



Geometric Characterization of the Eyring–Kramers Formula

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Abstract: In this paper we consider the mean transition time of an over-damped Brownian particle between local minima of a smooth potential. When the minima and saddles are non-degenerate this is in the low noise regime exactly characterized by the so called Eyring–Kramers law and gives the mean transition time as a quantity depending on the curvature of the minima and the saddle. In this paper we find an extension of the Eyring–Kramers law giving an upper bound on the mean transition time when both the minima/saddles are degenerate (flat) while at the same time covering multiple saddles at the same height. Our main contribution is a new sharp characterization of the capacity of two local minima as a ratio of two geometric quantities, i.e., the minimal cut and the geodesic distance.

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1. Introduction

In this paper we investigate the so called metastable exit times for the stochastic differential equation

$$dX_t = -\nabla F(X_t)dt + \frac{\sqrt{\varepsilon}}{2}dB_t, \quad (1.1)$$

where F is a smooth potential with many local minima and ε is a small number.

The main question of metastability is to determine how much time the process (1.1) takes from going from one local minima to another one. We call these the metastable exit times. This question has a rich history and in the double well case with non-degenerate minima and a saddle point this is characterized by a formula called Eyring–Kramers law [11, 17] which can be stated as follows: Assume that x and y are quadratic local minima of F , separated by a unique saddle z which is such that the Hessian has a single negative eigenvalue $\lambda_1(z)$. Then the expected transition time τ from x to y satisfies

$$\mathbb{E}^x[\tau] \simeq \frac{2\pi}{|\lambda_1(z)|} \sqrt{\frac{|\det(\nabla^2 F(z))|}{\det(\nabla^2 F(x))}} e^{(F(z)-F(x))/\varepsilon}, \quad (1.2)$$

where \simeq denotes that the comparison constant tends to 1 as $\varepsilon \rightarrow 0$.

The validity of the above formula has been studied, from a qualitative perspective, quite extensively, starting from the work of Freidlin and Wentzell. For more information, see the book [12]. Roughly 15 years ago, Bovier et. al. produced a series of papers [6–9] (see also [5]) which provided the first proof of (1.2) in the general setting of Morse functions. Specifically, they showed that the comparison function is like $1 + \mathcal{O}(\varepsilon^{1/2} |\log \varepsilon|^{3/2})$. In these papers, they utilized the connection to classical potential theory in order to reduce the problem of estimating metastable exit times to the problem of estimating certain capacities sharply. This approach was later used in [4] to generalize (1.2) to general polynomial type of degeneracies.

In this paper we are interested in estimating the metastable exit times in the case of general type of degenerate critical points. This requires new techniques and effective notation from geometric function theory which we will describe below. Our motivation comes from the field of non-convex optimization where we cannot expect the minima/saddles to be quadratic or even to have polynomial growth in any direction. In particular, such situations are well known in the context of neural networks, where the minima and saddles may be completely flat in some directions [15]. Furthermore, it seems that they are preferable, see [19] for a discussion, see also [3] for an explicit example.

The main goal is to estimate the dependency of the metastable exit times with respect to the geometry of the potential F . In the proof of (1.2) in [8] this is reduced to estimating the ratio of the L^1 norm of the hitting probability and the capacity. Thus in order to estimate the metastable exit times, one needs to produce

- (1) Estimates of the L^1 integral of the hitting probability, i.e. the integral of capacity potentials with respect to the Gibbs measure.
- (2) Estimates of the capacity itself, i.e. estimates of the energy of the capacity potentials.

The interesting point is that the influence of F on 1 and 2 is in a sense dual. Specifically, the shape of minima of F influence 1 while the shape of saddles between minima

influence 2. As is well known, the main difficulty is to estimate 2, which is an interesting topic of its own.

Our main contribution is a sharp capacity estimate for a very general class of degenerate saddle points. In order to achieve this, we phrase the problem in the language of geometric function theory, where the capacity estimates are a central topic [13,20]. We introduce two geometric quantities which allow us to estimate the capacity in a sharp and natural way. As a byproduct, we see that in the case of several saddle points at the same height, the topology dictates how the local capacities add up. Here we restrict ourselves in two topological cases, which we call the parallel and the serial case, and it turns out that the formulas for the total capacity have natural counterparts in electrical networks of capacitors, see Theorem 1. Even in the context of non-degenerate saddles, our formulas provide a generalization of the result of [8] where the authors consider only the parallel case. As we mentioned, we allow the saddle points to be degenerate but we have to assume that saddles are non-branching, see (1.6).

1.1. Assumptions and statement of the main results. In order to state our main results we first need to introduce our assumptions on the potential F . We also need to introduce notation from geometric function theory which might seem rather heavy at first, but it turns out to be robust enough for us to treat the potentials with possible degenerate critical points.

Let us first introduce some general terminology. We say that a critical point z of a function $f \in C^1(\mathbb{R}^n)$ is a local minimum (maximum) of f if $f(x) \geq f(z)$ ($f(x) \leq f(z)$) in a neighborhood of z . If f is not locally constant at a critical point z , then z is a saddle point if it is not a local minimum / maximum. For technical reasons we also allow saddle points to include points z where f is locally constant. We say that a local minimum at z is *proper* if there exists a $\hat{\delta} > 0$ such that for every $0 < \delta < \hat{\delta}$ there exists a ρ such that

$$f(x) \geq f(z) + \delta \quad \text{for all } x \in \partial B_\rho(z),$$

where $B_\rho(z)$ denotes an open ball with radius ρ centered at z . When the center is at the origin we use the short notation B_ρ .

Let us then proceed to our assumptions on the potential F . Throughout the paper we assume that $F \in C^2(\mathbb{R}^n)$ and satisfies the following quadratic growth condition

$$F(x) \geq \frac{|x|^2}{C_0} - C_0 \tag{1.3}$$

for a constant $C_0 \geq 1$. We assume that every local minimum point z of F is proper, as described above, and that there is a convex function $G_z : \mathbb{R}^n \rightarrow \mathbb{R}$ which has a proper minimum at 0 with $G(0) = 0$ such that

$$|F(x+z) - F(z) - G_z(x)| \leq \omega(G_z(x)), \tag{1.4}$$

where $\omega : [0, \infty) \rightarrow [0, \infty)$ is a continuous and increasing function with

$$\lim_{s \rightarrow 0} \frac{\omega(s)}{s} = 0. \tag{1.5}$$

We denote by δ_0 the largest number for which $\omega(\delta) \leq \frac{\delta}{8}$ for all $\delta \leq 4\delta_0$. We define a neighborhood of the local minimum point z and $\delta < \delta_0$ as

$$O_{z,\delta} := \{x \in \mathbb{R}^n : G_z(x) < \delta\} + \{z\}.$$

For the saddles, we assume that for every saddle point z of F there are convex functions $g_z : \mathbb{R} \rightarrow \mathbb{R}$ and $G_z : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ which have a proper minimum at 0 with $g_z(0) = G_z(0) = 0$, and that there exists an isometry¹ $T_z : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that, denoting $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$, it holds

$$\left| (F \circ T_z)(x) - F(z) + g_z(x_1) - G_z(x') \right| \leq \omega(g_z(x_1)) + \omega(G_z(x')), \tag{1.6}$$

where $\omega : [0, \infty) \rightarrow [0, \infty)$ is as in (1.5). The assumption (1.6) allows the saddle point to be degenerate, but we do not allow them to have many branches, i.e., the sets $\{F < F(z)\} \cap B_\rho(z)$ cannot have more than two components. Note that the convex functions g_z, G_z and the isometry T_z depend on z , while the function ω is the same for all saddle points. We define a neighborhood of the saddle point z and $\delta < \delta_0$ as

$$O_{z,\delta} := T_z \left(\{x_1 \in \mathbb{R} : g_z(x_1) < \delta\} \times \{x' \in \mathbb{R}^{n-1} : G_z(x') < \delta\} \right), \tag{1.7}$$

where T_z is the isometry in (1.6). Note that, since the saddle may be flat, we should talk about sets rather than points. However, we adopt the convention that we always choose a representative point from each saddle (set) and thus we may label the saddles by points z_1, z_2, \dots . Moreover, we assume that there is a $\delta_1 \leq \delta_0$ such that for $\delta < \delta_1$ we have that if z_1 and z_2 are two different saddle points, then their neighborhoods $O_{z_1,3\delta}$ and $O_{z_2,3\delta}$ defined in (1.7) are disjoint. We assume the same for local minima (or more precisely, the representative points of sets of local minima).

Let us then introduce the notation related to geometric function theory [13,20]. Let us fix two disjoint sets A and B in a domain Ω (open and connected set). We say that a smooth path $\gamma : [0, 1] \rightarrow \mathbb{R}^n$ connects A and B in the domain Ω if

$$\gamma(0) \in A, \quad \gamma(1) \in B \quad \text{and} \quad \gamma([0, 1]) \subset \Omega.$$

We denote the set of all paths connecting A and B inside Ω as $\mathcal{C}(A, B; \Omega)$. We follow the standard notation and define a dual object to this by saying that a smooth hypersurface $S \subset \mathbb{R}^n$ (possibly with boundary) separates A from B in Ω if every path $\gamma \in \mathcal{C}(A, B; \Omega)$ intersects S . We denote the set of smooth hypersurfaces, separating A and B inside Ω as $\mathcal{S}(A, B; \Omega)$. We define the geodesic distance between A and B in Ω as

$$d_\varepsilon(A, B; \Omega) := \inf \left(\int_0^1 |\gamma'(t)| e^{\frac{F(\gamma(t))}{\varepsilon}} : \gamma \in \mathcal{C}(A, B; \Omega) \right) \tag{1.8}$$

and its dual, which we call the minimal cut, by

$$V_\varepsilon(A, B; \Omega) := \inf \left(\int_S e^{-\frac{F(x)}{\varepsilon}} d\mathcal{H}^{n-1}(x) : S \in \mathcal{S}(A, B; \Omega) \right). \tag{1.9}$$

¹ Recall that a mapping T is an isometry if $|T(x) - T(y)| = |x - y|$. In \mathbb{R}^n , this implies that $T(x) = Ax + b$, where A is an orthogonal matrix. That is, T consists of translation b and a rotation A .

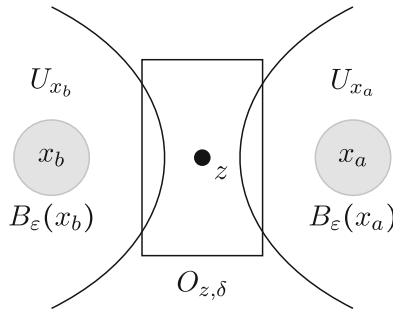


Fig. 1. The neighborhood $O_{z,\delta}$ of the saddle point z connects the sets U_{x_a} and U_{x_b}

Whenever $\Omega = \mathbb{R}^n$ we instead use the notation $d_\epsilon(A, B)$ and $V_\epsilon(A, B)$. Here \mathcal{H}^k denotes the k -dimensional Hausdorff measure. Finally, we define the communication height between the sets A and B as

$$F(A; B) := \inf_{\gamma \in \mathcal{C}(A, B; \mathbb{R}^n)} \sup_{t \in [0, 1]} F(\gamma(t)).$$

Let us then assume that x_a and x_b are local minimum points and denote the communication height between x_a and x_b as

$$F(x_a; x_b) := F(\{x_a\}; \{x_b\}).$$

Notice that $F(x_a), F(x_b) \leq F(x_a; x_b)$. For $s \in \mathbb{R}$, denote

$$U_s := \{x \in \mathbb{R}^n : F(x) < F(x_a; x_b) + s\}.$$

Assuming that $0 < \delta \leq \delta_1$, We note that the points x_a and x_b lie in different components of the set $U_{-\delta/3}$ while they are in the same component of the set $U_{\delta/3}$. We will always denote the components of $U_{-\delta/3}$ containing the points x_a and x_b by U_{x_a} and U_{x_b} , respectively. It is important to notice that if z is a saddle point and $F(z) < F(x_a; x_b) + \delta/3$, then the neighborhood $O_{z,\delta}$ defined in (1.7) intersects the set $U_{-\delta/3}$. We will sometimes call the components of the set $U_{-\delta/3}$ *islands* and the neighborhoods $O_{z,\delta}$ *bridges* since we may connect islands with bridges, see Fig. 1. (The terminology is obviously taken from the Seven Bridges of Königsberg). We say that the set of saddle points $Z_{x_a, x_b} = \{z_1, \dots, z_N\}$ *charge capacity* if it is the smallest set with the property that every $\gamma \in \mathcal{C}(B_\epsilon(x_a), B_\epsilon(x_b); U_{\delta/3})$ intersects the bridge $O_{z_i, \delta}$, defined in (1.7), for some $z_i \in Z_{x_a, x_b}$. In particular, it holds that $Z_{x_a, x_b} \subset U_{\delta/3}$.

We will focus on two different topological situations, where the saddle points in Z_{x_a, x_b} are either *parallel* or *in series*. We say that the points in Z_{x_a, x_b} are parallel if for every $z_i \in Z_{x_a, x_b}$ there is a path

$$\gamma \in \mathcal{C}(B_\epsilon(x_a), B_\epsilon(x_b); U_{\delta/3})$$

passing only through z_i . We say that the points in Z_{x_a, x_b} are in series if every path $\gamma \in \mathcal{C}(B_\epsilon(x_a), B_\epsilon(x_b); U_{\delta/3})$ passes through the bridge $O_{z_i, \delta}$, defined in (1.7), for all $z_i \in Z_{x_a, x_b}$. In other words, if the points in $Z_{x_a, x_b} = \{z_1, \dots, z_N\}$ are parallel, then the islands occupied by the points x_a and x_b respectively are connected with N bridges and we need to pass only one to get from x_a to x_b . If they are in series, then we have to pass all N bridges in order to get from x_a to x_b , see Fig. 2.

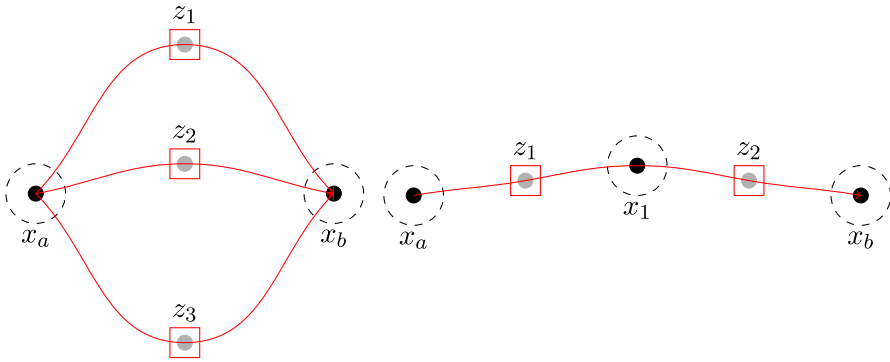


Fig. 2. Left picture is the parallel case and the right is the series case

Recall that U_{x_a} and U_{x_b} denote the islands, i.e., the components of $U_{-\delta/3}$, which contain the points x_a and x_b . If the points in $Z_{x_a, x_b} = \{z_1, \dots, z_N\}$ are parallel, then it follows from our assumptions on F that we may connect U_{x_a} and U_{x_b} with one bridge, i.e., for every z_i the set

$$U_{z_i, \delta} := O_{z_i, \delta} \cup U_{x_a} \cup U_{x_b} \tag{1.10}$$

is connected, again see Fig. 1. Then all paths $\gamma \in \mathcal{C}(B_\varepsilon(x_a), B_\varepsilon(x_b); U_{z_i, \delta})$ pass through the bridge $O_{z_i, \delta}$. If the points in Z_{x_a, x_b} are in series, then it is useful to order them $Z_{x_a, x_b} = \{z_1, \dots, z_N\}$ as follows. Let us consider a path $\gamma \in \mathcal{C}(B_\varepsilon(x_a), B_\varepsilon(x_b); U_{\delta/3})$ which passes through each point in Z_{x_a, x_b} precisely once. This means that there are

$$0 < t_1 < \dots < t_N < 1 \quad \text{such that} \quad \gamma(t_i) = z_i, \tag{1.11}$$

which gives a natural ordering for points in Z_{x_a, x_b} . By the assumption (1.6), we also deduce that there are s_1, \dots, s_{N-1} such that $t_i < s_i < t_{i+1}$ and $\min\{F(\gamma(s_i)), F(\gamma(s_{i+1}))\} < F(z_{i+1}) - \delta/3$. We denote

$$\gamma(s_i) = x_i, \quad x_0 = x_a \quad \text{and} \quad x_N = x_b. \tag{1.12}$$

The idea is that then every point x_i lie in a different island, i.e., component of $U_{-\delta/3}$ which we denote by U_{x_i} , see Fig. 2. We may also choose x_i such that they are local minimum points of F . Again it follows from our assumptions on F that the set $\Omega = \bigcup_{i=1}^N O_{z_i, \delta} \cup U_{x_i} \cup U_{x_a}$ is connected.

We are now ready to state our main results. The first result is a quantitative lower bound on the capacity between the sets $B_\varepsilon(x_a)$ and $B_\varepsilon(x_b)$, where x_a and x_b are two local minimum points of F . For a given domain $\Omega \subset \mathbb{R}^n$ we define the capacity of two disjoint sets $A, B \subset \Omega$ with respect to the domain Ω as

$$\text{cap}(A, B; \Omega) := \inf \left(\varepsilon \int_{\Omega} |\nabla u|^2 e^{-\frac{F}{\varepsilon}} dx : u = 1 \text{ in } A, u = 0 \text{ in } B \right).$$

Above, the infimum is taken over functions $u \in W_{loc}^{1,2}(\Omega)$. In the case $\Omega = \mathbb{R}^n$ we denote

$$\text{cap}(A, B) = \text{cap}(A, B; \mathbb{R}^n)$$

for short.

Finally, for functions f and g which depend continuously on $\varepsilon > 0$, we adopt the notation

$$f(\varepsilon) \simeq g(\varepsilon)$$

when there exists a constant C depending only on the data of the problem such that

$$(1 - \hat{\eta}(C, \varepsilon))f(\varepsilon) \leq g(\varepsilon) \leq (1 + \hat{\eta}(C, \varepsilon))f(\varepsilon),$$

where $\hat{\eta}(C, \cdot)$ is an increasing and continuous function $\hat{\eta}(C, \cdot) : [0, \infty) \rightarrow [0, \infty)$ with $\lim_{s \rightarrow 0} \hat{\eta}(C, \cdot) = 0$. In all our estimates, the function $\hat{\eta}$ is specified and depends only on the function ω from (1.4) and (1.6). In order to define it, we first let $0 < \varepsilon \leq \delta_0/2$ be fixed and let $\varepsilon_1(\varepsilon)$ be the unique solution to

$$\sqrt{\omega(\varepsilon_1)\varepsilon_1} = \varepsilon. \tag{1.13}$$

From the assumption that $\omega(s) < s/2$ for $s < \delta_0$ we see that $\varepsilon < \varepsilon_1$. Furthermore, since ω is increasing we get that $\varepsilon_1 \rightarrow 0$ as $\varepsilon \rightarrow 0$. Now, from the definition of ε_1 in (1.13) we see, using $\lim_{s \rightarrow 0} \frac{\omega(s)}{s} = 0$ and $\varepsilon_1 \rightarrow 0$ as $\varepsilon \rightarrow 0$, that

$$\frac{\varepsilon_1}{\varepsilon} = \frac{\sqrt{\varepsilon_1}}{\sqrt{\omega(\varepsilon_1)}} \rightarrow \infty \quad \text{as } \varepsilon \rightarrow 0.$$

On the other hand, again using the same facts, we see that

$$\frac{\omega(\varepsilon_1)}{\varepsilon} = \frac{\sqrt{\omega(\varepsilon_1)}}{\sqrt{\varepsilon_1}} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Thus

$$\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon_1}{\varepsilon} = \infty \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \frac{\omega(\varepsilon_1)}{\varepsilon} = 0. \tag{1.14}$$

In the following we will denote

$$\eta(x) = e^{-1/x} x^n \quad \text{and} \tag{1.15}$$

$$\hat{\eta}(C, \varepsilon) = \max \left\{ \varepsilon, \eta \left(C \frac{\varepsilon_1(\varepsilon)}{\varepsilon} \right) \right\}. \tag{1.16}$$

Finally, in our main theorems and our lemmas/propositions beyond Sect. 3 there is a ball B_R which contains all the level sets of interest. The existence of such a ball is given by the quadratic growth condition (1.3). The constants in the estimates in our main theorems and in Sect. 3 are unless otherwise stated, depending on $n, \|\nabla F\|_{B_R}, \delta, R, C_0$, specifically, this applies to the constants in $\hat{\eta}$, and as such, gives precise meaning to $a \simeq b$.

Theorem 1. *Assume that F satisfies the structural assumptions above. Let x_a and x_b be two local minimum points of F and let $Z_{x_a, x_b} = \{z_1, \dots, z_N\}$ be the set of saddle points which charge capacity as defined above, and let $0 < \delta \leq \delta_1$ be fixed. There exists an $0 < \varepsilon_0 \leq \delta$ such that if $0 < \varepsilon \leq \varepsilon_0$ the following holds:*

If the points in $Z_{x_a, x_b} = \{z_1, \dots, z_N\}$ are parallel, then, using the notation $U_{z_i, \delta}$ from (1.10), it holds

$$\text{cap}(B_\varepsilon(x_a), B_\varepsilon(x_b)) \simeq \sum_{i=1}^N \text{cap}(B_\varepsilon(x_a), B_\varepsilon(x_b); U_{z_i, \delta}). \tag{1.17}$$

Moreover for all $i = 1, \dots, N$ we have the estimate

$$\text{cap}(B_\varepsilon(x_a), B_\varepsilon(x_b); U_{z_i, \delta}) \simeq \varepsilon \frac{V_\varepsilon(B_\varepsilon(x_a), B_\varepsilon(x_b); U_{z_i, \delta})}{d_\varepsilon(B_\varepsilon(x_a), B_\varepsilon(x_b); U_{z_i, \delta})} e^{\frac{F(z_i)}{\varepsilon}},$$

where $d_\varepsilon(B_\varepsilon(x_a), B_\varepsilon(x_b); U_{z_i, \delta})$ and $V_\varepsilon(B_\varepsilon(x_a), B_\varepsilon(x_b); U_{z_i, \delta})$ are defined in (1.8) and (1.9).

If the points in Z_{x_a, x_b} are in series, then, using the ordering z_1, \dots, z_N from (1.11) for the points in Z_{x_a, x_b} and the points x_0, x_1, \dots, x_N defined in (1.12), it holds

$$\frac{1}{\text{cap}(B_\varepsilon(x_a), B_\varepsilon(x_b))} \simeq \sum_{i=1}^N \frac{1}{\text{cap}(B_\varepsilon(x_{i-1}), B_\varepsilon(x_i))}, \tag{1.18}$$

where we have the estimate

$$\text{cap}(B_\varepsilon(x_{i-1}), B_\varepsilon(x_i)) \simeq \varepsilon \frac{V_\varepsilon(B_\varepsilon(x_{i-1}), B_\varepsilon(x_i))}{d_\varepsilon(B_\varepsilon(x_{i-1}), B_\varepsilon(x_i))} e^{\frac{F(z_i)}{\varepsilon}}$$

for all $i = 1, \dots, N$.

Let us make a few remarks on the statement of the above theorem. First, in the case of a single saddle $Z_{x_a, x_b} = \{z\}$ the above capacity estimate reduces to

$$\text{cap}(B_\varepsilon(x_a), B_\varepsilon(x_b)) \simeq \varepsilon \frac{V_\varepsilon(B_\varepsilon(x_a), B_\varepsilon(x_b))}{d_\varepsilon(B_\varepsilon(x_a), B_\varepsilon(x_b))} e^{\frac{F(z)}{\varepsilon}},$$

where $d_\varepsilon(B_\varepsilon(x_a), B_\varepsilon(x_b))$ is the geodesic distance between $B_\varepsilon(x_a)$ and $B_\varepsilon(x_b)$, and $V_\varepsilon(B_\varepsilon(x_a), B_\varepsilon(x_b))$ is the area of the 'smallest cross section'. This is in accordance with the classical result on parallel plate capacitors, where the capacity depends linearly on the area and is inversely proportional to their distance.

The statement (1.17), when the saddle points are parallel, means that each saddle point z_1, \dots, z_N charges capacity and the total capacity is their sum. Again the situation is the same as in the case of parallel plate capacitors with capacity C_1, \dots, C_N , where the total capacity is the sum

$$C = C_1 + \dots + C_N.$$

On the other hand, if the plate capacitors are in series their total capacity satisfies

$$\frac{1}{C} = \frac{1}{C_1} + \dots + \frac{1}{C_N}$$

which is precisely the statement in (1.18).

Using the assumption (1.6) we calculate in Proposition 4.1 and in Proposition 4.2 more explicit, but less geometric, formulas for the single saddle case in a domain Ω . Namely, we have

$$d_\varepsilon(B_\varepsilon(x_a), B_\varepsilon(x_b); \Omega) \simeq e^{\frac{F(z)}{\varepsilon}} \int_{\mathbb{R}} e^{-\frac{g_z(x_1)}{\varepsilon}} dx_1$$

and

$$V_\varepsilon(B_\varepsilon(x_a), B_\varepsilon(x_b); \Omega) \simeq e^{-\frac{F(z)}{\varepsilon}} \int_{\mathbb{R}^{n-1}} e^{-\frac{G_z(x')}{\varepsilon}} dx',$$

and thus we recover the result in [4]. In particular, if the saddle point is non-degenerate, i.e., g_z and G_z are second order polynomials and the negative eigenvalue of $\nabla^2 F(z)$ is $-\lambda_1$, we may estimate

$$d_\varepsilon(B_\varepsilon(x_a), B_\varepsilon(x_b)) \simeq \sqrt{\frac{2\pi\varepsilon}{\lambda_1}} e^{\frac{F(z)}{\varepsilon}}$$

and

$$V_\varepsilon(B_\varepsilon(x_a), B_\varepsilon(x_b)) \simeq (2\pi\varepsilon)^{\frac{n-1}{2}} \frac{\sqrt{\lambda_1} e^{-\frac{F(z)}{\varepsilon}}}{\sqrt{\det(\nabla^2 F(z))}}.$$

In particular, we recover the classical formula (1.2).

Our second main theorem is an estimate on the so called metastable exit times. However, in order to state it we need some further definitions. Assume that the local minima of F are labelled x_i and ordered such that $F(x_i) \leq F(x_j)$ if $i \leq j$. We will group the minima at the same level using the sets $G_k, k = 1, \dots, K$, i.e. $x_i, x_j \in G_k$ if $F(x_i) = F(x_j)$, and $x \in G_i$ and $y \in G_j$, then $F(x) < F(y)$ for $i < j$. We also write $F(G_i) := F(x)$ with $x \in G_i$. Furthermore, we will denote $S_k = \bigcup_{i=1}^k G_i$ for $k = 1, \dots, K$. We will also consider $G_k^\varepsilon = \bigcup_{x \in G_k} B_\varepsilon(x)$ and $S_k^\varepsilon = \bigcup_{i=1}^k G_i^\varepsilon$.

In addition to the previous structural assumptions we assume further that for $\delta_2 \leq \delta_1$ small enough, it holds

$$F(G_{k+1}) - F(G_k) \geq \delta_2 \tag{1.19}$$

for all $k = 1, \dots, K$. For a set A we denote τ_A the first hitting time of A of the process (1.1), i.e. for X_t as in (1.1) we define

$$\tau_A := \inf\{t \geq 0 : X_t \in A\}.$$

In our second theorem we give an upper bound on the hitting time for the process defined by (1.1) to go from a local minimum point in G_{k+1}^ε to a lower one in S_k .

Theorem 2. *Assume that F satisfies the structural assumptions above, let Ω be a domain that contains S_{k+1}^ε . There exists an $0 < \varepsilon_0 \leq \delta_2$ such that if $0 < \varepsilon < \varepsilon_0$, the following holds:*

For $x \in G_{k+1}^\varepsilon$ we have

$$\mathbb{E}^x [\tau_{S_k^\varepsilon} \mathbb{I}_{\tau_{S_k^\varepsilon} < \tau_{\Omega^c}}] \leq \frac{C e^{-F(G_{k+1})/\varepsilon} \sum_{x \in G_{k+1}} |O_{x,\varepsilon}|}{\max_{x \in G_k, y \in G_{k+1}} \text{cap}(B_\varepsilon(x), B_\varepsilon(y); \Omega)} + C \varepsilon^{\alpha/2}.$$

Let $x_a \in G_k, x_b \in G_{k+1}$ be a pair that maximizes the pairwise capacity. Then, with the notation of Theorem 1, we get in the parallel case

$$\mathbb{E}^x[\tau_{S_k^\varepsilon} \mathbb{I}_{\tau_{S_k^\varepsilon} < \tau_{\Omega^c}}] \leq \varepsilon^{-1} \frac{C e^{-F(G_{k+1})/\varepsilon} \sum_{x \in G_{k+1}} |O_{x,\varepsilon}|}{\sum_{i=1}^N e^{-F(z_i)/\varepsilon} \frac{\mathcal{H}^{n-1}(\{G_{z_i} < \varepsilon\})}{\mathcal{H}^1(\{g_{z_i} < \varepsilon\})}} + C \varepsilon^{\alpha/2},$$

and in the series case we get

$$\mathbb{E}^x[\tau_{S_k^\varepsilon} \mathbb{I}_{\tau_{S_k^\varepsilon} < \tau_{\Omega^c}}] \leq \frac{C}{\varepsilon} \sum_{x \in G_{k+1}} \sum_{i=1}^N e^{(F(z_i) - F(G_{k+1}))/\varepsilon} \frac{|O_{x,\varepsilon}| \mathcal{H}^1(\{g_{z_i} < \varepsilon\})}{\mathcal{H}^{n-1}(\{G_{z_i} < \varepsilon\})} + C \varepsilon^{\alpha/2}.$$

The additive error in Theorem 2 can be removed for small ε as the right hand side of the above tends to ∞ as $\varepsilon \rightarrow 0$.

If both the minima and saddles are non-degenerate points, $\Omega = \mathbb{R}^n$ and there is only one saddle z connecting x_a, x_b (with $F(x_a) < F(z) < F(x_b)$), where x_a, x_b are the only minima of F , then the above estimate coincides with Eyring–Kramers formula (up to a constant)

$$\mathbb{E}^{x_b}[\tau_{B_\varepsilon(x_a)}] \leq C \frac{1}{\lambda_1} \sqrt{\frac{\det(\nabla^2 F(z))}{\det(\nabla^2 F(x_b))}} e^{(F(z) - F(x_b))/\varepsilon}.$$

Here λ_1 is the first eigenvalue of the Hessian of F at the saddle z .

2. Preliminaries

The generator of the process (1.1) is the following elliptic operator

$$L_\varepsilon = -\varepsilon \Delta + \nabla F \cdot \nabla. \tag{2.1}$$

In this section we study the potential and regularity theory associated with the operator (2.1). We provide the identities and pointwise estimates that we will need in the course of the proofs. Most of these are standard, but we provide them adapted to our situation for the reader’s convenience. We note that in this section we only require that the potential F is of class C^2 and satisfy the quadratic growth condition (1.3).

2.1. Potential theory.

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be a regular domain and let $G_\Omega(x, y)$ be the Green’s function for Ω , i.e., for every $f \in C(\Omega)$ the function

$$u_f = \int_\Omega G_\Omega(x, y) f(y) dy$$

is the solution of the Poisson equation

$$\begin{cases} L_\varepsilon u_f = f & \text{in } \Omega \\ u_f = 0 & \text{on } \partial\Omega. \end{cases}$$

The natural associated measures are the Gibbs measure $d\mu_\varepsilon = e^{-F/\varepsilon} dx$ and the Gibbs surface measure $d\sigma_\varepsilon = e^{-F/\varepsilon} d\mathcal{H}^{n-1}$.

Remark 2.2. Note that the Green’s function is symmetric w.r.t. the Gibbs measure, i.e.

$$G(x, y)e^{-F(x)/\varepsilon} = G(y, x)e^{-F(y)/\varepsilon}.$$

We also have the fundamental Green’s identities. Here we assume that Ω is a Lipschitz domain and denote the inner normal by n .

Lemma 2.3. *Let Ω be a smooth domain, ψ, ϕ be in $C^2(\overline{\Omega})$, and G_Ω be the Green’s function for Ω . Then the following Green’s identities holds (**Green’s first identity**)*

$$\int_\Omega \psi L_\varepsilon \phi - \varepsilon \nabla \psi \cdot \nabla \phi d\mu_\varepsilon = \varepsilon \int_{\partial\Omega} \psi \nabla \phi \cdot n d\sigma_\varepsilon, \tag{2.2}$$

and (**Green’s second identity**)

$$\int_\Omega \psi L_\varepsilon \phi - \phi L_\varepsilon \psi d\mu_\varepsilon = \varepsilon \int_{\partial\Omega} \psi \nabla \phi \cdot n - \phi \nabla \psi \cdot n d\sigma_\varepsilon. \tag{2.3}$$

Furthermore, the following (balayage) representation formula holds: for every $g \in C(\partial\Omega)$ the function

$$u(x) = \varepsilon e^{F(x)/\varepsilon} \int_{\partial\Omega} g \nabla_y G_\Omega(y, x) \cdot n d\sigma_\varepsilon(y) \tag{2.4}$$

is the solution of the Dirichlet problem

$$\begin{cases} L_\varepsilon u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Proof. Integration by parts gives

$$\begin{aligned} \int_\Omega \psi L_\varepsilon \phi d\mu_\varepsilon &= \int_\Omega \psi (-\varepsilon \Delta \phi + \nabla F \cdot \nabla \phi) d\mu_\varepsilon \\ &= \int_\Omega (\varepsilon \nabla \psi \cdot \nabla \phi - \frac{\varepsilon}{\varepsilon} \psi \nabla F \cdot \nabla \phi + \psi \nabla F \cdot \nabla \phi) d\mu_\varepsilon \\ &\quad + \varepsilon \int_{\partial\Omega} \psi \nabla \phi \cdot n d\sigma_\varepsilon \\ &= \int_\Omega \varepsilon \nabla \psi \cdot \nabla \phi d\mu_\varepsilon + \varepsilon \int_{\partial\Omega} \psi \nabla \phi \cdot n d\sigma_\varepsilon. \end{aligned}$$

The second Green’s identity follows from the first by applying it twice

$$\begin{aligned} \int_\Omega \psi L_\varepsilon \phi - \varepsilon \nabla \psi \cdot \nabla \phi d\mu_\varepsilon - \int_\Omega \phi L_\varepsilon \psi + \varepsilon \nabla \psi \cdot \nabla \phi d\mu_\varepsilon \\ = \varepsilon \int_{\partial\Omega} \psi \nabla \phi \cdot n d\sigma_\varepsilon - \varepsilon \int_{\partial\Omega} \phi \nabla \psi \cdot n d\sigma_\varepsilon. \end{aligned}$$

We may now obtain the representation formula for the Dirichlet problem. We choose $\phi(x) = G_\Omega(x, y)$ and obtain by Green’s second identity that

$$\psi(x)e^{-F(x)/\varepsilon} - \varepsilon \int_{\partial\Omega} \psi \nabla \phi \cdot n d\sigma_\varepsilon = \int_\Omega \phi L_\varepsilon \psi d\mu_\varepsilon.$$

Now relabeling $x \rightarrow y$, we get the representation formula (2.4). □

Recall the definition of capacity (variational): for $A, B \subset \Omega$ two disjoint compact sets

$$\text{cap}(A, B; \Omega) := \inf \left(\varepsilon \int_{\Omega} |\nabla h|^2 e^{-\frac{F}{\varepsilon}} dx : h \geq 1 \text{ in } A, u \in H_0^1(\Omega \setminus B) \right). \tag{2.5}$$

The extension of capacity to open sets follows in the classical way

$$\text{cap}(U, B; \Omega) := \sup\{\text{cap}(A, B; \Omega) : A \text{ compact and } A \subset U\}.$$

It is well known that for bounded sets with regular boundary the continuity of the capacity implies that $\text{cap}(U, B; \Omega) = \text{cap}(\overline{U}, B; \Omega)$. The extension w.r.t the second entry follows similarly. The variational definition of the capacity has many equivalent forms, one that we will need is the one below:

Lemma 2.4. *Let $A, B \subset \Omega$ be two disjoint compact sets. Then the variational formulation of capacity coincides with the balayage definition, i.e.,*

$$\text{cap}(A, B; \Omega) = \sup \left\{ \int_A e^{-F(y)/\varepsilon} d\mu(y) : \text{supp } \mu \subset A, \int_{\Omega} G_{\Omega \setminus B}(x, y) d\mu(y) \leq 1 \right\}.$$

The unique measure which maximizes the above, i.e., satisfying

$$\int_A e^{-F(y)/\varepsilon} d\mu_{A,B}(y) = \text{cap}(A, B; \Omega), \quad \int_{\Omega} G_{\Omega \setminus B}(x, y) d\mu(y) \leq 1,$$

is called the equilibrium measure $\mu_{A,B}$. The corresponding equilibrium potential is defined as $h_{A,B} = \int_{\Omega} G_{\Omega \setminus B}(x, y) d\mu_{A,B}(y)$ and is the minimizer of (2.5).

If in addition A, B are smooth, then we have

$$\begin{aligned} \text{cap}(A, B; \Omega) &= \int_A e^{-F(y)/\varepsilon} d\mu_{A,B}(y) = \varepsilon \int |\nabla h_{A,B}|^2 d\mu_{\varepsilon} \\ &= -\varepsilon \int_{\partial A} \nabla h_{A,B} \cdot nd\sigma_{\varepsilon}. \end{aligned} \tag{2.6}$$

Proof. The claim follows from the symmetry of the Green’s function, Remark 2.2, and the strong maximum principle that $h_{A,B} = 1$ in A , see [1, 10]. From (2.2) we get that

$$\int_{\Omega \setminus B} h_{A,B} L_{\varepsilon} h_{A,B} - \varepsilon |\nabla h_{A,B}|^2 d\mu_{\varepsilon} = \varepsilon \int_{\partial(\Omega \setminus B)} h_{A,B} \nabla h_{A,B} \cdot nd\sigma_{\varepsilon}.$$

Using $h_{A,B} = 0$ on $\partial(\Omega \setminus B)$ we see that the right hand side of the above is zero. Moreover, from $L_{\varepsilon} h_{A,B} = \mu_{A,B}$ and from the definition of $d\mu_{\varepsilon}$ we get

$$\int_A e^{-F(y)/\varepsilon} d\mu_{A,B}(y) = \varepsilon \int_{\Omega} |\nabla h_{A,B}|^2 d\mu_{\varepsilon}. \tag{2.7}$$

Note that since $h_{A,B} = 1$ in A and 0 on $\partial(\Omega \setminus B)$, and since $L_{\varepsilon} h_{A,B} = 0$ in $\Omega \setminus (A \cup B)$, we have by the uniqueness of the solution to the Dirichlet problem that $h_{A,B}$ coincides with the variational minimizer of (2.5). This establishes the first two equalities of (2.6).

To prove the last equality in (2.6) we insert $h_{A,B} = \phi = \psi$ into (2.2) (Green’s first identity) and get

$$\int_{\Omega \setminus (A \cup B)} h_{A,B} L_{\varepsilon} h_{A,B} - \varepsilon |\nabla h_{A,B}|^2 d\mu_{\varepsilon} = \varepsilon \int_{\partial(\Omega \setminus (A \cup B))} h_{A,B} \nabla h_{A,B} \cdot nd\sigma_{\varepsilon}.$$

Since $\text{supp } \mu_{A,B} \subset A$, and $h_{A,B} = 0$ on B and 1 on A , we get

$$-\varepsilon \int_{\Omega} |\nabla h_{A,B}|^2 d\mu_{\varepsilon} = \varepsilon \int_{\partial A} \nabla h_{A,B} \cdot n d\sigma_{\varepsilon}, \tag{2.8}$$

where n is the outward unit normal of A . The result now follows from (2.7) and (2.8). \square

Definition 2.5. Let Ω be a smooth domain and $A \subset \Omega$. Then we define the potential of the equilibrium potential as

$$w_{A,\Omega}(x) = \int_{\Omega} G_{\Omega \setminus A}(x, y) h_{A,\Omega^c}(y) dy.$$

The definition of the potential of the equilibrium potential might seem technical at first. However, $w_{A,\Omega}$ has a clear probabilistic interpretation as the expected hitting time of hitting A of a process killed at $\partial\Omega$. Indeed, the probabilistic interpretation of h_{A,Ω^c} is $\mathbb{P}(\tau_A < \tau_{\Omega^c})$ i.e. the probability of hitting A before Ω^c . By Dynkin’s formula we see that then

$$\begin{aligned} w_{A,\Omega}(x) &= \mathbb{E}^x [w_{A,\Omega}(X_{\tau_{A \cup \Omega^c}})] - \mathbb{E}^x \left[\int_0^{\tau_{A \cup \Omega^c}} L_{\varepsilon} w_{A,\Omega}(X_t) dt \right] \\ &= \int_0^{\infty} \mathbb{E}^x \left[\mathbb{I}_{t \leq \tau_{A \cup \Omega^c}} \mathbb{E}^{X_t} [\mathbb{I}_A(X_{\tau_{A \cup \Omega^c}})] \right] dt \\ &= \int_0^{\infty} \mathbb{E}^x \left[\mathbb{I}_{t \leq \tau_{A \cup \Omega^c}} \mathbb{I}_A(X_{\tau_{A \cup \Omega^c}}) \right] dt \\ &= \mathbb{E}^x [\tau_A \mathbb{I}_{\tau_A < \tau_{\Omega^c}}]. \end{aligned}$$

We also have the following integration by parts formula for the potential of the equilibrium potential:

Lemma 2.6. *Let Ω be a smooth domain, let $A \subset \Omega$ be a smooth set, and assume that $B_{2\rho}(x) \subset \Omega \setminus A$. Then*

$$\int_{\partial B_{\rho}(x)} w_{A,\Omega}(y) e^{-F(y)/\varepsilon} d\mu_{B_{\rho}(x),A}(y) = \int_{\Omega \setminus A} h_{A,\Omega^c}(z) h_{B_{\rho}(x),A}(z) d\mu_{\varepsilon}.$$

The above statement looks more familiar if we write it in the formal way as

$$\int_{\Omega \setminus A} w_{A,\Omega}(y) e^{-F(y)/\varepsilon} dL_{\varepsilon} h_{B_{\rho},A}(y) = \int_{\Omega \setminus A} L_{\varepsilon} w_{A,\Omega} e^{-F/\varepsilon} h_{B_{\rho},A} dz,$$

where $dL_{\varepsilon} h_{B_{\rho},A}(y)$ is the equilibrium measure $d\mu_{B_{\rho}(x),A}(y)$.

Proof. Using the definition of $w_{A,\Omega}$ and Fubini’s theorem

$$\begin{aligned} &\int_{\partial B_{\rho}(x)} w_{A,\Omega}(y) e^{-F(y)/\varepsilon} d\mu_{B_{\rho}(x),A}(y) \\ &= \int_{\partial B_{\rho}(x)} \left(\int_{\Omega \setminus A} G_{\Omega \setminus A}(y, z) h_{A,\Omega^c}(z) dz \right) e^{-F(y)/\varepsilon} d\mu_{B_{\rho}(x),A}(y) \\ &= \int_{\Omega \setminus A} h_{A,\Omega^c}(z) \int_{\partial B_{\rho}(x)} G_{\Omega \setminus A}(y, z) e^{-F(y)/\varepsilon} d\mu_{B_{\rho}(x),A}(y) dz. \end{aligned}$$

Using the symmetry of the Green’s function, Remark 2.2,

$$\begin{aligned} & \int_{\Omega \setminus A} h_{A, \Omega^c}(z) \int_{\partial B_\rho(x)} G_{\Omega \setminus A}(y, z) e^{-F(y)/\varepsilon} d\mu_{B_\rho(x), A}(y) dz \\ &= \int_{\Omega \setminus A} h_{A, \Omega^c}(z) e^{-F(z)/\varepsilon} \int_{\partial B_\rho(x)} G_{\Omega \setminus A}(z, y) d\mu_{B_\rho(x), A}(y) dz. \end{aligned}$$

Note that $\text{supp } \mu_{B_\rho(x), A} \subset \partial B_\rho(x)$ and as such

$$u(z) = \int_{\Omega} G_{\Omega \setminus A}(z, y) d\mu_{B_\rho(x), A}(y) = \int_{\partial B_\rho(x)} G_{\Omega \setminus A}(z, y) d\mu_{B_\rho(x), A}(y)$$

solves the equation $L_\varepsilon u = \mu_{B_\rho(x), A}$ in Ω and $u = 0$ on $\partial\Omega \setminus A$. Consequently, by the uniqueness result to the Dirichlet-Poisson problem, we get $u(z) = h_{B_\rho(x), A}(z)$. Hence

$$\begin{aligned} & \int_{\Omega \setminus A} h_{A, \Omega^c}(z) e^{-F(z)/\varepsilon} \int_{\partial B_\rho(x)} G_{\Omega \setminus A}(z, y) d\mu_{B_\rho(x), A}(y) dz \\ &= \int_{\Omega \setminus A} h_{A, \Omega^c}(z) h_{B_\rho(x), A}(z) d\mu_\varepsilon(z). \end{aligned}$$

Combining the equalities above yields the result. □

2.2. *Classical pointwise estimates.* In this section we recall classical pointwise estimates for functions which satisfy

$$L_\varepsilon u = f$$

in a domain Ω , where the operator L_ε is defined in (2.1). First, since we assume that $F \in C^2(\mathbb{R}^n)$, then for all Hölder continuous f the solutions of the above equation are $C^{2, \alpha}$ -regular, see [14]. However, these regularity estimates depend on ε and blow up as $\varepsilon \rightarrow 0$. The point is that we may obtain regularity estimates for constants independent of ε if we restrict ourselves on small enough scales. To this aim, for a given domain Ω we choose a positive number ν such that

$$\frac{\|\nabla F\|_{L^\infty(\Omega)}}{\varepsilon} \leq \nu.$$

We have the following two theorems from [14].

Lemma 2.7 (Harnack’s inequality). *Let Ω be a domain and let $u \in C^2(\Omega)$ be a non-negative function satisfying $L_\varepsilon u = 0$. Then for any $B_{3R}(x) \subset \Omega$ it holds that*

$$\sup_{B_R(x)} u \leq C \inf_{B_R(x)} u$$

for a constant $C = C(n, \nu R)$. In particular, if $\|\nabla F\|_{L^\infty(\Omega)} \leq L$, then for $R \leq \frac{\varepsilon}{L}$ the constant C is independent of ε . Furthermore, for $p \in (1, \infty)$ and any number k we have

$$\sup_{B_R(x)} |u - k| \leq C_p \left(\int_{B_{2R}(x)} |u - k|^p dx \right)^{1/p}$$

and

$$\left(\int_{B_{2R}(x)} |u|^p dx \right)^{1/p} \leq C_p \inf_{B_R(x)} u,$$

where the symbol \int denotes the average integral, and the constant C_p in addition to above depends also on p .

In the non-homogeneous case $L_\varepsilon u = f$ we have the following generalization of Harnack’s inequality.

Lemma 2.8 *Let Ω be a domain and let $u \in C^2(\Omega)$ be a non-negative function satisfying $L_\varepsilon u = f$. Then for any $B_{3R}(x) \subset \Omega$ it holds that*

$$\sup_{B_R(x)} u \leq C \left(\inf_{B_R(x)} u + \frac{R}{\varepsilon} \|f\|_{L^n(B_{2R}(x))} \right)$$

for a constant $C = C(n, \nu R)$. In particular, if $\|\nabla F\|_{L^\infty(\Omega)} \leq L$, then for $R \leq \frac{\varepsilon}{L}$ the constant C is independent of ε and we have

$$\sup_{B_R(x)} u \leq C \left(\inf_{B_R(x)} u + \|f\|_{L^n(B_{2R}(x))} \right).$$

The Harnack inequality in Lemma 2.7 holds also in the case of the punctured ball.

Lemma 2.9. *Let $u \in C^2(B_{3R}(x) \setminus \{x\})$ be a non-negative function satisfying $L_\varepsilon u = 0$ in $B_{3R}(x) \setminus \{x\}$. Then*

$$\sup_{\partial B_R(x)} u \leq C \inf_{\partial B_R(x)} u$$

for a constant $C = C(n, \nu R)$.

Proof. By translating the coordinates we may assume that $x = 0$. Let $x_0, y_0 \in \partial B_R$ be such that $\sup_{\partial B_R(x)} u = u(x_0)$ and $\inf_{\partial B_R(x)} u = u(y_0)$. We choose points $x_1, \dots, x_{N-1}, x_N \in \partial B_R$ such that $|x_i - x_{i-1}| \leq R/4$ and $x_N = y_0$. Note that the number N is bounded. Now we may use Harnack’s inequality Lemma 2.7 in balls $B_{R/4}(x_i)$ to get

$$u(x_{i-1}) \leq Cu(x_i).$$

We obtain the claim by applying the above over $i = 1, \dots, N$. □

Lemma 2.10. *Let Ω be a domain and let $u, h \in C^2(\Omega)$ be non-negative functions such that $L_\varepsilon u = h$ and h satisfies Harnack’s inequality with constant c_0 . Then the function $v = u + h$ satisfies Harnack’s inequality, i.e., for all $B_{3R}(x) \subset \Omega$ it holds that*

$$\sup_{B_R(x)} v \leq C \inf_{B_R(x)} v$$

for a constant $C = C(n, \nu R, R^2/\varepsilon, c_0)$. In particular, if $\|\nabla F\|_{L^\infty(\Omega)} \leq L$, then for $R \leq \min\{\varepsilon/L, \sqrt{\varepsilon}\}$ the constant C is independent of ε .

Proof. Again we may assume that $x = 0$. Using Lemma 2.8 and Harnack’s inequality for h yields

$$\begin{aligned} \sup_{B_R} u &\leq C \inf_{B_R} u + \frac{CR}{\varepsilon} \|h\|_{L^n(B_{2R})} \\ &\leq C \inf_{B_R} u + \frac{CR}{\varepsilon} |B_R|^{1/n} \inf_{B_R} h \\ &\leq C \inf_{B_R} u + \frac{CR^2}{\varepsilon} \inf_{B_R} h. \end{aligned}$$

Now, using Harnack’s inequality for h again, we obtain

$$\sup_{B_R} v \leq \sup_{B_R} u + \sup_{B_R} h \leq C \inf_{B_R} u + C \inf_{B_R} h \leq C \inf_{B_R} v$$

for a constant C as in the statement. This proves the claim. □

The Harnack’s inequality in Lemma 2.7 implies Hölder continuity for solutions of $L_\varepsilon u = 0$.

Lemma 2.11. *Let $u \in C^2(B_{3R}(x))$ be a function such that for any constant c , for which $v = u + c$ is non-negative, the function v satisfies Harnack’s inequality with constant C_0 , independent of c . Then there exists $C = C(C_0) > 1$ and $\alpha = \alpha(C_0) \in (0, 1)$ such that, for all $\rho \leq R$, it holds that*

$$\text{osc}_{B_\rho(x)} u \leq C \left(\frac{\rho}{R}\right)^\alpha \text{osc}_{B_R(x)} u.$$

In particular, if $u, h \in C^2(\Omega)$ are non-negative functions such that $L_\varepsilon u = h$ and h satisfies Harnack’s inequality with constant C_0 , then $u + h$ satisfies the estimate above.

Proof. The proof follows verbatim from the classical proof of Moser, see [14, Theorem 8.22]. □

3. Technical Lemmas

In this section we provide some preliminary results for the proofs of the main theorems. We recall that we assume that the potential F satisfies the structural assumptions from Sect. 1.1, and that from this moment on our constants are allowed to depend on the data, see paragraph after (1.16).

3.1. Rough estimates for potentials. In this subsection we provide estimates for the capacitary potential $h_{A,B}$, when A and B are two disjoint closed sets. The first estimate is the so called renewal estimate of [8]. In order to trace dependencies of constants, we provide a proof.

Lemma 3.1. *Let Ω be a smooth domain, let $A, B \subset \Omega$ be disjoint smooth sets, and consider $h_{A,B}$ as the capacitary potential in Ω . Assume that $B_{4\rho}(x) \subset (\Omega \setminus (A \cup B))$, and that $r \leq \min \left\{ \frac{\varepsilon}{\|\nabla F\|_{L^\infty(B_{2\rho}(x))}}, \rho \right\}$. Then there exists a constant $C = C(n, v) > 1$ such that*

$$h_{A,B}(x) \leq C \frac{\text{cap}(B_r(x), A; \Omega)}{\text{cap}(B_r(x), B; \Omega)}.$$

Proof. Again, without loss of generality, we may assume that $x = 0$. Since $h_{A, B \cup B_r} = h_{A, B}$ on $\partial(\Omega \setminus (A \cup B))$, we can use (2.4) to represent $h_{A, B}$ as follows

$$h_{A, B}(z) = \varepsilon e^{F(z)/\varepsilon} \int_{\partial(\Omega \setminus (A \cup B))} h_{A, B \cup B_r} \nabla G_{\Omega \setminus (A \cup B)}(y, z) \cdot nd\sigma_\varepsilon(y). \quad (3.1)$$

Now by Green’s second identity (2.3) in $\Omega \setminus (A \cup B \cup B_r)$ and (3.1) we see that, for $z \in \Omega$,

$$\begin{aligned} h_{A, B}(z) &= h_{A, B \cup B_r}(z) \\ &\quad - \varepsilon e^{F(z)/\varepsilon} \int_{\partial B_r} G_{\Omega \setminus (A \cup B)}(y, z) \nabla h_{A, B \cup B_r}(y) \cdot nd\sigma_\varepsilon(y), \end{aligned} \quad (3.2)$$

where n is the inward unit normal of B_r . First note that by (2.3) we can identify the equilibrium measure as

$$\mu_{B \cup B_r, A} = -\varepsilon \nabla h_{B \cup B_r, A} \cdot nd\sigma_\varepsilon = \varepsilon \nabla h_{A, B \cup B_r} \cdot nd\sigma_\varepsilon.$$

Using that $h_{A, B \cup B_r} = 1 - h_{B \cup B_r, A}$, together with the above and (3.2), we get for $z \in \overline{B_r}$ (since $h_{A, B \cup B_r}(z) = 0$) that

$$h_{A, B}(z) = \int_{\partial B_r} G_{\Omega \setminus (A \cup B)}(y, z) e^{(F(z) - F(y))/\varepsilon} d\mu_{B \cup B_r, A}(y). \quad (3.3)$$

First note that $\mu_{B_r \cup B, A}|_{\partial B_r}$ is an admissible measure for $\text{cap}(B_r, A; \Omega)$, which follows from the fact that by the comparison principle, the potentials for ordered measures are ordered and the support of $\mu_{B_r \cup B, A}|_{\partial B_r}$ is in $\overline{B_r}$. To bound $h_{A, B}$ from above, note that by the balayage representation of capacity (see Lemma 2.4) and the above, we obtain

$$\int_{\partial B_r} e^{-F(y)/\varepsilon} d\mu_{B \cup B_r, A}(y) \leq \text{cap}(B_r, A; \Omega).$$

Applying the above to (3.3) gives, for $z \in B_r$,

$$h_{A, B}(z) \leq \sup_{y \in \partial B_r} G_{\Omega \setminus (A \cup B)}(y, z) e^{F(z)/\varepsilon} \text{cap}(B_r, A; \Omega). \quad (3.4)$$

It remains to bound the Green’s function. For $z \in \overline{B_r}$ we have by (2.4) and (2.6) and Remark 2.2 that

$$\begin{aligned} 1 &= h_{B_r, A \cup B}(z) = \int_{\partial B_r} G_{\Omega \setminus (A \cup B)}(z, y) d\mu_{B_r, A \cup B}(y) \\ &= \int_{\partial B_r} G_{\Omega \setminus (A \cup B)}(y, z) e^{(F(z) - F(y))/\varepsilon} d\mu_{B_r, A \cup B}(y) \\ &\geq \inf_{\partial B_r} G_{\Omega \setminus (A \cup B)}(y, z) e^{F(z)/\varepsilon} \text{cap}(B_r, A \cup B; \Omega) \\ &\geq \inf_{\partial B_r} G_{\Omega \setminus (A \cup B)}(y, z) e^{F(z)/\varepsilon} \text{cap}(B_r, B; \Omega). \end{aligned} \quad (3.5)$$

Now putting together (3.4) and (3.5) and Lemma 2.9 we are done. □

The result below is a version of the rough capacity bound of [8], but we give a simplified proof. We will later use a similar argument in the proof of Theorem 1.

Lemma 3.2. *Let $D \subset B_R$ be a smooth closed set. Let $x \in B_R \setminus D$ be such that $B_{4\rho}(x) \subset B_R \setminus D$, for $\rho \leq \varepsilon$. Then there exists constants $q_1, q_2 \in \mathbb{R}$ and $C > 1$ such that*

$$\frac{1}{C} \varepsilon^{q_1} \rho^{n-1} e^{-F(x;D)/\varepsilon} \leq \text{cap}(B_\rho(x), D) \leq C \varepsilon \rho^{q_2} e^{-F(x;D)/\varepsilon}.$$

Proof. We assume without loss of generality that $F(B_\rho; D) = 0$, since the quantities can always be scaled back. Consider $\gamma \in \mathcal{C}(B_\rho(x), D; B_R)$ (i.e. a curve connecting $B_\rho(x)$ and D inside B_R) such that $\sup_t F(\gamma(t)) \leq C\varepsilon$ and let $u(z) = h_{D, B_\rho(x)}(z)$. We first note by Lemma 2.4 that

$$\text{cap}(B_\rho(x), D) = \varepsilon \int |\nabla u|^2 e^{-F(y)/\varepsilon} dy.$$

Fix an $n - 1$ dimensional disk D_ρ of radius ρ . Then by Cauchy-Schwarz

$$\int_{B_R} |\nabla u|^2 e^{-F(y)/\varepsilon} dy \geq \int_0^1 \int_{D_\rho} \left| \left\langle \frac{\dot{\gamma}}{|\dot{\gamma}|}, \nabla u(\gamma(t) + z) \right\rangle \right|^2 |\dot{\gamma}| d\sigma_\varepsilon(z) dt.$$

By the fundamental theorem of calculus and Cauchy-Schwarz, we have for a fixed point $z \in D_\rho$ that

$$\begin{aligned} 1 &= u(\gamma(1)) - u(\gamma(0)) = \int_0^1 \frac{d}{dt} u(\gamma(t) + z) dt \\ &= \int_0^1 \frac{d}{dt} u(\gamma(t) + z) \frac{\sqrt{|\dot{\gamma}|}}{\sqrt{|\dot{\gamma}|}} e^{-F(\gamma(t)+z)/(2\varepsilon)} e^{F(\gamma(t)+z)/(2\varepsilon)} dt \\ &\leq \left(\int_0^1 \left| \frac{d}{dt} u(\gamma(t) + z) \right|^2 \frac{1}{|\dot{\gamma}|} e^{-F(\gamma(t)+z)/\varepsilon} dt \right)^{1/2} \left(\int |\dot{\gamma}| e^{F(\gamma(t)+z)/\varepsilon} dt \right)^{1/2}. \end{aligned}$$

From the above we get

$$\begin{aligned} \int_{B_R} |\nabla u|^2 e^{-F(y)/\varepsilon} dy &\geq \int_0^1 \int_{D_\rho} \left| \left\langle \frac{\dot{\gamma}}{|\dot{\gamma}|}, \nabla u(\gamma(t) + z) \right\rangle \right|^2 |\dot{\gamma}| d\sigma_\varepsilon(z) dt \\ &\geq \int_{D_\rho} \left(\int_0^1 |\dot{\gamma}| e^{F(\gamma(t)+z)/\varepsilon} dt \right)^{-1} d\sigma_\varepsilon(z). \end{aligned}$$

Now since F is Lipschitz in B_R and $F(B_\rho; D) = 0$, we know that there exists a constant $C(\gamma)$ such that, for $z \in D_\rho$ and $\rho < 2\varepsilon$,

$$\int_0^1 |\dot{\gamma}| e^{F(\gamma(t)+z)/\varepsilon} dt \leq C(\gamma). \tag{3.6}$$

In the above the constant C depends on the length of γ , which can be assumed to be bounded. To see this, take an ε neighborhood of γ , E_ε and consider disjoint balls $B_\varepsilon(y_i) \subset E_\varepsilon$ such that $E_\varepsilon \subset \bigcup_i B_{5\varepsilon}(y_i)$ given e.g. by the Vitali covering lemma. The number of such balls is at most $C_1 R^n / \varepsilon^n$, for a dimensional constant C_1 . If we construct a piecewise linear curve γ_ε connecting the center of each ball in the covering, this curve will be inside $E_{10\varepsilon}$ and its length will be bounded by $C_2 R^n / \varepsilon^{n-1}$, for a dimensional constant C_2 . This newly constructed curve can be mollified to achieve a smooth curve without increasing the length by more than a factor. From the above and the Lipschitz

continuity of F it is clear that $\sup_t F(\gamma_\varepsilon(t)) \leq C\varepsilon$, and as such we can replace γ with γ_ε in the above and get from (3.6) that there is a constant $C > 1$ depending only on the data such that

$$\int_0^1 |\dot{\gamma}| e^{F(\gamma(t)+z)/\varepsilon} dt \leq \varepsilon^{1-n} C.$$

This implies that for a new constant C we have

$$\int_{B_R} |\nabla u|^2 e^{-F(y)/\varepsilon} dy \geq C\varepsilon^{n-1} \rho^{n-1},$$

which completes the proof of the lower bound after rescaling our potential F .

To prove the upper bound we have two possible cases: In the case when $F(x; D) = F(x)$ we can take a cutoff function $\chi_{B_\rho(x)} \leq \phi \leq \chi_{B_{2\rho}(x)}$ where $|\nabla \phi| \leq C/\rho$ as a competitor in the variational formulation of capacity (2.5). Then

$$\int_{B_R} |\nabla \phi|^2 e^{-F(y)/\varepsilon} dx = \int_{B_{2\rho}(x)} |\nabla \phi|^2 e^{-F(y)/\varepsilon} dx \leq C\rho^{n-2}.$$

In the case where $F(x; D) > F(x)$, consider the set $\hat{D} = \{z \in B_R : F(z) \leq F(x; D)\}$ and let \hat{D}_1 be the component that intersects D . We set $\tilde{D} = (\hat{D}_1 \cup D) \setminus B_{4\rho}(x)$. By the Lipschitz continuity, we know that $\inf_{\tilde{D}} F > -C\rho$. We take $\chi_{\tilde{D}+B_\rho} \leq 1 - \phi \leq \chi_{\tilde{D}+B_{2\rho}}$, where $|\nabla \phi| \leq C/\rho$, and get

$$\begin{aligned} \int_{B_R} |\nabla \phi|^2 e^{-F(y)/\varepsilon} dx &= \int_{(\tilde{D}+B_{2\rho}) \setminus \tilde{D}} |\nabla \phi|^2 e^{-F(y)/\varepsilon} dx \\ &\leq C\rho^{-2} |(\tilde{D} + B_{2\rho}) \setminus \tilde{D}|. \end{aligned}$$

Again, the upper bound follows from rescaling the potential F as in the case of the lower bound. This completes the whole proof. \square

Lemma 3.3. *Let $A, B \subset B_R$ be smooth disjoint sets, and let $x \in B_R$ be such that $B_\varepsilon(x) \subset B_R \setminus (A \cup B)$, $\varepsilon \in (0, 1)$. Then, if $F(x; B) \leq F(x; A)$, there exists constants q and C such that*

$$h_{A,B}(x) \leq C\varepsilon^q e^{-(F(x;A)-F(x;B))/\varepsilon}.$$

Proof. Let $L := \|\nabla F\|_{L^\infty(B_R)}$. By combining Lemmas 3.1 and 3.2 with $R = \varepsilon$, $r = \min\{\varepsilon/L, \varepsilon\}$ yields the result. \square

Remark 3.4. By relabeling A, B to B, A and using the fact that $h_{A,B} = 1 - h_{B,A}$, we get that if the reverse inequality holds, i.e. $F(x; B) > F(x; A)$, then

$$1 - h_{A,B}(x) \leq C\varepsilon^q e^{-(F(x;B)-F(x;A))/\varepsilon}.$$

Lemma 3.5. *Let Ω be a smooth domain and let $x_a, x_b \in \Omega \subset B_R$ be two local minimum points of F . Fix $0 < \delta < \delta_1$ and assume that $U_{-\delta/3} = \{x : F(x) < F(x_a; x_b) - \delta/3\} \subset \Omega$. Then there exists an $\varepsilon_0 \in (0, 1)$ and a constant $C = C > 1$ such that, for any $0 \leq \varepsilon \leq \varepsilon_0$ for which $B_{3\varepsilon}(x_a), B_{3\varepsilon}(x_b) \subset U_{-\delta/3}$, the following holds: If U_i is a component of $U_{-\delta/3}$, then*

$$\text{osc}_{U_i} h_{B_\varepsilon(x_a), B_\varepsilon(x_b)} \leq C\varepsilon.$$

Proof. Consider any component U_i of $U_{-\delta/3}$. We note that we can take ε small enough depending on the Lipschitz constant of F in B_R and δ such that there exists a Lipschitz domain D_i satisfying

$$U_i + B_\varepsilon \subset D_i \subset U_{-\delta/4}.$$

For simplicity, denote $u := h_{B_\varepsilon(x_a), B_\varepsilon(x_b)}$. Since D_i is Lipschitz we may use the Poincaré inequality to get

$$\int_{D_i} |u - u_{D_i}|^2 dx \leq C \int_{D_i} |\nabla u|^2 dx.$$

Using that $D_i \subset U_{-\delta/4}$ together with Lemma 2.4

$$\begin{aligned} \int_{D_i} |\nabla u|^2 dx &\leq e^{\sup_{D_i} F/\varepsilon} \int_{D_i} |\nabla u|^2 e^{-F(x)/\varepsilon} dx \\ &\leq \varepsilon^{-1} e^{\sup_{D_i} F/\varepsilon} \text{cap}(B_\varepsilon(x_a), B_\varepsilon(x_b); \Omega). \end{aligned}$$

Using the definition of $U_{-\delta/4}$ and Lemma 3.2, we get

$$\int_{D_i} |u - u_{D_i}|^2 dx \leq C\varepsilon^{q_1} e^{-\delta/4\varepsilon} \tag{3.7}$$

for some constant $q_1 \in \mathbb{R}$. Now, for any $x_0 \in U_i$ we have by Lemma 2.7 that

$$\sup_{B_\varepsilon(x_0)} |u - u_{D_i}|^2 \leq C \left(\int_{B_{2\varepsilon}} |u - u_{D_i}|^2 dx \right)$$

which together with (3.7) gives

$$\sup_{B_\varepsilon(x_0)} |u - u_{D_i}|^2 \leq C\varepsilon^{q_1-n} e^{-\delta/4\varepsilon}.$$

Since x_0 was an arbitrary point in U_i we conclude that there exists $\varepsilon_0 \in (0, 1)$ depending only on the data such that if $\varepsilon < \varepsilon_0$, the claim holds. \square

We conclude this subsection with an estimate relating the value of the potential of the equilibrium potential to the ratio of the L^1 norm of the equilibrium potential and the capacity.

Lemma 3.6. *Let Ω be a smooth domain and let $A \Subset \Omega$ be a smooth open set and consider $w_{A,\Omega}$ as the potential of the equilibrium potential in Ω (see Definition 2.5). Let $x \in \Omega$ be a critical point of F such that $B_{3\sqrt{\varepsilon}}(x) \subset \Omega$. Then there exists a constant $C > 1$ such that for $\rho < \sqrt{\varepsilon}$ we have*

$$\begin{aligned} w_{A,\Omega}(x) - C \left(\frac{\rho}{\sqrt{\varepsilon}} \right)^\alpha (w_{A,\Omega}(x) + 1) &\leq \frac{\int h_{A,\Omega} c h_{B_\rho(x), A} d\mu_\varepsilon}{\text{cap}(B_\rho(x), A; \Omega)} \\ &\leq w_{A,\Omega}(x) + C \left(\frac{\rho}{\sqrt{\varepsilon}} \right)^\alpha (w_{A,\Omega}(x) + 1). \end{aligned}$$

Proof. From Lemma 2.6 we get

$$\int_{\partial B_\rho(x)} w_{A,\Omega}(y) e^{-F(y)/\varepsilon} d\mu_{B_\rho(x),A}(y) = \int_{(A \cup B)^c} h_{A,\Omega^c}(z) h_{B_\rho(x),A}(z) d\mu_\varepsilon.$$

We can estimate the left hand side as

$$\begin{aligned} w_{A,\Omega}(x) - \text{osc}_{B_\rho} w_{A,\Omega} &\leq \inf_{B_\rho(x)} w_{A,\Omega} \leq \frac{\int_{\partial B_\rho(x)} e^{-F(y)/\varepsilon} w_{A,\Omega} d\mu_{B_\rho(x),A}(y)}{\int_{\partial B_\rho(x)} e^{-F(y)/\varepsilon} d\mu_{B_\rho(x),A}(y)} \\ &\leq \sup_{B_\rho(x)} w_{A,\Omega} \leq w_{A,\Omega}(x) + \text{osc}_{B_\rho} w_{A,\Omega}. \end{aligned}$$

We want to estimate the oscillation of $w_{A,\Omega}$ which we do by considering

$$\text{osc } w_{A,\Omega} = \text{osc}(w_{A,\Omega} + h_{A,\Omega^c} - h_{A,\Omega^c}) \leq \text{osc}(w_{A,\Omega} + h_{A,\Omega^c}) + \text{osc}(h_{A,\Omega^c}).$$

Now, the oscillation of $w_{A,\Omega} + h_{A,\Omega^c}$ and h_{A,Ω^c} can be estimated by Lemma 2.11 for $\rho \leq \frac{1}{C} \sqrt{\varepsilon}$. That is,

$$\begin{aligned} \text{osc}_{B_\rho}(w_{A,\Omega} + h_{A,\Omega^c}) + \text{osc}_{B_\rho}(h_{A,\Omega^c}) &\leq C \left(\frac{\rho}{\sqrt{\varepsilon}} \right)^\alpha \sup_{B_{\sqrt{\varepsilon}}} (w_{A,\Omega} + h_{A,\Omega^c}) \\ &\quad + C \left(\frac{\rho}{\sqrt{\varepsilon}} \right)^\alpha \sup_{B_{\sqrt{\varepsilon}}}(h_{A,\Omega^c}). \end{aligned}$$

We apply Lemma 2.7 to replace the supremums on the right hand side with the value at x as both $w_{A,\Omega} + h_{A,\Omega^c}$ and h_{A,Ω^c} satisfies the Harnack inequality (see Lemma 2.10). That is,

$$\begin{aligned} \text{osc}_{B_\rho}(w_{A,\Omega} + h_{A,\Omega^c}) + \text{osc}_{B_\rho}(h_{A,\Omega^c}) &\leq C \left(\frac{\rho}{\sqrt{\varepsilon}} \right)^\alpha (w_{A,\Omega}(x) + h_{A,\Omega^c}(x)) \\ &\leq C \left(\frac{\rho}{\sqrt{\varepsilon}} \right)^\alpha (w_{A,\Omega}(x) + 1). \end{aligned}$$

It is easily seen that the above can be extended to $\rho \leq \sqrt{\varepsilon}$ by applying Lemma 2.10 again and by enlarging the constant C . The proof is completed by using (2.6) and collecting the estimates above. \square

3.2. Laplace asymptotics for log-concave functions. The assumptions (1.4) and (1.6) ensure that near critical points the potential F is well approximated by convex functions. Therefore we will need basic estimates for log-concave functions, which rather surprisingly we did not find in the literature.

Lemma 3.7. *Assume $G : \mathbb{R}^n \rightarrow \mathbb{R}$ is a convex function which has a proper minimum at the origin and $G(0) = 0$. Then there exists a constant $C = C(n) > 1$ such that*

$$\frac{1}{C} |\{G < \varepsilon\}| \leq \int_{\mathbb{R}^n} e^{-\frac{G}{\varepsilon}} dx \leq C |\{G < \varepsilon\}|. \tag{3.8}$$

Moreover, there is a constant $C = C(n)$ such that for all $\Lambda > 1$, we have

$$\int_{\{G < \Lambda \varepsilon\}} e^{-\frac{G}{\varepsilon}} dx \geq (1 - \eta(C\Lambda^{-1})) \int_{\mathbb{R}^n} e^{-\frac{G}{\varepsilon}} dx, \tag{3.9}$$

with η as in (1.15).

Proof. By approximation we may assume that G is smooth. The lower bound in (3.8) follows immediately from

$$\int_{\mathbb{R}^n} e^{-\frac{G}{\varepsilon}} dx \geq \int_{\{G < \varepsilon\}} e^{-\frac{G}{\varepsilon}} dx \geq e^{-1} |\{G < \varepsilon\}|.$$

To prove the upper bound in (3.8) we first show that, for all $t > 0$, it holds

$$|\{G < 2t\}| \leq 2^n |\{G < t\}|. \tag{3.10}$$

In order to prove (3.10) it is enough to consider only the case $t = 1$ (the general case follows by considering $\tilde{G} = G/t$). Denote $E_1 = \{G < 1\}$ and $E_2 = \{G < 2\}$. Hence our goal is to show

$$E_2 \subset 2E_1 = \{2x : x \in E_1\}.$$

Fix $\hat{x} \in \partial E_1$ and define $g(t) = G(t\hat{x})$ for $t \geq 0$. By our assumptions, $g(t)$ is a smooth convex function satisfying $g(0) = 0$ and $g(1) = 1$. As such, both g, g' are increasing functions from which we can conclude that $g'(1) \geq 1$. Now, by the fundamental theorem of calculus,

$$g(2) - g(1) = \int_1^2 g'(t) dt \geq 1$$

which gives $g(2) \geq 2$. This means that for all $\hat{x} \in \partial E_1$ we have $G(2\hat{x}) \geq 2$. That is, we have $E_2 \subset 2E_1$. Thus

$$|E_2| \leq |2E_1| \leq 2^n |E_1|$$

and (3.10) follows. Iterating (3.10) gives

$$|\{G < 2^j \varepsilon\}| \leq 2^{jn} |\{G < \varepsilon\}|$$

and hence

$$|\{G < \varrho \varepsilon\}| \leq (2\varrho)^n |\{G < \varepsilon\}| \tag{3.11}$$

for all $\varrho \geq 1$. We conclude the proof of the upper bound in (3.8) by using (3.11) as

$$\begin{aligned} \int_{\mathbb{R}^n} e^{-\frac{G}{\varepsilon}} dx &\leq \sum_{j=0}^{\infty} \int_{\{j\varepsilon \leq G < (j+1)\varepsilon\}} e^{-\frac{G}{\varepsilon}} dx \\ &\leq \sum_{j=0}^{\infty} |\{j\varepsilon \leq G < (j+1)\varepsilon\}| e^{-j} \\ &\leq 2^n \sum_{j=0}^{\infty} e^{-j} (j+1)^n |\{G < \varepsilon\}| \leq C(n) |\{G < \varepsilon\}|. \end{aligned}$$

It remains to prove (3.9). Fix $\Lambda > 1$. Then, for every $x \in \{G \geq \Lambda\varepsilon\}$, it holds

$$e^{-\frac{G(x)}{\varepsilon}} = e^{-\frac{\Lambda G(x)}{\Lambda\varepsilon}} = \left(e^{-\frac{G(x)}{\Lambda\varepsilon}}\right)^\Lambda = \left(e^{-\frac{G(x)}{\Lambda\varepsilon}}\right)^{\Lambda-1} e^{-\frac{G(x)}{\Lambda\varepsilon}} \leq e^{-\Lambda+1} e^{-\frac{G(x)}{\Lambda\varepsilon}}. \tag{3.12}$$

Therefore we have, by (3.8), (3.11) and (3.12),

$$\begin{aligned} \int_{\{G \geq \Lambda\varepsilon\}} e^{-\frac{G}{\varepsilon}} dx &\leq e^{-\Lambda+1} \int_{\{G \geq \Lambda\varepsilon\}} e^{-\frac{G}{\Lambda\varepsilon}} dx \leq e^{-\Lambda+1} \int_{\mathbb{R}^n} e^{-\frac{G}{\Lambda\varepsilon}} dx \\ &\leq C e^{-\Lambda} |\{G < \Lambda\varepsilon\}| \\ &\leq C e^{-\Lambda} \Lambda^n |\{G < \varepsilon\}| \\ &\leq C e^{-\Lambda} \Lambda^n \int_{\mathbb{R}^n} e^{-\frac{G}{\varepsilon}} dx \end{aligned} \tag{3.13}$$

and the inequality (3.9) follows by using (1.15). □

Lemma 3.8. *Assume $G : \mathbb{R}^n \rightarrow \mathbb{R}$ is a function which has a proper global minimum at the origin and $G(0) = 0$. Furthermore, assume there is a constant C_0 such that, for all $a > 0$ and $\varepsilon > 0$, it holds that*

$$\int_{G>a} e^{-G/\varepsilon} dx < C_0 e^{-a/\varepsilon}. \tag{3.14}$$

If there is a level $\varepsilon_0 > 0$ such that G is convex on the component of $\{G(x) < \varepsilon_0\}$ that contains 0, then there is an $\varepsilon_1(n, |\{G < \varepsilon_0/2\}|) < \varepsilon_0$ and a constant $C = C(C_0, n) > 1$ such that, for all $\varepsilon < \varepsilon_1$, it holds that

$$C^{-1} |\{G < \varepsilon\}| \leq \int_{\mathbb{R}^n} e^{-\frac{G}{\varepsilon}} dx \leq C |\{G < \varepsilon\}|.$$

Proof. Since G is convex in the level set $\{G < \varepsilon_0\}$, we know that the level set $\{G \leq \varepsilon_0/2\}$ is convex and as such we can extend the function G outside that level set to a globally convex function. This allows us to apply Lemma 3.7 and obtain

$$\frac{1}{C} |\{G < \varepsilon\}| \leq \int_{\{G < \varepsilon_0/2\}} e^{-\frac{G}{\varepsilon}} dx \leq C |\{G < \varepsilon\}|. \tag{3.15}$$

Now, split the integral as

$$\int e^{-G/\varepsilon} dx = \int_{G \leq \varepsilon_0/2} e^{-G/\varepsilon} dx + \int_{G > \varepsilon_0/2} e^{-G/\varepsilon} dx.$$

From (3.15) it follows that it suffices to bound the second integral on the right hand side. Using (3.14) for $a = \varepsilon_0/2$ we get

$$\int_{G > \varepsilon_0/2} e^{-G/\varepsilon} dx \leq C_0 e^{-(\varepsilon_0/2)/\varepsilon}.$$

Since G is convex in the level set $\{G < \varepsilon_0/2\}$, which is again convex, it follows that we can construct a conical function \tilde{G} as follows: For any $\hat{x} \in \partial\{G < \varepsilon_0/2\}$ define $\tilde{G}(t\hat{x}/\|\hat{x}\|) = t\varepsilon_0/2$. The level sets of \tilde{G} satisfy, for $\varepsilon < \varepsilon_0/2$,

$$|\{\tilde{G} < \varepsilon\}| \leq |\{G < \varepsilon\}|.$$

However,

$$|\{\tilde{G} < \varepsilon\}| = \left(\frac{\varepsilon}{\varepsilon_0/2}\right)^n |\{\tilde{G} < \varepsilon_0/2\}| = \left(\frac{\varepsilon}{\varepsilon_0/2}\right)^n |\{G < \varepsilon_0/2\}|.$$

Now, we can choose $\varepsilon_1(n, C_0, |\{G < \varepsilon_0/2\}|) < \varepsilon_0/2$ such that for $\varepsilon < \varepsilon_1$ we have

$$e^{-(\varepsilon_0/2)/\varepsilon} \leq \left(\frac{\varepsilon}{\varepsilon_0}\right)^n |\{G < \varepsilon_0/2\}|.$$

This means that for $\varepsilon < \varepsilon_1$ we also have

$$\int_{G > \varepsilon_0/2} e^{-G/\varepsilon} dx \leq C_0 |\{G < \varepsilon\}|$$

which together with (3.15) completes the proof. □

We conclude this section with the following technical lemma which is useful when we study the potential near critical points.

Lemma 3.9. *Assume $G : \mathbb{R}^n \rightarrow \mathbb{R}$ is a convex function which has a proper minimum at the origin and $G(0) = 0$. Let $\omega : [0, \infty) \rightarrow [0, \infty)$ be as in (1.4) and (1.6). Then for all $\delta \leq \delta_0$, we have*

$$\int_{\{G < \delta\}} e^{-\frac{G(x)}{\varepsilon}} e^{\frac{\pm\omega(G(x))}{\varepsilon}} dx \simeq \int_{\mathbb{R}^n} e^{-\frac{G(x)}{\varepsilon}} dx.$$

Proof. Denote $\Lambda_\varepsilon = \frac{\varepsilon_1}{\varepsilon}$ with ε_1 as in (1.13). From (1.14) we know that $\Lambda_\varepsilon \rightarrow \infty$ as $\varepsilon \rightarrow 0$. Now, by (3.9) in Lemma 3.7 and (1.14), we get

$$\int_{\{G < \varepsilon_1\}} e^{-\frac{G(x)}{\varepsilon}} e^{\frac{\omega(G(x))}{\varepsilon}} dx \simeq \int_{\{G < \Lambda_\varepsilon \varepsilon\}} e^{-\frac{G(x)}{\varepsilon}} dx \simeq \int_{\mathbb{R}^n} e^{-\frac{G(x)}{\varepsilon}} dx. \tag{3.16}$$

The lower bound follows immediately from this. In order to prove the upper bound, note that $\omega(s) \leq s/2$ for all $s \leq \delta_0$ by assumption. Therefore we can repeat the argument in (3.13) to get

$$\int_{\{\varepsilon_1 < G < \delta\}} e^{-\frac{G(x)}{\varepsilon}} e^{\frac{\omega(G(x))}{\varepsilon}} dx \leq \int_{\{G > \Lambda_\varepsilon \varepsilon\}} e^{-\frac{G(x)}{2\varepsilon}} dx \leq \eta(C\Lambda_\varepsilon^{-1}) \int_{\mathbb{R}^n} e^{-\frac{G}{\varepsilon}} dx,$$

which together with Lemma 3.16 yields the upper bound. □

4. Proofs of Theorems 1 and 2

In this section we prove the capacity estimate in Theorem 1 and exit time estimate in Theorem 2. Before we begin, we would like to remind the reader that, as in Sect. 3, we will assume that F satisfies our structural assumptions and that all constants depend on the data, see the paragraph after (1.16).

We first study the geometric quantities $d_\varepsilon(A, B; \Omega)$ and $V_\varepsilon(A, B; \Omega)$ defined in (1.8) and (1.9) and give a more explicit, but less geometric, characterization. The characterization for the geodesic distance $d_\varepsilon(A, B; \Omega)$ turns out to be much easier than for the separating surface $V_\varepsilon(A, B; \Omega)$ and therefore we prove it first.

In the following two propositions we will first fix two local minimum points of F , say x_a and x_b . Their communication height $F(x_a, x_b)$ defines the island $U_{-\delta/3}$ which we recall are components of $\{F < F(x_a, x_b) - \delta/3\}$. We first study the geodesic distance between $B_\varepsilon(x_1)$ and $B_\varepsilon(x_2)$, where x_1 and x_2 are two local minima of F .

Proposition 4.1. *Let us fix local minimum points x_a and x_b of F . Let U_{x_1} and U_{x_2} be the islands, i.e., the components of the set $U_{-\delta/3} = \{F < F(x_a; x_b) - \delta/3\}$, containing $B_\varepsilon(x_1)$ and $B_\varepsilon(x_2)$ respectively, where x_1 and x_2 are two (possibly different) local minimum points. Assume that z is a saddle point in Z_{x_a, x_b} such that the bridge $O_{z, \delta}$ connects U_{x_1} and U_{x_2} , and denote $\Omega = U_{x_1} \cup U_{x_2} \cup O_{z, \delta}$. Then it holds for g_z given in (1.6) that*

$$d_\varepsilon(B_\varepsilon(x_1), B_\varepsilon(x_2); \Omega) \simeq e^{\frac{F(z)}{\varepsilon}} \int_{\mathbb{R}} e^{-\frac{g_z(y_1)}{\varepsilon}} dy_1.$$

Proof. Denote $g = g_z$. We begin by proving the lower bound, i.e.,

$$d_\varepsilon(B_\varepsilon(x_1), B_\varepsilon(x_2); \Omega) \geq (1 - \hat{\eta}(C, \varepsilon)) e^{\frac{F(z)}{\varepsilon}} \int_{\mathbb{R}} e^{-\frac{g(y_1)}{\varepsilon}} dy_1. \tag{4.1}$$

To this aim we choose a smooth curve $\gamma \in \mathcal{C}(B_\varepsilon(x_1), B_\varepsilon(x_2); \Omega)$ which, by assumptions, intersects the bridge $O_{z, \delta}$. We may choose the coordinates in \mathbb{R}^n such that $z = 0$ and

$$O_\delta = O_{z, \delta} = \{y_1 : g(y_1) < \delta\} \times \{y' \in \mathbb{R}^{n-1} : G(y') < \delta\}.$$

Let $\tau \in \mathbb{R}$ be such that $g(\tau) < \frac{\delta}{100}$ and denote $S_\tau = \{\tau\} \times \{G < \delta\} \subset O_\delta$. Let us next show that $S_\tau \in \mathcal{S}(B_\varepsilon(x_1), B_\varepsilon(x_2); \Omega)$. Since $0 \in Z_{x_a, x_b}$ then it follows that $|F(0) - F(x_a; x_b)| < \delta/3$. Now, consider a narrow cylinder of the form $\hat{O}_\delta = \{g(y_1) < \frac{\delta}{100}\} \times \{G(y') < \delta\} \subset O_\delta$, then, any surface of the form $S(\tau)$ for τ such that $g(\tau) < \delta/100$ lies inside \hat{O}_δ , and is parallel to the bottom/top of the cylinder. Thus, by the assumption (1.6) and $|F(0) - F(x_a; x_b)| < \delta/3$, we know that the cylindrical part of the boundary ($\{g < \delta/100\} \times \{G = \delta\}$) does not intersect $U_{x_1} \cup U_{x_2}$. Therefore any curve in $\mathcal{C}(B_\varepsilon(x_1), B_\varepsilon(x_2); \Omega)$ will pass through $S(\tau)$. In conclusion, $S_\tau \in \mathcal{S}(B_\varepsilon(x_1), B_\varepsilon(x_2); \Omega)$ for $g(\tau) < \frac{\delta}{100}$.

Let us fix $\gamma \in \mathcal{C}(B_\varepsilon(x_1), B_\varepsilon(x_2); \Omega)$ and denote the projection to the y_1 -axis by $\pi_1 : \mathbb{R}^n \rightarrow \mathbb{R}$, i.e. $\pi_1(y) = y_1$. Since $S_\tau \in \mathcal{S}(B_\varepsilon(x_1), B_\varepsilon(x_2); \Omega)$, γ intersects S_τ . Thus we conclude that $\tau \in \pi_1(\gamma([0, 1]) \cap O_\delta)$. This holds for every $\tau \in \{g < \delta/100\}$ and therefore

$$\{g < \delta/100\} \subset \pi_1(\gamma([0, 1]) \cap O_\delta). \tag{4.2}$$

Now the assumption (1.6) implies that, in the set O_δ , it holds that

$$\begin{aligned} F(y) - F(0) &\geq -g(y_1) - \omega(g(y_1)) + G(y') - \omega(G(y')) \\ &\geq -g(y_1) - \omega(g(y_1)) + \frac{1}{2}G(y') \geq -g(y_1) - \omega(g(y_1)). \end{aligned} \tag{4.3}$$

Then for $\gamma_1 = \pi_1(\gamma)$ we have by (4.2), (4.3) and Lemma 3.9 that

$$\begin{aligned} \int_{\{t: \gamma(t) \in O_\delta\}} |\gamma'| e^{\frac{F(\gamma)}{\varepsilon}} dt &\geq e^{\frac{F(0)}{\varepsilon}} \int_{\{t: \gamma(t) \in O_\delta\}} |\gamma'| e^{-\frac{g(\gamma_1)}{\varepsilon}} e^{-\frac{\omega(g(\gamma_1))}{\varepsilon}} dt \\ &\geq e^{\frac{F(0)}{\varepsilon}} \int_{\{g < \delta/100\}} e^{-\frac{g(y_1)}{\varepsilon}} e^{-\frac{\omega(g(y_1))}{\varepsilon}} dy_1 \\ &\geq (1 - \eta(C\varepsilon)) e^{\frac{F(0)}{\varepsilon}} \int_{\mathbb{R}} e^{-\frac{g(y_1)}{\varepsilon}} dy_1, \end{aligned}$$

proving (4.1).

To prove the upper bound, i.e.

$$d_\varepsilon(B_\varepsilon(x_1), B_\varepsilon(x_1); \Omega) \leq (1 + \hat{\eta}(C, \varepsilon)) \int_{\mathbb{R}} e^{-\frac{g(y_1)}{\varepsilon}} dy_1$$

with $\hat{\eta}$ as in (1.16), we denote by $\tau_- < 0 < \tau_+$ the numbers such that $g(\tau_-) = g(\tau_+) = \delta$. We first connect the points $x_3 = (\tau_-, 0)$ and $x_4 = (\tau_+, 0)$ by a segment $\gamma_0(t) = tx_4 + (1 - t)x_3$. Then it holds by the assumption (1.6) and by Lemma 3.9 that

$$\int_{\gamma_0} e^{\frac{F(\gamma_0)}{\varepsilon}} dt \leq e^{\frac{F(0)}{\varepsilon}} \int_{\{g < \delta\}} e^{-\frac{g(y_1)}{\varepsilon}} e^{\frac{\omega(g(y_1))}{\varepsilon}} dy_1 \leq (1 + \eta(C\Lambda_\varepsilon^{-1}))e^{\frac{F(0)}{\varepsilon}} \int_{\mathbb{R}} e^{-\frac{g(y_1)}{\varepsilon}} dy_1.$$

We then connect the points x_3, x_4 to x_1, x_2 with smooth curves $\gamma_1, \gamma_2 \subset \{x \in \Omega : F < F(0) - \delta/3\}$. Since it holds $g(t) \leq C|t|$ we have $|\{g < \varepsilon\}| \geq c\varepsilon$. Therefore it holds by Lemma 3.7 that

$$\begin{aligned} \int_{\gamma_i} |\gamma_i'| e^{\frac{F(\gamma_i)}{\varepsilon}} dt &\leq e^{\frac{F(0)}{\varepsilon}} e^{-\frac{\delta}{3\varepsilon}} \int_{\gamma_i} |\gamma_i'| dt \leq C e^{\frac{F(0)}{\varepsilon}} e^{-\frac{\delta}{3\varepsilon}} \\ &\leq e^{\frac{F(0)}{\varepsilon}} \hat{\eta}(C, \varepsilon) |\{g < \varepsilon\}| \leq e^{\frac{F(0)}{\varepsilon}} \hat{\eta}(C, \varepsilon) \int_{\mathbb{R}} e^{-\frac{g(y_1)}{\varepsilon}} dy_1. \end{aligned}$$

The constant in the last expression depends on the length of γ_i . We can use a similar argument as in the proof of Lemma 3.2 to bound the length of the curve. This time, we will however consider coverings with balls of size comparable to δ . As we are in the level set $\{F < F(0) - \delta/3\}$ we have some room to replace our curve with another curve which has a length depending on δ and R , while still retaining the same upper bound as above.

The upper bound now follows by joining the paths γ_1, γ_0 and γ_2 , thus, constructing a competitor for the geodesic length. \square

We need to prove similar result to Proposition 4.1, but for the minimal cut. This turns out to be trickier than the previous result for paths.

Proposition 4.2. *Assume that $x_a, x_b, z, U_{x_1}, U_{x_2}, O_{z,\delta}$ and Ω are as in Proposition 4.1. Then it holds for G_z from (1.6) that*

$$V_\varepsilon(B_\varepsilon(x_1), B_\varepsilon(x_2); \Omega) \simeq e^{-\frac{F(z)}{\varepsilon}} \int_{\mathbb{R}^{n-1}} e^{-\frac{G_z(y')}{\varepsilon}} dy'.$$

Proof. Denote $G_z = G$ for short. As in the proof of Proposition 4.1 we may assume that $z = 0$ and that

$$O_\delta = O_{z,\delta} = \{y_1 : g(y_1) < \delta\} \times \{y' \in \mathbb{R}^{n-1} : G(y') < \delta\}.$$

Let us begin by proving the upper bound. In the proof of Proposition 4.1 we already observed that the surface $S_0 = \{0\} \times \{G < \delta\}$ is in the family of separating surfaces $S_0 \in \mathcal{S}(B_\varepsilon(x_1), B_\varepsilon(x_2); \Omega)$. Therefore the assumption (1.6) and Lemma 3.9 together with the definition of V_ε imply

$$\begin{aligned} V_\varepsilon(B_\varepsilon(x_1), B_\varepsilon(x_2); \Omega) &\leq \int_{S_0} e^{-\frac{F}{\varepsilon}} d\mathcal{H}^{n-1} \leq e^{-\frac{F(0)}{\varepsilon}} \int_{\{G < \delta\}} e^{-\frac{G}{\varepsilon}} e^{\frac{\omega(G)}{\varepsilon}} dy' \\ &\simeq e^{-\frac{F(0)}{\varepsilon}} \int_{\mathbb{R}^{n-1}} e^{-\frac{G(y')}{\varepsilon}} dy'. \end{aligned}$$

The upper bound follows directly from this. Moreover by Lemma 3.7 it holds that

$$V_\varepsilon(B_\varepsilon(x_1), B_\varepsilon(x_2); \Omega) \leq C e^{-\frac{F(0)}{\varepsilon}} |\{G < \varepsilon\}| \leq C e^{-\frac{F(0)}{\varepsilon}}. \tag{4.4}$$

In order to prove the lower bound we fix a small $t > 0$ and choose a smooth hypersurface $S \in \mathcal{S}(B_\varepsilon(x_1), B_\varepsilon(x_2); \Omega)$ such that

$$\int_S e^{-\frac{F}{\varepsilon}} d\mathcal{H}^{n-1} \leq V_\varepsilon(B_\varepsilon(x_1), B_\varepsilon(x_2); \Omega) + t.$$

Then S divides the domain Ω into two different components, from which we denote the component containing x_1 by \hat{U}_{x_1} . Note that then $\partial\hat{U}_{x_1} \cap \Omega \subset S$. Denote $\rho = \varepsilon^2$. We use an idea from [16] and instead of studying the set \hat{U}_{x_1} , we study the density

$$v_\rho(x) := \frac{|B_\rho(x) \cap \hat{U}_{x_1}|}{|B_\rho|}$$

which can be written as a convolution, $v_\rho(x) = \frac{1}{|B_\rho|}(\chi_{\hat{U}_{x_1}} * \chi_{B_\rho})$. To see why studying v_ρ is relevant, we need some setup that we will present next.

We choose a subset \hat{O} of the bridge O_δ as

$$\hat{O} := \{x_1 : g(y_1) < \delta\} \times \{y' \in \mathbb{R}^{n-1} : G(y') < \delta/100\},$$

see Fig. 3, and denote its bottom/top boundaries by Γ_- and Γ_+ , i.e.,

$$\{g = \delta\} \times \{G < \delta/100\} = \Gamma_- \cup \Gamma_+.$$

Now since $0 \in Z_{x_a, x_b}$, we have $|F(0) - F(x_a; x_b)| < \delta/3$. Using this and (1.6) we deduce that $\Gamma_- \cup \Gamma_+ \subset \{F < F(x_a; x_b) - \delta/3\}$ and $\Gamma_- \cup \Gamma_+ \subset \{F < F(0) - \delta/3\}$, i.e.,

$$\Gamma_- \cup \Gamma_+ \subset U_{x_1} \cup U_{x_2} \cap \{F < F(0) - \delta/3\}. \tag{4.5}$$

Moreover, by relabeling we may assume that $\Gamma_+ \subset U_{x_1}$ and $\Gamma_- \subset U_{x_2}$. Furthermore, by the Lipschitz-continuity of F we have $|F(x) - F(y)| \leq c\varepsilon^2$ for all $y \in B_\rho(x)$. Note also that for all $x \in \hat{O}$ and $y \in B_\rho(x)$ it holds $x - y \in \Omega$.

We will now relate v_ρ to the surface integral of S as follows: Recall that the set \hat{U}_{x_1} has smooth boundary in Ω and thus its characteristic function is a BV-function. In particular, the derivative $|\nabla\chi_{\hat{U}_{x_1}}|$ is a Radon measure in Ω and

$$\int_\Omega |\nabla\chi_{\hat{U}_{x_1}}| e^{-\frac{F}{\varepsilon}} dx = \int_{\partial\hat{U}_{x_1} \cap \Omega} e^{-\frac{F}{\varepsilon}} d\mathcal{H}^{n-1} \leq \int_S e^{-\frac{F}{\varepsilon}} d\mathcal{H}^{n-1}. \tag{4.6}$$

Using the definition of v_ρ and the Lipschitzness of F inside $B_\rho(x)$, we may thus estimate

$$\begin{aligned} \int_{\hat{O}} |\nabla v_\rho| e^{-\frac{F(x)}{\varepsilon}} dx &\leq \frac{1}{|B_\rho|} \int_{\hat{O}} e^{-\frac{F(x)}{\varepsilon}} \int_{\mathbb{R}^n} |\nabla\chi_{\hat{U}_{x_1}}(y)| |\chi_{B_\rho}(x - y)| dy dx \\ &\leq \frac{1}{|B_\rho|} \int_{\hat{O}} e^{c\varepsilon} \int_{\mathbb{R}^n} e^{-\frac{F(y)}{\varepsilon}} |\nabla\chi_{\hat{U}_{x_1}}(y)| |\chi_{B_\rho}(x - y)| dy dx \\ &\leq (1 + C\varepsilon) \int_\Omega |\nabla\chi_{\hat{U}_{x_1}}(y)| e^{-\frac{F(y)}{\varepsilon}} dy. \end{aligned} \tag{4.7}$$

Putting together (4.6) and (4.7) we see that it is enough to establish a lower bound on the integral of $|\nabla v_\rho|$ in \hat{O} . In order to achieve this, we first claim that for all x such that $B_\rho(x) \subset \Omega \cap \{F < F(0) - \delta/3\}$ we have, when ε is small,

$$\begin{aligned} v_\rho(x) &\geq 1 - C\varepsilon && \text{for } x \in U_{x_1} \cap \{F < F(0) - \delta/3\} \quad \text{and} \\ v_\rho(x) &\leq C\varepsilon && \text{for } x \in U_{x_2} \cap \{F < F(0) - \delta/3\}. \end{aligned} \tag{4.8}$$

We now complete the proof of the lower bound, using (4.8), and postpone the proof of (4.8) to the end. Assume now that (4.8) holds. Then we can use the fundamental theorem of calculus and (4.5) to get that for all $y' \in \{G < \delta/100\}$

$$1 - 2C\varepsilon \leq \int_{\{g < \delta\}} \partial_{y_1} v_\rho(y_1, y') dy_1. \tag{4.9}$$

Now, arguing as in (4.3) we conclude that

$$F(y) - F(0) \leq G(y') + \omega(G(y')) \quad \text{for } y \in O_\delta. \tag{4.10}$$

Multiplying and dividing with $e^{-\frac{F(y)}{\varepsilon}}$ inside the integral in (4.9) and using (4.10) we get

$$(1 - 2C\varepsilon)e^{-F(0)} e^{-\frac{G(y') - \omega(G(y'))}{\varepsilon}} \leq \int_{\{G < \delta/100\}} \int_{\{g < \delta\}} |\partial_{y_1} v_\rho(y_1, y')| e^{-F(y)/\varepsilon} dy_1.$$

Integrating over $y' \in \{G < \delta/100\}$ we obtain

$$\begin{aligned} &(1 - 2C\varepsilon)e^{-F(0)} \int_{\{G < \delta/100\}} e^{-\frac{G(y')}{\varepsilon}} e^{-\frac{\omega(G(y'))}{\varepsilon}} dy' \\ &\leq \int_{\{G < \delta/100\}} \int_{\{g < \delta\}} |\partial_{y_1} v_\rho(y_1, y')| e^{-\frac{F(y)}{\varepsilon}} dy_1 dy' \\ &\leq \int_{\hat{O}} |\nabla v_\rho| e^{-\frac{F(y)}{\varepsilon}} dx. \end{aligned}$$

The lower bound on the integral on the right hand side follows by Lemma 3.9, i.e. we have

$$(1 - \eta(\varepsilon))e^{-F(0)} \int_{\mathbb{R}^{n-1}} e^{-\frac{G(y')}{\varepsilon}} dy' \leq \int_{\hat{O}} |\nabla v_\rho| e^{-\frac{F(y)}{\varepsilon}} dy. \tag{4.11}$$

Now, assuming (4.8), we may use (4.6), (4.7) and (4.11) to get the lower bound from

$$\begin{aligned} (1 - \eta(\varepsilon))e^{-F(0)} \int_{\mathbb{R}^{n-1}} e^{-\frac{G(y')}{\varepsilon}} dy' &\leq (1 + C\varepsilon) \int_S e^{-\frac{F}{\varepsilon}} d\mathcal{H}^{n-1} \\ &\leq (1 + C\varepsilon)(V_\varepsilon(B_\varepsilon(x_1), B_\varepsilon(x_2); \Omega) + t) \end{aligned}$$

as t is arbitrarily small. Thus we obtain the lower bound, and hence in order to complete the proof it remains to prove (4.8).

To this aim, we fix $x \in U_{x_1} \cup U_{x_2} \cap \{F < F(0) - \delta/3\}$ such that $B_\rho(x) \subset \Omega$. By the relative isoperimetric inequality (See for instance [2, Theorem 3.40] or [18]) and by $\rho = \varepsilon^2$ it holds

$$\begin{aligned} \mathcal{H}^{n-1}(\partial \hat{U}_{x_1} \cap B_\rho(x)) &\geq c \min \{ |B_\rho(x) \cap \hat{U}_{x_1}|^{\frac{n-1}{n}}, |B_\rho(x) \setminus \hat{U}_{x_1}|^{\frac{n-1}{n}} \} \\ &\geq c \varepsilon^{2(n-1)} \min \{ v_\rho(x), 1 - v_\rho(x) \}^{\frac{n-1}{n}}. \end{aligned}$$

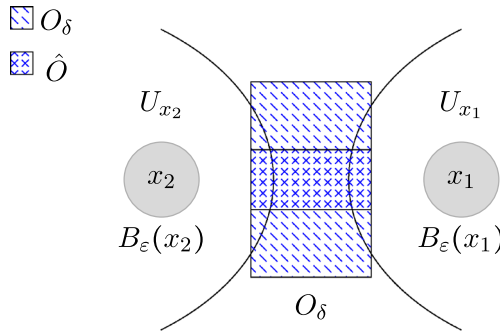


Fig. 3. The bridge O_δ connects the sets U_{x_1} and U_{x_2} . The smaller cylindrical bridge \hat{O} has its top and bottom inside $U_{x_1} \cup U_{x_2}$

On the other hand, since $x \in \{F < F(0) - \delta/3\}$ and thus $B_\rho(x) \subset \{F < F(0) - \delta/4\}$, we have by (4.4) that

$$\begin{aligned} \mathcal{H}^{n-1}(\partial \hat{U}_{x_1} \cap B_\rho(x)) &\leq e^{\frac{F(0)}{\epsilon}} e^{-\frac{\delta}{4\epsilon}} \int_{\partial \hat{U}_{x_1} \cap B_\rho(x)} e^{-\frac{F}{\epsilon}} d\mathcal{H}^{n-1} \\ &\leq e^{\frac{F(0)}{\epsilon}} e^{-\frac{\delta}{4\epsilon}} \int_S e^{-\frac{F}{\epsilon}} d\mathcal{H}^{n-1} \\ &\leq e^{\frac{F(0)}{\epsilon}} e^{-\frac{\delta}{4\epsilon}} (V_\epsilon(B_\epsilon(x_1), B_\epsilon(x_2); \Omega) + t) \\ &\leq C e^{-\frac{\delta}{4\epsilon}}. \end{aligned}$$

By combining the two inequalities above we obtain (4.8) which completes the whole proof. \square

Proof of Theorem 1. We consider parallel case and series case separately.

Parallel case Assume that the saddle points in $F_{x_a, x_b} = \{z_1, \dots, z_N\}$ are parallel, see Fig. 2. Let us fix a saddle point $z_i \in F_{x_a, x_b}$ and recall the definition of the bridge $O_{z_i, \delta}$ in (1.7). By considering $F - F(z_i)$ instead of F , we may assume

$$z_i = 0, \quad F(0) = 0$$

and

$$O_\delta := O_{0, \delta} = \{x_1 \in \mathbb{R} : g(x_1) < \delta\} \times \{x' \in \mathbb{R}^{n-1} : G(x') < \delta\}.$$

We also recall the notation $U_{-\delta/3} = \{F < F(x_a; x_b) - \delta/3\}$. We denote the island, i.e., the component of $U_{-\delta/3}$, which contains the point x_a by U_{x_a} and the island which contains the point x_b by U_{x_b} . Since the saddle points are parallel, the bridge O_δ connects the islands U_{x_a} and U_{x_b} which means that the set $\Omega = U_{x_a} \cup U_{x_b} \cup O_\delta$ is open and connected. By Lemma 3.5 we have $\text{osc}_{U_{x_a}}(h_{A,B}) + \text{osc}_{U_{x_b}}(h_{A,B}) \leq C\epsilon$. Since $h_{A,B} = 1$ in $B_\epsilon(x_a) \subset U_{x_a}$ and $h_{A,B} = 0$ in $B_\epsilon(x_b) \subset U_{x_b}$, it follows that

$$h_{A,B} \geq 1 - C\epsilon \text{ in } U_{x_a} \quad \text{and} \quad h_{A,B} \leq C\epsilon \text{ in } U_{x_b}. \tag{4.12}$$

Let us choose a subset \hat{O} of the bridge O_δ as in the proof of Proposition 4.2 (see Fig. 3), i.e.

$$\hat{O} := \{x_1 : g(x_1) < \delta\} \times \{x' \in \mathbb{R}^{n-1} : G(x') < \delta/100\},$$

and denote its bottom/top boundaries by $\Gamma_- \subset \{x_1 < 0\}$ and $\Gamma_+ \subset \{x_1 > 0\}$, i.e.,

$$\{g = \delta\} \times \{G < \delta/100\} = \Gamma_- \cup \Gamma_+.$$

Then by using (1.6) and arguing as in the proof of Proposition 4.2 we deduce that $\Gamma_-, \Gamma_+ \subset \{F < F(x_a; x_b) - \delta/3\}$ and we may assume $\Gamma_+ \subset U_{x_a}$ and $\Gamma_- \subset U_{x_b}$. Therefore, by (4.12), we have that $h_{A,B} \leq C\varepsilon$ on Γ_- and $h_{A,B} \geq 1 - C\varepsilon$ on Γ_+ . Now, by the fundamental theorem of calculus and Cauchy-Schwarz inequality, it holds that

$$\begin{aligned} 1 - 2C\varepsilon &\leq \int_{\{g < \delta\}} \partial_{x_1} h_{A,B}(x) dx_1 = \int_{\{g < \delta\}} \partial_{x_1} h_{A,B}(x) e^{-\frac{F(x)}{2\varepsilon}} e^{\frac{F(x)}{2\varepsilon}} dx_1 \\ &\leq \left(\int_{\{g < \delta\}} |\nabla h_{A,B}(x)|^2 e^{-\frac{F(x)}{\varepsilon}} dx_1 \right)^{\frac{1}{2}} \left(\int_{\{g < \delta\}} e^{\frac{F(x)}{\varepsilon}} dx_1 \right)^{\frac{1}{2}}. \end{aligned} \tag{4.13}$$

Let us next estimate the last term above. By assumption (1.6) we have, for $x \in \hat{O}$,

$$F(x) \leq -g(x_1) + \omega(g(x_1)) + G(x') + \omega(G(x')).$$

Therefore, by Lemma 3.9 and Proposition 4.1, we can estimate

$$\begin{aligned} \int_{\{g < \delta\}} e^{\frac{F(x)}{\varepsilon}} dx_1 &\leq (1 + \hat{\eta}(C, \varepsilon)) e^{\frac{G(x')}{\varepsilon}} e^{\frac{\omega(G(x'))}{\varepsilon}} \int_{\{g < \delta\}} e^{-\frac{g(x_1)}{\varepsilon}} e^{\frac{\omega(g(x_1))}{\varepsilon}} dx_1 \\ &\leq (1 + \hat{\eta}(C, \varepsilon)) e^{\frac{G(x')}{\varepsilon}} e^{\frac{\omega(G(x'))}{\varepsilon}} \int_{\mathbb{R}} e^{-\frac{g(x_1)}{\varepsilon}} dx_1 \\ &\leq (1 + \hat{\eta}(C, \varepsilon)) e^{\frac{G(x')}{\varepsilon}} e^{\frac{\omega(G(x'))}{\varepsilon}} d_\varepsilon(B_\varepsilon(x_a), B_\varepsilon(x_b); \Omega). \end{aligned} \tag{4.14}$$

We combine the inequalities (4.13) and (4.14) leading to (for another constant C)

$$\int_{\{g < \delta\}} |\nabla h_{A,B}(x)|^2 e^{-\frac{F(x)}{\varepsilon}} dx_1 \geq \frac{(1 - \hat{\eta}(C, \varepsilon))}{d_\varepsilon(B_\varepsilon(x_a), B_\varepsilon(x_b); \Omega)} e^{-\frac{G(x')}{\varepsilon}} e^{-\frac{\omega(G(x'))}{\varepsilon}}$$

for all $x' \in \{G < \delta/100\}$. By integrating over $x' \in \{G < \delta/100\}$ we have by Fubini's theorem, Lemma 3.9 and Proposition 4.2, that

$$\begin{aligned} \int_{\hat{O}} |\nabla h_{A,B}|^2 e^{-\frac{F(x)}{\varepsilon}} dx &\geq \frac{(1 - \hat{\eta}(C, \varepsilon))}{d_\varepsilon(B_\varepsilon(x_a), B_\varepsilon(x_b); \Omega)} \int_{\{G < \delta/100\}} e^{-\frac{G(x')}{\varepsilon}} e^{-\frac{\omega(G(x'))}{\varepsilon}} dx' \\ &\geq \frac{(1 - \hat{\eta}(C, \varepsilon))}{d_\varepsilon(B_\varepsilon(x_a), B_\varepsilon(x_b); \Omega)} \int_{\mathbb{R}^{n-1}} e^{-\frac{G(x')}{\varepsilon}} dx' \\ &\geq (1 - \hat{\eta}(C, \varepsilon)) \frac{V_\varepsilon(B_\varepsilon(x_a), B_\varepsilon(x_b); \Omega)}{d_\varepsilon(B_\varepsilon(x_a), B_\varepsilon(x_b); \Omega)}. \end{aligned} \tag{4.15}$$

Therefore, by repeating the argument for every saddle $z_i \in Z_{x_a, x_b}$ and using the fact that the bridges $O_{z_i, \delta}$ are disjoint, we obtain after scaling back the potential

$$\int_{\mathbb{R}^n} |\nabla h_{A,B}|^2 e^{-\frac{F(x)}{\varepsilon}} dx \geq (1 - \hat{\eta}(C, \varepsilon)) \sum_{i=1}^N \frac{V_\varepsilon(B_\varepsilon(x_a), B_\varepsilon(x_b); \Omega)}{d_\varepsilon(B_\varepsilon(x_a), B_\varepsilon(x_b); \Omega)} e^{\frac{F(z_i)}{\varepsilon}}.$$

This yields the lower bound when the saddle points are parallel.

For the upper bound, we only give a sketch of the argument as it is fairly straightforward. The idea is to construct a competitor h in the variational characterization of the capacity, see (2.5). Let us first define h in the set $U_{\delta/3} = \{F < F(x_a; x_b) + \delta/3\}$. Since the saddle points $Z_{x_a, x_b} = \{z_1, \dots, z_N\}$ are parallel, it follows that the points x_a and x_b lie in different components of the set

$$\tilde{U} = U_{\delta/3} \setminus \bigcup_{i=1}^N O_{z_i, \delta},$$

where $O_{z_i, \delta}$ is defined in (1.8). Denote the components of \tilde{U} containing x_a and x_b by \tilde{U}_{x_a} and \tilde{U}_{x_b} , respectively. We define first

$$h = 1 \text{ in } \tilde{U}_{x_a} \text{ and } h = 0 \text{ in } \tilde{U}_{x_b}.$$

Let us next fix a saddle point $z_i \in Z_{x_a, x_b}$. As before, we may again assume that

$$z_i = 0, \quad F(0) = 0$$

and

$$O_\delta := O_{0, \delta} = \{x_1 \in \mathbb{R} : g(x_1) < \delta\} \times \{x' \in \mathbb{R}^{n-1} : G(x') < \delta\}.$$

Moreover, we may assume that

$$\tilde{U}_{x_a} \cap \partial O_\delta \subset \{x_1 > 0\} \quad \text{and} \quad \tilde{U}_{x_b} \cap \partial O_\delta \subset \{x_1 < 0\}.$$

Let $\tau_- < 0 < \tau_+$ be numbers such that $g(\tau_-) = g(\tau_+) = \delta/100$. We define $h(x) = \varphi(x_1)$ in O_δ such that the function $\varphi : [\tau_-, \tau_+] \rightarrow \mathbb{R}$ is a solution of the ordinary differential equation

$$\frac{d}{ds} \left(\varphi'(s) e^{\frac{g(s)}{\varepsilon}} \right) = 0 \quad \text{in } (\tau_-, \tau_+)$$

with boundary values $\varphi(\tau_-) = 0$ and $\varphi(\tau_+) = 1$. Note that then

$$\varphi'(s) e^{\frac{g(s)}{\varepsilon}} = \left(\int_{\tau_-}^{\tau_+} e^{-\frac{g(x_1)}{\varepsilon}} dx_1 \right)^{-1}.$$

We extend φ into \mathbb{R} by setting $\varphi(s) = 0$ for $s \leq \tau_-$ and $\varphi(s) = 1$ for $s \geq \tau_+$. It follows that for the function h we have, by construction, Lemma 3.9, and an argument similar to the one leading to (4.13), that

$$\begin{aligned} & \int_{O_\delta} |\nabla h|^2 e^{-\frac{F(x)}{\varepsilon}} dx \\ & \leq \int_{\{g < \delta\}} |\varphi'(x_1)|^2 e^{\frac{g(x_1)}{\varepsilon}} e^{\frac{\omega(g(x_1))}{\varepsilon}} dx_1 \int_{\{G < \delta\}} e^{-\frac{G(x')}{\varepsilon}} e^{\frac{\omega(G(x'))}{\varepsilon}} dx' \\ & \leq (1 + \hat{\eta}(C, \varepsilon)) \left(\int_{\mathbb{R}} e^{-\frac{g(x_1)}{\varepsilon}} dx_1 \right)^{-1} \left(\int_{\mathbb{R}^{n-1}} e^{-\frac{G(x')}{\varepsilon}} dx' \right). \end{aligned}$$

By repeating the construction for every saddle point $z_i \in Z_{x_a, x_b}$, we obtain a function which is defined in $U_{\delta/3}$. We denote this function by $h : U_{\delta/3} \rightarrow \mathbb{R}$. Note that now for h

the estimate (4.15) is optimal. Moreover, h is Lipschitz continuous. We extend h to \mathbb{R}^n without increasing the Lipschitz constant L , e.g., by defining

$$h(x) = \sup_{y \in U_{\delta/3}} (h(y) - L|x - y|) \quad \text{for } x \in \mathbb{R}^n \setminus U_{\delta/3}.$$

This finally leads to the upper bound completing the proof of the parallel case, while we leave the final details on the upper bound for the reader.

Series case Assume that the saddle points $Z_{x_a, x_b} = \{z_1, \dots, z_N\}$ are in series, see Fig. 2. We use the ordering as in (1.11) and denote the points x_i as in (1.12). We also fix the islands, $U_{x_{i-1}}$ and U_{x_i} (components of $\{F < F(x_a; x_b) - \delta/3\}$), which are connected by the bridge $O_{z_i, \delta}$. Again we may assume that $z_i = 0$, $F(0) = 0$ and that

$$O_\delta = O_{z_i, \delta} = \{y_1 : g(y_1) < \delta\} \times \{y' : G(y') < \delta\}.$$

By Lemma 3.5 we have $\text{osc}_{U_{x_{i-1}}}(h_{A,B}) + \text{osc}_{U_{x_i}}(h_{A,B}) \leq C\varepsilon$. Therefore there are numbers c_{i-1}, c_i such that

$$|h_{A,B} - c_{i-1}| \leq C\varepsilon \text{ in } U_{x_{i-1}} \quad \text{and} \quad |h_{A,B} - c_i| \leq C\varepsilon \text{ in } U_{x_i}.$$

Then, using the fundamental theorem of calculus as in (4.13), we obtain

$$|c_{i-1} - c_i| - 2C\varepsilon \leq \left(\int_{\{g < \delta\}} |\nabla h_{A,B}(y)|^2 e^{-\frac{F(y)}{\varepsilon}} dy_1 \right)^{\frac{1}{2}} \left(\int_{\{g < \delta\}} e^{\frac{F(y)}{\varepsilon}} dy_1 \right)^{\frac{1}{2}}$$

for

$$(y_1, y') \in \{g < \delta\} \times \{G < \delta/100\}.$$

Moreover, arguing as in (4.14), we have

$$\int_{\{g < \delta\}} e^{\frac{F(y)}{\varepsilon}} dy_1 \leq (1 + \hat{\eta}(C, \varepsilon)) e^{\frac{G(y')}{\varepsilon}} e^{\frac{\omega(G(y'))}{\varepsilon}} d_\varepsilon(x_{i-1}, x_i).$$

These together imply

$$\int_{\{g < \delta\}} |\nabla h_{A,B}(y)|^2 e^{-\frac{F(y)}{\varepsilon}} dy_1 \geq \frac{(1 - \hat{\eta}(C, \varepsilon))(c_{i-1} - c_i)^2}{d_\varepsilon(x_{i-1}, x_i)} e^{-\frac{G(y')}{\varepsilon}} e^{-\frac{\omega(G(y'))}{\varepsilon}}.$$

By integrating over $y' \in \{G < \delta/100\}$ we have, by Fubini's theorem, Lemma 3.9, and Proposition 4.2, that

$$\int_{O_\delta} |\nabla h_{A,B}|^2 e^{-\frac{F(y)}{\varepsilon}} dy \geq (1 - \hat{\eta}(C, \varepsilon))(c_{i-1} - c_i)^2 \frac{V_\varepsilon(x_{i-1}, x_i)}{d_\varepsilon(x_{i-1}, x_i)}.$$

By repeating the argument for every saddle $z_i \in Z_{x_a, x_b}$ and using the fact that the sets $O_{z_i, \delta}$ are disjoint we obtain

$$\int_{\mathbb{R}^n} |\nabla h_{A,B}|^2 e^{-\frac{F(y)}{\varepsilon}} dy \geq (1 - \hat{\eta}(C, \varepsilon)) \sum_{i=1}^N (c_{i-1} - c_i)^2 \frac{V_\varepsilon(x_{i-1}, x_i)}{d_\varepsilon(x_{i-1}, x_i)}.$$

Recall that the numbers c_i are the approximate values of $h_{A,B}$ in the components U_{x_i} . Therefore we may choose them such that $1 = c_0$ and $c_N = 0$. By denoting $y_i = c_{i-1} - c_i$ and $a_i = \frac{V_\varepsilon(x_{i-1}, x_i)}{d_\varepsilon(x_{i-1}, x_i)}$ we may write

$$\sum_{i=1}^N (c_{i-1} - c_i)^2 \frac{V_\varepsilon(x_{i-1}, x_i)}{d_\varepsilon(x_{i-1}, x_i)} = \sum_{i=1}^N a_i y_i^2,$$

where we have a constraint $\sum_{i=1}^N y_i = 1$. By a standard optimization argument (using Lagrange multipliers) we get that under such a constraint it holds that

$$\sum_{i=1}^N a_i y_i^2 \geq \left(\sum_{i=1}^N \frac{1}{a_i} \right)^{-1}.$$

This yields the lower bound in the case when the saddle points are in series. The upper bound on the other hand follows from a similar argument than in the parallel case, and we leave the details for the reader. This completes the proof in the series case, and hence the whole proof. \square

Proof of Theorem 2. Let us first recall the notation related to Theorem 2. We assume that the local minima x_i of F are ordered such that $F(x_i) \leq F(x_j)$ if $i \leq j$, and they are grouped into sets G_i such that $x_i, x_j \in G_k$ if $F(x_i) = F(x_j)$ and $x \in G_i, y \in G_j$ with $i < j$ if $F(x) < F(y)$. We also denoted $F(G_i) := F(x)$ with $x \in G_i, S_k = \bigcup_{i=1}^k G_i, G_k^\varepsilon = \bigcup_{x \in G_k} B_\varepsilon(x)$, and $S_k^\varepsilon = \bigcup_{i=1}^k G_i^\varepsilon$.

The proof of Theorem 2 follows from the following lemma together with Lemma 3.6 and Theorem 1.

Lemma 4.3. *Under the assumptions of Theorem 2, there exists constants $C = C(F) > 1$ and $\varepsilon = \varepsilon_0(F) \in (0, 1)$ such that, for all $0 < \varepsilon \leq \varepsilon_0$, we have*

$$\frac{1}{C} \sum_{x \in G_{k+1}} |O_{x,\varepsilon}| \leq e^{\frac{F(G_{k+1})}{\varepsilon}} \int h_{G_{k+1}^\varepsilon, S_k^\varepsilon} d\mu_\varepsilon \leq C \sum_{x \in G_{k+1}} |O_{x,\varepsilon}|.$$

We prove Theorem 2 first, while the proof of Lemma 4.3 is given later on.

Proof of Theorem 2. Using Lemma 3.6 and choosing $\rho = \varepsilon$ we obtain that, for ε small enough and $x \in G_{k+1}^\varepsilon$, that

$$\mathbb{E}^x [\tau_{S_k^\varepsilon} \mathbb{I}_{\tau_{S_k^\varepsilon} < \tau_{\Omega^c}}] \leq C \frac{\int h_{S_k^\varepsilon, \Omega^c} h_{G_{k+1}^\varepsilon, S_k^\varepsilon} d\mu_\varepsilon}{\text{cap}(G_{k+1}^\varepsilon, S_k^\varepsilon)} + C\varepsilon^{\alpha/2}.$$

The ratio above can be estimated by using Lemma 4.3 and the monotonicity of the capacity. That is, the numerator can be bounded by Lemma 4.3, while for the capacity we have

$$\text{cap}(G_{k+1}^\varepsilon, S_k^\varepsilon) \geq \text{cap}(G_{k+1}^\varepsilon, G_k^\varepsilon) \geq \max_{x \in G_k, y \in G_{k+1}} \text{cap}(B_\varepsilon(x), B_\varepsilon(y)). \tag{4.16}$$

The claim for the parallel and series cases now follows by assuming that the maximum is attained for a pair of minima $x_a \in G_k, x_b \in G_{k+1}$ and applying Theorem 1. \square

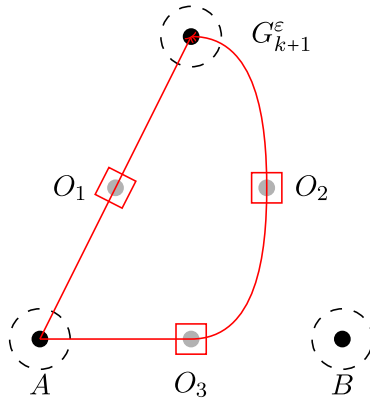


Fig. 4. Geometric view of multiple minima at same height and multiple saddles at the same height

Remark 4.4. We note that in the general case, the last inequality in (4.16) has the optimal dependence with respect to ε but the inequalities may differ by a constant. Essentially the inequality is sharp only in the case where only one saddle contributes to the total value of the capacity. Hence we have the sharp estimate when saddle points are parallel or in series, but in general the situation might be more complicated than that. We have illustrated this in Fig. 4, where each gray dot is a saddle at the same height, and A, B produces G_k . Then the precise value of $\text{cap}(G_{k+1}^\varepsilon, A \cup B)$ is already non-trivial to calculate.

Proof of Lemma 4.3. First we will prove a localization estimate for exponential integrals. Consider a set $O \in \mathcal{O}$ and a function f such that $f(0) = l$ is a proper local minimum and that f is locally convex around 0. Then there exists an ε_0 such that, for any $\varepsilon < \varepsilon_0$,

$$\int_O e^{-f(x)/\varepsilon} dx \leq C e^{-l/\varepsilon} |\{f < \varepsilon\} \cap O|. \tag{4.17}$$

We will first prove (4.17) and then repeatedly apply it to prove Lemma 4.3.

In order to prove (4.17), we begin by rescaling such that $l = 0$. Then we extend f outside O as $+\infty$ and call this extended function \hat{f} . We first prove

$$\int_{\{\hat{f} > a\}} e^{-\hat{f}(x)/\varepsilon} dx \leq c e^{-a/\varepsilon}$$

which, by the definition of \hat{f} , is equivalent to

$$\int_{\{f > a\} \cap O} e^{-f(x)/\varepsilon} dx \leq c e^{-a/\varepsilon}.$$

This now follows from Lemma 3.8 by using $|O| < \infty$ and observing that \hat{f} satisfies the assumptions of Lemma 3.8. Hence we observe (4.17).

Consider now the set

$$U_{-\delta_2/3} \equiv \{y : F(y) \leq F(G_{k+1}; S_k) - \delta_2/3\}$$

and let U_i be the component of $U_{-\delta_2/3}$ containing x_i . We split

$$\int h_{G_{k+1}^\varepsilon, S_k^\varepsilon} e^{-F/\varepsilon} dx = \int_{U_{-\delta_2/3}^c} h_{G_{k+1}^\varepsilon, S_k^\varepsilon} e^{-F/\varepsilon} dx + \int_{U_{-\delta_2/3} \setminus S_k^\varepsilon} h_{G_{k+1}^\varepsilon, S_k^\varepsilon} e^{-F/\varepsilon} dx,$$

where complement is understood with respect to the domain Ω . By assumptions (1.4) and (1.6) on F it holds that

$$F(G_{k+1}; S_k) \geq F(G_{k+1}) + \frac{2}{3}\delta_2. \tag{4.18}$$

Also by the quadratic growth (1.3) we can bound the first integral as

$$\int_{U_{-\delta_2/3}^c} h_{G_{k+1}^\varepsilon, S_k^\varepsilon} e^{-F/\varepsilon} dx \leq C e^{-(F(G_{k+1}; S_k) - \delta_2/3)/\varepsilon} \leq C e^{-\frac{\delta_2}{3\varepsilon}} e^{-F(G_{k+1})/\varepsilon},$$

which shows that the first integral is negligible in the final estimate. For the second integral we further split

$$\int_{U_{-\delta_2/3} \setminus S_k^\varepsilon} h_{G_{k+1}^\varepsilon, S_k^\varepsilon} e^{-F/\varepsilon} dx = \sum_i \int_{U_i \setminus S_k^\varepsilon} h_{G_{k+1}^\varepsilon, S_k^\varepsilon} e^{-F/\varepsilon} dx.$$

We will now consider all the different components U_i depending on what minima they contain. We start with the components U_i that do not intersect $S_k^\varepsilon \cup G_{k+1}^\varepsilon$. Then all local minima in U_i are larger than $F(G_{k+1})$, and hence from (4.17) we get that there exists a constant C such that

$$\int_{U_i} h_{G_{k+1}^\varepsilon, S_k^\varepsilon} e^{-F/\varepsilon} dx \leq C e^{-F(G_{k+2})/\varepsilon} \leq C e^{-\delta_2/\varepsilon} e^{-F(G_{k+1})/\varepsilon},$$

where the last inequality follows from (1.19). This shows that also this term is negligible. Consider next the component U_i that intersects G_{k+1}^ε but do not intersect S_k^ε . In this case, by (4.17) and Lemmas 3.5 and 3.7, we have

$$\frac{1}{C} \sum_{x \in G_{k+1} \cap U_i} |O_{x, \varepsilon}| \leq e^{\frac{F(G_{k+1})}{\varepsilon}} \int_{U_i} h_{G_{k+1}^\varepsilon, S_k^\varepsilon} e^{-F/\varepsilon} dx \leq C \sum_{x \in G_{k+1} \cap U_i} |O_{x, \varepsilon}| \tag{4.19}$$

providing us the leading term that contributes to the final estimate.

Consider next a component U_i such that $U_i \cap S_k^\varepsilon \neq \emptyset$. Since U_i is a component of $U_{-\delta_2/3}$, it follows from $U_i \cap S_k \neq \emptyset$ that $F(y; S_k) \leq F(y; G_{k+1}) - \delta_2/3 \leq F(y; G_{k+1})$ in U_i . Therefore we have, by Lemma 3.3, in U_i that

$$h_{G_{k+1}^\varepsilon, S_k^\varepsilon} \leq C \varepsilon^q e^{-(F(y; G_{k+1}) - F(y; S_k))/\varepsilon}.$$

Hence, for $q \in \mathbb{R}$, we obtain

$$\int_{U_i} h_{G_{k+1}^\varepsilon, S_k^\varepsilon} e^{-F/\varepsilon} dx \leq \varepsilon^q \int_{U_i} e^{-(F(y; G_{k+1}) - F(y; S_k))/\varepsilon} e^{-F(y)/\varepsilon} dy.$$

In order to compute the integral on the right hand side we study the infimum value of the function $f(y) = F(y; G_{k+1}) - F(y; S_k) + F(y)$. Clearly, the infimum is attained at an interior point of U_i , denoted by x_i . It follows that then x_i is necessarily a local minimum point of F . By above considerations, we also have $F(y; S_k) < F(y; G_{k+1})$ for all $y \in U_i$,

and thus we may deduce that $x_i \notin G_{k+1}$. If now $x_i \in S_k$, then $F(x_i) = F(x_i; S_k)$ and thus, by the definition of f and by (4.18),

$$\inf_{U_i} f(y) = f(x_i) \geq F(x_i; G_{k+1}) \geq F(S_k; G_{k+1}) \geq F(G_{k+1}) + \frac{2}{3}\delta_2.$$

It remains to study the case where $x_i \in G_j$ for some $j \geq k + 2$. In this case we apply

$$\inf_{U_i} f(y) = f(x_i) = \overbrace{F(x_i; G_{k+1}) - F(x_i; S_k)}^{\geq 0} + F(G_j) \geq F(G_{k+2}) \geq F(G_{k+1}) + \delta_2,$$

where the last inequality follows from (1.19). Therefore we can conclude that, for $\delta_3 = \frac{2}{3}\delta_2$, it holds that

$$\int_{U_i} \varepsilon^q e^{-(F(y; G_{k+1}) - F(y; S_k))/\varepsilon} e^{-F(y)/\varepsilon} dy \leq C\varepsilon^q e^{-\frac{\delta_3}{\varepsilon}} e^{-F(G_{k+1})/\varepsilon}.$$

Consequently, the component U_i satisfying $U_i \cap S_k^\varepsilon \neq \emptyset$ does not contribute either. The proof is hence completed by (4.19) and by the fact that the integral over the remaining components are negligible whenever ε is small enough. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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References

1. Aikawa, H., Essén, M.: Potential Theory-Selected Topics. Springer, Berlin (1996)
2. Ambrosio, L., Fusco, N., Pallara, D.: Functions of Bounded Variation and Free Discontinuity Problems. Oxford University Press, New York (2000)
3. Avelin, B., Julin, V.: Approximation of BV functions by neural networks: A regularity theory approach (2020). [arXiv:2012.08291](https://arxiv.org/abs/2012.08291)
4. Berglund, N., Gentz, B.: The Eyring–Kramers law for potentials with nonquadratic saddles. *Markov Process. Relat. Fields* **16**(3), 549–598 (2010)
5. Bovier, A., Den Hollander, F.: *Metastability: A Potential-Theoretic Approach*, vol. 351. Springer, Berlin (2016)

6. Bovier, A., Eckhoff, M., Gaynard, V., Klein, M.: Metastability in stochastic dynamics of disordered mean-field models. *Probab. Theory Relat. Fields* **119**(1), 99–161 (2001)
7. Bovier, A., Eckhoff, M., Gaynard, V., Klein, M.: Metastability and low lying spectra in reversible Markov chains. *Commun. Math. Phys.* **228**(2), 219–255 (2002)
8. Bovier, A., Eckhoff, M., Gaynard, V., Klein, M.: Metastability in reversible diffusion processes I. Sharp asymptotics for capacities and exit times. *J. Eur. Math. Soc. (JEMS)* **6**, 399–424 (2004)
9. Bovier, A., Gaynard, V., Klein, M.: Metastability in reversible diffusion processes II. Precise asymptotics for small eigenvalues. *J. Eur. Math. Soc.* **7**, 69–99 (2005)
10. Carleson, L.: Selected problems on exceptional sets (1967)
11. Eyring, H.: The activated complex in chemical reactions. *J. Chem. Phys.* **3**(2), 107–115 (1935)
12. Freidlin, M.I., Wentzell, A.D.: Random perturbations of dynamical systems. Translated from the Russian by Joseph Szücs. *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, vol 260, viii+326 pp. Springer, New York (1984)
13. Gehring, F.W.: Extremal length definitions for the conformal capacity of rings in space. *Mich. Math. J.* **9**, 137–150 (1962)
14. Gilbarg, D., Trudinger, N.S.: *Elliptic Partial Differential Equations of Second Order*. Springer, Berlin (2001)
15. Hochreiter, S., Jürgen, S.: Flat minima. *Neural Comput.* **9**(1), 1–42 (1997)
16. Julin, V.: Generalized Harnack inequality for semilinear elliptic equations. *J. Math. Pures Appl.* **9**(106), 877–904 (2016)
17. Kramers, H.A.: Brownian motion in a field of force and the diffusion model of chemical reactions. *Physica* **7**(4), 284–304 (1940)
18. Maz'ya, Vladimir: Lectures on isoperimetric and isocapacity inequalities in the theory of Sobolev spaces. *Contemp. Math.* **338**, 307–340 (2003)
19. Petzka, H., Kamp, M., Adilova, L.: Relative flatness and generalization. *Adv. Neural. Inf. Process. Syst.* **34**, 18420–18432 (2021)
20. Ziemer, W.P.: Extremal length and conformal capacity. *Trans. Am. Math. Soc.* **126**, 460–473 (1967)

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