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Nonuniqueness of Solutions to the Euler Equations with Vorticity in a Lorentz Space

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Abstract: For the two dimensional Euler equations, a classical result by Yudovich states that solutions are unique in the class of bounded vorticity; it is a celebrated open problem whether this uniqueness result can be extended in other integrability spaces. We prove in this note that such uniqueness theorem fails in the class of vector fields u with uniformly bounded kinetic energy and vorticity in the Lorentz space $L^{1,\infty}$.

1. Introduction

Let us consider the 2-dimensional Euler equations

$$\begin{cases} \partial_t u + \operatorname{div}(u \otimes u) + \nabla p = 0 \\ \operatorname{div} u = 0 \end{cases} \tag{1}$$

where $u : [0,1] \times \mathbb{T}^2 \to \mathbb{R}^2$ is the velocity of a fluid and $p : [0,1] \times \mathbb{T}^2 \to \mathbb{R}$ the pressure. This system can be equivalently rewritten as the two dimensional Euler system in vorticity formulation, which is a transport equation for the vorticity $\omega = \operatorname{curl}(u)$, i.e.

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0 \\ u = \nabla^\perp \Delta^{-1} \omega. \end{cases} \qquad \text{in } \mathbb{T}^2 \times [0,1]. \tag{2}$$

In the latter formulation, it is clear that L^p norms of the vorticity are formally conserved for any $p \in [1,\infty]$. For $p > 1$, this was used in [11] to prove the existence of distributional solutions starting from an initial datum with vorticity in L^p. A similar existence result is much more involved for $p = 1$, and it was obtained by Delort [10] (see also [11,12,22]), improving the existence theory up to measure initial vorticities in H^{-1} (this latter condition guarantees finiteness of the energy) whose positive (or negative) part is absolutely continuous. As regards uniqueness, the classical result of Yudovich [15,16] (see also the proof in [17]) states that, given an initial datum $\omega_0 \in L^\infty$, there exists a unique bounded solution to (2) starting from ω_0. However, the classical problem raised by Yudovich about the sharpness of his result is still open. Let u_0 be an initial datum in L^2 with $\operatorname{curl} u_0$ in some function space X. Is the solution of the Euler equations in vorticity formulation unique in the class $L^\infty(X)$?

The main result of this paper provides a negative answer when X is the Lorentz space $L^{1,\infty}$.

Theorem 1.1. *There exists a nontrivial solution* $u \in C^0([0,1]; L^2(\mathbb{T}^2))$ *to* (1) *satisfying*

(i) $\omega = \operatorname{curl} u \in C^0([0,1]; L^{1,\infty}(\mathbb{T}^2))$*;*
(ii) $u(0,\cdot) = 0$.

Moreover, $u \in C^0([0,1]; W^{s,p}(\mathbb{T}^2))$ *for any* $s \in (0,1)$ *and* $p \in (1, \frac{2}{1+s})$.

Remark 1.2. The conclusion (i) in Theorem 1.1 has to be interpreted as follows. There exist a function $\operatorname{curl} u \in C^0([0,1]; L^{1,\infty}(\mathbb{T}^2))$ and a sequence $u_n \in C^\infty([0,1] \times \mathbb{T}^2)$ solving the Euler equations with an error term R_n in the right hand side (see (3)) such that

$$\|R_n\|_{C^0(L^1)} + \|u_n - u\|_{C^0(L^2)} + \|\operatorname{curl} u_n - \operatorname{curl} u\|_{C^0(L^{1,\infty})} \to 0,$$

as $n \to 0$.

Recently, there have been formidable attempts to disprove this conjecture for $X = L^p$, none of which has by now fully solved it. Vishik [23,24], see also [1], proposed a complex line of approach to this problem, which however has the price of showing nonuniqueness only with an additional degree of freedom, namely a forcing term in the right-hand side of the equation (2) in the integrability space $L^1(L^p)$. The nonuniqueness suggested by this work is of symmetry breaking type and, in contrast with the ideas of this paper, his nonuniqueness stems from the linear part of the equation, by carefully choosing an initial datum that sees the instability directions of a linearized operator.

A second attempt has been pursued by Bressan and Shen [2], based on numerical experiments which share the symmetry breaking type of nonuniqueness of Vishik. Their work is a first step in the direction of a computer assisted proof.

Our approach is instead of different nature and stems from the convex integration technique. The latter was introduced by De Lellis and Székelyhidi [9] in the context of nonlinear PDEs, inspired by the work of Nash on isometric embeddings [20], which found striking applications in recent years to different PDEs (see for instance [5–7, 14,18,19] and the references quoted therein). As such, our proof would probably be less constructive with respect to the strategies of [2,23,24], where an initial datum for which nonuniqueness is expected is described fairly explicitly as well as the mechanism for the creation of two different singularities. Conversely, the latter approaches see the drawbacks described above and are by no means "generic" in the initial data, whereas it is known (see for instance [8,21]) that convex integration methods yield not only the lack of uniqueness/smoothness for certain specific initial data, but also that solutions are typical (in the Baire category sense).

1.1. Strategy of proof. The guiding thread of this construction is an iterative procedure, where one starts from a solution (u_q, p_q, R_q) of the Euler equations with an error term in the right-hand side, namely

$$\begin{cases} \partial_t u_q + \operatorname{div}(u_q \otimes u_q) + \nabla p_q = \operatorname{div} R_q \\ \operatorname{div} u_q = 0, \end{cases} \tag{3}$$

and iteratively corrects this error by adding a fastly oscillating perturbation to the approximate solution. The nonlinear interaction of this perturbation with itself generates a resonance which allows for the cancellation of the previous error; the other terms are mainly seen as new error terms, with smaller size with respect to the previous error. More precisely, we define the new solution $(u_{q+1}, p_{q+1}, R_{q+1})$ by setting

$$u_{q+1}(x) = u_q(x) + a_q(x) w_{\lambda_{q+1}}(x), \quad w_{\lambda_{q+1}}(x) := w_{q+1}(\lambda_{q+1} x), \quad \lambda_{q+1} \in \mathbb{Z},$$

where $\lambda_{q+1} \gg \lambda_q$ is a higher frequency with respect to the typical frequencies in u_q, w_{q+1} is called building block of the construction and enjoys suitable integrability properties, a_q is a slowly varying coefficient.

The cancellation of error happens because the low frequency term in $a_q^2 w_{\lambda_{q+1}} \otimes w_{\lambda_{q+1}}$ satisfies

$$a_q^2 \int w_{\lambda_{q+1}} \otimes w_{\lambda_{q+1}} \sim R_q \, .$$

This forces us to require that

$$\begin{aligned} \int_{\mathbb{T}^2} |w_{q+1}|^2 &\sim 1 \\ a_q^2 &\sim |R_q|. \end{aligned} \tag{4}$$

On the contrary, we wish to control the quantity $\|Du_q\|_{L^{1,\infty}}$ and to this end we need

$$\|D(u_{q+1} - u_q)\|_{L^{1,\infty}} \sim \|a_q\|_{L^\infty} \|\lambda_{q+1} w_{\lambda_{q+1}}\|_{L^{1,\infty}},$$

arbitrarily small. This will be achieved by designing a new family of intermittent building blocks with gradients small in Lorentz spaces, see Sect. 1.3 below.

1.2. Limitations of current convex integration schemes. We now justify why, with the current method, getting a vorticity in L^p, $p \geq 1$, cannot be expected. The new error R_{q+1}, generated after correcting the old error R_q with the low frequency term in $a_q^2 w_{\lambda_{q+1}} \otimes w_{\lambda_{q+1}}$, contains the high frequencies of $a_q^2 w_{\lambda_{q+1}} \otimes w_{\lambda_{q+1}}$, hence its size is at least

$$\begin{aligned} \|R_{q+1}\|_{L^1} &\geq \left\| \nabla(a_q^2) \mathrm{div}^{-1} \Big(w_{\lambda_{q+1}} \otimes w_{\lambda_{q+1}} - \int w_{\lambda_{q+1}} \otimes w_{\lambda_{q+1}} \Big) \right\|_{L^1} \\ &\geq \lambda_{q+1}^{-1} \|\nabla R_q\|_{L^1} \geq \lambda_{q+1}^{-1} \lambda_q \|R_q\|_{L^1}. \end{aligned}$$

Here we used (4), i.e. $a^2 \sim |R_q|$ and that the intermittent term $w_{\lambda_{q+1}}$ has unitary norm in L^2. In particular, we have that $\|R_q\|_{L^1} \lambda_q$ is a nondecreasing sequence in q and hence

$$\|R_q\|_{L^1} \gtrsim \lambda_q^{-1}. \tag{5}$$

In order to control $\|\nabla u_q\|_{L^1}$ we need $\|\nabla(u_{q+1} - u_q)\|_{L^1}$ arbitrarily small, but the Sobolev inequality gives that

$$\|\nabla w_{\lambda_{q+1}}\|_{L^1} = \lambda_{q+1} \|\nabla w_{q+1}\|_{L^1} \gtrsim \lambda_{q+1} \|w_{q+1}\|_{L^2} \sim \lambda_{q+1}.$$

Hence,

$$\|\nabla(u_{q+1} - u_q)\|_{L^1} \sim \|a_q\|_{L^\infty} \|\lambda_{q+1} w_{\lambda_{q+1}}\|_{L^1} \gtrsim \lambda_{q+1} \sqrt{\|R_q\|_{L^1}} \gtrsim \lambda_{q+1}^{1/2}, \tag{6}$$

where we used (5) in the last step.

In particular, with the current way to cancel the error in the iteration, we cannot expect $\nabla u \in L^1$.

Note that the inequality (6), where we end up with a term of size $\sim \sqrt{\lambda_{q+1}}$, is in agreement with a known limitation of the convex integration scheme, see [5, Section 2.4.1]. In fact, it is shown there that the quadratic error term allows to control at most half of the derivative of the solution u. The only way to overcome this is to exploit the intermittency, but in our context we cannot because of the Sobolev inequality.

1.3. Building blocks with Lorentz integrability. To push the convex integration scheme to its boundaries and obtain $X = L^{1,\infty}$, we need to introduce a new family of building blocks $\{W_i\}_{i=1,\dots,4}$. The latter is the most important novelty of this paper, and its construction requires a new idea. In a nutshell, we design W_i so that its atomic decomposition, as a Lorentz function, is made up of "almost solutions" to the Euler equations. To this aim, we bundle together a family of intermittent jets [4] with different sizes and characteristic velocities. This structure allows sharpening the intermittency mechanism reaching the critical $L^{1,\infty}$ integrability of ∇W_i.

To put forward this idea, there are several technical challenges to overcome, let us mention a few.

The high velocity of each jet, needed to force the bundle to almost solve the Euler equations, makes the term $\partial_t W_i$ big. The latter should be treated as an error, hence we have to make its anti-divergence small. To do so, we exploit a special structure: we build the profiles of our jets in such a way that $\partial_t W_i = \operatorname{div}(A_i)$ where A_i is a small symmetric potential.

The bundle structure and the 2-dimensional constraint make it very difficult to keep the supports of W_i disjoint in space-time. It requires a new, delicate, combinatorial argument.

We refer the reader to Sect. 4 for the precise construction and more explanations on our choice of building blocks.

Remark 1.3. The proof of Theorem 1.1 is flexible enough, due to the exponential convergence of the iterative sequence, to give $\omega \in L^{1,q}$ for some $q \gg 1$. A technical refinement of the current proof, based on Remark 4.4, would give $q > 4$.

2. Iteration and Euler–Reynolds System

We consider the system of equations (3) in $[0, 1] \times \mathbb{T}^2$, where R is a traceless symmetric tensor.

As already remarked, our solution to (1) is obtained by passing to the limit solutions of (3) with suitable constraints on u and R. The latter are built by means of an iterative procedure based on the following.

Proposition 2.1. *There exists $M > 0$ such that the following holds. For any smooth solution (u_0, p_0, R_0) of (3), there exists another smooth solution (u_1, p_1, R_1) of (3) such that*

(i) $\|R_1\|_{L^\infty(L^1)} \le \frac{1}{3}\|R_0\|_{L^\infty(L^1)}$;
(ii) $\|u_1 - u_0\|_{C^0(L^2)} + \|\operatorname{curl}(u_1 - u_0)\|_{C^0(L^{1,\infty})} \le M\|R_0\|_{L^\infty(L^1)}$;

(iii) *for any* $s \in (0,1)$, $p \in (1, \frac{2}{1+s})$ *there exists* $c(p,s) > 0$ *s.t.*

$$\|D^s(u_1 - u_0)\|_{C^0(L^p)} \le (M\|R_0\|_{L^\infty(L^1)})^{c(p,s)};$$

(iv) *if* $R_0(\cdot,t) = 0$ *in* $[0,t_0]$, *then* $R_1(\cdot,t) = 0$ *and* $u_1(\cdot,t) = u_0(\cdot,t)$ *in* $[0,t_0/2]$.

Proof of Theorem 1.1, given Proposition 2.1. Fix $\lambda > 0$. We start the iteration scheme with

$$u_0(x,t) := \chi(t)\sin(x_2\lambda)e_1$$

where $\chi \in C_c^\infty([0,1])$, $\chi = 0$ in $[0,1/2]$ and $\chi = 1$ in $[3/2,1]$. Notice that $-\operatorname{div} R_0 = \chi'(t)\sin(x_2\lambda)e_1 + \nabla p$, hence we can choose a traceless symmetric tensor R_0 such that $\|R_0\|_{L^1} \le C\lambda^{-1}$.

Applying iteratively Proposition 2.1 with $t_0 = 1/2$ we build a sequence $\{(u_n, p_n, R_n) : n \in \mathbb{N}\}$ of smooth solutions to (1) such that, for any $n \ge 0$, it holds

$$\begin{aligned}\|R_n\|_{L^\infty(L^1)} \le C3^{-n}\lambda^{-1}, \quad \|u_{n+1} - u_n\|_{C^0(L^2)} &+ \|\operatorname{curl}(u_{n+1} - u_n)\|_{C^0(L^{1,\infty})} \\ &\le CM3^{-n+1}\lambda^{-1},\end{aligned}$$

and $u_n(\cdot,t) = 0$ for any $t \in [0, 2^{-n-1}]$. Moreover, for any $s \in (0,1)$ and $p \in \left(1, \frac{2}{1+s}\right)$ it holds

$$\|D^s(u_{n+1} - u_n)\|_{C^0(L^p)} \le C(M,\lambda)3^{-nc(p,s)}. \tag{7}$$

It follows that $R_n \to 0$ in $L^\infty(L^1)$ and $u_n \to u$ in $C^0(L^2)$, where u satisfies the assumptions of Theorem 1.1. Moreover $u \in C^0(W^{s,p})$ for $s \in (0,1)$ and $p \in (1, \frac{2}{1+s})$ as a consequence of (7).

We now prove that there exists $\operatorname{curl} u \in C^0(L^{1,\infty})$ with the property that $\|\operatorname{curl} u_n - \operatorname{curl} u\|_{C^0(L^{1,\infty})} \to 0$ as $n \to \infty$. A bit of care is needed since only the weak triangle inequality $\|f+g\|_{L^{1,\infty}} \le 2\|f\|_{L^{1,\infty}} + 2\|g\|_{L^{1,\infty}}$ holds true. However, the latter is enough for our purposes

$$\begin{aligned}\|\operatorname{curl} u_N\|_{C^0(L^{1,\infty})} &= \left\|\operatorname{curl} u_0 + \operatorname{curl}\left(\sum_{n=0}^{N-1} u_{n+1} - u_n\right)\right\|_{C^0(L^{1,\infty})} \\ &\le 2\|\operatorname{curl} u_0\|_{C^0(L^{1,\infty})} + \sum_{n=0}^{N-1} 2^{n+1}\|\operatorname{curl}(u_{n+1} - u_n)\|_{C^0(L^{1,\infty})} \\ &\le 2\|\operatorname{curl} u_0\|_{C^0(L^{1,\infty})} + CM\lambda^{-1}\sum_{n=0}^{N-1} 2^{n+1}3^{-n+1} < \infty,\end{aligned}$$

hence setting $\operatorname{curl} u := \operatorname{curl} u_0 + \sum_{n=0}^\infty (\operatorname{curl} u_{n+1} - \operatorname{curl} u_n)$ we get the sought conclusion. □

The remaining part of this note is devoted to the proof of Proposition 2.1. In Sect. 4 we introduce the building blocks of our construction, in Sect. 5 we use them to define the perturbation $u_1 - u_0$, finally in Sect. 6, we introduce the new error term R_1 and show that it can be made arbitrarily small.

3. Preliminary Lemmas

3.1. Lorentz spaces. For every measurable function $f : \mathbb{T}^d \to \mathbb{R}$ we recall the definition

$$\|f\|_{L^{r,q}} := r^{1/q} \left\| \lambda \mathscr{L}^d(\{|f| \geq \lambda\})^{1/r} \right\|_{L^q\left((0,\infty), \frac{d\lambda}{\lambda}\right)},$$

(see e.g. [13]) and we define the Lorentz space $L^{r,q}$ with $r \in [1, \infty)$, $q \in [1, \infty]$, as the space of those functions f such that $\|f\|_{L^{r,q}} < \infty$. Note that, in spite of the notation, $\|\cdot\|_{L^{r,q}}$ is in general not a norm but for $(r, q) \neq (1, \infty)$ the topological vector space $L^{r,q}$ is locally convex and there exists a norm $|||\cdot|||_{r,q}$ which is equivalent to $\|\cdot\|_{L^{r,q}}$ in the sense that the inequality $C^{-1}|||f|||_{r,q} \leq \|f\|_{L^{r,q}} \leq C|||f|||_{r,q}$ holds.

3.2. Improved Hölder inequality. We recall the following improved Hölder inequality, stated as in [18, Lemma 2.6] (see also [3, Lemma 3.7]). If $\lambda \in \mathbb{N}$ and $f, g : \mathbb{T}^2 \to \mathbb{R}$ are smooth functions, then we have

$$\|f(x)g(\lambda x)\|_{L^p} \leq \|f\|_{L^p}\|g\|_{L^p} + C(p)\lambda^{-1/p}\|f\|_{C^1}\|g\|_{L^p} \quad \text{for any } p \in [1, \infty]. \tag{8}$$

When $\fint_{\mathbb{T}^2} g = 0$, then

$$\left| \int_{\mathbb{T}^2} f(x)g(\lambda x)\, dx \right| \leq C\lambda^{-1}\|f\|_{C^1}\|g\|_{L^1}. \tag{9}$$

3.3. Anti-divergence operators. Let now us introduce the anti-divergence operator

$$\begin{aligned} &\mathcal{R}_0 : C^\infty(\mathbb{T}^2; \mathbb{R}^2) \to C^\infty(\mathbb{T}^2; \mathrm{Sym}_2),\\ &\mathcal{R}_0(v) := (D\Delta^{-1} + (D\Delta^{-1})^T - I \cdot \operatorname{div} \Delta^{-1})\left(v - \fint_{\mathbb{T}^2} v\right). \end{aligned}$$

Here Sym_2 denotes the space of symmetric matrices in $\mathbb{R}^2$. It is simple to check that $\operatorname{div}(\mathcal{R}_0(v)) = v - \fint_{\mathbb{T}^2} v$, and that $D\mathcal{R}_0$ is a Calderon-Zygmund operator, in particular it holds

$$\|\mathcal{R}_0(v)\|_{L^p} \leq C\|\Delta^{-1/2}v\|_{L^p} \quad \text{for any } p \in (1, \infty)\,, \tag{10}$$

$$\|\mathcal{R}_0(v)\|_{L^p} \leq C(p)\|v\|_{L^p} \quad \text{for any } p \in [1, \infty]\,. \tag{11}$$

Notice that (10) and (11) imply

$$\|\mathcal{R}_0(v_\lambda)\|_{L^p} \leq C(p)\lambda^{-1}\|v\|_{L^p} \quad \text{for any } p \in [1, \infty], \tag{12}$$

where $v_\lambda(x) := v(\lambda x)$ for some $\lambda \in \mathbb{N}$. The latter is immediate for $p \in (1, \infty)$, since

$$\|\mathcal{R}_0(v_\lambda)\|_{L^p} \leq C\|\Delta^{-1/2}v_\lambda\|_{L^p} \leq C\lambda^{-1}\|v\|_{L^p},$$

in the case $p = 1$ and $p = \infty$ we need to take advantage of the Sobolev embedding theorem:

$$\begin{aligned} \|\mathcal{R}_0(v_\lambda)\|_{L^\infty} &= \lambda^{-1}\|\mathcal{R}_0(v)\|_{L^\infty} \leq C\lambda^{-1}\|\nabla\mathcal{R}_0(v)\|_{L^{3/2}}\\ &\leq C\lambda^{-1}\|v\|_{L^{3/2}} \leq C\lambda^{-1}\|v\|_{L^\infty} = C\lambda^{-1}\|v\|_{L^\infty}. \end{aligned}$$

The same argument applies to $\mathcal{R}_0^*$, the adjoint of $\mathcal{R}_0$, then case $p = 1$ follows by duality.

Lemma 3.1. *Let $\lambda \in \mathbb{N}$ and $f \in C^\infty(\mathbb{T}^2;\mathbb{R})$, $v \in C^\infty(\mathbb{T}^2;\mathbb{R}^2)$ with $\int_{\mathbb{T}^2} v = 0$, and $v_\lambda = v(\lambda x)$. If we set*

$$\mathcal{R}(f v_\lambda) = f\mathcal{R}_0 v_\lambda - \mathcal{R}_0\left(\nabla f \cdot \mathcal{R}_0 v_\lambda + \int_{\mathbb{T}^2} f v_\lambda\right) \in C^\infty(\mathbb{T}^2; Sym_2)$$

then,

$$\begin{aligned} &\mathrm{div}\mathcal{R}(f v_\lambda) = f v_\lambda - \int_{\mathbb{T}^2} f v_\lambda\,, \\ &\|\mathcal{R}(f v_\lambda)\|_{L^p} \leq C(p)\lambda^{-1}\|f\|_{C^1}\|v\|_{L^p} \quad \textit{for every } p \in [1,\infty]\,. \end{aligned} \tag{13}$$

Proof. The verification of $\mathrm{div}\,\mathcal{R}(f v_\lambda) = f v_\lambda - \int_{\mathbb{T}^2} f v_\lambda$ is immediate. To prove (13) we use (12) and (9):

$$\begin{aligned} &\|f\mathcal{R}_0 v_\lambda\|_{L^p} \leq \|f\|_{C^0}\|\mathcal{R}_0 v_\lambda\|_{L^p} \leq C\lambda^{-1}\|f\|_{C^0}\|v\|_{L^p}\,, \\ &\left\|\mathcal{R}_0\left(\nabla f \cdot \mathcal{R}_0 v_\lambda + \int_{\mathbb{T}^2} f v_\lambda\right)\right\|_{L^p} \\ &\quad \leq C\left\|\nabla f \cdot \mathcal{R}_0 v_\lambda + \int_{\mathbb{T}^2} f v_\lambda\right\|_{L^p} \leq C\lambda^{-1}\|f\|_{C^1}\|v\|_{L^p} + C\lambda^{-1}\|f\|_{C^1}\|v\|_{L^1}\,. \end{aligned}$$

□

Remark 3.2. The operator $\mathcal{R}$ can be also defined on scalar functions $f : \mathbb{T}^2 \to \mathbb{R}$, $v : \mathbb{T}^2 \to \mathbb{R}$ as

$$\mathcal{R}(f v_\lambda) = f\nabla\Delta^{-1}v_\lambda - \nabla\Delta^{-1}\left(\nabla f \cdot \mathcal{R}_0 v_\lambda + \int_{\mathbb{T}^2} f v_\lambda\right) \in C^\infty(\mathbb{T}^2;\mathbb{R}^2),$$

and arguing as in Lemma 3.1 we can easily show that $\mathrm{div}\mathcal{R}(f v_\lambda) = f v_\lambda - \int_{\mathbb{T}^2} f v_\lambda$ and

$$\|\mathcal{R}(f v_\lambda)\|_{L^p} \leq C(p)\lambda^{-1}\|f\|_{C^1}\|v\|_{L^p} \qquad \text{for every } p \in [1,\infty].$$

Lemma 3.3. *For any $a \in C^\infty(\mathbb{T}^2)$ and $A \in C^\infty(\mathbb{T};\mathbb{R}^{2\times 2})$ with $\int_{\mathbb{T}^2} A = 0$, it holds*

$$\|\mathcal{R}_0\mathcal{R}(\nabla a \cdot \mathrm{div}\, A)\|_{L^1} \leq C(\|a\|_{C^3})\|A\|_{L^1}. \tag{14}$$

Proof. Set $T(A) := \mathcal{R}(\nabla a \cdot \mathrm{div}\, A)$. By duality, it suffices to show that

$$\|T^*\mathcal{R}_0^*(B)\|_{L^\infty} \leq C(\|a\|_{C^3})\|B\|_{L^\infty},$$

where T^* and $\mathcal{R}_0^*$ denote the adjoint of T and $\mathcal{R}_0$, respectively. To this aim we employ the Sobolev embedding and the fact that $DT^*\mathcal{R}_0^*(B)$ maps L^p into L^p for any $p \in (1,\infty)$:

$$\|T^*\mathcal{R}_0^*(B)\|_{L^\infty} \leq C\|DT^*\mathcal{R}_0^*(B)\|_{L^3} \leq C(\|a\|_{C^3})\|B\|_{L^3} \leq C(\|a\|_{C^3})\|B\|_{L^\infty}.$$

□

4. Building Blocks

In this section we introduce the building blocks of our construction. They will be employed in Sect. 5 to define the principal term of $u_1 - u_0$ in Proposition 2.1.

Proposition 4.1. (Building blocks) *Set $\xi_1 := e_1$, $\xi_2 := e_2$, $\xi_3 := e_1 + e_2$ and $\xi_4 := e_1 - e_2$. Then, for any $\varepsilon > 0$ there exist $W_i^p, W_i^c, Q_i \in C^\infty((-1,1)\times\mathbb{T}^2;\mathbb{R}^2)$, $A_i \in C^\infty((-1,1)\times\mathbb{T}^2;\mathrm{Sym}_2)$ for $i = 1,\dots,4$, such that*

(i) $\mathrm{div}\,(W_i^p + W_i^c) = 0$, $\partial_t Q_i = \mathrm{div}\,(W_i^p \otimes W_i^p)$, *and* $\partial_t(W_i^p + W_i^c) = \mathrm{div}\,(A_i)$;
(ii) $\int_{\mathbb{T}^2} A_i = 0$, $\int_{\mathbb{T}^2} W_i^p = \int_{\mathbb{T}^2} W_i^c = 0$, *and W_i^p, W_i^c, A_i are λ^{-1}-periodic functions for some $\lambda \in \mathbb{Z}$ with $\lambda \geq \varepsilon^{-1}$;*
(iii) $\int_{\mathbb{T}^2} W_i^p \otimes W_i^p = \frac{\xi_i}{|\xi_i|} \otimes \frac{\xi_i}{|\xi_i|}$;
(iv) *the following estimates hold*

$$
\begin{aligned}
&\varepsilon\|W_i^p\|_{L^2} + \|W_i^p\|_{L^1} + \|W_i^c\|_{L^2} \leq \varepsilon\,,\\
&\|D(W_i^p + W_i^c)\|_{L^{1,\infty}} + \|Q_i\|_{L^2} + \|DQ_i\|_{L^{1,\infty}} + \|A_i\|_{L^1} \leq \varepsilon\,,\\
&\|D^s(W_i^p + W_i^c)\|_{L^p} + \|D^s Q_i\|_{L^p} \leq \varepsilon^{c(p,s)} \quad \text{for any } s \in (0,1) \text{ and } p \in \left(1, \frac{2}{1+s}\right);
\end{aligned}
\tag{15}
$$

(v) *for $i \neq i'$ the union of the supports of W_i^p, W_i^c, Q_i, is disjoint in space-time from the union of the supports of $W_{i'}^p$, $W_{i'}^c$, $Q_{i'}$.*

The velocity field W_i^p is the principal term, it has zero mean, high frequency $\lambda \geq \varepsilon^{-1}$, is controlled in the relevant norms (cf. (iv)), and satisfies the fundamental property (iii): the quadratic interaction $W_i^p \otimes W_i^p$ produces the lower order term $\frac{\xi_i}{|\xi_i|} \otimes \frac{\xi_i}{|\xi_i|}$. The latter, combined with slow coefficients $a_i \in C^\infty(\mathbb{T}^2)$, is used to cancel the error R_0 out. To achieve the crucial bound $\|DW_i^p\|_{L^{1,\infty}}$ we design the principal term as

$$
W_i^p(x,t) = W^p_{\xi_i,K,n_0}(x,t) := \frac{1}{K^{1/2}} \sum_{k=n_0+1}^{K+n_0} W^k_{(\xi_i)}(x,t), \tag{16}
$$

where $K, n_0 \gg 1$ are big parameters and ξ_i is one of the four directions appearing in the statement of Proposition 4.1. In a first stage, we build $W_i^p(x,t)$ for a fixed parameter i, ignoring the issue that, for different parameters, such functions will not have disjoint support; only in Sect. 4.6 we make sure to suitably time-translate them, making substantial use of their special structure, to guarantee that Proposition 4.1 (v) holds. The vector fields $W^k_{(\xi_i)}(x,t)$, $k = n_0+1,\dots,n_0+K$, are the 2-dimensional counterpart of the intermittent jets introduced in [4]. They have L^2 norm equal to 1, and are supported on disjoint balls of radius $2^{-k}r$, for some $r \ll 1$, which move in direction ξ_i with speed $\mu 2^k$, where $\mu \gg 1$. The fast time translation is used to make $W^k_{(\xi)}$ "almost divergence free" and "almost solutions to the Euler equations". In more rigorous terms, it means that there exist vector fields $(W^k_{(\xi)})^c$, $(Q^k_{(\xi)})^c$, that are smaller than $W^k_{(\xi)}$ satisfying $\mathrm{div}\,(W^k_{(\xi)} + (W^k_{(\xi)})^c) = 0$ and $\partial_t(Q^k_{(\xi)})^c = \mathrm{div}\,(W^k_{(\xi)} \otimes W^k_{(\xi)})$. The vector fields W_i^c and Q_i are defined bundling together $(W^k_{(\xi)})^c$ and $(Q^k_{(\xi)})^c$ as we did in (16).

Another important property we need is that $W_i^p \otimes W_j^p = 0$ when $i \neq j$. It is ensured by (v) in Proposition 4.1, which builds upon a delicate combinatorial lemma presented in Sect. 4.6.

We finally explain the role of the matrix A_i in our construction. Let us begin by noticing that the principal term W_i^p has big time derivative, being fast translating in time. Hence, the term $\partial_t W_i^p$ cannot be treated as an error. To overcome this difficulty we impose an extra structure on W_i^p and W_i^c. We construct them in order to have the identity $\partial_t(W_i^p + W_i^c) = \operatorname{div}(A_i)$, for some symmetric matrix A_i which has small L^1-norm. The latter can be added to the new error term R_1.

4.1. General notation. Given a velocity field $u := (u_1, u_2) : \mathbb{R}^2 \to \mathbb{R}^2$ we write

$$u^\perp := (-u_2, u_1), \quad \operatorname{curl}(u) := \partial_1 u_2 - \partial_2 u_1 \quad \operatorname{div}(u) := \partial_1 u_1 + \partial_2 u_2.$$

Let us fix $r_\perp \ll r_\parallel \ll 1$ and $k \in \mathbb{N}, k \geq 1$. We adopt the following convention: given any $\rho : \mathbb{R} \to \mathbb{R}$ supported in $(-1, 1)$ we write

$$\rho^k_{r_\perp}(x) := \left(\frac{1}{2^{-k} r_\perp}\right)^{1/2} \rho\left(\frac{x - 2^{2-k} r_\perp}{2^{-k} r_\perp}\right),$$

$$\rho^k_{r_\parallel}(x) := \left(\frac{1}{2^{-k} r_\parallel}\right)^{1/2} \rho\left(\frac{x}{2^{-k} r_\parallel}\right).$$

Notice that $\operatorname{supp}(\rho^k_{r_\perp}) \subset (3 \cdot 2^{-k} r_\perp, 5 \cdot 2^{-k} r_\perp)$, in particular

$$\operatorname{supp}(\rho^k_{r_\perp}) \cap \operatorname{supp}(\rho^{k'}_{r_\perp}) = \emptyset \quad \text{for } k \neq k',$$

and

$$\bigcup_{k \geq 1} \operatorname{supp}(\rho^k_{r_\perp}) \subset (0, 5 r_\perp 2^{-n_0}).$$

With a slight abuse of notation we keep denoting by $\rho^k_{r_\perp}, \rho^k_{r_\parallel} : \mathbb{T} \to \mathbb{R}$ their periodized version.

4.2. Construction of the principal block. We consider $\Phi, \psi : \mathbb{R} \to \mathbb{R}$ supported in $(-1, 1)$, we set $\phi := -\Phi'''$ and assume $\int \psi^2 = \int \phi^2 = 1$. Given $r_\perp \ll r_\parallel \ll 1$ and $k \in \mathbb{N}$ we have

$$\begin{aligned} \operatorname{supp}(\phi^k_{r_\perp}) \cap \operatorname{supp}(\phi^{k'}_{r_\perp}) &= \operatorname{supp}((\Phi')^k_{r_\perp}) \cap \operatorname{supp}((\Phi')^{k'}_{r_\perp}) \\ &= \operatorname{supp}((\Phi'')^k_{r_\perp}) \cap \operatorname{supp}((\Phi'')^{k'}_{r_\perp}) = \emptyset \quad \text{for } k \neq k', \end{aligned}$$

and

$$\bigcup_k \operatorname{supp}(\phi^k_{r_\perp}), \bigcup_k \operatorname{supp}(\Phi^k_{r_\perp}) \subset (0, 5 r_\perp 2^{-n_0}). \tag{17}$$

We periodize $(\Phi')^k_{r_\perp}, (\Phi'')^k_{r_\perp}, \phi^k_{r_\perp}, \psi^k_{r_\parallel}$ keeping the same notation.

Given a vector $\xi \in \mathbb{Q}^2$, and parameters $\lambda, \mu \gg 1$ we set

$$\begin{aligned} (\Phi')^k_{(\xi)}(x) &:= (\Phi')^k_{r_\perp}(\lambda x \cdot \xi^\perp), \quad (\Phi'')^k_{(\xi)}(x) := (\Phi'')^k_{r_\perp}(\lambda x \cdot \xi^\perp), \\ \phi^k_{(\xi)}(x) &:= \phi^k_{r_\perp}(\lambda x \cdot \xi^\perp), \\ \psi^k_{(\xi)}(x, t) &:= \psi^k_{r_\parallel}(\lambda(x \cdot \xi + \mu 2^k t)), \\ W^k_{(\xi)}(x, t) &:= \frac{\xi}{|\xi|} \psi^k_{(\xi)}(x, t) \phi^k_{(\xi)}(x). \end{aligned}$$

We finally fix $K, n_0 \in \mathbb{N}$, and define the principal block

$$W^p_{\xi,K,n_0}(x,t) := \frac{1}{K^{1/2}} \sum_{k=n_0+1}^{K+n_0} W^k_{(\xi)}(x, t+t_k), \tag{18}$$

where t_k are time translations that will be chosen later. The following fundamental identity holds

$$\begin{aligned} \int_{\mathbb{T}^2} W^p_{\xi,K,n_0} \otimes W^p_{\xi,K,n_0} &= \frac{1}{K} \sum_{k=n_0+1}^{K+n_0} \int_{\mathbb{T}^2} W^k_{(\xi)} \otimes W^k_{(\xi)} \\ &= \frac{\xi}{|\xi|} \otimes \frac{\xi}{|\xi|} \int_{\mathbb{T}^2} (\psi^k_{(\xi)} \phi^k_{(\xi)})^2 = \frac{\xi}{|\xi|} \otimes \frac{\xi}{|\xi|}. \end{aligned} \tag{19}$$

4.3. Correction of the divergence. Observe that

$$\operatorname{div} W^k_{(\xi)}(x,t) = \frac{\lambda}{2^{-k} r_\parallel} (\dot{\psi})^k_{(\xi)}(x,t) \phi^k_{(\xi)}(x).$$

Setting

$$(W^k_{(\xi)})^c(x,t) := \frac{r_\perp}{r_\parallel} \frac{\xi^\perp}{|\xi|} (\dot{\psi})^k_{(\xi)}(x,t) (\Phi'')^k_{(\xi)}(x),$$

and using the identity $2^{-k} r_\perp \partial_{x_1} (\Phi'')^k_{r_\perp} = -\phi^k_{r_\perp}$ we get $\operatorname{div}(W_{(\xi)} + W^c_{(\xi)}) = 0$.

To correct the divergence of W_{ξ,K,n_0} we introduce

$$W^c_{\xi,K,n_0}(x,t) := \frac{1}{K^{1/2}} \sum_{k=n_0+1}^{K+n_0} (W^k_{(\xi)})^c(x, t+t_k),$$

and set

$$W_{\xi,K,n_0}(x,t) := W^p_{\xi,K,n_0}(x,t) + W^c_{\xi,K,n_0}(x,t).$$

4.4. Time correction. Let us now set

$$Q^k_{(\xi)}(x,t) := \frac{1}{2^k \mu} \xi (\psi^k_{(\xi)}(x, t+t_k) \phi^k_{(\xi)}(x))^2,$$

and observe that

$$\begin{aligned} \operatorname{div}(W^k_{(\xi)} \otimes W^k_{(\xi)}) &= 2(W^k_{(\xi)} \cdot \nabla \psi^k_{(\xi)}) \phi^k_{(\xi)} \frac{\xi}{|\xi|} = \frac{1}{2^k \mu} 2 \left(W^k_{(\xi)} \cdot \partial_t \psi^k_{(\xi)} \right) \phi^k_{(\xi)} \frac{\xi}{|\xi|} \\ &= \frac{1}{2^k \mu} \partial_t \left(\psi^k_{(\xi)} \phi^k_{(\xi)} \right)^2 \frac{\xi}{|\xi|} = \partial_t Q^k_{(\xi)}. \end{aligned}$$

Hence

$$\operatorname{div}(W^p_{\xi,K,n_0} \otimes W^p_{\xi,K,n_0}) = \frac{1}{K} \sum_{k=n_0+1}^{K+n_0} \operatorname{div}(W^k_{(\xi)} \otimes W^k_{(\xi)}) = \partial_t \left(\frac{1}{K} \sum_{k=n_0+1}^{K+n_0} Q^k_{(\xi)} \right). \tag{20}$$

The time corrector is defined as

$$Q_{\xi,K,n_0}(x,t) := \frac{1}{K}\sum_{k=n_0+1}^{K+n_0} Q^k_{(\xi)}(x,t).$$

4.5. Estimates on building blocks. In this section we collect the relevant estimates on the building blocks. Given $N \geq 0$ we write $D^N = D^{\lfloor N\rfloor}\Delta^{s/2}$ where $\lfloor N\rfloor$ is the integer part of N, $s := N - \lfloor N\rfloor$ and $D^{\lfloor N\rfloor}$ is the standard derivative operator.

Lemma 4.2. *For any $N \in [0,\infty)$, $M \geq 0$ integer and $p \in [1,\infty]$ there exists $C = C(N, M, p, |\xi|, \Phi, \psi) > 0$ such that the following hold.*

$$\|D^N\partial_t^M \psi^k_{(\xi)}\|_{L^p(\mathbb{T})} \leq C2^{k(N+2M+1/2-1/p)} r_\parallel^{1/p-1/2}\left(\frac{\lambda}{r_\parallel}\right)^N\left(\frac{\lambda\mu}{r_\parallel}\right)^M,$$

$$\begin{aligned}&\|D^N(\Phi')^k_{(\xi)}\|_{L^p(\mathbb{T})} + \|D^N(\Phi'')^k_{(\xi)}\|_{L^p(\mathbb{T})} + \|D^N\phi^k_{(\xi)}\|_{L^p(\mathbb{T})}\\ &\quad\leq C2^{k(N+1/2-1/p)} r_\perp^{1/p-1/2}\left(\frac{\lambda}{r_\perp}\right)^N,\end{aligned}$$

$$\begin{aligned}&\|D^N\partial_t^M W^k_{(\xi)}\|_{L^p(\mathbb{T}^2)} + \frac{r_\parallel}{r_\perp}\|D^N\partial_t^M (W^k_{(\xi)})^c\|_{L^p(\mathbb{T}^2)}\\ &\quad\leq C2^{k(N+2M+1-2/p)}(r_\parallel r_\perp)^{1/p-1/2}\left(\frac{\lambda}{r_\perp}\right)^N\left(\frac{\lambda\mu}{r_\parallel}\right)^M,\end{aligned}$$

$$2^k\mu\|D^N\partial_t^M Q^k_{(\xi)}\|_{L^p(\mathbb{T}^2)} \leq C2^{k(N+2M+2-2/p)}(r_\parallel r_\perp)^{1/p-1}\left(\frac{\lambda}{r_\perp}\right)^N\left(\frac{\lambda\mu}{r_\parallel}\right)^M.$$

The proof of Lemma 4.2 is a simple computation, so we omit it. Let us draw some useful consequence. Summing on k and reminding that then terms in the sum in (18) have disjoint support, we get

$$\|W^p_{\xi,K,n_0}\|_{L^2(\mathbb{T}^2)} + \frac{r_\parallel}{r_\perp}\|W^c_{\xi,K,n_0}\|_{L^2(\mathbb{T}^2)} \leq C \tag{21}$$

(in particular, this says that the principal part is much smaller than the corrector),

$$\|Q_{\xi,K,n_0}\|_{L^2(\mathbb{T}^2)} \leq \frac{C}{\mu(r_\parallel r_\perp)^{1/2}}, \tag{22}$$

and

$$\left\|W^p_{\xi,K,n_0}\right\|_{L^p(\mathbb{T}^2)} + \frac{r_\parallel}{r_\perp}\|W^c_{\xi,K,n_0}\|_{L^p(\mathbb{T}^2)} \leq C\frac{(r_\perp r_\parallel)^{1/p-1/2}}{K^{1/2}}, \quad \text{for any } p \in [1,2). \tag{23}$$

Moreover, for $s \in (0,1)$ and $p < \frac{2}{1+s}$ it holds

$$\left\|D^s W^k_{(\xi)}\right\|_{L^p(\mathbb{T}^2)} + \frac{r_\parallel}{r_\perp}\|D^s(W^k_{(\xi)})^c\|_{L^p(\mathbb{T}^2)} \leq C2^{-k\gamma}(r_\parallel r_\perp)^{1/p-1/2}\left(\frac{\lambda}{r_\perp}\right)^s,$$

$$\mu\|D^s Q^k_{(\xi)}\|_{L^p(\mathbb{T}^2)} \leq C2^{-k\gamma}(r_\parallel r_\perp)^{1/p-1}\left(\frac{\lambda}{r_\perp}\right)^s,$$

where $\gamma := -s - 1 + 2/p > 0$. In particular

$$\left\| D^s W^p_{\xi,K,n_0} \right\|_{L^p(\mathbb{T}^2)} + \frac{r_\parallel}{r_\perp} \| D^s W^c_{\xi,K,n_0} \|_{L^p(\mathbb{T}^2)} \le C 2^{-n_0\gamma} K^{-1/2} (r_\parallel r_\perp)^{1/p-1/2} \left(\frac{\lambda}{r_\perp} \right)^s , \tag{24}$$

$$\| D^s Q_{\xi,K,n_0} \|_{L^p(\mathbb{T}^2)} \le C 2^{-n_0\gamma} \mu^{-1} K^{-1} (r_\parallel r_\perp)^{1/p-1} \left(\frac{\lambda}{r_\perp} \right)^s . \tag{25}$$

Lemma 4.3. (Lorentz estimates) *There exists* $C = C(|\xi|, \Phi, \psi) > 0$ *such that*

$$\| D W_{\xi,K,n_0} \|_{L^{1,\infty}} \le C \frac{\lambda}{K^{1/2}} \left(\frac{r_\parallel}{r_\perp} \right)^{1/2} ,$$
$$\| D Q_{\xi,K,n_0} \|_{L^{1,\infty}} \le C \frac{\lambda}{\mu r_\perp K} .$$

Proof. Observe that

$$\begin{aligned}
|D W^k_{(\xi)}| &= \lambda 2^k \left| r_\parallel^{-1} \frac{\xi}{|\xi|} \otimes \frac{\xi}{|\xi|} (\psi')^k_{(\xi)}(x,t) \phi^k_{(\xi)}(x) + r_\perp^{-1} \frac{\xi}{|\xi|} \otimes \frac{\xi^\perp}{|\xi|} \psi^k_{(\xi)}(x,t) (\phi')^k_{(\xi)}(x) \right| \\
&\le \lambda 2^k r_\perp^{-1} (|(\psi')^k_{(\xi)}(x,t)||\phi^k_{(\xi)}(x)| + |\psi^k_{(\xi)}(x,t)||(\phi')^k_{(\xi)}(x)|) \\
&= \lambda \left(\frac{r_\parallel}{r_\perp} \right)^{1/2} \frac{1}{2^{-k}(r_\perp r_\parallel)^{1/2}} (|(\psi')^k_{(\xi)}(x,t)||\phi^k_{(\xi)}(x)| + |\psi^k_{(\xi)}(x,t)||(\phi')^k_{(\xi)}(x)|) \\
&:= \lambda \left(\frac{r_\parallel}{r_\perp} \right)^{1/2} \Omega^k_1(x,t),
\end{aligned}$$

and similarly

$$|D Q^k_{(\xi)}| \le \frac{\lambda}{\mu r_\perp} \Omega^k_2(x,t) ,$$

where for $i = 1, 2$

$$\begin{aligned}
|\Omega^k_i| \le C 2^{2k} (r_\perp r_\parallel)^{-1}, \quad \mathscr{L}^2(\operatorname{supp}(\Omega^k_i)) \le C 2^{-2k} r_\perp r_\parallel, \\
\operatorname{supp}(\Omega^k_i) \cap \operatorname{supp}(\Omega^{k'}_i) = \emptyset, \quad \text{for } k \ne k'.
\end{aligned} \tag{26}$$

Let us now fix $s \ge 1$ and k_* the smallest integer satisfying $k_* \ge n_0 + 1$ and $C 2^{2k_*} \ge s K^{1/2} r_\perp r_\parallel$. It holds

$$\begin{aligned}
\mathscr{L}^2 \left(\left\{ \frac{1}{K^{1/2}} \sum_{k=n_0+1}^{K+n_0} \Omega^k_1 \ge s \right\} \right) &= \sum_{k=n_0+1}^{K+n_0} \mathscr{L}^2(\{\Omega^k_1 \ge s K^{1/2}\}) \\
&\le \sum_{k=k_*}^{K+n_0} \mathscr{L}^2(\{\Omega^k_1 \ge s K^{1/2}\}).
\end{aligned}$$

From (26) and the choice of k_* we get

$$\sum_{k=k_*}^{K+n_0} \mathscr{L}^2(\{\Omega_k \ge s K^{1/2}\}) \le \sum_{k=k_*}^{K+n_0} C 2^{-2k} r_\perp r_\parallel \le \frac{C}{s K^{1/2}} \sum_{k \ge k_*} 2^{2k_* - 2k} \le \frac{C}{s K^{1/2}},$$

hence

$$\|DW^p_{\xi,K,n_0}\|_{L^{1,\infty}} \leq \lambda \left(\frac{r_\|}{r_\perp}\right)^{1/2} \left\| \frac{1}{K^{1/2}} \sum_{k=n_0+1}^{K+n_0} \Omega^k \right\|_{L^{1,\infty}} \leq C^2 \frac{\lambda}{K^{1/2}} \left(\frac{r_\|}{r_\perp}\right)^{1/2},$$

the estimate on $\|DW^c_{\xi,K,n_0}\|_{L^{1,\infty}}$ can be obtained following the same strategy. An analogous argument gives

$$\mathscr{L}^2 \left(\left\{ \frac{1}{K} \sum_{k=n_0+1}^{K+n_0} \Omega_2^k \geq s \right\} \right) \leq \frac{C}{sK},$$

yielding

$$\|DQ_{\xi,K,n_0}\|_{L^{1,\infty}} \leq C \frac{\lambda}{\mu r_\perp} \left\| \frac{1}{K} \sum_{k=n_0+1}^{K+n_0} \Omega^k \right\|_{L^{1,\infty}} \leq C^2 \frac{\lambda}{\mu r_\perp K}.$$

□

Remark 4.4. It is not hard to prove the following extension of Lemma 4.3. For any $q \geq 1$ it holds

$$\|DW_{\xi,K,n_0}\|_{L^{1,q}} \leq C \frac{\lambda}{K^{1/2-1/q}} \left(\frac{r_\|}{r_\perp}\right)^{1/2},$$
$$\|DQ_{\xi,K,n_0}\|_{L^{1,\infty}} \leq C \frac{\lambda}{\mu r_\perp K^{1-1/q}}.$$

Lemma 4.5. *There exists a smooth* λ*-periodic function* $A_{\xi,K,n_0} : \mathbb{T}^2 \to \mathrm{Sym}_2$ *such that*

$$\partial_t W_{\xi,K,n_0} = \operatorname{div}(A_{\xi,K,n_0}), \tag{27}$$
$$\|A_{\xi,K,n_0}\|_{L^1} \leq C(|\xi|, \Phi, \psi)\mu K^{1/2} r_\perp^{3/2} r_\|^{-1/2}. \tag{28}$$

Proof. Setting

$$\begin{aligned} A_{(\xi),k} &:= -2^k \left(\frac{r_\perp}{r_\|}\right) \mu \left(\left(\frac{\xi}{|\xi|} \otimes \frac{\xi^\perp}{|\xi|} + \frac{\xi^\perp}{|\xi|} \otimes \frac{\xi}{|\xi|} \right) (\psi')^k_{(\xi)} (\Phi'')^k_{(\xi)} \right. \\ &\quad \left. + \frac{r_\perp}{r_\|} \frac{\xi^\perp}{|\xi|} \otimes \frac{\xi^\perp}{|\xi|} (\psi'')^k_{(\xi)} (\Phi')^k_{(\xi)} \right), \\ A^c_{(\xi),k} &:= 2^k \left(\frac{r_\perp}{r_\|}\right)^2 \mu \left(\left(\frac{\xi}{|\xi|} \otimes \frac{\xi^\perp}{|\xi|} + \frac{\xi^\perp}{|\xi|} \otimes \frac{\xi}{|\xi|} \right) (\psi')^k_{(\xi)} (\Phi')^k_{(\xi)} \right. \\ &\quad \left. - \frac{r_\perp}{r_\|} \frac{\xi^\perp}{|\xi|} \otimes \frac{\xi^\perp}{|\xi|} (\psi'')^k_{(\xi)} (\Phi)^k_{(\xi)} \right), \end{aligned}$$

it holds

$$\begin{aligned} \partial_t W^k_{(\xi)} &= 2^{2k} \mu \lambda r_\|^{-1} \xi (\psi')^k_{(\xi)}(x,t) \phi^k_{(\xi)}(x) \\ &= -2^k \mu r_\|^{-1} r_\perp \operatorname{div} \left(\left(\frac{\xi}{|\xi|} \otimes \frac{\xi^\perp}{|\xi|} + \frac{\xi^\perp}{|\xi|} \otimes \frac{\xi}{|\xi|} \right) (\psi')^k_{(\xi)} (\phi'')^k_{(\xi)} \right. \\ &\quad \left. + \frac{\xi^\perp}{|\xi|} \otimes \frac{\xi^\perp}{|\xi|} (\psi'')^k_{(\xi)} (\phi')^k_{(\xi)} \right) \\ &= \operatorname{div}(A_{(\xi),k}), \end{aligned}$$

and similarly

$$\partial_t (W^k_{(\xi)})^c = \frac{r_\perp}{r_\parallel} 2^{2k} \mu\lambda r_\parallel^{-1} \xi (\psi')^k_{(\xi)}(x,t)(\Phi'')^k_{(\xi)}(x) = \operatorname{div}(A^c_{(\xi),k}).$$

Hence (27) is satisfied. Defining

$$A_{\xi,K,n_0} := \frac{1}{K^{1/2}} \sum_{k=n_0+1}^{K+n_0} (A_{(\xi),k} + A^c_{(\xi),k})$$

and arguing as in Lemma 4.2, we obtain that

$$\|A_{(\xi),k}\|_{L^1} + \|A^c_{(\xi),k}\|_{L^1} \le C(|\xi|, \Phi, \psi)\mu K^{1/2} r_\perp r_\parallel^{-1}(r_\perp r_\parallel)^{1/2},$$

which yields (28). □

4.6. Combinatorial lemma. The following proposition shows that, up to a suitable (time) translation of each element in the bundle, the building blocks associated to different directions can be taken disjoint.

Proposition 4.6. *Let $\xi_1 = e_1$, $\xi_2 = e_2$, $\xi_3 = e_1 + e_2$ and $\xi_4 = e_1 - e_2$. Then for $n_0 = 5K$ the functions in the family $\{W^k_{(\xi_{i+1})}(x, t + i\mu^{-1}2^{-5K})\}_{k=n_0,\dots,n_0+K;\ i=0,1,2,3}$ have all supports mutually disjoint in space-time.*

Proof. We apply Lemma 4.7 below to the families $\{W^k_{(\xi_2)}(x, t+i\mu^{-1}2^{-5K})\}_{k=n_0,\dots,n_0+K}$ and $\{W^k_{(\xi_2)}(x, t + j\mu^{-1}2^{-5K})\}_{k=n_0,\dots,n_0+K}$; up to shifting the time axis, we can assume that $i = 0$ and that $j \in \{1, 2, 3\}$ and conclude the proof. □

Lemma 4.7. *Let $\xi_1, \xi_2 \in \{e_1, e_2, e_1 + e_2, e_1 - e_2\}$ be two different vector fields. Let us consider two families $\{W^k_{(\xi_1)}(x, t)\}_{k=n_0,\dots,n_0+K}$ and $\{W^k_{(\xi_2)}(x, t + t_0)\}_{k=n_0,\dots,n_0+K}$ for some $t_0 \in [\mu^{-1}2^{-7K}, \mu^{-1}2^{-7K+2}]$ and for $n_0 = 5K$. Then the supports of all these functions are disjoint in space-time, namely*

$$W^k_{(\xi_1)}(x,t) \otimes W^h_{(\xi_2)}(x, t + t_0) = 0 \qquad \textit{for all } k, h \in \{1, \dots, K\}.$$

Proof. The family $\{W^k_{(\xi_1)}(x,t)\}_{k=n_0,\dots,n_0+K}$ is supported by (17) in space in a tube along ξ_1 of size $r_\parallel 2^{-n_0}$ and similarly the family $\{W^k_{(\xi_2)}(x, t+t_0)\}_{k=n_0,\dots,n_0+K}$ is supported in the tube along ξ_2 of size $r_\parallel 2^{-n_0}$. Since these two thin tubes intersect only in a neighborhood of the origin, we deduce that the supports of $W^k_{(\xi_1)}(x,t)$ and $W^h_{(\xi_2)}(x,t)$, where $h, k \in \{n_0, \dots, n_0 + K\}$, can intersect for some time $t > 0$ only if they both belong to $B_R(0)$, where $R := r_\parallel 2^{-n_0+1}$.

We claim the following: *suppose that for a certain $t > 0$ and $k \in \{n_0, \dots, n_0 + K\}$ we have* $\operatorname{supp} W^k_{(\xi_1)}(\cdot, t) \cap B_{r_\parallel 2^{-n_0+1}} \neq \emptyset$. *Then* $\operatorname{supp} W^h_{(\xi_2)}(\cdot, t + t_0) \cap B_R = \emptyset$ *for every $h \in \{n_0, \dots, n_0 + K\}$.*

The previous claim excludes the simultaneous presence at any $t > 0$ of the support of $W^k_{(\xi_1)}(\cdot, t)$ and the support of $W^h_{(\xi_2)}(\cdot, t + t_0)$ in $B_R(0)$, thereby concluding the proof of the lemma.

We now prove the claim. Let us fix a time t such that $\operatorname{supp} W^k_{(\xi_1)}(\cdot, t) \cap B_R \neq \emptyset$. Since $\operatorname{supp} W^k_{(\xi_1)}(\cdot, t)$ is moving at constant speed $\mu 2^k$ along the tube on the torus, there exists $\bar{t}$ such that $|t-\bar{t}| \le R\mu^{-1}2^{-k}$ and $\operatorname{supp} W^k_{(\xi_1)}(\cdot,\bar{t}) = \operatorname{supp} W^k_{(\xi_1)}(x,0)$. At time $\bar{t}$ we have information about the position of $\operatorname{supp} W^h_{(\xi_2)}(\cdot,\bar{t}+t_0)$; more precisely, we have that

$$\operatorname{supp} W^h_{(\xi_2)}(\cdot,\bar{t}+t_0) \subseteq \bigcup_{n\in\mathbb{N}} \left(\operatorname{supp} W^h_{(\xi_2)}(\cdot,t_0) + n\frac{\xi_2}{2^K}\right) \tag{29}$$

because the ratio between the (constant) velocity of $\operatorname{supp} W^k_{(\xi_1)}(\cdot,t)$ and the velocity of $\operatorname{supp} W^k_{(\xi_2)}(\cdot,t)$ is of the form 2^j for some $j \in \{-K,\dots,K\}$.

In the union in the right-hand side of (29), thanks to the upper bound on t_0, the choice $n=0$ identifies the ball of the (finite) union at minimal distance from the origin for every k. By the lower bound on t_0 and the fact that the minimal velocity is $\mu 2^{n_0}$, we get that this distance is greater than 2^{n_0-7K}. At time t the distance between $\operatorname{supp} W^h_{(\xi_2)}(\cdot,t+t_0)$ and $B_R(0)$ is therefore bigger than

$$\begin{aligned}2^{n_0-7K} - |t-\bar{t}|\mu 2^h - R &\ge 2^{n_0-7K} - R2^{h-k} - R \ge 2^{n_0-7K} - R2^K - R\\ &\ge 2^{n_0-7K} - 2^{-n_0+K+1} = 2^{-2K} - 2^{-4K+1} > 0.\end{aligned}$$

This concludes the proof of the claim. □

4.7. Proof of Proposition 4.1. Let $n_0 = 5K$ and $\{W^k_{(\xi_{i+1})}(x, t+i\mu^{-1}2^{-5K})\}_{k=n_0,\dots,n_0+K;\ i=0,1,2,3}$ be as in Proposition 4.6. Since $\operatorname{supp} W^k_{\xi_{i+1}} = \operatorname{supp}(W^k_{\xi_{i+1}})^c = \operatorname{supp} Q^k_{\xi_{i+1}}$, by translating in time $(W^k_{\xi_{i+1}})^c$ and $Q^k_{\xi_{i+1}}$ with $t_{k,i} := i\mu^{-1}2^{-5K}$ we deduce that $W^p_{i+1} := W^p_{\xi_{i+1},K,n_0}$, $W^c_{i+1} := W^c_{\xi_{i+1},K,n_0}$, $Q_{i+1} := Q_{\xi_{i+1},K,n_0}$ and $A_{i+1} := A_{\xi,K,n_0}$ satisfy (v) in Lemma 4.1. We refer the reader to Lemma 4.5 for the construction of A_{ξ,K,n_0}. Properties (i) and (ii) in Lemma 4.1 are now immediate from (19), (20) and Lemma 4.5. We are left with the proof of (iii) and (iv) in Lemma 4.1. To do so we have to choose appropriately the parameters λ, μ, K, $r_\perp$ and $r_\parallel$. Let $\delta < 1/2$ to be chosen later in terms of $\varepsilon > 0$, we set

$$\lambda = \left(\frac{r_\perp}{r_\parallel}\right)^{-1/2}\delta^4 \quad K = \left(\frac{r_\perp}{r_\parallel}\right)^{-2}\delta^4 \quad \mu = (r_\perp r_\parallel)^{-1/2}\delta^{-1},$$

leaving $r_\perp \ll r_\parallel \ll 1$ free. From Lemma 4.3, Lemma 4.5, (21), (22) and (23) we deduce

$$\begin{aligned}&\|D(W^c_i + W^p_i)\|_{L^{1,\infty}} \le C\frac{\lambda}{K^{1/2}}\left(\frac{r_\parallel}{r_\perp}\right)^{1/2} = C\delta^2,\\ &\|DQ_i\|_{L^{1,\infty}} \le C\frac{\lambda}{\mu K r_\perp} = C\delta\frac{r_\perp}{r_\parallel} \le C\delta,\\ &\|A_i\|_{L^1} \le C\mu K^{1/2}(r_\parallel^{-1}r_\perp)(r_\parallel r_\perp)^{1/2} = C\delta,\\ &\|Q_i\|_{L^2(\mathbb{T}^2)} \le \frac{C}{\mu(r_\parallel r_\perp)^{1/2}} = C\delta,\\ &\|W^p_i\|_{L^2} + \frac{r_\parallel}{r_\perp}\|W^c_i\|_{L^2} \le 1.\end{aligned}$$

Moreover, from (24) and (25) we deduce

$$\|D^s W^p_{\xi,K,n_0}\|_{L^p} + \frac{r_\parallel}{r_\perp}\|D^s W^c_{\xi,K,n_0}\|_{L^p} + \|D^s Q_{\xi,K,n_0}\|_{L^p} \le C\exp\left\{-C(p,s)\delta^4\left(\frac{r_\perp}{r_\parallel}\right)^{-2}\right\},$$

for any $s \in (0,1)$ and $p \in \left(1, \frac{2}{1+s}\right)$. The conclusions (iii) and (iv) in Lemma 4.1 follow by choosing first δ small enough so that $C\delta \le \varepsilon$, and after $r_\perp \ll r_\| \ll 1$ so that $\frac{r_\perp}{r_\|} \le \varepsilon$, $\lambda = \delta^4 r_\|^{1/2} r_\perp^{-1/2} \ge \varepsilon^{-1}$ and $C \exp\left\{-C(p,s)\delta^4 \left(\frac{r_\perp}{r_\|}\right)^{-2}\right\} \le \varepsilon^{c(p,s)}$.

5. Definition of the Perturbations

Let us begin by observing that there exist $\Gamma_i \in C^\infty(\mathrm{Sym}_2, \mathbb{R})$, $i = 1, \dots, 4$ such that

$$R = \sum_{i=1}^{4} \Gamma_i(R)^2 e_i \otimes e_i, \quad \text{for any } R \in \mathrm{Sym}_2 \text{ such that } |R - I| < 1/8,$$

where $e_1 := (1,0)$, $e_2 := (0,1)$, $e_3 := (1/\sqrt{2}, 1/\sqrt{2})$ and $e_4 := (1/\sqrt{2}, -1/\sqrt{2})$.

We can define, for instance,

$$\Gamma_1(R)^2 := R_{1,1} - R_{1,2} - \frac{1}{2}, \quad \Gamma_2(R)^2 := R_{2,2} - R_{1,2} - \frac{1}{2},$$
$$\Gamma_3(R)^2 := 2R_{1,2} + \frac{1}{2}, \quad \Gamma_4(R)^2 := \frac{1}{2}.$$

It is immediate to show the identity $R = \sum_{i=1}^4 \Gamma_i(R)^2 e_i \otimes e_i$. Moreover, using that $|R - I| < 1/8$, we deduce

$$\begin{aligned}
\Gamma_1(R)^2 &= \frac{1}{2} + (R_{1,1} - 1) - R_{1,2} \ge \frac{1}{2} - |R_{1,1} - 1| - |R_{1,2}| \ge \frac{1}{4}\,, \\
\Gamma_2(R)^2 &= \frac{1}{2} + (R_{2,2} - 1) - R_{1,2} \ge \frac{1}{2} - |R_{2,2} - 1| - |R_{1,2}| \ge \frac{1}{4}\,, \\
\Gamma_3(R)^2 &\ge \frac{1}{2} - 2|R_{1,2}| \ge 1/4\,,
\end{aligned}$$

which implies that Γ_i are smooth functions.

We define

$$a_i(x,t) := (10\chi(t)(|R_0(x,t)| + \|R_0\|_{L^1}))^{1/2}\, \Gamma_i\left(I - \frac{10^{-1}}{|R_0(x,t)| + \|R_0\|_{L^\infty(L^1)}} R(x,t)\right),$$

where $\chi \in C^\infty(\mathbb{R})$ satisfies $0 \le \chi \le 1$, $\chi = 0$ on $[0, t_0/2]$, and $\chi = 1$ on $[t_0, \infty)$. Our choice leads to

$$\sum_{i=1}^{4} a_i(x,t)^2 \frac{\xi_i}{|\xi_i|} \otimes \frac{\xi_i}{|\xi_i|} = -R_0(x,t) + \chi(t) 10(|R_0(x,t)| + \|R_0\|_{L^\infty(L^1)}) I,$$

where $\xi_1 = (1,0)$, $\xi_2 = (0,2)$, $\xi_3 = (1,1)$ and $\xi_4 = (1,-1)$. The latter implies that

$$-\mathrm{div}\,(R_0) = \mathrm{div}\left(\sum_{i=1}^{4} a_i(x,t)^2 \frac{\xi_i}{|\xi_i|} \otimes \frac{\xi_i}{|\xi_i|}\right) + \nabla P, \tag{30}$$

for some smooth pressure function P.

We observe that the coefficient a_i is a “slow function”, namely its derivatives are estimated only in terms of the smoothness of R_0

$$\|\partial_t^M \nabla^N a_i\|_{L^\infty} \leq C(t_0, \|R_0\|_{C^{N+M}}, N, M)\,,$$
$$\|a_i\|_{L^\infty(L^2)} \leq 5\|R_0\|^{1/2}_{L^\infty(L^1)}\,.$$

For $\varepsilon > 0$ to be chosen later, we consider the functions W_i^p, W_i^c, Q_i, A_i from Proposition 4.1. We define the new velocity field as the sum of the previous one, a principal perturbation, a divergence corrector and a temporal corrector

$$u_1 := u_0 + u_1^{(p)} + u_1^{(c)} + u_1^{(t)},$$

where

$$u_1^{(p)} = \sum_{i=1}^4 a_i (W_i^p + W_i^c), \quad u_1^{(c)} = -\sum_{i=1}^4 \mathcal{R}\left(\nabla a_i \cdot (W_i^p + W_i^c)\right), \quad u_1^{(t)} = -\mathbb{P}(\sum_{i=1}^4 a_i^2 Q_i),$$

where $\mathbb{P} = \nabla^\perp \Delta^{-1}\mathrm{div} : C^\infty(\mathbb{T}^2;\mathbb{R}^2) \to C^\infty(\mathbb{T}^2;\mathbb{R}^2)$ is the Leray projector.

We refer the reader to Remark 3.2 for the definition of $\mathcal{R}$. From now on, in order to simplify our notation, for any function space X and any map f which depends on t and x, we will write $\|f\|_X$ meaning $\|f\|_{L^\infty(X)}$.

5.1. Estimate on $\|u_1 - u_0\|_{L^2}$ and on $\|u_1 - u_0\|_{L^1}$. By the triangular inequality,

$$\|u_1 - u_0\|_{L^2} \leq \|u_1^{(p)}\|_{L^2} + \|u_1^{(c)}\|_{L^2} + \|u_1^{(t)}\|_{L^2}$$

and we estimate the right-hand side separately as

$$\begin{aligned}
\|u_1^{(p)}\|_{L^2} &\leq \sum_{i=1}^4 \|a_i(W_i^p + W_i^c)\|_{L^2} \\
&\leq \sum_{i=1}^4 \left(\|a_i\|_{L^2}\|W_i^p + W_i^c\|_{L^2} + C\frac{\|a_i\|_{C_1}\|W_i^p + W_i^c\|_{L^2}}{\lambda^{1/2}} \right) \\
&\leq \|R_0\|_{L^1} + \varepsilon^{1/2} C(t_0, \|R_0\|_{C_1}),
\end{aligned}$$

where in the second line we used the improved Holder inequality (8) and (iii) in Proposition 4.1.

From Remark 3.2 we deduce

$$\|u_1^{(c)}\|_{L^2} \leq C\varepsilon \sum_{i=1}^4 \|a_i\|_{C_2}\|W_i^p + W_i^c\|_{L^2} \leq \varepsilon C(t_0, \|R_0\|_{C^2}).$$

Finally we employ (iv) in Proposition 4.1 to get

$$\|u_1^{(t)}\|_{L^2} \leq \sum_{i=1}^4 \|a_i\|_{L^\infty}\|Q_i\|_{L^2} \leq \varepsilon C(t_0, \|R_0\|_{L^\infty}).$$

Analogously

$$\begin{aligned}\|u_1-u_0\|_{L^1} &\le \sum_{i=1}^{4}\left(\|u_1^{(p)}\|_{L^1}+\|u_1^{(c)}\|_{L^1}+\|u_1^{(t)}\|_{L^1}\right)\\ &\le C\sum_{i=1}^{4}(\|a_i\|_{L^\infty}\|W_i^p+W_i^c\|_{L^1}+\|u_1^{(c)}\|_{L^2}+\|u_1^{(t)}\|_{L^2})\\ &\le \varepsilon C(t_0,\|R_0\|_{C^2}). \end{aligned} \tag{31}$$

5.2. Estimate on $\|\operatorname{curl}(u_1-u_0)\|_{L^{1,\infty}}$ *and* $\|D^s(u_1-u_0)\|_{L^p}$. By triangular inequality,

$$\begin{aligned}&\|\operatorname{curl}(u_1-u_0)\|_{L^{1,\infty}}\\ &\le C\sum_{i=1}^{4}\left(\|D(a_i(W_i^p+W_i^c))\|_{L^{1,\infty}}+\|D\mathcal{R}(\nabla a_i\cdot(W_i^p+W_i^c)\|_{L^{1,\infty}}+\|\operatorname{curl}\mathbb{P}(a_iQ_i)\|_{L^{1,\infty}}\right),\end{aligned}$$

we estimate the right-hand side separately as

$$\begin{aligned}\|D(a_i(W_i^p+W_i^c))\|_{L^{1,\infty}} &\le \|a_i\|_{C_1}\|(W_i^p+W_i^c)\|_{L^1}+\|a_i\|_{L^\infty}\|D(W_i^p+W_i^c)\|_{L^{1,\infty}}\\ &\le \varepsilon C(t_0,\|R_0\|_{C^1}),\\ \|\operatorname{curl}\mathbb{P}(a_iQ_i)\|_{L^{1,\infty}} &= \|\operatorname{curl}(a_iQ_i)\|_{L^{1,\infty}}\le C\|a_i\|_{C_1}\|Q_i\|_{L^1}+\|a_i\|_{L^\infty}\|DQ_i\|_{L^{1,\infty}}\\ &\le \varepsilon C(t_0,\|R_0\|_{C^1}),\end{aligned}$$

where we employed (iv) in Proposition 4.1. Using that $D\mathcal{R}$ is a Calderon-Zygmund operator we deduce

$$\|D\mathcal{R}(\nabla a_i\cdot(W_i^p+W_i^c))\|_{L^{1,\infty}}\le C\|\nabla a_i\cdot(W_i^p+W_i^c)\|_{L^1}\le \varepsilon C(t_0,\|R_0\|_{C^1}).$$

Let us now fix $s\in(0,1)$ and $p\in(1,\frac{2}{1+s})$. Arguing as above employing (15) we get

$$\begin{aligned}\|D^s(u_1-u_0)\|_{L^p} &\le C\sum_{i=1}^{4}\left(\|D^s(a_i(W_i^p+W_i^c))\|_{L^p}+\|D^s\mathcal{R}(\nabla a_i\cdot(W_i^p+W_i^c)\|_{L^p}\right.\\ &\quad\left.+\|D^s\mathbb{P}(a_iQ_i)\|_{L^p}\right)\\ &\le \varepsilon^{c(p,s)}C(t_0,\|R_0\|_{C^1}).\end{aligned}$$

6. New Error

We define R_1 in such a way that

$$\partial_t u_1+\operatorname{div}(u_1\otimes u_1)+\nabla p_1=\operatorname{div}(R_1),$$

which, by subtracting the equation for u_0, is equivalent to

$$\begin{aligned}\operatorname{div}(R_1)&=\operatorname{div}(u_0\otimes(u_1-u_0)+(u_1-u_0)\otimes u_0+(u_1-u_0)\otimes(u_1-u_0)+R_0)\\ &\quad+\partial_t(u_1-u_0)+\nabla(p_1-p_0).\end{aligned}\tag{32}$$

We are going to define

$$R_1 := R_1^{(l)} + R_1^{(t)} + R_1^{(q)} ,$$

where the various addends are defined in the following paragraphs, and show that

$$\|R_1^{(l)}\|_{L^1} + \|R^{(t)}\|_{L^1} + \|R^{(q)}\|_{L^1} \leq \varepsilon C(t_0, \|R_0\|_{C^3}).$$

The proof of Proposition 2.1 will follow by choosing ε small enough.

6.1. Linear error. Let us set

$$R_1^{(l)} := u_0 \otimes (u_1 - u_0) + (u_1 - u_0) \otimes u_0, \tag{33}$$

thanks to (31) it holds

$$\|R_1^{(l)}\|_{L^1} \leq 2\|u_0\|_{L^\infty}\|u_1 - u_0\|_{L^1} \leq \varepsilon C(t_0, \|R_0\|_{C^2}) .$$

6.2. Temporal error. Let us set

$$\begin{aligned} R_1^{(t)} \quad &:= \mathcal{R}(\partial_t a_i \cdot (W_i^p + W_i^c)) + a_i A_i - \mathcal{R}(\nabla a_i \cdot A_i) \\ &+\mathcal{R}_0\mathcal{R}(\partial_t(\nabla a_i) \cdot (W_i^p + W_i^c)) + \mathcal{R}_0\mathcal{R}(\nabla a_i \cdot \operatorname{div}(A_i)) - \mathcal{R}_0\mathbb{P}\left(\textstyle\sum_{i=1}^4 \partial_t a_i^2 Q_i\right). \end{aligned}$$

Using that

$$\begin{aligned} \partial_t u_1^{(t)} &= -\mathbb{P}(\sum_{i=1}^4 \partial_t a_i^2 Q_i) - \mathbb{P}(\sum_{i=1}^4 a_i^2 \operatorname{div}(W_i^p \otimes W_i^p)) \\ &= -\mathbb{P}(\sum_{i=1}^4 \partial_t a_i^2 Q_i) - \sum_{i=1}^4 a_i^2 \operatorname{div}(W_i^p \otimes W_i^p) - \nabla P, \end{aligned}$$

for some pressure term P, it is immediate to verify the identity

$$\partial_t(u_1 - u_0) = \operatorname{div}(R_1^{(t)}) - \sum_{i=1}^4 a_i^2 \operatorname{div}(W_i^p \otimes W_i^p) - \nabla P. \tag{34}$$

Since $\mathcal{R}$ and $\mathcal{R}_0$ send L^1 to L^1 (cf. Lemma 3.1 and Remark 3.2), we have that

$$\begin{aligned} &\|\mathcal{R}(\partial_t a_i \cdot (W_i^p + W_i^c))\|_{L^1} + \|\mathcal{R}_0\mathcal{R}(\partial_t \nabla a_i \cdot (W_i^p + W_i^c))\|_{L^1} \\ &\qquad \leq 2\|a\|_{C^2}\|W_i^p + W_i^c\|_{L^1} \leq \varepsilon C(t_0, \|R_0\|_{C^2}). \end{aligned}$$

$$\|\mathcal{R}_0\mathbb{P}\left(\sum_{i=1}^4 \partial_t a_i^2 Q_i\right)\|_{L^1} \leq \sum_{i=1}^4 \|\partial_t a_i^2 Q_i\|_{L^2} \leq \sum_{i=1}^4 \|\partial_t a_i^2\|_{L^\infty}\|Q_i\|_{L^2} \leq \varepsilon C(t_0, \|R_0\|_{C^1}).$$

From (iv) in Proposition 4.1 we get

$$\|a_i A_i\|_{L^1} + \|\mathcal{R}(\nabla a_i \cdot A_i)\|_{L^1} \leq 2\|a_i\|_{C^1}\|A_i\|_{L^1} \leq \varepsilon C(t_0, \|R_0\|_{C^1}).$$

By employing (14) we bound

$$\|\mathcal{R}_0\mathcal{R}(\nabla a_i \cdot \operatorname{div}(A_i))\|_{L^1} \leq C\|a_i\|_{C^3}\|A_i\|_{L^1} \leq \varepsilon C(t_0, \|R_0\|_{C^3}) .$$

6.3. Quadratic error terms. Let us set

$$R_1^{(q)} = (u_1 - u_0)\otimes(u_1-u_0) - \sum_{i=1}^4 a_i^2 W_i^p \otimes W_i^p + \sum_{i=1}^4 \mathcal{R}\Big(\nabla a_i^2 \cdot \Big(W_i^p\otimes W_i^p - \fint_{\mathbb{T}^2} W_i^p \otimes W_i^p\Big)\Big),$$

and show that (32) holds. In view of (33), (30) and (34) it amounts to check that

$$\begin{aligned}\operatorname{div}(R_1^{(q)}) &= \operatorname{div}\Big((u_1-u_0)\otimes(u_1-u_0) - \sum_{i=1}^4 a_i^2\Big(\frac{\xi_i}{|\xi_i|}\otimes\frac{\xi_i}{|\xi_i|}\Big)\Big)\\ &\quad - \sum_{i=1}^4 a_i^2 \operatorname{div}(W_i^p\otimes W_i^p) + \nabla(p_1-p_2).\end{aligned}$$

The latter easily follows by noticing that, as a consequence of (ii) in Proposition 4.1, one has

$$\begin{aligned}\sum_{i=1}^4 \nabla a_i^2\cdot\Big(W_i^p\otimes W_i^p - \fint_{\mathbb{T}^2} W_i^p\otimes W_i^p\Big) &= \sum_{i=1}^4 \nabla a_i^2\cdot\Big(W_i^p\otimes W_i^p - \fint_{\mathbb{T}^2} W_i^p\otimes W_i^p\Big)\\ = \sum_{i=1}^4 \operatorname{div}\Big(a_i^2\Big(W_i^p\otimes W_i^p - \frac{\xi_i}{|\xi_i|}\otimes\frac{\xi_i}{|\xi_i|}\Big)\Big) &- \sum_{i=1}^4 \operatorname{div}(a_i^2 W_i^p\otimes W_i^p).\end{aligned}$$

Let us finally prove that $\|R_1^{(q)}\|_{L^1} \le \varepsilon C(t_0, \|R_0\|_{C^2})$. We begin by observing that

$$\begin{aligned}&(u_1-u_0)\otimes(u_1-u_0) - \sum_{i=1}^4 a_i^2 W_i^p\otimes W_i^p\\ &\quad= \sum_{i=1}^4 (a_i^2 W_i^p\otimes W_i^c + a_i^2 W_i^c\otimes W_i^p + a_i^2 W_i^c\otimes W_i^c)\\ &\qquad + (u_1^{(c)} + u_1^{(t)})\otimes(u_1-u_0) + (u_1-u_0)\otimes(u_1^{(c)}+u_1^{(t)}),\end{aligned}$$

From (iv) in Proposition 4.1, the estimates in Sect. 5.1 on $\|u_1^{(c)}\|_{L^2}$, $\|u_1^{(t)}\|_{L^2}$, $\|u_1 - u_0\|_{L^2}$ and Lemma 3.1 we deduce

$$\begin{aligned}&\|a_i^2 W_i^p\otimes W_i^c + a_i^2 W_i^c\otimes W_i^p + a_i^2 W_i^c\otimes W_i^c\|_{L^1} \le \|a_i\|_{L^\infty}(2\|W_i^p\|_{L^2}\|W_i^c\|_{L^2} + \|W_i^c\|_{L^2}^2)\\ &\qquad \le \varepsilon C(t_0, \|R_0\|_{L^\infty}),\\ &\|(u_1^{(c)}+u_1^{(t)})\otimes(u_1-u_0) + (u_1-u_0)\otimes(u_1^{(c)}+u_1^{(t)})\|_{L^1} \le 2\|u_1^{(c)}+u_1^{(t)}\|_{L^2}\|u_1-u_0\|_{L^2}\\ &\qquad \le \varepsilon C(t_0, \|R_0\|_{C_2}),\\ &\Big\|\mathcal{R}\Big(\nabla a_i^2\cdot(W_i^p\otimes W_i^p - \fint_{\mathbb{T}^2} W_i^p\otimes W_i^p)\Big)\Big\|_{L^1} \le C\varepsilon\|\nabla a_1\|_{C^1}\|W_i^p\otimes W_i^p\|_{L^1}\\ &\qquad \le \varepsilon C(t_0, \|R_0\|_{C^2}).\end{aligned}$$

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