/4$. This matrix is always invertible since $12 + 4s_{72} - s_{82} > 12 - 2s_{72} - s_{82} > 0$.

**Example:** $B = \tilde{\mathbb{F}}_3$. As another specific example consider the case $n = 3$, for which

$$[[M_{C_{\text{na}}(1\alpha)(1\beta)}]] = -\frac{1}{2}\begin{pmatrix} 3(12s_{71} - 4s_{72} - 3s_{81} + s_{82}) & -12s_{71} + 4s_{72} + 3s_{81} - s_{82} \\ -12s_{71} + 4s_{72} + 3s_{81} - s_{82} & 4s_{71} - s_{81} + 8 \end{pmatrix}. \tag{7.27}$$

Generically the determinant of this matrix is non-vanishing, but there is a family of allowed choices of characteristic data for which the determinant vanishes. For example, making the choices

$$S_7 = S_8 = -K \quad \Leftrightarrow \quad (s_{71}, s_{72}) = (s_{81}, s_{82}) = (2, 6) \tag{7.28}$$

leads to a singular matrix. $M$ does not develop any additional null vectors as a result of the above specialization, so it is possible to fully solve the U(1) symmetry conditions by eliminating distinctive parameters. In contrast to the previous specific example $B = \tilde{\mathbb{F}}_0$, this choice for the characteristic data is not forbidden by the constraints described at the beginning of this section and hence it appears that such a choice of parameters describes a consistent F-theory flux vacuum in which the U(1) gauge symmetry can be preserved in 4D in spite of $M_{C_{\text{na}}(1\alpha)(1\beta)}$ being singular; therefore, an explicit solution must include at least one nontrivial flux background other than $\phi^{1\beta}$ as in e.g. the polynomial solution (7.14). This provides an explicit example of the kind of situation mentioned at the end of Sect. 3.2.

*7.1.4. Resolution independence of $M_{red}$* We collect some evidence supporting the conjecture that $M_{\text{red}}$ (and hence $H^{\text{vert}}_{2,2}(X, \mathbb{Z})$) is also resolution independent in the the more general setting of models with U(1) gauge factors. Here, we compare the resolution of the $(\text{SU}(2) \times \text{U}(1))/\mathbb{Z}_2$ model studied in the previous subsections, which we denote by $X_2$, and an alternative resolution $X'_3$ defined by the sequence of blowups

$$X'_3 \xrightarrow{(e_2, s_8|e_3)} X'_2 \xrightarrow{(u,v|e_2)} X'_1 \xrightarrow{(u, s_4 v + s_7 w|e_1)} X_0 \tag{7.29}$$

where we follow the notation of [28].

For simplicity, let us specialize again to the case $B = \mathbb{P}^3$, where we again denote the SU(2) gauge divisor by $S_8 = nH$. In a common basis $S_{0H}$, $S_{HH}$, $S_{H2}$, $S_{1H}$, $S_{11}$ we find

$$M_{\text{red}}(X_2) = \begin{pmatrix} -4 & 1 & 0 & s_7 & s_7(s_7 - n) \\ 1 & 0 & 0 & 1 & -4 \\ 0 & 0 & -2n & n & -4n \\ s_7 & 1 & n & -4 & ns_7 - s_7^2 - 4s_7 + 16 \\ s_7(s_7 - n) & -4 & -4n & ns_7 - s_7^2 - 4s_7 + 16 & -n^2 s_7 + 3ns_7^2 - 4ns_7 - 2s_7^3 + 32s_7 - 64 \end{pmatrix}$$

$$M_{\text{red}}(X_3') = \begin{pmatrix} -4 & 1 & 0 & s_7 & s_7^2 \\ 1 & 0 & 0 & 1 & -4 \\ 0 & 0 & -2n & n & -4n \\ s_7 & 1 & n & -4 & -ns_7 - s_7^2 - 4s_7 + 16 \\ s_7^2 & -4 & -4n & -ns_7 - s_7^2 - 4s_7 + 16 & -n^2 s_7 - 3ns_7^2 + 12ns_7 - 2s_7^3 + 32s_7 - 64 \end{pmatrix}. \tag{7.30}$$

These two matrices are related by a change of basis

$$M_{\text{red}}(X_2) = U^{\text{t}} M_{\text{red}}(X_3') U, \quad U = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 2s_7 & ns_7 & -s_7 & 2(4 - s_7) & 1 \end{pmatrix}. \tag{7.31}$$

An analogous change of basis holds for other choices of base we have checked.

*7.2.* $(SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$ *model.* One of the initial motivations of this paper was to analyze the 4D massless chiral spectrum of the universal $(\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_6$ model of [27]. This model is believed to be the most general F-theory model with tuned $(\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_6$ gauge symmetry and generic matter spectrum, consisting of the representations appearing in the MSSM as well as three additional "exotic" matter representations. The gauge sector of the 4D $\mathcal{N} = 1$ supergravity describing this theory at low energies admits three linearly independent families of anomaly free combinations of chiral matter representations, so a flux compactification of the $(\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_6$ F-theory model can be expected to yield at most three independent combinations of chiral indices.

While the universal $(\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_6$ model can be defined by means of a Weierstrass model, due to the presence of a U(1) gauge factor (much like the $(\mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_2$ model), for the purpose of computing a resolution it proves to be more convenient to start with a construction of the singular F-theory background as a hypersurface $X_0$ of an ambient $\mathbb{P}^2$ bundle where the elliptic fiber of $X_0$ is realized as a general cubic in the $\mathbb{P}^2$ fibers of the ambient space. The hypersurface equation for $X_0$ can be obtained by unHiggsing the U(1) model with charge $q = 4$ matter constructed in [102]. The characteristic data of this model consists of the classes $K$, $\Sigma_2$, $\Sigma_3$, $Y$ where $\Sigma_m$ is the gauge divisor class of the nonabelian factor $\mathrm{SU}(m)$ and $Y =: W_{01}$ pulls back to the intersection of the (rational) zero and generating sections of a resolution of $X_0$. One special subclass of these models are those with $Y = 0$, which have only MSSM-type matter and have been studied using the toric $F_{11}$ fiber [28,29].

In a forthcoming publication [30], following the approach of this paper we present a complete analysis of the lattice of 4D symmetry-preserving vertical fluxes and associated 4D chiral multiplicities of the universal $(\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_6$ model over an arbitrary threefold base $B$. Consistent with the emerging picture of the landscape of F-theory vertical flux vacua described in this paper, one of the main results of [30] is that all three families of chiral matter representations can be realized in F-theory—further evidence suggesting that the linear constraints on 4D chiral matter multiplicities imposed by F-theory geometry coincide with the linear constraints implied by 4D anomaly cancellation.

These results are found for arbitrary bases using the simplified analysis associated with the restricted class of flux backgrounds $\phi^{1\alpha} = 0$, as well as for specific bases using the full analysis of $M_{\text{red}}$ and keeping quantization conditions intact.

## 8. Conclusions and Future Directions

*8.1. Summary of results.* We have described a novel and coherent approach for analyzing 4D vertical flux compactifications in F-theory (that is, flux backgrounds belonging to $H^{\text{vert}}_{2,2}(X, \mathbb{Z})$) that preserve 4D local Lorentz and gauge symmetry. Our approach both offers unique computational advantages, and sheds light on the geometric nature of some of the resolution-invariant physics encoded in the singularities of the F-theory background related to the 4D massless chiral spectrum that has so far proven difficult to analyze directly in the type IIB duality frame.

One of the key elements of our analysis is the integral lattice of vertical 4-cycles, with symmetric bilinear form given by the symmetric matrix of quadruple intersection numbers of the smooth CY fourfold $X$ interpreted as an intersection pairing on surfaces corresponding to the pairwise intersections of divisors. By Poincaré duality the nondegenerate part of this lattice is equivalent to the lattice $H^{2,2}_{\text{vert}}(X, \mathbb{Z})$ of vertical flux backgrounds. We conjecture that this lattice, along with its nondegenerate inner product given by the matrix $M_{\text{red}}$, is a resolution-invariant structure for any singular elliptic CY fourfold encoding an F-theory compactification. This conjecture seems natural from the point of view of type IIB string geometry, and is satisfied by a wide range of explicit examples that we have considered in this paper. The resolution-independence of $M_{\text{red}}$ also implies that the symmetric bilinear form $M$ on the formal space of intersection surfaces $S_{IJ}$, which contains a nullspace corresponding to homologically trivial surfaces, is resolution-independent. This further implies the existence of nontrivial relations among the set of quadruple intersection numbers of the resolved CY fourfold, even though these quadruple intersection numbers are not in general resolution-independent (i.e., equivalent under an integral linear change of basis of the divisors.) Understanding the geometry of this conjecture better and its ramifications for the intersection structure of singular CY fourfolds is an interesting problem for further investigation.

The resolution-independence of $M$ and $M_{\text{red}}$ is a sufficient condition for the chiral matter content of a given class of F-theory flux compactifications to be resolution-invariant, but as far as we can tell is not directly provable from this geometric condition. The structure of $M_{\text{red}}$ we have studied here could be used to further study F-theory flux compactifications both in situations where the geometric gauge group remains unbroken in 4D by the fluxes, which is the primary focus here, as well for cases where the geometric gauge group is broken by vertical fluxes, which seems like an interesting direction for further research. In cases where the flux does not break the gauge group, additional constraints are placed on the fluxes. Conceptually, the approach we have taken here to studying such vacua involves the interplay between two operations applied to the formal intersection pairing matrix $M$. The first of these two operations entails restricting to a sublattice of flux backgrounds satisfying the constraints necessary and sufficient to preserve 4D local Lorentz and gauge symmetry. This operation is central to the standard approach used in much of the previous literature to analyze vertical flux backgrounds in F-theory; our methods for computing intersection numbers combined with the rigid structure of the elliptic fibration enable us to write a formal expression for the elements of this sublattice. In contrast, the second of these operations involves taking the quotient of the lattice of vertical 4-cycles by homologically trivial cycles, the result of which is the lattice of vertical homology classes $H_{2,2}^{\mathrm{vert}}(X, \mathbb{Z})$ with inner product given by $M_{\mathrm{red}}$. While these two operations commute, studying the interplay between the two different orders of these operations gives insight into the structure of the connection between chiral matter and F-theory fluxes.

Computationally, the approach presented here is a synthesis of various techniques that have appeared in the literature. Notably, we apply recently developed algebrogeometric techniques for computing intersection numbers of divisors in smooth elliptically fibered CY varieties to classes of resolutions that can more easily be obtained from geometric constructions of singular F-theory backgrounds in which the elliptic fiber is realized as a general cubic in $\mathbb{P}^2$—this procedure therefore provides a means to analyze a broader class of F-theory constructions than is encompassed by the usual Weierstrass model construction. Moreover, since these techniques (like those used in [23,32]) express the intersection numbers of divisors in terms of triple intersections of certain divisors in the base of the elliptic fibration (i.e., the characteristic data), this approach can be used to conveniently organize the landscape of F-theory vertical flux compactifications into families of vacua with fixed gauge symmetry and matter representations over an arbitrary base.

We have demonstrated the utility of this approach by analyzing vertical flux backgrounds in numerous examples with simple gauge symmetry group and generic matter. We have also analyzed several examples of models with a U(1) gauge factor, to illustrate the straightforward generalization of these methods to models with U(1) gauge factors; in principle a similar analysis is possible for models with an arbitrary number of U(1) factors. Of particular note among models with U(1) gauge factors is the universal $(\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_6$ model [27] whose 4D massless chiral spectrum we analyze in a forthcoming publication [30] using the methods described in this paper. We find in all examples that the linear constraints on the chiral matter multiplicities imposed by F-theory geometry exactly match the 4D anomaly cancellation conditions, which suggests that it may be possible to realize all anomaly-free combinations of 4D chiral matter in F-theory, at least at the level of allowed linearly independent families of the generic matter types for a given gauge group, although of course tadpole constraints will impose a limit on the magnitude of the number of families possible in any given direction, giving a finite bound on the set of allowed F-theory models.

*8.2. Future directions.* The existence of a resolution independent structure such as $H_{2,2}^{\mathrm{vert}}(X, \mathbb{Z})$ is consistent with the expectation that the kinematics of F-theory vacua are captured entirely in the singular elliptic CY geometry encoded by the axiodilaton over a general base in type IIB string theory. To our knowledge the conjecture that $H_{2,2}^{\mathrm{vert}}(X, \mathbb{Z})$ is resolution independent has not previously been explored in the literature and would be useful to prove rigorously, as this points to several potential future avenues of investigation related to the physics of F-theory flux compactifications:

- One of the outstanding challenges of F-theory is to give a complete and mathematically precise definition of this formulation of string theory. While this is often done by taking a limit of M-theory (see, e.g., [66,103]), a more intrinsic definition may be possible from the point of view of type IIB string theory. The progress made here in understanding the resolution-independent aspects of the singular elliptic fourfold geometry $X_0$ may help in better understanding how matter surfaces and chiral matter may be formulated and computed directly from the type IIB point of view.

- From a mathematical point of view, the resolution-independence of $M_{\rm red}$ indicates that there is some intrinsic meaning to the lattice $H^{2,2}_{\rm vert}(X_0, \mathbb{Z})$ of integral vertical surfaces and their intersection form on the singular fourfold geometry $X_0$. This is particularly intriguing as the surfaces $S_{ij}$ most relevant for chiral matter in F-theory flux vacua project to trivial surfaces in the base and thus are hidden in the singular elliptic CY given by the F-theory Weierstrass model. Developing a clear mathematical picture of this aspect of intersection theory of singular complex fourfolds poses an interesting challenge on the mathematical side.
- More concretely, the resolution-independence of $H^{2,2}_{\rm vert}(X, \mathbb{Z})$ suggests that there should be some way of directly computing the intersection matrix $M_{\rm red}$ without explicitly performing any blowups at all. While many of the intersection numbers that form this matrix are resolution-independent, others are not, so identifying an organizing principle that would make possible a resolution-independent statement of the form of this matrix would be a significant step forward for the intrinsic understanding of singular F-theory flux vacua.
- We have focused in this paper on the intersection structure of CY fourfolds, which is relevant for 4D F-theory vacua. We may speculate, however, that the analogous homology group $H_{2,2}^{\rm vert}(X^{(3)}, \mathbb{Z})$ for a CY threefold $X^{(3)}$ is also resolution-invariant. It may be possible to prove this resolution-invariance in a more direct and explicit way, and this may further shed light on the structure of $H_{2,2}^{\rm vert}(X, \mathbb{Z})$ for a CY fourfold.
- While in this paper we have focused on fluxes that preserve the geometric gauge group, so that the gauge invariance constraints $\Theta_{I\alpha} = 0$ are all satisfied, it would be interesting to study flux vacua in which this condition is weakened. In particular, as discussed in e.g. [27], while direct tuning of the Standard Model gauge group in F-theory is one way to get semi-realistic physics models, the bulk of the moduli space of CY fourfolds, and apparently the vast majority of the flux vacua, are dominated by bases that force large numbers of non-Higgsable gauge factors such as $E_6$, $E_7$, $E_8$ (see e.g. [39,40,104]); for these bases it is difficult or impossible to tune the Standard Model gauge group, but the group $(\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_6$ may be realized by turning on fluxes that break the gauge symmetry. Some preliminary work in this direction for $E_8$ breaking was done in [105], but the methods developed here may provide a very useful tool in more systematically pursuing this kind of analysis for flux breaking of non-Higgsable groups like $E_6$ and $E_7$.
- Intriguingly, in all models we study we find that the symmetry-preserving fluxes appear to depend on resolution-invariant linear combinations of triple intersections of characteristic divisor classes in the base of the elliptic fibration, so that the minimum magnitude of the fluxes appears to be controlled by certain numbers of special points lying in the discriminant locus. Since the chiral indices themselves can be expressed as linear combinations of the fluxes, this suggests that the chiral indices in some sense "count" special points in the F-theory base. One very clear illustration of this idea is given by (4, 6) points, as the symmetry-preserving fluxes in the models we have studied receive contributions proportional to the numbers of (4, 6) points in the base.

More generally, in many cases the multiplicity of chiral matter in fixed representations is proportional to the number of points in the base in the intersection of the associated matter curve and another characteristic divisor, suggesting some explicit direct connection between chiral matter fields and base geometry. While these observations are not necessarily unique to our analysis, our computational methods have enabled us to survey a large enough number of examples to reveal patterns among different families of models, see, e.g., the expressions for the fluxes in Table 1. Along these lines, it would be interesting and quite useful to understand how to associate the chiral indices to certain types of singularities visible directly from the F-theory limit, and it is possible that the resolution independence of $H_{2,2}^{\text{vert}}(X, \mathbb{Z})$ will prove useful in this capacity.

- Another direction in which this work could naturally be extended involves the question of whether or not all families of anomaly-free matter can be realized in F-theory. In all the examples we have studied, of simple gauge groups and groups with a single U(1) factor, we have found by considering both generic and specific choices of base that F-theory imposes no linear constraints on the chiral matter multiplicities beyond those expected by anomaly cancellation. This has implications for the analysis of the "swampland," suggesting that at the level of linear families of matter F-theory naturally realizes the full set of possibilities that are consistent with low-energy constraints. It would be good to check whether this continues to hold for more complicated models with more abelian factors, or even to find some general principle based on the resolution-invariance of $M_{\text{red}}$ that can match the rank of this intersection form with the number of expected families of chiral matter.
- At finer level of detail, there are questions related to the quantization and multiplicities of chiral matter that could be explored further both mathematically and through more concrete physics models. As we have discussed here (see in particular Sect. 2.8), the quantization conditions on matter from purely vertical fluxes may be weakened when the other components of middle homology are incorporated and/or fractional vertical flux coefficients are included, since by Poincaré duality there should be in principle cycles with a single unit of flux through any primitive matter surface, even though in general the determinant of $M_{\text{red}}$ has magnitude greater than 1. Further analysis of the geometry and associated physics of these kinds of questions could help elucidate more detailed swampland type questions regarding which precise multiplicities of matter can arise in given 4D supergravity models realized from F-theory.
- While in this paper we have focused on chiral matter in 4D theories, a full understanding of the low-energy physics of a given F-theory compactification also requires understanding the vector-like matter. Though vector-like matter multiplicities are subtler than chiral matter, some recent progress has been made in this direction [82,106–109]. It would be interesting to investigate whether there is resolution-independent structure, analogous to that studied here, that can be used to describe such vector-like multiplicities.
- Finally, we note that for a pair of CY fourfolds related by mirror symmetry, their respective vertical and horizontal cohomologies are isomorphic [49,79]. In this paper we restricted our focus to vertical flux backgrounds and did not attempt to explore the space of horizontal fluxes associated to a given 4D F-theory model. However, a more complete analysis of F-theory flux compactifications generically requires horizontal fluxes to be included in the picture. If it turns out that the vertical homology of a given CY fourfold is indeed resolution invariant, this would suggest that the corresponding horizontal homology of the mirror CY fourfold is also an invariant structure across certain regions of moduli space and may provide a strategy for studying horizontal fluxes, which have received comparatively less attention in the literature, and which may also give insight into the quantization issues mentioned above. The intersection

form on the horizontal part of $H^4(X, \mathbb{Z})$ also plays an important role in recent work that uses asymptotic Hodge theory to describe string vacua in large field limits [110, 111], and it would be interesting to understand if similar resolution-independent structure is relevant there.

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## A. 4D Anomaly Cancellation

We review anomaly cancellation in four dimensions, following primarily [19].

Consider a 4D $\mathcal{N} = 1$ theory with gauge algebra of the form

$$\mathfrak{g} = \bigoplus_s \mathfrak{g}_s \oplus \bigoplus_{\bar{a}} \mathfrak{u}(1)_{\bar{a}}, \tag{A.1}$$

with the $\mathfrak{g}_s$ being simple nonabelian gauge algebra factors, indexed by $s$, and with $\mathfrak{u}(1)_{\bar{a}}$ being abelian gauge factors, indexed by $\bar{a}$. As we are only considering local gauge anomalies, we need not specify the global structure of the gauge group $\mathsf{G}$ here. Matter in chiral multiplets transforms in irreducible representations of the form

$$\mathsf{r} = \bigotimes_s \mathsf{r}_s \otimes \bigotimes_{\bar{a}} q_{\mathsf{r},\bar{a}} =: (\mathsf{r}_1, \mathsf{r}_2, \dots)_{(q_{\mathsf{r},1}, q_{\mathsf{r},2}, \dots)}. \tag{A.2}$$

In four dimensions, the gauge and gauge–gravitational mixed anomalies have contributions from chiral Weyl fermions via the familiar triangle diagrams, and additionally from Green–Schwarz tree-level diagrams exchanging U(1)-gauged scalar axion fields

$\rho_\alpha$. The conditions for the anomaly to cancel reduce to[35][36]

$$
\begin{aligned}
0 &= \sum_{\mathsf{r}_s} \Big( \prod_{s' \neq s} \dim \mathsf{r}_{s'} \Big) \frac{\mathrm{tr}_{\mathsf{r}_s} F_s^3}{\mathrm{tr}_{\mathsf{f}} F_s^3}, \\
b_s^\alpha \Theta_{\alpha \bar{a}} &= 2\lambda_s \sum_{\mathsf{r}_s} \Big( \prod_{s' \neq s} \dim \mathsf{r}_{s'} \Big) w_{\bar{a}}^{\mathsf{r}_s} \frac{\mathrm{tr}_{\mathsf{r}_s} F_s^2}{\mathrm{tr}_{\mathsf{f}} F_s^2}, \\
a^\alpha \Theta_{\alpha \bar{a}} &= -\frac{1}{4} \sum_{\mathsf{r}} (\dim \mathsf{r}) w_{\bar{a}}^{\mathsf{r}}, \\
b_{\bar{a}\bar{b}}^\alpha \Theta_{\alpha \bar{c}} + b_{\bar{a}\bar{c}}^\alpha \Theta_{\alpha \bar{b}} + b_{\bar{b}\bar{c}}^\alpha \Theta_{\alpha \bar{a}} &= 2 \sum_{\mathsf{r}} (\dim \mathsf{r}) w_{\bar{a}}^{\mathsf{r}} w_{\bar{b}}^{\mathsf{r}} w_{\bar{c}}^{\mathsf{r}},
\end{aligned}
\tag{A.3}
$$

where $\Theta_{\alpha\bar{a}}$ is the gauging of the axion $\rho_\alpha$ under the abelian vector $A^a$ associated with the gauge factor $\mathfrak{u}(1)_{\bar{a}}$,

$$
\mathrm{D}\rho_\alpha = \mathrm{d}\rho_\alpha + \Theta_{\alpha\bar{a}} A^{\bar{a}}, \tag{A.4}
$$

while the $a^\alpha$, $b_s^\alpha$, $b_{\bar{a}\bar{b}}^\alpha$ are anomaly coefficients that specify the Green–Schwarz couplings of $\rho_\alpha$,

$$
S_{\mathrm{GS}} = -\frac{1}{8} \int \frac{2}{\lambda_s} b_s^\alpha \rho_\alpha \mathrm{tr}_{\mathsf{f}}(F_s \wedge F_s) + 2 b_{\bar{a}\bar{b}}^\alpha \rho_\alpha F_{\bar{a}} \wedge F_{\bar{b}} - \frac{1}{2} a^\alpha \rho_\alpha \mathrm{tr}(R \wedge R). \tag{A.5}
$$

Here, $\lambda_s = 2c_s^\vee / A_{\mathrm{Adj}_s}$, with $c_s^\vee$ the dual Coxeter number of $\mathfrak{g}_s$.

## B. Tensor Structures in Intersection Products of Divisors

This appendix is an overview of various tensor structures characterizing the pushforwards of intersection numbers of divisors in a resolution $X$ of a singular elliptically fibered CY variety defining an F-theory model with gauge group $\mathsf{G}$. Although the structures we describe in this appendix are to our knowledge not rigorously proven, we expect them to apply for the full class of CY fourfolds we describe in this paper (see below for a precise statement of our assumptions about the type of CY manifold for which these structures apply). Furthermore, these structures have been verified in a vast number of examples of intersection products for resolutions of F-theory models studied in the literature. We include relevant references where appropriate; however, since much of this structure has been described in numerous places in the literature, we do not attempt to be exhaustive. Note also that the explicit computations of intersection numbers that we carry out in this paper using the techniques of Appendix E match with these general tensor structures in all cases we have computed where the structure is known, and appear to extend to other cases (e.g., 4-Cartan index intersection numbers) where the general structure is not understood.

---

[35] The symbol '$\mathrm{tr}_{\mathsf{f}}$' denotes a trace taken over the field strength $F_s$ transforming in the fundamental (i.e. defining) representation $\mathsf{f}$ of the gauge factor $\mathfrak{g}_s$ (see Eq. (A.5)), and similarly for '$\mathrm{tr}_{\mathsf{r}_s}$'. Note also that the traces over products of field strengths can be expressed as, e.g., $\mathrm{tr}_{\mathsf{r}_s} F_s^p = \sum_{w \in \mathsf{r}_s} \left( \sum_{i_s} \varphi^{i_s} w_{i_s} \right)^p$.

[36] Note that in F-theory flux compactifications with abelian gauge factors associated to rational sections $\hat{D}_{\bar{a}}$, we must impose $\Theta_{\alpha\bar{a}} = 0$ in order to ensure that the associated abelian gauge symmetry is not rendered massive in the low energy effective 4D theory by the Stückelberg mechanism.

Let $X$ be a resolution of a singular elliptic CY $n$-fold $X_0$ with canonical projection

$$\pi : X \to B\,. \tag{B.1}$$

For simplicity we assume that $X_0$ has a rational zero section and an additional rational section, although we stress that much of the structure described here is generalizable to cases in which there are an arbitrary number of rational sections. Following the Shioda–Tate–Wazir formula [112], we use the basis of divisors $\hat{D}_I$, where $I = 0, a, \alpha, i_s$ respectively labels a choice of rational zero section, additional rational sections, pullbacks of divisors in the base, and Cartan divisors (i.e., the irreducible components of the pullback to $X$ of the irreducible components $\Sigma_s$ of the discriminant locus $\{\Delta = 0\} \subset B$). For simplicity, we assume that the intersections $\Sigma_s \cap \Sigma_{s'}$ are pairwise transverse. We refer to the divisor classes

$$\Sigma_s, \quad \Sigma_{s'}, \quad \ldots \quad K = -c_1(B), \quad W_{0a} := \pi_*(\hat{D}_0 \cdot \hat{D}_a) \tag{B.2}$$

as the characteristic data of $X$. Note that $\hat{D}$ denotes a divisor class in the Chow ring of $X$, whereas $D$ denotes a divisor class in the Chow ring of $B$ (and similarly for higher codimension). Unless the distinction is otherwise unclear from the context, we generically use the same symbol for both a divisor and its class in the Chow ring, and moreover we typically do not explicitly indicate pullback maps. Repeated indices are summed over when one index is raised and the other is lowered.

A convenient method for evaluating intersection numbers in the Chow ring of a smooth elliptic variety $X$ is to compute the pushforward $\pi_*$ of the intersection product to the Chow ring of $B$, where $\pi$ is defined in Eq. (B.1). This method is particularly useful because the projection formula (see, e.g., [113])

$$f_*(f^*(C) \cdot \hat{D}) = C \cdot f_*(\hat{D}) \tag{B.3}$$

for classes[37] $C$, $\hat{D}$ and $f$ an appropriate map implies that intersection products involving divisors $\hat{D}_\alpha$ that are the pullbacks of divisors $D_\alpha$ in the base inherit the intersection structure of $D_\alpha \subset B$. Hence, we can anticipate the pushforwards of intersection products to exhibit the general structure (for concreteness assume that $X$ is a CY fourfold, i.e., $n = 4$)

$$\begin{aligned}
\hat{D}_I \cdot \hat{D}_J \cdot \hat{D}_K \cdot \hat{D}_L &= W_{IJKL} := W^{\alpha\beta\gamma}_{IJKL} D_\alpha \cdot D_\beta \cdot D_\gamma \\
\hat{D}_\alpha \cdot \hat{D}_J \cdot \hat{D}_K \cdot \hat{D}_L &= W_{JKL} \cdot D_\alpha := W^{\beta\gamma}_{JKL} D_\alpha \cdot D_\beta \cdot D_\gamma \\
\hat{D}_\alpha \cdot \hat{D}_\beta \cdot \hat{D}_K \cdot \hat{D}_L &= W_{KL} \cdot D_\alpha \cdot D_\beta := W^{\gamma}_{KL} D_\alpha \cdot D_\beta \cdot D_\gamma \\
\hat{D}_\alpha \cdot \hat{D}_\beta \cdot \hat{D}_\gamma \cdot \hat{D}_L &= W_L \cdot D_\alpha \cdot D_\beta \cdot D_\gamma := W_L D_\alpha \cdot D_\beta \cdot D_\gamma\,,
\end{aligned} \tag{B.4}$$

where $W_{IJKL}, W_{JKL}, W_{KL}, W_L$ can be expressed as intersection products in the Chow ring of $B$—note that these intersection products only involve the characteristic data $K, \Sigma_s, W_{01}$.

To make contact with low-energy effective field theoretic descriptions of the low-energy effective field theory describing M-theory compactified on $X$, we can change our

[37] Note that $\hat{D}$ is a divisor class in the Chow ring of the space in the preimage of the map $f$.

basis of divisors to the "gauge" basis $\hat{D}_{\bar{I}} = \sigma_{\bar{I}}^{J} \hat{D}_J$ defined by [18,93]

$$\begin{aligned}
\sigma_{\bar{0}}^{I} &= (1, 0, -\frac{1}{2} W_{00}^{\alpha}, 0)^I \\
\sigma_{\bar{1}}^{I} &= (-1, 1, W_{00}^{\alpha} - W_{01}^{\alpha}, -W_{1|j_{s'}} W^{j_{s'}|i_s})^I \\
\sigma_{\bar{J}}^{I} &= \delta_{\bar{J}}^{I} \quad \text{for } \bar{J} \neq 0, 1 .
\end{aligned} \tag{B.5}$$

By linearity, the fluxes in the gauge basis are then linear combinations of the fluxes in the standard geometric basis. For example, $\hat{D}_{\bar{1}} = \sigma_{\bar{1}}^{I} \hat{D}_I$ is the image of the Shioda map described in [64]. Using

$$\begin{aligned}
(\sigma^{-1})_{0}^{\bar{I}} &= (1, 0, \frac{1}{2} W_{00}^{\alpha}, 0)^{\bar{I}} \\
(\sigma^{-1})_{1}^{\bar{I}} &= (1, 1, -\frac{1}{2} W_{00}^{\alpha} + W_{01}^{\alpha}, W_{1|j_{s'}} W^{j_{s'}|i_s})^{\bar{I}} \\
(\sigma^{-1})_{J}^{\bar{I}} &= \delta_{J}^{\bar{I}} \quad \text{for } J \neq 0, 1 .
\end{aligned} \tag{B.6}$$

one can invert the above linear transformation:

$$(\sigma^{-1})_{J}^{\bar{I}} \sigma_{\bar{I}}^{K} = \delta_{J}^{K}, \quad \sigma_{\bar{K}}^{J} (\sigma^{-1})_{J}^{\bar{I}} = \delta_{\bar{K}}^{\bar{I}} . \tag{B.7}$$

We sometimes make the abuse of notation

$$\bar{I} = i = (\bar{a}, i_s) \tag{B.8}$$

to collectively denote all abelian gauge indices as opposed to distinguishing between pure U(1) and nonabelian Cartan indices (note $\hat{D}_{\bar{a}} := \sigma_{\bar{a}}^{I} \hat{D}_I$, where for our purposes we need only consider a single generating section, i.e., $a = 1$.)

Throughout the paper we make extensive use of the fact that intersection numbers of the form $W_{IJKL}$ where $I, J, K, L \neq 0, a, \alpha$ exhibit special tensor structures. (Note that various aspects of the structure of the pushforwards $W_{IJKL}$ have been pointed out and used extensively in the string theory literature, see, e.g., Section 3 of [19] and references therein.) In particular, it is useful to grade the intersection numbers $W_{IJKL}$ by their number of nonabelian Cartan indices $I = i_s$, which corresponds to the number of nonabelian Cartan divisors $\hat{D}_{i_s}$ appearing in the expression $W_{IJKL} = \hat{D}_I \cdot \hat{D}_J \cdot \hat{D}_K \cdot \hat{D}_L$. We now summarize some features of these tensor structures:

- Four nonabelian Cartan indices. Without introducing specific assumptions about the chiral matter content of the 4D theory engineered by $X$, a general characterization of the four Cartan index tensor structure is to our knowledge presently unknown. It may be possible to combine assumptions about 4D anomaly cancellation and the existence of particular matter surfaces $S_{\mathsf{r}}$ to predict a subset of the $W_{ijkl}$.
- Three nonabelian Cartan indices. Intersection products involving three nonabelian Cartan divisors in elliptically fibered CY manifolds have been the subject of a great deal of string theory literature. For example, over a twofold base $B^{(2)}$, these intersection products are intersection numbers of a CY threefold $X^{(3)}$, and they encode various aspects of the kinematics of 5D M-theory compactifications on $X^{(3)}$ dual to the Coulomb branch of 6D F-theory compactifications on $X^{(3)} \times S^1$. In our case (i.e., CY fourfolds $X$), intersection products involving three Cartan divisors are divisor

classes and hence push forward to divisor classes in $B$, but nonetheless carry much of the same tensor structure as in the CY threefold case, as we now demonstrate. Combining various results on the relationship between these intersection numbers and the corresponding low-energy effective physics (see e.g. [85,90,93]) with known results on the geometry of matter curves $C_{\mathsf{r}}$ for local matter arising from transverse intersections of divisors in $B$ [94], we can infer[38] the following structure:

$$\begin{aligned} W_{i_s j_{s'} k_{s''}} &= \rho^{\mathsf{R}_{ss'}}_{i_s j_{s'} k_{s''}} C_{\mathsf{R}_{ss'}} \\ C_{\mathsf{R}_{ss'}} &= \Sigma_s \cdot \Delta_{\mathsf{R}_{ss'}} = \Sigma_s \cdot (a_{\mathsf{R}_{ss'}} K + b_{\mathsf{R}_{ss'}} \Sigma_s + c_{\mathsf{R}_{ss'}} \Sigma_{s'}) \,, \end{aligned} \tag{B.9}$$

where $a_{\mathsf{R}_{ss'}}, b_{\mathsf{R}_{ss'}}, c_{\mathsf{R}_{ss'}} \in \frac{1}{2}\mathbb{Z}$ and $\mathsf{R}_{ss'}$ denotes a hypermultiplet representation transforming under the product gauge group $\mathsf{G}_s \times \mathsf{G}_{s'}$ (by convention $\mathsf{R}_{ss}$ only transforms under $\mathsf{G}_s$). For a fixed gauge group and any of the generic matter types, the coefficients $a_{\mathsf{R}_{ss'}}, b_{\mathsf{R}_{ss'}}, c_{\mathsf{R}_{ss'}}$ can be computed directly from the associated universal Weierstrass model.

In the above expressions we sum over 5D $\mathcal{N} = 1$ hypermultiplet[39] representations $\mathsf{R}_{ss'}$, and $\rho^{\mathsf{R}_{ss'}}_{i_s j_{s'} k_{s''}}$ are triple intersection numbers that can be extracted from the pure Cartan expression for the prepotential $\mathcal{F}$ of a 5D M-theory compactification (see [92]):

$$6\mathcal{F}_{\text{Cartan}} = \varphi^{i_s} \varphi^{j_{s'}} \varphi^{k_{s''}} W_{i_s j_{s'} k_{s''}} = -\frac{1}{2} \sum_{\mathsf{R}_{ss'}} C_{\mathsf{R}_{ss'}} \sum_{w \in \mathsf{R}_{ss'}} \operatorname{sign}(\varphi \cdot w)(w_{i_{s''}} \varphi^{i_{s''}})^3 \,. \tag{B.10}$$

For example, $\Delta_{\mathbf{adj}_s} = (\Sigma_s + K)/2$ and $C_{\mathsf{R}_{ss'}} = \Sigma_s \cdot \Sigma_{s'}$. In the above expression $w_l$ are the components of the weight $w$ in the basis of fundamental weights, i.e., the basis canonically dual to simple coroots $\alpha_i^\vee$ satisfying $\alpha_i^\vee \cdot w = w_i$. Note that $W_{i_s j_{s'} k_{s''}} \propto \partial_{\phi^{i_s}} \partial_{\phi^{j_{s'}}} \partial_{\phi^{k_{s''}}} \mathcal{F}_{\text{Cartan}}$ is manifestly resolution-dependent, since the right hand side of Eq. (B.10) depends explicitly on $\operatorname{sign}(\varphi \cdot w)$, which in turn depends on the specific phase of the Coulomb branch to which the intersection numbers correspond.

- Two nonabelian Cartan indices. For intersection numbers of the form $W_{i_s j_{s'} \alpha\beta} = W_{i_s j_{s'}} \cdot D_\alpha \cdot D_\beta$, we have the resolution-independent structure [19]

$$W_{i_s j_{s'}} = W_{i_s | j_{s'}} \Sigma_s = -\delta_{ss'} \kappa^{(s)}_{ij} \Sigma_s \,, \tag{B.11}$$

where $W_{i_s|j_{s'}}$ is (minus) the inverse Killing form[40] associated to $\mathsf{G}_{\text{na}}$ and satisfies the relation

$$W_{i_s|j_{s'}} W^{j_{s'}|k_{s''}} = \delta^{k_{s''}}_{i_s} \,. \tag{B.12}$$

- One or fewer nonabelian Cartan indices. For intersection numbers of the form $W_{i_s\alpha\beta\gamma} = W_{i_s} D_\alpha \cdot D_\beta \cdot D_\gamma$ the tensor structure trivializes:

$$W_{i_s} = 0 \,. \tag{B.13}$$

38 For instance, the most direct method to derive Eq. (B.9) is to match 5D 1-loop CS terms to intersection products involving three nonabelian Cartan divisors as in Eq. (5.8) and below, and then to identify the "coefficients" $C_{\mathsf{R}}$ as the classes of matter curves described in [94].

39 In this notation, $\mathsf{R} = \mathsf{r} \oplus \mathsf{r}^*$ is a quaternionic representation, and hence sums over representations do not distinguish between a complex representation $\mathsf{r}$ and its conjugate $\mathsf{r}^*$.

40 The tensor $\kappa^{(s)}_{ij}$ is the inverse of the metric tensor of the simple Lie algebra $\mathfrak{g}_s \subset \mathfrak{g}_{\text{na}} = \oplus_s \mathfrak{g}_s$ and appears, e.g., in the 5D scalar kinetic term $\int \kappa^{(s)}_{ij} \, d\varphi^{i_s} \wedge * d\varphi^{j_s}$.

Understanding these tensor structures turns out to be crucial for characterizing the general form of the constrained fluxes for F-theory models with gauge group $\mathsf{G} = (\mathrm{U}(1) \times \mathsf{G}_{\mathrm{na}})/\Gamma$. In particular, the fact that $W_{i_s|j_t}$ can be inverted is crucial to the calculation in Appendix C.

Note that we also make liberal use of the formal identity

$$W_{i_s JKL} = \Sigma_s \cdot W_{JKL|i_s}\,, \tag{B.14}$$

of which the definition $W_{i_s j_{s'}} = \Sigma_s W_{i_s|j_{s'}}$ in Eq. (B.11) is a special case. Finally, when the zero section $\hat{D}_0$ (or any section, for that matter) is holomorphic, we have

$$\begin{aligned} W_{0000} &= K^3\,, \\ W_{000\alpha} &= K^2 \cdot D_\alpha\,, \\ W_{00\alpha\beta} &= K \cdot D_\alpha \cdot D_\beta\,, \\ W_{0\alpha\beta\gamma} &= D_\alpha \cdot D_\beta \cdot D_\gamma\,. \end{aligned} \tag{B.15}$$

## C. Solution to the Symmetry Constraints

In this appendix we show that, given a resolution $X$ of a singular fourfold $X_0$ corresponding to a 4D F-theory compactification with gauge symmetry $\mathsf{G} = (\mathrm{U}(1) \times \mathsf{G}_{\mathrm{na}})/\Gamma$, flux backgrounds $G$ that preserve the full Poincaré and gauge symmetry in the F-theory limit can typically be parametrized entirely by distinctive parameters $\phi^{\hat{I}\hat{J}}$, where $\hat{I}, \hat{J} = 0, 1, i_s$.

Our starting point is the unconstrained expression for a flux $\Theta_{IJ}$, which can be split into terms that depend separately on distinctive and non-distinctive parameters as follows:

$$\Theta_{IJ} = \Theta^{\mathrm{d}}_{IJ} + \phi^{0\beta} W_{0\beta IJ} + \phi^{1\beta} W_{1\beta IJ} + \phi^{\beta\gamma} W_{\beta\gamma IJ} + \phi^{\beta k_{s''}} W_{\beta k_{s''} IJ}\,. \tag{C.1}$$

The term $\Theta^{\mathrm{d}}_{IJ}$ in the above expression only depends explicitly on distinctive parameters.

Our goal is to explicitly constrain the above expression to lie a subspace in which the symmetry constraints (2.10) and (2.11) are satisfied, by solving for the non-distinctive parameters in terms of the distinctive parameters. To see how this works, we separate the local Lorentz and gauge symmetry constraints[41] into distinctive and non-distinctive contributions, leading to the following linear system[42]

$$0 = \Theta_{0\alpha} = \Theta^{\mathrm{d}}_{0\alpha} + \phi^{0\beta} W_{0\beta 0\alpha} + \phi^{1\beta} W_{1\beta 0\alpha} + \phi^{\beta\gamma} W_{\beta\gamma 0\alpha}\,, \tag{C.2}$$

$$0 = \Theta_{1\alpha} = \Theta^{\mathrm{d}}_{1\alpha} + \phi^{0\beta} W_{0\beta 1\alpha} + \phi^{1\beta} W_{1\beta 1\alpha} + \phi^{\beta\gamma} W_{\beta\gamma 1\alpha} + \phi^{\beta j_{s'}} W_{\beta j_{s'} 1\alpha}\,, \tag{C.3}$$

$$0 = \Theta_{\alpha\beta} = \Theta^{\mathrm{d}}_{\alpha\beta} + \phi^{0\gamma} W_{0\gamma\alpha\beta} + \phi^{1\gamma} W_{1\gamma\alpha\beta}\,, \tag{C.4}$$

$$0 = \Theta_{\alpha i_s} = \Theta^{\mathrm{d}}_{\alpha i_s} + \phi^{1\beta} W_{1\beta\alpha i_s} + \phi^{\beta j_{s'}} W_{\beta j_{s'}\alpha i_s}\,. \tag{C.5}$$

41 For generic characteristic data, the set of fluxes $\Theta_{I\alpha}$ are linearly independent. However, for some special choices of characteristic data it is possible for certain linear combinations of the fluxes to vanish, say $\nu^{\alpha I}\Theta_{I\alpha} = 0$, indicating the existence of additional null vectors for the intersection matrix $M$. In such cases, some of the constraints become redundant; in practice we drop these redundant constraints so that we only solve a linearly independent subset $\Theta_{I\alpha} = 0$.

42 Note that $W_{\alpha\beta\gamma\delta} = 0$ by definition. Moreover, we assume the (unproven) property $W_{\alpha\beta\gamma i_s} := W_{\alpha\beta 0 i_s} = 0$ for the smooth fourfolds $X$ we consider.

There are $3+2\,h^{1,1}(B)+h^{1,1}(B)^2+2\mathrm{rk}\,\mathsf{G}_{\mathrm{na}}+h^{1,1}(B)\mathrm{rk}\,\mathsf{G}_{\mathrm{na}}+(\mathrm{rk}\,\mathsf{G}_{\mathrm{na}})^2$ parameters $\phi^{IJ}$ in total. Since the number of independent constraints is $2h^{1,1}(B)+h^{1,1}(B)^2+\mathrm{rk}\,\mathsf{G}_{\mathrm{na}}h^{1,1}(B)$, subtracting this number from the number of independent parameters $\phi^{IJ}$ generically leaves behind $3+2\mathrm{rk}\,\mathsf{G}_{\mathrm{na}}+(\mathrm{rk}\,\mathsf{G}_{\mathrm{na}})^2$ independent parameters, precisely equal to the number of distinctive parameters $\phi^{\hat{I}\hat{J}}$. Thus our task can be reduced to solving the above linear system in such a way that the non-distinctive terms in Eq. (C.1) are replaced by linear combinations of the terms $\Theta^{\mathrm{d}}_{0\alpha}, \Theta^{\mathrm{d}}_{1\alpha}, \Theta^{\mathrm{d}}_{\alpha\beta}, \Theta^{\mathrm{d}}_{\alpha i_s}$.

We now derive an explicit algebraic expression for the symmetry-constrained fluxes. First, observe that $\phi^{\beta j_{s'''}}\Sigma_{s'''}\cdot D_\alpha\cdot D_\beta = W^{i_{s'}|j_{s'''}}\phi^{\beta k_{s''}}W_{\beta k_{s''}\alpha i_{s'}}$, so that the final set of terms in Eq. (C.3) are given by $\phi^{\beta j_{s'}}W_{\beta j_{s'}1\alpha} = W_{1|j_{s'}}(\phi^{\beta j_{s'}}\Sigma_{s'}\cdot D_\alpha\cdot D_\beta) = W_{1|j_{s'}}W^{i_{s'''}|j_{s'}}\phi^{\beta k_{s''}}W_{\beta k_{s''}\alpha i_{s'''}}$. This allows us to replace the final set of terms in Eq. (C.3) with the first two sets of terms on the right-hand side of Eq. (C.5). Next, by observing that $W_0 = W_1 = 1$ and $W_{00} = W_{11} = K$, as well as $W_{01\beta\gamma} = W^{\alpha}_{01}W_{0\alpha\beta\gamma}$, we able to use the constraint (C.4) to eliminate the first two sets of non-distinctive terms from the linear combination $(\Theta_{0\alpha}+\Theta_{1\alpha})/2$. Finally, the linear combination $(\Theta_{0\alpha}-\Theta_{1\alpha})/2$ can be used to simplify the resulting expressions as well as constrain the parameters $\phi^{1\alpha}$. In summary we find that the symmetry constraints can be re-expressed as

$$\phi^{0\gamma}W_{0\alpha\beta\gamma} = -\Theta^{\mathrm{d}}_{\alpha\beta} - \phi^{1\gamma}W_{0\alpha\beta\gamma} \tag{C.6}$$

$$\phi^{\beta\gamma}W_{0\alpha\beta\gamma} = -\Theta^{\mathrm{d}}_{0\alpha} + W^{\beta}_{00}\Theta^{\mathrm{d}}_{\alpha\beta} + (W^{\beta}_{00} - W^{\beta}_{01})\phi^{1\gamma}W_{0\alpha\beta\gamma} \tag{C.7}$$

$$\phi^{\beta j_{s'}}W_{\beta j_{s'}\alpha i_s} = -\Theta^{\mathrm{d}}_{\alpha i_s} - \phi^{1\beta}W_{1\beta\alpha i_s} \tag{C.8}$$

$$\begin{aligned} W_{1|j_{s'}}W^{j_{s'}|i_s}\phi^{\beta k_{s''}}W_{\beta k_{s''}\alpha i_s} &= -\Theta^{\mathrm{d}}_{1\alpha} + \Theta^{\mathrm{d}}_{0\alpha} + (W^{\beta}_{01} - W^{\beta}_{00})\Theta^{\mathrm{d}}_{\alpha\beta} \\ &\quad + 2(W^{\beta}_{01} - W^{\beta}_{00})\phi^{1\gamma}W_{0\alpha\beta\gamma}. \end{aligned} \tag{C.9}$$

The equations (C.8) and (C.9) can be combined to recover the U(1) gauge symmetry constraint equations,

$$\phi^{IJ}W_{\bar{1}IJ}\cdot D_\alpha = (\phi^{1\beta}W_{\bar{1}1\beta} + \phi^{\hat{K}\hat{L}}W_{\bar{1}\hat{K}\hat{L}})\cdot D_\alpha = \Theta_{\bar{1}\alpha} = \sigma^{J}_{\bar{1}}\Theta_{J\alpha} = 0 \tag{C.10}$$

where in the above equation (compare to Eq. (B.5))

$$W_{\bar{1}KL} = -W_{1|k_{s''}}W^{k_{s''}|i_s}W_{i_sKL} + W_{1KL} - W_{0KL} + (W_{00} - W_{01})\cdot W_{KL}\,. \tag{C.11}$$

Using Eqs. (C.6) to (C.9) to eliminate all dependence on non-distinctive parameters, we find that the symmetry-constrained fluxes $\Theta_{\hat{I}\hat{J}} = M_{C(\hat{I}\hat{J})(KL)}\phi^{KL}$ are defined by

$$M_{C\,(\hat{I}\hat{J})(\hat{K}\hat{L})} = M_{C_{\mathrm{na}}\,(\hat{I}\hat{J})(\hat{K}\hat{L})} - M_{C_{\mathrm{na}}\,(\hat{I}\hat{J})(1\alpha)}M^{+(1\alpha)(1\beta)}_{C_{\mathrm{na}}}M_{C_{\mathrm{na}}\,(1\beta)(\hat{K}\hat{L})}\,, \tag{C.12}$$

where $M_{C_{\mathrm{na}}} = C^{\mathrm{t}}_{\mathrm{na}}MC_{\mathrm{na}}$ is the restriction of $M$ to the sublattice $\Lambda_{C_{\mathrm{na}}}$ of backgrounds only satisfying the purely nonabelian constraints $\Theta_{i_s\alpha} = 0$. The components of $M_{C_{\mathrm{na}}}$ are

$$\begin{aligned} M_{C_{\mathrm{na}}\,(IJ)(KL)} &= W_{IJKL} - W_{IJ|i_s}\cdot W^{i_s|j_{s'}}W_{KLj_{s'}} - W_{0IJ}\cdot W_{KL} - W_{IJ}\cdot W_{0KL} \\ &\quad + W_{00}\cdot W_{IJ}\cdot W_{KL} \end{aligned} \tag{C.13}$$

where in particular

$$\begin{aligned} M_{C_{\mathrm{na}}(1\alpha)(KL)} &= D_\alpha \cdot W_{\bar{1}KL} \\ &= D_\alpha \cdot (-W_{1|k_{s''}} W^{k_{s''}|i_s} W_{i_s KL} + W_{1IJ} - W_{0KL} + (W_{00} - W_{01}) \cdot W_{KL}) \end{aligned} \tag{C.14}$$

$$\begin{aligned} M_{C_{\mathrm{na}}(1\alpha)(1\beta)} &= D_\alpha \cdot D_\beta \cdot W_{\bar{1}\bar{1}} \\ &= D_\alpha \cdot D_\beta \cdot (-W_{1|k_{s''}} W^{k_{s''}|i_s} W_{1i_s} + 2(W_{00} - W_{01}))\,. \end{aligned} \tag{C.15}$$

Note that $W_{\bar{1}\bar{1}}$ is equal to (minus) the height pairing divisor associated to the factor $\mathrm{U}(1) \subset \mathsf{G}$, and that $M^{+(1\alpha)(1\beta)}_{C_{\mathrm{na}}}$ is the inverse (when it exists) of $M_{C_{\mathrm{na}}(1\alpha)(1\beta)}$.

## D. Resolution-Independence of $M_{\mathrm{red}}$ in Nonabelian Models

In this appendix we show that the matrices $M_{\mathrm{red}}$ associated with different resolutions can be related through a basis change $U$ when a physically-motivated condition is satisfied. This does not completely prove that $M_{\mathrm{red}}$ is resolution-independent; in particular, $U$ may in general be rational, although in all cases we have considered the change of basis is integral, and we suspect but have not proven that this is always the case.

Given a singular F-theory model characterized by a nonabelian group $\mathsf{G} = \mathsf{G}_{\mathrm{na}} = \prod_s \mathsf{G}_s$ and matter spectrum $\oplus \mathsf{r}^{\oplus n_{\mathsf{r}}}$, we consider the set of possible F-theory resolutions for which the chiral indices can be expressed as linear combinations of the fluxes, $\chi_{\mathsf{r}} = x_{\mathsf{r}}^{i_s j_t} \Theta_{i_s j_t}$, i.e., those resolutions where the matter surfaces for all chiral matter representations contain a vertical component. (Thus for this analysis we ignore the potential existence of unusual resolutions, such as those described in Appendix G, for which a subset of the matter surfaces do not contain a vertical component). We make the assumption that, for each pair of resolutions of the same singular geometry, the resulting $M_{\mathrm{phys}}$ is the same up to a choice of integral basis. While we do not have a general proof that this must always be the case it is not much stronger than the statement that the set of allowed flux backgrounds and chiral multiplicities are the same for both resolutions, which we expect on physical grounds since the physics of any F-theory model should be resolution-invariant.[43]

Consider a pair of resolutions $X, \tilde{X}$ satisfying this criterion. For simplicity we assume that the gauge group has a single nonabelian factor $G$, though a very similar analysis can be carried out for groups with multiple nonabelian factors. For each of these resolutions, we consider again the general form of $M_{\mathrm{red}}$ (4.16), namely

$$M_{\mathrm{red}} = \begin{pmatrix} [[D_{\alpha'} \cdot K \cdot D_\alpha]] & [[D_{\alpha'} \cdot D_\alpha \cdot D_\beta]] & 0 & 0 \\ [[D_{\alpha'} \cdot D_{\beta'} \cdot D_\alpha]] & 0 & 0 & [[W_{\alpha'\beta' j_t k_u}]] \\ 0 & 0 & [[W_{\alpha' i'_{s'} \alpha i_s}]] & [[W_{\alpha' i'_{s'} j_t k_u}]] \\ 0 & [[W_{j'_{t'} k'_{u'} \alpha\beta}]] & [[W_{j'_{t'} k'_{u'} \alpha i_s}]] & [[W_{j'_{t'} k'_{u'} j_t k_u}]] \end{pmatrix}\,, \tag{D.1}$$

where we recall that unprimed indices denote columns and primed indices denote rows. This matrix generally has the schematic structure

$$M_{\mathrm{red}} = \begin{pmatrix} M' & Q \\ Q^{\mathrm{t}} & M'' \end{pmatrix}, \tag{D.2}$$

43 This does not rule out the possibility of situations where, despite the fact that the fluxes for two resolutions are the same, the intersection pairings on their respective lattices of flux backgrounds differ (e.g. $M_{\mathrm{phys}} = (4), (1)$ and $\chi_{\mathsf{r}} = 1, 4$.) However, we have not encountered such situations in any of the F-theory models we have studied.

where $M'$ takes the form (4.15) and is non-degenerate and thus invertible. We can then carry out a change of basis using the matrix

$$U_1 = \begin{pmatrix} \mathrm{Id} & u_1 \\ 0 & \mathrm{Id} \end{pmatrix}, \tag{D.3}$$

namely

$$U_1^{\mathrm{t}} M_{\mathrm{red}} U_1 = \begin{pmatrix} M' & 0 \\ 0 & M_1'' \end{pmatrix}, \tag{D.4}$$

where explicitly $u_1 = -(M')^{-1} Q$ and $M_1'' = M'' - u_1^{\mathrm{t}} M' u_1$. Note that $u_1$, and hence $U_1$ are generically rational since $(M')^{-1}$ contains a factor of $\det \kappa$ in the denominator as discussed abstractly in Sect. 2.8 and more explicitly in Sect. 6.2.2. Because $M'$ is non-degenerate, the symmetry constraints are imposed by simply setting the first set of coordinates to vanish. Thus, $M_1''$ is essentially $M_{\mathrm{phys}}$. The subtlety here is that since $(M')^{-1}$ has a denominator of $\det \kappa$, in most cases, like the example described explicitly in Sect. 6.4.3, we have

$$M_1'' = M_{\mathrm{phys}} / (\det \kappa)^2. \tag{D.5}$$

In such a situation, the condition that the change of basis Eq. (D.3) gives an integer vector means that $M_1''$ only acts on the sublattice of vectors $\phi''$ such that $u_1 \phi''$ is integer-valued in all components.

From this analysis we can now immediately see that the equivalence of $M_{\mathrm{phys}}$ between the two resolutions allows us to relate the forms of $M_{\mathrm{red}}$. Assuming that Eq. (D.5) holds for both the resolutions $X$, $\tilde{X}$, and that the resulting $M_{\mathrm{phys}}$ are related through an integral linear transformation

$$\tilde{M}_{\mathrm{phys}} = U_p^{\mathrm{t}} M_{\mathrm{phys}} U_p, \tag{D.6}$$

we have

$$\tilde{M}_{\mathrm{red}} = U^{\mathrm{t}} M_{\mathrm{red}} U, \tag{D.7}$$

where

$$U = U_1 \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & U_p \end{pmatrix} \tilde{U}_1^{-1} = \begin{pmatrix} 1 & u_1 U_p - \tilde{u}_1 \\ 0 & U_p \end{pmatrix}. \tag{D.8}$$

Note that the matrix $U_p$ used in Eq. (D.6) is not uniquely defined, and has an ambiguity up to the set of automorphisms of the lattice $M_{\mathrm{phys}}$. Thus, we have a number of candidate transformations $U$ of the form Eq. (D.8). As an explicit class of examples note that even if $M_{\mathrm{phys}} = \tilde{M}_{\mathrm{phys}}$ and both matrices are diagonal with distinct eigenvalues, the matrix $U_p$ may be any diagonal matrix composed of elements $\pm 1$.

This argument is almost enough to prove that the two forms of $M_{\mathrm{red}}$ are equivalent under an integral linear change of basis, whenever the $M_{\mathrm{phys}}$ forms are equivalent. The possible obstructions to this result of resolution-independence of $M_{\mathrm{red}}$ are related to the rational form of $u_1, \tilde{u}_1$. In general, we expect both $u_1$ and $\tilde{u}_1$ to have rational terms with a denominator of $\det \kappa$. The fractional parts need to cancel for Eq. (D.8) to be an integer transformation. This implies a certain compatibility condition, whereby the elements of $M_{\mathrm{red}}$ lie in the weight lattice but not the root lattice of $\mathsf{G}$. Because there is some ambiguity in the choice of $U_p$, as mentioned above, for there to be an integral $U$ satisfying Eq. (D.8), only one of the possible $U_p$ choices needs to satisfy this necessary compatibility condition. In all cases we have considered this condition is satisfied for at least one choice of $U_p$, and the resulting $U$ is an integral change of basis that explicitly demonstrates the resolution-invariance of $M_{\text{red}}$ for the resolutions we have studied. We do not have a general proof that this must always occur, but note that even if this compatibility condition is not satisfied, Eq. (D.7) still holds for the matrix $U$ defined in Eq. (D.8). Finally, note that there could also be a subtlety if in one of the resolutions the denominator in Eq. (D.5) is cancelled but not in the other resolution, although it is difficult to imagine a circumstance where the resulting $M_{\text{phys}}$ would still match. We have not encountered any such situations.

The analysis presented here assumes a single nonabelian gauge factor. This argument can easily be generalized to multiple nonabelian gauge factors. For theories with an abelian factor, the story is less clear as we do not have as general a way of understanding $M_{\text{red}}$, and there are potential issues with the invertibility of the part of the matrix analogous to $M'$. Nonetheless, we suspect that a similar approach will shed light on the resolution-independence of theories with more general gauge groups.

Note that when the matrices $M_{\text{red}}$ associated with two different resolutions are related by an integral change of basis, the same is also true for the general intersection matrices $M$ associated with the two resolutions. This follows since, as discussed in Appendix H, in each case there is an integral transformation putting $M$ in the form Eq. (H.1), so composing these transformations with the appropriate integral $U$ on the subspace containing $M_{\text{red}}$ gives an integral transformation relating the two versions of $M$.

## E. Pushforward Formulae

In this Appendix we describe some details of the computational approach we use to evaluate intersection products of divisor classes in resolutions of singular elliptically fibered projective varieties over a smooth base $B$, $X \to B$. Since it is in practice rather cumbersome to compute intersection products of divisors in blowups of elliptically fibered spaces like $X$ directly (i.e. in the Chow ring of $X$), we circumvent this difficulty by pushing these intersection products down to the Chow ring of $B$ (see [113], Remark 3.2.4, p. 55), where the intersection form is by assumption known explicitly.

The computational methods we describe here are essentially a simple and straightforward adaptation of the formulae presented in [20] (see also [114] for related discussions of pushforward formulas, as well as [115,116] for more recent work that uses pushforward formulas to compute various characteristic numbers of elliptic fibrations). Many of the foundational results in intersection theory, algebraic and complex geometry upon which these formulae are based can be found in classic texts such as [55,113].

*E.1. Pushforward maps for resolutions of singular elliptic fibrations.* The pushforward maps we describe in this appendix can be realized explicitly as a composition of pushforward maps associated to two types of projection maps:

Canonical projection of the elliptic fibration. The first type of projection map is the canonical projection of the singular elliptic fibration, $X_0 \to B$. We can determine the pushforward $\varpi_*$ associated to this projection by exploiting the fact that the singular elliptic varieties $X_0$ we consider this paper are all realized as hypersurfaces inside an ambient projective bundle, $Y_0$, which can be viewed as the projectivization of a direct sum of line bundles $\mathscr{L}_a \to B$,

$$Y_0 = \mathbb{P}(\mathscr{V}) \overset{\varpi}{\to} B, \quad \mathscr{V} = \oplus_{a=1}^{3} \mathscr{L}_a \,. \tag{E.1}$$

Since $\mathscr{V}$ is a direct sum of (complex) line bundles, standard results in complex geometry imply that the total Chern class is given by

$$c(\mathscr{V}) = \prod_a (1 + \boldsymbol{L}_a) \quad \implies \quad \boldsymbol{L}_a = c_1(\mathscr{L}_a). \tag{E.2}$$

The above characterization of $Y_0$ as the projectivization of a direct sum of line bundles provides sufficient information for us to specify all types of divisors of $Y_0$ in which we are interested. The different types of divisors are as follows: One type of divisor we wish to consider is the pullback $\boldsymbol{D}_\alpha$ of a divisor $D_\alpha$ (which lives in the Chow ring of $B$) to the Chow ring of $Y_0$. Of particular importance is a special subset of the divisor classes, namely the first Chern classes $c_1(\mathscr{L}_a) = \boldsymbol{L}_a = f(\boldsymbol{D}_\alpha)$ (here, $f$ is an unspecified linear function of certain divisors we refer to characteristic data of the elliptic CY $X_0$, which can themselves be expressed as linear combinations of the pullbacks $\boldsymbol{D}_\alpha$.) The second type of divisor class we consider is $\boldsymbol{H} := c_1(\mathscr{O}_{Y_0}(1))$ where $\mathscr{O}_{Y_0}(1)$ is the twisting sheaf (i.e. the dual of the tautological line bundle) associated to the projectivization of $\mathscr{V}$. It turns out that all characteristic classes of $X_0$ can be associated to formal power series of the classes $\boldsymbol{H}, \boldsymbol{D}_\alpha$, hence our first goal is to explain how to compute intersection numbers of these divisor classes by pushing them forward to the Chow ring of $B$ via the map $\varpi_*$; since the divisors $\boldsymbol{D}_\alpha$ are pullbacks of divisor classes living in the Chow ring of $B$, this task reduces to computing pushforwards of intersection products of the class $\boldsymbol{H}$, as we now describe.

Given a formal power series $\tilde{Q}(t) = \tilde{Q}_n t^n$, one can derive an explicit expression for the pushforward of $\tilde{Q}(\boldsymbol{H})$ to the Chow ring of $B$ by exploiting well known properties of the total Segre class $s(\mathscr{V}) = c(\mathscr{V})^{-1}$. In particular, we use the fact that (see Chapter 3 of [113])

$$\varpi_* \frac{1}{1 - \boldsymbol{H}} = s(\mathscr{V}) = \prod_a \frac{1}{1 + \boldsymbol{L}_a}, \tag{E.3}$$

along with the degree of the map $\varpi$, to obtain the formula[44]

$$\varpi_* \tilde{Q}(\boldsymbol{H}) = \lim_{t_c \to \boldsymbol{L}_c} \sum_{a=1}^{3} \frac{Q(-t_a)}{\prod_{b \neq a}(t_a - t_b)} \,. \tag{E.4}$$

In the above equation $t_a$ are distinct formal variables and (setting $\tilde{Q}_n = \varpi^* Q_n$) the power series $\tilde{Q}(t) = \varpi^* Q_i t^i$ implicitly defines $Q(t) = Q_i t^i$ via the projection formula Eq. (B.3). Observe that Eq. (E.4) is a simple adaptation of the derivation in Theorem 1.11 of [20] to the case where the line bundles $\mathscr{L}_a$ are all distinct. Note that we present the right hand side of Eq. (E.4) as a limit to accommodate special cases where $\boldsymbol{L}_a = \boldsymbol{L}_b$ for some subset of the first Chern classes $\boldsymbol{L}_a$.

Although in the above discussion we have assumed that the line bundles $\mathscr{L}_a$ are generically all distinct, it is important to keep in mind that the scaling symmetry of the projectivization of the bundle $\mathscr{V}$ implies that $\mathbb{P}(\mathscr{V}) \cong \mathbb{P}(\mathscr{V} \otimes \mathscr{I})$ where $\mathscr{I}$ is any invertible line bundle. In particular, without loss of generality we may set $\mathscr{I} = \mathscr{L}_c^{-1}$ and reduce to the standard case

$$\mathbb{P}(\mathscr{V}) \cong \mathbb{P}(\mathscr{L} \oplus \mathscr{L}' \oplus \mathscr{O}) \,, \tag{E.5}$$

which in turn brings the formula Eq. (E.4) into contact with equivalent formulae that have appeared in related literature on elliptic fibrations—see for example (7.3) in Lemma 2.8 of [117]. The above standard form is as specific as we can be about the choice of line bundles $\mathscr{L}, \mathscr{L}'$ while still accommodating the full scope of $\mathsf{G}$ models we wish to describe. For example, models that include U(1) gauge factors such as the $(\mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_2$ model of Sect. 7.1 typically have $\mathscr{L} \neq \mathscr{L}'$. Nonetheless in some cases we can specialize further in order to obtain more succinct formulae that are in practice easier to implement. One possibility entails specializing further to the case $\boldsymbol{L} = p\boldsymbol{L}''$, $\boldsymbol{L}' = p'\boldsymbol{L}''$ where $p, p'$ are non-negative integers, for which it is possible to obtain even simpler expressions for Eq. (E.4). A notable set of examples of this specialization are the Tate models, for which $\boldsymbol{L} = -2\boldsymbol{K}$, $\boldsymbol{L}' = -3\boldsymbol{K}$; in these cases Eq. (E.4) reduces to the formula presented at the beginning of Theorem 1.11 in [20].

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44 In more detail, we expand both sides of the pushforward identity

$$\varpi_* \frac{1}{1 - \boldsymbol{H}} = \frac{1}{\prod_{a=1}^{3}(1 + \boldsymbol{L}_a)}$$

as formal power series, and then (using the fact that $\varpi$ is a degree two map, hence $\varpi_* 1 = \varpi_* \boldsymbol{H} = 0$) match terms of equal degree to obtain a general relation of the form $\varpi_* \boldsymbol{H}^p = f_p(\boldsymbol{L}_a)$ that can be applied to any formal power series term-by-term in the expansion $\tilde{Q}(\boldsymbol{H}) = \sum_{p=0}^{\infty} \varpi^* Q_p \boldsymbol{H}^p$. Explicitly, we make the substitutions $\boldsymbol{H} \to \epsilon \boldsymbol{H}$, $\boldsymbol{L}_a \to \epsilon t_a$ so that

$$\sum_{n=0}^{\infty} \varpi_*(\boldsymbol{H}^n)\epsilon^n = \frac{1}{\prod_{a=1}^{3}(1 + \epsilon t_a)} = \sum_{n=0}^{\infty} s_n(-t_1, -t_2, -t_3)\epsilon^n = \sum_{n=0}^{\infty} \left( \sum_{a=1}^{3} \frac{(-t_a)^{n+2}}{\prod_{b \neq a}(t_a - t_b)} \right) \epsilon^n$$

where $s_n(-t_1, -t_2, -t_3)$ are totally symmetric polynomials of degree $n$ in the three variables $-t_a$. Note that the right hand side of the above equation makes use of the identity $s_n(t_1, \ldots, t_d) = \sum_{a=1}^{d} t_a^{n+d-1} \prod_{b \neq a} \frac{1}{t_a - t_b}$, see Lemma 1.10 of [20], although for fixed $d$ the result above can easily be obtained by computing a partial fraction decomposition of the expression $\frac{1}{\prod_a (1+\epsilon t_a)}$. We then match the $\mathcal{O}(\epsilon)$ terms and re-sum the resulting formal power series to obtain the succinct expression inside the limit on the right hand side of Eq. (E.4).

Blowdowns. We now turn to the second type of projection map, namely the projection associated to a blowup (i.e. the blowdown). The elliptic fibrations $X_0$ we study are typically singular and in practice require a resolution $X \to X_0$ implemented by sequence of blowups in order for the intersection products of divisors to be well-defined and calculable. We specifically consider blowups of the form

$$X_{i+1} \overset{(g_{i+1,1},\ldots,g_{i+1,n_{i+1}}|e_{i+1})}{\longrightarrow} X_i \tag{E.6}$$

where the notation $(g_{i+1,1},\ldots,g_{i+1,n_{i+1}}|e_{i+1})$ is shorthand for the blowup $Y_{i+1} \to Y_i$ of the ambient space $Y_i$ along the center $\{g_{i+1,1} = \cdots = g_{i+1,n_{i+1}} = 0\} \subset Y_i$ with exceptional divisor $e_{i+1} = 0$, and $X_{i+1} \subset Y_{i+1}$ is the proper transform of $X_i$ under the blowup. Importantly, note that we must restrict to blowups where the centers $g_i$ are (at most) linear polynomials in the homogeneous coordinates of the ambient space of the fiber. We abuse notation and make the replacements

$$g_{i,j} \to e_i g_{i,j} \tag{E.7}$$

to implement the $i$th blowup. Each blowup (chosen appropriately) introduces a new divisor class

$$\boldsymbol{E}_i = [e_i] \tag{E.8}$$

and thus it is desirable to be able to compute pushforwards of formal multivariate power series depending on the classes $\boldsymbol{E}_i$ (again, as was the case with the first type of pushforward map $\varpi_*$ described above, the projection formula Eq. (B.3) implies that we are free to ignore divisor classes that are pullbacks and simply focus on the action of the pushforward map $f_i$ on classes $\boldsymbol{E}_i$). Fortunately, there is a similar formula to Eq. (E.4), derived by an analogous procedure, that can be used to compute pushforwards $f_{i+1*}$—we refer the interested reader to Section 3.1 of [20] for details of the derivation. Given a blowup $Y_{i+1} \overset{f_{i+1}}{\to} Y_i$ along the center $g_{i+1,1} = \cdots = g_{i+1,n_{i+1}} = 0$ and a formal power series $\tilde{Q}(\boldsymbol{E}_{i+1})$ in the Chow ring of $Y_{i+1}$, the pushforward of $\tilde{Q}$ to the Chow ring of $Y_i$ is given by (see [20], Theorem 1.8)

$$f_{i+1*}\tilde{Q}(\boldsymbol{E}_{i+1}) = \sum_{k=1}^{n_{i+1}} Q(\boldsymbol{g}_{i+1,k})M_k, \quad M_k = \prod_{\substack{m=1\\ m\neq k}}^{n_{i+1}} \frac{\boldsymbol{g}_{i+1,m}}{\boldsymbol{g}_{i+1,m} - \boldsymbol{g}_{i+1,k}} \tag{E.9}$$

where in the above formula $n_{i+1}$ is the number of generators of the center of the $(i+1)$th blowup and we assume $\boldsymbol{g}_{i+1,k} = [g_{i+1,k}]$ are all distinct.

Equations (E.4) and (E.9) can be composed to compute the pushforward of any intersection product in the Chow ring of $X$ to the Chow ring of $B$. We briefly sketch how this computation works in practice before presenting some explicit examples. Recall that the resolutions we study in this paper are hypersurfaces $X \subset Y$ inside ambient projective (in fact, toric) bundles $Y$ equipped with the projection $\pi : Y \to B$. In such cases, the divisors of $X$ can be realized concretely as the restriction of divisors in the Chow ring of $Y$ to the hypersurface $X$, namely

$$\hat{D}_I = \hat{\boldsymbol{D}}_I \cap \boldsymbol{X}, \quad \hat{\boldsymbol{D}}_I = \ell_I(\boldsymbol{H}, \boldsymbol{E}_i) \tag{E.10}$$

where $\ell_I$ is a linear polynomial and $\boldsymbol{X}$ is the divisor class of the resolved hypersurface $X \subset Y$. For example, a quadruple intersection product takes the form

$$\hat{D}_I \cdot \hat{D}_J \cdot \hat{D}_K \cdot \hat{D}_L = \hat{\boldsymbol{D}}_I \cdot \hat{\boldsymbol{D}}_I \cdot \hat{\boldsymbol{D}}_I \cdot \hat{\boldsymbol{D}}_I \cdot \hat{\boldsymbol{X}} =: \tilde{Q}(\boldsymbol{E}_i, \boldsymbol{H}, \boldsymbol{D}_\alpha), \tag{E.11}$$

where $\cdot$ in the middle expression above (i.e. on the right hand side of the first equality) should be understood as the intersection product in the Chow ring of $Y$, and $\tilde{Q}$ on the far right hand side should be viewed as a formal power series in the Chow ring of $Y$. Assuming that the resolution $X \to X_0$ is obtained by means of a sequence of, say $r$ blowups, the pushforward $\pi_*$ of the projection $\pi : X \to B$ can be viewed as shorthand for a composition of pushforwards,

$$\pi_* = \varpi_* \circ f_{1*} \circ \cdots \circ f_{(r-1)*} \circ f_{r*}\,, \tag{E.12}$$

where the first pushforward $f_{*r}$ maps the expression from the Chow ring of $Y_r$ to the Chow ring of $Y_{r-1}$, the second pushforward maps the resulting expression to the Chow ring of $Y_{r-2}$, and so on.

Ultimately, all of the characteristic classes of $X$ in which we are interested can be expressed as formal power series in the Chow ring of $Y$. For example, the total Chern class $c(X)$ is given by

$$c(X) = \left(\prod_i (1+\boldsymbol{E}_i) \cdot \prod_{j=1}^{n_i} \frac{(1+\boldsymbol{g}_{i,j} - \boldsymbol{E}_i)}{(1+\boldsymbol{g}_{i,j})}\right) \cdot \frac{c(Y_0)}{1+\boldsymbol{X}} \cap \boldsymbol{X}\,. \tag{E.13}$$

where in the above expression

$$c(Y_0) = \prod_k (1+\boldsymbol{H}+\boldsymbol{L}_k) \cdot c(B) \tag{E.14}$$

and we recall that $\boldsymbol{L}_k$ is defined in Eq. (E.1). The Chern polynomial $c_t(X) = 1 + c_1(X)t + c_2(X)t^2 + \cdots$ can be used to extract terms of different degree from the above expression, either in the Chow ring of $Y$ or (after computing the pushforward) in the Chow ring of $B$.

*E.2. Example: SU(2) model.* We illustrate the pushforward technology by way of an example, namely the SU(2) model (this is the $N = 2$ case of the SU($N$) Tate models described in Appendix F.1). Although this example has already been worked out explicitly in [20], we reproduce some of the details here in order to clarify our particular choice of notation.

The Weierstrass equation defining the singular SU(2) model $X_0$ is

$$y^2 z + a_1 xyz + a_{3,1}\sigma yz = x^3 + a_{2,1}\sigma x^2 + a_{4,1}\sigma xz^2 + a_{6,2}\sigma^2 z^3 = 0\,. \tag{E.15}$$

We resolve this model by means of the blowup

$$X_1 \xrightarrow{(x,y,\sigma|e_1)} X_0\,, \tag{E.16}$$

meaning that we make the replacements

$$x \to e_1 x, \quad y \to e_1 y, \quad \sigma \to e_1 \sigma\,. \tag{E.17}$$

Factoring out two powers of $e_1$ from Weierstrass equation of the total transform (i.e., subtracting two copies of the exceptional divisor), we see that the proper transform $X_1 \subset Y_1$ is described by

$$y^2 z + a_1 xyz + e_1 a_{3,1}\sigma yz = e_1 x^3 + e_1 a_{2,1}\sigma x^2 + a_{4,1}\sigma xz^2 + a_{6,2}\sigma^2 z^3 = 0\,. \tag{E.18}$$

The divisor class of the proper transform $X_1$ in the Chow ring of $Y_1$ is

$$\boldsymbol{X}_1 = 3\boldsymbol{H} - 6\boldsymbol{K} - \boldsymbol{E}_1\,. \tag{E.19}$$

The sole Cartan divisor of $X_1$ is

$$\hat{D}_i = \boldsymbol{E}_1 \cap \boldsymbol{X}_1\,. \tag{E.20}$$

We use the pushforward technology described at the beginning of this section to compute the quadruple intersection number $W_{iiii}$. The first step is to evaluate the pushforward of $W_{iiii}$ to the Chow ring of $X_0$, which we denote explicitly by $f_{1*}$ (here we indicate the pushforward and pullback maps explicitly, keeping in mind that $\varpi_*$ is the pullback of the projection $\varpi : X_0 \to B$):

$$\begin{aligned} f_{1*}(\hat{D}_i^4) &= f_{1*}(\boldsymbol{E}_1^4 \cdot \boldsymbol{X}_1) \\ &= f_{1*}(f_1^*(3\boldsymbol{H} - 6\varpi^* K) \cdot \boldsymbol{E}_1^4 - \boldsymbol{E}_1^5) \\ &= (3\boldsymbol{H} - 6\varpi^* K) \cdot f_{1*}(\boldsymbol{E}_1)^4 - f_{1*}(\boldsymbol{E}_1)^5 \\ &= (3\boldsymbol{H} - 6\varpi^* K) \cdot \sum_{k=1}^{3} \boldsymbol{g}_{1,k}^4 \cdot \prod_{\substack{m=1 \\ m\neq k}}^{3} \frac{\boldsymbol{g}_{1,m}}{\boldsymbol{g}_{1,m} - \boldsymbol{g}_{1,k}} - \sum_{k=1}^{3} \boldsymbol{g}_{1,k}^5 \cdot \prod_{\substack{m=1 \\ m\neq k}}^{3} \frac{\boldsymbol{g}_{1,m}}{\boldsymbol{g}_{1,m} - \boldsymbol{g}_{1,k}} \\ &=: \tilde{Q}(\boldsymbol{H})\,. \end{aligned} \tag{E.21}$$

In the above expression $\boldsymbol{g}_{1,k}$ are the classes of the generators of the blowup center:

$$\begin{aligned} \boldsymbol{g}_{1,1} &= [e_1 x] = \boldsymbol{H} - 2\varpi^* K \\ \boldsymbol{g}_{1,2} &= [e_1 y] = \boldsymbol{H} - 3\varpi^* K \\ \boldsymbol{g}_{1,3} &= [e_1 \sigma] = \varpi^* \Sigma\,. \end{aligned} \tag{E.22}$$

Thus far, we have computed the pushforward of the quadruple intersection $W_{iiii}$ to the Chow ring of $X_0$. In order to compute the pushforward of $W_{iiii}$ to the base, we now expand $\tilde{Q}(\boldsymbol{H})$ as a formal power series in the variable $\boldsymbol{H}$ (with coefficients consisting of polynomials in the classes $\varpi^* K$, $\varpi^* \Sigma$) and evaluate the pushforward of each power of $\boldsymbol{H}$ to the Chow ring of $B$ using the formula Eq. (E.4). We do not include details of this computation here as it is completely analogous to the pushforward computation illustrated above in Eq. (E.21). In the end, we obtain

$$W_{iiii} = 2\Sigma \cdot (-4K^2 + 2K \cdot \Sigma - \Sigma^2)\,. \tag{E.23}$$

## F. Resolutions of Some Tate Models

The Tate form of the Weierstrass model $X_0$ is defined by the hypersurface equation

$$y^2 z + a_1 xyz + a_3 yz^2 - (x^3 + a_2 x^2 z + a_4 xz^2 + a_6 z^3) = 0 \tag{F.1}$$

in the ambient projective bundle $Y_0 = \mathbb{P}(\mathscr{V}) \to B$ with $\mathbb{P}^2$ fibers parametrized by homogeneous coordinates $[x : y : z]$. Tate tunings of simple nonabelian gauge groups $\mathsf{G} = \mathrm{SU}(N), \mathrm{SO}(4k+2), \mathrm{E}_6$ are characterized by Kodaira singularities of (resp.) types $\mathrm{I}_N^{\text{split}}, \mathrm{I}_{2k-3}^{*\text{split}}, \mathrm{IV}^{*\text{split}}$ over the codimension-one locus $\Sigma \subset B$. Since the main feature of such models is the existence of a particular type of Kodaira singularity in codimension-one, the divisor class $\Sigma$ together with the canonical class $K$ are sufficient to characterize the features of the elliptic fibration $X$ in which we are primarily interested, and most other relevant mathematical quantities can be defined in terms of $K$ and $\Sigma$ (or their dual line bundles). In particular, the divisor classes

$$[a_n] = -n\boldsymbol{K} \tag{F.2}$$

and the classes of the divisors $x, y, z = 0$ in the ambient space $Y_0$ are given by[45]

$$\begin{aligned} [x] &= \boldsymbol{H} - 2\boldsymbol{K} \\ [y] &= \boldsymbol{H} - 3\boldsymbol{K} \\ [z] &= \boldsymbol{H} \end{aligned} \tag{F.3}$$

where $\boldsymbol{H} = c_1(\mathscr{O}_{\mathbb{P}(\mathscr{V})}(1))$ denotes the hyperplane class of the fibers of $Y_0$ and $\boldsymbol{K}$ is the pullback of the canonical class $K$. This implies that the divisor class of the zero locus of the Weierstrass equation is

$$\boldsymbol{X}_0 = 3\boldsymbol{H} - 6\boldsymbol{K}\,. \tag{F.4}$$

Note that $X_0$ is equipped with a holomorphic zero section $x = z = 0$.

Our aim is to compute intersection numbers of divisors in resolutions $X \to X_0$ of the singular model defined by (F.1). There is a vast literature on crepant resolutions of CY singularities in the context of F-theory compactifications analyzed from various perspectives; see, e.g., [69,70,118,119] for more comprehensive explorations of the networks of possible resolutions associated to the F-theory Coulomb branch. As noted in Footnote 8, our resolutions are in general only partial resolutions in that we do not attempt to resolve all singular fibers that appear over codimension-three loci universally in certain Tate models, nor do we consider cases where additional tunings leading to singular fibers over loci of codimension two (or higher) in the base are forced by the specific choice of singular elliptic fibration. The resolutions we study are composed of a sequence of blowups of the ambient space $Y_0$ of the form (E.6); these blowups restrict to blowups of various loci on the CY hypersurface $X_0$. To carry out the computation of intersection numbers, we select both a basis of divisors for the resolved space, $\hat{D}_{I=0,\alpha,i} \subset X$, and a basis of divisors $\boldsymbol{H}, \boldsymbol{D}_\alpha, \boldsymbol{E}_i \subset Y$ for the proper transform $Y$ of the ambient space under the sequence of blowups, where $\boldsymbol{E}_i$ is the class of the exceptional divisor associated to the $i$th blowup $Y_i \to Y_{i-1}$. We now present some specific examples.

45 We use bold symbols to denote divisor classes in the Chow ring of the ambient space.

*F.1. SU(N) model.* F-theory models with SU($N$) gauge group have been the subject of much attention in the literature, especially low rank examples [53,54,70,95,120,121]. Apart from special exceptions such as SU(6) with three-index antisymmetric matter, SU($N$) models in F-theory are characterized by a $\mathrm{I}_N^{\mathrm{split}}$ singularity over $\Sigma = \{\sigma = 0\} \subset B$ and can be realized explicitly using a Weierstrass model

$$y^2 z + a_1 xyz + a_3 yz^2 - (x^3 + a_2 x^2 z + a_4 xz^2 + a_6 z^3) = 0 \tag{F.5}$$

together with the Tate tuning

$$\mathrm{I}_N^{\mathrm{split}} \; : \; a_1 = a_1, \quad a_2 = a_{2,1}\sigma, \quad a_{3,\lfloor \frac{N}{2} \rfloor}\sigma^{\lfloor \frac{N}{2} \rfloor}, \quad a_{4,\lceil \frac{N}{2} \rceil}\sigma^{\lceil \frac{N}{2} \rceil}, \quad a_6 = a_{6,N}\sigma^N \, . \tag{F.6}$$

These models have a holomorphic zero section $x = z = 0$, and contain matter in the adjoint, fundamental ($\boldsymbol{N}$), and two-index antisymmetric ($N(N-1)/2$) representations. The discriminant locus takes the form

$$\Delta = \sigma^N(\Delta^{(2)} + \mathcal{O}(\sigma)), \quad \Delta^{(2)} = -a_1^4 p_N \, , \tag{F.7}$$

where the polynomial $p_N$ is defined to be

$$p_N = \begin{cases} -a_{4,\lfloor \frac{N}{2} \rfloor}^2 + \mathcal{O}(a_1) & N \text{ even} \\ a_{2,1} a_{3,\lfloor \frac{N}{2} \rfloor}^2 + \mathcal{O}(a_1) & N \text{ odd.} \end{cases} \tag{F.8}$$

Thus, the fundamental and antisymmetric matter multiplets are localized on (resp.) the codimension-two loci $\sigma = p_N = 0$, $\sigma = a_1 = 0$, whose divisor classes in the Chow ring of $B$ are

$$C_{\mathbf{N}} = \begin{cases} \Sigma \cdot (-8K - N\Sigma), & N \neq 3 \\ \Sigma \cdot (-9K - 3\Sigma), & N = 3 \end{cases}, \quad C_{\frac{1}{2}N(N-1)} = \Sigma \cdot (-K) \, . \tag{F.9}$$

We primarily study the family of resolutions $X_{N-1} \to X_0$ realized in [121] by the sequence of blowups

$$X_{N-1} \xrightarrow{(*,e_{N-2}|e_{N-1})} \cdots \xrightarrow{(x,e_2|e_3)} X_2 \xrightarrow{(y,e_1|e_2)} X_1 \xrightarrow{(x,y,\sigma|e_1)} X_0, \quad * = \begin{cases} x & (N \text{ even}) \\ y & (N \text{ odd}) \end{cases} . \tag{F.10}$$

See (E.6) and the discussion immediately below for an explanation of the notation used to indicate blowup maps in the above equation. The class of the holomorphic zero section is

$$\hat{D}_0 = \frac{\boldsymbol{H}}{3} \cap \boldsymbol{X}_{N-1} \, . \tag{F.11}$$

Let $\hat{D}_i$ denote the Cartan divisors of $X_{N-1}$. Using the fact that $\boldsymbol{E}_i = \{e_i = 0\}$ is the class of the exceptional divisor of the $i$th blowup in the ambient space $Y_i$, we may write [121]

$$\hat{D}_i = \begin{cases} (\boldsymbol{E}_{2i-1} - \boldsymbol{E}_{2i}) \cap \boldsymbol{X}_{N-1} & i < \lceil \frac{N}{2} \rceil \\ \boldsymbol{E}_{N-1} \cap \boldsymbol{X}_{N-1} & i = \lceil \frac{N}{2} \rceil \\ (\boldsymbol{E}_{2N-2i} - \boldsymbol{E}_{2N-2i+1}) \cap \boldsymbol{X}_{N-1} & i > \lceil \frac{N}{2} \rceil . \end{cases} \tag{F.12}$$

The Cartan divisors correspond to simple coroots of $\mathrm{SU}(N)$, in a basis where the nonzero pushforwards $\pi_*(\hat{D}_i \cdot \hat{D}_j) = W_{i|j}\Sigma$ are given by

$$[[W_{ij}]] = [[\pi_*(\hat{D}_i \cdot \hat{D}_j)]] = -\begin{pmatrix} 2 & 1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \end{pmatrix}\Sigma \tag{F.13}$$

corresponding to the usual presentation of the $\mathrm{SU}(N)$ Cartan matrix.

The signs and Dynkin coefficients of the weights in the fundamental and antisymmetric representations of SU(5) and SU(6), needed to compute the field theoretic expressions for the 3D Chern–Simons terms associated to these models, are given in Table 3.[46]

*F.2. $SU(6)^\circ$ model.* The exotic SU(6) Tate tuning is

$$a_1 = a_1\,, \quad a_2 = a_{2,2}\sigma^2\,, \quad a_3 = a_{3,\frac{N}{2}-1}\sigma^{\frac{N}{2}-1}\,, \quad a_4 = a_{4,\frac{N}{2}+1}\sigma^{\frac{N}{2}+1}\,, \quad a_6 = a_{6,N}\sigma^N \tag{F.14}$$

for $N$ even and where $\sigma = 0$ is the codimension-one locus in the base over which there is a $\mathrm{I}_N^{\mathrm{split}}$ singularity. We study the resolution $X_5 \to X_0$ obtained by the following sequence of blowups:

$$X_5 \xrightarrow{(y,e_4|e_5)} X_4 \xrightarrow{(y,e_3|e_4)} X_3 \xrightarrow{(x,e_2|e_3)} X_2 \xrightarrow{(y,e_1|e_2)} X_1 \xrightarrow{(x,y,\sigma|e_1)} X_0\,. \tag{F.15}$$

The signs to which the above resolution corresponds are given in Table 4. The classes of the Cartan divisors $\hat{D}_{i=2,\ldots,6}$ are

$$\begin{aligned} \hat{D}_2 &= (\boldsymbol{E}_1 - \boldsymbol{E}_2) \cap \boldsymbol{X}_5\,, \\ \hat{D}_3 &= (\boldsymbol{E}_3 - \boldsymbol{E}_4) \cap \boldsymbol{X}_5\,, \\ \hat{D}_4 &= (\boldsymbol{E}_4 - \boldsymbol{E}_5) \cap \boldsymbol{X}_5\,, \\ \hat{D}_5 &= \boldsymbol{E}_5 \cap \boldsymbol{X}_5\,, \\ \hat{D}_6 &= (\boldsymbol{E}_2 - \boldsymbol{E}_3) \cap \boldsymbol{X}_5\,. \end{aligned} \tag{F.16}$$

[46] Since a pair of irreps $\mathsf{r}$, $\mathsf{r}^*$ related by conjugation are characterized by the same partial ordering, a labeling convention assigning indices to the weights of $\mathsf{r}$ automatically determines a identical convention for the conjugate irrep $\mathsf{r}^*$.

**Table 3.** Signs and Cartan charges of the BPS particles associated to the representations $\boldsymbol{N}$ and $\boldsymbol{N(N-1)/2}$ in the SU$(N)$ model resolutions (F.10) for $N = 5, 6$

| G | $\boldsymbol{N}$ | $\boldsymbol{N(N-1)/2}$ |
|---|---|---|
| SU(5) | $\left(\begin{array}{ccccc} \frac{\varphi\cdot w}{\lvert\varphi\cdot w\rvert} & w_2 & w_3 & w_4 & w_5 \\ \hline + & 1 & 0 & 0 & 0 \\ + & -1 & 1 & 0 & 0 \\ + & 0 & -1 & 1 & 0 \\ - & 0 & 0 & -1 & 1 \\ - & 0 & 0 & 0 & -1 \end{array}\right)$ | $\left(\begin{array}{ccccc} \frac{\varphi\cdot w}{\lvert\varphi\cdot w\rvert} & w_2 & w_3 & w_4 & w_5 \\ \hline + & 0 & 1 & 0 & 0 \\ + & 1 & -1 & 1 & 0 \\ + & -1 & 0 & 1 & 0 \\ + & 1 & 0 & -1 & 1 \\ + & -1 & 1 & -1 & 1 \\ - & 1 & 0 & 0 & -1 \\ - & -1 & 1 & 0 & -1 \\ - & 0 & -1 & 0 & 1 \\ - & 0 & -1 & 1 & -1 \\ - & 0 & 0 & -1 & 0 \end{array}\right)$ |
| SU(6) | $\left(\begin{array}{cccccc} \frac{\varphi\cdot w}{\lvert\varphi\cdot w\rvert} & w_2 & w_3 & w_4 & w_5 & w_6 \\ \hline + & 1 & 0 & 0 & 0 & 0 \\ + & -1 & 1 & 0 & 0 & 0 \\ + & 0 & -1 & 1 & 0 & 0 \\ - & 0 & 0 & -1 & 1 & 0 \\ - & 0 & 0 & 0 & -1 & 1 \\ - & 0 & 0 & 0 & 0 & -1 \end{array}\right)$ | $\left(\begin{array}{cccccc} \frac{\varphi\cdot w}{\lvert\varphi\cdot w\rvert} & w_2 & w_3 & w_4 & w_5 & w_6 \\ \hline + & 0 & 1 & 0 & 0 & 0 \\ + & 1 & -1 & 1 & 0 & 0 \\ + & -1 & 0 & 1 & 0 & 0 \\ + & 1 & 0 & -1 & 1 & 0 \\ + & -1 & 1 & -1 & 1 & 0 \\ + & 1 & 0 & 0 & -1 & 1 \\ + & -1 & 1 & 0 & -1 & 1 \\ + & 0 & -1 & 0 & 1 & 0 \\ - & 1 & 0 & 0 & 0 & -1 \\ - & -1 & 1 & 0 & 0 & -1 \\ - & 0 & -1 & 1 & -1 & 1 \\ - & 0 & -1 & 1 & 0 & -1 \\ - & 0 & 0 & -1 & 0 & 1 \\ - & 0 & 0 & -1 & 1 & -1 \\ - & 0 & 0 & 0 & -1 & 0 \end{array}\right)$ |

The charges are the Dynkin $w_i$ coefficients of the weights $w$ and the signs correspond to the signs of the BPS central charges $\varphi \cdot w$ for a given choice of Coulomb branch moduli $\varphi^i$. The indices $i$ of the weights are chosen to match the indices labeling the Cartan divisors $\hat{D}_i$ in (F.12), which are associated to the simple coroots of $\mathfrak{su}(N)$

**Table 4.** Signs and Cartan charges of the BPS particles associated to the weights of the **6** in the exotic SU(6) model resolution (F.15)

$$\left(\begin{array}{cccccc} \frac{\varphi\cdot w}{\lvert\varphi\cdot w\rvert} & w_2 & w_3 & w_4 & w_5 & w_6 \\ \hline + & 1 & 0 & 0 & 0 & 0 \\ + & -1 & 1 & 0 & 0 & 0 \\ + & 0 & -1 & 1 & 0 & 0 \\ + & 0 & 0 & -1 & 1 & 0 \\ - & 0 & 0 & 0 & -1 & 1 \\ - & 0 & 0 & 0 & 0 & -1 \end{array}\right)$$

The charges are the Dynkin coefficients of the weights of the **6** and the signs correspond to the signs of the BPS central charges $\varphi \cdot w$ for a given choice of Coulomb branch moduli $\varphi^i$

*F.3. SO*$(4k+2)$ *model.* The SO$(4k+2)$ Tate model is characterized by a $\mathrm{I}^{*\text{split}}_{2k-3}$ Kodaira singularity singularity over a gauge divisor $\Sigma = \{\sigma = 0\} \subset B$, and can be realized explicitly using a Weierstrass model

$$y^2z + a_1xyz + a_3yz^2 - (x^3 + a_2x^2z + a_4xz^2 + a_6z^3) = 0 \tag{F.17}$$

together with the Tate tuning

$$\mathrm{I}_{2k-3}^{*\mathrm{split}} \,:\, a_1 = a_{1,1}\sigma, \quad a_2 = a_{2,1}\sigma, \quad a_{3,k}\sigma^k, \quad a_{4,k+1}\sigma^{k+1}, \quad a_6 = a_{6,2k+1}\sigma^{2k+1}\,. \tag{F.18}$$

These models have matter in the adjoint, fundamental ($\mathbf{4k+2}$), and spinor ($\mathbf{4^k}$). The discriminant locus takes the form

$$\Delta = \sigma^{2k+3}(\Delta^{(2)} + \mathcal{O}(\sigma)), \quad \Delta^{(2)} = 16a_2^3a_3^2\,. \tag{F.19}$$

The classes of the matter curves of the fundamental and spinor representations are

$$C_{\mathbf{4k+2}} = \Sigma \cdot (-3K - k\Sigma), \quad C_{\mathbf{4^k}} = \Sigma \cdot (-2K - \Sigma)\,. \tag{F.20}$$

We study the family of resolutions $X_{2k+1} \to X_0$ realized in [122] by the sequence of blowups

$$\begin{aligned} X_{2k+1} &\xrightarrow{(e_{2k-2},e_{2k-1}|e_{2k+1})} X_{2k} \xrightarrow{(y,e_{2k-1}|e_{2k})} X_{2k-1} \xrightarrow{(x,e_{2k-2}|e_{2k-1})} \cdots \\ &\cdots \xrightarrow{(y,e_1|e_2)} X_1 \xrightarrow{(x,y,\sigma|e_1)} X_0\,. \end{aligned} \tag{F.21}$$

The class of the holomorphic zero section is

$$\hat{D}_0 = \frac{\boldsymbol{H}}{3} \cap \boldsymbol{X}_{2k+1}\,. \tag{F.22}$$

Specifically, we restrict our attention to the specific case SO(10) (i.e., $k = 2$). For the SO(10) model we choose the basis of Cartan divisors ($i = 2, \ldots, 6$),

$$\begin{aligned} \hat{D}_2 &= (-\boldsymbol{E}_1 + 2\boldsymbol{E}_2 - \boldsymbol{E}_3 - \boldsymbol{E}_5) \cap \boldsymbol{X}_5 \\ \hat{D}_3 &= (\boldsymbol{E}_1 - \boldsymbol{E}_2) \cap \boldsymbol{X}_5 \\ \hat{D}_4 &= \boldsymbol{E}_5 \cap \boldsymbol{X}_5 \\ \hat{D}_5 &= \boldsymbol{E}_4 \cap \boldsymbol{X}_5 \\ \hat{D}_6 &= (\boldsymbol{E}_3 - \boldsymbol{E}_4 - \boldsymbol{E}_5) \cap \boldsymbol{X}_5 \end{aligned} \tag{F.23}$$

in which the Cartan matrix is represented as

$$[[W_{ij}]] = [[\pi_*(\hat{D}_i \cdot \hat{D}_j)]] = -\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{pmatrix} \Sigma\,. \tag{F.24}$$

**Table 5.** Signs and Cartan charges associated to BPS particles associated to the weights of the **16** in the SO(10) model resolution (F.21)

| $\frac{\varphi\cdot w}{\lvert\varphi\cdot w\rvert}$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ | $w_6$ |
|---|---|---|---|---|---|
| + | 0 | 0 | 0 | 0 | 1 |
| + | 0 | 0 | 1 | 0 | −1 |
| + | 0 | 1 | −1 | 1 | 0 |
| + | 0 | 1 | 0 | −1 | 0 |
| + | 1 | −1 | 0 | 1 | 0 |
| + | −1 | 0 | 0 | 1 | 0 |
| + | 1 | −1 | 1 | −1 | 0 |
| − | −1 | 0 | 1 | −1 | 0 |
| − | 1 | 0 | −1 | 0 | 1 |
| − | −1 | 1 | −1 | 0 | 1 |
| − | 1 | 0 | 0 | 0 | −1 |
| − | −1 | 1 | 0 | 0 | −1 |
| − | 0 | −1 | 0 | 0 | 1 |
| − | 0 | −1 | 1 | 0 | −1 |
| − | 0 | 0 | −1 | 1 | 0 |
| − | 0 | 0 | 0 | −1 | 0 |

The charges are the Dynkin coefficients of the weights of the **16** and the signs correspond to the signs of the BPS central charges $\varphi \cdot w$ for given Coulomb branch moduli $\varphi^i$. The indices $i$ of the weights are chosen to match the indices labeling the Cartan divisors $\hat{D}_i$ in (F.23), which are associated to the simple coroots of $\mathfrak{so}(10)$

*F.4. $E_6$ model.* The $E_6$ Tate model $X_0$ is characterized by a IV$^{*\text{split}}$ singularity over divisor $\Sigma = \{\sigma = 0\} \subset B$ and can be defined by the following Weierstrass equation

$$y^2 z + a_{3,2}\sigma^2 y z^2 - (x^3 + a_{4,3}\sigma^2 x z^2 + a_{6,5}\sigma^5 z^3) = 0\,. \tag{F.25}$$

This model has a holomorphic zero section $x = z = 0$ and matter spectrum $\mathbf{27} \oplus \mathbf{78}$. We study the resolution [20]

$$X_6 \xrightarrow{(y,e_4|e_6)} X_5 \xrightarrow{(y,e_3|e_5)} X_4 \xrightarrow{(e_2,e_3|e_4)} X_3 \xrightarrow{(x,e_2|e_3)} X_2 \xrightarrow{(y,e_1|e_2)} X_1 \xrightarrow{(x,y,\sigma|e_1)} X_0\,. \tag{F.26}$$

The signs to which the above resolution corresponds are given in Table 6. Again, the notation for the blowup maps in the above equation is explained in (E.6) and the discussion immediately below. The class of the holomorphic zero section is

$$\hat{D}_0 = \frac{\boldsymbol{H}}{3} \cap \boldsymbol{X}_{N-1}\,. \tag{F.27}$$

The classes of the Cartan divisors $\hat{D}_{i=2,\ldots,7}$ are

$$\begin{aligned}
\hat{D}_2 &= \boldsymbol{E}_5 \cap \boldsymbol{X}_6\\
\hat{D}_3 &= \boldsymbol{E}_6 \cap \boldsymbol{X}_6\\
\hat{D}_4 &= (-\boldsymbol{E}_1 + 2\boldsymbol{E}_2 - \boldsymbol{E}_3 - \boldsymbol{E}_4) \cap \boldsymbol{X}_6\\
\hat{D}_5 &= (\boldsymbol{E}_1 - 2\boldsymbol{E}_2 + \boldsymbol{E}_3 + 2\boldsymbol{E}_4 - \boldsymbol{E}_6) \cap \boldsymbol{X}_6\\
\hat{D}_6 &= (\boldsymbol{E}_3 - \boldsymbol{E}_4 - \boldsymbol{E}_5) \cap \boldsymbol{X}_6\\
\hat{D}_7 &= (\boldsymbol{E}_1 - \boldsymbol{E}_2) \cap \boldsymbol{X}_6\,.
\end{aligned} \tag{F.28}$$

**Table 6.** Signs and Cartan charges associated to BPS particles associated to the weights of the **27** in the $E_6$ model resolution (F.26)

| $\frac{\varphi\cdot w}{\lvert\varphi\cdot w\rvert}$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ | $w_6$ | $w_7$ |
|---|---|---|---|---|---|---|
| + | 1 | 0 | 0 | 0 | 0 | 0 |
| + | −1 | 1 | 0 | 0 | 0 | 0 |
| + | 0 | −1 | 1 | 0 | 0 | 0 |
| + | 0 | 0 | −1 | 1 | 0 | 1 |
| + | 0 | 0 | 0 | −1 | 1 | 1 |
| + | 0 | 0 | 0 | 1 | 0 | −1 |
| + | 0 | 0 | 0 | 0 | −1 | 1 |
| + | 0 | 0 | 1 | −1 | 1 | −1 |
| + | 0 | 0 | 1 | 0 | −1 | −1 |
| + | 0 | 1 | −1 | 0 | 1 | 0 |
| + | 0 | 1 | −1 | 1 | −1 | 0 |
| + | 1 | −1 | 0 | 0 | 1 | 0 |
| − | −1 | 0 | 0 | 0 | 1 | 0 |
| + | 0 | 1 | 0 | −1 | 0 | 0 |
| + | 1 | −1 | 0 | 1 | −1 | 0 |
| − | −1 | 0 | 0 | 1 | −1 | 0 |
| − | 1 | −1 | 1 | −1 | 0 | 0 |
| − | −1 | 0 | 1 | −1 | 0 | 0 |
| − | 1 | 0 | −1 | 0 | 0 | 1 |
| − | −1 | 1 | −1 | 0 | 0 | 1 |
| − | 1 | 0 | 0 | 0 | 0 | −1 |
| − | −1 | 1 | 0 | 0 | 0 | −1 |
| − | 0 | −1 | 0 | 0 | 0 | 1 |
| − | 0 | −1 | 1 | 0 | 0 | −1 |
| − | 0 | 0 | −1 | 1 | 0 | 0 |
| − | 0 | 0 | 0 | −1 | 1 | 0 |
| − | 0 | 0 | 0 | 0 | −1 | 0 |

The charges are the Dynkin coefficients of the weights of the **27** and the signs correspond to the signs of the BPS central charges $\varphi \cdot w$ for given Coulomb branch moduli $\varphi^i$. The indices $i$ of the weights are chosen to be match the indices of the Cartan divisors $\hat{D}_i$ in Eq. (F.28), which are associated to the simple coroots of $\mathfrak{e}_6$

The above Cartan divisors are labeled such that

$$[[W_{ij}]] = [[\pi_*(\hat{D}_i \cdot \hat{D}_j)]] = -\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix} \Sigma\,. \tag{F.29}$$

## G. A Puzzle: SU(5) Model Resolutions

In this paper we restrict our attention to vertical M-theory flux backgrounds $G \in H^{2,2}_{\text{vert}}(X, \mathbb{Z})$ where $X$ is an arbitrary smooth CY fourfold. In order for such backgrounds to lift to consistent F-theory backgrounds, we require that $X$ is a resolution of a singular CY fourfold $X_0$ and that $G$ satisfies certain conditions both necessary and sufficient to lift to an F-theory flux background.

Our procedure for computing the chiral indices $\chi_{\mathsf{r}}$ is predicated on the assumption that given an M-theory flux background $X, G$ satisfying the above conditions, the full set of chiral multiplicities $\chi_{\mathsf{r}}$ can be extracted in the M-theory duality frame by identifying an appropriate collection of matter surfaces $S_{\mathsf{r}}$ and computing integrals of the form

$\chi_{\mathsf{r}} = \int_{S_{\mathsf{r}}} G$. Although the equation $\chi_{\mathsf{r}} = \int_{S_{\mathsf{r}}} G$ is expected to hold true for general $X$ and $G \in H^{2,2}(X,\mathbb{R}) \cap H^4(X,\mathbb{Z})$ lifting to consistent F-theory flux vacua, in our case the restriction $G \in H^{2,2}_{\text{vert}}(X,\mathbb{Z})$ necessitates further assumptions about the matter surfaces, namely that (recalling the orthogonal decomposition Eq. (2.12)) since a complete basis of vertical fluxes is given by

$$\Theta_{ij} = \int_{S_{ij}} G, \tag{G.1}$$

there exists a non-trivial choice of coefficients $x^{ij}_{\mathsf{r}}$ satisfying

$$\chi_{\mathsf{r}} = x^{ij}_{\mathsf{r}} \Theta_{ij} = \int_{x^{ij}_{\mathsf{r}} S_{Cij}} G \tag{G.2}$$

for all $\mathsf{r}$ in the 4D spectrum. Equivalently, Eq. (2.12) suggests that our procedure only yields a non-trivial answer for the chiral index $\chi_{\mathsf{r}}$ provided the corresponding matter surface $S_{\mathsf{r}}$ has components in the vertical homology $H^{\text{vert}}_{2,2}(X,\mathbb{Z})$:

$$S_{\mathsf{r}} = x^{ij}_{\mathsf{r}} S_{Cij} = x^{ij}_{\mathsf{r}} S_{ij} + \cdots, \quad x^{ij}_{\mathsf{r}} S_{ij} \neq 0\,, \tag{G.3}$$

where $\cdots$ indicates other components with indices $IJ \neq ij$. If no $x^{ij}_{\mathsf{r}}$ exist such that the above equation is satisfied, this suggests $S_{\mathsf{r}}$ does not have any components in $H^{\text{vert}}_{2,2}(X,\mathbb{Z})$.

A useful test of these assumptions is to compute the full set of gauge symmetry-preserving vertical fluxes all available resolutions of a given $\mathsf{G}$ model; if the assumptions are valid, then for any such resolution $X$ it will always be possible to find such a choice of $x^{ij}_{\mathsf{r}}$, so that the chiral indices may be expressed as a sum over $\Theta_{ij}$. One valuable model for which the full set of resolutions (up to codimension-three singularities in the base $B$) has been computed is the universal SU(5) model with generic matter [54].

Unfortunately, the pushforward technology used to compute quadruple intersection numbers as described in this paper cannot be applied to all of these resolutions, so we do not have a direct geometric computation of $\Theta_{ij}$ for the full network of resolutions of the SU(5) model. However, the analysis of [54] not only includes explicit descriptions of the resolutions, but also the Dynkin coefficients $\hat{D}_i \cdot C_w = w_i$ of the matter curves $C_w$ whose volumes $\text{vol}(C_w) = \varphi \cdot w$ shrink to zero as a result of the flop transitions connecting pairs of resolutions. Since we know for certain resolutions the field theoretic expressions Eq. (5.4) for the fluxes $\Theta_{ij} = -\Theta^{3\text{D}}_{ij}$ in terms of the Dynkin coefficients $w_i$ and the signs of the BPS central charges $\varphi \cdot w$ (see e.g. Table 3), by starting with the known collection of signs we can use the fact that the sign of a single central charge $\varphi \cdot w$ flips as we move to an adjacent resolution, to determine the signs of the full set of central charges in the adjacent resolution—in other words, we use the fact that the signs of the central charges of a pair of resolutions related by a flop transition differ by a single sign flip. In this manner, by flipping the signs of appropriate central charges as we move around the graph in Appendix 3, we can determine the signs of all the SU(5) resolutions, which in turn permits us to compute the field theoretic expressions for $\Theta_{ij}$ for all SU(5) model resolutions, at least in principle.

At face value, simply knowing the field theoretic expressions $\Theta^{3\text{D}}_{ij}$ for all resolutions of a given $\mathsf{G}$ model does not appear to be particularly illuminating, because it tells us nothing about the corresponding geometric expressions for $\Theta_{ij}$ in the stringy UV completion. Nevertheless, one reasonable assumption we can make is that 4D anomaly

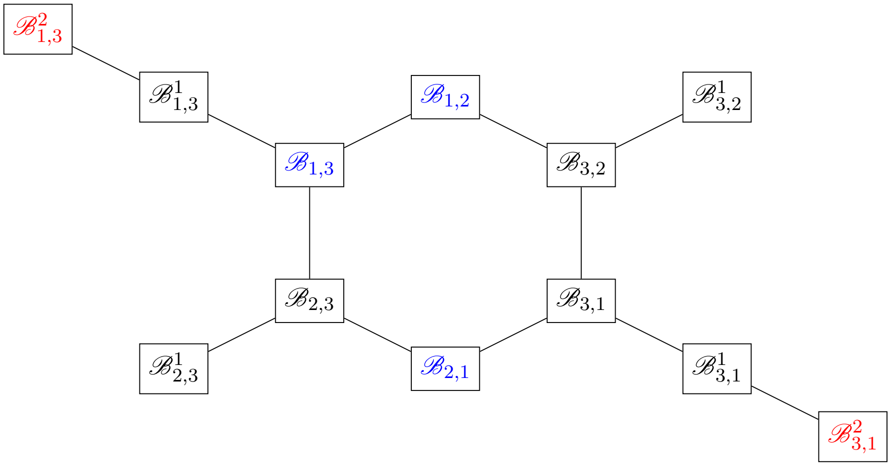


**Fig. 3.** Network of resolutions for the SU(5) Tate model, as presented in Figure 5 of [54] (see also Figure 6). Each boxed node represents a particular resolution of the singular SU(5) model and each edge connecting a pair of nodes indicates a flop transition between two resolutions. A flop transition between two resolutions $\mathscr{B}$, $\mathscr{B}'$ is characterized by all curves belonging to a homology class $C$ of $\mathscr{B}$ collapsing to zero volume and a new curve $C'$ being blown up whose volume in $\mathscr{B}'$ is (formally) minus the volume of $C$ in $\mathscr{B}$. In the context of the SU(5) model the curve classes of interest $C_w$ correspond to weights $w$ transforming in some representation of SU(5); in particular, the volume of such a curve is $\varphi^i w_i$, where $w_i$ are the Dynkin coefficients of the weight $w$. The resolutions in red are those for which the 3D CS terms $\Theta^{3D}_{ij} = 0$

cancellation is satisfied and hence the chiral indices $\chi_{\mathsf{r}}$ appearing in the field theoretic expressions

$$\Theta^{3D}_{ij} = x^{\mathsf{r}}_{ij}\chi_{\mathsf{r}} \tag{G.4}$$

may freely be constrained to obey 4D cancellation while remaining consistent with the geometric expressions $\Theta_{ij} = \int_{S_{ij}} G$. This observation potentially leads to a puzzle: Suppose there exists a resolution (equivalently, a collection of signs of central charges) such that the coefficients $x^{\mathsf{r}}_{ij}$ are proportional to the coefficients of the pure gauge anomaly condition for all $ij$. In such cases, for anomaly cancellation to be satisfied we must have $\Theta_{ij} = 0$ for all $ij$, and hence it appears our assumption about the matter surfaces having non-trivial vertical components somehow fails.

In fact, this seems to be precisely the case for the two resolutions $\mathscr{B}^2_{1,3}$, $\mathscr{B}^2_{3,1}$ of the SU(5) model. We can see this in the case of $\mathscr{B}^2_{1,3}$ by following the procedure described above. First note the signs appearing in Table 3 correspond to the resolution $\mathscr{B}_{1,3}$, for which we find

$$\begin{aligned}
\Theta^{3D}_{22} &= -\chi_{\mathbf{5}} - \chi_{\mathbf{10}} \\
\Theta^{3D}_{23} &= \frac{1}{2}(\chi_{\mathbf{5}} + \chi_{\mathbf{10}}) \\
\Theta^{3D}_{33} &= -\chi_{\mathbf{5}} \\
\Theta^{3D}_{24} &= 0 \\
\Theta^{3D}_{34} &= \frac{1}{2}(\chi_{\mathbf{5}} + \chi_{\mathbf{10}}) \\
\Theta^{3D}_{44} &= -\chi_{\mathbf{10}} \\
\Theta^{3D}_{25} &= 0 \\
\Theta^{3D}_{35} &= -\chi_{\mathbf{10}} \\
\Theta^{3D}_{45} &= \frac{1}{2}(\chi_{\mathbf{10}} - \chi_{\mathbf{5}}) \\
\Theta^{3D}_{55} &= \chi_{\mathbf{5}} + \chi_{\mathbf{10}}
\end{aligned} \tag{G.5}$$

To go from the resolution $\mathscr{B}_{1,3}$ to $\mathscr{B}^1_{1,3}$, we must flip the sign of the central charge corresponding to the weight $w^{\mathbf{10}} = (0, -1, 0, 1)$, which according to Table 3 is negative and hence must become positive. Then, to go from the resolution $\mathscr{B}^1_{1,3}$ to the resolution $\mathscr{B}^2_{1,3}$, we must flip the sign of the central charge corresponding to the weight $w^{\mathbf{5}} = (0, 0, -1, 1)$ from negative to positive. Computing $\Theta^{3D}_{ij}$, we find

$$\begin{aligned}
\Theta^{3D}_{22} &= -(\chi_{\mathbf{5}} + \chi_{\mathbf{10}}) \\
\Theta^{3D}_{23} &= \frac{1}{2}(\chi_{\mathbf{5}} + \chi_{\mathbf{10}}) \\
\Theta^{3D}_{33} &= -(\chi_{\mathbf{5}} + \chi_{\mathbf{10}}) \\
\Theta^{3D}_{24} &= 0 \\
\Theta^{3D}_{34} &= \frac{1}{2}(\chi_{\mathbf{5}} + \chi_{\mathbf{10}}) \\
\Theta^{3D}_{44} &= -(\chi_{\mathbf{5}} + \chi_{\mathbf{10}}) \\
\Theta^{3D}_{25} &= 0 \\
\Theta^{3D}_{35} &= 0 \\
\Theta^{3D}_{45} &= \frac{1}{2}(\chi_{\mathbf{5}} + \chi_{\mathbf{10}}) \\
\Theta^{3D}_{55} &= 0.
\end{aligned} \tag{G.6}$$

Comparing the above expressions to the anomaly cancellation condition $\chi_{\mathbf{5}} + \chi_{\mathbf{10}} = 0$, we therefore expect $\Theta_{ij} = -\Theta^{3D}_{ij} = 0$ for all $ij$ in the resolution $\mathscr{B}^2_{1,3}$. Analogous results hold for $\mathscr{B}^2_{3,1}$.

One possible interpretation of the above computation is that $\Theta^{3D}_{ij} = -\Theta_{ij} = 0$ and hence according to our above reasoning the matter surfaces $S_{\mathsf{r}}$ do not contain any vertical components. This would in turn suggest that in general only a proper subset of possible resolutions can be used to access information about the 4D massless chiral spectrum, given purely vertical flux backgrounds. If this conclusion is true, the resolutions $\mathscr{B}^2_{1,3}$, $\mathscr{B}^2_{3,1}$ are counterexamples to the conjecture that $H^{\text{vert}}_{2,2}(X, \mathbb{Z})$ is a resolution-independent structure and they furthermore contradict the assumption that all resolutions can be used to compute the 4D massless chiral spectrum (at least for strictly vertical flux backgrounds). On the other hand, since we do not currently have the means to compute lattice of vertical fluxes for $\mathscr{B}^2_{1,3}$, $\mathscr{B}^2_{3,1}$, it is possible that these resolutions exhibit some unusual features that might explain why $\Theta^{3\text{D}}_{ij} = 0$. We leave a satisfactory explanation of this puzzle to future work, although we stress that it is important to understand this aspect of the computation since the explanation could significantly affect key assumptions underlying our analysis.

## H. Reducing the Intersection Pairing Matrix

We now present the systematic approach to finding $M_{\text{red}}$ from the degenerate intersection pairing matrix $M$. We seek to find a basis for the integral lattice defined by $M$ such that $M_{\text{red}}$ is embedded as a submatrix and all other entries are zero, i.e., we wish to find a unimodular matrix $P \in \text{GL}(\dim M, \mathbb{Z})$ such that

$$P^{\text{t}} M P = \begin{pmatrix} M_{\text{red}} & 0 \\ 0 & 0 \end{pmatrix}. \tag{H.1}$$

If we have a (degenerate) basis matrix for $M$, i.e., a matrix $B$ satisfying $M = B^{\text{t}} B$ (which can be found using the Cholesky algorithm generalized to positive semidefinite symmetric matrices), then this can be carried out using standard lattice reduction algorithms such as the LLL algorithm [123,124]. However, although $M$ defines an integral lattice, there may be no integral basis $B$ for this lattice, and so determining $B$ can require extracting square roots. In practice, one may wish to avoid this due to issues with floating point arithmetic.

An alternate approach is to begin by finding the LDLT decomposition of $M$, yielding a lower unitriangular matrix $L$ and a diagonal matrix $D$ such that $M = LDL^{\text{t}}$. (The basis matrix $B$ could then be found as $B = \sqrt{D} L^{\text{t}}$, which can clearly introduce square roots.) A basis satisfying our desired properties can be found by putting $B^{\text{t}}$ in (row-style) Hermite normal form $H = UB^{\text{t}}$. Two basis matrices $B$ and $B'$ describe the same integral lattice if and only if their transposes have the same Hermite normal form $H$, and thus the transpose of the Hermite normal form $H^{\text{t}}$ itself serves as an appropriate choice of basis matrix. From the definition of the basis matrix, we see then that $P = U^{\text{t}}$ provides us with the congruence we were seeking. While we can put the potentially real-valued matrix $B$ into Hermite normal form using integer Gaussian elimination or a modification of the LLL algorithm [125], the benefit of this approach is that using LDLT decomposition allows us to find the appropriate $U$ without needing to extract the square roots in $B$. Specifically, the $U$ that puts $B^{\text{t}}$ into Hermite normal form is also the matrix that puts $L$ into Hermite normal form, $\tilde{H} = UL$. Thus, the desired unimodular congruence matrix $P = U^{\text{t}}$ can be found from the LDLT decomposition using only rational matrices.

To summarize, the desired unimodular congruence $P$ can be found efficiently either by using the LLL algorithm directly on the potentially real-valued basis matrix $B$ (which may be found using the Cholesky decomposition), or by finding the LDLT decomposition $M = LDL^{\text{t}}$ and then using the LLL algorithm to find the matrix $U$ putting $L$ into Hermite normal form and setting $P = U^{\text{t}}$; the latter approach avoids ever introducing real-valued matrices.[47] In Mathematica, we can implement the latter approach to find $U$ simply as `Transpose@First@HermiteDecomposition[First@LDLT[mat]]`, where `mat` is the input matrix and `LDLT` is a user-defined function computing the LDLT decomposition of `mat` and returning `{L, D}`.

As an example of this procedure, consider

$$M = \begin{pmatrix} 9 & 0 & -3 & -21 \\ 0 & 9 & -6 & -42 \\ -3 & -6 & 5 & 35 \\ -21 & -42 & 35 & 245 \end{pmatrix}, \tag{H.2}$$

which has two independent null vectors, (7, 14, 0, 3) and (1, 2, 3, 0). The LDLT decomposition of this matrix is

$$M = LDL^{\mathrm{t}}, \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{3} & -\frac{2}{3} & 1 & 0 \\ -\frac{7}{3} & -\frac{14}{3} & 0 & 1 \end{pmatrix}, \quad D = \mathrm{diag}(9, 9, 0, 0). \tag{H.3}$$

The matrix $L$ can be put into Hermite normal form via a unimodular matrix $U$ as

$$H = UL, \quad H = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \quad U = \begin{pmatrix} 5 & 10 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 5 & 10 & 1 & 2 \\ 7 & 14 & 0 & 3 \end{pmatrix}. \tag{H.4}$$

We find then that

$$UMU^{\mathrm{t}} = \begin{pmatrix} 5 & 10 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 5 & 10 & 1 & 2 \\ 7 & 14 & 0 & 3 \end{pmatrix} \begin{pmatrix} 9 & 0 & -3 & -21 \\ 0 & 9 & -6 & -42 \\ -3 & -6 & 5 & 35 \\ -21 & -42 & 35 & 245 \end{pmatrix} \begin{pmatrix} 5 & 0 & 5 & 7 \\ 10 & 1 & 10 & 14 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 6 & 0 & 0 \\ 6 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{H.5}$$

as desired.

Although the method above provides a systematic approach to find $M_{\mathrm{red}}$, for the cases we consider in this paper the result can be easily read off by inspection. We now briefly discuss two situations that one may encounter when carrying out this process: in some cases, $M_{\mathrm{red}}$ cannot be found as a submatrix in $M$ and so a nontrivial basis change is required, and in some cases there may be rational (as opposed to integral) components of the lattice coordinates at intermediate stages of the process, with the final result nevertheless involving only integer values for the lattice coordinates. As an example of the kind of issue that arises in the former case, consider the integral lattice $\Gamma = \mathbb{Z}^2$ with the symmetric bilinear form

$$\begin{pmatrix} 9 & 6 \\ 6 & 4 \end{pmatrix} \tag{H.6}$$

[47] It is worth noting that while this approach does provide a systematic method to find $M_{\mathrm{red}}$, which amounts to finding a lattice basis for the null space of $M$, it does not provide a general method for checking if two integral lattices are congruent to one another, as there is ambiguity in the determination of the basis matrix $B$ (or equivalently in using the LDLT decomposition). Specifically, there are in general multiple valid choices of basis matrix $B$ that span different integral lattices in $\mathbb{R}^n$ but nevertheless reproduce the same Gram matrix $M$. Relating two lattices $M$, $M'$ via the approach outlined here is thus sufficient but not necessary to prove congruence over the integers.

and null vector $(2, -3)$. The quotient lattice $\Gamma' = \Gamma/\sim$ with $(x, y) \sim (x+2n, y-3n)$ can be described as a lattice $\mathbb{Z}$ with inner product (1), but this cannot be realized by simply dropping one of the coordinates $x$ or $y$; rather, the generator of the quotient lattice must be a vector of the form $(1, -1) + k(2, -3)$ (or its negative). As mentioned in the main text, in all the cases we have considered, we find that it is possible to choose a proper basis as a subset of the original basis vectors when the full nullspace is considered. At intermediate steps to reach such a basis, however, we may find it useful to, e.g., project out a null vector by dropping a coordinate in which the null vector has a non-unit value, which naively would suggest fractional values for the coordinates in the reduced lattice, but this generally is compensated by further nullspace vector removal. This is the second issue raised above. As an example of this kind of procedure, consider a 4D lattice with null vectors $(7, 14, 0, 3)$ and $(1, 2, 3, 0)$ (as is the case for the example (H.2)). If we first project out the first of these vectors by dropping the fourth coordinate, we are left with a lattice of points $(x, y, z)$ where $x, y$ may have non-integer parts $m/3, 2m/3$. We can then, however, use the second null vector to drop the first coordinate, subtracting a multiple $m/3$ of this vector from the lattice vectors with non-integer parts, and this automatically removes the non-integer components from the second variable as well, so that $(y, z)$ are good coordinates for the quotient lattice.[48] An example of a situation where the more complicated kind of intermediate fractional lattice arises is described explicitly in the case of the $(\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_6$ gauge group in a followup paper that applies the methods developed here to universal Weierstrass models with that gauge group. This kind of explicit computational approach for finding the reduced basis is not essential in any way to our results but it makes the explicit analysis of various cases easier. In general, the basis of the quotient lattice and the resulting $M_{\text{red}}$ can always be determined efficiently via the method described at the start of this section.

## I. Condition to Lie in the Root Lattice

Here, we briefly discuss the conditions for an integer vector $v_i$ to lie in the root lattice of a simple group $\mathsf{G}_{\text{na}}$ in the basis of fundamental weights. The constraints can be simply summarized as

$$(C^{-\mathrm{t}})_{ij} v_j \in \mathbb{Z}^n, \tag{I.1}$$

with $C_{ij}$ the Cartan matrix and $n$ the rank of $\mathsf{G}_{\text{na}}$. This follows because the rows of $C_{ij}$ are precisely the roots of $\mathsf{G}_{\text{na}}$ expressed in the basis of fundamental weights, and so a vector $v_i$ lies in the root lattice if and only if $v_i = (C^{\mathrm{t}})_{ij} u_j$ for some integer vector $u_j$. Due to the appearance of the inverse Cartan matrix, the conditions following from Eq. (I.1) are all modular congruence conditions mod $\det C$; note that $\det C$ is the order of the center of $\mathsf{G}_{\text{na}}$. Because $v_i$ must be an integer vector, these conditions can be reduced to a single condition for all cases except $\mathsf{G}_{\text{na}} = \mathrm{SO}(4k + 2)$, for which there are two independent conditions mod $\mathbb{Z}_2$ (this is related to the fact that the center of $\mathrm{SO}(4k + 2)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$, rather than $\mathbb{Z}_4$). The conditions are summarized in Table 7.

[48] In this simple case, this of course can also be seen easily by first projecting out the second null vector and then recomputing the null vectors in the reduced space, so that the first null vector in $(y, z, w)$ coordinates becomes $(0, -21, 3)$ indicating that $(0, -7, 1)$ is also in that second partially reduced lattice. We have found it simplest, however, in specific cases of interest to simply start with the list of null vectors of the original matrix and go through intermediate non-integral lattice bases as described above.

**Table 7.** List of modular congruence conditions that must be satisfied for an integer vector $v_i$ to lie in the root lattice (in the basis of fundamental weights) for each of the compact simple Lie groups

| Group | Conditions |
|---|---|
| $A_n$ | $\sum_{j=1}^{n} j v_j \in (n+1)\mathbb{Z}$ |
| $B_n$ | $\sum_{j=1}^{n} j v_j \in 2\mathbb{Z}$ |
| $C_n$ | $v_n \in 2\mathbb{Z}$ |
| $D_{2k}$ | $\sum_{j=1}^{2k} j v_j \in 2\mathbb{Z}, \quad v_{2k-1} + v_{2k} \in 2\mathbb{Z}$ |
| $D_{2k+1}$ | $(2k-1)v_{2k} + (2k+1)v_{2k+1} \in 4\mathbb{Z}$ |
| $E_6$ | $\sum_{j=1}^{6} j v_j \in 3\mathbb{Z}$ |
| $E_7$ | $v_4 + v_6 + v_7 \in 2\mathbb{Z}$ |
| $E_8$ | – |
| $F_4$ | – |
| $G_2$ | – |

We use the conventions of [126] for the ordering of simple roots

## J. Notation

Below is a list of notation commonly used throughout this document:

- $\boxed{X_0}$: Singular elliptic Calabi–Yau (CY) fourfold (i.e., complex dimension four) defining the compactification space of a 4D F-theory model. For the more general case of a $n$-fold where $n \neq 4$, we write $X_0^{(n)}$.
- $\boxed{Y_0}$: Ambient fivefold projective bundle (i.e., a bundle over the base $B$ in which the fibers are projective spaces) in which the singular CY fourfold is realized as a hypersurface, $X_0 \subset Y_0$.
- $\boxed{B}$: Threefold base of the singular elliptic CY fourfold, $X_0 \to B$. More generally, when $B$ is an $(n-1)$-fold with $n \neq 4$, we write $B^{(n-1)}$.
- $\boxed{D_\alpha}$: Basis of primitive divisors of $B$. We use the same symbol to denote a divisor and its class in the Chow ring.
- $\boxed{D \cdot D'}$: Intersection product of pair divisors $D$, $D'$.
- $\boxed{[x]}$: Class of the divisor $x = 0$ in the appropriate Chow ring. For products $xy = 0$, we have $[xy] = [x] + [y]$.
- $\boxed{K}$: Canonical divisor of $B$, $K = K^\alpha D_\alpha$.
- $\boxed{\Delta}$: Discriminant of the Weierstrass equation. The locus $\Delta = 0$ in $B$ is the discriminant locus, over which the elliptic fibers of $X_0$ develop singularities.
- $\boxed{\Sigma_s}$: The divisor class of the codimension-one locus $\sigma_s = 0$ in the base supporting the simple gauge algebra $\mathfrak{g}_s$, i.e., $\Sigma_s = [\sigma_s] = \Sigma_s^\alpha D_\alpha$.
- $\boxed{a_n}$: Sections of the anticanonical class, i.e., $[a_n] = n(-K)$. When the sections are tuned to vanish over a gauge divisor $\sigma = 0$, we write $a_n = a_{n,m_n}\sigma^m$ with $[a_{n,m_n}] = n(-K) - m_n\Sigma$.
- $\boxed{\Delta^{(2)}}$: Residual codimension-two components of the discriminant when restricted to a particular codimension-one component. For example, for Tate models with a simple gauge algebra over the codimension-one locus $\sigma = 0$ the discriminant can be written $\Delta = \sigma^m(\Delta^{(2)} + \mathcal{O}(\sigma))$.
- $\boxed{X}$: Smooth elliptic CY fourfold $X \to X_0$ resolving the singular fourfold $X_0$. When the smooth fourfold is the result of an explicit finite sequence of blowups, we write $X_i$ to denote the proper transform of $X_0$ under the $i$th blowup.

- $\boxed{\pi}$: Canonical projection map from the smooth fourfold to the base, $\pi : X \to B$.
- $\boxed{\hat{D}_I}$: Standard geometric basis of primitive divisors in $X$, where the "hat" decoration distinguishes divisors in $X$ (more generally $X^{(n)}$) from divisors in $B$ (more generally $B^{(n-1)}$). For elliptic CY fourfolds, the indices $I = 0, a, \alpha, i_s$ label the zero section $\hat{D}_0$, generating sections $\hat{D}_a$, the pullbacks of divisors $\hat{D}_\alpha = \pi^* D_\alpha$ in the base $B$, and Cartan divisors $\hat{D}_{i_s}$.
- $\boxed{\hat{D}_{\hat{I}}}$: Distinctive divisors, which exclude the pullbacks of base divisors, i.e., $\hat{I} = 0, a, i_s$.
- $\boxed{\hat{D}_{\bar{I}}}$: Divisors in the "gauge" basis, $\hat{D}_{\bar{I}} = \sigma_{\bar{I}}^{I} \hat{D}_I$, where $\bar{I} = \bar{0}$ is associated to $\mathrm{U}(1)_{\mathrm{KK}}$ and $\bar{I} = \bar{1}$ is associated to $\mathrm{U}(1) \subset \mathsf{G}$. In particular, $\hat{D}_{\bar{1}}$ is the image of $\hat{D}_1$ under the Shioda map.
- $\boxed{\sigma_{\bar{I}}^{I}}$: Change of basis matrix, from the standard geometric basis $\hat{D}_I$ to the "gauge field" basis $\hat{D}_{\bar{I}}$. The inverse matrix is $(\sigma^{-1})_{I}^{\bar{I}}$. See (B.5) and below.
- $\boxed{W_{IJKL}}$: Pushforward with respect to the projection $\pi$ of the quadruple intersection product, i.e., $W_{IJKL} = \pi_*(\hat{D}_I \cdot \hat{D}_J \cdot \hat{D}_K \cdot \hat{D}_L)$. Since the pushforward is evaluated in the Chow ring of the base, we may write $W_{IJKL} = W_{IJKL}^{\alpha\beta\gamma} D_\alpha \cdot D_\beta \cdot D_\gamma$. See Appendix B.
- $\boxed{W_{JKL|i_s}}$: Factor in the intersection product $W_{i_s JKL} = W_{JKL|i_s} \cdot \Sigma_s$. Note that $W_{JKL|i_s} = W_{JKL|i_s}^{\alpha\beta} D_\alpha \cdot D_\beta$.
- $\boxed{k_{\bar{I}\bar{J}\bar{K}}^{\mathrm{5D}}}$: 5D one-loop Chern–Simons coupling.
- $\boxed{\Theta_{\bar{I}\bar{J}}^{\mathrm{3D}}}$: 3D one-loop Chern–Simons coupling.
- $\boxed{W_{i_s|j_s}}$: (Minus the) elements of the inverse Killing form of the simple nonabelian subalgebra $\mathfrak{g}_s$, i.e., $W_{i_s|j_s} = -\kappa_{ij}^{(s)}$.
- $\boxed{\kappa_{ij}}$: Matrix elements of the inverse Killing form $\kappa$.
- $\boxed{W_{\bar{1}\bar{1}}}$: Minus the height pairing divisor in $B$ associated to the $\mathrm{U}(1)$ gauge factor.
- $\boxed{[A_a]}$: A vector $A$ whose components are $A_a$. Not to be confused with the class of a divisor, as should hopefully be clear from the context.
- $\boxed{[[A_{ab}]]}$: A matrix $A$ whose elements are $A_{ab}$.
- $\boxed{Y}$: Ambient fivefold bundle in which the resolution $X$ is realized as a hypersurface, $X \subset Y$. In practice, $Y = Y_i$ is the total transform of the ambient projective bundle $Y_0$ under a composition of blowups, $f_i : Y_{i-1} \to Y_i$.
- $\boxed{\boldsymbol{D}}$: Divisor in the ambient fivefold $Y$ whose restriction to the hypersurface $X$ is a divisor in $X$, i.e., $D = \boldsymbol{D} \cap \boldsymbol{X}$.
- $\boxed{e_i}$: Local coordinate whose zero locus in $Y$ is (the proper transform of) the exceptional divisor $\boldsymbol{E}_i$.
- $\boxed{\varpi}$: Canonical projection map from the ambient fivefold to the base, $\varpi : Y \to B$.
- $\boxed{\mathsf{G}}$: F-theory gauge symmetry group encoded in the singularities of $X_0$.

- $\boxed{\mathsf{G}_{\text{na}}}$: Nonabelian subgroup of $\mathsf{G}$. We abuse notation and write $\mathsf{G}_{\text{na}} = \prod_s \mathsf{G}_s$ where the index $s$ labels the simple subgroups of $\mathsf{G}_{\text{na}}$.
- $\boxed{\mathfrak{g}}$: Lie algebra of the gauge group, $\mathfrak{g} = \text{Lie}(\mathsf{G})$.
- $\boxed{\mathfrak{g}_{\text{na}}}$: Nonabelian subalgebra of $\mathfrak{g}$. Analogously, we write $\mathfrak{g}_{\text{na}} = \oplus_s \mathfrak{g}_s$.
- $\boxed{\mathsf{r}}$: Irreducible complex representation of the gauge symmetry group $\mathsf{G}$.
- $\boxed{n_{\mathsf{r}}}$: Multiplicity of irreps $\mathsf{r}$ appearing in a representation, i.e., $\mathsf{r} \oplus \cdots \oplus \mathsf{r} = \mathsf{r}^{\oplus \mathsf{r}}$.
- $\boxed{\chi_{\mathsf{r}}}$: Chiral multiplicity of matter representations $\mathsf{r}$, i.e., $\chi_{\mathsf{r}} = n_{\mathsf{r}} - n_{\mathsf{r}^*} = -\chi_{\mathsf{r}^*}$.
- $\boxed{\mathsf{R}}$: Quaternionic representation, $\mathsf{R} = \mathsf{r} \oplus \mathsf{r}^*$.
- $\boxed{C_{\mathsf{R}}}$: Class of the codimension-two locus in $B$ over which local matter transforming in the representation $\mathsf{r}$ or $\mathsf{r}^*$ is supported. Given a gauge divisor $\Sigma_s$, $C_{\mathsf{R}_s} = \Sigma_s \cdot (a_{\mathsf{R}_s} K + b_{\mathsf{R}_s} \Sigma_s)$ for some coefficients $a_{\mathsf{R}_s}, b_{\mathsf{R}_s} \in \mathbb{Q}$.
- $\boxed{w_i^{\mathsf{r}}}$: Dynkin coefficients (i.e., coefficients in a basis of fundamental weights, which are canonically dual to the simple coroots) of the weight $w^{\mathsf{r}}$ of the representation $\mathsf{r}$.
- $\boxed{\varphi^i}$: Real Coulomb branch moduli. Equivalently, these are the coefficients of a scalar $\varphi$ expanded in a basis of simple coroots. Since the simple coroots of an algebra are canonically dual to the fundamental weights, given a weight $w^{\mathsf{r}}$, we have $\varphi \cdot w^{\mathsf{r}} = \varphi^i w_i^{\mathsf{r}}$.
- $\boxed{H^{2,2}_{\text{vert}}(X, \mathbb{Z})}$: Vertical cohomology subgroup of the orthogonal decomposition $H^{2,2}(X, \mathbb{C}) = H^{2,2}_{\text{vert}}(X, \mathbb{C}) \oplus H^4_{\text{hor}}(X, \mathbb{C}) \oplus H^{2,2}_{\text{rem}}(X, \mathbb{C})$. Note that $H^{2,2}_{\text{vert}}(X, \mathbb{Z})$ is the linear span with integer coefficients of wedge products of the elements of $H^{1,1}(X, \mathbb{Z})$. Given a basis of divisors $\hat{D}_I$, we write $\text{PD}(S_{IJ}) = \text{PD}(\hat{D}_I) \wedge \text{PD}(\hat{D}_J)$.

Equivalently, $H^{\text{vert}}_{2,2}(X, \mathbb{Z})$ is spanned by $S_{IJ}$ modulo homological equivalence, $\phi \sim \psi \Leftrightarrow M(\phi - \psi) = 0$. As a lattice, $H^{\text{vert}}_{2,2}(X, \mathbb{Z}) = \Lambda_S/\sim$.

- $\boxed{\text{PD}(\hat{D})}$: Poincaré dual of the divisor $\hat{D}$.
- $\boxed{\Lambda_S}$: Lattice of 4-cycles $S_{IJ} = \hat{D}_I \cap \hat{D}_J$ equipped with the bilinear form $M$.
- $\boxed{\phi}$: Poincaré dual vertical flux background, $\phi = \text{PD}(G) \in \Lambda_S$. We frequently abuse terminology and refer to $\phi$ as a flux background, rather than the Poincaré dual of a flux background $G$.
- $\boxed{\phi'}$: "Non-distinctive" flux backgrounds, i.e., backgrounds spanning the directions $IJ$ where $I = \alpha$ or $J = \alpha$.
- $\boxed{\phi''}$: "Distinctive" flux backgrounds, i.e., backgrounds spanning the directions $IJ$ where $I, J \neq \alpha$.
- $\boxed{M}$: Intersection pairing $M : \Lambda_S \times \Lambda_S \to Z$. As a matrix, we write $M_{(IJ)(KL)} = (\hat{D}_I \cdot \hat{D}_J) \cdot (\hat{D}_K \cdot \hat{D}_L)$.
- $\boxed{\Theta_{IJ}}$: Integral of a vertical flux background over the cycle $S_{IJ}$, i.e., $\Theta_{IJ} = \int_{S_{IJ}} G = \phi \cdot (\hat{D}_I \cdot \hat{D}_J) = M_{(IJ)(KL)} \phi^{KL}$.
- $\boxed{\Theta^{\text{d}}_{IJ}}$: The terms in the expansion of $\Theta_{IJ}$ that only depend on distinctive flux backgrounds, i.e., $\Theta^{\text{d}}_{IJ} = M_{(IJ)(\hat{K}\hat{L})} \phi^{\hat{K}\hat{L}}$.
- $\boxed{\Lambda_C}$: The sublattice $\Lambda_C \subset \Lambda_S$ of flux backgrounds $\phi$ satisfying the symmetry constraints $\Theta_{I\alpha} = 0$. See (2.10) and (2.11). We sometimes write $\Lambda_C \cong P\Lambda_S$, where

$P$ is an idempotent matrix, when we can solve the symmetry constraints for all $\phi'$ in terms of $\phi''$.

- $\boxed{M_C}$: Restriction of $M$ to the sublattice $\Lambda_C \subset \Lambda_S$.
- $\boxed{MP}$: Restriction of the intersection pairing $M$ to the sublattice $\Lambda_C$ via right action of the idempotent matrix $P$, which is defined in certain circumstances, and simplifies some computations. In these cases, $M_C$ can be written explicitly as a submatrix in $MP = \begin{pmatrix} 0 & \\ & M_C \end{pmatrix}$ that only acts on distinctive flux backgrounds $\phi'' \subset \Lambda_S$, given the embedding $\phi = \begin{pmatrix} \phi' \\ \phi'' \end{pmatrix}$ Note the isomorphism $M_C \Lambda_C \cong MP\Lambda_S$, and furthermore that $(MP)^{\mathrm{t}} = P^{\mathrm{t}}M = MP$.
- $\boxed{\nu}$: Null vectors of $M$, i.e., $M\nu = 0$. Equivalently, vectors satisfying $\Theta_{IJ}\nu^{IJ} = 0$ or $S_{IJ}\nu^{IJ} = 0$. Each independent $\nu$ represents a homological equivalence relation.
- $\boxed{M_{\mathrm{red}}}$: "Reduced intersection pairing", i.e., the restriction of the intersection pairing $M$ to $H^{\mathrm{vert}}_{2,2}(X, \mathbb{Z}) = \Lambda_S/\!\sim$.
- $\boxed{\Lambda_{\mathrm{phys}}}$: Sublattice $\Lambda_{\mathrm{phys}} \subset H^{\mathrm{vert}}_{2,2}(X, \mathbb{Z}) = \Lambda_S/\!\sim$ of flux backgrounds both satisfying the symmetry constraints and quotiented by homological equivalence. We sometimes write $\Lambda_{\mathrm{phys}} = \Lambda_C/\!\sim$.
- $\boxed{M_{\mathrm{phys}}}$: Restriction of the intersection pairing $M$ to $\Lambda_{\mathrm{phys}}$. Schematically, $M_{\mathrm{phys}} = M_C/\!\sim$.

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