Commun. Math. Phys. 403, 917–1003 (2023)
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-023-04812-8

Communications in **Mathematical Physics**

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Gravitational Blocks, Spindles and GK Geometry

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Received: 21 November 2022 / Accepted: 13 July 2023
Published online: 4 August 2023 –

Abstract: We derive a gravitational block formula for the supersymmetric action for a general class of supersymmetric AdS solutions, described by GK geometry. Extremal points of this action describe supersymmetric AdS_3 solutions of type IIB supergravity, sourced by D3-branes, and supersymmetric AdS_2 solutions of $D = 11$ supergravity, sourced by M2-branes. In both cases, the branes are also wrapped over a two-dimensional orbifold known as a spindle, or a two-sphere. We develop various geometric methods for computing the gravitational block contributions, allowing us to recover previously known results for various explicit supergravity solutions, and to significantly generalize these results to other compactifications. For the AdS_3 solutions we give a general proof that our off-shell supersymmetric action agrees with an appropriate off-shell c-function in the dual field theory, establishing a very general exact result in holography. For the AdS_2 solutions our gravitational block formula allows us to obtain the entropy for supersymmetric, magnetically charged and accelerating black holes in AdS_4.

Contents

1. Introduction

The programme of identifying and studying novel geometric structures associated with supersymmetric AdS solutions of string/M-theory has led to enormous progress in our understanding of the AdS/CFT correspondence. A particularly rich arena is provided by supersymmetric $\mathrm{AdS}_3 \times Y_7$ solutions of type IIB supergravity [1] and $\mathrm{AdS}_2 \times Y_9$ solutions of $D = 11$ supergravity [2]. The $\mathrm{AdS}_3 \times Y_7$ solutions, which are dual to $d = 2$, $\mathcal{N} = (0, 2)$ SCFTs, are supported by five-form flux and are associated with D3-branes wrapping two-dimensional, compact surfaces. The $\mathrm{AdS}_2 \times Y_9$ solutions are supported by electric four-form flux and are associated with M2-branes wrapping such surfaces and, furthermore, arise as the near horizon limit of supersymmetric black hole solutions. The geometry on Y_7 and Y_9 was further understood in [3] and, moreover, extended to general odd dimensions Y_{2n+1}; it is referred to as GK geometry.

Initial progress in studying GK geometry arose from constructing explicit solutions [3–6]. However, such explicit constructions represent only a small fraction of a much larger landscape of solutions, and new techniques are needed in order to study the whole family and to elucidate properties of the dual field theories. Inspired by techniques that were developed to study Sasaki–Einstein (SE) manifolds some time ago [7,8], a geometric extremal problem characterizing GK-geometries was formulated in [9] and further developed in [10–14]. In particular, these new techniques enable one to obtain quantities of physical interest for $\mathrm{AdS}_3 \times Y_7$ and $\mathrm{AdS}_2 \times Y_9$ backgrounds, including the central charge of the dual to $d = 2$, $\mathcal{N} = (0, 2)$ SCFTs and the entropy of the supersymmetric black holes, respectively, without needing explicit solutions.

A rich sub-class of such $\mathrm{AdS}_3 \times Y_7$ and $\mathrm{AdS}_2 \times Y_9$ solutions are closely associated with $\mathrm{AdS}_5 \times SE_5$ and $\mathrm{AdS}_4 \times SE_7$ solutions, respectively. Recalling that the latter solutions arise from D3-branes and M2-branes sitting at the apex of the corresponding Calabi-Yau CY_3 and CY_4 cone geometries, one can further wrap the D3 and M2-branes on a compact Riemann surface. Generically this results in a GK geometry which consists of the SE geometry fibred over the Riemann surface. This setup has been studied in [10–14] and, in particular, significant progress has been made in the context of toric SE manifolds, for which additional algebraic tools have been developed [10,13,14]. In the type IIB context, one can interpret the $\mathrm{AdS}_3 \times Y_7$ solutions as being dual to the $d = 4$, $\mathcal{N} = 1$ SCFTs which are associated with $\mathrm{AdS}_5 \times SE_5$, that are then reduced on the Riemann surface and flow to $d = 2$, $\mathcal{N} = (0, 2)$ SCFTs in the IR. The geometric extremization of the GK geometry in the toric context has then been precisely identified with c-extremization [15,16] of the dual field theory in [10,11]. In the $D = 11$ context there is an analogous picture, with the $\mathrm{AdS}_2 \times Y_9$ solutions also being associated with the near horizon limit of supersymmetric black holes living in $\mathrm{AdS}_4 \times SE_7$. In this context the geometric extremization of the GK geometry corresponds to I-extremization [17,18] in the field theory, as discussed in [12,13,19], and this has increased the scope of recovering the entropy of supersymmetric AdS black holes from the dual field theory, vastly extending [17].

Until recently, the construction of supersymmetric AdS solutions associated with branes wrapping two-dimensional surfaces has focussed on compact Riemann surfaces, as discussed in the previous paragraph, with supersymmetry preserved via a topological twist [20]. However, it has recently been shown [21,22] that one can wrap D3 and M2-branes on certain two-dimensional orbifolds known as spindles, and moreover, the supergravity solutions preserve supersymmetry in a novel way, called the anti-twist. The spindle solutions of [21,22] were constructed in minimal $D = 5, 4$ gauged supergravity theories and then uplifted on Sasaki–Einstein manifolds SE_5, SE_7, respectively, to obtain $\mathrm{AdS}_3 \times Y_7$ solutions of type IIB and $\mathrm{AdS}_2 \times Y_9$ solutions of $D = 11$ supergravity. More general solutions have been found in STU-type gauged supergravities which can be uplifted on spheres, $SE_5 = S^5$ or $SE_7 = S^7$ (or specific orbifolds, thereof) [23–28]. For the STU class it was shown in [27] that in addition to the anti-twist solutions there are also topological twist solutions, which we will simply refer to as twist solutions.[1] The fact that supersymmetry is preserved in just one of these two ways is related to the global properties of spin^c spinors on spindles with an azimuthal rotation symmetry [27].

The GK geometry for the $\mathrm{AdS}_3 \times Y_7$ and $\mathrm{AdS}_2 \times Y_9$ solutions obtained by uplifting these gauged supergravity solutions involving spindles, consists of fibrations of Sasaki–

[1] Note that for the usual AdS solutions based on Riemann surfaces with a topological twist, the metric on the Riemann surface has constant curvature and the Killing spinors are constant on the Riemann surface. Neither of these features is present for twist solutions associated with spindles.

Einstein manifolds over spindles. A most striking feature is that the orbifold singularities of the spindles get resolved in the uplift, and the resulting GK geometries are completely smooth. In fact, these smooth solutions had already been constructed some time ago using a quite different perspective [4,6]. While not relevant for this paper, we note that other constructions of solutions associated with various branes wrapped on spindles as well as higher dimensional orbifolds, have also been made.[2]

The principal aim of this paper is to synthesize the general geometric extremization techniques for studying GK geometries that have been developed with the recent progress in constructing explicit solutions associated to branes wrapped on spindles. The formalism we develop will allow us to study a much broader class of configurations of D3 and M2-branes wrapping spindles, for which supergravity solutions are very unlikely to be ever found in explicit form. In particular, by solving the geometric extremization problem one can extract key properties of the solutions and of the holographic dual field theories.

The key object in GK geometric extremization [9] is the supersymmetric action, S_{SUSY}, which has to be suitably extremized in the space of R-symmetry Killing vectors. For $\text{AdS}_3 \times Y_7$ solutions S_{SUSY} is proportional to the "trial central charge", which after extremization becomes the central charge of the dual $d = 2$ SCFT. For $\text{AdS}_2 \times Y_9$ solutions S_{SUSY} is proportional to the "trial entropy" associated with the dual $d = 1$ SCFT; on-shell it is associated with the entropy of supersymmetric black holes which have the AdS_2 solution as a near horizon limit. One of our main results is to demonstrate that for GK geometry consisting of fibrations of Sasaki–Einstein manifolds over spindles, we can write S_{SUSY} in a "gravitational block" form. Furthermore, this rewriting is also applicable for fibrations of Sasaki–Einstein manifolds over two-spheres, thus providing a new perspective on some of the results for GK geometry discussed in [10–14].

The idea of gravitational blocks was first proposed in [46] in the context of supersymmetric black holes and black strings in $\text{AdS}_4 \times S^7$ and $\text{AdS}_5 \times S^5$, respectively, which carry electric and magnetic charges as well as non-trivial rotation and with spherical horizons. There it was shown that the entropy can be obtained by extremizing certain entropy functions that are obtained by summing ("gluing") basic building blocks. This observation was inspired by the factorization of partition functions of $\mathcal{N} = 2$ field theories in $d = 3$, proven in [47] (for later developments in field theory see also [48–50]). In subsequent work [23,30] it was shown that for the class of explicit supergravity solutions associated with M2 and D3-branes[3] wrapping spindles, obtained by uplifting solutions in $D = 4, 5$ STU gauged supergravity theories on S^7, S^5, respectively, it is possible to write the off-shell trial entropy and trial central charge in the form of gravitational blocks, both for the twist and the anti-twist classes. Here we will systematically derive the results of [23,30], and moreover show that they can be extended to the whole class of GK geometry consisting of fibrations of arbitrary Sasaki–Einstein manifolds over spindles, and, furthermore, over two-spheres.

We now turn to an outline of the paper, also highlighting some of the key results. We begin in Sect. 2 by reviewing some key aspects of GK geometry on Y_{2n+1}, $n \geq 3$, including summarizing the extremal problem involving the supersymmetric action S_{SUSY} as a function of the R-symmetry Killing vector ξ. In particular, the extremal

[2] Solutions corresponding to branes wrapped on spindles appear in [27,29] (M5-branes), [30,31] (D4-branes), [32] (D2-branes), while [33] discusses compactifying the Leigh–Strassler theory on a spindle. Solutions for D4 or M5-branes wrapped on four-dimensional orbifolds are contained in [30,31,34–37]. A related construction, involving branes wrapped on disks with orbifold singularities, has been presented in [38–45]. We also highlight that some of these solutions still have orbifold singularities in $D = 10, 11$.

[3] D4 and M5-branes were also considered in [30].

problem involves imposing some flux quantization conditions, which in the case of AdS_3 and AdS_2 solutions is associated with flux quantization in the type IIB and $D = 11$ supergravity solutions, respectively. In the remainder of the paper we focus on GK geometry of the fibred form

$$X_{2n-1} \hookrightarrow Y_{2n+1} \xrightarrow{\pi} \Sigma, \tag{1.1}$$

where X_{2n-1} are Sasakian fibers and $\Sigma = \mathbb{WCP}^1_{[m_-,m_+]}$ is a spindle, i.e. a weighted projective space with co-prime weights $m_\pm \in \mathbb{N}$, with an azimuthal symmetry. While we focus throughout on spindles, our main results are also applicable to replacing the spindle with a smooth two-sphere by setting $m_\pm = 1$. We assume that the fibers have a $U(1)^s$ isometry (the "flavour symmetry") so that we can write

$$\xi = \sum_{\mu=0}^{s} b_\mu \, \partial_{\varphi_\mu}, \tag{1.2}$$

where $(b_\mu) = (b_0, b_1, b_2, \ldots, b_s) \in \mathbb{R}^{s+1}$. Here ∂_{φ_0} denotes the Killing vector generating azimutal rotations of the spindle (uplifted to Y_{2n+1}), while ∂_{φ_i} with $i = 1, \ldots, s$ is a basis for the $U(1)^s$ action on the fibers X_{2n-1}. The two fibers located at the north and south poles of the spindle Σ, denoted by $X_\pm$, respectively, are orbifolds $X_\pm \equiv X_{2n-1}/\mathbb{Z}_{m_\pm}$ and they play an important role in our analysis.

We analyse this setup in Sect. 3 where we prove our main result

$$\boxed{S_{\text{SUSY}} = \frac{2\pi b_1}{b_0} \left(\mathcal{V}^+_{2n-1} - \mathcal{V}^-_{2n-1}\right)} \tag{1.3}$$

where $\mathcal{V}^\pm_{2n-1}$ are "master volumes" [10,13,14] of the covering spaces X_{2n-1} of the fibers $X_\pm$, respectively. With $\xi_\pm$ the orthogonal projection of the R-symmetry vector ξ onto the directions tangent to the fibres $X_\pm$ over the two poles and, similarly $J_\pm \equiv J|_{X_\pm}$ the transverse Kähler class of the GK geometry restricted to the fibres at the poles, we have $\mathcal{V}^\pm_{2n-1} = \mathcal{V}_{2n-1}(\xi_\pm; [J_\pm])$. We refer to (1.3) as the "gravitational block" decomposition of S_{SUSY}.

We also define geometric R-charges, $R^\pm_a$, which are associated with certain supersymmetric submanifolds $S^\pm_a$ of dimension $(2n-3)$ in Y_{2n+1}. More precisely, the latter are defined as $U(1)^s$-invariant codimension two submanifolds $S_a \subset X_{2n-1}$, whose cones are divisors in the Calabi-Yau cone X_{2n-1}.[4] These in turn define codimension four submanifolds $S^\pm_a \subset Y_{2n+1}$, as the copies of S_a in the fibres $X_\pm = X_{2n-1}/\mathbb{Z}_{m_\pm}$ over the two poles of the spindle. For $AdS_3 \times Y_7$ solutions when $n = 3$, the latter are three-dimensional supersymmetric submanifolds in the fibres $X_5/\mathbb{Z}_{m_\pm}$ and the geometric R-charges are dual to the R-charges of baryonic operators associated with D3-branes wrapping these submanifolds. Similarly, for $AdS_2 \times Y_9$ solutions when $n = 4$, they are five-dimensional supersymmetric submanifolds in the fibres $X_7/\mathbb{Z}_{m_\pm}$ and the geometric R-charges are dual to the R-charges of baryonic operators associated with M5-branes wrapping these submanifolds.

[4] Later in Sect. 7 we will take these to be precisely the toric divisors when $C(X_{2n-1})$ is a toric Calabi-Yau cone, with the index $a = 1, \ldots, d$ running over the number of facets d of the associated polyhedral cone in $\mathbb{R}^n$. But for now we may take a to be a general index, labelling any set of $U(1)^s$-invariant divisors in $C(X_{2n-1})$.

The remainder of the paper considers further special cases where we can make additional progress. In Sects. 4 and 5 we consider what we refer[5] to as the "flavour twist", which is defined by imposing a certain restriction on the quantized fluxes so that they are determined by the fibration structure and the choice of ξ. A particular case is provided by examples for which the fibre X_{2n-1} has no "baryonic symmetries" *i.e.* $H^2(X_{2n-1}, \mathbb{R}) \cong 0$. For the flavour twist we show that for $n = 3$ the off-shell trial central charge for $\mathrm{AdS}_3 \times Y_7$ solutions can be written as

$$\mathscr{Z} = \frac{1}{b_0}\left(\frac{1}{\mathrm{Vol}_S(X_5)|_{b_i^{(+)}}} - \frac{1}{\mathrm{Vol}_S(X_5)|_{b_i^{(-)}}}\right) 3\pi^3 N^2, \tag{1.4}$$

while for $n = 4$ the off-shell entropy for $\mathrm{AdS}_2 \times Y_9$ solutions has the form

$$\mathscr{S} = \frac{1}{b_0}\left(\frac{1}{\sqrt{\mathrm{Vol}_S(X_7)|_{b_i^{(+)}}}} - \frac{\sigma}{\sqrt{\mathrm{Vol}_S(X_7)|_{b_i^{(-)}}}}\right) \frac{8\pi^3 N^{3/2}}{3\sqrt{6}}. \tag{1.5}$$

Here $\mathrm{Vol}_S(X_{2n-1})|_{b_i}$ is the Sasaki-volume of X_{2n-1} as a function of the Reeb vector b_i and $\sigma = \pm 1$ is associated with the two ways of preserving supersymmetry on the spindle, the twist and the anti-twist [27]. Also $N = m_+ m_- \mathcal{N}_0$ with $\mathcal{N}_0 \in \mathbb{N}$, can be understood as the flux through the Sasaki–Einstein space for the associated $\mathrm{AdS}_5 \times SE_5$ and $\mathrm{AdS}_4 \times SE_7$ solutions. A special sub-case of the flavour twist is what we shall call the universal anti-twist; for $n = 3$ and 4 this corresponds to the explicit $\mathrm{AdS}_3 \times Y_7$ and $\mathrm{AdS}_2 \times Y_9$ solutions constructed using minimal gauged supergravity in $D = 5$ and $D = 4$ in [21] and [22], respectively, both of which are in the anti-twist class, and we find exact agreement. Section 5 illustrates with some specific examples of the flavour twist.

We then switch gears in Sects. 6–8 to study the case when the fibres X_{2n-1}, and hence Y_{2n+1}, are toric. Section 6 reviews some basic features of toric GK geometry on Y_{2n+1} [10,13,14]. In Sect. 7 we discuss how S_{SUSY} in (1.3) can be written algebraically in terms of the toric data of the fibre X_{2n-1}. We also obtain similarly explicit expressions for the geometric R-charges, $R_a^{\pm}$, in terms of $\mathcal{V}_{2n-1}$ (see (7.46)). In Sect. 8 we focus on $\mathrm{AdS}_3 \times Y_7$ solutions and prove the remarkable result

$$\mathscr{Z} = \frac{1}{b_0} \sum_{a<b<c} (\vec{v}_a, \vec{v}_b, \vec{v}_c) \left(R_a^+ R_b^+ R_c^+ - R_a^- R_b^- R_c^-\right) 3N^2, \tag{1.6}$$

where $R_a^{\pm} = R_a^{\pm}(b_i^{\pm})$ are the geometric R-charges and $(\vec{v}_a, \vec{v}_b, \vec{v}_c)$ is a 3×3 determinant, with $\vec{v}_a$ the toric data for the fibre X_{2n-1}. The $\mathrm{AdS}_3 \times Y_7$ solutions can be interpreted as being dual to the $\mathcal{N} = 1$, $d = 4$ SCFT, which is dual to $\mathrm{AdS}_5 \times SE_5$, that is then compactified on the spindle with magnetic fluxes switched on. We determine the explicit map between the field theory variables involved in c-extremization and those appearing in the extremization of the GK geometry. We also illustrate some of our formalism explicitly by considering some examples.

In Sect. 9 we discuss some features of the $\mathrm{AdS}_2 \times Y_9$ solutions when interpreted as the near horizon limit of supersymmetric and accelerating black holes in $\mathrm{AdS}_4 \times SE_7$. This complements the recent discussion in [51].

[5] This was called "mesonic twist" in [12].

Section 10 concludes with some discussion. Appendix A and B contain some technical material relevant for Sects. 7 and 8, respectively. Appendix C discusses some aspects of Kaluza–Klein reduction of type IIB and $D = 11$ supergravity on SE_5 and SE_7 spaces, respectively, which illuminates some subtleties concerning flavour and baryonic symmetries discussed in Sect. 9.

2. AdS Solutions from GK Geometry

From a physics perspective, we are interested in a class of supersymmetric $\mathrm{AdS}_3 \times Y_7$ solutions of type IIB string theory supported by D3-brane flux [1] and $\mathrm{AdS}_2 \times Y_9$ solutions of M-theory supported by M2-brane flux [2]. In both cases the internal space Y_{2n+1} is equipped with a GK geometry [3] with $n = 3$, $n = 4$, respectively. In [9] solutions to the supergravity equations of motion were shown to be critical points of a certain finite-dimensional extremal problem for GK geometry. In the next two subsections we briefly review these constructions, in general (odd) dimension for Y_{2n+1}, before then specializing to the above two cases of physical interest.

2.1. GK geometry. We begin by discussing some salient features of GK geometry, referring to [3] for more details.

GK geometry [3] is defined on odd-dimensional Riemannian manifolds $Y = Y_{2n+1}$, of dimension $2n + 1$, where $n \geq 3$. The metric on Y_{2n+1} is equipped with a unit norm Killing vector field ξ, called the R-symmetry vector. Since ξ is nowhere vanishing, it defines a foliation $\mathcal{F}_\xi$ of Y_{2n+1}, and in local coordinates we may write

$$\xi = \frac{1}{c}\partial_z, \quad \eta = c\,(\mathrm{d}z + P), \tag{2.1}$$

where we have defined $c \equiv \frac{1}{2}(n-2)$, and η is the Killing one-form dual to ξ. The metric on Y_{2n+1} then takes the form

$$\mathrm{d}s^2_{Y_{2n+1}} = \eta^2 + \mathrm{e}^B\,\mathrm{d}s^2_T, \tag{2.2}$$

where $\mathrm{d}s^2_T$ is a Kähler metric transverse to $\mathcal{F}_\xi$. This Kähler metric has a transverse Kähler two-form J, Ricci two-form $\rho = \mathrm{d}P$, and Ricci scalar R. Moreover, the conformal factor e^B in (2.2) is given by

$$\mathrm{e}^B = \frac{c^2}{2}R. \tag{2.3}$$

In particular this means that we require the Kähler metric to have positive scalar curvature, $R > 0$.

The metric cone over Y_{2n+1} is by definition $C(Y_{2n+1}) \equiv \mathbb{R}_{>0} \times Y_{2n+1}$, with conical metric $\mathrm{d}s^2_{C(Y_{2n+1})} = \mathrm{d}r^2 + r^2\,\mathrm{d}s^2_{Y_{2n+1}}$. There is also an equivalent characterization of GK geometry in terms of the cone geometry [3]. For a GK geometry on Y_{2n+1} the cone has an integrable complex structure, and moreover we require there to exist a nowhere vanishing holomorphic $(n+1, 0)$-form Ψ, which is closed $\mathrm{d}\Psi = 0$. In this sense $C(Y_{2n+1})$ is then Calabi-Yau, having vanishing first Chern class, although the conical metric $\mathrm{d}s^2_{C(Y_{2n+1})}$

is neither Kähler nor Ricci-flat. We also require that Ψ has definite charge under ξ, satisfying

$$\mathcal{L}_\xi \Psi = \frac{\mathrm{i}}{c}\,\Psi. \tag{2.4}$$

In particular this condition implies that ξ is a holomorphic vector field on $C(Y_{2n+1})$.

A GK geometry becomes "on-shell", satisfying the supergravity equations of motion in the string/M-theory applications described in Sects. 2.3 and 2.4, if the transverse Kähler metric satisfies the non-linear partial differential equation

$$\Box R = \frac{1}{2}R^2 - R_{ab}R^{ab}, \tag{2.5}$$

where R_{ab} denotes the Ricci tensor for the Kähler metric, and $\Box$ is the Laplacian operator.

2.2. The extremal problem. We are interested in the following extremal problem in GK geometry, introduced in [9]. We fix a complex cone $C(Y_{2n+1}) = \mathbb{R}_{>0} \times Y_{2n+1}$ with holomorphic volume form Ψ, together with a holomorphic $U(1)^{s+1}$ action. Here $s \geq 0$ necessarily, as ξ generates at least a $U(1)$ action, and in this paper we will be interested in the case that $s \geq 1$ and there is at least a $U(1)^2$ action. We take corresponding generating vector fields ∂_{φ_μ}, $\mu = 0, 1, \ldots, s$, with each φ_μ having period 2π. Moreover, we choose this basis so that the holomorphic volume form has unit charge under ∂_{φ_1}, and is uncharged under $\partial_{\varphi_{\hat{\mu}}}$, $\hat{\mu} = 0, 2, \ldots, s$. Notice here that we have singled out the ∂_{φ_0} direction, as well as the ∂_{φ_1} direction. The reason for this notation will become clear in Sect. 3. A choice of holomorphic R-symmetry vector ξ may then be written as

$$\xi = \sum_{\mu=0}^{s} b_\mu\, \partial_{\varphi_\mu}, \tag{2.6}$$

where we may regard the coefficients as defining a vector $(b_\mu) = (b_0, b_1, b_2, \ldots, b_s) \in \mathbb{R}^{s+1}$.[6] Given the condition (2.4), we must then set

$$b_1 = \frac{1}{c} = \frac{2}{n-2}. \tag{2.7}$$

A choice of ξ determines the foliation $\mathcal{F}_\xi$, and we then further choose a transverse Kähler metric with basic cohomology class $[J] \in H^{1,1}_B(\mathcal{F}_\xi)$. It will often be convenient to note from (2.1) that

$$\mathrm{d}\eta = c\rho = \frac{1}{b_1}\rho, \tag{2.8}$$

where $[\rho] \in H^{1,1}_B(\mathcal{F}_\xi)$ also defines a basic cohomology class.

Given this data, we may define the following *supersymmetric action*

$$S_{\mathrm{SUSY}} \equiv \int_{Y_{2n+1}} \eta \wedge \rho \wedge \frac{J^{n-1}}{(n-1)!}. \tag{2.9}$$

[6] As is standard in the physics literature, b_μ here denotes both the vector, and the μ-th component of this vector. When this abuse of notation might lead to potential confusion, we write $(b_\mu) \in \mathbb{R}^{s+1}$ for the vector.

It is straightforward to show [9] that this is a positive multiple of the integral of the scalar curvature of the transverse metric $R > 0$ over Y_{2n+1}, and thus $S_{\rm SUSY} > 0$ is a necessary condition for a regular on-shell solution. A necessary condition for the transverse Kähler metric to solve the PDE (2.5) is the constraint equation

$$\int_{Y_{2n+1}} \eta \wedge \rho^2 \wedge \frac{J^{n-2}}{(n-2)!} = 0, \tag{2.10}$$

which is equivalent to imposing that the integral of (2.5) over Y_{2n+1} holds. We also impose the *flux quantization conditions*

$$\int_{\Sigma_\alpha} \eta \wedge \rho \wedge \frac{J^{n-2}}{(n-2)!} = \nu_n \mathcal{N}_\alpha. \tag{2.11}$$

Here $\Sigma_\alpha \subset Y_{2n+1}$ are codimension two submanifolds, tangent to ξ, which form a basis for the free part of $H_{2n-1}(Y_{2n+1}; \mathbb{Z})$, ν_n are certain real constants that are given in the physical cases of interest of $n = 3$, $n = 4$ in Sects. 2.3, 2.4 below, and $\mathcal{N}_\alpha \in \mathbb{Z}$ are the quantized fluxes. For a given ξ, it is important to notice that the quantities (2.9), (2.10), (2.11) depend only on the basic cohomology classes $[J], [\rho] \in H^{1,1}_B(\mathcal{F}_\xi)$, and not on the choice of Kähler metric itself.

The extremal problem we are interested in is to extremize the supersymmetric action (2.9), subject to imposing the constraints (2.10), (2.11), for fixed flux numbers $\mathcal{N}_\alpha$. Here the parameter space is the choice of ξ, parametrized by $(b_\mu) \in \mathbb{R}^{s+1}$ via (2.6), and the choice of transverse Kähler class. The number of constraints (2.10), (2.11) is the same as the number of transverse Kähler class parameters. Assuming one can eliminate the latter, the supersymmetric action (2.9) then effectively becomes a function only of b_μ and the flux numbers $\mathcal{N}_\alpha \in \mathbb{Z}$. While we don't have a general theorem to this effect, we shall see later that this is the case in various examples. The main result of [9] is that GK geometries that solve the PDE (2.5) are necessarily solutions to this extremal problem. It is an important outstanding problem to determine sufficient conditions for when solutions of the extremal problem guarantee that one in fact has a GK geometry solving the PDE; for some further discussion see [9,10].

2.3. AdS_3 solutions from D3-branes. Setting $n = 3$ in the above gives internal spaces Y_7 that are associated to supersymmetric solutions of type IIB supergravity of the form [1]

$$\begin{aligned} \mathrm{d}s^2_{10} &= L^2\, \mathrm{e}^{-B/2} \left(\mathrm{d}s^2_{\mathrm{AdS}_3} + \mathrm{d}s^2_{Y_7}\right), \\ F_5 &= -L^4 \left(\mathrm{vol}_{\mathrm{AdS}_3} \wedge F + *_{Y_7} F\right). \end{aligned} \tag{2.12}$$

Here we have introduced the closed two-form

$$F \equiv -\frac{1}{c} J + \mathrm{d}(\mathrm{e}^{-B}\eta), \tag{2.13}$$

where $c = \frac{1}{2}(n-2) = \frac{1}{2}$, $\mathrm{d}s^2_{\mathrm{AdS}_3}$ denotes the unit radius metric on AdS_3, and $L > 0$ is a constant. In fact, if desired one can absorb L in the transverse Kähler geometry via the rescaling $J \to L^{-4} J$, which implies $\mathrm{e}^{-B} \to L^{-4}\mathrm{e}^{-B}$.

The fact that only the five-form flux F_5 in type IIB is non-zero implies that these solutions are in some sense supported only by D3-branes. This flux is properly quantized via (2.11), satisfying

$$\frac{1}{(2\pi\ell_s)^4 g_s}\int_{\Sigma_\alpha} F_5 = \mathcal{N}_\alpha, \tag{2.14}$$

provided we take the constant ν_3 to be

$$\nu_3 = \frac{2(2\pi\ell_s)^4 g_s}{L^4}, \tag{2.15}$$

where ℓ_s is the string length, and g_s denotes the string coupling constant. Furthermore, the extremal value of the supersymmetric action (2.9) determines the central charge c_{sugra} of the dual field theory. In fact [9] introduced a "trial central charge" $\mathscr{Z}$, defined by

$$\mathscr{Z} \equiv \frac{3L^8}{(2\pi)^6 g_s^2 \ell_s^8} S_{\text{SUSY}} = \frac{12(2\pi)^2}{\nu_3^2} S_{\text{SUSY}}, \tag{2.16}$$

and on-shell, i.e. after extremizing, one has has

$$\mathscr{Z}_{\text{os}} = c_{\text{sugra}}. \tag{2.17}$$

2.4. AdS$_2$ solutions from M2-branes. Setting instead $n = 4$ gives internal spaces Y_9 that are associated to supersymmetric solutions of $D = 11$ supergravity of the form [2]

$$\begin{aligned} \mathrm{d}s^2_{11} &= L^2\,\mathrm{e}^{-2B/3}\left(\mathrm{d}s^2_{\mathrm{AdS}_2} + \mathrm{d}s^2_{Y_9}\right), \\ G &= -L^3\,\mathrm{vol}_{\mathrm{AdS}_2}\wedge F, \end{aligned} \tag{2.18}$$

where F is again given by (2.13), but with now $c = \frac{1}{2}(n-2) = 1$. Again $L > 0$ is a constant which can be absorbed into the transverse Kähler geometry via $J \to L^{-3}J$ (this is what is done in [51]).

We may interpret the fact that the flux G is zero when restricted to Y_9, as meaning there is only M2-brane flux, and no M5-brane flux, sourcing these solutions. The Hodge dual seven-form $*_{11}G$ is properly quantized via (2.11), satisfying

$$\frac{1}{(2\pi\ell_p)^6}\int_{\Sigma_\alpha} *_{11}G = \mathcal{N}_\alpha, \tag{2.19}$$

provided we take the constant ν_4 to be

$$\nu_4 = \frac{(2\pi\ell_p)^6}{L^6}, \tag{2.20}$$

where ℓ_p is the eleven-dimensional Planck length. In this case we define a "trial entropy" $\mathscr{S}$ via

$$\mathscr{S} \equiv \frac{4\pi L^9}{(2\pi)^8\ell_p^9} S_{\text{SUSY}} = \frac{2(2\pi)^2}{\nu_4^{3/2}} S_{\text{SUSY}}, \tag{2.21}$$

where the supersymmetric action is given by (2.9) with $n = 4$. When the $D = 11$ solution arises as the near horizon limit of a supersymmetric black hole, it was argued in [9] that the on-shell value, $\mathscr{S}_{\text{os}}$, is the entropy of the black hole. More generally, one expects this quantity to be the logarithm of a supersymmetric partition function of the dual superconformal quantum mechanics.

3. Spindly Gravitational Blocks

In the remainder of the paper we will be interested in studying the extremal problem described in Sect. 2, in the special case that the internal space Y_{2n+1} takes the fibred form

$$X_{2n-1} \hookrightarrow Y_{2n+1} \xrightarrow{\pi} \Sigma. \tag{3.1}$$

Here Y_{2n+1} projects under the projection map π to a two-dimensional surface Σ, with Sasakian fibre X_{2n-1}.

Physically this corresponds to the following set-up. When the fibres $X = X_{2n-1}$ of $Y = Y_{2n+1}$ are Sasaki–Einstein manifolds, taking $n = 3$, $n = 4$ leads to associated $\mathrm{AdS}_5 \times X_5$ and $\mathrm{AdS}_4 \times X_7$ solutions of type IIB supergravity and $D = 11$ supergravity, respectively. These are the near horizon limits of N D3-branes or N M2-branes placed at the Calabi-Yau cone singularities of $C(X_{2n-1})$, respectively. We may then interpret the fibration (3.1) as wrapping the D3-branes or M2-branes over the two-dimensional surface Σ, with a general partial topological twist/fibration. The resulting low-energy effective theories on these wrapped branes, in dimensions $d = 2$ and $d = 1$ respectively, then flow to superconformal fixed points, with near horizon holographic duals given by $\mathrm{AdS}_3 \times Y_7$ and $\mathrm{AdS}_2 \times Y_9$, respectively.

The case where $\Sigma = \Sigma_g$ is a smooth Riemann surface of genus g, and where the R-symmetry vector ξ is tangent to toric fibres X, was studied in [10,13]. Here we are interested in generalizing this set-up by taking $\Sigma = \mathbb{WCP}^1_{[m_-,m_+]}$ to be a spindle, or equivalently a weighted projective space with co-prime weights $m_\pm \in \mathbb{N}$. Moreover, we take general fibres (not just toric), where crucially ξ now has a component that is also tangent to Σ.[7] The action of ξ on Σ is simply rotation about the poles, which are orbifold points modelled locally by $\mathbb{C}/\mathbb{Z}_{m_\pm}$. This includes the special case that $\Sigma = S^2$ is a two-sphere,[8] when $m_\pm = 1$.

The main result of this section will be that the supersymmetric action (2.9), together with the associated constraint (2.10) and flux quantization conditions (2.11), *localize* to integrals over the fibres $X_\pm$ over the poles of the spindle.[9] We refer to these contributions as *gravitational blocks*, since they generalize similar (conjectured) formulas that have already appeared in the literature – see [23,25,30,46,49,52,53]. As well as deriving these gravitational block formulas, we will also relate physical quantities in the parent $d = 4$ and $d = 3$ field theories on the D3-branes and M2-branes, respectively, to physical quantities in the compactified $d = 2$ and $d = 1$ theories in the IR. In turn, this will allow us to prove certain relations in holography.

3.1. Fibrations over a spindle. We may construct a spindle Σ straightforwardly by gluing copies of $\mathbb{C}/\mathbb{Z}_{m_+}$ and $\mathbb{C}/\mathbb{Z}_{m_-}$, along their common S^1 boundary. The fibred geometries Y_{2n+1} that we are interested in may then be realized via a modification of this gluing construction.

We begin with the product $\mathbb{C} \times X_{2n-1}$, where X_{2n-1} will be the base of a Calabi-Yau cone $C(X_{2n-1})$ with $\dim_{\mathbb{C}} C(X_{2n-1}) = n$. The conical metric is $\mathrm{d}s^2_{C(X_{2n-1})} =$

7 More precisely, $\pi_*\xi$ is a non-zero vector field on Σ.

8 In practice, for the S^2 case the extremal ξ does not have a component tangent to S^2, as in [10,13]. From a physics perspective, for $n = 3$ one can understand this as being associated with the lack of mixing of the non-abelian isometries of the S^2 with the R-symmetry vector in c-extremization.

9 In general these fibres will be topologically $X_\pm \equiv X_{2n-1}/\mathbb{Z}_{m_\pm}$, rather than copies of the generic fibre X_{2n-1}, as we will explain in Sect. 3.1 below. Strictly speaking, (3.1) is then not a fibration.

$\mathrm{d}r^2 + r^2\,\mathrm{d}s^2_{X_{2n-1}}$, with $r > 0$, and we denote by Ω the closed holomorphic $(n, 0)$-form on $C(X_{2n-1})$. We also suppose that $C(X_{2n-1})$ is equipped with a holomorphic $U(1)^s$ action, with the Reeb vector field $\mathcal{J}(r\partial_r)$ generating a subgroup of this action, where $\mathcal{J}$ is the complex structure tensor of $C(X_{2n-1})$. We take a basis ∂_{ψ_i}, $i = 1, \ldots, s$, of holomorphic vector fields that generates the $U(1)^s$ action, such that the holomorphic volume form Ω has charge 1 under ∂_{ψ_1}, and is uncharged under ∂_{ψ_i}, $i = 2, \ldots, s$. The corresponding coordinates ψ_i on $X_{2n-1} = \{r = 1\} \subset C(X_{2n-1})$ here have period 2π.[10]

Given this set-up, the space Y_{2n+1} may be constructed by taking $(\mathbb{C} \times X_{2n-1})/\mathbb{Z}_{m_+}$ and gluing it to $(\mathbb{C} \times X_{2n-1})/\mathbb{Z}_{m_-}$, where the local orbifold groups $\mathbb{Z}_{m_\pm}$ act on both $\mathbb{C}$ and the "fibres" X_{2n-1}. It then remains to specify this action, and also make precise how we glue. We may accomplish both together by first choosing two homomorphisms $h_\pm : U(1) \to U(1)^s$. This is equivalent to specifying integers $\alpha_i^\pm \in \mathbb{Z}$, $i = 1, \ldots, s$, so that explicitly

$$h_\pm(\omega) = (\omega^{\alpha_1^\pm}, \ldots, \omega^{\alpha_s^\pm}) \in U(1)^s, \tag{3.2}$$

with $\omega \in U(1)$ a unit norm complex number. Given a point $(z, x) \in \mathbb{C} \times X_{2n-1}$, where $z \in \mathbb{C}$ and x is a point in X_{2n-1}, the $\mathbb{Z}_{m_\pm}$ orbifold actions are then defined by taking the generators $\Gamma_\pm$ to be

$$\Gamma_\pm(z, x) \equiv (\omega_\pm \cdot z, h_\pm(\omega_\pm) \cdot x) \in \mathbb{C} \times X_{2n-1}, \tag{3.3}$$

respectively. Here $\omega_\pm \equiv \mathrm{e}^{2\pi \mathrm{i}/m_\pm}$ are primitive $m_\pm$-th roots of unity, and the action of $h_\pm \in U(1)^s$ on the point $x \in X$ is via the assumed holomorphic $U(1)^s$ action on X_{2n-1}. Quotienting $\mathbb{C} \times X_{2n-1}$ by (3.3) then defines $(\mathbb{C} \times X_{2n-1})/\mathbb{Z}_{m_\pm}$. Notice here that any two choices of $\alpha_i^\pm$ that agree modulo $m_\pm$ (respectively for upper and lower signs) give the same $\mathbb{Z}_{m_\pm}$ quotient. That is, $(\mathbb{C} \times X_{2n-1})/\mathbb{Z}_{m_\pm}$ depends only on $\alpha_i^\pm \in \mathbb{Z}_{m_\pm}$, where the latter then specify homomorphisms from $\mathbb{Z}_{m_\pm} \to U(1)^s$. However, in the gluing construction of the two local models we describe next it will be important to regard $\alpha_i^\pm \in \mathbb{Z}$, and not just defined mod $m_\pm$.

The homomorphisms $h_\pm$ also specify diffeomorphisms $\Phi_\pm$ (which may be thought of as large $U(1)^s$ gauge transformations) of $S^1 \times X_{2n-1} \subset \mathbb{C} \times X_{2n-1}$, where $S^1 = \{z \in \mathbb{C} \mid |z| = 1\} \cong U(1)$ are the unit norm complex numbers. Specifically,

$$\Phi_\pm(z, x) \equiv (z, h_\pm^{-1}(z) \cdot x) \in S^1 \times X_{2n-1}. \tag{3.4}$$

Notice that the composition $\Phi_\pm \circ \Gamma_\pm \circ \Phi_\pm^{-1}$ maps $(z, x) \in S^1 \times X_{2n-1}$ to $(\omega_\pm \cdot z, x)$. Thus, after composing with the diffeomorphism $\Phi_\pm$, the quotient acts only on the z coordinate, showing that $(S^1 \times X_{2n-1})/\mathbb{Z}_{m_\pm} \cong S^1/\mathbb{Z}_{m_\pm} \times X_{2n-1} \cong S^1 \times X_{2n-1}$, which is the boundary of $(\mathbb{C} \times X_{2n-1})/\mathbb{Z}_{m_\pm}$. Both models then have the same boundary $S^1 \times X_{2n-1}$, and we glue these boundaries with the identity diffeomorphism, after reversing the orientation of $(\mathbb{C} \times X_{2n-1})/\mathbb{Z}_{m_-}$. This constructs the total space Y_{2n+1}.

We may make the above discussion more explicit by introducing coordinates. We write $z_\pm = |z_\pm|\mathrm{e}^{\mathrm{i}\hat{\phi}^\pm}$ as complex coordinates on each copy of $\mathbb{C}$, where $\hat{\phi}^\pm$ have period 2π before quotienting. The diffeomorphism/large gauge transformation (3.4) is then implemented by the coordinate transformation

$$\phi^\pm \equiv \hat{\phi}^\pm, \quad \varphi_i^\pm \equiv \psi_i^\pm - \alpha_i^\pm \hat{\phi}_\pm. \tag{3.5}$$

[10] Compare to the discussion at the start of Sect. 2.2. The reason for using the notation ψ_i here, rather than φ_i, will become clear momentarily.

Here $\psi_i^{\pm}$ are 2π-period angular coordinates on each copy of X_{2n-1}, and $(\phi^{\pm}, \varphi_i^{\pm})$ are the new angular coordinates on each copy of $S^1 \times X_{2n-1}$. It follows that on the quotients $(\mathbb{C} \times X_{2n-1})/\mathbb{Z}_{m_\pm}$ the $\phi^{\pm}$ have periods $2\pi/m_\pm$, respectively. The gluing is then accomplished by identifying (the minus sign due to the orientation change of the second copy)

$$\varphi \equiv m_+\phi^+ = -m_-\phi^-, \tag{3.6}$$

so that φ is a 2π-period coordinate for the azimuthal direction of the spindle. The fibres X_{2n-1} are glued with the identity diffeomorphism, meaning we identify[11]

$$\varphi_i^+ = \varphi_i^-. \tag{3.7}$$

We then define $\varphi_i \equiv \varphi_i^+ = \varphi_i^-$ as the angular coordinates on the fibres X_{2n-1}.

3.2. Twist and anti-twist. The previous subsection gives a self-contained description for how to construct the fibration (3.1) over a spindle Σ. However, we may also make contact with the discussion in [27], which focused on $U(1)$ fibrations over a spindle, by focusing on the ith factor of $U(1) \subset U(1)^s$. The corresponding pair of integers (α_i^+, α_i^-) specify a $U(1)$ orbibundle over the spindle base Σ.[12] One can introduce a corresponding connection one-form A_i, and the discussion in [27] computes the total flux/Chern number

$$\frac{1}{2\pi}\int_\Sigma F_i = \frac{\alpha_i^-}{m_-} + \frac{\alpha_i^+}{m_+} = \frac{p_i}{m_+m_-}, \tag{3.8}$$

where $F_i \equiv \mathrm{d}A_i$, and we have defined

$$p_i \equiv \alpha_i^- m_+ + \alpha_i^+ m_-. \tag{3.9}$$

The corresponding complex line bundle over Σ is denoted $\mathcal{O}(p_i)$. In particular, different choices of $\alpha_i^{\pm}$ with the same p_i in (3.9) give isomorphic line bundles, as we shall see.

Conversely, given a choice of $(p_1, \dots, p_s) \in \mathbb{Z}^s$, we may specify the local model data above by first picking coprime integers $a_\pm$ satisfying

$$a_-m_+ + a_+m_- = 1, \tag{3.10}$$

which exist by Bezout's lemma for coprime m_+, m_-, and then defining

$$\alpha_i^+ \equiv a_+p_i, \quad \alpha_i^- \equiv a_-p_i, \quad i = 2, \dots, s. \tag{3.11}$$

[11] The fibres X_{2n-1} can be glued with the diffeomorphism $\varphi_i^+ = \varphi_i^- - t_i\varphi$ where the parameters $t_i \in \mathbb{Z}$ give a further diffeomorphism/large gauge transformation when we glue, as described in [27]. However, in terms of the original coordinates this leads to the identification $\psi_i^+ - \alpha_i^+\hat{\phi}_+ = \psi_i^- - \alpha_i^-\hat{\phi}_- + t_i m_-\hat{\phi}_-$, from which we see that we can simply absorb t_i into a redefinition of α_i^- via $\alpha_i^- \to \alpha_i^- - t_i m_-$ (notice this leaves α_i^- mod m_- invariant, and so preserves the local model $(\mathbb{C} \times X_{2n-1})/\mathbb{Z}_{m_-}$). Thus we can set $t_i = 0$ without loss of generality. We could also remove the remaining redundancy in this construction/description by requiring $\alpha_i^+ \in \{1, \dots, m_+\}$, and then take $\alpha_i^- \in \mathbb{Z}$.

[12] This pair of integers was denoted $(m_N, -m_S)$ in [27], where recall we have without loss of generality set the variable $p = 0$ in [27], absorbing this into m_N or m_S (*cf.* equation (3.7) and the discussion after where p in [27] should be identified with one of the t_i). We also note that the analogue of the ψ_i coordinates here were denoted by χ (for a single $U(1)$) in [27].

These satisfy (3.9), by virtue of (3.10). Notice that here we treat the $i = 1$ direction differently, as discussed below. One can check that different choices in the above construction result in equivalent spaces Y_{2n+1}. Explicitly, given a solution $(a_+, a_-) \in \mathbb{Z}^2$ to (3.10), another solution is given by taking

$$a_+ \mapsto a_+ - \kappa m_+, \quad a_- \mapsto a_- + \kappa m_-, \tag{3.12}$$

where $\kappa \in \mathbb{Z}$ is arbitrary. Via (3.11) this in turn shifts $\alpha_i^\pm \to \alpha_i^\pm \mp \kappa p_i m_\pm \equiv \alpha_i^\pm \bmod m_\pm$. It follows that different choices of κ lead to local models at each pole with equivalent quotient spaces $(\mathbb{C} \times X_{2n-1})/\mathbb{Z}_{m_\pm}$. Moreover, the fact that $\alpha_i^\pm/m_\pm$ change by $\mp\kappa p_i$, with opposite signs, then implies this change to each local model simply cancels in the gluing construction, to obtain Y_{2n+1}.

The $i = 1$ copy of $U(1)$ is special, as we chose the basis so that the holomorphic $(n, 0)$-form Ω on X_{2n-1} has charge 1 under this isometry. Equivalently, the Killing spinor associated to the GK geometry is charged under this direction. This was analysed in some detail in [27], with the conclusion being that there are precisely two possibilities, called the *twist* and *anti-twist*:

$$\begin{aligned} \text{twist}: &\quad \alpha_1^+ = -1, \ \alpha_1^- = -1, \\ \text{anti-twist}: &\quad \alpha_1^+ = -1, \ \alpha_1^- = +1\,. \end{aligned} \tag{3.13}$$

Via (3.9) we then have[13]

$$\begin{aligned} \text{twist}: &\quad p_1 = -m_+ - m_-, \\ \text{anti-twist}: &\quad p_1 = +m_+ - m_-. \end{aligned} \tag{3.14}$$

In particular for the twist case this gives $\mathcal{O}(p_1) = \mathcal{O}(-m_+ - m_-) = K_\Sigma$ as the canonical line orbibundle of the spindle Σ. It will be convenient for the remainder of the paper to write

$$\alpha_1^- = -\sigma, \qquad \boxed{p_1 = -\sigma m_+ - m_-}\,, \tag{3.15}$$

where we have introduced

$$\sigma \equiv \begin{cases} +1 & \text{twist}, \\ -1 & \text{anti-twist}. \end{cases} \tag{3.16}$$

The result for the twist case $\sigma = +1$ may also be obtained via the following construction. Recall that Y_{2n+1} is a fibration of X_{2n-1} over Σ, and that Ψ denotes the holomorphic $(n+1, 0)$-form on $C(Y_{2n+1})$. The twist case arises precisely when $C(X_{2n-1})$ fibred over Σ is also Calabi-Yau, in the sense that it admits a holomorphic $(n + 1, 0)$-form. This space is in general only a partial resolution of the cone singularity $C(Y_{2n+1})$, and we denote its holomorphic volume form also by Ψ. This may then be constructed locally as the wedge product of a $(1, 0)$-form on the spindle Σ with the holomorphic $(n, 0)$-form Ω on $X_{2n-1} \subset C(X_{2n-1})$, being careful to check this glues together appropriately.

[13] Fixing the *overall* sign of p_1 here involves a choice of convention, which we have fixed in writing both (3.13) and (3.14), and also in the discussion of the complex structure and holomorphic volume form in the remainder of this subsection. The result of [27] more generally fixes $|\alpha_1^\pm| = 1$, rather than (3.13), and it is the *relative* sign of α_1^+ and α_1^- that distinguishes the twist and anti-twist cases.

We begin by constructing Ψ in each of the two patches. Recall that $z_\pm = |z_\pm|\mathrm{e}^{\mathrm{i}\hat{\phi}_\pm}$ are the complex coordinates on each copy of $\mathbb{C}$. We denote by $\Omega^\pm$ the $(n,0)$-forms on the corresponding copies of X_{2n-1}, before quotienting. These may further be written as $\Omega^\pm = \mathrm{e}^{\mathrm{i}\psi_1^\pm}\hat{\Omega}^\pm$, where $\hat{\Omega}^\pm$ is uncharged under all $\partial_{\psi_i^\pm}$, $i = 1,\ldots,s$. We may then define

$$\Psi^\pm \equiv \mathrm{d}z_\pm \wedge \Omega^\pm, \tag{3.17}$$

which are holomorphic $(n+1,0)$-forms, before quotienting by $\mathbb{Z}_{m_\pm}$. Combining the angular changes of coordinates in Sect. 3.1, we find

$$\Psi^\pm = \mathrm{d}\left(|z_\pm|\mathrm{e}^{\mathrm{i}\phi^\pm}\right) \wedge \mathrm{e}^{\mathrm{i}\alpha_1^\pm\phi^\pm}\mathrm{e}^{\mathrm{i}\varphi_1}\hat{\Omega}^\pm = \mathrm{d}\left(|z_\pm|\mathrm{e}^{\pm\mathrm{i}\varphi/m_\pm}\right) \wedge \mathrm{e}^{\pm\mathrm{i}\alpha_1^\pm\varphi/m_\pm}\mathrm{e}^{\mathrm{i}\varphi_1}\hat{\Omega}^\pm. \tag{3.18}$$

In order for the first expression to be invariant under the $\mathbb{Z}_{m_\pm}$ quotient, which recall acts only on the $\phi^\pm$ coordinates, we see that we immediately require $\alpha_1^\pm \equiv -1 \mod m_\pm$. Furthermore, in the second expression both φ and φ_1 are globally defined angular coordinates (on the complement of fixed points of the torus action), where recall that (3.6) implements the gluing of the spindle, while $\varphi_i = \varphi_i^+ = \varphi_i^-$ identifies angular coordinates on the two copies of X_{2n-1}. Requiring $\Psi \equiv \Psi^+ = \Psi^-$ to agree on the overlap in (3.18), and moreover be charged only under ∂_{φ_1}, with unit charge, then imposes precisely the twist condition $\alpha_1^+ = \alpha_1^- = -1$ in (3.13). Thus, the twist condition is equivalent to the fibration of $C(X_{2n-1})$ over Σ admitting this global holomorphic $(n+1,0)$-form. In the literature this is often then called a (*partial*) *topological twist*. At present there is no similar geometric interpretation of the anti-twist in (3.13).

Finally, for both the twist and the anti-twist case, notice that defining $\partial_{\varphi_0} \equiv \partial_\varphi$, together with ∂_{φ_i}, $i = 1,\ldots,s$, these vector fields generate a $U(1)^{s+1}$ action on the total space Y_{2n+1}, where this basis then satisfies the conditions imposed in the general set-up described in Sect. 2.2. In particular, the holomorphic $(n+1,0)$-form Ψ is *uncharged* under ∂_{φ_0}. In terms of the coordinates introduced in the construction in Sect. 3.1, it is helpful to note here that

$$\partial_{\varphi_0} \equiv \partial_\varphi = \pm\frac{1}{m_\pm}\partial_{\hat{\phi}^\pm} \pm \sum_{i=1}^{s}\frac{\alpha_i^\pm}{m_\pm}\partial_{\varphi_i}, \quad \partial_{\varphi_i} = \partial_{\varphi_i^\pm} = \partial_{\psi_i^\pm}. \tag{3.19}$$

Later in our discussion we will be particularly interested in the vector fields $\zeta_\pm$, that by definition rotate the normal directions to the fibres over the poles of the spindle; that is, the copies of $X_\pm \equiv X_{2n-1}/\mathbb{Z}_{m_\pm}$ located at $z_\pm = 0$, respectively, in the construction of Sect. 3.1. These are given by $\zeta_\pm = \partial_{\hat{\phi}^\pm}$, and (3.19) immediately gives

$$\zeta_\pm \equiv \partial_{\hat{\phi}^\pm} = \begin{cases} m_+\partial_{\varphi_0} + \partial_{\varphi_1} - \sum_{i=2}^{s}\alpha_i^+\partial_{\varphi_i} \equiv \sum_{\mu=0}^{s} v_{+\mu}\partial_{\varphi_\mu} \\ -m_-\partial_{\varphi_0} + \sigma\partial_{\varphi_1} - \sum_{i=2}^{s}\alpha_i^-\partial_{\varphi_i} \equiv \sigma\sum_{\mu=0}^{s} v_{-\mu}\partial_{\varphi_\mu}, \end{cases} \tag{3.20}$$

where we have used (3.13) and (3.15). We may then also read off the "toric data" vectors

$$(v_{+\mu}) = (m_+, 1, -\vec{\alpha}^+), \quad (v_{-\mu}) = (-\sigma m_-, 1, -\sigma\vec{\alpha}^-), \tag{3.21}$$

where $\vec{\alpha}^\pm \equiv (\alpha_2^\pm,\ldots,\alpha_s^\pm)$. We will make use of this result in Sect. 7.

For later use, we define $\xi_\pm$ to be the orthogonal projection of the R-symmetry vector ξ given in (2.6) onto directions tangent to the fibres over the two poles. These may then be viewed as R-symmetry vectors for the fibres $X_\pm$. From (3.20) we have

$$\xi_+ \equiv \xi - \frac{b_0 \zeta_+}{m_+} = \sum_{i=1}^{s} b_i^{(+)} \partial_{\varphi_i}, \quad \xi_- \equiv \xi + \frac{b_0 \zeta_-}{m_-} = \sum_{i=1}^{s} b_i^{(-)} \partial_{\varphi_i}, \tag{3.22}$$

where we have defined the shifted vectors

$$b_i^{(+)} \equiv b_i - \frac{b_0}{m_+} v_{+i}, \quad b_i^{(-)} \equiv b_i + \frac{b_0}{\sigma m_-} v_{-i}. \tag{3.23}$$

For later use note that

$$b_i^{(+)} - b_i^{(-)} = \frac{b_0 p_i}{m_+ m_-}, \tag{3.24}$$

where we used (3.9).

3.3. Gravitational block lemma. In this section we prove the following general formula

$$\int_{Y_{2n+1}} \eta \wedge \rho \wedge \Gamma = \frac{2\pi b_1}{b_0} \left(m_+ \int_{X_+} \eta \wedge \Gamma - m_- \int_{X_-} \eta \wedge \Gamma \right). \tag{3.25}$$

Here Y_{2n+1} is a GK geometry fibering over a spindle Σ, as in (3.1), where the fibres over the two poles of the spindle are respectively $X_\pm \cong X_{2n-1}/\mathbb{Z}_{m_\pm}$, with the quotient given by (3.3). The orientations of $X_\pm$ here are those naturally induced via the Stokes' theorem argument we introduce shortly, but we note that these may not agree with the natural orientation induced by the complex structure on $C(Y_{2n+1})$ (we shall return to discuss this later). Recall that we write the R-symmetry vector on Y_{2n+1} as

$$\xi = \sum_{\mu=0}^{s} b_\mu \partial_{\varphi_\mu}, \tag{3.26}$$

where the vector field ∂_{φ_0} was defined in the previous subsection, and rotates the spindle Σ, while ∂_{φ_i} for $i = 1, \ldots, s$ are tangent to the fibres X_{2n-1} of (3.1). The differential form Γ in (3.25) is then any closed form on Y_{2n+1} that is basic with respect to the foliation defined by ξ; that is, $\mathcal{L}_\xi \Gamma = 0$, $\xi \lrcorner \Gamma = 0$.

Both the supersymmetric action (2.9) and constraint equation (2.10) take the form of the left hand side of (3.25), but so too does the flux quantization condition (2.11), for submanifolds $\Sigma_\alpha \subset Y_{2n+1}$ that themselves fibre over the spindle direction Σ. Equation (3.25) will form the basis for much of the rest of the paper. Here we provide a general differential-geometric proof for arbitrary fibre X_{2n-1}. In later sections we will obtain an alternative proof for the case of toric X_{2n-1}.

We begin by noting from (2.8) that $\rho = b_1 \mathrm{d}\eta$, so that (3.25) is equivalent to

$$\int_{Y_{2n+1}} \eta \wedge \mathrm{d}\eta \wedge \Gamma = \frac{2\pi}{b_0} \left(m_+ \int_{X_+} \eta \wedge \Gamma - m_- \int_{X_-} \eta \wedge \Gamma \right). \tag{3.27}$$

In Sect. 3.1 we introduced on Y_{2n+1} a set of 2π-period coordinates φ_μ, $\mu = 0, \ldots, s$, which are globally well-defined away from the fixed points of the $U(1)^{s+1}$ action on

Y_{2n+1}. In particular, φ_0 is a 2π-period coordinate for the azimuthal direction of the spindle Σ, while φ_i, $i = 1, \ldots, s$, are coordinates on the fibres X_{2n-1}. φ_0 is then well-defined on Σ, except at the poles. Each pole of the spindle is locally modelled as a quotient $\mathbb{C}/\mathbb{Z}_{m_\pm}$, where we introduced complex coordinates $z_\pm = |z_\pm|\mathrm{e}^{\mathrm{i}\hat{\phi}_\pm}$ on the covering space copies of $\mathbb{C}$. Via (3.5), (3.6), we then have

$$\hat{\phi}_+ = \frac{1}{m_+}\varphi_0, \quad \hat{\phi}_- = -\frac{1}{m_-}\varphi_0. \tag{3.28}$$

Given the above notation, we next define the one-form

$$\upsilon_0 \equiv \frac{\mathrm{d}\varphi_0}{2\pi}. \tag{3.29}$$

A priori this is a one-form on Σ that is well-defined except at the poles. We may then pull this back to a one-form $\pi^*\upsilon_0$ on Y_{2n+1}, which in an abuse of notation we simply refer to also as υ_0. The latter is correspondingly then well-defined on Y_{2n+1}, except at the fibres $X_\pm$ over the poles, i.e. precisely where we integrate on the right hand side of (3.27).

The form υ_0 is manifestly closed, but we must be careful at the poles. Indeed, if we take the standard complex coordinate $z = |z|\mathrm{e}^{\mathrm{i}\phi}$ on $\mathbb{C}$, where ϕ has period 2π, then $\upsilon \equiv \mathrm{d}\phi/2\pi$ is a smooth one-form on $\mathbb{C} \setminus \{0\}$. Its exterior derivative is a two-form that is zero on $\mathbb{C} \setminus \{0\}$, but we also have

$$\int_{S^1_\epsilon} \upsilon = 1, \tag{3.30}$$

where $S^1_\epsilon \equiv \{|z| = \epsilon\}$ is the circle of radius $\epsilon > 0$. We may then identify

$$\mathrm{d}\upsilon = \delta \equiv \delta(x, y)\, \mathrm{d}x \wedge \mathrm{d}y, \tag{3.31}$$

as a distribution-valued two-form, where $z = x + \mathrm{i}y$, and $\delta(x, y)$ is the usual Dirac delta function. More abstractly, $\mathrm{d}\upsilon = \delta$ is a delta function representative of the Poincaré dual to the origin $0 \in \mathbb{C}$. We then have

$$1 = \int_{D_\epsilon} \delta = \int_{D_\epsilon} \mathrm{d}\upsilon = \int_{S^1_\epsilon} \upsilon, \tag{3.32}$$

as in (3.30), where $D_\epsilon \equiv \{|z| \le \epsilon\}$ is the disc of radius ϵ.

We may apply the above analysis to υ_0 on the spindle, where $\mathrm{d}\upsilon_0$ is zero except at the poles of Σ. These are both orbifold singularities, where recall that we introduced a local model at each pole, starting with $\mathbb{C} \times X_{2n-1}$ and then quotienting by the $\mathbb{Z}_{m_\pm}$ action generated by (3.3). We may then identify

$$\mathrm{d}\upsilon_0 = m_+\delta_+ - m_-\delta_-. \tag{3.33}$$

Here the factors of $\pm m_\pm$ come from the corresponding factors in (3.28), and $\delta_\pm$ are defined precisely as in (3.31), in local coordinates $z_\pm$ for each covering space $\mathbb{C}$. Notice that the origin of $\mathbb{C}$ is fixed under the $\mathbb{Z}_{m_\pm}$ action generated by (3.3), so this group acts purely on the fibres X_{2n-1} over the origin in $\mathbb{C} \times X_{2n-1}$, meaning the fibres over the poles are $X_\pm = X_{2n-1}/\mathbb{Z}_{m_\pm}$. Using (3.33) we may then immediately write

$$\int_{Y_{2n+1}} \eta \wedge \mathrm{d}\upsilon_0 \wedge \Gamma = m_+ \int_{X_+} \eta \wedge \Gamma - m_- \int_{X_-} \eta \wedge \Gamma. \tag{3.34}$$

Here $\mathrm{d}\upsilon_0$ is zero except near to $X_\pm$, which are at the origins of each local model $(\mathbb{C} \times X_{2n-1})/\mathbb{Z}_{m_\pm}$. On the other hand, the delta functions in (3.33) precisely restrict the integral to these copies of $X_\pm$. Starting with the left hand side of (3.34), we may then integrate by parts, using the fact that $\mathrm{d}\Gamma = 0$, to obtain[14]

$$\int_{Y_{2n+1}} \eta \wedge \mathrm{d}\upsilon_0 \wedge \Gamma = \int_{Y_{2n+1}} \upsilon_0 \wedge \mathrm{d}\eta \wedge \Gamma = \int_{Y_{2n+1}} (\xi \lrcorner \upsilon_0)\, \eta \wedge \mathrm{d}\eta \wedge \Gamma. \tag{3.35}$$

Here in the last step we have used the fact that both $\mathrm{d}\eta$ and Γ are basic, so $\xi \lrcorner \mathrm{d}\eta = 0 = \xi \lrcorner \Gamma$, while $\xi \lrcorner \eta = 1$. On the other hand, from (3.26) we immediately have $\xi \lrcorner \upsilon_0 = b_0/2\pi$. Combining (3.35) with (3.34), we have thus proven (3.27).

3.4. Gravitational block formula. We can now use these results to refine the extremal problem discussed in Sect. 2.2 for GK geometries of the fibred form (3.1). Specifically we want to reconsider the supersymmetric action (2.9), the constraint equation (2.10) and the expression for the fluxes (2.11).

We first observe that the supersymmetric action (2.9) takes the form of (3.25), with $\Gamma = J^{n-1}/(n-1)!$ and hence we may write

$$S_{\mathrm{SUSY}} = \frac{2\pi b_1}{b_0}\left(m_+ \int_{X_+} \eta \wedge \frac{J^{n-1}}{(n-1)!} - m_- \int_{X_-} \eta \wedge \frac{J^{n-1}}{(n-1)!} \right), \tag{3.36}$$

or

$$\boxed{S_{\mathrm{SUSY}} = \frac{2\pi b_1}{b_0}\left[m_+ \mathrm{Vol}(X_+) - m_- \mathrm{Vol}(X_-) \right]} \tag{3.37}$$

where $\mathrm{Vol}(X_\pm)$ are the induced volumes of the fibres over the poles. Indeed, since $X_\pm \cong X_{2n-1}/\mathbb{Z}_{m_\pm}$, we have in both cases $m_\pm \mathrm{Vol}(X_\pm) = \mathrm{Vol}(X_{2n-1})$, although the induced volume forms for each copy of X_{2n-1} are in general different, and the volumes are different. We shall obtain an explicit formula for this for toric X_{2n-1} in Sect. 7. As remarked after equation (3.25), the orientations of $X_\pm$ in (3.37) may not agree with those induced from the complex structure on $C(Y_{2n+1})$ (or equivalently the transverse complex structure on Y_{2n+1}), and correspondingly the "volumes" $\mathrm{Vol}(X_\pm)$ will then be *negative*. In fact we shall find that we can take $\mathrm{Vol}(X_+) > 0$ and $\sigma\, \mathrm{Vol}(X_-) > 0$.

We next note that there are two preferred fluxes, $N^{X_\pm}$, associated with the fibres $X_\pm \cong X_{2n-1}/\mathbb{Z}_{m_\pm}$ at the north and south poles, respectively. Following (2.11) we define

$$\int_{X_\pm} \eta \wedge \rho \wedge \frac{J^{n-2}}{(n-2)!} \equiv \nu_n\, N^{X_\pm}. \tag{3.38}$$

We can then apply (3.25) to the constraint equation (2.10). We set $\Gamma = \rho \wedge \frac{J^{n-2}}{(n-2)!}$, so that the constraint becomes

$$m_+ \int_{X_+} \eta \wedge \rho \wedge \frac{J^{n-2}}{(n-2)!} - m_- \int_{X_-} \eta \wedge \rho \wedge \frac{J^{n-2}}{(n-2)!} = 0. \tag{3.39}$$

[14] One can circumvent the use of delta functions by instead cutting out small neighbourhoods of $X_\pm$ in Y_{2n+1}, to obtain a manifold with two boundary components, $(S^1 \times X_{2n-1})/\mathbb{Z}_{m_+} \sqcup (-S^1 \times X_{2n-1})/\mathbb{Z}_{m_-}$, and then integrating $\upsilon_0 \wedge \eta \wedge \Gamma$ over this boundary. The integral of υ_0 over the circle factors gives $m_\pm$, resulting in the right hand side of (3.34). Instead using Stokes' theorem, we have $\mathrm{d}\upsilon_0 = 0$ on the interior of the manifold with boundary, and the only contribution is the right hand side of (3.35).

Thus, the constraint equation simply relates the two fluxes $N^{X_\pm}$ in (3.38). It is convenient to then define $N \in \mathbb{Z}$ via

$$\boxed{N \equiv m_+ N^{X_+} = m_- N^{X_-}} \tag{3.40}$$

Notice that since m_+ and m_- are assumed co-prime, this means that $N = m_+ m_- \mathcal{N}_0$, where $\mathcal{N}_0 \in \mathbb{N}$. Indeed, without loss of generality we will take

$$N > 0, \tag{3.41}$$

and hence with $m_\pm > 0$, we also have $N^{X_\pm} > 0$.

Before continuing, we now define the geometric R-charges for GK geometries of the fibred form (3.1). The R-charges, $R_a^\pm$, are associated with certain supersymmetric submanifolds $S_a^\pm$ of dimension $(2n-3)$ in Y_{2n+1}. More precisely, these may be defined as follows. We begin with a set of $U(1)^s$-invariant codimension two submanifolds $S_a \subset X_{2n-1}$, whose cones are divisors in the Calabi-Yau cone X_{2n-1}.[15] These in turn define codimension four submanifolds $S_a^\pm \subset Y_{2n+1}$, as the copies of S_a in the fibres $X_\pm = X_{2n-1}/\mathbb{Z}_{m_\pm}$ over the two poles of the spindle. We then define

$$R_a^\pm \equiv \frac{4\pi}{\nu_n N^{X_\pm}} \int_{S_a^\pm} \eta \wedge \frac{J^{n-2}}{(n-2)!}. \tag{3.42}$$

For $\mathrm{AdS}_3 \times Y_7$ solutions when $n = 3$, the latter are three-dimensional supersymmetric sub-manifolds in the fibres $X_5/\mathbb{Z}_{m_\pm}$ and the geometric R-charges are dual to the R-charges of baryonic operators associated with D3-branes wrapping these submanifolds. Similarly, for $\mathrm{AdS}_2 \times Y_9$ solutions when $n = 4$, they are five-dimensional supersymmetric submanifolds in the fibres $X_7/\mathbb{Z}_{m_\pm}$ and the geometric R-charges are dual to the R-charges of baryonic operators associated with M5-branes wrapping these submanifolds. We emphasize that these supersymmetric submanifolds exist even in cases with $H^2(X_{2n-1}, \mathbb{R}) \cong 0$, which we discuss further in Sect. 4. We also note that there is again an issue of orientation on S_a and we will see later in the toric case that we have $R_a^+ > 0$ and $\sigma R_a^- > 0$.

We now compute the fluxes through another preferred class of $(2n-1)$-dimensional submanifolds $\Sigma_a \subset Y_{2n+1}$. Specifically, we consider Σ_a which are the total spaces of the supersymmetric submanifolds S_a of the fibres X_{2n-1} defined above, that are then fibred over the spindle Σ:

$$S_a \hookrightarrow \Sigma_a \to \Sigma. \tag{3.43}$$

The associated flux through this class of cycles is[16]

$$\int_{\Sigma_a} \eta \wedge \rho \wedge \frac{J^{n-2}}{(n-2)!} \equiv \nu_n M_a, \tag{3.44}$$

[15] Later in Sect. 7 we will take these to be precisely the toric divisors when $C(X_{2n-1})$ is a toric Calabi-Yau cone, with the index $a = 1, \ldots, d$ running over the number of facets d of the associated polyhedral cone in $\mathbb{R}^n$. But for now we may take a to be a general index, labelling any set of $U(1)^s$ invariant divisors in $C(X_{2n-1})$.

[16] We emphasize that the fluxes M_a are, in general, distinct from the fluxes $\mathcal{N}_\alpha$ in (2.11) – see (7.37). We also note that even when $H_{2n-3}(X_{2n-1}, \mathbb{R}) = 0$, so that the homology classes of S_a in the fibres X_{2n-1} are necessarily trivial, it does not follow that the M_a are zero. Indeed, these are integrals of the flux over Σ_a, whose homology classes are generically non-trivial due to the fibration.

where $M_a \in \mathbb{Z}$. In a completely analogous fashion to how we derived the block formula (3.25), we can split the integral in (3.44) into two pieces as

$$\int_{\Sigma_a} \eta \wedge \rho \wedge \frac{J^{n-2}}{(n-2)!} = \frac{2\pi b_1}{b_0} \left(m_+ \int_{S_a^+} \eta \wedge \frac{J^{n-2}}{(n-2)!} \right. \\ \left. - m_- \int_{S_a^-} \eta \wedge \frac{J^{n-2}}{(n-2)!} \right), \tag{3.45}$$

where recall that $S_a^{\pm}$ are the copies of the S_a in the fibres over the north and south poles of the spindle, respectively. We thus conclude that these preferred fluxes are related to the R-charges associated to these submanifolds via

$$\boxed{M_a = \frac{b_1}{2b_0} \left(R_a^+ - R_a^- \right) N} \tag{3.46}$$

4. Block Formula for Some Simpler Sub-classes

We can make further progress by making some additional assumptions on the GK geometry. In this section we first consider what we call a "flavour twist" (called a "mesonic twist" in the toric setting in [12]) followed by the "universal anti-twist", which involves an additional assumption.

4.1. Block formula for X_{2n-1} with a flavour twist. First consider the special class of geometries where the fibre X_{2n-1} has no baryonic symmetries, *i.e.* $H^2(X_{2n-1}, \mathbb{R}) \cong 0$. The nomenclature comes from the fact that for $n = 3, 4$ the field theories dual to the associated $\mathrm{AdS}_5 \times X_5$ and $\mathrm{AdS}_4 \times X_7$ geometries then have no baryonic $U(1)$ flavour symmetries. For this case the transverse Kähler class of the GK geometry restricted to the fibres at the poles, $[J|_{X_\pm}]$, must necessarily be proportional to $[\rho]$ and we can write

$$[J|_{X_\pm}] = \Lambda_\pm [\rho] \in H_B^2(\mathcal{F}_{\xi_\pm}), \tag{4.1}$$

with $\Lambda_\pm \in \mathbb{R}$, and recall that $\xi_\pm$ are the R-symmetry vectors of the fibres $X_\pm$, introduced in (3.22).

We can also consider a more general class of geometries where we don't assume $H^2(X_{2n-1}, \mathbb{R}) \cong 0$, but nevertheless (4.1) is still satisfied. We call this the "flavour twist". We will see below that this corresponds to fixing the GK geometry Y_{2n+1}, including certain constraints on the fluxes M_a, in terms of X_{2n-1}, $m_\pm$ and p_i.

Substituting (4.1) into the supersymmetric action (3.37) we obtain

$$S_{\text{SUSY}} = \frac{2\pi b_1}{b_0} \left(m_+ \Lambda_+^{n-1} \int_{X_+} \eta \wedge \frac{\rho^{n-1}}{(n-1)!} - m_- \Lambda_-^{n-1} \int_{X_-} \eta \wedge \frac{\rho^{n-1}}{(n-1)!} \right) \\ = \frac{\pi (2b_1)^n}{b_0} \Big[m_+ \Lambda_+^{n-1} \operatorname{Vol}_S(X_+) - m_- \Lambda_-^{n-1} \operatorname{Vol}_S(X_-) \Big]. \tag{4.2}$$

Here $\operatorname{Vol}_S(X_\pm)$ is the Sasakian volume, obtained from $\operatorname{Vol}(X_\pm)$ by setting $[J] = \frac{1}{2b_1}[\rho]$, namely

$$\operatorname{Vol}_S(X_\pm) = \frac{1}{(2b_1)^{n-1}} \int_{X_\pm} \eta \wedge \frac{\rho^{n-1}}{(n-1)!}. \tag{4.3}$$

These Sasaki volumes can be considered to be functions of trial Reeb vectors $\xi_\pm$ tangent to $X_\pm$. In practice, $\mathrm{Vol}_S(X_\pm)$ can be computed by taking the expression for the Sasakian volume of the fibre manifold X_{2n-1} and taking the trial Reeb vector to be the orthogonal projection of the R-symmetry vector ξ on Y_{2n-1} onto the fibres over the poles. Recall $\xi_\pm$ were defined in (3.22), where we have

$$\xi_+ \equiv \xi - \frac{b_0 \zeta_+}{m_+} = \sum_{i=1}^{s} b_i^{(+)} \partial_{\varphi_i}, \quad \xi_- \equiv \xi + \frac{b_0 \zeta_-}{m_-} = \sum_{i=1}^{s} b_i^{(-)} \partial_{\varphi_i}. \tag{4.4}$$

Finally, we should divide by $m_\pm$ in order to take into account the quotient/orbifold singularities, and hence we can write

$$\mathrm{Vol}_S(X_\pm) = \frac{1}{m_\pm} \mathrm{Vol}_S(X_{2n-1})|_{\xi=\xi_\pm} = \frac{1}{m_\pm} \mathrm{Vol}_S(X_{2n-1})|_{b_i^{(\pm)}}. \tag{4.5}$$

Here $\mathrm{Vol}_S(X_{2n-1})|_{b_i}$ is the standard positive[17] Sasaki volume on X_{2n-1} as a function of b_i (leading to a positive Sasaki–Einstein volume after extremization). We thus have

$$S_{\mathrm{SUSY}} = \frac{\pi(2b_1)^n}{b_0} \left[\Lambda_+^{n-1} \, \mathrm{Vol}_S(X_{2n-1})|_{b_i^{(+)}} - \Lambda_-^{n-1} \, \mathrm{Vol}_S(X_{2n-1})|_{b_i^{(-)}} \right]. \tag{4.6}$$

To make further progress, we now consider the expression for the flux quantization (3.38) through the fibres over the poles. Using the same type of argument we find

$$N^{X_\pm} = \frac{1}{\nu_n} \int_{X_\pm} \eta \wedge \rho \wedge \frac{J^{n-2}}{(n-2)!} = \frac{(n-1)}{m_\pm} \frac{(2b_1)^{n-1}}{\nu_n} \Lambda_\pm^{n-2} \, \mathrm{Vol}_S(X_{2n-1})|_{b_i^{(\pm)}}, \tag{4.7}$$

and we also recall the constraint equation (3.40) given by $N = m_+ N^{X_+} = m_- N^{X_-}$. We can therefore solve for $\Lambda_\pm$ in terms of N. After substituting into (4.6) we obtain an expression for the off-shell action as a function of the R-symmetry vector ξ given in (2.6) and the integer N. At this point, we can set $b_1 = \frac{2}{n-2}$ and the extremization of the GK geometry with respect to the remaining components of b_μ can be carried out. There is some difference in signs depending on whether n is even or odd, which we will discuss further below.

Before doing that we derive an expression for the geometric R-charges $R_a^\pm$, defined in (3.42). Recall that $S_a^\pm$ are the copies of the supersymmetric $(2n-3)$-dimensional submanifolds S_a in the fibres over the north and south poles of the spindle, respectively. We can express $R_a^\pm$ in terms of the Sasakian volume of the submanifolds $S_a^\pm$, defined as

$$\mathrm{Vol}_S(S_a^\pm) \equiv \frac{1}{(2b_1)^{n-2}} \int_{S_a^\pm} \eta \wedge \frac{\rho^{n-2}}{(n-2)!} = \frac{1}{m_\pm} \mathrm{Vol}_S(S_a)|_{b_i^{(\pm)}}. \tag{4.8}$$

Here $\mathrm{Vol}_S(S_a)|_{b_i}$ is the standard positive Sasaki volume of the submanifold $S_a \subset X_{2n-1}$ as a function of b_i (leading to a positive result on the Sasaki–Einstein manifold after extremization). Hence, after eliminating $\Lambda_\pm$ using (4.7), we can conclude that

$$\boxed{R_a^\pm = \frac{2\pi}{(n-1)b_1} \frac{\mathrm{Vol}_S(S_a)}{\mathrm{Vol}_S(X_{2n-1})}\bigg|_{b_i^{(\pm)}}} \tag{4.9}$$

[17] The sign is consistent with the toric formalism in Sect. 7 as well as with the known explicit supergravity solutions.

Notice that using (3.46) we can therefore write the fluxes in the form

$$\boxed{M_a = \frac{\pi N}{(n-1)b_0}\left[\frac{\mathrm{Vol}_S(S_a)}{\mathrm{Vol}_S(X_{2n-1})}\bigg|_{b_i^{(+)}} - \frac{\mathrm{Vol}_S(S_a)}{\mathrm{Vol}_S(X_{2n-1})}\bigg|_{b_i^{(-)}}\right]} \tag{4.10}$$

This equation must be interpreted carefully. The fluxes M_a on the left hand side are integers, while on the right hand side for fixed X_{2n-1} this is a function of the spindle data $m_\pm$, the flavour twisting variables p_i, and the R-symmetry vector $(b_\mu) = (b_0, b_1, \ldots, b_s) \in \mathbb{R}^{s+1}$. Fixing X_{2n-1}, $m_\pm$ and p_i fixes the internal space Y_{2n+1}, while the extremization involves varying over the R-symmetry vector, which of course is not compatible with holding the M_a fixed in (4.10). However, the correct interpretation is to extremize the supersymmetric action, detailed below in (4.17), (4.20), to obtain the critical R-symmetry b^*_μ, and then use this to compute the associated fluxes for this flavour twist solution using (4.10); in this sense the GK geometry Y_{2n+1} and the fluxes M_a are fixed by X_{2n-1}, $m_\pm$ and p_i. A necessary and sufficient condition for this to make sense is that the right hand side of (4.10) is rational (hence integer for appropriate choice of N). On the other hand, this is guaranteed to be true if $(b^*_\mu) \in \mathbb{Q}^{s+1}$ is itself rational, as the Sasakian volumes are (up to overall powers of π which cancel) rational functions of b_i with rational coefficients [8]. The fluxes are thus determined in this way for the flavour twist. We shall also expand upon this point in Sect. 7.3, in the case that X_{2n-1} is toric.

We now return to the issue of signs. Recall that $N = m_+ N^{X_+} = m_- N^{X_-} > 0$ and hence from (4.7) we can conclude that for odd n we have $\Lambda_\pm \ \mathrm{Vol}_S(X_{2n-1})|_{b_i^{(\pm)}} > 0$ while for even n we have $\mathrm{Vol}_S(X_{2n-1})|_{b_i^{(\pm)}} > 0$. Next consider $\mathrm{Vol}(X_\pm)$:

$$\mathrm{Vol}(X_\pm) \equiv \int_{X_\pm} \eta \wedge \frac{J^{n-1}}{(n-1)!} = \frac{(2b_1)^{n-1}}{m_\pm} \Lambda_\pm^{n-1} \mathrm{Vol}_S(X_{2n-1})|_{b_i^{(\pm)}}. \tag{4.11}$$

Recalling our discussion in Sect. 3.4, by considerations of the toric case (see (7.16)), the orientation on $X_\pm$ that we need to take are such that

$$\mathrm{Vol}(X_+) > 0, \quad \sigma\, \mathrm{Vol}(X_-) > 0. \tag{4.12}$$

Thus, for odd n we have:

$$\Lambda_+ > 0, \quad \sigma\Lambda_- > 0, \qquad \mathrm{Vol}_S(X_{2n-1})|_{b_i^{(+)}} > 0, \quad \sigma\ \mathrm{Vol}_S(X_{2n-1})|_{b_i^{(-)}} > 0, \tag{4.13}$$

while for even n we have:

$$\Lambda_+ > 0, \quad \sigma\Lambda_- > 0, \qquad \mathrm{Vol}_S(X_{2n-1})|_{b_i^{(+)}} > 0, \quad \mathrm{Vol}_S(X_{2n-1})|_{b_i^{(-)}} > 0, \tag{4.14}$$

which fixes the sign ambiguity in solving (4.7).

We now further illustrate using the two cases of physical interest when $n = 3$ and $n = 4$. The case $n = 3$ is associated with $\mathrm{AdS}_3 \times Y_7$ solutions. From (4.13) we have

$$\mathrm{Vol}_S(X_5)|_{b_i^{(+)}} > 0, \qquad \sigma\ \mathrm{Vol}_S(X_5)|_{b_i^{(-)}} > 0, \tag{4.15}$$

and

$$\Lambda_\pm = \frac{\nu_3 N}{8b_1^2 \mathrm{Vol}_S(X_5)|_{b_i^{(\pm)}}}. \tag{4.16}$$

Using (4.6) we therefore find that the off-shell central charge (2.16) can be written

$$\boxed{\mathscr{Z} = \frac{6\pi^3 N^2}{b_1 b_0}\left(\frac{1}{\mathrm{Vol}_S(X_5)|_{b_i^{(+)}}} - \frac{1}{\mathrm{Vol}_S(X_5)|_{b_i^{(-)}}}\right)} \tag{4.17}$$

We should set $b_1 = 2$ when carrying out the extremization.

We now consider the case of $n = 4$, associated with $\mathrm{AdS}_2 \times Y_9$ solutions. From (4.14) we now have

$$\mathrm{Vol}_S(X_7)|_{b_i^{(\pm)}} > 0, \tag{4.18}$$

and

$$\Lambda_+ = \left[\frac{\nu_4 N}{24 b_1^3 \mathrm{Vol}_S(X_7)|_{b_i^{(+)}}}\right]^{1/2}, \qquad \Lambda_- = \sigma\left[\frac{\nu_4 N}{24 b_1^3 \mathrm{Vol}_S(X_7)|_{b_i^{(-)}}}\right]^{1/2} \tag{4.19}$$

The off-shell entropy (2.21) now reads

$$\boxed{\mathscr{S} = \frac{8\pi^3 N^{3/2}}{3 b_0\sqrt{6 b_1}}\left(\frac{1}{\sqrt{\mathrm{Vol}_S(X_7)|_{b_i^{(+)}}}} - \frac{\sigma}{\sqrt{\mathrm{Vol}_S(X_7)|_{b_i^{(-)}}}}\right)} \tag{4.20}$$

We should set $b_1 = 1$ when carrying out the extremization.

It is interesting to make a comparison with fibrations with no baryonic symmetries over round two spheres. To do this we want to consider the twist case, i.e. $\sigma = 1$, set $m_\pm = 1$ and take the limit $b_0 \to 0$. First observe that with $\sigma = 1$ we have

$$\lim_{b_0\to 0}\frac{f(b_i^{(+)}) - f(b_i^{(-)})}{b_0} = -\sum_{i=1}^{s}\left(\frac{v_{+i}}{m_+} + \frac{v_{-i}}{m_-}\right)\frac{\partial f}{\partial b_i} = \sum_{i=1}^{s}\frac{p_i}{m_+ m_-}\frac{\partial f}{\partial b_i}, \tag{4.21}$$

where $f(b_i)$ is a generic function of the Reeb vector. Defining the operator $\nabla = -\sum_{i=1}^{s}\frac{p_i}{m_+m_-}\frac{\partial}{\partial b_i}$ we thus have

$$\lim_{b_0\to 0}\mathscr{Z} = -\frac{6\pi^3 N^2}{b_1}\nabla\frac{1}{\mathrm{Vol}_S(X_{2n-1})}, \tag{4.22}$$

and

$$\lim_{b_0\to 0}\mathscr{S} = -\frac{4N^{3/2}}{\sqrt{b_1}}\nabla\sqrt{\frac{2\pi^6}{27\,\mathrm{Vol}_S(X_{2n-1})}}. \tag{4.23}$$

In these expressions one should take derivatives with respect to b_1 before setting $b_1 = 2$ or $b_1 = 1$, respectively. We also observe that after setting $m_\pm = 1$ (4.23) is the same as equation (3.10) of [12], who considered the case of X_7 with no baryonic symmetries fibred over a round two-sphere

4.2. Universal anti-twist. The universal anti-twist case is associated with Sasaki–Einstein X_{2n-1} fibred over the spindle and, moreover, with the twisting in the fibration only along the Reeb Killing vector of the Sasaki–Einstein space. We therefore impose the condition

$$[J|_{X_\pm}] = \Lambda_\pm[\rho] \in H^2_B(\mathcal{F}_{\xi_\pm}), \tag{4.24}$$

where $\Lambda_\pm$ are constants, to ensure that X_{2n-1} is Sasaki–Einstein. Notice that this is the same condition as the flavour twist (4.1) imposed in the previous subsection. However, in addition we demand that the fluxes, p_i, determining the fibration satisfy

$$p_i = \frac{p_1}{b_1^{(+)}} b_i^{(+)} = \frac{p_1}{b_1^{(-)}} b_i^{(-)}. \tag{4.25}$$

We need to check *a posteriori* that after extremization this is consistent with the p_i being integer, much as with the discussion of fluxes after equation (4.10). We comment on this further below. Note that (4.25) is consistent with (3.24) and, in particular, implies that the twisting parameters can be written as

$$p_i = \frac{p_1}{\left[b_1 + b_0(-a_- + \sigma a_+)\right]} b_i, \qquad i = 2, \dots, s. \tag{4.26}$$

As we will see we shall need to set $\sigma = -1$, and thus just have the anti-twist case, as well as take the Sasaki–Einstein manifold X_{2n-1} to be quasi-regular.

Since the universal anti-twist is a special case of the previous subsection, we can carry over all of the previous results of Sect. 4.1 and write, for general n,

$$S_{\mathrm{SUSY}} = \frac{\pi(\nu_n N)^{\frac{n-1}{n-2}}}{[2b_1(n-1)^{n-1}]^{\frac{1}{n-2}} b_0} \left[\frac{1}{[\mathrm{Vol}_S(X_{2n-1})|_{b_i^{(+)}}]^{\frac{1}{n-2}}} - \frac{k_n}{[\mathrm{Vol}_S(X_{2n-1})|_{b_i^{(-)}}]^{\frac{1}{n-2}}} \right], \tag{4.27}$$

where $k_n = +1$ if n is odd and $k_n = \sigma$ if is n even. As usual, to extremize we should set $b_1 = 2/(n-2)$. To proceed we extremize over b_i and then b_0. For the former, it is useful to use (4.25) to introduce a new variable

$$r_i \equiv \frac{n}{b_1^{(+)}} b_i^{(+)} = \frac{n}{b_1^{(-)}} b_i^{(-)}, \tag{4.28}$$

and we note that $r_1 = n$. We next recall that $\mathrm{Vol}_S(X_{2n-1})(r_i)$, as a function of the Reeb vector r_i, is homogeneous of degree $-n$ in r_i. Hence, we can write

$$\mathrm{Vol}_S(X_{2n-1})|_{b_i^{(\pm)}} = \left(\frac{b_1^{(\pm)}}{n}\right)^{-n} \mathrm{Vol}_S(X_{2n-1})|_{r_i}, \tag{4.29}$$

and after substituting into (4.27) we obtain

$$S_{\mathrm{SUSY}} = \frac{\pi(\nu_n N)^{\frac{n-1}{n-2}}}{[2b_1 n^n (n-1)^{n-1}]^{\frac{1}{n-2}} b_0} \frac{1}{[\mathrm{Vol}_S(X_{2n-1})|_{r_i}]^{\frac{1}{n-2}}} \left[(b_1^{(+)})^{\frac{n}{n-2}} - k_n (b_1^{(-)})^{\frac{n}{n-2}} \right]. \tag{4.30}$$

At this point we can set $b_1 = 2/(n-2)$ and proceed with the extremization. It is clear that extremizing over b_i, $i = 2, 3, \dots$, is the same as extremizing over r_i, $i = 2, 3, \dots$.

Thus, we immediately deduce that S_{SUSY} is extremized over b_i when $\text{Vol}_S(X_{2n-1})|_{r_i}$ is extremized over r_i and, recalling that $r_1 = n$, this results in nothing but the Sasaki–Einstein volume i.e. $\text{Vol}_{SE}(X_{2n-1})$. This leads to the off-shell action

$$S_{\text{SUSY}} = \left(\frac{n-2}{4n^n(n-1)^{n-1}}\right)^{\frac{1}{n-2}} \frac{\pi(\nu_n N)^{\frac{n-1}{n-2}}}{\text{Vol}_{SE}(X_{2n-1})^{\frac{1}{n-2}}} \frac{1}{b_0}\left[(b_1^{(+)})^{\frac{n}{n-2}} - k_n(b_1^{(-)})^{\frac{n}{n-2}}\right], \tag{4.31}$$

with the extremization over b_0 still to be carried out. We will explicitly extremize over b_0 for the two cases of physical interest below and favourably compare with known results for explicit supergravity solutions. In particular we find that only the anti-twist case with $\sigma = -1$ gives rise to a positive on-shell value for S_{SUSY}.

For the geometric R-charges, from (4.9) and (4.28) we have

$$R_a^{\pm} = \frac{2\pi}{(n-1)b_1} \frac{\text{Vol}_S(S_a)}{\text{Vol}_S(X_{2n-1})}\bigg|_{\frac{b_1^{(\pm)} r_i}{n}}, \tag{4.32}$$

where $\text{Vol}_S(S_a)$ is the volume of S_a with respect to the Sasakian metric. Now $\text{Vol}_S(X_{2n-1})(r_i)$ and $\text{Vol}_S(S_a)(r_i)$ are homogeneous of degree $-n$ and $-(n-1)$ in r_i, respectively. Thus, by a similar scaling argument we deduce that after extremizing over b_i the off-shell R-charges, as a function of b_0, are given by

$$R_a^{\pm} = \frac{2\pi b_1^{(\pm)}}{n(n-1)b_1} \frac{\text{Vol}_{SE}(S_a)}{\text{Vol}_{SE}(X_{2n-1})} = \frac{b_1^{(\pm)}}{b_1} R_a, \tag{4.33}$$

where we introduced

$$R_a \equiv \frac{2\pi}{n(n-1)} \frac{\text{Vol}_{SE}(S_a)}{\text{Vol}_{SE}(X_{2n-1})}. \tag{4.34}$$

For $n = 3$ and $n = 4$ we have $R_a \equiv R_a^{\text{4d}}$ and $R_a \equiv R_a^{\text{3d}}$, respectively, where R_a^{4d} and R_a^{3d} are the R-charges associated with the $\text{AdS}_5 \times SE_5$ and $\text{AdS}_4 \times SE_7$ solutions, respectively. Note that combining (3.46) with (3.24), the equations above for $R_a^{\pm}$ allows us to find the fluxes (with $\sigma = -1$)

$$M_a = \frac{b_1}{2b_0}(R_a^+ - R_a^-)N = \frac{1}{2b_0}(b_1^{(+)} - b_1^{(-)})R_a N = \frac{m_+ - m_-}{2m_+m_-} R_a N, \tag{4.35}$$

and we also have (with $\sigma = -1$)

$$R_a^+ + R_a^- = \left(2 - \frac{b_0}{b_1}\chi\right) R_a, \tag{4.36}$$

where $\chi = (m_- + m_+)/(m_- m_+)$ is the orbifold Euler character of the spindle. In both (4.35), (4.36) we have set $\sigma = -1$, as it is needed for the universal anti-twist case. Note that in the special case that the SE space is toric, we have $\sum_a R_a = 2$ and hence we then also have (with $\sigma = -1$)

$$\frac{1}{2}\sum_a (R_a^+ + R_a^-) = 2 - \frac{b_0}{b_1}\chi. \tag{4.37}$$

Finally, we should return to the issue of everything being properly quantized. Combining (4.26) with (4.28) and evaluating on-shell, we have

$$p_i = \frac{p_1}{n} r_i^*, \tag{4.38}$$

where r_i^* is the critical Reeb vector for the Sasaki–Einstein metric on X_{2n-1}. The latter is thus necessarily quasi-regular, meaning that $(r_i^*) \in \mathbb{Q}^s$ is rational. One may then ensure that $p_i \in \mathbb{Z}$ in (4.38) by appropriately choosing $p_1 = m_+ - m_- \in \mathbb{Z}$, which is a constraint on the choice of spindle Σ. Note that a similar conclusion was reached for the universal twist solutions in [10], where there $b_0 = 0$ and $\Sigma = \Sigma_g$ is a smooth Riemann surface. In that case the universal twist requires an appropriate divisibility property for the genus g of the Riemann surface. Such issues were also effectively discussed in [21] for the explicit $\mathrm{AdS}_3 \times Y_7$ universal anti-twist solutions described in the next subsection, where $\mathbb{Z}_k$ quotients along the Reeb $U(1)$ isometry of the Sasaki–Einstein metric on X_{2n-1} were also considered.[18]

Finally, since the R-charges R_a are rational for rational Reeb vector r_i^*, one may ensure the fluxes M_a in (4.35) are integer by appropriately choosing N. Again, *cf.* the discussion of the universal twist in [10].

4.2.1. $AdS_3 \times Y_7$ case For the $n = 3$ case, from (2.16) and (4.31), we therefore have

$$\mathscr{Z} = \frac{\pi^3 N^2}{9\mathrm{Vol}(SE_5)} \frac{1}{b_0} \left[(b_1^{(+)})^3 - (b_1^{(-)})^3 \right]. \tag{4.39}$$

Notice that this expression is quadratic in b_0. If we extremize over b_0 we find that it is only in the anti-twist case, $\sigma = -1$, that we get a positive on-shell central charge. So, setting $\sigma = -1$ we find the extremal value

$$b_{0*} = \frac{3m_- m_+ (m_- + m_+)}{m_-^2 + m_- m_+ + m_+^2}, \tag{4.40}$$

and one can check that (4.15) is satisfied, provided that we take $m_+ > m_-$ (which is associated with taking $N > 0$ and (4.12)). We can express the on-shell central charge in terms of, $a_{4\mathrm{d}}$, the central charge of the $d = 4$ SCFT, which is given by

$$a_{4\mathrm{d}} = \frac{\pi^3 N^2}{4\mathrm{Vol}(SE_5)}, \tag{4.41}$$

and we find

$$\mathscr{Z}_{\mathrm{os}} = \frac{4(m_+ - m_-)^3}{3m_- m_+ (m_-^2 + m_- m_+ + m_+^2)} a_{4\mathrm{d}}, \tag{4.42}$$

which is positive for $m_+ > m_-$. The results (4.42) and (4.40) are in precise alignment with the results (21), (22) for the explicit supergravity solutions constructed in minimal gauged supergravity in [21] (after identifying $m_\pm$ with $n_\mp$).

From (4.33), the (off-shell) geometric R-charges can be written in the form

$$R_a^{\pm} = \frac{1}{2} b_1^{(\pm)} R_a^{4\mathrm{d}}, \tag{4.43}$$

[18] For such quotients the effective $U(1)^s$ action is not the same as on the unquotiented space.

and hence

$$R_a^+ = \frac{(m_+ - m_-)(m_- + 2m_+)}{2(m_-^2 + m_- m_+ + m_+^2)} R_a^{4d}, \quad R_a^- = -\frac{(m_+ - m_-)(2m_- + m_+)}{2(m_-^2 + m_- m_+ + m_+^2)} R_a^{4d}, \quad (4.44)$$

where R_a^{4d} are the four-dimensional R-charges associated with the $AdS_5 \times SE_5$ solution. Notice that with $R_a^{4d} > 0$ and $m_+ > m_-$ we have $R_a^+ > 0$ and $\sigma R_a^- > 0$ (with $\sigma = -1$ for this case as we have seen). We will argue more generally in the toric setting that we always have $R_a^+ > 0$ and $\sigma R_a^- > 0$.

4.2.2. $AdS_2 \times Y_9$ case For $n = 4$ using (2.21) and (4.31) we get the off-shell entropy

$$\mathscr{S} = \frac{N^{3/2}\pi^3}{6^{3/2}\mathrm{Vol}(SE_7)^{1/2}} \frac{1}{b_0} \left[(b_1^{(+)})^2 - \sigma (b_1^{(-)})^2 \right], \quad (4.45)$$

which is to be extremized over b_0. If $\sigma = +1$ there are no extrema, so we again conclude that we again must have the anti-twist case with $\sigma = -1$.

We can express the off-shell result in terms of the four-dimensional Newton constant $G_{(4)}$ (in the conventions of [22]), defined by

$$\frac{1}{G_{(4)}} = \frac{2^{3/2}\pi^2}{3^{3/2}\mathrm{Vol}(SE_7)^{1/2}} N^{3/2}. \quad (4.46)$$

Recall that the free energy on the three-sphere, $\mathcal{F}_{S^3}$, of the $d = 3$ SCFT dual to the $AdS_4 \times SE_7$ solution is given by

$$\mathcal{F}_{S^3} = \frac{\pi}{2G_{(4)}}. \quad (4.47)$$

We then can express the off-shell entropy (with $\sigma = -1$) as

$$\mathscr{S} = \frac{\pi}{8G_{(4)}} \frac{1}{b_0} \left[(b_1^{(+)})^2 + (b_1^{(-)})^2 \right]. \quad (4.48)$$

Extremizing over b_0 (and setting $b_1 = 1$) we find two solutions but only one gives a positive entropy. Focusing on this extremum we find

$$b_{0*} = \frac{\sqrt{2} m_- m_+}{\sqrt{m_-^2 + m_+^2}}, \quad (4.49)$$

with the associated on-shell entropy given by

$$\mathscr{S}_{\mathrm{os}} = \frac{\pi}{4G_{(4)}} \frac{\sqrt{2}\sqrt{m_-^2 + m_+^2} - m_+ - m_-}{m_- m_+}. \quad (4.50)$$

This agrees with the entropy for the supersymmetric, accelerating and magnetically charged black holes that were constructed in minimal gauged supergravity in (1.1) of [22].

From (4.33), the (off-shell) geometric R-charges can be written in the form

$$R_a^{\pm} = b_1^{(\pm)} R_a^{3d}, \quad (4.51)$$

and hence

$$R_a^+ = \frac{\sqrt{m_-^2 + m_+^2} - \sqrt{2}m_-}{\sqrt{m_-^2 + m_+^2}} R_a^{3\mathrm{d}}, \quad R_a^- = \frac{\sqrt{m_-^2 + m_+^2} - \sqrt{2}m_+}{\sqrt{m_-^2 + m_+^2}} R_a^{3\mathrm{d}}, \tag{4.52}$$

where $R_a^{3\mathrm{d}}$ are the four-dimensional R-charges associated with the $\mathrm{AdS}_4 \times SE_7$ solution. Notice that with $m_+ > m_-$ and $R_a^{3\mathrm{d}} > 0$ we again have $R_a^+ > 0$ and $\sigma R_a^- > 0$ (with $\sigma = -1$).

Finally, as in [51] we can also make a comparison of the off-shell entropy (4.48) with an off-shell entropy function that was constructed in [22]. More specifically, [22] considered a complex locus of supersymmetric, accelerating and magnetically[19] charged black holes that were non-extremal and then holographically calculated the on-shell action I at the AdS_4 boundary. After suitable extremization I gives rise to the on-shell entropy. To make the connection, note that we can also write the off-shell action (4.48) with $b_1 = 1$ as

$$\mathscr{S} = \mp \frac{1}{2\mathrm{i}G_{(4)}} \left(\frac{\varphi^2}{\omega} + (G_{(4)} Q_m)^2 \omega \right), \tag{4.53}$$

with

$$(G_{(4)} Q_m) = \frac{m_- - m_+}{4m_- m_+}, \qquad \varphi = \frac{\chi}{4}\omega \pm \pi \mathrm{i}, \qquad \omega = \mp 2\pi \mathrm{i}\, b_0, \tag{4.54}$$

and we see this is precisely the same as (4.41) of [53] with $\mathscr{S} = -I$. We also highlight that in the special case that the SE_7 is toric, using (4.37) we can also write

$$\varphi = \pm \frac{\pi \mathrm{i}}{4} \sum_a (R_a^+ + R_a^-). \tag{4.55}$$

The off-shell action (4.48) also agrees with (5.20) of [53], with $\mathscr{S} = -I$, after setting $a_1 = 1$ and $a_2 = -b_0$.

5. Some Examples

In this section we use the general results of the previous section to calculate some physical quantities for particular examples of $\mathrm{AdS}_3 \times Y_7$ and $\mathrm{AdS}_2 \times Y_9$ solutions. For the $\mathrm{AdS}_3 \times Y_7$ solutions we consider $X_5 = S^5$ and compare with known results associated with explicit supergravity solutions. For the $\mathrm{AdS}_2 \times Y_9$ solutions we consider $X_7 = S^7$, for which we can also compare with known supergravity results, as well as $X_7 = V^{5,2}$ for which explicit supergravity solutions are not known. Note that the cases $X_5 = S^5$ and $X_7 = S^7$ can also be treated with the toric methods that we develop in Sects. 6 and 7.

[19] In fact the calculation in [22] was carried out for a larger family of black hole solutions with additional electric charge and rotation.

5.1. $AdS_3 \times Y_7$ solutions with $X_5 = S^5$. Since $H^2(S^5, \mathbb{R}) \cong 0$ the case of $X_5 = S^5$ is an example with no baryonic symmetries, as in Sect. 4. In a suitable basis, compatible with the toric data we will use for this example later, the Sasakian volume of S^5 as a function of the Reeb vector can be written as

$$\mathrm{Vol}_S(S^5) = \frac{\pi^3}{b_2\, b_3\, (b_1 - b_2 - b_3)}. \tag{5.1}$$

As is well known there are three supersymmetric three-submanifolds S_a which are $U(1)^3$ invariant and associated with baryonic operators in $\mathcal{N} = 4$ SYM theory dual to $\mathrm{AdS}_5 \times S^5$. The Sasakian volume of these submanifolds are given by

$$\mathrm{Vol}_S(S_1) = \frac{2\pi^2}{b_2\, b_3}, \qquad \mathrm{Vol}_S(S_a) = \frac{2\pi^2}{b_a\, (b_1 - b_2 - b_3)}, \quad a = 2, 3. \tag{5.2}$$

It will be convenient below to briefly recall that we obtain the Sasaki–Einstein volume by setting $b_1 = 3$ in (5.1) and then extremizing over b_2, b_3. The extremal values are $b_2 = b_3 = 1$ with $\mathrm{Vol}_{SE}(S^5) = \pi^3$ and $\mathrm{Vol}_{SE}(S_a) = 2\pi^2$.

We now consider the GK geometry associated with the $\mathrm{AdS}_3 \times Y_7$ solutions. The off-shell central charge can be obtained from (4.17) and reads

$$\mathscr{Z} = \frac{6N^2}{b_1 b_0}\left[b_2^{(+)} b_3^{(+)}\left(b_1^{(+)} - b_2^{(+)} - b_3^{(+)}\right) - b_2^{(-)} b_3^{(-)}\left(b_1^{(-)} - b_2^{(-)} - b_3^{(-)}\right)\right]. \tag{5.3}$$

Similarly, the off-shell geometric R-charges are obtained from (4.9) and are given by

$$R_1^{\pm} = \frac{2}{b_1}\left(b_1^{(\pm)} - b_2^{(\pm)} - b_3^{(\pm)}\right), \qquad R_a^{\pm} = \frac{2b_a^{(\pm)}}{b_1}, \quad a = 2, 3. \tag{5.4}$$

Observe that for this example, which is also toric, we have

$$\begin{aligned} \sum_{a=1}^{d} R_a^{\pm} &= \frac{2b_1^{(\pm)}}{b_1}, \\ \frac{1}{2}\sum_{a=1}^{d}\left(R_a^{+} + R_a^{-}\right) &= 2 - \frac{m_- - \sigma m_+}{m_+ m_-}\frac{b_0}{b_1}, \end{aligned} \tag{5.5}$$

in agreement with the general toric results (7.47), (7.49).

Recall that there are two preferred fluxes, $N^{X_\pm}$, associated with the fibres $X_\pm \cong S^5/\mathbb{Z}_{m_\pm}$ at the north and south poles, respectively, and from (3.40) we have $N \equiv m_+ N^{X_+} = m_- N^{X_-}$. There are no further cohomologically non-trivial five-form fluxes. However, we also have three five-form fluxes M_a defined as the integral of the five-form flux through the three submanifolds Σ_a, obtained as the three supersymmetric submanifolds S_a of S^5 fibred over the spindle, as in (3.44). Using (3.46) we obtain the following expression for the fluxes M_a:

$$\mathfrak{p}_a \equiv \frac{M_a}{N} = \frac{1}{m_+ m_-}\left\{p_1 - (p_2 + p_3),\, p_2,\, p_3\right\}, \tag{5.6}$$

where as usual, $p_1 = -\sigma m_+ - m_-$. For this example, we have $\sum_{a=1}^{d} M_a = \frac{p_1}{m_+ m_-} N$, which one can check is in agreement with the general result (7.34) for toric X_7. We also note that the variables $\mathfrak{p}_a$ we have introduced can be identified with the background magnetic fluxes of the dual $d = 4$ SCFT, in this case $\mathcal{N} = 4$ SYM theory, as we discuss in more detail in the toric setting in Sect. 8.

To obtain the on-shell values for the central charge and R-charges we should set $b_1 = 2$ and then extremize $\mathscr{Z}$ over b_0, b_2 and b_3. The expression for $\mathscr{Z}$ is quadratic in b_0, b_2, b_3 and we find that the extremal values for b_0, b_2 and b_3 depend on the spindle data $m_\pm, \sigma$ as well as $a_\pm$ (subject to (3.11)). However, the on-shell values of $\mathscr{Z}$ and $R_a^\pm$ are independent of $a_\pm$.

There is a $\mathbb{Z}_3$ symmetry permuting the $\mathfrak{p}_a$ but not (p_1, p_2, p_3): the latter is a consequence of the fact that we singled out the p_1 direction in constructing the fibration. It is therefore illuminating to express the on-shell values of $\mathscr{Z}$ and $R_a^\pm$ in terms of $\mathfrak{p}_a$. For the on-shell central charge we find

$$c_{\text{sugra}} \equiv \mathscr{Z}_{\text{os}} = \frac{6m_-^2 m_+^2 \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3}{m_-^2 + m_+^2 - m_-^2 m_+^2(\mathfrak{p}_1^2 + \mathfrak{p}_2^2 + \mathfrak{p}_3^2)} N^2. \tag{5.7}$$

In the case of the twist $\sigma = +1$ we have exact agreement with the supergravity solutions of [27].[20] For the anti-twist case $\sigma = -1$ we likewise have agreement with the supergravity solutions found in [24,27], and for the special anti-twist case when $\mathfrak{p}_1 = \mathfrak{p}_2 = \mathfrak{p}_3 = (\frac{m_+ - m_-}{3m_- m_+})$ we recover the result that we derived in (4.42) for the universal anti-twist after using $a_{4\text{d}} = N^2/4$; this is also the result found for the supergravity solutions constructed in [21].

Starting with (4.9), the on-shell expressions for the R-charges can be written

$$\begin{aligned} R_a^+ &= -Cm_+ \left(\mathfrak{p}_1[\sigma + m_-\mathfrak{p}_1], \mathfrak{p}_2[\sigma + m_-\mathfrak{p}_2], \mathfrak{p}_3[\sigma + m_-\mathfrak{p}_3]\right), \\ R_a^- &= -Cm_- \left(\mathfrak{p}_1[1 + m_+\mathfrak{p}_1], \mathfrak{p}_2[1 + m_+\mathfrak{p}_2], \mathfrak{p}_3[1 + m_+\mathfrak{p}_3]\right), \end{aligned} \tag{5.8}$$

where

$$C = \frac{2m_- m_+}{m_-^2 + m_+^2 - m_-^2 m_+^2(\mathfrak{p}_1^2 + \mathfrak{p}_2^2 + \mathfrak{p}_3^2)}. \tag{5.9}$$

This is a new result which could be checked using the explicit supergravity solutions. It is also interesting to observe that if we demand, on-shell, $c_{\text{sugra}} > 0$ as well as $R_a^+ > 0$ and $\sigma R_a^- > 0$ then, for the anti-twist case ($\sigma = -1$) we find we must have $\mathfrak{p}_1 > 0$, $\mathfrak{p}_2 > 0$ and $\mathfrak{p}_3 = -\mathfrak{p}_1 - \mathfrak{p}_2 + (m_+ - m_-)/(m_+ m_-) > 0$ (which, in particular, implies $m_+ > m_-$); this is in alignment with the result found for the explicit supergravity solutions as in (3.30) of [27]. Similarly for the twist case ($\sigma = +1$) we find that we must have any two of $(\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3)$ to be positive, now with $\mathfrak{p}_3 = -\mathfrak{p}_1 - \mathfrak{p}_2 - (m_+ + m_-)/(m_+ m_-) > 0$ which is also consistent[21] with the result found for the explicit supergravity solutions, see (3.31) of [27].

[20] We should identify $\mathfrak{p}_a$ with $p_a/(m_- m_+)$ there, as well as m_-, m_+ with n_1, n_2, respectively, and notice that the constraint (8.28) on our $\mathfrak{p}_a$ is twist case A of [27] when $\sigma = +1$ and anti-twist case B when $\sigma = -1$.

[21] Here we do not find a condition $m_+ > m_-$ as stated in (3.31) of [27]; the resolution is that twist solutions can also be found using the analysis of [27] with $m_+ < m_-$ after relabelling.

For the universal anti-twist case with $\mathfrak{p}_1 = \mathfrak{p}_2 = \mathfrak{p}_3 = (\frac{m_+ - m_-}{3m_- m_+})$ and $\sigma = -1$, the geometric R-charges simplify to give

$$
\begin{aligned}
R_a^+ &= \frac{(m_+ - m_-)(m_- + 2m_+)}{3(m_-^2 + m_- m_+ + m_+^2)}, \\
R_a^- &= -\frac{(m_+ - m_-)(m_+ + 2m_-)}{3(m_-^2 + m_- m_+ + m_+^2)}.
\end{aligned} \tag{5.10}
$$

This is in exact agreement with the general result of (4.43) after using the fact that $R_a^{4\mathrm{d}} = 2/3$.

Finally, we can prove a conjecture for the central charge stated in [23,30]. To see this, observe that we can use (5.3), (5.4) to write the off-shell central charge as

$$
\mathscr{Z} = \frac{3N^2}{b_0}\frac{2}{b_1}\left[\prod_{a=1}^{3}\left(\varphi_a + \mathfrak{p}_a \frac{b_0}{2}\right) - \prod_{a=1}^{3}\left(\varphi_a - \mathfrak{p}_a \frac{b_0}{2}\right)\right], \tag{5.11}
$$

where

$$
\varphi_a \equiv \frac{b_1}{4}\left(R_a^+ + R_a^-\right), \tag{5.12}
$$

and $\sum_a \varphi_a = \frac{1}{2}(b_1^{(+)} + b_1^{(-)}) = b_1 - \frac{b_0}{2}\frac{m_- - \sigma m_+}{m_- m_+}$. We also recall $\sum \mathfrak{p}_a = -\frac{\sigma m_+ + m_-}{m_+ m_-}$ and that we should set $b_1 = 2$ before extremizing. This is then in agreement with e.g. [30] provided that we identify our $(m_\pm, \varphi_a, b_0, \mathfrak{p}_a)$ with their $(n_\mp, \varphi_a, -2\epsilon, -\mathfrak{n}_a)$.

5.2. Examples of $AdS_2 \times Y_9$ solutions. We now consider some examples of $\mathrm{AdS}_2 \times Y_9$ solutions of $D = 11$ supergravity with Y_9 a fibration of X_7 over a spindle, specifically, $X_7 = S^7$ and also $X_7 = V^{5,2}$. Both of these cases are examples with no baryonic symmetries with $H^2(X_7, \mathbb{R}) \cong 0$ and can be analysed using the results of Sect. 4. The S^7 case is toric and can also be treated using the toric formalism, similar to the analysis for S^5 later in Sect. 8.4.1.

5.2.1. $X_7 = S^7$ example In a suitable basis, the Sasakian volume of S^7, as a function of the Reeb vector, is given by

$$
\mathrm{Vol}_S(S^7) = \frac{\pi^4}{3\, b_2 b_3 b_4 (b_1 - b_2 - b_3 - b_4)}. \tag{5.13}
$$

There are four supersymmetric five-submanifolds S_a, which are $U(1)^4$ invariant and associated with baryonic operators in the SCFT dual to $\mathrm{AdS}_4 \times S^7$. The Sasakian volumes of these supersymmetric submanifolds are given by

$$
\begin{aligned}
\mathrm{Vol}_S(S_1) &= \frac{\pi^3}{b_2 b_3 b_4}, \\
\mathrm{Vol}_S(S_a) &= \frac{\pi^3 b_a}{b_2 b_3 b_4 (b_1 - b_2 - b_3 - b_4)}, \quad a = 2, 3, 4.
\end{aligned} \tag{5.14}
$$

The Sasaki–Einstein volume can be obtained by setting $b_1 = 4$ and then extremizing (5.13) over b_2, b_3, b_4. The extremal point has $b_2 = b_3 = b_4 = 1$ with $\mathrm{Vol}_{SE}(S^7) = \pi^4/3$ and $\mathrm{Vol}_{SE}(S_a) = \pi^3$.

We now consider the GK geometry associated with $\mathrm{AdS}_2 \times Y_9$ solutions. The off-shell entropy function can be obtained from (4.20) and takes the form

$$\mathscr{S} = \frac{8\pi N^{3/2}}{3b_0\sqrt{2b_1}}\left[\sqrt{b_2^{(+)}b_3^{(+)}b_4^{(+)}\left(b_1^{(+)}-b_2^{(+)}-b_3^{(+)}-b_4^{(+)}\right)}\right.$$
$$\left.-\sigma\sqrt{b_2^{(-)}b_3^{(-)}b_4^{(-)}\left(b_1^{(-)}-b_2^{(-)}-b_3^{(-)}-b_4^{(-)}\right)}\right]. \tag{5.15}$$

The off-shell geometric R-charges are obtained from (4.9) and are given by

$$R_1^{\pm} = \frac{2}{b_1}\left(b_1^{(\pm)}-b_2^{(\pm)}-b_3^{(\pm)}-b_4^{(\pm)}\right), \qquad R_a^{\pm} = \frac{2b_a^{(\pm)}}{b_1}, \quad a = 2, 3, 4. \tag{5.16}$$

Observe that for this example, which is also toric, we have

$$\sum_{a=1}^{d} R_a^{\pm} = \frac{2b_1^{(\pm)}}{b_1},$$
$$\frac{1}{2}\sum_{a=1}^{d}\left(R_a^{+}+R_a^{-}\right) = 2 - \frac{m_- - \sigma m_+}{m_+m_-}\frac{b_0}{b_1}, \tag{5.17}$$

in agreement with the general toric results (7.47), (7.49). Also, using (3.46) we can find the following expression for the preferred five-form fluxes M_a defined in (3.44).

$$\mathfrak{p}_a \equiv \frac{M_a}{N} = \frac{1}{m_+m_-}\left\{p_1-(p_2+p_3+p_4),\ p_2,\ p_3,\ p_4\right\}, \tag{5.18}$$

where, as usual, $p_1 = -\sigma m_+ - m_-$. For this example, we have $\sum_{a=1}^{d} M_a = \frac{p_1}{m_+m_-}N$, which one can check is in agreement with the general result (7.34) for toric X_7.

To obtain the on-shell values for $\mathscr{S}$ and $R_a^{\pm}$, we should set $b_1 = 1$, as appropriate for an $\mathrm{AdS}_2 \times Y_9$ solution, and then extremize (5.15) as a function of b_2, b_3, b_4, and b_0. Performing such a computation analytically with general fluxes is difficult due to the presence of the square roots. We will look at a simpler case when the fluxes are pairwise equal, i.e. $M_1 = M_2$ and $M_3 = M_4$, which amounts to setting

$$p_2 = \frac{1}{2}(p_1 - 2p_3), \qquad p_4 = p_3. \tag{5.19}$$

This choice corresponds to considering fibrations of the S^7 that maintain $U(2)\times U(2) \subset SO(8)$ symmetry and will allow us to compare with some explicit supergravity solutions. Notice in passing that this implies that $p_1 = -\sigma m_+ - m_-$ must be even. Consistent with this $U(2)\times U(2)$ symmetry we now make the further assumption that we can carry out the extremization over the locus

$$b_2 = \frac{1}{2}(b_1 - 2b_3) + \frac{b_0}{2}(\sigma a_+ - a_-), \qquad b_4 = b_3, \tag{5.20}$$

which implies

$$b_2^{(\pm)} = \frac{1}{2}(b_1^{(\pm)} - 2b_3^{(\pm)}), \qquad b_4^{(\pm)} = b_3^{(\pm)}. \tag{5.21}$$

With these restrictions, the arguments of the square roots in (5.15) become perfect squares, so that

$$\mathscr{S} = \frac{4\pi N^{3/2}}{3b_0\sqrt{2}}\left[\delta_+ b_3^{(+)}(b_1^{(+)} - 2b_3^{(+)}) - \delta_- b_3^{(-)}(b_1^{(-)} - 2b_3^{(-)})\right], \tag{5.22}$$

where $\delta_\pm$ are signs defined by

$$\delta_+ \equiv \text{sign}\left[b_3^{(+)}(b_1^{(+)} - 2b_3^{(+)})\right], \qquad \delta_- \equiv \sigma\,\text{sign}\left[b_3^{(-)}(b_1^{(-)} - 2b_3^{(-)})\right]. \tag{5.23}$$

We now need to extremize over b_0, b_3 with $b_1 = 2$. By an explicit calculation we find that when $\delta_+ = \delta_-$, (5.22) is linear in b_3 and b_0 and so it does not have any extremum. Hence we set

$$\delta_+ = -\delta_- \equiv \delta. \tag{5.24}$$

We then find two different set of values for $(b_0,\ b_3)$ that extremize the off-shell entropy (5.22)

$$b_0^* = \mp\frac{2\sigma m_+ m_-}{\mathcal{D}}, \qquad b_3^* = \frac{1}{4} \pm \frac{\sigma m_- - m_+ + 4p_3\sigma(1 - 2a_- m_+)}{4\mathcal{D}}, \tag{5.25}$$

where

$$\mathcal{D} = \sqrt{-16p_3^2 - 8p_3(\sigma m_+ + m_-) + (\sigma m_- - m_+)^2}, \tag{5.26}$$

with the corresponding entropy given by

$$\mathscr{S}_{\text{os}} = -\delta\frac{\pi N^{3/2}}{3\sqrt{2}}\left(\frac{m_- - \sigma m_+ \pm \sigma\mathcal{D}}{m_- m_+}\right). \tag{5.27}$$

Clearly there are restrictions on $m_\pm > 0$, p_3 and σ to ensure that $\mathscr{S}_{\text{os}}$ is real and positive. The conditions (5.23), (5.24), and the positivity of the entropy imply

$$\delta = 1 \quad \Rightarrow \quad \begin{cases} b_3^{(+)}(b_1^{(+)} - 2b_3^{(+)})|_{b_0^*, b_3^*} > 0 \\ \sigma\, b_3^{(-)}(b_1^{(-)} - 2b_3^{(-)})|_{b_0^*, b_3^*} < 0 \\ m_- - \sigma m_+ \pm \sigma\mathcal{D} < 0, \end{cases} \tag{5.28}$$

$$\delta = -1 \quad \Rightarrow \quad \begin{cases} b_3^{(+)}(b_1^{(+)} - 2b_3^{(+)})|_{b_0^*, b_3^*} < 0 \\ \sigma\, b_3^{(-)}(b_1^{(-)} - 2b_3^{(-)})|_{b_0^*, b_3^*} > 0 \\ m_- - \sigma m_+ \pm \sigma\mathcal{D} > 0. \end{cases} \tag{5.29}$$

For the twist case, $\sigma = +1$, we find that it is not possible to satisfy these conditions. Hence, there are no twist solutions with $X_7 = S^7$ and pairwise equal fluxes. This conclusion was also reached within the context of a specific class of local supergravity solutions in [27,28].

We now consider the anti-twist case, $\sigma = -1$. In this case there are no possibilities for the lower sign extremal solution in (5.28), (5.29), but there are possibilities for the upper sign. In particular we find

$$b_0^* = \frac{2m_+ m_-}{\mathcal{D}}, \qquad b_3^* = \frac{1}{4} - \frac{m_- + m_+ + 4p_3(1 - 2a_- m_+)}{4\mathcal{D}}, \tag{5.30}$$

and

$$\mathscr{S}_{\rm os} = -\delta \frac{\pi N^{3/2}}{3\sqrt{2}} \left(\frac{m_- + m_+ - \mathcal{D}}{m_- m_+} \right), \tag{5.31}$$

with

$$\begin{aligned}
\delta = 1 \quad &\text{for:} \quad \frac{1}{2}(m_+ - m_-) < p_3 < 0 \quad \text{or} \quad 0 < p_3 < \frac{1}{2}(m_+ - m_-), \\
\delta = -1 \quad &\text{for:} \quad -\frac{m_-}{2} < p_3 < \min\left[0, \frac{1}{2}(m_+ - m_-)\right] \\
&\text{or} \quad \max\left[0, \frac{1}{2}(m_+ - m_-)\right] < p_3 < \frac{m_+}{2}.
\end{aligned} \tag{5.32}$$

Notice that in both cases $m_+ - m_-$ is even. For these classes of anti-twist solutions we can also compute the on-shell R-charges and find

$$\begin{aligned}
R_1^+ = R_2^+ &= \frac{1}{2} + \frac{1}{2\mathcal{D}}(m_+ - 3m_- - 4p_3), \\
R_3^+ = R_4^+ &= \frac{1}{2} - \frac{1}{2\mathcal{D}}(m_+ + m_- - 4p_3), \\
R_1^- = R_2^- &= \frac{1}{2} - \frac{1}{2\mathcal{D}}(3m_+ - m_- - 4p_3), \\
R_3^- = R_4^- &= \frac{1}{2} - \frac{1}{2\mathcal{D}}(m_+ + m_- + 4p_3).
\end{aligned} \tag{5.33}$$

Note that (5.31) is consistent with the analysis of explicit supergravity solutions studied in [25].[22] The expressions for the geometric R-charges (5.33) are a new result, which could, in principle, be compared with a calculation using the supergravity solutions. If we now impose the conditions $R_a^+ > 0$ and $\sigma R_a^- > 0$ (here with $\sigma = -1$) we find that we are only left with the case

$$\delta = 1 \quad \text{for:} \quad 0 < p_3 < \frac{1}{2}(m_+ - m_-). \tag{5.34}$$

We can also make a comparison with the analysis of explicit supergravity solution in [28]. An expression for the on-shell entropy for general fluxes was given in equation (6.4) of [28] and reads, in their notation,

$$\mathscr{S}_{[28]}^{\rm os} = \frac{2\pi}{3m_+m_-} N^{3/2} \sqrt{-\sigma m_+ m_- + \hat{P}^{(2)} + \sigma \sqrt{\left(-\sigma m_+ m_- + \hat{P}^{(2)}\right)^2 - 4\hat{P}^{(4)}}}, \tag{5.35}$$

where

$$\hat{P}^{(2)} = (m_+ m_-)^2 \sum_{a<b} \mathfrak{p}_a \mathfrak{p}_b, \qquad \hat{P}^{(4)} = (m_+ m_-)^4 \prod_{a=1}^{4} \mathfrak{p}_a. \tag{5.36}$$

[22] The fluxes there are 2 times the fluxes here.

We expect that this agrees with our general result; we will explicitly check it agrees in the special case of pairwise equal fluxes. Specifically, from (5.18), (5.19) we substitute

$$\mathfrak{p}_1 = \mathfrak{p}_2 = \frac{m_+ - m_- - 2p_3}{2m_+m_-}, \qquad \mathfrak{p}_3 = \mathfrak{p}_4 = \frac{p_3}{m_+m_-}, \tag{5.37}$$

to find that (5.35) can be simplified to give

$$\mathscr{S}^{\text{os}}_{[28]} = \frac{2\pi}{3m_+m_-} N^{3/2} \sqrt{\frac{(m_- + m_+ - \mathcal{D})^2}{8}}. \tag{5.38}$$

Taking the square root we get agreement with the two distinct branches, $\delta = \pm 1$, above. Here we ruled out the $\delta = -1$ branch and one can do the same using the analysis of [28]. In addition we believe that our result (5.34) arising from the positivity conditions R-charges is in accord with the existence of supergravity solutions of [28] (up to a relabelling of the parameters in the solutions and using (2.17) of [28]).

We can now consider the further special sub-case of the universal anti-twist which has, with $\sigma = -1$,

$$p_2 = p_3 = p_4 = \frac{1}{4} p_1 = \frac{1}{4}(m_+ - m_-). \tag{5.39}$$

In this case the fluxes in (5.18) read

$$\mathfrak{p}_a \equiv \frac{M_a}{N} = \frac{p_1}{4m_+m_-}\{1, 1, 1, 1\}. \tag{5.40}$$

We can also check that the resulting extremal values for b_2, b_3, b_4 and b_0 are consistent with (4.26). The on-shell entropy (5.31) and R-charges (5.33) also agree with those given in Sect. 4.2.2.

Finally, we can prove a conjecture for the entropy stated in [30] (see also [23]). To see this observe that we can use (5.15), (5.16) to write the off-shell entropy function, not assuming the $U(2)^2$ symmetry condition (5.19), as

$$\mathscr{S} = \frac{2\pi N^{3/2}}{3b_0\sqrt{2b_1}}\left[\sqrt{\prod_{a=1}^{4}(\varphi_a + \mathfrak{p}_a b_0)} - \sigma \sqrt{\prod_{a=1}^{4}(\varphi_a - \mathfrak{p}_a b_0)}\right], \tag{5.41}$$

where

$$\varphi_a \equiv \frac{b_1}{2}\left(R_a^+ + R_a^-\right), \tag{5.42}$$

and $\sum_a \varphi_a = b_1^{(+)} + b_1^{(-)} = 2b_1 - b_0 \frac{m_- - \sigma m_+}{m_- m_+}$. We also recall $\sum \mathfrak{p}_a = -\frac{\sigma m_+ + m_-}{m_+ m_-}$ and that we should set $b_1 = 1$ before extremizing. This is then in agreement with [30] provided that we identify our $(m_\pm, \varphi_a, b_0, \mathfrak{p}_a)$ with their $(n_\mp, \varphi_a, -\epsilon, -\mathfrak{n}_a)$.

5.2.2. $X_7 = V^{5,2}$ example We now consider an example of X_7 where there is currently no known explicit supergravity solution. Specifically, we consider the (non-toric) Sasaki manifold $V^{5,2} = SO(5)/SO(3)$. This manifold has $SO(5) \times U(1)$ isometry, with a maximal torus $U(1)^3$, and hence this is an example with $s = 3$. The Sasakian volume can be computed using the Hilbert series method and reads [12]

$$\mathrm{Vol}_S(V^{5,2}) = \frac{54\pi^4}{(b_1 - b_2)(b_1 + b_2)(b_1 - b_3)(b_1 + b_3)}. \tag{5.43}$$

There are five supersymmetric five-submanifolds S_a, which are $U(1)^3$ invariant, and are associated with baryonic operators in the SCFT dual to $\mathrm{AdS}_4 \times V^{5,2}$. The Sasakian volumes of these supersymmetric submanifolds are given by

$$\begin{aligned} \mathrm{Vol}_S(S_1) &= \frac{b_1\pi^3}{(b_1 - b_2)(b_1 + b_2)(b_1 - b_3)(b_1 + b_3)}, \\ \mathrm{Vol}_S(S_2) &= \frac{\pi^3}{(b_1 - b_2)(b_1 + b_2)(b_1 + b_3)}, \quad \mathrm{Vol}_S(S_3) = \frac{\pi^3}{(b_1 + b_2)(b_1 - b_3)(b_1 + b_3)}, \\ \mathrm{Vol}_S(S_4) &= \frac{\pi^3}{(b_1 - b_2)(b_1 - b_3)(b_1 + b_3)}, \quad \mathrm{Vol}_S(S_5) = \frac{\pi^3}{(b_1 - b_2)(b_1 + b_2)(b_1 - b_3)}. \end{aligned} \tag{5.44}$$

The Sasaki–Einstein volume can be obtained by setting $b_1 = 4$ and then extremizing (5.13) over b_2, b_3. The extremal point has $b_2 = b_3 = 0$ with $\mathrm{Vol}_{SE}(V^{5,2}) = (27/128)\pi^4$ and $\mathrm{Vol}_{SE}(S_a) = \pi^3/64$.

We now consider the GK geometry associated with $\mathrm{AdS}_2 \times Y_9$ solutions. The off-shell entropy function can be obtained from (4.20) and takes the form

$$\begin{aligned} \mathscr{S} = \frac{4\pi N^{3/2}}{27 b_0 \sqrt{b_1}} \Bigg[& \sqrt{(b_1^{(+)} - b_2^{(+)})(b_1^{(+)} + b_2^{(+)})(b_1^{(+)} - b_3^{(+)})(b_1^{(+)} + b_3^{(+)})} \\ & - \sigma \sqrt{(b_1^{(-)} - b_2^{(-)})(b_1^{(-)} + b_2^{(-)})(b_1^{(-)} - b_3^{(-)})(b_1^{(-)} + b_3^{(+)})} \Bigg]. \end{aligned} \tag{5.45}$$

The off-shell geometric R-charges are obtained from (4.9) and are given by

$$\begin{aligned} R_1^{\pm} &= \frac{2}{3}\frac{b_1^{(\pm)}}{b_1}, \quad R_2^{\pm} = \frac{2}{3}\frac{b_1^{(\pm)} - b_3^{(\pm)}}{b_1}, \quad R_3^{\pm} = \frac{2}{3}\frac{b_1^{(\pm)} - b_2^{(\pm)}}{b_1}, \\ R_4^{\pm} &= \frac{2}{3}\frac{b_1^{(\pm)} + b_2^{(\pm)}}{b_1}, \quad R_5^{\pm} = \frac{2}{3}\frac{b_1^{(\pm)} + b_3^{(\pm)}}{b_1}. \end{aligned} \tag{5.46}$$

From (3.46) we obtain the fluxes

$$\mathfrak{p}_a \equiv \frac{M_a}{N} = \frac{1}{3m_+ m_-}\left\{p_1,\ p_1 - p_3,\ p_1 - p_2,\ p_1 + p_2,\ p_1 + p_3\right\}, \tag{5.47}$$

with $p_1 = -\sigma m_+ - m_-$. Note that

$$M_1 + M_2 + M_5 = M_1 + M_3 + M_4 = \frac{p_1}{m_+ m_-} N. \tag{5.48}$$

These equations play an analogous role to the general formula (7.34) valid in the toric case, that we discuss later. Indeed, both (7.34) and (5.48) may be interpreted in field theory as the condition that the superpotential is twisted via the line bundle $\mathcal{O}(p_1)$, where the flux terms on the left hand side correspond to the twistings of individual fields that appear in the superpotential. For $V^{5,2}$ the particular combination of terms in (5.48) may be determined from the field theory dual worked out in [54], with particular fields being associated with particular divisors $C(S_a) \subset C(V^{5,2})$.

To obtain the on-shell values for $\mathscr{S}$ and $R_a^{\pm}$, we should set $b_1 = 1$ and then extremize (5.45) as a function of b_2, b_3 and b_0. As for the S^7 case, the case of general fluxes is involved, hence we will consider the case when four of the fluxes are pairwise equal, namely $M_2 = M_3$ and $M_4 = M_5$, which holds when

$$p_2 = p_3. \tag{5.49}$$

This choice corresponds to considering fibrations of the $V^{5,2}$ that maintain $SU(2) \times SU(2) \times U(1) \subset SO(5) \times U(1)$ isometry. In this case the entropy (5.45) becomes symmetric under the exchange $b_2 \leftrightarrow b_3$, and we may assume that at the extremal point

$$b_2 = b_3. \tag{5.50}$$

With (5.49) and restricting to the locus (5.50) we find that the (5.45) simplifies to

$$\mathscr{S} = \frac{4\pi N^{3/2}}{27 b_0}\left[\delta_+ (b_1^{(+)} - b_2^{(+)})(b_1^{(+)} + b_2^{(+)}) - \delta_- (b_1^{(-)} - b_2^{(-)})(b_1^{(-)} + b_2^{(-)})\right], \tag{5.51}$$

where the signs are defined as

$$\begin{aligned} \delta_+ &\equiv \operatorname{sign}\left[(b_1^{(+)} - b_2^{(+)})(b_1^{(+)} + b_2^{(+)})\right], \\ \delta_- &\equiv \sigma \operatorname{sign}\left[(b_1^{(-)} - b_2^{(-)})(b_1^{(-)} + b_2^{(-)})\right]. \end{aligned} \tag{5.52}$$

By an explicit calculation we find that when $\delta_+ = \delta_-$, (5.51) is linear in b_2 and b_0 and so it does not have any extremum. Hence we take

$$\delta_+ = -\delta_- \equiv \delta. \tag{5.53}$$

We then find that (5.51) is extremized for

$$b_0^* = \pm \frac{2 m_+ m_-}{\sqrt{2(m_+^2 + m_-^2) - p_2^2}}, \qquad b_2^* = \mp \frac{p_2 (a_+ m_- - a_- m_+)}{\sqrt{2(m_+^2 + m_-^2) - p_2^2}}. \tag{5.54}$$

Interestingly, b_0^* and b_2^* do not depend on whether we are in the twist or in the anti-twist case. The on-shell entropy reads

$$\mathscr{S}_{\text{os}} = \delta \, \frac{8\pi N^{3/2}}{27 m_+ m_-}\left[\sigma m_+ - m_- \pm \sqrt{2\left(m_-^2 + m_+^2\right) - p_2^2}\right]. \tag{5.55}$$

Clearly we must have $p_2^2 \leq 2\left(m_-^2 + m_+^2\right)$ and there are additional restrictions that are needed in order to have a positive entropy. Mimicking what we did for the case $X_7 = S^7$, we need to analyse the cases of $\delta = \pm 1$ and the two signs (5.54), for each case of the twist, $\sigma = +1$, and the anti-twist, $\sigma = -1$. We find that there are no possibilities for choosing $m_\pm > 0$, p_2 for the twist case and hence there are no $\mathrm{AdS}_2 \times Y_9$ solutions in the twist class for $X_7 = V^{5,2}$ and $p_2 = p_3$.

For the anti-twist case with $\sigma = -1$ we find there are no possibilities associated with the lower sign in the extremal values in (5.54), (5.55). However, there are possibilities associated with the upper sign. In particular we find

$$b_0^* = \frac{2m_+m_-}{\sqrt{2(m_+^2+m_-^2)-p_2^2}}, \qquad b_2^* = -\frac{p_2(a_+m_- - a_-m_+)}{\sqrt{2(m_+^2+m_-^2)-p_2^2}}, \tag{5.56}$$

and

$$\mathscr{S}_{\mathrm{os}} = -\delta\,\frac{8\pi N^{3/2}}{27m_+m_-}\left[m_+ + m_- - \sqrt{2\left(m_-^2+m_+^2\right)-p_2^2}\right], \tag{5.57}$$

with

$$\begin{aligned} \delta &= 1 \qquad \text{for} \qquad -|m_+-m_-| < p_2 < |m_+-m_-|, \\ \delta &= -1 \qquad \text{for} \qquad -(m_++m_-) < p_2 < -|m_+-m_-| \\ &\qquad\qquad\quad \text{or} \qquad |m_+-m_-| < p_2 < (m_++m_-). \end{aligned} \tag{5.58}$$

Notice that the condition $p_2^2 \leq 2\left(m_-^2+m_+^2\right)$ is automatically satisfied. For this class, with the upper sign, the on-shell R-charges are given by

$$\begin{aligned} R_1^\pm &= \frac{2}{3} - \frac{4m_\mp}{3\sqrt{2(m_+^2+m_-^2)-p_2^2}}, \\ R_2^+ = R_3^+ &= \frac{2}{3} - \frac{2(2m_- + p_2)}{3\sqrt{2(m_+^2+m_-^2)-p_2^2}}, \\ R_4^+ = R_5^+ &= \frac{2}{3} - \frac{2(2m_- - p_2)}{3\sqrt{2(m_+^2+m_-^2)-p_2^2}}, \\ R_2^- = R_3^- &= \frac{2}{3} - \frac{2(2m_+ - p_2)}{3\sqrt{2(m_+^2+m_-^2)-p_2^2}}, \\ R_4^- = R_5^- &= \frac{2}{3} - \frac{2(2m_+ + p_2)}{3\sqrt{2(m_+^2+m_-^2)-p_2^2}}. \end{aligned} \tag{5.59}$$

If we now impose the conditions $R_a^+ > 0$ and $\sigma R_a^- > 0$ (here with $\sigma = -1$) we find that we are only left with the case of the upper sign with

$$\delta = 1 \qquad \text{for:} \qquad 0 < p_3 < \frac{1}{2}(m_+ - m_-). \tag{5.60}$$

It is natural to conjecture that these are sufficient constraints on the parameters in order for the $\mathrm{AdS}_2 \times Y_9$ solutions to actually exist.

We can now consider the further special sub-case of the universal anti-twist which has

$$p_2 = p_3 = 0. \tag{5.61}$$

In this case the fluxes in (5.47) read

$$\mathfrak{p}_a \equiv \frac{M_a}{N} = \frac{p_1}{3m_+m_-}\big\{1, 1, 1, 1, 1\big\}. \tag{5.62}$$

From (5.56) we see that the extremal value has $b_2^* = b_3^* = 0$, consistent with (4.26). The extremal value of b_0, the on-shell entropy (5.57) and R-charges (5.59) agree with those given in Sect. 4.2.2.

6. Toric GK Geometry

In this section we recall some general properties of the GK geometry associated with toric Y_{2n+1}, and in particular discuss the master volume $\mathcal{V}_{2n+1}$ of Y_{2n+1} which allows one to carry out the geometric extremization algebraically. We follow [10,13,14], where further details can be found.

6.1. Toric Kähler cones. We begin with a toric Kähler cone $C(Y_{2n+1})$ in real dimension $2(n+1)$. Thus we have a Kähler metric of the form

$$\mathrm{d}s^2_{C(Y_{2n+1})} = \mathrm{d}r^2 + r^2\mathrm{d}s^2_{2n+1} \ , \tag{6.1}$$

with a $U(1)^{n+1}$ action generated by holomorphic Killing vectors ∂_{φ_μ}, $\mu = 0, 1, \dots, n$, with each φ_μ having period 2π. We assume that $C(Y_{2n+1})$ is Gorenstein meaning that it admits a global holomorphic $(n+1, 0)$-form $\Psi_{(n+1,0)}$. As before we choose a basis so that this holomorphic volume form has unit charge under ∂_{φ_1} and is uncharged under $\partial_{\varphi_{\hat\mu}}$, $\hat\mu = 0, 2, \dots, n$.

The manifold Y_{2n+1} is embedded at $r = 1$. The complex structure of the cone pairs the radial vector $r\partial_r$ with the Killing vector field ξ tangent to Y_{2n+1}, i.e. $\xi = \mathcal{J}(r\partial_r)$. We can write

$$\xi = \sum_{\mu=0}^{n} b_\mu \partial_{\varphi_\mu} \ , \tag{6.2}$$

and the vector $(b_\mu) = (b_0, b_1, \dots, b_n) \in \mathbb{R}^{n+1}$ then parametrizes the choice of R-symmetry vector ξ. A given vector field ξ defines a foliation of Y_{2n+1} which we denote by $\mathcal{F}_\xi$. We also have

$$\mathcal{L}_\xi \Psi_{(n+1,0)} = \mathrm{i}b_1 \Psi_{(n+1,0)}. \tag{6.3}$$

Similarly, the complex structure pairs the one-form η, dual to the Killing vector ξ, with $\mathrm{d}r/r$. For Kähler cones we have

$$\mathrm{d}\eta = 2J_S \ , \tag{6.4}$$

where J_S is the transverse Kähler form. In this case η is a contact one-form on Y_{2n+1} and ξ, satisfying $\xi \lrcorner \eta = 1$, is then also called the Reeb vector field. The associated Sasakian metric on Y_{2n+1} is given by

$$\mathrm{d}s^2_{Y_{2n+1}} = \eta^2 + \mathrm{d}s^2_{2n}(J_S) \ . \tag{6.5}$$

We also note that that (2.8) immediately gives the cohomology relation

$$[\mathrm{d}\eta] = \frac{1}{b_1}[\rho] \in H^2_B(\mathcal{F}_\xi), \tag{6.6}$$

where ρ denotes the Ricci two-form of the transverse Kähler metric $\mathrm{d}s^2_{2n}(J_S)$.

We next define the moment map coordinates

$$y^\mu \equiv \tfrac{1}{2} r^2 \partial_{\varphi_\mu} \lrcorner\, \eta\,, \qquad \mu = 0, 1, \ldots, n. \tag{6.7}$$

These span the moment map polyhedral cone $\mathcal{C} \subset \mathbb{R}^{n+1}$, where the y^μ are Euclidean coordinates on $\mathbb{R}^{n+1}$. The polyhedral cone $\mathcal{C}$, which is convex, is determined by D vectors $v_{A\mu}$ with $A = 1, \ldots D$ and for each A we have $(v_{A\mu}) \in \mathbb{Z}^{n+1}$. Specifically, these are inward pointing and primitive normals to the D facets of the cone and we have

$$\mathcal{C} = \left\{ (y^\mu) \in \mathbb{R}^{n+1} \mid \sum_{\mu=0}^{n} y^\mu v_{A\mu} \geq 0\,, \quad A = 1, \ldots, D \right\}. \tag{6.8}$$

If we change basis for the $U(1)^{n+1}$ action with an $SL(n+1, \mathbb{Z})$ transformation, then there is a corresponding $SL(n+1, \mathbb{Z})$ transformation on the moment map coordinates and hence on the toric data, i.e. the vectors $v_{A\mu}$. In the basis above, satisfying (6.3), the Gorenstein condition implies that $(v_{A\mu=1}) = 1$ for all A.

Geometrically, $C(Y_{2n+1})$ fibres over the polyhedral cone $\mathcal{C}$. There is a trivial $U(1)^{n+1}$ fibration over the interior $\mathcal{C}_{\mathrm{int}}$ of $\mathcal{C}$, with the D normal vectors $v_{A\mu}$ to each bounding facet $\{\sum_{\mu=0}^{n} y^\mu v_{A\mu} = 0\} \subset \partial\mathcal{C}$ specifying which $U(1) \subset U(1)^{n+1}$ collapses along that facet. Each facet is also the image under the moment map of a toric divisor $\mathcal{D}_A$ in $C(Y_{2n+1})$, where $\mathcal{D}_A$ is a complex codimension one submanifold that is invariant under the action of the $U(1)^{n+1}$ torus. We define T_A to be the $U(1)^{n+1}$ invariant $(2n-1)$-cycle in Y_{2n+1} associated with the toric divisor $\mathcal{D}_A$ on the cone. Since $\dim H_{2n-1}(Y_{2n+1}, \mathbb{R}) = D - (n+1)$, the toric $(2n-1)$-cycles $[T_A] \in H_{2n-1}(Y_{2n+1}, \mathbb{Z})$ are not independent in $H_{2n-1}(Y_{2n+1}, \mathbb{Z})$, satisfying the $(n+1)$ relations

$$\sum_{A=1}^{D} v_{A\mu}[T_A] = 0\,, \qquad \mu = 0, 1, \ldots, n. \tag{6.9}$$

We next define the Reeb cone, $\mathcal{C}^*$, to be the dual cone to $\mathcal{C}$. Then for a Kähler cone metric on $C(Y_{2n+1})$ we necessarily have $(b_\mu) \in \mathcal{C}^*_{\mathrm{int}}$, where $\mathcal{C}^*_{\mathrm{int}}$ is the open interior of the Reeb cone [7]. The image of $Y_{2n+1} = \{r = 1\}$ under the moment map is then the compact, convex n-dimensional polytope

$$P = P(b_\mu) = \mathcal{C} \cap H(b_\mu), \tag{6.10}$$

where the Reeb hyperplane defined by

$$H = H(b_\mu) \equiv \left\{ (y^\mu) \in \mathbb{R}^{n+1} \mid \sum_{\mu=0}^{n} y^\mu b_\mu = \tfrac{1}{2} \right\}. \tag{6.11}$$

6.2. The master volume. We first fix a choice of toric Kähler cone metric on the complex cone $C(Y_{2n+1})$. As described in the previous subsection, this allows us to introduce coordinates (y^μ, φ_μ) on $C(Y_{2n+1})$. For a fixed choice of such complex cone, with Reeb vector ξ given by (2.6), we would then like to study a more general class of transversely Kähler metrics of the form (6.5) with J, in general, no longer equal to J_S.

Of central interest is the "master volume" defined by

$$\mathcal{V}_{2n+1} \equiv \int_{Y_{2n+1}} \eta \wedge \frac{J^n}{n!}\,, \tag{6.12}$$

which is considered to be a function both of the vector ξ, specified by b_μ, and the transverse Kähler class $[J] \in H^2_B(\mathcal{F}_\xi)$. We can introduce $c_A \in H^2_B(\mathcal{F}_\xi)$ to be basic representatives of integral classes in $H^2(Y_{2n+1}, \mathbb{Z})$, which are Poincaré dual to the D toric divisors $\mathcal{D}_A$ on $C(Y_{2n+1})$. This allows us to write

$$[J] = -2\pi \sum_{A=1}^{D} \lambda_A c_A \in H^2_B(\mathcal{F}_\xi), \tag{6.13}$$

with the real parameters λ_A determining the transverse Kähler class. The c_A are not all independent and $[J]$ in fact only depends on $D-n$ of the D parameters $\{\lambda_A\}$, as we shall see shortly. It is also useful to note that the first Chern class of the foliation can be written in terms of the c_A as

$$[\rho] = 2\pi \sum_{A=1}^{D} c_A \in H^2_B(\mathcal{F}_\xi)\,. \tag{6.14}$$

We then see that in the special case in which

$$\lambda_A = -\frac{1}{2b_1}, \qquad A = 1, \ldots D, \tag{6.15}$$

we recover the Sasakian Kähler class $[\rho] = 2b_1[J_S]$ and the master volume (6.12) reduces to the Sasakian volume.

The master volume (6.12) may be written as

$$\mathcal{V}_{2n+1} = \frac{(2\pi)^{n+1}}{|b|}\mathrm{Vol}(\mathcal{P})\,, \tag{6.16}$$

where $|b| \equiv (\sum_{\mu=0}^{n} b_\mu b_\mu)^{1/2}$. The factor of $(2\pi)^{n+1}$ arises by integrating over the torus $U(1)^{n+1}$, while $\mathrm{Vol}(\mathcal{P})$ is the Euclidean volume of the compact, convex n-dimensional polytope

$$\begin{aligned}\mathcal{P} &= \mathcal{P}(b_\mu; \lambda_a)\\ &\equiv \left\{ (y^\mu) \in H(b_\mu) \;\middle|\; \sum_{\mu=0}^{n}(y^\mu - y^\mu_*)v_{A\mu} \geq \lambda_A\,,\ A = 1, \ldots, D \right\},\end{aligned} \tag{6.17}$$

with y^μ_* an arbitrary point in the Reeb hyperplane. Different choices for y^μ_* can be reabsorbed into a re-definition of the λ_A. It is convenient to take

$$(y^\mu_*) \equiv \left(0, \frac{1}{2b_1}, 0, \ldots, 0\right) \in H. \tag{6.18}$$

Notice then that with $\lambda_A = -\frac{1}{2b_1}$ the polytope $\mathcal{P}$ is the same as the Sasaki polytope P given in (6.10).

The master volume $\mathcal{V}_{2n+1}$ is homogeneous of degree n in the λ_A, and we have

$$\mathcal{V}_{2n+1} \equiv \int_{Y_{2n+1}} \eta \wedge \tfrac{1}{n!} J^n = (-2\pi)^n \sum_{A_1,\ldots,A_n=1}^{D} \tfrac{1}{n!} I_{A_1\ldots A_n} \lambda_{A_1} \ldots \lambda_{A_n}, \tag{6.19}$$

where the "intersection" numbers $I_{A_1\ldots A_n}$ are defined as

$$I_{A_1\ldots A_n} \equiv \int_{Y_{2n+1}} \eta \wedge c_{A_1} \wedge \cdots \wedge c_{A_n} = \frac{1}{(-2\pi)^n} \frac{\partial^n \mathcal{V}_{2n+1}}{\partial\lambda_{A_1} \ldots \partial\lambda_{A_n}} . \tag{6.20}$$

We may then calculate

$$\int_{Y_{2n+1}} \eta \wedge \rho^k \wedge \tfrac{1}{(n-k)!} J^{n-k} = (-1)^k \sum_{A_1,\ldots,A_k=1}^{D} \frac{\partial^k \mathcal{V}_{2n+1}}{\partial\lambda_{A_1} \ldots \partial\lambda_{A_k}}. \tag{6.21}$$

We are also interested in integrating over T_A, the $(2n-1)$-dimensional and $U(1)^{n+1}$ invariant cycle in Y_{2n+1} associated with a toric divisor $\mathcal{D}_A$ on the cone and Poincaré dual to c_A. We have

$$\begin{aligned}\int_{T_A} \eta \wedge \rho^k \wedge \tfrac{1}{(n-k-1)!} J^{n-k-1} &= \int_{Y_{2n+1}} \eta \wedge \rho^k \wedge \tfrac{1}{(n-k-1)!} J^{n-k-1} \wedge c_A \\ &= \frac{(-1)^{k+1}}{2\pi} \sum_{B_1,\ldots,B_k=1}^{D} \frac{\partial^{k+1} \mathcal{V}_{2n+1}}{\partial\lambda_A \partial\lambda_{B_1} \ldots \partial\lambda_{B_k}}.\end{aligned} \tag{6.22}$$

We can obtain similar formulas for integrals over higher-dimensional cycles that are Poincaré dual to products of the c_A. For example, consider the codimension four cycle $T_{B_1B_2}$ that is Poincaré dual to $c_{B_1} \wedge c_{B_2}$. We then find, for example,

$$\begin{aligned}\int_{T_{A_1A_2}} \eta \wedge \rho^k \wedge \tfrac{1}{(n-k-2)!} J^{n-2} &= \int_{Y_{2n+1}} \eta \wedge \tfrac{1}{(n-2)!} J^{n-2} \wedge c_{A_1} \wedge c_{A_2} \\ &= \frac{(-1)^k}{(2\pi)^2} \sum_{B_1,\ldots,B_k=1}^{D} \frac{\partial^{s+2} \mathcal{V}_{2n+1}}{\partial\lambda_{A_1} \partial\lambda_{A_2} \partial\lambda_{B_1} \ldots \partial\lambda_{B_k}}\end{aligned} \tag{6.23}$$

It can also be shown that the master volume $\mathcal{V}_{2n+1}$ is homogeneous of degree -1 in b_μ.

The master volume satisfies the important identity

$$\sum_{A=1}^{D} \left(v_{A\mu} - \frac{b_\mu}{b_1} \right) \frac{\partial \mathcal{V}_{2n+1}}{\partial\lambda_A} = 0, \qquad \mu = 0, 1, \ldots, n. \tag{6.24}$$

It follows from this, together with homogeneity, that the master volume is invariant under the "gauge" transformations

$$\lambda_A \;\to\; \lambda_A + \sum_{\mu=0}^{n} \gamma^\mu (v_{A\mu} b_1 - b_\mu), \tag{6.25}$$

for arbitrary constants γ^μ. Since the transformation parametrized by γ^1 is trivial, we see that the master volume only depends on $D-n$ of the D parameters $\{\lambda_A\}$, as noted above.[23]

It is possible to obtain very explicit formulas for the master volume in low dimensions in terms of $v_{A\mu}$, b_μ and λ_A. Specifically, in dimensions $n=1,2$ and 3 the relevant formulae for Y_3, Y_5 and Y_7 were derived in [10,14] and [13], respectively.[24]

Finally, we note that the formulae in this section assume that the polyhedral cone $\mathcal{C}$ is convex, since we started the section with a cone that admits a toric Kähler cone metric. However, this convexity condition is, in general, too restrictive. Indeed, for applications to AdS_2 and AdS_3 solutions of interest many explicit supergravity solutions are known that are associated with "non-convex toric cones", as defined in [9], which in particular have toric data which do not define a convex polyhedral cone. Based on the consistent picture that has emerged, including recovering results for the known explicit supergravity solutions, it is expected that the key formulae in this section are also applicable to non-convex toric cones, and we will assume this is the case in the sequel.

6.3. Geometric extremization for toric Y_{2n+1}. When Y_{2n+1} is toric, the geometric extremization of Sect. 2.2 for the associated GK geometry on Y_{2n+1} can be carried out using the master volume $\mathcal{V}_{2n+1}(b_\mu;\lambda_A)$. In order to carry this out we need expressions for the off-shell supersymmetric action, the constraint equation and flux quantization conditions. These can all be expressed in terms of derivatives of the master volume as follows:

$$\begin{aligned} S_{\rm SUSY} &= -\sum_{A=1}^{D}\frac{\partial\mathcal{V}_{2n+1}}{\partial\lambda_A}, \\ 0 &= \sum_{A,B=1}^{D}\frac{\partial^2\mathcal{V}_{2n+1}}{\partial\lambda_A\partial\lambda_B}, \\ \nu_n M_A &= -\frac{1}{(2\pi)}\frac{\partial S_{\rm SUSY}}{\partial\lambda_A} = \frac{1}{2\pi}\sum_{B=1}^{D}\frac{\partial^2\mathcal{V}_{2n+1}}{\partial\lambda_A\partial\lambda_B}, \end{aligned} \tag{6.26}$$

with $M_A\in\mathbb{Z}$. The M_A are not all independent[25] and, as a consequence of (6.9), satisfy $(n+1)$ linear relations given by

$$\sum_{A=1}^{D} v_{A\mu}M_A = 0\,, \qquad \mu=0,1,\ldots,n. \tag{6.27}$$

Notice that the $\mu=1$ component in (6.27), $\sum_A M_A = 0$, is actually the constraint equation given in (6.26). Since the master volume is homogeneous of degree n in the

[23] Note that since the λ_A specify a Kähler class, they satisfy some positivity constraints, and some care is required in utilizing these gauge transformations.

[24] Note that the minus sign in (2.23) of [13] should be a plus sign.

[25] Note that the M_A are a linear combination of the (independent) quantized fluxes $\mathcal{N}_\alpha$ in (2.11), which are associated with the flux through a basis for the free part of $H_{2n-1}(Y_{2n+1};\mathbb{Z})$.

λ_A, it is also possible to write

$$S_{\text{SUSY}} = -\frac{2\pi}{n-1}\nu_n \sum_{A=1}^{D} \lambda_A M_A. \tag{6.28}$$

The geometric extremization principle that is to be implemented for the GK geometry on Y_{2n+1} is now simple to state. To obtain the on-shell supersymmetric action, we need to fix $b_1 = \frac{2}{n-2}$ and then extremize S_{SUSY} with respect to $b_0, b_2, \ldots, b_n$ as well as the $D-n$ independent λ_A, subject to the constraint equation and flux quantization conditions in (6.26).

7. Toric Geometry Fibred Over a Spindle

With the toric formalism of Sect. 6 to hand, we now return to studying GK geometries Y_{2n+1} of the fibred form

$$X_{2n-1} \hookrightarrow Y_{2n+1} \longrightarrow \Sigma, \tag{7.1}$$

introduced in Sect. 3, where now the fibres X_{2n-1} are toric Sasaki–Einstein manifolds and $\Sigma = \mathbb{WCP}^1_{[m_-,m_+]}$ is a spindle. There is then a toric $U(1)^n$ action on X_{2n-1}, and when combined with the azimuthal rotation symmetry of the spindle $\Sigma = \mathbb{WCP}^1_{[m_-,m_+]}$, this gives rise to a toric $U(1)^{n+1}$ action on Y_{2n+1}. We will utilize the toric geometry summarized in the previous section both for Y_{2n+1} and, with suitable notational changes, for the fibres X_{2n-1}. Our main goal is to relate the quantities appearing in the geometric extremal problem for Y_{2n+1} in Sect. 2.2 to the master volume $\mathcal{V}_{2n-1}$ of the toric fibres X_{2n-1}.

7.1. Toric data for Y_{2n+1}. As described in Sect. 3, the fibration of X_{2n-1} over Σ is specified by n integers $(p_i) = (p_1, \ldots, p_n)$.[26] These are effectively Chern numbers for the associated $U(1)^n$ fibration over Σ, and may be identified in terms of the local gluing data $\alpha_i^{\pm}$ of Sect. 3 via (3.9). Moreover, as explained in Sect. 3.2, we have from (3.14)

$$p_1 = -(\sigma m_+ + m_-)\,, \tag{7.2}$$

where $\sigma = \pm 1$ corresponds to the twist or anti-twist case, respectively:

$$\sigma = \begin{cases} +1 & \text{twist}\,, \\ -1 & \text{anti-twist}\,. \end{cases} \tag{7.3}$$

Recall here that we are using a basis for the $U(1)^n$ action on X_{2n-1} for which the holomorphic $(n,0)$-form Ω on the cone $C(X_{2n-1})$ is only charged with respect to the first $U(1)$. This $U(1)$, associated with the integer Chern number p_1 in (7.2), may then be viewed as a fiducial choice of Reeb action on the Calabi-Yau cone $C(X_{2n-1})$, which in the holographic context is dual to a fiducial choice of R-symmetry of the dual SCFT (i.e.

[26] Setting $m_+ = m_+ = 1$ is the case of the round S^2, which arises in [10] as a genus $g = 0$ Riemann surface case. We should identify p_i here with $-n_i$ there.

not necessarily the superconformal R-symmetry). The remaining $U(1)^{n-1}$, associated with the integer Chern numbers

$$\vec{p} \equiv (p_2, \dots, p_n), \tag{7.4}$$

are dual to flavour symmetries.

The toric data for $C(Y_{2n+1})$ may be determined in terms of the toric data for the Calabi-Yau cone $C(X_{2n-1})$, together with the spindle data $m_\pm$, and the fibration data p_i, σ. To begin, let v_{ai} with $a = 1, \dots, d$ and $i = 1, \dots, n$ be the toric data for $C(X_{2n-1})$, with $(v_{ai})_{i=1}^n \in \mathbb{Z}^n$ for each a. In the basis for the $U(1)^n$ action described above, where the holomorphic $(n, 0)$-form Ω on the cone $C(X_{2n-1})$ has charge 1 under the first $U(1) \subset U(1)^n$, and is uncharged under the remaining $U(1)^{n-1}$, we have $(v_{ai})_{i=1}^n = (1, \vec{w}_a)$, with $\vec{w}_a \in \mathbb{Z}^{n-1}$ describing a convex lattice polytope. Following the discussion in Sect. 3, the toric data for the cone $C(Y_{2n+1})$ is then given by $D = d + 2$ vectors $v_{A\mu}$ with $A = +, -, a$ and $\mu = 0, 1, \dots, n$ and $(v_{A\mu})_{\mu=0}^n \in \mathbb{Z}^{n+1}$ for each A. Explicitly, in the basis of vector fields ∂_{φ_μ}, $\mu = 0, 1, \dots, n$, generating the $U(1)^{n+1}$ action introduced in Sect. 3, the $v_{A\mu}$ can be written

$$\begin{aligned} v_{+\mu} &= (m_+, v_{+i}), && \text{where } v_{+i} \equiv (1, -a_+\vec{p}) , \\ v_{-\mu} &= (-\sigma m_-, v_{-i}), && \text{where } v_{-i} \equiv (1, -\sigma a_-\vec{p}) , \\ v_{a\mu} &= (0, v_{ai}), && \text{where } v_{ai} \equiv (1, \vec{w}_a) . \end{aligned} \tag{7.5}$$

Here, as in (3.10), the $a_\pm \in \mathbb{Z}$ satisfy

$$a_- m_+ + a_+ m_- = 1 . \tag{7.6}$$

Recall that ∂_{φ_0} is a vector field on Y_{2n+1} that rotates the spindle direction, with the holomorphic $(n+1, 0)$-form Ψ on $C(Y_{2n+1})$ having charge zero under ∂_{φ_0}, while ∂_{φ_i}, $i = 1, \dots, n$, are vector fields that generate the toric $U(1)^n$ action on the fibres X_{2n-1}. This immediately leads to the result $v_{a\mu} = (0, v_{ai})$ in (7.5). Geometrically, the v_{ai} specify the d $U(1) \subset U(1)^n$ subgroups that fix d corresponding toric divisors in $C(X_{2n-1})$, and in $C(Y_{2n+1})$ there are then correspondingly d toric divisors, fixed by $U(1) \subset U(1)^{n+1}$ and specified by $v_{a\mu}$; more precisely, these are the d toric divisors of $C(X_{2n-1})$ fibred over Σ. On the other hand, in $C(Y_{2n+1})$ there are two additional toric divisors, fixed by the $U(1) \subset U(1)^{n+1}$ subgroups specified by $v_{\pm\mu}$, respectively. Geometrically, these are cones over the fibres $X_\pm = X_{2n-1}/\mathbb{Z}_{m_\pm}$ over the two poles of the spindle, and the vector fields $\zeta_+ = (v_{+\mu})$, $\zeta_- = \sigma(v_{-\mu})$ were determined in (3.20), (3.21).

Recall that the integers $a_\pm$ in (7.6) are unique only up to shifts

$$a_+ \mapsto a_+ - \kappa\, m_+, \qquad a_- \mapsto a_- + \kappa\, m_- , \tag{7.7}$$

where $\kappa \in \mathbb{Z}$ is arbitrary. The toric data $v_{A\mu}$ for Y_{2n+1} given in (7.5) *a priori* depends on the choice of $a_\pm$, but recall that we are free to make $SL(n+1, \mathbb{Z})$ transformations, corresponding to a change of basis for the torus $U(1)^{n+1}$. One can check that the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \kappa p_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & 0 & \ddots & \vdots \\ \kappa p_n & 0 & 0 & \dots & 1 \end{pmatrix} \in SL(n+1, \mathbb{Z}), \tag{7.8}$$

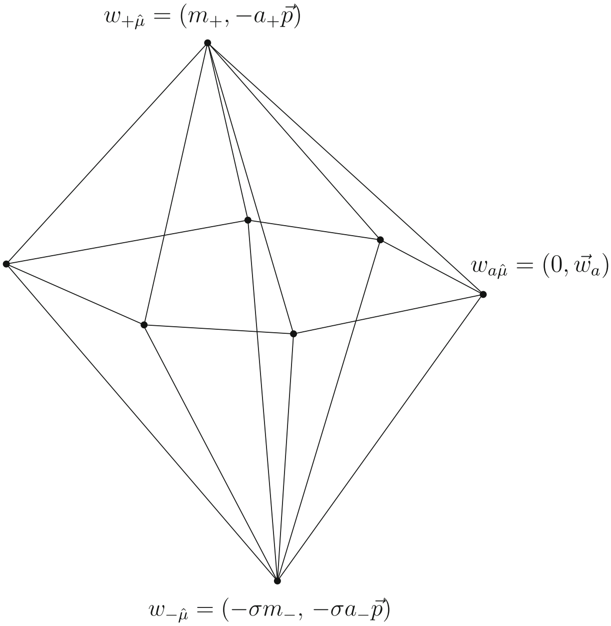

Fig. 1. Toric data for a representative twist case ($\sigma = +1$) with $n = 3$, $d = 6$. The toric diagram is a suspension of the toric diagram for X_5, the two-dimensional polytope at height zero. For the anti-twist case ($\sigma = -1$) notice that the figure is no longer convex (recall $m_\pm \in \mathbb{N}$)

acting on the μ index of $v_{A\mu}$, maps the vectors $v_{A\mu}$ in (7.5) to the same set of vectors with the replacements (7.7). The toric data we have presented is thus well-defined, with all choices of $a_\pm$ satisfying (7.6) describing the same toric geometry.

The toric data $v_{A\mu}$ for Y_{2n+1} is given by a "suspension" of the toric data for X_{2n-1}, at least in the twist case with $\sigma = +1$. Dropping the $\mu = 1$ component in (7.5) we can define truncated vectors

$$w_{+\hat{\mu}} = (m_+, -a_+\vec{p}), \quad w_{-\hat{\mu}} = (-\sigma m_-, -\sigma a_-\vec{p}), \quad w_{a\hat{\mu}} = (0, \vec{w}_a), \tag{7.9}$$

with $\hat{\mu} = 0, 2, \ldots, n$. In Fig. 1 we have provided an illustration of the polytope that these vectors define for the twist case $\sigma = +1$, of dimension $n = 3$, $d = 7$. In particular the two-dimensional polytope at height zero is the "toric diagram" for X_5, defined by the vectors $\vec{w}_a$; we see that the toric diagram for Y_7 is the suspension of this, adding two additional vectors $w_{\pm\hat{\mu}}$. For the twist case, when $\sigma = +1$, the polytope is convex, as illustrated in the figure, but it is not for the anti-twist case when $\sigma = -1$.

That the toric data (7.5) describes a fibration (7.1) directly follows from our discussion in Sect. 3. An alternative point of view is provided by utilizing the Delzant construction (see e.g. [55]), as discussed in Appendix A.

7.2. Relating the master volumes of Y_{2n+1} and X_{2n-1}. We now show that we can relate the master volume $\mathcal{V}_{2n+1}$ for $C(Y_{2n+1})$, or more precisely certain derivatives of it, to the master volume $\mathcal{V}_{2n-1}$ associated with $C(X_{2n-1})$.

Recall from (6.16) that the master volume for Y_{2n+1} can be written

$$\mathcal{V}_{2n+1} = \frac{(2\pi)^{n+1}}{|b|}\mathrm{Vol}(\mathcal{P}) > 0\,, \tag{7.10}$$

where $|b| \equiv (\sum_{\mu=0}^{n} b_\mu b_\mu)^{1/2}$ and $\mathrm{Vol}(\mathcal{P})$ is the Euclidean volume of the compact, convex n-dimensional polytope

$$\begin{aligned}\mathcal{P} &= \mathcal{P}(b_\mu;\lambda_a)\\ &\equiv \left\{ (y^\mu) \in H(b_\mu) \;\mid\; \sum_{\mu=0}^{n}(y^\mu - y_*^\mu)v_{A\mu} \geq \lambda_A\,, \quad A = 1,\ldots,D \right\}\,,\end{aligned} \tag{7.11}$$

with $(y_*^\mu) \in H = H(b_\mu)$ given by

$$(y_*^\mu)_{\mu=0}^{n} = (0, y_*^i), \qquad y_*^i \equiv \left(\frac{1}{2b_1}, 0, \ldots, 0\right)\,. \tag{7.12}$$

The defining condition for the $(n+1)$-dimensional Reeb hyperplane H given in (6.11) can then be written in the form

$$\sum_{\mu=0}^{n} y^\mu b_\mu = \tfrac{1}{2} \quad \Leftrightarrow \quad y^0 b_0 + \sum_{i=1}^{n}(y^i - y_*^i)b_i = 0. \tag{7.13}$$

We now want to exploit the fact that the polytope $\mathcal{P}$ takes a special form, namely a "truncated prism" with upper and lower faces, $\mathcal{P}_\pm$, each of which is a polytope of one lower dimension specified by d vectors; see Fig. 2.

Specifically, these polytopes are defined by

$$\mathcal{P}_\pm \;\equiv\; \mathcal{P} \cap \left\{ \sum_{\mu=0}^{n}(y^\mu - y_*^\mu)v_{\pm\mu} = \lambda_\pm \right\}\,, \tag{7.14}$$

with $v_{\pm\mu}$ given in (7.5). Geometrically, these correspond to the $X_{2n-1}/\mathbb{Z}_{m_\pm}$ fibres over the two poles of the spindle, associated with orbifold singularities $\mathbb{Z}_{m_\pm}$, respectively, as described in Sect. 3. The faces $\mathcal{P}_\pm$ of the truncated prism are the images of toric divisors $\mathcal{D}_\pm$ in $C(Y_{2n+1})$ under the moment map. We denote the corresponding $U(1)^{n+1}$ invariant $(2n-1)$-dimensional manifolds as $T_\pm$, which are copies of the fibre $T_\pm \equiv X_{2n-1}/\mathbb{Z}_{m_\pm}$ over the poles of the spindle. These submanifolds were denoted $X_\pm$ in Sect. 3, and in particular in Sect. 3.3, but as commented there we must be careful with orientations. From (6.22), the volume of these manifolds are given by

$$\mathrm{Vol}(T_\pm) \equiv \int_{T_\pm} \eta \wedge \tfrac{1}{(n-1)!} J^{n-1} = -\frac{1}{2\pi}\frac{\partial \mathcal{V}_{2n+1}}{\partial \lambda_\pm}\,, \tag{7.15}$$

where the orientation on $T_\pm$ is induced from the complex structure and $\mathrm{Vol}(T_\pm) > 0$. We shall see later that these satisfy the homology relation $m_+[T_+] = \sigma m_-[T_-] \in H_{2n-1}(Y_{2n+1}, \mathbb{R})$, leading us to identify

$$T_+ = X_+, \qquad T_- = \sigma X_-. \tag{7.16}$$

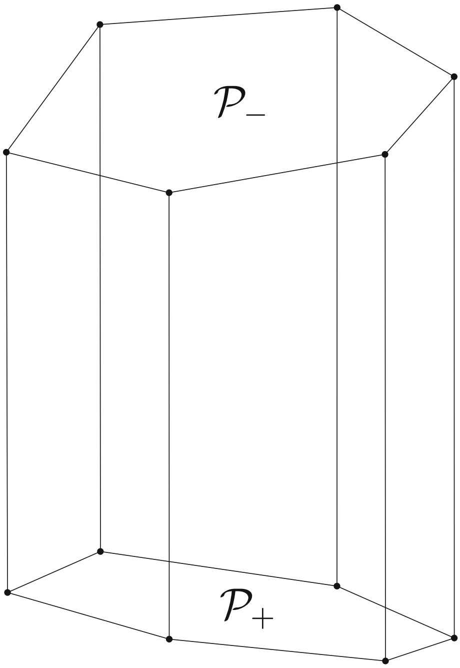

Fig. 2. Toric polytope for a representative twist case with $n = 3$, $d = 6$

In particular, $\mathrm{Vol}(T_\pm)$ should always be positive for a *bona fide* GK geometry, where note that in Sect. 3 (see also (4.12)) we then have $\mathrm{Vol}(X_-) = \sigma\mathrm{Vol}(T_-)$.

We now want to re-express $\mathrm{Vol}(T_\pm)$ in terms of master volume $\mathcal{V}_{2n-1}$ for the fibres X_{2n-1}. First consider T_+. We begin by noting that the volume of T_+ is obtained from the Euclidean volume of the $(n-1)$-dimensional polytope $\mathcal{P}_+$, where one imposes

$$\sum_{\mu=0}^{n}(y^\mu - y_*^\mu)v_{+\mu} = \lambda_+ \,, \qquad \sum_{\mu=0}^{n}(y^\mu - y_*^\mu)v_{a\mu} \geq \lambda_a \,. \tag{7.17}$$

Notice that since $v_{a\mu} = (0, v_{ai})$, this last inequality is exactly the same as the one for the n-dimensional polytope:

$$\sum_{i=1}^{n}(y^i - y_*^i)v_{ai} \geq \lambda_a \,. \tag{7.18}$$

On the other hand, the first equation in (7.17) reads

$$y^0 m_+ + \sum_{i=1}^{n}(y^i - y_*^i)v_{+i} \;=\; \lambda_+ \,. \tag{7.19}$$

We can solve this for y^0 and then substitute into (7.13) to write the Reeb hyperplane condition in the form

$$\sum_{i=1}^{n}(y^i - y_*^i)\left(b_i - \frac{b_0}{m_+}v_{+i}\right) = -\frac{b_0}{m_+}\lambda_+ \,. \tag{7.20}$$

When $\lambda_+ = 0$, notice this is an n-dimensional Reeb hyperplane equation, where we have shifted the n-dimensional Reeb vector $b_i \to b_i - \frac{b_0}{m_+} v_{+i}$, which was already introduced in (3.23). We can also absorb the right hand side of (7.20) into a new y_*^i, which recall is an arbitrary point in the n-dimensional Reeb hyperplane. It is then convenient to make the shift

$$y_*^i \;\to\; y_*^i + \left(\frac{b_0}{m_+ b_1 - b_0} \lambda_+, 0, \ldots, 0 \right) \equiv y_*^{(+)i} \; , \tag{7.21}$$

to then find that the two conditions (7.20) and (7.18), and hence (7.17) along with the Reeb hyperplane condition, can be written in the equivalent form:

$$\sum_{i=1}^{n} (y^i - y_*^{(+)i}) b_i^{(+)} = 0 \; , \qquad \sum_{i=1}^{n} (y^i - y_*^{(+)i}) v_{ai} \geq \lambda_a^{(+)} \; , \tag{7.22}$$

where we have also defined

$$b_i^{(+)} \equiv b_i - \frac{b_0}{m_+} v_{+i} , \qquad \lambda_a^{(+)} \equiv \lambda_a + \frac{b_0}{m_+ b_1^{(+)}} \lambda_+ . \tag{7.23}$$

Now (7.22) are precisely the conditions for specifying an $(n-1)$-dimensional polytope $\mathcal{P}_+$ associated with the fibre $X_{2n-1}/\mathbb{Z}_{m_+}$ sitting at the pole of the spindle with orbifold singularity $\mathbb{Z}_{m_+}$, as a function of the Reeb hyperplane vector $b_i^{(+)}$ and Kähler class parameters $\lambda_a^{(+)}$. If $\mathcal{V}_{2n-1}(b_i; \lambda_a)$ denotes the master volume of X_{2n-1} as a function of Reeb vector b_i and Kähler class parameters λ_a, then the above analysis shows that

$$\mathrm{Vol}(T_+) = -\frac{1}{2\pi} \frac{\partial \mathcal{V}_{2n+1}}{\partial \lambda_+} = \frac{1}{m_+} \mathcal{V}_{2n-1}^+ , \tag{7.24}$$

where we have defined

$$\mathcal{V}_{2n-1}^+ \equiv \mathcal{V}_{2n-1}(b_i^{(+)}; \lambda_a^{(+)}) . \tag{7.25}$$

Here the factor of $1/m_+$ on the right hand side of (7.24) is because the corresponding fibre is not X_{2n-1}, but rather $T_+ = X_{2n-1}/\mathbb{Z}_{m_+}$.

A nearly identical analysis goes through for T_-, and we can summarize the final results for both cases as follows[27]

$$\begin{aligned} \mathcal{V}_{2n-1}^+ \equiv \mathcal{V}_{2n-1}(b_i^{(+)}; \lambda_a^{(+)}) &= -\frac{m_+}{2\pi} \frac{\partial \mathcal{V}_{2n+1}}{\partial \lambda_+} = m_+ \mathrm{Vol}(T_+) \; , \\ \mathcal{V}_{2n-1}^- \equiv \mathcal{V}_{2n-1}(b_i^{(-)}; \lambda_a^{(-)}) &= -\frac{\sigma m_-}{2\pi} \frac{\partial \mathcal{V}_{2n+1}}{\partial \lambda_-} = \sigma m_- \mathrm{Vol}(T_-) \; , \end{aligned} \tag{7.26}$$

where we have defined the shifted vectors

$$\begin{aligned} b_i^{(+)} &= b_i - \frac{b_0}{m_+} v_{+i} , \qquad & \lambda_a^{(+)} &= \lambda_a + \frac{b_0}{m_+ b_1^{(+)}} \lambda_+ , \\ b_i^{(-)} &= b_i + \frac{b_0}{\sigma m_-} v_{-i} , \qquad & \lambda_a^{(-)} &= \lambda_a - \frac{b_0}{\sigma m_- b_1^{(-)}} \lambda_- . \end{aligned} \tag{7.27}$$

[27] As already remarked, the polytope $\mathcal{P}$ is convex only in the twist case with $\sigma = 1$, with the anti-twist formulae with $\sigma = -1$ formally obtained from the twist formulae by replacing m_- with $-m_-$ in the toric data (7.5).

with $b_i^{(\pm)}$ already introduced in (3.23). From (7.26) we also immediately deduce that

$$
\begin{aligned}
&\frac{\partial^2\mathcal{V}_{2n+1}}{\partial\lambda_+\partial\lambda_a} = -\frac{2\pi}{m_+}\frac{\partial\mathcal{V}^+_{2n-1}}{\partial\lambda_a}, &&\frac{\partial^2\mathcal{V}_{2n+1}}{\partial\lambda_-\partial\lambda_a} = -\frac{2\pi}{\sigma m_-}\frac{\partial\mathcal{V}^-_{2n-1}}{\partial\lambda_a},\\
&\frac{\partial^2\mathcal{V}_{2n+1}}{\partial\lambda_+^2} = -\frac{2\pi b_0}{m_+^2 b_1^{(+)}}\sum_{a=1}^{d}\frac{\partial\mathcal{V}^+_{2n-1}}{\partial\lambda_a}, &&\frac{\partial^2\mathcal{V}_{2n+1}}{\partial\lambda_-^2} = +\frac{2\pi b_0}{m_-^2 b_1^{(-)}}\sum_{a=1}^{d}\frac{\partial\mathcal{V}^-_{2n-1}}{\partial\lambda_a},\\
&\frac{\partial^2\mathcal{V}_{2n+1}}{\partial\lambda_+\partial\lambda_-} = 0.
\end{aligned}
\tag{7.28}
$$

The last result is the statement that T_+ and T_- do not intersect, which is obvious geometrically as they are distinct fibres of a fibration. Since $\mathrm{Vol}(T_\pm) > 0$ we also note from (7.26) that $\mathcal{V}^+_{2n-1} > 0$ and $\sigma\mathcal{V}^-_{2n-1} > 0$.

We next consider the identity (6.24) satisfied by the master volume $\mathcal{V}_{2n+1}$. The $\mu = 1$ component of this vector equation is trivial. The $\mu = 0$ component, which will play a key role in a moment, can be written as

$$
b_0\sum_{a=1}^{d}\frac{\partial\mathcal{V}_{2n+1}}{\partial\lambda_a} = \Big(m_+b_1 - b_0\Big)\frac{\partial\mathcal{V}_{2n+1}}{\partial\lambda_+} - \Big(\sigma m_- b_1 + b_0\Big)\frac{\partial\mathcal{V}_{2n+1}}{\partial\lambda_-}. \tag{7.29}
$$

We do not explicitly write out the remaining components here.

7.3. Geometric extremization for $X_{2n-1} \hookrightarrow Y_{2n+1} \to \Sigma$. We now have all the ingredients to translate the geometric extremization procedure for Y_{2n+1}, summarized in Sect. 6.3, in terms of the shifted master volumes on the fibres X_{2n-1}. We first consider the supersymmetric action. From the definition (6.26) and then using the identity satisfied by the master volume (7.29) we immediately have

$$
\begin{aligned}
S_{\mathrm{SUSY}} = -\sum_{A=1}^{D}\frac{\partial\mathcal{V}_{2n+1}}{\partial\lambda_A} &= -\frac{b_1}{b_0}\left(m_+\frac{\partial\mathcal{V}_{2n+1}}{\partial\lambda_+} - \sigma m_-\frac{\partial\mathcal{V}_{2n+1}}{\partial\lambda_-}\right)\\
&= \frac{2\pi b_1}{b_0}\left[m_+\mathrm{Vol}(T_+) - \sigma m_-\mathrm{Vol}(T_-)\right],
\end{aligned}
\tag{7.30}
$$

and we notice that the last expression is in exact alignment with the general result (3.25) that we proved earlier, taking into account the orientations in (7.16). Hence, using (7.26), we obtain the key result

$$
\boxed{S_{\mathrm{SUSY}} = 2\pi\frac{b_1}{b_0}\left(\mathcal{V}^+_{2n-1} - \mathcal{V}^-_{2n-1}\right)} \tag{7.31}
$$

and recall that $S_{\mathrm{SUSY}} > 0$.

We next consider the fluxes on Y_{2n+1}. We denote these by $M_A = (M_+, M_-, M_a)$, which are the fluxes over the corresponding toric codimension two submanifolds $T_A \subset Y_{2n+1}$, where $T_A = (T_+, T_-, T_a)$. In particular

$$
M_\pm \equiv \frac{1}{\nu_n}\int_{T_\pm}\eta\wedge\rho\wedge\frac{J^{n-2}}{(n-2)!}. \tag{7.32}
$$

From (7.16) we take

$$M_+ = N^{X_+}, \qquad M_- = \sigma N^{X_-}, \tag{7.33}$$

where $N^{X_\pm} > 0$ were the fluxes introduced in Sect. 3. The T_a may be identified with Σ_a in (3.43), and are the total spaces of toric codimension two submanifolds $S_a \subset X_{2n-1}$, fibred over the spindle Σ, with $a = 1, \ldots, d$.

We first note that the linear relations satisfied by these fluxes given in (6.27) can be written in the equivalent form

$$\begin{aligned} m_+ M_+ &= \sigma m_- M_- \equiv N\,, \\ \sum_{a=1}^{d} M_a &= -\frac{\sigma m_+ + m_-}{m_+ m_-} N, \\ \sum_{a=1}^{d} \vec{w}_a M_a &= \frac{N}{m_+ m_-} \vec{p}, \end{aligned} \tag{7.34}$$

and these include the constraint equation $\sum_A M_A = 0$. The first line implies $m_+[T_+] = \sigma m_-[T_-] \in H_{2n-1}(Y_{2n+1}, \mathbb{R})$ that we mentioned earlier. As an aside notice that for the twist case, $\sigma = +1$, the second line implies that some of the $M_a < 0$. We can also easily find expressions for the fluxes M_A in terms of $\mathcal{V}_{2n-1}$. Indeed from the expression for the M_A in terms of derivatives of S_{SUSY} given in (6.26) we immediately deduce

$$\boxed{\begin{aligned} m_+ M_+ &= -\frac{1}{\nu_n} \frac{b_1}{b_1^{(+)}} \sum_{a=1}^{d} \frac{\partial \mathcal{V}^+_{2n-1}}{\partial \lambda_a}, \\ \sigma m_- M_- &= -\frac{1}{\nu_n} \frac{b_1}{b_1^{(-)}} \sum_{a=1}^{d} \frac{\partial \mathcal{V}^-_{2n-1}}{\partial \lambda_a}, \\ M_a &= -\frac{1}{\nu_n} \frac{b_1}{b_0} \left(\frac{\partial \mathcal{V}^+_{2n-1}}{\partial \lambda_a} - \frac{\partial \mathcal{V}^-_{2n-1}}{\partial \lambda_a} \right) \end{aligned}} \tag{7.35}$$

The constraint equation can be recast in the form

$$\boxed{\frac{1}{b_1^{(+)}} \sum_{a=1}^{d} \frac{\partial \mathcal{V}^+_{2n-1}}{\partial \lambda_a} = \frac{1}{b_1^{(-)}} \sum_{a=1}^{d} \frac{\partial \mathcal{V}^-_{2n-1}}{\partial \lambda_a}} \tag{7.36}$$

which ensures $m_+ M_+ = \sigma m_- M_- \equiv N$.

Due to the constraints (7.34), the set of toric fluxes $M_A = (M_+, M_-, M_a)$ is overdetermined, and it is sometimes convenient to instead work with a minimal set of unconstrained quantized fluxes $\mathcal{N}_\alpha$, as originally introduced in equation (2.11). We may make contact between the two descriptions using the results of Appendix A. Here the index on $\mathcal{N}_\alpha$ may be identified as $\alpha = \hat{I} = 0, 1, \ldots, d-n$, labelling a basis for the free part of $H_{2n-1}(Y_{2n+1}, \mathbb{Z})$, while recall that $A = 1, \ldots, D = d+2$. We then have the general homology relation in toric geometry (see e.g. [56])

$$M_A = \sum_{I=0}^{d-n} Q^A_{\hat{I}} \mathcal{N}_{\hat{I}}. \tag{7.37}$$

Written out explicitly using the charge matrix $Q^A_{\hat{I}}$, satisfying $\sum_{A=1}^{D} Q^A_{\hat{I}}\, v_{A\mu} = 0$, and deduced in Appendix A, this reads

$$M_+ = m_- \mathcal{N}_0, \qquad M_- = \sigma m_+ \mathcal{N}_0, \qquad M_a = q^a_0 \mathcal{N}_0 + \sum_{I=1}^{d-n} q^a_I \mathcal{N}_I. \tag{7.38}$$

Here q^a_I is the corresponding charge matrix for the toric fibres X_{2n-1}, satisfying $\sum_{a=1}^{d} q^a_I\, v_{ai} = 0$, and q^a_0 satisfies $\sum_{a=1}^{d} q^a_0\, v_{ai} = p_i$, where recall the integers p_i determine the fibration of X_{2n-1} over Σ. Notice that the q^a_0 are not unique, in general, and can be replaced with $q^a_0 \to \sum_{I=1}^{d-n} c_I q^a_I$ where c_I are constants. In turn this shifts the "baryonic fluxes" $\mathcal{N}_I$, via $\mathcal{N}_I \to c_I \mathcal{N}_0$ showing that they are not unique either, and we comment on this further in Sect. 9.

Given that in (7.34) we identify $m_+ M_+ = \sigma m_- M_- = N$, where recall $\sigma^2 = 1$, the first two equations in (7.38) allow us to identify

$$\mathcal{N}_0 = \frac{N}{m_+ m_-}. \tag{7.39}$$

Since $\mathcal{N}_0$ is required to be a positive integer (2.11), note this shows that N is divisible by $m_+ m_-$, as commented earlier, below (3.40). The toric flux numbers M_a thus satisfy the last equation in (7.38). In Appendix A we further introduce the (non-unique) integers α^j_a that satisfy

$$\sum_{a=1}^{d} v_{ai} \alpha^j_a = \delta^j_i. \tag{7.40}$$

In terms of the gauged linear sigma model description, α^j_a is the charge of the ath coordinate on $\mathbb{C}^d$ under the jth $U(1)$ in the toric $U(1)^n$ action on the fibres X_{2n-1}. Using this, we may then write $q^a_0 = \sum_{i=1}^{n} p_i \alpha^i_a$, and hence the last equation in (7.38) reads

$$M_a = \frac{N}{m_+ m_-} \sum_{i=1}^{n} p_i \alpha^i_a + \sum_{I=1}^{d-n} q^a_I \mathcal{N}_I. \tag{7.41}$$

This expresses the toric flux numbers M_a directly in terms of the "flavour fluxes" p_i, and a set of "baryonic fluxes" $\mathcal{N}_I$, one for each independent cycle, together with N and the toric data. Although as we comment further in Sect. 9, the $\mathcal{N}_I$ depend on an (arbitrary) choice of the α^j_a satisfying (7.40), and different choices lead to different $\mathcal{N}_I$.

Although we shall not work through the details, the flavour twist case of Sect. 4 is obtained by imposing the condition (4.1), which in the toric setting amounts to set all the $\lambda_a \equiv \lambda$ to be equal in a particular gauge. To see this, from (6.13) and (6.14) we can write the transverse Kähler class for Y_{2n+1} in the form $[J] = -2\pi(\lambda_+ c_+ + \lambda_+ c_- + \lambda \sum_a c_a)$, while the first Chern class of the foliation is $[\rho] = 2\pi(c_+ + c_- + \sum_a c_a)$. In particular it is clear that (4.1) is satisfied on the toric fibres at the poles.

7.4. R-charges of baryonic operators. We now provide an expression for the geometric R-charges that are associated with certain supersymmetric cycles $T_{\pm a}$, $a = 1, \ldots, d$, of codimension four on Y_{2n+1}, which were introduced in Sect. 3.4. Recall that these are defined as copies of $U(1)^n$ invariant codimension two submanifolds $S_a \subset X_{2n-1}$, whose cones are divisors in the Calabi-Yau cone X_{2n-1}, over each of the two poles of the spindle. In the notation of Sect. 3 we have

$$T_{+a} = S_a^+, \qquad T_{-a} = \sigma S_a^-, \tag{7.42}$$

precisely as in (7.16). In the case of $n = 3, 4$ the geometric R-charges are dual to the R-charges of baryonic operators associated with D3, M5-branes wrapping these cycles, respectively.

In Sect. 6.2 we introduced $c_A \in H_B^2(\mathcal{F}_\xi)$ to be basic representatives of integral classes in $H^2(Y_{2n+1}, \mathbb{Z})$, which are Poincaré dual to the D toric divisors $\mathcal{D}_A$ on $C(Y_{2n+1})$. We write $c_A = \{c_\pm, c_a\}$, with $c_\pm$ Poincaré dual to the divisors $\mathcal{D}_\pm$. The codimension four supersymmetric cycles $T_{\pm a}$ are then Poincaré dual to $c_\pm \wedge c_a$. Then using the result (6.23) in the definition of the R-charges (3.42) we deduce[28]

$$\begin{aligned} R_a^\pm &\equiv \frac{4\pi}{\nu_n M_\pm} \int_{T_{\pm a}} \eta \wedge \frac{J^{n-2}}{(n-2)!} \\ &= \frac{1}{M_\pm} \frac{2}{\nu_n} \frac{1}{(2\pi)} \frac{\partial^2 \mathcal{V}_{2n+1}}{\partial \lambda_\pm \partial \lambda_a} \,. \end{aligned} \tag{7.43}$$

Notice that these agree with the expressions defined in (3.42), with the different choices of orientation in the cycles in (7.42) cancelling due to the division by $M_\pm$ in (7.43), rather than $N^{X_\pm}$ in (3.42), where these are similarly related via (7.33). Since $M_+ > 0$ and $\sigma M_- > 0$ from the first line of (7.43) we have

$$\boxed{R_a^+ > 0, \qquad \sigma R_a^- > 0} \tag{7.44}$$

We will prove in a moment that these R-charges satisfy the identities

$$\boxed{\sum_{a=1}^d R_a^+ v_{a\mu} = \frac{2}{b_1}\left(b_\mu - \frac{b_0}{m_+} v_{+\mu}\right), \qquad \sum_{a=1}^d R_a^- v_{a\mu} = \frac{2}{b_1}\left(b_\mu + \frac{b_0}{\sigma m_-} v_{-\mu}\right)} \tag{7.45}$$

Using the results in the previous subsection we can now express the geometric R-charges in terms of $\mathcal{V}_{2n-1}$. Indeed using (7.28) we have

$$\boxed{R_a^\pm = -\frac{1}{N} \frac{2}{\nu_n} \frac{\partial \mathcal{V}_{2n-1}^\pm}{\partial \lambda_a}} \tag{7.46}$$

[28] Note that the normalization of the geometric R-charges here is slightly different to what has appeared before in the literature. For example, setting $m_\pm = 1$, $\sigma = +1$ and $b_0 = 0$, we have the set-up associated with a Sasaki–Einstein space fibred over a sphere. In this case $M_\pm = N$ and the geometric R-charges here differ from those in [10,13] by a factor of N.

In addition, using the expressions for the M_A given in (7.35) we immediately derive the result

$$\boxed{\sum_{a=1}^{d} R_a^{+} = 2 - \frac{2b_0}{b_1 m_+}\,, \qquad \sum_{a=1}^{d} R_a^{-} = 2 + \frac{2b_0}{b_1 \sigma m_-}} \tag{7.47}$$

and we also note that

$$\boxed{R_a^{+} - R_a^{-} = \frac{2b_0}{b_1}\frac{M_a}{N}} \tag{7.48}$$

agreeing with the general formula (3.46) derived earlier. It is also interesting to highlight that in the toric case, from (7.47) we can write

$$\boxed{\frac{1}{2}\sum_{a=1}^{d}\left(R_a^{+} + R_a^{-}\right) = 2 - \frac{m_- - \sigma m_+}{m_+ m_-}\frac{b_0}{b_1}} \tag{7.49}$$

Note that for the toric anti-twist case, with $\sigma = -1$, this expression is the same as what we saw in the universal anti-twist case in (4.37).

Observe that (7.47) is actually the $\mu = 1$ component of (7.45). Notice that the $\mu = 0$ component of (7.45) is trivially satisfied. Thus, to complete the proof of (7.45) we therefore just need to check the remaining components which read

$$\sum_{a=1}^{d} R_a^{+}\vec{v}_a = \frac{2}{b_1}\vec{b}^{(+)}\,, \qquad \sum_{a=1}^{d} R_a^{-}\vec{v}_a = \frac{2}{b_1}\vec{b}^{(-)}\,. \tag{7.50}$$

Using the expressions for $R_a^{\pm}$ given in (7.46), we find that these are identities after using the fact that the $(2n-1)$-dimensional master volumes $\mathcal{V}_{2n-1}(\vec{b}^{(\pm)}; \lambda_a^{(\pm)})$ satisfy the identity

$$\sum_{a=1}^{d}\left(\vec{v}_a - \frac{\vec{b}^{(\pm)}}{b_1^{(\pm)}}\right)\frac{\partial \mathcal{V}^{\pm}_{(2n-1)}}{\partial \lambda_a} = 0\,, \tag{7.51}$$

which is the $(2n-1)$-dimensional version of (6.24).

7.5. The limit when $b_0 = 0$, $m_\pm = 1$. It is interesting to consider taking the limit $b_0 \to 0$ in the expressions for the geometric extremization problem above. In particular, if we set $m_\pm = 1$, and $\sigma = +1$ the spindle is then a two-sphere and we could expect to recover the results for GK geometry fibred over a Riemann surface of genus $g = 0$ with a topological twist. In particular, in this set-up there is no mixing of the R-symmetry vector with the $U(1)$ action on the two sphere and hence $b_0 = 0$.

We first note that setting $m_\pm = 1$ and taking the $b_0 \to 0$ limit of the off-shell supersymmetric action (7.31) we find

$$\lim_{b_0\to 0} S_{\mathrm{SUSY}} = -2\pi b_1 \sum_{i=1}^{n}(-p_i)\frac{\partial \mathcal{V}_{2n-1}}{\partial b_i} - A\sum_{a=1}^{d}\frac{\partial \mathcal{V}_{2n-1}}{\partial \lambda_a}, \tag{7.52}$$

with $-p_1 = +2$, and where we have defined

$$A \equiv -2\pi (\lambda_+ + \lambda_-) . \tag{7.53}$$

Note that in these expressions we should take derivatives before setting $b_1 = 2/(n-2)$. In addition, from (7.35) for the fluxes, in the $b_0 \to 0$ limit we obtain

$$\begin{aligned} N &= -\frac{1}{\nu_n} \sum_{a=1}^{d} \frac{\partial \mathcal{V}_{2n-1}}{\partial \lambda_a}, \\ M_a &= \frac{1}{\nu_n} \left[\frac{A}{2\pi} \sum_{b=1}^{d} \frac{\partial^2 \mathcal{V}_{2n-1}}{\partial \lambda_a \partial \lambda_b} + b_1 \sum_{i=1}^{n} (-p_i) \frac{\partial^2 \mathcal{V}_{2n-1}}{\partial \lambda_a \partial b_i} \right], \end{aligned} \tag{7.54}$$

and hence the constraint equation $\sum_A M_A = 0$ becomes in this limit

$$0 = -2\pi(-p_1) \sum_{a=1}^{d} \frac{\partial \mathcal{V}_{2n-1}}{\partial \lambda_a} + A \sum_{a,b=1}^{d} \frac{\partial^2 \mathcal{V}_{2n-1}}{\partial \lambda_a \partial \lambda_b} + 2\pi b_1 \sum_{a=1}^{d} \sum_{i=1}^{n} (-p_i) \frac{\partial^2 \mathcal{V}_{2n-1}}{\partial \lambda_a \partial b_i} . \tag{7.55}$$

Finally, the R-charges (7.46) are given by

$$\lim_{b_0 \to 0} R_a^{\pm} = -\frac{1}{N} \frac{2}{\nu_n} \frac{\partial \mathcal{V}_{2n-1}}{\partial \lambda_a} . \tag{7.56}$$

These are precisely the formulae derived in [10,13,14], associated with a Sasaki–Einstein space fibred over a Riemann surface of genus $g = 0$, after taking into account that the R-charges differ by an extra factor of N as noted before, as well as identifying $-p_i$ here with n_i there.

We expect that setting $m_{\pm} = 1$ implies that we must have $b_0 = 0$ (which we assumed above). For the case $n = 3$ we can explicitly show that this is true in Sect. 8.3.

8. Matching AdS$_3$ $\times$ Y_7 Solutions with Field Theory

This section will focus on the toric set up of the previous section with $n = 3$. This is associated with AdS$_3$ $\times$ Y_7 solutions of type IIB supergravity, where Y_7 is a toric GK geometry consisting of a fibration of a toric Sasaki–Einstein space X_5 over a spindle. These solutions can be interpreted as being dual to the $\mathcal{N} = 1$, $d = 4$ SCFT, which is dual to AdS$_5$ $\times$ X_5, that is then compactified on the spindle with magnetic fluxes switched on. We determine the explicit map between the field theory variables involved in c-extremization and those appearing in the extremization of the GK geometry. We also illustrate some of our formalism explicitly by considering the examples when $X_5 = S^5$ and also $T^{1,1}$.

8.1. $\mathcal{N}=1$, $d=4$ SCFT on $\mathbb{R}^{1,1}\times\Sigma$. Consider the four-dimensional $\mathcal{N}=1$ SCFT that is associated with N D3-branes sitting at the apex of the Calabi-Yau 3-fold cone $C(X_5)$, with toric vectors v_{ai}, $a=1,\dots,d$ and $i=1,\dots 3$. Recall that the SCFT has, generically,[29] $U(1)^d$ symmetry with $U(1)^{d-3}$ baryonic symmetry associated with the number of independent three cycles $d-3=\dim H_3(X_5,\mathbb{R})$ on the Sasaki–Einstein manifold X_5. We then place the SCFT on a spindle with background magnetic fluxes for the $U(1)^d$ symmetry, associated with either a twist or an anti-twist to preserve supersymmetry. Assuming that the resulting field theory flows to a $d=2$, $\mathcal{N}=(0,2)$ SCFT in the IR, we can extract the central charge, as well as the R-charges of the baryonic operators, using c-extremization [15]. The extremization can be carried out in two stages by first extremizing over the baryonic directions first. The resulting central charge after the first stage, which is still off-shell, is what can be matched with the off-shell gravitational computation, as we shall show in detail.

Let Δ_a be the trial R-charges of the fields that are associated with the toric divisors of $C(X_5)$ – see, for example, [56,57] for a general discussion of quiver gauge theories dual to toric $C(X_5)$, and this trial R-charge assignment. The Δ_a satisfy the constraint

$$\sum_{a=1}^{d}\Delta_a=2\,, \tag{8.1}$$

which is equivalent to the statement that the superpotential of the theory has R-charge 2. We also turn on background magnetic fluxes for the $U(1)^d$ symmetry, with field strengths F_a, $a=1,\dots,d$, given by

$$\mathfrak{p}_a=\frac{1}{2\pi}\int_\Sigma F_a, \tag{8.2}$$

where the $\mathfrak{p}_a$ are integers divided by (m_-m_+). These may be viewed as twisting the associated fields into sections of $\mathcal{O}(\mathfrak{p}_a)$ over the spindle, where to preserve supersymmetry we have the constraint

$$\sum_{a=1}^{d}\mathfrak{p}_a=-\frac{\sigma m_++m_-}{m_+m_-}\,, \tag{8.3}$$

associated with the twist and the anti-twist case when $\sigma=\pm1$, respectively.

We next define the field theory quantities[30]

$$\begin{aligned}\Delta_a^+&\equiv\Delta_a+\frac{1}{2}\varepsilon\left(\mathfrak{p}_a-\frac{r_a}{2}\frac{m_--\sigma m_+}{m_-m_+}\right),\\ \Delta_a^-&\equiv\Delta_a-\frac{1}{2}\varepsilon\left(\mathfrak{p}_a+\frac{r_a}{2}\frac{m_--\sigma m_+}{m_-m_+}\right),\end{aligned} \tag{8.4}$$

with r_a a set of arbitrary variables satisfying the constraint

$$\sum_{a=1}^{d}r_a=2\,. \tag{8.5}$$

[29] In some cases this is enlarged to a non-Abelian symmetry.

[30] These were labelled $\Delta_a^{(1)}$, $\Delta_a^{(2)}$ in [23].

Different choices of r_a give different gauges, and so will drop out of final results.[31] To see this, notice that the freedom we have is

$$r_a \;\mapsto\; r_a + \delta r_a \;, \qquad \text{with} \qquad \sum_{a=1}^{d} \delta r_a = 0 \,. \tag{8.6}$$

On the other hand, from (8.4) we see that such a shift may be absorbed into the trial R-charges Δ_a via

$$\Delta_a \;\mapsto\; \Delta_a + \frac{\delta r_a}{2}\frac{m_- - \sigma m_+}{m_- m_+}\varepsilon \,. \tag{8.7}$$

and, moreover, this shift preserves the constraint (8.1). Notice that (8.1)–(8.4) imply that $\Delta_a^\pm$ satisfy

$$\sum_{a=1}^{d} \Delta_a^+ = 2 - \frac{\varepsilon}{m_+} \,, \qquad \sum_{a=1}^{d} \Delta_a^- = 2 + \frac{\varepsilon}{\sigma m_-} \,. \tag{8.8}$$

Using anomaly polynomials, it has been shown that the off-shell trial central charge in the large N limit is given by [23]

$$c_{\text{trial}} = \frac{3}{\varepsilon}\left(\sum_{a<b<c} (\vec{v}_a, \vec{v}_b, \vec{v}_c)\Delta_a^+\Delta_b^+\Delta_c^+ - \sum_{a<b<c} (\vec{v}_a, \vec{v}_b, \vec{v}_c)\Delta_a^-\Delta_b^-\Delta_c^-\right) N^2 \,, \tag{8.9}$$

where $(\vec{v}_a, \vec{v}_b, \vec{v}_c) \equiv \det(\vec{v}_a, \vec{v}_b, \vec{v}_c)$. Associated with the $d-3$ baryonic directions, we can first impose a partial set of extremization conditions

$$\sum_{a=1}^{d} q_I^a \frac{\partial c_{\text{trial}}}{\partial \Delta_a} = 0 \,, \qquad I = 1, \ldots, d-3, \tag{8.10}$$

where we recall that q_I^a are the kernel vectors for the toric data for $C(X_5)$ satisfying

$$\sum_{a=1}^{d} q_I^a \, v_{ai} = 0. \tag{8.11}$$

which specifies the embedding $U(1)^{d-3} \subset U(1)^d$ (see Appendix A). After solving these conditions, and substituting back into c_{trial}, one obtains an off-shell function, $c_{\text{trial}}|_{\text{baryonic}}$, of $d-(d-3)=3$ variables: we start with the $d+1$ variables $\Delta_1, \ldots, \Delta_d, \varepsilon$, with one constraint (8.1), and end up with (say) $\Delta_1, \Delta_2, \varepsilon$, where we have eliminated $\Delta_3, \ldots, \Delta_d$ using the constraints (8.10). After extremizing over the baryonic directions, the resulting $\Delta_a^\pm|_{\text{baryonic}}$ are then also functions of trial R-charges $\Delta_1, \Delta_2, \varepsilon$. The on-shell results are then obtained by further extremizing over $\Delta_1, \Delta_2, \varepsilon$. This procedure is carried out for fixed magnetic fluxes $\mathfrak{p}_a$.

It is interesting to highlight that the baryon mixing constraints (8.10) can be recast as an expression that is linear in either Δ_a and ε. We prove this in Appendix B and, for example, we can write the conditions as

$$\sum_{b=1}^{d}\sum_{a=1}^{b}\sum_{c=1}^{d} q_I^a (\vec{v}_a, \vec{v}_b, \vec{v}_c) \left(\mathfrak{p}_b \Delta_c^+ + \mathfrak{p}_c \Delta_b^+ + \varepsilon\, \mathfrak{p}_b \mathfrak{p}_c\right) = 0. \tag{8.12}$$

[31] In various papers the results for Δ_a in different gauges have been reported.

8.2. Matching with GK geometry. Recalling (7.34), the rank of the field theory gauge group, N, is identified on the gravity side via

$$N \equiv m_+ M_+ = \sigma m_- M_- . \tag{8.13}$$

Furthermore the background magnetic fluxes for the SCFT $\mathfrak{p}_a$ are identified on the gravity side via

$$\mathfrak{p}_a \equiv \frac{M_a}{N}, \tag{8.14}$$

where M_a characterize the quantized five-form flux through the d toric five-cycles on Y_7 associated with the toric divisors $\mathcal{D}_a$. Importantly, the second condition in (7.34) on the M_a implies that this identification is consistent with the field theory constraint (8.3) on the $\mathfrak{p}_a$. Note also from the third condition in (7.34), for consistency, we deduce that the background magnetic fluxes of the SCFT are related to the geometric data $\vec{p}$ which specifies the fibration of the SE_5 over the spindle, via

$$\sum_a \vec{w}_a \mathfrak{p}_a = \frac{1}{m_- m_+} \vec{p}. \tag{8.15}$$

This particular relation may be understood in field theory as follows. Recall that turning on the magnetic flux $\mathfrak{p}_a$, the corresponding fields Z_a become sections of $\mathcal{O}(\mathfrak{p}_a)$ over the spindle Σ. On the other hand, in the gauged linear sigma model description of the fibres $C(X_5)$, the Z_a are also sections of non-trivial bundles over the fibres X_5, as they are (typically) charged under the baryonic $U(1)^{d-3}$ symmetries. However, from (8.11) one can verify that $\prod_{a=1}^{d} Z_a^{v_{ai}}$ are sections of trivial bundles on the fibres X_5, that moreover have charge δ_{ij} under the jth toric $U(1) \subset U(1)^3$. The variables $\vec{p}$ then precisely describe the twisting of this ith flavour direction over the spindle Σ, for $i = 2, 3$, which is the equality (8.15). A more detailed discussion of this may be found in appendix B of [13], albeit in the case that $\Sigma = \Sigma_g$ is a smooth Riemann surface of genus g.

We also make the identification

$$\varepsilon = b_0 , \tag{8.16}$$

as well as

$$\Delta_a^+\big|_{\text{baryonic}} = R_a^+ , \qquad \Delta_a^-\big|_{\text{baryonic}} = R_a^- , \tag{8.17}$$

where the notation on the left hand side means that we have extremized the trial c-function in field theory over the baryonic directions, as discussed in the last subsection. We immediately notice that the conditions on the $\Delta_a^\pm\big|_{\text{baryonic}}$ arising from (8.8) are consistent with the conditions on the $R_a^\pm$ in (7.47). Observe that $R_a^\pm$ are functions of the trial R-symmetry vector (b_0, b_2, b_3) (with $b_1 = 2$, as required for a GK geometry) while the $\Delta_a^\pm|_{\text{baryonic}}$ are functions of trial R-charges Δ_1, Δ_2, ε, after extremizing over the baryonic directions.

The claim is that (8.17) gives the change of variables between field theory and GK geometry. Notice this is *a priori* over-determined, as there are $2d$ equations in (8.17), but only three independent variables. However, we can prove that the identification is consistent; the proof is somewhat involved so we have presented it in Appendix B. We will also show there, using the key results of the previous section, that the off-shell trial central charge in gravity (7.31) can be expressed in the form

$$\mathscr{Z} = \frac{3N^2 b_1^2}{4b_0} \sum_{a<b<c} (\vec{v}_a, \vec{v}_b, \vec{v}_c) \left(R_a^+ R_b^+ R_c^+ - R_a^- R_b^- R_c^- \right). \tag{8.18}$$

With the above dictionary, this is then in exact agreement with the field theory result (8.9).

8.3. New variables. In field theory, instead of using the variables Δ_a subject to the constraints (8.1) and (8.10) we can use a slightly different set of variables φ_a defined via

$$\varphi_a \equiv \frac{1}{2} (\Delta_a^+ + \Delta_a^-) = \Delta_a + \frac{\varepsilon}{4} \frac{\sigma m_+ - m_-}{m_+ m_-} r_a, \tag{8.19}$$

where we used (8.4) and we recall the gauge parameters satisfy $\sum_a r_a = 2$. The inverse relation is $\Delta_a^\pm = \varphi_a \pm \frac{\varepsilon}{2} \mathfrak{p}_a$. The constraints (8.1), (8.10) then become the following constraints on the φ_a:

$$\begin{aligned} &\sum_{a=1}^{d} \varphi_a = 2 + \frac{\varepsilon}{2} \frac{\sigma m_+ - m_-}{m_+ m_-}, \\ &\qquad \sum_{b=1}^{d} \sum_{a=1}^{b} \sum_{c=1}^{d} q_I^a (\vec{v}_a, \vec{v}_b, \vec{v}_c) (\varphi_b \mathfrak{p}_c + \varphi_c \mathfrak{p}_b) = 0, \end{aligned} \tag{8.20}$$

where we used (B.26) to get the second expression. We also highlight that while $(\vec{v}_a, \vec{v}_b, \vec{v}_c)$ is antisymmetric in b, c and $(\varphi_b \mathfrak{p}_c + \varphi_c \mathfrak{p}_b)$ is symmetric, the sum is not trivial due to middle summation running from $a = 1, \ldots, b$. The off-shell trial central charge (8.9) can be written in terms of the φ_a as:

$$c_{\text{trial}} = 3N^2 \sum_{a<b<c} (\vec{v}_a, \vec{v}_b, \vec{v}_c) \left(\mathfrak{p}_a \varphi_b \varphi_c + \mathfrak{p}_b \varphi_a \varphi_c + \mathfrak{p}_c \varphi_a \varphi_b + \frac{\varepsilon^2}{4} \mathfrak{p}_a \mathfrak{p}_b \mathfrak{p}_c \right). \tag{8.21}$$

These formulae generalize those of [23,30], who studied the case of $X_5 = S^5$, to arbitrary X_5. Indeed for the special case of $X_5 = S^5$, which we will study in the next subsection, there is no baryonic symmetry and we just have the first constraint in (8.20). Furthermore, the trial central charge (8.21) reads

$$c_{\text{trial}} = 3N^2 \left(\mathfrak{p}_1 \varphi_2 \varphi_3 + \mathfrak{p}_2 \varphi_1 \varphi_3 + \mathfrak{p}_3 \varphi_1 \varphi_2 + \frac{\varepsilon^2}{4} \mathfrak{p}_1 \mathfrak{p}_2 \mathfrak{p}_3 \right), \tag{8.22}$$

and we have recovered the result (5.26) of [30].

For general X_5, using the φ_a variables it is straightforward to show that when the spindle becomes a two-sphere, then necessarily $\varepsilon = 0$ (recall that in Sect. 7.5 we assumed $\varepsilon = 0$). Indeed, setting $m_\pm = 1$ and $\sigma = +1$, both of the constraints (8.20) are independent of ε. Thus, the only ε dependence in c_{trial} is the quadratic dependence in the last term in (8.21) and hence the extremal point will necessarily have $\varepsilon = 0$.

8.4. Examples. We finish this section by discussing two toric examples when $X_5 = S^5$ and $T^{1,1}$, making the relation between the quantities appearing in field theory and GK geometry that was discussed in Sect. 8.2 very explicit. One can also check that the general formula for the associated toric Y_7 GK geometry discussed in Sect. 6, based on the master volume $\mathcal{V}_7$, agrees with the formula that we derived in Sect. 7 using $\mathcal{V}_5^{\pm}$. In fact, studying these examples was very helpful in elucidating the general results of Sect. 7. For the S^5 example, we also recover all of the results of Sect. 5.1, where we analysed this case as an example of an X_5 with no baryonic symmetries.

8.4.1. S^5 fibred over a spindle The toric data for S^5 can be specified by $d = 3$ inward pointing normal vectors given by

$$v_{1i} = (1, \vec{w}_1)\ , \qquad v_{2i} = (1, \vec{w}_2)\ , \qquad v_{3i} = (1, \vec{w}_3), \tag{8.23}$$

where

$$\vec{w}_1 = (0, 0)\ , \qquad \vec{w}_2 = (1, 0)\ , \qquad \vec{w}_3 = (0, 1). \tag{8.24}$$

There are no baryonic directions associated with S^5 and the kernel q_I^a for the toric data is trivial.

For S^5 fibred over a spindle, the toric data for the associated toric GK geometry Y_7 is then given by $D = 5$ inward pointing normal vectors given in (7.5):

$$v_{+\mu} = (m_+, 1, -a_+\vec{p}), \quad v_{-\mu} = (-\sigma m_-, 1, -\sigma a_-\vec{p}), \quad v_{a\mu} = (0, 1, \vec{w}_a), \tag{8.25}$$

where $\vec{p} = (p_2, p_3) \in \mathbb{Z}^2$ and $a_\pm \in \mathbb{Z}$ satisfy $a_-m_+ + a_+m_- = 1$. The three integers $p_i = (p_1, p_2, p_3)$, with $p_1 = -(m_- + \sigma m_+)$, specify the fibration. There is a one-dimensional kernel Q_0^A, satisfying $\sum_{A=1}^5 Q_0^A v_{A\mu} = 0$ (see Appendix A), given by

$$Q_0^A = (m_-, \sigma m_+, -(m_- + \sigma m_+) - p_2 - p_3, p_2, p_3)\ . \tag{8.26}$$

Using the vectors (8.25) we can now obtain an explicit, and lengthy, expression for the master volume $\mathcal{V}_7$ for Y_7 using the formula given in (2.27) of [13].

We can now carry out the geometric extremization using the procedure summarized in Sect. 6.3. The master volume $\mathcal{V}_7$ depends on just $5 - 3 = 2$ of the 5 Kähler class parameters λ_A. We therefore solve the constraint equation and one of the five-form flux quantization conditions in (6.26) for two of the λ_A. For example, we can solve the constraint equation and the expression for M_+ for (λ_+, λ_-). It must be the case, and indeed it is, that the remaining Kähler class parameters $\lambda_1, \lambda_2, \lambda_3$ then drop out of any final formula. The flux vector M_A in (6.26) is now expressed in terms of M_+, p_2, p_3 and, in particular, we find $\sigma m_- M_- = m_+ M_+ \equiv N$.

Instead of using these three variables, it is illuminating to express the flux vector in terms of M_1, M_2, M_3 to get

$$M_A = \left\{M_+, \frac{\sigma m_+}{m_-}M_+, M_1, M_2, M_3\right\}\ , \qquad M_+ = -\frac{m_-}{m_- + \sigma m_+}\sum_{a=1}^{3} M_a. \tag{8.27}$$

We can further rescale the M_a by a factor of $N = \sigma m_- M_- = m_+ M_+$ and introduce $\mathfrak{p}_a$, which are to be identified with the background magnetic fluxes of the SCFT shortly, via

$$\mathfrak{p}_a \equiv \frac{M_a}{N}\ , \qquad \sum_{a=1}^{3}\mathfrak{p}_a = -\frac{m_- + \sigma m_+}{m_+ m_-}. \tag{8.28}$$

Note from (8.15), the fibration data $\vec{p} = (p_2, p_3)$ is related to the $\mathfrak{p}_a$ via

$$p_2 = m_- m_+ \mathfrak{p}_2, \qquad p_3 = m_- m_+ \mathfrak{p}_3. \tag{8.29}$$

Also, importantly, (8.28) implies

$$\mathfrak{p}_1 = -(\mathfrak{p}_2 + \mathfrak{p}_3) - \frac{m_- + \sigma m_+}{m_- m_+} = \frac{p_1 - p_2 - p_3}{m_- m_+}, \tag{8.30}$$

where recall $p_1 = -(m_- + \sigma m_+)$. Observe that there is a $\mathbb{Z}_3$ symmetry permuting the $\mathfrak{p}_a$ but not (p_1, p_2, p_3): the latter is a consequence of the fact that we singled out the p_1 direction in constructing the fibration.

At this point we can now obtain an expression for the off-shell central charge $\mathscr{Z}$ from (2.16) and the expression for S_{SUSY} in (6.26). This is expressed in terms of $m_\pm$, $a_\pm$, $\mathfrak{p}_a$, b_0, b_1, b_2, b_3 and we should set $b_1 = 2$. The expression for $\mathscr{Z}$ is quadratic in b_0, b_2, b_3 and after extremizing over these variables we find that the on-shell central charge, c_{sugra}, as calculated from the GK geometry, can be expressed as

$$c_{\mathrm{sugra}} \equiv \mathscr{Z}_{\mathrm{os}} = \frac{6 m_-^2 m_+^2 \mathfrak{p}_1 \mathfrak{p}_2 \mathfrak{p}_3}{m_-^2 + m_+^2 - m_-^2 m_+^2 (\mathfrak{p}_1^2 + \mathfrak{p}_2^2 + \mathfrak{p}_3^2)} N^2, \tag{8.31}$$

exactly as in Sect. 5.1. Notice that $a_\pm$ have dropped out of the final expression. With the master volume $\mathcal{V}_7$ in hand, it is also straightforward to obtain explicit expressions for both the off-shell and on-shell R-charges $R_a^\pm$ using (7.43). The on-shell expressions can be written

$$\begin{aligned} R_a^+ &= -C m_+ \left(\mathfrak{p}_1[\sigma + m_- \mathfrak{p}_1], \mathfrak{p}_2[\sigma + m_- \mathfrak{p}_2], \mathfrak{p}_3[\sigma + m_- \mathfrak{p}_3]\right), \\ R_a^- &= -C m_- \left(\mathfrak{p}_1[1 + m_+ \mathfrak{p}_1], \mathfrak{p}_2[1 + m_+ \mathfrak{p}_2], \mathfrak{p}_3[1 + m_+ \mathfrak{p}_3]\right), \end{aligned} \tag{8.32}$$

where

$$C = \frac{2 m_- m_+}{m_-^2 + m_+^2 - m_-^2 m_+^2 (\mathfrak{p}_1^2 + \mathfrak{p}_2^2 + \mathfrak{p}_3^2)}, \tag{8.33}$$

again in agreement with Sect. 5.1. Demanding $c_{\mathrm{sugra}} > 0$ as well $R_a^+ > 0$ and $\sigma R_a^- > 0$ gives the restrictions on the parameters in alignment with the explicit supergravity solutions as discussed in Sect. 5.1.

We can now compare these results with the corresponding calculations in field theory using anomaly polynomials and c-extremization. In fact these calculations were carried out for the case of the anti-twist already in [21,24,27]. We consider $\mathcal{N} = 4$ SYM theory dual to $\mathrm{AdS}_5 \times S^5$, with $SU(N)$ gauge group. We place the theory on a spindle with background magnetic fluxes, parametrized by $\mathfrak{p}_a$, for the $U(1)^3 \subset SU(4)$ global symmetry with $\sum_{a=1}^3 \mathfrak{p}_a = -\frac{m_- + \sigma m_+}{m_+ m_-}$. The trial field theory central charge, c_{trial}, is parametrized by Δ_a, satisfying $\sum_{a=1}^3 \Delta_a = 2$. From (8.9) we find that c_{trial} is explicitly given by

$$
\begin{aligned}
c_{\text{trial}} = \Bigg\{ & 3(\Delta_1\Delta_2\mathfrak{p}_3 + \Delta_1\Delta_3\mathfrak{p}_2 + \Delta_2\Delta_3\mathfrak{p}_1) \\
& - \epsilon \frac{3(m_- - \sigma m_+)[(r_3\Delta_2 + r_2\Delta_3)\mathfrak{p}_1 + (r_3\Delta_1 + r_1\Delta_3)\mathfrak{p}_2 + (r_2\Delta_1 + r_1\Delta_2)\mathfrak{p}_3]}{4m_-m_+} \\
& + 3\epsilon^2 \left[\frac{(r_2r_3\,\mathfrak{p}_1 + r_1r_3\,\mathfrak{p}_2 + r_1r_2\,\mathfrak{p}_3)(m_- - \sigma m_+)^2 + 9\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3 m_-^2 m_+^2)}{16 m_-^2 m_+^2} \right] \Bigg\} N^2.
\end{aligned} \tag{8.34}
$$

We then find that c_{trial} as a function of the three independent variables $\Delta_2, \Delta_3, \varepsilon$ (say) exactly agrees with the off-shell gravitational charge $\mathscr{Z}$ as a function of b_0, b_2, b_3 (with $b_1 = 2$), provided that we utilise the dictionary given in Sect. 8.2:

$$
\begin{aligned}
b_0 &= \varepsilon, \\
\mathfrak{p}_a \quad &\leftrightarrow \quad \frac{M_a}{N}, \\
\Delta_a^{\pm} \equiv \Delta_a &\pm \frac{1}{2}\varepsilon\left(\mathfrak{p}_a \mp \frac{r_a}{2}\frac{m_- - \sigma m_+}{m_-m_+}\right) \quad \leftrightarrow \quad R_a^{\pm},
\end{aligned} \tag{8.35}
$$

as well as identifying N on each side. Explicitly, for this example the dictionary reads

$$
\begin{aligned}
b_0 &= \varepsilon, \\
b_2 &= \Delta_2 + \frac{1}{2}\mathfrak{p}_2\varepsilon - \frac{r_2}{4}\left(\frac{m_- - \sigma m_+}{m_-m_+}\right)\varepsilon - a_+\mathfrak{p}_2 m_-\varepsilon, \\
b_3 &= \Delta_3 + \frac{1}{2}\mathfrak{p}_3\varepsilon - \frac{r_3}{4}\left(\frac{m_- - \sigma m_+}{m_-m_+}\right)\varepsilon - a_+\mathfrak{p}_3 m_-\varepsilon.
\end{aligned} \tag{8.36}
$$

Notice that the dictionary involves the parameters a_+. Recall that a_+ is only defined up the transformation given in (7.7). Making this shift in (8.36), and using (8.29), induces a transformation on the vector (b_0, b_1, b_2, b_3) which is precisely given by the $SL(4, \mathbb{Z})$ transformation in (7.8) with $b_2 \to b_2 + \kappa p_2 b_0$ and $b_3 \to b_2 + \kappa p_3 b_0$.

8.4.2. $T^{1,1}$ fibred over a spindle The toric data for $T^{1,1}$ can be specified by $d = 4$ inward pointing normal vectors given by

$$
v_{1i} = (1, \vec{w}_1)\,, \qquad v_{2i} = (1, \vec{w}_2)\,, \qquad v_{3i} = (1, \vec{w}_3), \qquad v_{4i} = (1, \vec{w}_4), \tag{8.37}
$$

where

$$
\vec{w}_1 = (0,0)\,, \qquad \vec{w}_2 = (1,0)\,, \qquad \vec{w}_3 = (1,1), \qquad \vec{w}_4 = (0,1). \tag{8.38}
$$

There is a one-dimensional kernel q_1^a, satisfying $\sum_{a=1}^4 q_1^a\, v_{ai} = 0$ (see Appendix A), given by

$$
q_1^a = (-1, 1, -1, 1)\,. \tag{8.39}
$$

The toric data for $T^{1,1}$ fibred over a spindle, Y_7, is then given by $D = 6$ inward pointing normal vectors given in (7.5):

$$v_{+\mu} = (m_+, 1, -a_+\vec{p}), \quad v_{-\mu} = (-\sigma m_-, 1, -\sigma a_-\vec{p}), \quad v_{a\mu} = (0, 1, \vec{w}_a), \tag{8.40}$$

where $\vec{p} = (p_2, p_3) \in \mathbb{Z}^2$ and $a_\pm \in \mathbb{Z}$ satisfy $a_- m_+ + a_+ m_- = 1$. The three integers $p_i = (p_1, p_2, p_3)$, with $p_1 = -(m_- + \sigma m_+)$, specify the fibration. There is a two-dimensional kernel $Q^A_{\hat{I}} = (Q^A_0, Q^A_1)$, satisfying $\sum_{A=1}^{6} Q^A_{\hat{I}} v_{A\mu} = 0$ (see Appendix A), given by

$$\begin{aligned} Q^A_0 &= (m_-, \sigma m_+, -(m_- + \sigma m_+) - p_2 - p_3, p_2, 0, p_3) \,, \\ Q^A_1 &= (0, 0, 1, -1, 1, -1) = (0, 0, q^1_1, q^2_1, q^3_1, q^4_1) \,. \end{aligned} \tag{8.41}$$

Using the vectors (8.25) we can now obtain an explicit, and lengthy, expression for the master volume $\mathcal{V}_7$ for Y_7 using the formula given in (2.27) of [13].

The master volume $\mathcal{V}_7$ depends on just $6 - 3 = 3$ of the 6 Kähler class parameters λ^A. We therefore solve the constraint equation and two of the five-form flux quantization conditions in (6.26) for three of the λ^A. For example, we can solve the constraint equation and the expressions for M_+ and M_1 for $(\lambda_+, \lambda_-, \lambda_1)$. It must be the case, and indeed it is, that the remaining Kähler class parameters $\lambda_2, \lambda_3, \lambda_4$ then drop out of any final formula. The flux vector M_A in (6.26) is now expressed in terms of M_+, M_1, p_2, p_3 and, in particular, we find $m_+ M_+ = \sigma m_- M_- \equiv N$. Instead of using these four variables, it is most illuminating to express the flux vector in terms of M_1, M_2, M_3, M_4 to obtain

$$M_A = \left\{ M_+, \frac{\sigma m_+}{m_-} M_+, M_1, M_2, M_3, M_4 \right\} , \qquad M_+ = -\frac{m_-}{m_- + \sigma m_+} \sum_{a=1}^{4} M_a . \tag{8.42}$$

We can further rescale the M_a by a factor of $N = \sigma m_- M_- = m_+ M_+$ and introduce $\mathfrak{p}_a$, which are to be identified with the background magnetic fluxes of the SCFT shortly, via

$$\mathfrak{p}_a \equiv \frac{M_a}{N} \,, \qquad \sum_{a=1}^{4} \mathfrak{p}_a = -\frac{m_- + \sigma m_+}{m_+ m_-} , \tag{8.43}$$

and also, from (8.15), we have the fibration data $\vec{p} = (p_2, p_3)$ is related to the $\mathfrak{p}_a$ via

$$p_2 = m_- m_+ (\mathfrak{p}_2 + \mathfrak{p}_3), \qquad p_3 = m_- m_+ (\mathfrak{p}_3 + \mathfrak{p}_4). \tag{8.44}$$

We can now obtain an expression for the off-shell central charge $\mathscr{Z}$ from (2.16) and the expression for S_{SUSY} in (6.26). This is expressed in terms of $m_\pm, a_\pm, \mathfrak{p}_a, b_0, b_1, b_2, b_3$ and we should set $b_1 = 2$. It is quadratic in b_0, b_2, b_3 and after extremizing $\mathscr{Z}$ over these variables we find that the on-shell central charge as calculated from gravity can be expressed as[32]

$$\begin{aligned} c_{\mathrm{sugra}} &\equiv \mathscr{Z}_{\mathrm{os}} \\ &= \frac{3(m_- + \sigma m_+)^2 \sum_{a<b, c\neq a,b} \mathfrak{p}_a \mathfrak{p}_b \mathfrak{p}_c^2 \sum_{a<b<c} \mathfrak{p}_a \mathfrak{p}_b \mathfrak{p}_c}{(m_-^2 - \sigma m_- m_+ + m_+^2) \prod_{a<b} (\mathfrak{p}_a + \mathfrak{p}_b) - \sigma m_+ m_- \Theta_{KW}} N^2, \end{aligned} \tag{8.45}$$

[32] This corrects some typos in [23] who analysed this from field theory as explained below.

where

$$\Theta_{KW} = \sum_{a<b,c\neq a,b} \mathfrak{p}_a\mathfrak{p}_b\mathfrak{p}_c^4 - 2\sum_{a<b}\mathfrak{p}_a\mathfrak{p}_b\prod_c \mathfrak{p}_c. \tag{8.46}$$

Notice that $a_\pm$ have dropped out of the final expression. The extremal values b_0, b_2, b_3 are lengthy and we don't give them here explicitly. With the master volume in hand, it is also straightforward to obtain explicit expressions for both the off-shell and on-shell R-charges $R_a^\pm$ using (7.43). The on-shell expressions are expressed in terms of $m_\pm$ and $\mathfrak{p}_a$ as expected. One then determine the ranges of allowed parameters by imposing, on-shell, $c_{\text{sugra}} > 0$, $R_a^+ > 0$ and $\sigma R_a^- > 0$.

Now $H^2(T^{1,1},\mathbb{Z}) \cong \mathbb{Z}$, associated with there being a single baryonic $U(1)$ symmetry in the Klebanov-Witten field theory dual to $\text{AdS}_5 \times T^{1,1}$. Thus, $H_5(Y_7,\mathbb{Z}) = \mathbb{Z}^2$ and there are two independent fluxes $\mathcal{N}_0, \mathcal{N}_1$. We can therefore express M_A in terms of $\mathcal{N}_0, \mathcal{N}_1$, given the spindle data $m_\pm, \sigma$ and the fibration data p_i. Recall from (7.39) we have $\mathcal{N}_0 = \frac{N}{m_+m_-}$. Also, from (8.41) we have $q_0^a = (-(m_-+\sigma m_+)-p_2-p_3,\, p_2, 0, p_3)$ and $q_1^a = (-1, 1, -1, 1)$. Hence from (7.38) we can write

$$\begin{aligned}(M_1, M_2, M_3, M_4) = \frac{N}{m_+m_-}&(-(m_- + \sigma m_+) - p_2 - p_3, p_2, 0, p_3)\\ &+\mathcal{N}_1(1, -1, 1, -1).\end{aligned} \tag{8.47}$$

and $\mathcal{N}_1$ is the baryonic charge. As we emphasized earlier, the definition of baryonic charge is not unique. In particular, for a given M_a, we could also write (7.38) in the form $M_a = (q_0^a + q_1^a\mathcal{N}_1/\mathcal{N}_0)\mathcal{N}_0$ and then using $\tilde{q}_0^a \equiv q_0^a + q_1^a\mathcal{N}_1/\mathcal{N}_0$ instead of q_0^a to define the baryonic charge, which still satisfies $\sum_{a=1}^d \tilde{q}_0^a\, v_{ai} = p_i$, we would conclude that the baryonic charge vanishes.

It is interesting to examine the special sub-class associated with the flavour twist that we discussed in Sects. 4 and 5. We first recall that the Sasakian volume of $T^{1,1}$ is given by

$$\text{Vol}_S(T^{1,1}) = \frac{\pi^3 b_1}{b_2b_3(b_1-b_2)(b_1-b_3)}, \tag{8.48}$$

while the Sasakian volume of the three-dimensional supersymmetric submanifolds are

$$\begin{aligned}\text{Vol}_S(S_1) &= \frac{2\pi^2}{b_2b_3}, & \text{Vol}_S(S_2) &= \frac{2\pi^2}{b_3(b_1-b_2)},\\ \text{Vol}_S(S_3) &= \frac{2\pi^2}{(b_1-b_2)(b_1-b_3)}, & \text{Vol}_S(S_4) &= \frac{2\pi^2}{b_2(b_1-b_3)}.\end{aligned} \tag{8.49}$$

The Sasaki volume (8.48) is extremized for $b_2 = b_3 = \frac{b_1}{2}$. Setting $b_1 = 3$ we get the Sasaki–Einstein volumes which are $\text{Vol}_{SE}(T^{1,1}) = \frac{16\pi^3}{27}$ and $\text{Vol}_{SE}(S_a) = \frac{8\pi^2}{9}$.

The flavour twist is obtained by demanding the condition (4.1), which in the toric setting amounts to set all the λ_a to be equal in a particular gauge. Using the gauge transformations (6.25), we can fix γ_2 and γ_3 in such a way that three of the four λ_a are equal. In order to fix the last λ_a, we need to fix one of the fluxes M_a, e.g.

$$M_3 = \frac{\pi N}{2b_0}\left[\left.\frac{\text{Vol}_S(S_3)}{\text{Vol}_S(T^{1,1})}\right|_{\vec{b}^{(+)}} - \left.\frac{\text{Vol}_S(S_3)}{\text{Vol}_S(T^{1,1})}\right|_{\vec{b}^{(-)}}\right], \tag{8.50}$$

(*c.f.* (8.47)). All the other fluxes are fixed by (7.34), and likewise they read

$$M_a = \frac{\pi N}{2b_0}\left[\frac{\mathrm{Vol}_S(S_a)}{\mathrm{Vol}_S(T^{1,1})}\bigg|_{\vec{b}^{(+)}} - \frac{\mathrm{Vol}_S(S_a)}{\mathrm{Vol}_S(T^{1,1})}\bigg|_{\vec{b}^{(-)}}\right], \tag{8.51}$$

exactly as we derived earlier in (4.10).

The universal anti-twist case is obtained by setting (4.24) as well as (4.25). The latter combined with (4.28) tells us that the on-shell twisting parameters are fixed to be, with $\sigma = -1$,

$$p_2 = p_3 = \frac{p_1}{2} = \frac{1}{2}(m_+ - m_-)\,. \tag{8.52}$$

Furthermore, the condition (8.50) implies

$$\mathfrak{p}_1 = \mathfrak{p}_2 = \mathfrak{p}_3 = \mathfrak{p}_4 = \frac{m_+ - m_-}{4m_+m_-}. \tag{8.53}$$

We then recover, as expected, all the general results of Sect. 4.2 for the universal anti-twist, in particular the on-shell results for $\mathscr{L}$ given in (4.42) and $R_a^{\pm}$ given in (4.44) with

$$a_{4\mathrm{d}} = \frac{27N^2}{64}, \qquad R_a^{4\mathrm{d}} = \frac{1}{2}, \quad a = 1, 2, 3, 4. \tag{8.54}$$

We can now compare with field theory. For the case of the anti-twist the relevant field theory calculations were carried out already in [23]; here we also include the twist case. We consider the quiver gauge theory dual to $\mathrm{AdS}_5 \times T^{1,1}$, with rank N gauge groups. This gauge theory has $U(1)^3 \times U(1)_B$ symmetry where $U(1)_B$ is a baryonic symmetry associated with the kernel (8.39). We now put this gauge theory on a spindle with background magnetic fluxes, parametrized by $\mathfrak{p}_a$, for the $U(1)^3 \times U(1)_B$ symmetry with $\sum_{a=1}^4 \mathfrak{p}_a = -\frac{m_- + \sigma m_+}{m_+m_-}$. The trial central charge is parametrized by Δ_a, satisfying $\sum_a \Delta_a = 2$ and from the general result (8.9) we find (setting here $r_1 = r_2 = r_3 = r_4 = \frac{1}{2}$ for simplicity)

$$\begin{aligned} c_{\mathrm{trial}} = N^2\Big\{&3[\mathfrak{p}_1(\Delta_2\Delta_3 + \Delta_2\Delta_4 + \Delta_3\Delta_4) + \mathfrak{p}_2(\Delta_1\Delta_3 + \Delta_1\Delta_4 + \Delta_3\Delta_4) \\ &+ \mathfrak{p}_3(\Delta_1\Delta_2 + \Delta_1\Delta_4 + \Delta_2\Delta_4) + \mathfrak{p}_4(\Delta_1\Delta_2 + \Delta_1\Delta_3 + \Delta_2\Delta_3)] \\ &- \varepsilon\frac{3(m_- - \sigma m_+)}{4m_-m_+}(\mathfrak{p}_1(2-\Delta_1) + \mathfrak{p}_2(2-\Delta_2) + \mathfrak{p}_3(2-\Delta_3) + \mathfrak{p}_4(2-\Delta_4)) \\ &+ \varepsilon^2[-\frac{9(m_- + \sigma m_+)(m_- - \sigma m_+)^2}{64m_-^3m_+^3} \\ &+ \frac{3}{4}(\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3 + \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_4 + \mathfrak{p}_1\mathfrak{p}_3\mathfrak{p}_4 + \mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_4)]\Big\}N^2. \end{aligned} \tag{8.55}$$

The on-shell central charge calculated in field theory is obtained by extremizing with respect to Δ_a and ϵ, subject to $\sum_a \Delta_a = 2$. We can do this in two steps, first extremizing over the baryonic direction defined by $q_1^a\Delta_a = \Delta_1 - \Delta_2 + \Delta_3 - \Delta_4$, and then over the remaining three independent variables, which can taken to be, for example, $\Delta_1, \Delta_2, \varepsilon$. After the first step we get $c_{\mathrm{trial}}|_{\mathrm{baryonic}}$ as a function of $\Delta_1, \Delta_2, \varepsilon$ and we find that there is an exact agreement with the off-shell gravitational charge $\mathscr{X}$, as a function of b_0, b_2, b_3. The dictionary between the two variables is as described in Sect. 8.2:

$$\begin{aligned} b_0 &= \varepsilon, \\ \mathfrak{p}_a &\leftrightarrow \frac{M_a}{N}, \\ \Delta_a^{\pm}|_{\text{baryonic}} \equiv \Delta_a \pm \frac{1}{2}\varepsilon\left(\mathfrak{p}_a \mp \frac{r_a}{2}\frac{m_- - \sigma m_+}{m_- m_+}\right) &\leftrightarrow R_a^{\pm}, \end{aligned} \tag{8.56}$$

as well as identifying N on each side. Explicitly, for this example we find

$$\begin{aligned} b_0 &= \varepsilon, \\ b_2 &= \frac{1}{\mathfrak{p}_1 + \mathfrak{p}_2}\left[2\mathfrak{p}_2 - (\mathfrak{p}_2 + \mathfrak{p}_3)\,\Delta_1 + (\mathfrak{p}_1 + \mathfrak{p}_4)\,\Delta_2\right] + \frac{1}{2}\,\varepsilon(\mathfrak{p}_2 + \mathfrak{p}_3) \\ &\quad \frac{1}{4}\varepsilon\left(\frac{m_- - \sigma m_+}{m_+ m_-}\right)\frac{(-r_2\mathfrak{p}_1 + (r_1 - 2)\mathfrak{p}_2 + r_1\mathfrak{p}_3 - r_2\mathfrak{p}_4)}{\mathfrak{p}_1 + \mathfrak{p}_2} - a_+ m_- \varepsilon(\mathfrak{p}_2 + \mathfrak{p}_3)\,, \\ b_3 &= 2 - \Delta_1 - \Delta_2 + \frac{1}{2}\,\varepsilon(\mathfrak{p}_3 + \mathfrak{p}_4) - \frac{1}{4}\,\varepsilon(r_3 + r_4)\left(\frac{m_- - \sigma m_+}{m_+ m_-}\right) \\ &\quad - a_+ m_- \varepsilon(\mathfrak{p}_3 + \mathfrak{p}_4)\,. \end{aligned} \tag{8.57}$$

Notice that the dictionary involves the parameters a_+. Recall that a_+ is only defined up the transformation given in (7.7). Making this shift in (8.57), and using (8.44), induces a transformation on the vector (b_0, b_1, b_2, b_3) which is precisely given by the $SL(4, \mathbb{Z})$ transformation in (7.8) with $b_2 \to b_2 + \kappa p_2 b_0$ and $b_3 \to b_2 + \kappa p_3 b_0$.

9. Black Holes in AdS$_4 \times SE_7$

For $n = 4$ we expect the AdS$_2 \times Y_9$ solutions, with Y_9 fibred as in (1.1), can arise as the near horizon limit of supersymmetric, accelerating black holes that carry magnetic charge and asymptotically approach AdS$_4 \times X_7$, with X_7 a Sasaki–Einstein manifold. Such solutions, and our associated entropy function (2.21), have recently been discussed in [51]. In this section we make contact with that work, along with some related observations.

As explained in [51], from a four-dimensional perspective the near horizon black hole solutions take the form AdS$_2 \times \Sigma$, with the conical deficits of the spindle horizon Σ being related to a non-zero acceleration parameter for the black hole [22]. Such black holes can carry two types of magnetic charge, associated to massless gauge fields in AdS$_4$ with a different $D = 11$ origin: one type arise from isometries of X_7 (“flavour symmetries”), while the other type arise from homology cycles of X_7 (“baryonic symmetries”). The flavour magnetic charges can immediately be identified with the twisting parameters $p_i \in \mathbb{Z}$: this follows directly from their definition in (3.8), where the A_i are connection one-forms for the twisting of the $U(1)^s$ action on X_7 over the spindle Σ. As already alluded to in Sect. 7.3, the baryonic magnetic charges, or equivalently “baryonic fluxes” $\mathcal{N}_I$, instead require (arbitrary) choices to define unambiguously. This is a reflection of the well-known mixing between baryonic and flavour symmetries in field theory, and can also be seen directly at the level of Kaluza–Klein reduction.

Specifically, as described in [51], in considering a linear Kaluza–Klein reduction of $D = 11$ supergravity on an $\mathrm{AdS}_4 \times X_7$ background, one has the perturbation of the six-form potential

$$\delta C_6 = \left[\sum_{i=1}^{s} A_i \wedge \omega_i + \sum_{I=1}^{b_5(X_7)} A_I \wedge \omega_I \right] \frac{\nu_4 L^6}{2\pi} N. \tag{9.1}$$

Here both ω_i and ω_I are co-closed five-forms on X_7, but ω_I is closed while $\mathrm{d}\omega_i = \partial_{\varphi_i} \lrcorner \, \mathrm{vol}_{X_7}$, with vol_{X_7} a suitably normalized volume form [58] (see also [59,60]). We are then free to shift $\omega_i \to \omega_i + \sum_I c_i^I \omega_I$, for arbitrary constants c_i^I, which is the freedom to mix baryonic symmetries into flavour symmetries. This is the same freedom discussed after equation (7.38), or equivalently the freedom to shift $\alpha_a^i \to \alpha_a^i + \sum_I c_i^I q_I^a$. This correspondingly shifts the four-dimensional gauge field as $A_I \to A_I - \sum_{i=1}^{s} c_i^I A_i$, and hence also the flux $\int_\Sigma \mathrm{d}A_I/2\pi$ through the spindle horizon Σ. Although one might (naively) define this as the "baryonic flux", as shown in Appendix C, this is *not* in general the same as the "baryonic flux" $\mathcal{N}_I$ already defined.

Recall that the original flux integrals $\mathcal{N}_\alpha$, defined in (2.11), depend only on the homology classes of the seven-cycles $\Sigma_\alpha \subset Y_9$. The ambiguity in correspondingly defining the fluxes $\mathcal{N}_I$, with associated homology cycles labelled by $I = 1, \ldots, b_5(X_7)$ in the fibre X_7, arises precisely because of the twisting. To explain this, pick representatives $\mathcal{C}_I \subset X_7$ of the five-cycles in the fibre, which are invariant under the $U(1)^s$ action that we twist by. The $\mathcal{C}_I$ then fibre over the spindle Σ to give seven-cycles in Y_9, with the seven-form flux through these cycles (2.11) then defining a set of fluxes. However, when the flavour magnetic charges/twisting parameters p_i are non-zero, the homology classes of these seven-cycles in general depend on the representative of $\mathcal{C}_I$: specifically, two representatives of the same cycle in X_7, but with different $U(1)^s$ actions, will twist differently, resulting in different seven-form fluxes.[33] This is just another manifestation of the baryonic/flavour mixing problem described in the previous paragraph. A precise relation between the Kaluza–Klein point of view and fluxes of the AdS_2 solution, and in particular equation (7.41), is spelled out in Appendix C.

Rather than trying to separate out a set of "baryonic magnetic charges", it is therefore preferable to work directly with the seven-form fluxes, with a chosen basis of seven-cycles, which are then defined unambiguously. Or, even better, in the toric case to work with the toric fluxes M_a, given by (7.38) and which are subject to the constraints in the last two lines of (7.34). It is the latter that most directly describe how the field theory is twisted over $\mathbb{R} \times \Sigma$, on the conformal boundary of the AdS_4 black hole solutions. Indeed, more generally it is presumably possible to define a set of "equivariant" fluxes, utilizing a form of equivariant cohomology that generalizes toric geometry appropriately, so that the flavour and baryonic charges may be combined naturally. We leave further development of this for future work, anticipating that the results of [59,60] could play an important role.

[33] For a concrete illustration of this in the AdS_3 case, see the formula (8.47) for the fluxes (M_1, M_2, M_3, M_4) for $T^{1,1}$ internal space. In this case there is only one internal three-cycle, with the four toric representatives all being ± 1 times this cycle in homology in $T^{1,1}$. But this flux vector depends explicitly on the twisting variables p_i, showing that the homology classes of the corresponding four five-cycles in Y_7 (the fibration of $T^{1,1}$ over Σ) are distinct i.e. in (8.47) we see that the four M_a are not simply related by $\pm$ signs, but instead are different linear combinations of $\mathcal{N}_0$ and $\mathcal{N}_1$.

In the case of the universal anti-twist such black hole solutions have been explicitly constructed in $D = 4$ minimal gauged supergravity[34] and then uplifted on Sasaki–Einstein X_7 to obtain black hole solutions in $D = 11$ [22]. As discussed in Sect. 4, the universal anti-twist is a special case of the flavour twist. In fact a richer class of black hole solutions were constructed in [22] that have non-vanishing rotation and also electric flavour charge. The entropy of these black holes was computed in [22] and, for vanishing rotation and electric charge, precisely agrees with the result (4.50) obtained using the formalism of this paper. However, a curious feature of these black hole solutions is that while setting the electric charge and rotation to zero leads to a regular $\mathrm{AdS}_2 \times Y_9$ horizon, the $\mathbb{R} \times \Sigma$ conformal boundary degenerates into a singular geometry.

It is possible that this is a feature of purely magnetically charged black holes in the anti-twist class more generally,[35] i.e. not just in the universal anti-twist class. If this is the case, then we expect that they should be viewed as limiting, degenerate cases of larger families of black hole solutions that also have non-vanishing electric charge and angular momentum. On the other hand we do not expect such restrictions in the twist class, and we expect, generically, that one will be able to construct purely magnetic charged and accelerating black holes with regular $\mathbb{R} \times \Sigma$ conformal boundaries and regular $\mathrm{AdS}_2 \times Y_9$ horizons.

The results of this paper allow us to calculate the entropy of supersymmetric, accelerating and purely magnetically charged black holes. In the flavour twist class, the results of Sect. 5.2 for the $X_7 = S^7$ case precisely agree with the results for explicit black hole solutions constructed using STU gauged supergravity [25]. The results for $V^{5,2}$ provide a precise new prediction for the entropy. A much richer family of cases arises when X_7 is toric and the techniques we developed in Sect. 7 allow one to obtain the entropy very explicitly[36] by solving a system of algebraic equations. It would be very interesting to recover these results from a field theory calculation and hence obtain a microstate counting interpretation of the black hole entropy.

A similar discussion to the above can be made for $\mathrm{AdS}_3 \times Y_7$ solutions with Y_7 fibred as in (1.1). One now expects these to arise as the near horizon limit of supersymmetric accelerating black strings in AdS_5, carrying flavour and baryonic magnetic charges. Many of the above points for black holes are also applicable to black strings, but there are some differences. For example, less is known about explicit solutions: indeed accelerating black strings in $D = 5$ minimal gauged supergravity are not known, which would be the analogue of the $D = 4$ accelerating black holes of [22]. Another important difference is that one cannot add electric charge (flavour or baryonic) and preserve the $\mathrm{AdS}_3 \times \Sigma$ horizon, which could be particularly relevant for anti-twist constructions. On the other hand, within the toric case one can make a precise connection with a field theory computation using c-extremization and anomaly polynomials, as we discussed in Sect. 8.

10. Discussion

In this paper we have studied various aspects of GK geometry, consisting of a Sasaki–Einstein space fibred over a spindle. One of our main results is the formula (3.25) which, in particular, gives rise to the gravitational block form for the supersymmetric

[34] Note that N_{SE} in e.g. (4.46) of [22] is the same as N in this paper.

[35] Indeed, it is challenging to solve the conformal Killing spinor equation on the boundary [22,33].

[36] Obtaining the final result in closed form can be challenging since the extremization procedure requires finding roots of polynomials.

action as given in (3.37). By setting $m_\pm = 1$, this formula is also valid if the spindle is replaced with a two-sphere, giving a new perspective on some of the results on GK geometry presented in [10–14]. These results provide a concrete demonstration that gravitational blocks appear very generally in the context of M2 and D3-branes reduced on two-dimensional spaces with an azimuthal symmetry.

Since the gravitational block formula (3.25) receives contributions from the two poles of the spindle, one might imagine it arises by applying an appropriate fixed point formula, with the associated action generated by azimuthal rotations of the spindle (or sphere). A technical observation is that there are some (implicit) signs in the gravitational block formula, related to the choice of whether we are in the twist or anti-twist class. As shown in [27], this is in turn related to the chirality of the Killing spinor at the fixed poles of the spindle. Thus, if a standard fixed point formula, such as the Berline–Vergne formula, can be used to derive the gravitational block result, the equivariant forms must be sensitive to these signs, related to spinor chirality.

On the other hand, in the specific case of the universal anti-twist, the supersymmetric action/entropy function given by (4.48) has already been related to the localization result in minimal gauged supergravity [61] in reference [53]. However, this relation is for now somewhat indirect, as in the present paper we have shown by direct computation that the supersymmetric action for this class of AdS_2 GK geometries is equal to the on-shell action of a corresponding class of supersymmetric accelerating black holes in AdS_4, for which the former arise as the near horizon limits of the latter. It is then more specifically the holographically renormalized on-shell action of these black holes that the localization formula of [61] applies to, as described in [53]. This intriguing relation between these two approaches to black hole entropy functions is discussed further in [51].

More generally one might imagine that conjectured gravitational block formulas for other classes of solutions, in different theories and in different dimensions, could be derived using a similar approach to this paper, and/or by an appropriate fixed point theorem. For example, the various black holes in AdS_4 and AdS_5 discussed in [46], the class of branes wrapped on spindles and higher-dimensional orbifolds in [30,31,34–37], and the general higher derivative gravitational block formula conjectured in [52] for black holes in AdS_4 and asymptotically flat space. We note that the latter formula was inspired by the localization formula in minimal gauged supergravity in [61]. These generalizations might also include adding various internal fluxes to the GK geometries focused on in the present paper, for example the solutions constructed recently in [33], as well the more general extensions of GK geometry considered in [62,63]. Another possibility would be to consider including higher derivative terms in the effective action. The geometric structures imposed by supersymmetry in different theories and different dimensions are often quite distinct, but the universality of this class of gravitational block/black hole entropy functions suggest there is a more universal approach to deriving these formulas, indeed perhaps utilizing an appropriate fixed point formula that is largely insensitive to the detailed geometry. It will be very interesting to pursue this in future work.

For the $\mathrm{AdS}_3 \times Y_7$ class of GK geometries in type IIB string theory discussed in Sect. 8, we have obtained a very general proof that our off-shell supersymmetric action agrees with the off-shell trial c-function in field theory, thus obtaining a very general exact result in AdS/CFT. Here Y_7 is an arbitrary fibration of a toric Sasaki–Einstein five-manifold X_5 over a spindle Σ. However, a key assumption on the gravity side is that the corresponding extremal AdS_3 supergravity solutions actually exist – provided they do, the extremal value of the supersymmetric action computes the central charge, but there might be obstructions to existence. Ultimately one needs an existence result for the corresponding PDE – see, for example, the discussion in [9,10]. However, it is natural to conjecture that this existence is guaranteed if and only if the R-charges $R_a^{\pm}$ satisfy the positivity conditions $R_a^+ > 0$, $\sigma R_a^- > 0$ (in our conventions). That this is a necessary condition can be seen from the first equality in 7.43: when $n = 3$ these R-charges are proportional to the Kähler class integrated over a basis of toric submanifolds in the fibres over the two poles of the spindle, and these should all be positive. On the field theory side this question is related to whether the $d = 4$, $\mathcal{N} = 1$ SCFT, dual to the $\mathrm{AdS}_5 \times X_5$ solution, indeed flows to a $d = 2$, (0, 2) SCFT in the IR when the theory is wrapped on a spindle.

Acknowledgements We thank Chris Couzens for discussions. This work was supported in part by STFC Grants ST/T000791/1 and ST/T000864/1. JPG is supported as a Visiting Fellow at the Perimeter Institute.

A. Delzant Construction

To see that the toric data (7.5) indeed describes a fibration (7.1), we can utilize the Delzant construction for toric cones (as summarized, for example, in [55]).

We first recall this construction for $C(X_{2n-1})$ and, for simplicity, we assume that X_{2n-1} is simply-connected. There is a linear map

$$\mathcal{A}^{(n)} : \mathbb{R}^d \to \mathbb{R}^n \,, \qquad \mathcal{A}^{(n)}(e_a) = v_a \in \mathbb{R}^n \,, \tag{A.1}$$

where $\{e_a\}$ denotes the standard orthonormal basis for $\mathbb{R}^d$, with components $e_{ab} = \delta_{ab}$, and $v_a = (v_{ai})_{i=1}^n$ are the toric data for X_{2n-1}, where in what follows it will be convenient to suppress the vector index i on v_a (which are the components of v_a in a basis for the Lie algebra of $U(1)^n$). Since $\mathcal{A}^{(n)}$ maps $\mathbb{Z}^d$ to $\mathbb{Z}^n$, there is an induced map of tori $U(1)^d = \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{R}^n/\operatorname{span}_{\mathbb{Z}}\{v_a\}$, with a kernel $U(1)^{d-n}$.[37] The latter is generated by an integer $d \times (d-n)$ matrix q_I^a, $I = 1, \ldots, d-n$, satisfying

$$\sum_{a=1}^{d} q_I^a \, v_{ai} = 0, \qquad i = 1, \ldots, n, \tag{A.2}$$

which specifies the embedding $U(1)^{d-n} \subset U(1)^d$. The toric $U(1)^n$ action on $C(X_{2n-1})$ is then via the quotient $U(1)^n = U(1)^d/U(1)^{d-n}$. In physics language, the above construction describes a gauged linear sigma model (GLSM) with d complex fields and $U(1)^{d-n}$ charges specified by q_I^a, with $C(X_{2n-1})$ being the vacuum moduli space (see e.g. [55]).

For a toric $C(Y_{2n+1})$ there is then a similar construction, with linear map

$$\mathcal{A}^{(n+1)} : \mathbb{R}^D \to \mathbb{R}^{n+1} \,, \qquad \mathcal{A}^{(n+1)}(e_A) = v_A \in \mathbb{R}^{n+1} \,, \tag{A.3}$$

[37] When X_{2n-1} is not simply-connected there is also a finite group as part of this kernel.

with standard orthonormal basis $\{e_A\}$ of $\mathbb{R}^D$, with components $e_{AB} = \delta_{AB}$, toric data $v_A = (v_{A\mu})_{\mu=0}^n$ for $C(Y_{2n+1})$, and with a kernel specified by an integer $D \times (D-n-1)$ matrix $Q_{\hat{I}}^A$, $\hat{I} = 0, 1, \ldots, d-n$, satisfying

$$\sum_{A=1}^{D} Q_{\hat{I}}^A v_{A\mu} = 0, \qquad \mu = 0, 1, \ldots, n. \tag{A.4}$$

From (7.5), which recall relates the toric data for $C(Y_{2n+1})$ in terms of that for $C(X_{2n-1})$, the spindle data $m_\pm$, and the twisting variables p_i, $a_\pm$, we can immediately identify part of this kernel:

$$Q_I = (0, 0, q_I^1, \ldots, q_I^d)\,, \qquad I = 1, \ldots, d. \tag{A.5}$$

Note here that this satisfies (A.4) by virtue of (A.2), where $v_a = (1, \vec{w}_a)$. Since $\hat{I}$ takes $D - n - 1 = d + 1 - n$ values and $I = 1, \ldots, d-n$, there is one more kernel vector to identify, which we label as the $\hat{I} = 0$ component of $Q_{\hat{I}}^A$. We write this charge vector as

$$Q_0 = (m_-, \sigma m_+, q_0^1, \ldots, q_0^d)\,. \tag{A.6}$$

Notice here that the first two entries are fixed by the zero'th components of the vectors in (7.5). From (7.5) and (A.4) we see that Q_0 is a kernel vector provided that the vector q_0^a satisfies

$$\sum_{a=1}^{d} q_0^a v_{ai} \;=\; p_i\,, \qquad i = 1, \ldots, n, \tag{A.7}$$

where we recall from (7.2) that $p_1 = -(m_- + \sigma m_+)$, with $\sigma = +1$ being the twist, and $\sigma = -1$ being the anti-twist. In fact the above equations are precisely describing the fibration (7.1) in a two step process, as we now describe.

To begin, recall that to fibre $C(X_{2n-1})$ over a spindle $\Sigma = \mathbb{WCP}^1_{[m_-,m_+]}$, we may first fibre $\mathbb{C}^d$ over Σ. To do so we must first lift the $U(1)^n$ action on $C(X_{2n-1})$ to $\mathbb{C}^d$, which means specifying n vectors $\alpha^i = (\alpha_a^i)_{a=1}^d \in \mathbb{Z}^d$, $i = 1, \ldots, n$ satisfying

$$\mathcal{A}^{(n)}(\alpha^i) = e_i \in \mathbb{Z}^n\,, \tag{A.8}$$

where $\{e_i\}$ is the standard orthonormal basis for $\mathbb{R}^n$. In components this condition reads

$$\sum_{a=1}^{d} v_{ai}\, \alpha_a^j = \delta_i^j\,. \tag{A.9}$$

Of course the choice of each $\alpha^i \in \mathbb{Z}^d$, $i = 1, \ldots, n$, is unique only up to the kernel of $\mathcal{A}^{(n)}$, generated by q_I^a. Geometrically, this is because $C(X_{2n-1})$ is precisely a Kähler quotient of $\mathbb{C}^d$ via the torus $U(1)^{d-n}$ generated by this kernel. Put more simply, the charge of the ath coordinate z^a of $\mathbb{C}^d$ under the ith $U(1) \subset U(1)^n$ is precisely α_a^i.

With this in hand, we now re-examine (A.7) and the kernel $Q_{\hat{I}}^A$. We first consider the twist case with $\sigma = +1$. The GLSM for $C(Y_{2n+1})$ begins with $\mathbb{C}^D = \mathbb{C}^{d+2}$. Taking the quotient by $U(1)$ with charge vector Q_0 given by (A.6) then describes the total space of a $\mathbb{C}^d$ bundle over $\Sigma = \mathbb{WCP}^1_{[m_-,m_+]}$, where the charge vector q_0^a describes the twisting. That is, as a first step we construct the $(d+1)$-dimensional space

$$Z \equiv \mathcal{O}(q_0^a)_\Sigma \times_{U(1)^d} \mathbb{C}^d \; . \tag{A.10}$$

Here the notation means we twist the ath coordinate of $\mathbb{C}^d$ by the line bundle $O(q_0^a)_\Sigma$. The zeroth component of the condition (A.7) then simply says that Z is a Calabi-Yau $(d+1)$-fold. Notice here that $q_0^a < 0$ is necessary for the convex toric geometry description we have given to be applicable, although the space Z defined by (A.10) exists as a complex manifold with zero first Chern class irrespective of the signs of the charges.

In the anti-twist case, with $\sigma = -1$, the charge of the second coordinate on $\mathbb{C}^D = \mathbb{C}^{d+2}$ under Q_0 in (A.6) is negative. Taking the Kähler quotient by this $U(1)$ then does not result in a space with the topology given in (A.10). Formally we may complex conjugate the second coordinate on $\mathbb{C}^D$, to see that topologically Z given by (A.10) is a partial resolution of the conical geometry in the anti-twist case. However, the space Z is then no longer Calabi-Yau, since $\sum_{a=1}^d q_0^a$ is not equal to $-(m_- + m_+)$ when $\sigma = -1$.[38] This lack of any clear relation to a partially resolved Calabi-Yau geometry, with a blown up copy of Σ on which the branes may wrap, is one reason why the anti-twist solutions are more difficult to interpret physically.

Returning to the twist case with $\sigma = +1$, at this stage we have quotiented by $U(1)$ out of $U(1)^{d+1-n}$ to obtain a Calabi-Yau $(d+1)$-fold Z, which is a $\mathbb{C}^d$ fibration over the spindle Σ. Quotienting by the remaining $U(1)^{d-n} = U(1)^{d+1-n}/U(1)$ then precisely turns the fibre $\mathbb{C}^d$ into $C(X_{2n-1})$ – this is clear from (A.5), which only acts on the $\mathbb{C}^d$ fibre direction of Z. Since $\mathbb{C}^d$ is fibred non-trivially over the spindle, the $C(X_{2n-1})$ will also be fibred, and equation (A.7) is telling us how. Specifically, this condition is solved by writing

$$q_0^a = \sum_{i=1}^n p_i \, \alpha_a^i \; . \tag{A.11}$$

The fact this solves (A.7) follows from (A.9):

$$\sum_{a=1}^d q_0^a \, v_{ai} = \sum_{a=1}^d \sum_{j=1}^n p_j \, \alpha_a^j \, v_{ai} = p_i \; , \qquad i = 1, \ldots, n. \tag{A.12}$$

Geometrically, recall that α_a^i gives the charge of the ath coordinate on $\mathbb{C}^d$ under the ith $U(1)$ in $U(1)^n$. Thus $\vec{p} = (p_2, \ldots, p_n)$ precisely has the interpretation of twisting $\mathbb{C}^d$ by the line bundle $O(\vec{p})_\Sigma$, so that we may also write

$$Z = O(\vec{p})_\Sigma \times_{U(1)^n} \mathbb{C}^d \; , \tag{A.13}$$

where q_0 in (A.10) is specified by (A.11). The vacuum moduli space with the full quotient is then

$$O(\vec{p})_\Sigma \times_{U(1)^n} C(X_{2n-1}) \; . \tag{A.14}$$

In the anti-twist case this last quotient of $\mathbb{C}^d$ by $U(1)^{d-n}$ to obtain the Calabi-Yau cone fibres $C(X_{2n-1})$ is still valid, but the space in (A.14), while being a partial resolution of $C(Y_{2n+1})$, is no longer Calabi-Yau.

This concludes our proof that the toric data (7.5) describes the fibration (3.1).

[38] This fact was discussed in some detail in [27] for various explicit supergravity solutions where $d = n$ and $C(X_{2n-1}) = \mathbb{C}^n$.

B. Matching GK Geometry for $\mathrm{AdS}_3 \times Y_7$ with Field Theory

Here we consider GK geometry for toric Y_7 comprised of toric X_5 fibred over a spindle. We show that the off-shell central charge $\mathscr{Z}$ for the $\mathrm{AdS}_3 \times Y_7$ solution matches with the off-shell central charge $c_{\text{trial}}|_{\text{baryonic}}$ in the field theory, where the extremization over the baryonic directions has been carried out. The key results of Sects. 7.3 and 7.4, are central to the proof and we also use some results of [11].

It will be convenient to utilise different choices of gauge for the Kähler class parameters λ^A. Recall that the master volume for toric Y_7, $\mathcal{V}_7$, is invariant under the gauge transformations given in (6.25):

$$\lambda_A \;\to\; \lambda_A + \sum_{\mu=0}^{3} \gamma^\mu (v_{A\mu} b_1 - b_\mu). \tag{B.1}$$

Although γ^μ has four components only three of them yield a non-trivial transformation since the dependence on γ^1 drops out. Any function which is homogeneous in λ_A is invariant under (B.1) and, in particular, $\mathcal{V}_7$, $\mathcal{V}_5^\pm$, $R_a^\pm$, and $\mathscr{Z}$ are all gauge-invariant quantities.

We will use three different gauge choices. We first note that the transformation of $\lambda_a^{(+)}$ only depends on γ^2, γ^3. We can therefore choose γ^2, γ^3 to set two of the $\lambda_a^{(+)}$ to vanish, for example

$$\textit{Plus gauge:} \qquad \lambda_1^{(+)} = \lambda_2^{(+)} = 0. \tag{B.2}$$

Similarly, we can instead choose γ^2, γ^3 to set two of the $\lambda_a^{(-)}$ to vanish, for example

$$\textit{Minus gauge:} \qquad \lambda_1^{(-)} = \lambda_2^{(-)} = 0. \tag{B.3}$$

Finally, noting that the transformation of $\lambda^\pm$ depends on $\gamma^0, \gamma^2, \gamma^3$ we can choose γ^2, γ^3 (say) to obtain

$$\textit{Symmetric gauge:} \qquad \lambda_+ = \lambda_- = 0 \quad \Rightarrow \quad \lambda_a^{(\pm)} = \lambda_a. \tag{B.4}$$

Note that these three gauge choices leave a residual one-parameter gauge invariance parametrised by γ^0, which we will not need to fix.

B.1. $\mathscr{Z}$ as a function of $R_a^\pm$. The master volume for toric X_5, $\mathcal{V}_5$, is homogeneous of degree 2 in the λ_a and hence we can write

$$\mathcal{V}_5^\pm = \frac{1}{2}\sum_{a=1}^{d} \frac{\partial \mathcal{V}_5^\pm}{\partial \lambda_a} \lambda_a^{(\pm)} = -\frac{N\nu_3}{4}\sum_{a=1}^{d} R_a^\pm \lambda_a^{(\pm)}, \tag{B.5}$$

where we used the expressions for the shifted R-charges, $R_a^\pm$, given in (7.46). We now show that it is possible to express $\lambda_a^{(\pm)}$ in (B.5) in terms of $R_a^\pm$ and hence obtain an expression for the supersymmetric action S_{SUSY}, and hence the off-shell central charge $\mathscr{Z}$, in terms of $R_a^\pm$ using (7.31) and (2.16). We will use different gauges to treat the $\pm$ cases, but the final results are gauge invariant.

It is helpful to recall the explicit formula for $\mathcal{V}_5 = \mathcal{V}_5(\vec{b};\{\lambda_a\})$ given in (1.3) of [10]:

$$\mathcal{V}_5 = \frac{(2\pi)^3}{2}\sum_{a=1}^{d}\lambda_a \frac{\lambda_{a-1}(\vec{v}_a,\vec{v}_{a+1},\vec{b}) - \lambda_a(\vec{v}_{a-1},\vec{v}_{a+1},\vec{b}) + \lambda_{a+1}(\vec{v}_{a-1},\vec{v}_a,\vec{b})}{(\vec{v}_{a-1},\vec{v}_a,\vec{b})(\vec{v}_a,\vec{v}_{a+1},\vec{b})}\,. \tag{B.6}$$

We now define

$$J_{ab} \equiv \frac{\partial \mathcal{V}_5}{\partial\lambda_a\partial\lambda_b}, \qquad J^{\pm}_{ab} \equiv J_{ab}\big|_{\vec{b}=\vec{b}^{(\pm)}}, \tag{B.7}$$

with J_{ab} independent of λ_a and $\vec{b}=(b_1,b_2,b_3)$. The only non-vanishing components of the matrix J_{ab} are J_{aa} and $J_{a,a+1}=J_{a+1,a}$ given by

$$J_{aa} = -(2\pi)^3\frac{(\vec{v}_{a-1},\vec{v}_{a+1},\vec{b})}{(\vec{v}_{a-1},\vec{v}_a,\vec{b})(\vec{v}_a,\vec{v}_{a+1},\vec{b})}, \qquad J_{a,a+1} = (2\pi)^3\frac{1}{(\vec{v}_a,\vec{v}_{a+1},\vec{b})}. \tag{B.8}$$

The homogeneity of the master volume $\mathcal{V}_5$ implies that we can recast $R^{\pm}_a$ in (7.46) as

$$\begin{aligned} R^{\pm}_a &= -\frac{2}{N\nu_3}\sum_{b=1}^{d} J^{\pm}_{ab}\,\lambda^{(\pm)}_b, \\ &= -\frac{16\pi^3}{N\nu_3}\left[\frac{\lambda_{a-1}}{(\vec{v}_{a-1},\vec{v}_a,\vec{b}^{(\pm)})} - \frac{(\vec{v}_{a-1},\vec{v}_{a+1},\vec{b}^{(\pm)})\,\lambda_a}{(\vec{v}_{a-1},\vec{v}_a,\vec{b}^{(\pm)})(\vec{v}_a,\vec{v}_{a+1},\vec{b}^{(\pm)})} + \frac{\lambda_{a+1}}{(\vec{v}_a,\vec{v}_{a+1},\vec{b}^{(\pm)})}\right]. \end{aligned} \tag{B.9}$$

We now focus on the + case and choose the "plus gaug" (B.2), $\lambda^{(+)}_1=\lambda^{(+)}_2=0$. In this gauge we have

$$\begin{aligned} R^+_1 &= -\frac{2}{N\nu_3}\left(J^+_{1d}\,\lambda^{(+)}_d\right), \\ R^+_2 &= -\frac{2}{N\nu_3}\left(J^+_{23}\,\lambda^{(+)}_3\right), \\ R^+_3 &= -\frac{2}{N\nu_3}\left(J^+_{33}\,\lambda^{(+)}_3 + J^+_{34}\,\lambda^{(+)}_4\right), \\ R^+_4 &= -\frac{2}{N\nu_3}\left(J^+_{43}\,\lambda^{(+)}_3 + J^+_{44}\,\lambda^{(+)}_4 + J^+_{45}\,\lambda^{(+)}_5\right), \\ &\ldots, \end{aligned} \tag{B.10}$$

and these equations can be solved recursively for the $\lambda^{(+)}_a$ to get (as in [11])

$$\lambda^{(+)}_a = -\frac{N\nu_3}{16\pi^3}\sum_{b=2}^{a}(\vec{v}_b,\vec{v}_a,\vec{b}^{(+)})\,R^+_b, \qquad a=3,\ldots,d. \tag{B.11}$$

The identity for R^+_a given in (7.50) ensures that the expression for $\lambda^{(+)}_d$ we get from the first equation in (B.10) is consistent with (B.11). If we now substitute (B.11) back into (B.5) we get

$$\mathcal{V}^+_5 = \frac{N^2\nu_3^2}{64\pi^3}\sum_{a=3}^{d}\sum_{b=2}^{a}(\vec{v}_b,\vec{v}_a,\vec{b}^{(+)})R^+_a\,R^+_b, \tag{B.12}$$

and then after again using the identity (7.50) we can write

$$\begin{aligned} \mathcal{V}_5^+ &= \frac{N^2 \nu_3^2 b_1}{128\pi^3} \sum_{a=3}^{d} \sum_{b=2}^{a} \sum_{c=1}^{d} (\vec{v}_b, \vec{v}_a, \vec{v}_c) R_a^+ R_b^+ R_c^+ \\ &= \frac{N^2 \nu_3^2 b_1}{128\pi^3} \sum_{a<b<c} (\vec{v}_a, \vec{v}_b, \vec{v}_c)\, R_a^+ R_b^+ R_c^+ . \end{aligned} \tag{B.13}$$

This expression does not depend on the gauge choice that we used and hence it holds in all gauges.

Analogously, for the $-$ case we can utilise the "minus gauge" (B.3), and obtain an equivalent expression for $\mathcal{V}_5^-$ with R_a^+ replaced with R_a^-. Putting these results together, we can write the supersymmetric action (7.31) and hence the trial central charge $\mathscr{Z}$ defined in (2.16) in the form

$$\mathscr{Z} = \frac{3N^2 b_1^2}{4b_0} \sum_{a<b<c} (\vec{v}_a, \vec{v}_b, \vec{v}_c) \left(R_a^+ R_b^+ R_c^+ - R_a^- R_b^- R_c^- \right) . \tag{B.14}$$

In this expression we note that $R_a^\pm$ can be viewed as functions of b_μ as well as the five-form fluxes M_A (after eliminating the λ's using the second two lines of (6.26) or, equivalently, (7.35) and (7.36)).

We now observe that after setting $b_1 = 2$, which is required in the GK extremization procedure, the expression for $\mathscr{Z}$ in (B.14) has precisely the same form as the field theory expression for the central charge (8.9) using the dictionary between field theory and geometric quantities discussed in Sect. 8.2:

$$\begin{aligned} \varepsilon &= b_0\,, \\ \Delta_a^+\big|_{\text{baryonic}} = R_a^+\,, &\qquad \Delta_a^-\big|_{\text{baryonic}} = R_a^-\,. \end{aligned} \tag{B.15}$$

We also identify the rank of the field theory gauge group, N, with the flux $N \equiv m_+ M_+ = \sigma m_- M_-$ and the background magnetic fluxes for the SCFT $\mathfrak{p}_a$ with the five-from fluxes via $\mathfrak{p}_a \equiv \frac{M_a}{N}$ as in (8.14)(8.16).

We now argue that the identification (B.15) is possible. On the GK geometry side we have three independent variables, b_0, b_2, b_3. We also have three independent variables on the field theory side, the $d+1$ variables ε, Δ_a satisfying the $d-2$ constraints

$$\sum_{a=1}^{d} \Delta_a = 2, \qquad \sum_{a=1}^{d} q_I^a \frac{\partial c_{\text{trial}}}{\partial \Delta_a} = 0\,, \qquad I = 1, \ldots, d-3\,. \tag{B.16}$$

Interestingly, we will show below (see (B.22)) that the latter constraints are actually $d-2$ linear constraints in terms of the variables Δ_a, ε. We will also show in Sect. B.3 that the $R_a^\pm$, after eliminating the λ_a variables, are linear functions of b_2, b_3 and b_0. Thus, (B.15) is $2d+1$ linear equations for 3 variables and hence the system seems to be overdetermined. However, only 3 of the equations are independent, and the other $2d-2$ can be obtained as linear combinations of these.

To see this, we introduce the notation

$$E_a^\pm \equiv \Delta_a^\pm\big|_{\text{baryonic}} - R_a^\pm , \tag{B.17}$$

so that the $2d$ linear equations in the second line of (B.15) are simply $E_a^{\pm} = 0$. Then, combining the identities (7.48) and the definition (8.4), we conclude

$$E_a^+ - E_a^- = 0, \tag{B.18}$$

which eliminates d equations, say those encoded by E_a^-, from the original system. Another equation can be eliminated because

$$\sum_{a=1}^{d} E_a^+ = 0, \tag{B.19}$$

where we used (7.47) and (8.8). Finally, the last set of $d-3$ linear constraints comes from the baryonic extremization condition (B.16).

To see this we will prove in the next subsection that

$$\sum_{a=1}^{d} q_I^a \left.\frac{\partial c_{\text{trial}}}{\partial \Delta_a}\right|_{\Delta_a^{\pm}=R_a^{\pm}(\vec{B},M_A)} = 0, \qquad I = 1, \ldots, d-3. \tag{B.20}$$

Furthermore, to do so we will also prove that the baryonic extremization condition (B.16) can be re-expressed in the equivalent form

$$\sum_{b=1}^{d}\sum_{a=1}^{b}\sum_{c=1}^{d} q_I^a(\vec{v}_a, \vec{v}_b, \vec{v}_c)\left(\Delta_b^+\Delta_c^+ - \Delta_b^-\Delta_c^-\right) = 0. \tag{B.21}$$

Writing (B.21) purely in terms of Δ_a^+, say, yields

$$\sum_{b=1}^{d}\sum_{a=1}^{b}\sum_{c=1}^{d} q_I^a(\vec{v}_a, \vec{v}_b, \vec{v}_c)\left(\mathfrak{p}_b\Delta_c^+ + \mathfrak{p}_c\Delta_b^+ + \varepsilon\,\mathfrak{p}_b\mathfrak{p}_c\right) = 0, \tag{B.22}$$

from which we see that the baryonic constraints are linear in the Δ_a and ε as noted above. Hence, given (B.20), the baryonic extremization condition implies that the system of linear equations E_a^+ is, indeed, also subject to $d-3$ linear constraints

$$\sum_{b=1}^{d}\sum_{a=1}^{b}\sum_{c=1}^{d} q_I^a(\vec{v}_a, \vec{v}_b, \vec{v}_c)\left(\mathfrak{p}_b E_c^+ + \mathfrak{p}_c E_b^+\right) = 0. \tag{B.23}$$

Thus, the linear identifications (B.15) are actually not overdetermined. In practice we can take the identification to be, for example,

$$\varepsilon = b_0\,, \qquad \Delta_1^+\big|_{\text{baryonic}} = R_1^+, \qquad \Delta_2^+\big|_{\text{baryonic}} = R_2^+. \tag{B.24}$$

B.2. Proof of baryonic extremization conditions. We now prove (B.20) and (B.21). We begin with the latter and show that we can rewrite the field theory baryon mixing condition

$$\sum_{a=1}^{d} q_I^a \frac{\partial c_{\text{trial}}}{\partial \Delta_a} = 0\,, \qquad I = 1, \ldots, d-3\,, \tag{B.25}$$

in the form

$$\sum_{b=1}^{d} \sum_{a=1}^{b} \sum_{c=1}^{d} q_I^a (\vec{v}_a, \vec{v}_b, \vec{v}_c) \left(\Delta_b^+ \Delta_c^+ - \Delta_b^- \Delta_c^-\right) = 0. \tag{B.26}$$

To see this, we begin by using the expression for c_{trial} in (8.9) to write

$$\sum_{a=1}^{d} q_I^a \frac{\partial c_{\text{trial}}}{\partial \Delta_a} = \frac{3}{\varepsilon} \sum_{e<b<c} (\vec{v}_e, \vec{v}_b, \vec{v}_c) \\ \left[q_I^e \Delta_b^+ \Delta_c^+ + q_I^b \Delta_e^+ \Delta_c^+ + q_I^c \Delta_b^+ \Delta_e^+ - (+ \leftrightarrow -)\right]. \tag{B.27}$$

We can rewrite the triple sum in the following way

$$\sum_{e<b<c} = \sum_{e=1}^{d} \sum_{b=e}^{d} \sum_{c=b}^{d} = \sum_{b=1}^{d} \sum_{e=1}^{b} \sum_{c=b}^{d}, \tag{B.28}$$

where due to the asymmetry of the determinant the sum gets no contribution if we set any two of e, b, c equal. Using (A.2), we can rewrite the third term in (B.27) as

$$-\sum_{b=1}^{d} \sum_{e=1}^{b} \sum_{c=1}^{b} (\vec{v}_e, \vec{v}_b, \vec{v}_c) q_I^c \Delta_b^+ \Delta_e^+ = \sum_{b=1}^{d} \sum_{e=1}^{b} \sum_{c=1}^{b} (\vec{v}_e, \vec{v}_b, \vec{v}_c) q_I^e \Delta_b^+ \Delta_c^+, \tag{B.29}$$

where we swapped the indices e and c to get the second expression. Hence, summing the first and the third term of (B.27) yields

$$\sum_{b=1}^{d} \sum_{e=1}^{b} \sum_{c=1}^{d} (\vec{v}_e, \vec{v}_b, \vec{v}_c) q_I^e \Delta_b^+ \Delta_c^+. \tag{B.30}$$

For the second term in (B.27), we can exploit the skew-symmetry of the summand to extend the range of c and then massage it via some relabelling and again using (A.2) to find

$$\begin{aligned}
&\sum_{b=1}^{d} \sum_{e=1}^{b} \sum_{c=b}^{d} (\vec{v}_e, \vec{v}_b, \vec{v}_c) q_I^b \Delta_e^+ \Delta_c^+ \\
&\quad = \sum_{b=1}^{d} \sum_{e=1}^{b} \sum_{c=1}^{d} (\vec{v}_e, \vec{v}_b, \vec{v}_c) q_I^b \Delta_e^+ \Delta_c^+ \\
&\quad = \sum_{e=1}^{d} \sum_{b=e}^{d} \sum_{c=1}^{d} (\vec{v}_e, \vec{v}_b, \vec{v}_c) q_I^b \Delta_e^+ \Delta_c^+ = -\sum_{e=1}^{d} \sum_{b=1}^{e} \sum_{c=1}^{d} (\vec{v}_e, \vec{v}_b, \vec{v}_c) q_I^b \Delta_e^+ \Delta_c^+ \\
&\quad = \sum_{b=1}^{d} \sum_{e=1}^{b} \sum_{c=1}^{d} (\vec{v}_e, \vec{v}_b, \vec{v}_c) q_I^e \Delta_b^+ \Delta_c^+.
\end{aligned} \tag{B.31}$$

Hence we find that the second term in (B.27) is equal to the sum of the first and the third term. We can obtain a similar result for the terms involving Δ_a^- and hence we have proven (B.26).

We now prove (B.20):

$$\sum_{a=1}^{d} q_I^a \left.\frac{\partial c_{\text{trial}}}{\partial \Delta_a}\right|_{\Delta_a^\pm = R_a^\pm(\vec{B}, M_A)} = 0, \qquad I = 1, \ldots, d-3. \tag{B.32}$$

Using what we just proved in (B.26), as well as the conditions satisfied by $R_c^\pm$ given in (7.50) to carry out the sum over c, this condition can be written equivalently as

$$W \equiv \sum_{a=1}^{d}\sum_{b=a}^{d} q_I^a \left[(\vec{v}_a, \vec{v}_b, \vec{b}^{(+)}) R_b^+ - (\vec{v}_a, \vec{v}_b, \vec{b}^{(-)}) R_b^-\right] = 0, \tag{B.33}$$

where we have also swapped the sums over a and b. To proceed we exploit the determinant identity $(\vec{a}, \vec{b}, \vec{c}) = \vec{a} \cdot (\vec{b} \wedge \vec{c})$ to write W in the form

$$W = \sum_{a=1}^{d} q_I^a\, \vec{v}_a \cdot \vec{u}_a, \qquad \vec{u}_a \equiv \sum_{b=a}^{d} \left[(\vec{v}_b \wedge \vec{b}^{(+)}) R_b^+ - (\vec{v}_b \wedge \vec{b}^{(-)}) R_b^-\right]. \tag{B.34}$$

Note that the expression for W is unchanged under the following shift of the vectors $\vec{u}_a$:

$$\vec{u}_a \to \vec{u}_a' = \vec{u}_a + \vec{v}_a \wedge \vec{z}_a, \tag{B.35}$$

for an arbitrary vector $\vec{z}_a$. The strategy is to show that we can choose $\vec{z}_a$ in such a way that all of the $\vec{u}_a'$ are actually the same vector $\vec{u}$, since if this is the case we then have

$$W = \sum_{a=1}^{d} q_I^a\, \vec{v}_a \cdot \vec{u}_a' = \sum_{a=1}^{d} q_I^a\, \vec{v}_a \cdot \vec{u} = 0, \tag{B.36}$$

where in the last step we again used (A.2).

We now show that such a choice of $\vec{z}_a$ is possible. We first compute the difference between two adjacent $\vec{u}_a'$ vectors:

$$\vec{u}_{a+1}' - \vec{u}_a' = -(\vec{v}_a \wedge \vec{b}^{(+)}) R_a^+ + (\vec{v}_a \wedge \vec{b}^{(-)}) R_a^- + \vec{v}_{a+1} \wedge \vec{z}_{a+1} - \vec{v}_a \wedge \vec{z}_a. \tag{B.37}$$

We thus want to choose the variables $\vec{z}_a$, $a = 1, \ldots, d$ to solve the system of equations

$$-\vec{v}_a \wedge \left[\vec{b}^{(+)} R_a^+ - \vec{b}^{(-)} R_a^- + \vec{z}_a\right] + \vec{v}_{a+1} \wedge \vec{z}_{a+1} = 0, \qquad a = 1, \ldots, d, \tag{B.38}$$

where, as usual, the indices are identified cyclically i.e. $d + 1 = 1$. Now since $\vec{v}_a$ and $\vec{v}_{a+1}$ are not parallel vectors the only way to solve this is if each of the two terms are proportional to $\vec{v}_a \wedge \vec{v}_{a+1}$ which implies that we can write

$$\begin{aligned} \vec{b}^{(+)} R_a^+ - \vec{b}^{(-)} R_a^- + \vec{z}_a &= k_a \vec{v}_{a+1} + k_a' \vec{v}_a, \\ \vec{z}_{a+1} &= -k_a \vec{v}_a + k_{a+1}'' \vec{v}_{a+1}, \end{aligned} \tag{B.39}$$

for some real constants k_a, k'_a, and k''_a, $a = 1, \ldots, d$. This is equivalent to

$$\begin{aligned} \vec{z}_a &= k_a \vec{v}_{a+1} + k'_a \vec{v}_a - \vec{b}^{(+)} R_a^+ + \vec{b}^{(-)} R_a^- , \\ \vec{z}_a &= -k_{a-1} \vec{v}_{a-1} + k''_a \vec{v}_a . \end{aligned} \tag{B.40}$$

Now recall from (B.35) that $\vec{z}_a$ can be shifted by a vector proportional to $\vec{v}_a$ without changing $\vec{u}'_a$. Hence, we can reabsorb, for example, the term $k''_a \vec{v}_a$ inside $\vec{y}_a$ with an associated redefinition of k'_a. Thus, consistency of the two expressions for each $\vec{z}_a$ in (B.40) amounts to solving the following system of equations for k_a and k'_a

$$k_a \vec{v}_{a+1} + k'_a \vec{v}_a + k_{a-1} \vec{v}_{a-1} = \vec{b}^{(+)} R_a^+ - \vec{b}^{(-)} R_a^- , \qquad a = 1, \ldots, d. \tag{B.41}$$

This is $3(d-1)$ independent equations for $2d$ variables k_a, k'_a. A solution can exist provided that a set of $d-3$ constraints are satisfied. Let us introduce the notation

$$\vec{b}^{(+)} R_a^+ - \vec{b}^{(-)} R_a^- \equiv (\beta_a^1,\, \vec{\beta}_a), \tag{B.42}$$

where $\vec{\beta}_a$ is a two dimensional vector. Then the first component of (B.41) can be used to solve for k'_a as follows

$$k'_a = \beta_a^1 - k_a - k_{a-1}. \tag{B.43}$$

Substituting this back into (B.41) we get the set of two-dimensional equations

$$k_a (\vec{w}_{a+1} - \vec{w}_a) - k_{a-1} (\vec{w}_a - \vec{w}_{a-1}) = \vec{\beta}_a - \beta_a^1 \vec{w}_a . \tag{B.44}$$

Now if we project this two-dimensional equation onto vectors orthogonal to $(\vec{w}_{a+1} - \vec{w}_a)$ and $(\vec{w}_a - \vec{w}_{a-1})$, we get two simple linear equations for k_{a+1} and k_a respectively.[39] We therefore deduce

$$\begin{aligned} & k_a \left[(\vec{w}_{a+1}, \vec{w}_a) - (\vec{w}_{a+1}, \vec{w}_{a-1}) + (\vec{w}_a, \vec{w}_{a-1}) \right] \\ &\quad = (\vec{\beta}_a, \vec{w}_a) - (\vec{\beta}_a, \vec{w}_{a-1}) + \beta_a^1 (\vec{w}_a, \vec{w}_{a-1}), \\ & - k_{a-1} \left[(\vec{w}_a, \vec{w}_{a+1}) - (\vec{w}_{a-1}, \vec{w}_{a+1}) + (\vec{w}_{a-1}, \vec{w}_a) \right] \\ &\quad = (\vec{\beta}_a, \vec{w}_{a+1}) - (\vec{\beta}_a, \vec{w}_a) - \beta_a^1 (\vec{w}_a, \vec{w}_{a+1}), \end{aligned} \tag{B.45}$$

where here $(\vec{a}, \vec{c})$ denotes the determinant of the 2×2 matrix built with the vectors $\vec{a}$ and $\vec{c}$. So, we obtain two different expressions for k_a and the constraint is that they must be equal

$$\begin{aligned} & \frac{(\vec{\beta}_a, \vec{w}_a) - (\vec{\beta}_a, \vec{w}_{a-1}) + \beta_a^1 (\vec{w}_a, \vec{w}_{a-1})}{(\vec{w}_{a+1}, \vec{w}_a) - (\vec{w}_{a+1}, \vec{w}_{a-1}) + (\vec{w}_a, \vec{w}_{a-1})} \\ &\quad = - \frac{(\vec{\beta}_{a+1}, \vec{w}_{a+2}) - (\vec{\beta}_{a+1}, \vec{w}_{a+1}) - \beta_{a+1}^1 (\vec{w}_{a+1}, \vec{w}_{a+2})}{(\vec{w}_{a+1}, \vec{w}_{a+2}) - (\vec{w}_a, \vec{w}_{a+2}) + (\vec{w}_a, \vec{w}_{a+1})} . \end{aligned} \tag{B.46}$$

[39] Note that we can do this since by construction $(\vec{w}_{a+1} - \vec{w}_a)$ and $(\vec{w}_a - \vec{w}_{a-1})$ are linearly independent.

We are left to show that β_a^1 and $\vec{\beta}_a$ are such that (B.46) is satisfied. Notice that this equation can be rewritten in terms of 3×3 determinants as

$$\begin{aligned}&\frac{(\vec{v}_a,\vec{v}_{a-1},\vec{b}^{(+)})R_a^+ - (\vec{v}_a,\vec{v}_{a-1},\vec{b}^{(-)})R_a^-}{(\vec{v}_{a+1},\vec{v}_a,\vec{v}_{a-1})}\\ &\quad = \frac{(\vec{v}_{a+2},\vec{v}_{a+1},\vec{b}^{(+)})R_{a+1}^+ - (\vec{v}_{a+2},\vec{v}_{a+1},\vec{b}^{(-)})R_{a+1}^-}{(\vec{v}_{a+2},\vec{v}_{a+1},\vec{v}_a)}.\end{aligned} \tag{B.47}$$

This equation is gauge invariant, so we can choose a convenient gauge to show that it is satisfied. In particular it is convenient to pick the "symmetric gauge" (B.4):

$$\lambda_+ = \lambda_- = 0 \quad \Rightarrow \quad \lambda_a^{(\pm)} = \lambda_a. \tag{B.48}$$

We substitute the expression (B.9) for $R_a^\pm$ into (B.47). The terms involving λ_{a-1} and λ_{a+2} cancel and we are left with an expression involving λ_a and λ_{a+1} given by[40]

$$\begin{aligned}&\lambda_a\Bigg[\frac{(\vec{v}_{a-1},\vec{v}_{a+1},\vec{b}^{(+)})(\vec{v}_a,\vec{v}_{a+1},\vec{b}^{(-)}) - (\vec{v}_{a-1},\vec{v}_{a+1},\vec{b}^{(-)})(\vec{v}_a,\vec{v}_{a+1},\vec{b}^{(+)})}{(\vec{v}_{a+1},\vec{v}_a,\vec{v}_{a-1})}\\ &\quad - \frac{(\vec{v}_{a+2},\vec{v}_{a+1},\vec{b}^{(+)})(\vec{v}_a,\vec{v}_{a+1},\vec{b}^{(-)}) - (\vec{v}_{a+2},\vec{v}_{a+1},\vec{b}^{(-)})(\vec{v}_a,\vec{v}_{a+1},\vec{b}^{(+)})}{(\vec{v}_{a+2},\vec{v}_{a+1},\vec{v}_a)}\Bigg]\\ &\quad + \lambda_{a+1}\Bigg[\frac{(\vec{v}_a,\vec{v}_{a-1},\vec{b}^{(+)})(\vec{v}_a,\vec{v}_{a+1},\vec{b}^{(-)}) - (\vec{v}_a,\vec{v}_{a-1},\vec{b}^{(-)})(\vec{v}_a,\vec{v}_{a+1},\vec{b}^{(+)})}{(\vec{v}_{a+1},\vec{v}_a,\vec{v}_{a-1})}\\ &\quad - \frac{(\vec{v}_a,\vec{v}_{a+2},\vec{b}^{(+)})(\vec{v}_a,\vec{v}_{a+1},\vec{b}^{(-)}) - (\vec{v}_a,\vec{v}_{a+2},\vec{b}^{(-)})(\vec{v}_a,\vec{v}_{a+1},\vec{b}^{(+)})}{(\vec{v}_{a+2},\vec{v}_{a+1},\vec{v}_a)}\Bigg] = 0.\end{aligned} \tag{B.49}$$

We can show that both the terms inside the square brackets vanish independently as a result of the vector quadruple product identity. Specifically for any four vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ in $\mathbb{R}^3$ we have

$$\vec{a}(\vec{b},\vec{c},\vec{d}) - \vec{b}(\vec{c},\vec{d},\vec{a}) + \vec{c}(\vec{d},\vec{a},\vec{b}) - \vec{d}(\vec{a},\vec{b},\vec{c}) = 0, \tag{B.50}$$

which immediately implies the following identity involving products of determinants:

$$(\vec{a},\vec{d},\vec{e})(\vec{b},\vec{d},\vec{c}) + (\vec{b},\vec{d},\vec{e})(\vec{c},\vec{d},\vec{a}) + (\vec{c},\vec{d},\vec{e})(\vec{a},\vec{d},\vec{b}) = 0. \tag{B.51}$$

Using this identity we can write

$$\begin{aligned}&(\vec{v}_{a-1},\vec{v}_{a+1},\vec{b}^{(+)})(\vec{v}_a,\vec{v}_{a+1},\vec{b}^{(-)}) - (\vec{v}_{a-1},\vec{v}_{a+1},\vec{b}^{(-)})(\vec{v}_a,\vec{v}_{a+1},\vec{b}^{(+)})\\ &\quad = (\vec{v}_{a+1},\vec{v}_a,\vec{v}_{a-1})(\vec{v}_{a+1},\vec{b}^{(-)},\vec{b}^{(+)}),\\ &(\vec{v}_{a+2},\vec{v}_{a+1},\vec{b}^{(+)})(\vec{v}_a,\vec{v}_{a+1},\vec{b}^{(-)}) - (\vec{v}_{a+2},\vec{v}_{a+1},\vec{b}^{(-)})(\vec{v}_a,\vec{v}_{a+1},\vec{b}^{(+)})\\ &\quad = (\vec{v}_{a+2},\vec{v}_{a+1},\vec{v}_a)(\vec{v}_{a+1},\vec{b}^{(-)},\vec{b}^{(+)}),\end{aligned} \tag{B.52}$$

which implies that the coefficient of λ_a in (B.49) vanishes. A similar argument shows that the coefficient of λ_{a+1} in (B.49) also vanishes and this concludes the proof of (B.32).

[40] We could in principle have exploited the residual gauge invariance to set e.g. $\lambda_{a+1} = 0$ but this is not necessary to conclude the proof.

B.3. $R_a^{\pm}$ *as functions of* b_2, b_3. Here we show that $R_a^{\pm}$, after eliminating the λ_a variables, are linear functions of b_2, b_3 and b_0. This shows that the second line in (B.15) are linear equations in these variables and also that (B.22), after substituting $\Delta_a^+ \to R_a^+$, are linear constraints.

The key equation from the previous subsection is (B.47). After eliminating R_a^- via (7.48) and using also (3.24), we find (B.47) can be rewritten as

$$\begin{aligned}&\frac{(\vec{v}_a, \vec{v}_{a-1}, \vec{p})\frac{R_a^+}{m_+m_-} + (\vec{v}_a, \vec{v}_{a-1}, \vec{b}^{(-)})\frac{M_a}{N}}{(\vec{v}_{a+1}, \vec{v}_a, \vec{v}_{a-1})}\\ &\quad = \frac{(\vec{v}_{a+2}, \vec{v}_{a+1}, \vec{p})\frac{R_{a+1}^+}{m_+m_-} + (\vec{v}_{a+2}, \vec{v}_{a+1}, \vec{b}^{(-)})\frac{M_{a+1}}{N}}{(\vec{v}_{a+2}, \vec{v}_{a+1}, \vec{v}_a)}.\end{aligned} \tag{B.53}$$

Schematically we can write this in the form

$$R_a^+ = R_{a+1}^+ A_a + B_a, \tag{B.54}$$

where A_a is independent of b_2, b_3 and b_0, while B_a is linear in b_2, b_3 and b_0. By recursively substituting the expression for R_{a+1}^+ into the expression for R_a^+, we can eventually express all R_a^+ in terms of the last one, R_d^+, as

$$R_a^+ = R_d^+ A_a' + B_a', \tag{B.55}$$

with A_a', B_a' having the same properties. Next we can use (7.47) to find

$$R_d^+ = \left(\sum_{a=1}^{d} A_a'\right)^{-1}\left[2 - \frac{b_0}{m_+} - \sum_{a=1}^{d} B_a'\right], \tag{B.56}$$

to conclude that R_d^+ is a linear function of b_2, b_3 and b_0. From (B.54) we find that all of the R_a^+ are linear functions of b_2, b_3 and b_0 and, recalling that R_a^- are related to R_a^+ via (3.46), the same is true for R_a^-.

C. Fluxes from Kaluza–Klein Reduction

In this appendix we expand further on how flavour and baryonic fluxes arise from Kaluza–Klein reduction in $\mathrm{AdS}_4 \times X_7/\mathrm{AdS}_5 \times X_5$ solutions with $n = 4, n = 3$, respectively, and in particular analyse the relation (7.41) for the toric fluxes M_a from this point of view. Essentially this expands on the discussion in [58] (using some of their results), to make contact with some flux formulas that appear in the present paper. Notice that here we are relating fluxes in a linearized Kaluza–Klein analysis around $\mathrm{AdS}_4 \times X_7/\mathrm{AdS}_5 \times X_5$ solutions, to fluxes in near horizon $\mathrm{AdS}_2 \times Y_9/\mathrm{AdS}_3 \times Y_7$ solutions – while one cannot literally compare the solutions at this level, the fluxes are quantized, and hence one should be able to make such a matching; and indeed, we find this leads to a consistent picture.

We work in general dimension n, with $n = 3$ and $n = 4$ relevant to the $\mathrm{AdS}_5 \times X_5$ and $\mathrm{AdS}_4 \times X_7$ solutions, respectively. We may then consider Kaluza–Klein reduction of, respectively, Type IIB and $D = 11$ supergravity on a Sasaki–Einstein manifold X_{2n-1} with $U(1)^s$ isometry. As explained in [58], we obtain a supergravity theory with an $\mathrm{AdS}_5/\mathrm{AdS}_4$ vacuum, which has massless gauge fields A_a, where the index a runs from 1 to $d \equiv s + b_{2n-3}$, where b_{2n-3} is the $(2n-3)$-th Betti number, i.e. $b_{2n-3} \equiv$

$\dim H_{2n-3}(X_{2n-1}, \mathbb{R})$. Such gauge fields appear in the expansion of the $(2n-2)$-form potential

$$C_{2n-2} = \left[\frac{2\pi}{V} \operatorname{vol}(X_{2n-1}) + \sum_{a=1}^{d} A_a \wedge \omega_a + \cdots \right] \frac{\nu_n L^{2n-2}}{(5-n)2\pi} N. \tag{C.1}$$

Here $V \equiv \mathrm{Vol}_{SE}(X_{2n-1})$ and $\operatorname{vol}(X_{2n-1})$ is the volume form on X_{2n-1}, coming from the AdS background, and the terms in the $\cdots$ in (C.1) are not relevant for our discussion. The remaining term $\sum_{a=1}^{d} A_a \wedge \omega_a$ is the Kaluza–Klein fluctuation term (called δC_6 in (9.1) with $n = 4$), where ω_a are $(2n-3)$-forms satisfying [58]

$$\mathrm{d}\omega_a = \sum_{i=1}^{s} v_{ai}\, \partial_{\varphi_i} \lrcorner \operatorname{vol}(X_{2n-1}), \tag{C.2}$$

$$\mathrm{d} *_{2n-1} \omega_a = 0. \tag{C.3}$$

The v_{ai} are constants, but in the case that X_{2n-1} is toric, i.e. $s = n$, the v_{ai} are precisely the toric data [58]. While we expect everything in this appendix to be true in general, for concreteness we assume that X_{2n-1} is toric so as to be able to make use of various toric formulae that appear in the main text, and Appendix A.

In general, quantities labelled by the index a can be split into linear combinations of quantities labelled by an index i, $i = 1, \ldots, s$, and those labelled by an index I, $I = 1, \ldots, b_{2n-3}$. We thus begin by writing

$$A_a = \sum_{i=1}^{s} \alpha_a^i\, A_i + \sum_{I=1}^{b_{2n-3}} q_I^a A_I, \tag{C.4}$$

$$\omega_a = \sum_{i=1}^{s} v_{ai}\, \omega_i + \sum_{I=1}^{b_{2n-3}} m_I^a\, \omega_I. \tag{C.5}$$

Here in (C.4) we take α_a^i and the baryonic charge matrix q_I^a to be the quantities introduced in Appendix A. This is then precisely what one means by a splitting into "flavour" and "baryonic" symmetries: by definition α_a^i is the charge of the ath coordinate z^a of the GLSM $\mathbb{C}^d$ under the ith $U(1) \subset U(1)^s$, while q_I^a is the charge of this coordinate under the Ith baryonic symmetry. By definition, these then couple to the flavour gauge field A_i and baryonic gauge field A_I, respectively, as in (C.4). Similarly, in (C.5) we take the ω_I to be harmonic $(2n-3)$-forms, constituting a basis of $H^{2n-3}(X_{2n-1}, \mathbb{R})$ and such that $\int_{\mathcal{C}_I} \omega_J = \delta_{IJ}$ for $\mathcal{C}_I$ a basis for the free part of $H_{2n-3}(X_{2n-1}, \mathbb{Z})$, while ω_i satisfy

$$\mathrm{d}\omega_i = \partial_{\varphi_i} \lrcorner \operatorname{vol}(X_{2n-1}). \tag{C.6}$$

This then ensures that (C.2) holds. Moreover, taking the m_I^a to satisfy the relations

$$\sum_{a=1}^{d} v_{ai}\, \alpha_a^j = \delta_i^j, \qquad \sum_{a=1}^{d} v_{ai}\, q_I^a = 0, \tag{C.7}$$

$$\sum_{a=1}^{d} q_I^a\, m_J^a = \delta_{IJ}, \qquad \sum_{a=1}^{d} \alpha_a^i\, m_I^a = 0, \tag{C.8}$$

then gives the projections

$$A_i = \sum_{a=1}^{d} v_{ai}\, A_a, \qquad A_I = \sum_{a=1}^{d} m_I^a A_a, \tag{C.9}$$

$$\omega_i = \sum_{a=1}^{d} \alpha_a^i\, \omega_a, \qquad \omega_I = \sum_{a=1}^{d} q_I^a\, \omega_a. \tag{C.10}$$

The forms ω_i and ω_I are then precisely the projections onto flavour and baryonic directions, respectively. The relations in (C.7) are precisely those in Appendix A, where v_{ai} give the toric data for X_{2n-1}. In particular, recall that α_a^j are not unique, precisely due to the kernel generated by q_I^a. On the other hand, in the first equation in (C.8) the baryonic charges q_I^a, $I = 1, \dots, b_{2n-3}$, are by definition linearly independent, and m_J^a is simply a choice of inverse. Due to the second equation in (C.7) we are free to shift $m_I^a \to m_I^a - \sum_{i=1}^{s} c_i^I v_{ai}$, and using this freedom one can then always impose the second equation in (C.8).

Notice, however, that we are still free to make the following simultaneous shifts, without changing A_a and ω_a:

$$A_I \to A_I - \sum_{i=1}^{s} c_i^I\, A_i, \qquad \omega_i \to \omega_i + \sum_{I=1}^{b_{2n-3}} c_i^I\, \omega_I, \tag{C.11}$$

$$\alpha_a^i \to \alpha_a^i + \sum_{I=1}^{b_{2n-3}} c_i^I\, q_I^a, \qquad m_I^a \to m_I^a - \sum_{i=1}^{s} c_i^I\, v_{ai}, \tag{C.12}$$

where c_i^I are arbitrary constants. This is the same freedom discussed in Sects. 7.3 and 9. With these decompositions, we can then rewrite (C.1) as

$$C_{2n-2} = \left[\frac{2\pi}{V} \operatorname{vol}(X_{2n-1}) + \sum_{i=1}^{s} A_i \wedge \omega_i + \sum_{I=1}^{b_{2n-3}} A_I \wedge \omega_I + \cdots \right] \frac{\nu_n L^{2n-2}}{(5-n)2\pi} N. \tag{C.13}$$

Up until this point, all we have done is expand upon the Kaluza–Klein analysis discussed in [58]. However, consider now putting the lower-dimensional theory on a spindle Σ, fibering the internal manifold X_{2n-1} over it and building an $(2n+1)$-dimensional internal space $X_{2n-1} \hookrightarrow Y_{2n+1} \to \Sigma$. The fibration is achieved by turning on the flavour magnetic charges

$$\frac{1}{2\pi} \int_{\Sigma} \mathrm{d}A_i = \frac{p_i}{m_+ m_-}, \tag{C.14}$$

as explained in Sect. 3.2. Next consider integrating $\mathrm{d}C_{2n-2}$ over supersymmetric $(2n-1)$-submanifolds Σ_a, that are fibrations $S_a \hookrightarrow \Sigma_a \to \Sigma$, where S_a is defined to be the (toric) submanifold such that $\int_{S_a} \omega_b = \delta_{ab}$ (see (3.43) in the main text). Such integrals give the fluxes M_a

$$M_a = \frac{5-n}{\nu_n L^{2n-2}} \int_{\Sigma_a} \mathrm{d}C_{2n-2}. \tag{C.15}$$

These fluxes depend only on the homology classes of Σ_a in Y_{2n+1} which, as explained in Sect. 7.3, may be decomposed in terms of the homology classes of the cycles Σ_I obtained by fibering $\mathcal{C}_I$ over the spindle plus that of the fibre X_{2n-1} itself as

$$\begin{aligned}[][\Sigma_a] &= \frac{q_0^a}{m_+m_-}[X_{2n-1}] + \sum_{I=1}^{b_{2n-3}} q_I^a [\Sigma_I] \\ &= \frac{1}{m_+m_-}\sum_{i=1}^{s} p_i\, \alpha_a^i\, [X_{2n-1}] + \sum_{I=1}^{b_{2n-3}} q_I^a [\Sigma_I].\end{aligned} \tag{C.16}$$

In particular this is the homology content of the last equation in (7.38), where $[X_{2n-1}]$ is the class of the fibre in $H_{2n-1}(Y_{2n+1}, \mathbb{Z})$. However, note that given the homology class $[\mathcal{C}_I] \in H_{2n-3}(X_{2n-1}, \mathbb{Z})$ in the fibre, the homology class $[\Sigma_I] \in H_{2n-1}(Y_{2n+1}, \mathbb{Z})$ in the total space is not uniquely identified but it depends on the specific representative we use for $[\mathcal{C}_I]$, because these will be twisted in different ways by the fibration. Indeed, in order for $[\Sigma_a]$ to be invariant under the shift of α_a^i in (C.11), correspondingly there must be a shift in $[\Sigma_I]$

$$[\Sigma_I] \;\to\; [\Sigma_I] - \frac{1}{m_+m_-}\sum_{i=1}^{s} p_i\, c_i^I\, [X_{2n-1}], \tag{C.17}$$

which is exactly what parametrizes the ambiguity in choosing a representative for $[\mathcal{C}_I]$.

We now have everything we need to compute the fluxes (C.15):

$$\begin{aligned} M_a &= \frac{5-n}{\nu_n L^{2n-2}}\left[\frac{1}{m_+m_-}\sum_{i=1}^{s} p_i\, \alpha_a^i \int_{X_{2n-1}} \mathrm{d}C_{2n-2} + \sum_{I=1}^{b_{2n-3}} q_I^a \int_{\Sigma_I} \mathrm{d}C_{2n-2}\right] \\ &= \frac{N}{m_+m_-V}\sum_{i=1}^{s} p_i\, \alpha_a^i \int_{X_{2n-1}} \mathrm{vol}(X_{2n-1}) + \sum_{I=1}^{b_{2n-3}} q_I^a\, \mathcal{N}_I \\ &= \frac{N}{m_+m_-}\sum_{i=1}^{s} p_i\, \alpha_a^i + \sum_{I=1}^{b_{2n-3}} q_I^a\, \mathcal{N}_I, \end{aligned} \tag{C.18}$$

where, in agreement with (2.14), (2.19), we introduced

$$\begin{aligned} \mathcal{N}_I \equiv \frac{5-n}{\nu_n L^{2n-2}}\int_{\Sigma_I} \mathrm{d}C_{2n-2} &= \frac{N}{2\pi}\left[\sum_{i=1}^{s}\int_{\Sigma}\mathrm{d}A_i \int_{\mathcal{C}_I}\omega_i + \sum_{J=1}^{b_{2n-3}}\int_{\Sigma}\mathrm{d}A_J\int_{\mathcal{C}_I}\omega_J\right] \\ &= \frac{N}{m_+m_-}\sum_{i=1}^{s} p_i \int_{\mathcal{C}_I}\omega_i + N\int_{\Sigma}\frac{\mathrm{d}A_I}{2\pi}. \end{aligned} \tag{C.19}$$

In particular this gives a formula for the "baryonic fluxes" $\mathcal{N}_I$ in terms of the Kaluza–Klein reduction. Notice that the second term $N\int_\Sigma \mathrm{d}A_I/2\pi$ is what one would naturally call the "baryonic flux", but that in general also the first term contributes, coming from the flavour twisting p_i.

Equation (C.18) shows how to split the fluxes M_a into a "flavour" part and a "baryonic" part. It is clear that, while the fluxes M_a are unambiguous and dictate how the fields are twisted over the spindle in the dual field theory,[41] on the other hand applying the shift invariance (C.11) to (C.18) results in an ambiguity in the flavour/baryonic splitting

$$\begin{aligned} \mathcal{N}_I &\to \mathcal{N}_I - \frac{N}{m_+ m_-} \sum_{i=1}^{s} c_i^I \, p_i \,, \\ \alpha_a^i &\to \alpha_a^i + \sum_{I=1}^{b_{2n-3}} c_i^I \, q_I^a \,. \end{aligned} \tag{C.20}$$

The transformations above encode the freedom to choose $s \times b_{2n-3}$ constants c_i^I and, given that there are b_{2n-3} fluxes $\mathcal{N}_I$, we can always choose them in such a way to fix the $\mathcal{N}_I$ to whatever we want (e.g. to zero) and we will actually still have some freedom left, parametrized by $(s-1)\, b_{2n-3}$ constants. The shift in the $\mathcal{N}_I$ can also be understood from (C.19) by noticing that $\mathrm{d}C_{2n-2}$ is invariant under the transformations (C.11) but the cycle Σ_I is not, and transforms as (C.17).[42]

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[41] In the toric case there is a general prescription to identify these M_a with the twisting of the dual gauge theory, since this has a known relation to the GLSM. There is no such known general prescription in the non-toric setting.

[42] One cannot see the shift in $\mathcal{N}_I$ directly from the final line of (C.19), as the information on how the cycle $\mathcal{C}_I$ has been fibred over Σ has been lost in this expression.

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Communicated by S. Nameki