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A Superunitary Fock Model of the Exceptional Lie Supergroup $\mathbb{D}(2, 1; \alpha)$

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Abstract: We construct a Fock model of the minimal representation of the exceptional Lie supergroup $\mathbb{D}(2, 1; \alpha)$. Explicit expressions for the action are given by integrating to group level a Fock model of the Lie superalgebra $D(2, 1; \alpha)$ constructed earlier by the authors. It is also shown that the representation is superunitary in the sense of de Goursac–Michel.

1. Introduction

The main result of this paper is the construction of a Fock model of the minimal representation of the Lie supergroup $\mathbb{D}(2, 1; \alpha)$. We do this by integrating to group level the representation of the Lie superalgebra $D(2, 1; \alpha)$ considered in [1]. In that sense, this paper can be seen as a sequel to [1]. We also show that this Fock model is superunitary in the sense of [2].

Minimal representations of Lie groups have a long tradition and can be constructed in many settings [3–9]. In the philosophy of the orbit method, minimal representations correspond to the minimal nilpotent orbit of the coadjoint action of the Lie group on the dual Lie algebra. They are 'small' infinite-dimensional representations, or more technically, they attain the smallest Gelfand–Kirillov dimension of all possible infinite-dimensional representations [5]. This implies that there are a lot of symmetries in their realisations which leads to a rich representation theory.

Recently, there has been an effort to generalize the framework of minimal representations to the setting of Lie supergroups and Lie superalgebras [1,10–12]. Although this is a logical next step, there are lot of technical and conceptual hurdles. For instance, a lot of tools used for Lie groups are not yet developed or become much more complex in the super setting. Another obstacle is the fact that in [13] it is shown that there are no superunitary representations for a large class of Lie supergroups in the standard definition [14] of a superunitary representation. This has lead to the development of alternative definitions of what should be a superunitary representation [2,15,16]. However, at the moment no satisfactory definition has been found. Therefore it is important to construct concrete models of representations that 'ought' to be superunitary, as we will do in this paper.

For the orthosymplectic Lie supergroup $OSp(p, q|2n)$ a Schrödinger model of the minimal representation was constructed in [11] using the framework of Jordan (super)algebras developed in [6]. This generalizes the minimal representation of $O(p, q)$ considered in [7,17–19]. Later, also a Fock model and intertwining Segal-Bargmann transform for $OSp(p, q|2n)$ were obtained in [12].

Recently, a Schrödinger model, Fock model and Segal-Bargmann transform of the Lie superalgebra $D(2, 1; \alpha)$ were constructed [1]. The paper [1] works entirely on algebra level. In particular it does not say anything about unitarity. It does, however, show that there exists a superhermitian product for which the Fock model is invariant. The goal of this paper is to integrate the Fock model considered in [1] to group level. We will show that the superhermitian product can be extended to a Hilbert space and that our representation extend to a superunitary representation in the sense of [2].

1.1. Contents. Let us now describe the contents of this paper. We start in Sect. 2 by recalling the definition of the Lie superalgebra $D(2, 1; \alpha)$ and giving an explicit expression of the Fock model considered in [1]. In Sect. 3, we introduce the Lie supergroup $\mathbb{D}(2, 1; \alpha)$ and deduce some properties we need to integrate the representation, while in Sect. 4, we recall the necessary properties of the polynomial Fock space considered in [1] and complete it to a Hilbert superspace.

Section 5 contains the main content of this paper. We start by giving an explicit form of the representation of the Lie supergroup $\mathbb{D}(2, 1; \alpha)$ in Theorem 5.1. We also give two alternatives way to present this representation (Corollaries 5.2 and 5.3). Note, however, that for one generating element of $\mathbb{D}(2, 1; \alpha)$ we were only able to give an explicit form if $\alpha > 0$.

In Sect. 5.3 we recall the definition of a superunitary representation (SUR) as introduced by de Goursac and Michel in [2] and show that the Fock model is such a SUR if $\alpha < 0$ (Theorem 5.10). The definition of a SUR of de Goursac and Michel has a major drawback, namely too many representations become SUR in the non-super case. We do not only recover unitary representations, but also Krein-unitary representations.

To remedy this situation, the authors in [2] introduce the concept of a strong SUR. However, we show that the Fock model we construct is never a strong SUR (Theorem 5.12).

1.2. Notations. The field $\mathbb{K}$ will always mean the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. Function spaces will always be defined over $\mathbb{C}$. We use the convention $\mathbb{N} = \{0, 1, 2, \ldots\}$ and denote the complex unit by $\imath$.

A supervector space is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. An element $v \in V$ is called homogeneous if $v \in V_i$, $i \in \mathbb{Z}/2\mathbb{Z}$. We call i its parity and denote it by $|v|$. When we use $|v|$ in a formula, we are considering homogeneous elements, with the implicit convention that the formula has to be extended linearly for arbitrary elements. If $\dim(V_i) = d_i$, then we write $\dim(V) = (d_{\bar{0}}|d_{\bar{1}})$. We denote the super vector space V with $V_{\bar{0}} = \mathbb{K}^m$ and $V_{\bar{1}} = \mathbb{K}^n$ as $\mathbb{K}^{m|n}$. A superalgebra is a supervector space $A = A_{\bar{0}} \oplus A_{\bar{1}}$ for which A is an algebra and $A_i A_j \subseteq A_{i+j}$.

2. The Lie Superalgebra $D(2, 1; \alpha)$

2.1. The construction of $D(2,1;\alpha)$. We can deform $D(2, 1) = \mathfrak{osp}(4|2)$ to obtain a one-parameter family of $(9|8)$-dimensional Lie superalgebras of rank 3. We will define these Lie superalgebras in the same way as we did in [1] using a construction of Scheunert. We will use the same notations as in [20].

Consider a two-dimensional vector space V with basis u_+ and u_-. Introduce a non-degenerate skew-symmetric bilinear form ψ by $\psi(u_+, u_-) = 1$. We will need three copies $(V_i, \psi_i), i = 1, 2, 3$ of (V, ψ) and the corresponding Lie algebra $\mathfrak{sl}(V_i) = \mathfrak{sp}(\psi_i)$ of linear transformations preserving ψ_i.

We use the following data to define a Lie superalgebra:

- a Lie algebra $\mathfrak{g}_{\bar{0}}$,
- a $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$,
- a $\mathfrak{g}_{\bar{0}}$-morphism $p : S^2(\mathfrak{g}_{\bar{1}}) \to \mathfrak{g}_{\bar{0}}$, with $S^2(\mathfrak{g}_{\bar{1}})$ the symmetric tensor power,
- for all $a, b, c \in \mathfrak{g}_{\bar{1}}$ the morphism p satisfies

$$[p(a, b), c] + [p(b, c), a] + [p(c, a), b] = 0, \tag{2.1}$$

where we denoted the $\mathfrak{g}_{\bar{0}}$-action on $\mathfrak{g}_{\bar{1}}$ by $[\cdot, \cdot]$.

Then $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a Lie superalgebra [21, Remark 1.5].

We set

$$\mathfrak{g}_{\bar{0}} = \mathfrak{sp}(\psi_1) \oplus \mathfrak{sp}(\psi_2) \oplus \mathfrak{sp}(\psi_3)$$
$$\mathfrak{g}_{\bar{1}} = V_1 \otimes V_2 \otimes V_3$$

and define the action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ by the outer tensor product

$$(A, B, C) \cdot x \otimes y \otimes z = A(x) \otimes y \otimes z + x \otimes B(y) \otimes z + x \otimes y \otimes C(z).$$

The $\mathfrak{g}_{\bar{0}}$-morphism p is given by

$$\begin{aligned} p(x_1 \otimes x_2 \otimes x_3, y_1 \otimes y_2 \otimes y_3) = {} & \sigma_1 \psi_2(x_2, y_2)\psi_3(x_3, y_3)p_1(x_1, y_1) \\ & + \sigma_2 \psi_3(x_3, y_3)\psi_1(x_1, y_1)p_2(x_2, y_2) \\ & + \sigma_3 \psi_1(x_1, y_1)\psi_2(x_2, y_2)p_3(x_3, y_3), \end{aligned}$$

where $\sigma_i \in \mathbb{K}$ and $p_i : V_i \times V_i \to \mathfrak{sp}(\psi_i)$ is defined by

$$p_i(x, y)z = \psi_i(y, z)x - \psi_i(z, x)y.$$

Then $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a Lie superalgebra if the morphism p satisfies the Jacobi identity (2.1). This is the case if and only if $\sigma_1 + \sigma_2 + \sigma_3 = 0$, see [20, Lemma 4.2.1]. If we denote $\mathfrak{g}$ by $\Gamma(\sigma_1, \sigma_2, \sigma_3)$ then we have

$$\Gamma(\sigma_1, \sigma_2, \sigma_3) \cong \Gamma(\sigma_1', \sigma_2', \sigma_3')$$

if and only if there is a non-zero scalar c and a permutation π of $(1, 2, 3)$ such that $\sigma_i' = c\sigma_{\pi(i)}$ [20, Lemma 5.5.16].

We set

$$D(2, 1; \alpha) := \Gamma\left(\frac{1 + \alpha}{2}, \frac{-1}{2}, \frac{-\alpha}{2}\right).$$

The Lie superalgebra $D(2,1;\alpha)$ is simple unless $\alpha = 0$ or $\alpha = -1$. Furthermore, we have the isomorphism

$$D(2,1;\alpha) \cong D(2,1;\beta)$$

if and only if α and β are in the same orbit under the transformations $\alpha \mapsto \alpha^{-1}$ and $\alpha \mapsto -1-\alpha$.

Consider the matrices

$$E_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F_i = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H_i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They give a realisation of $\mathfrak{sl}(V_i)$ where the vector space V_i is given by

$$u_+^i = (1,0)^t, \quad u_-^i = (0,1)^t.$$

We also obtain p from

$$p_i(u_+^i, u_+^i) = 2E_i, \quad p_i(u_+^i, u_-^i) = -H_i, \quad p_i(u_-^i, u_-^i) = -2F_i.$$

For the odd basis elements $u_\pm^1 \otimes u_\pm^2 \otimes u_\pm^3$ of $\mathbb{D}(2,1;\alpha)$ we introduce a more compact notation

$$u_{\pm\pm\pm} := u_\pm^1 \otimes u_\pm^2 \otimes u_\pm^3.$$

We have the following realisation of $\mathfrak{sl}(2)$ in $\mathbb{D}(2,1;\alpha)$

$$\{E_2 + E_3, H_2 + H_3, F_2 + F_3\}.$$

The corresponding three-grading by the eigenspaces of $\mathrm{ad}(H_2 + H_3)$ is given by

$$\begin{aligned} \mathfrak{g}_+ &= \{E_3, E_2, u_{-++}, u_{+++}\} \\ \mathfrak{g}_- &= \{F_3, F_2, u_{+--}, u_{---}\} \\ \mathfrak{g}_0 &= \{H_1, H_2, H_3, E_1, F_1, u_{-+-}, u_{++-}, , u_{+-+}, u_{--+}\}. \end{aligned} \tag{2.2}$$

2.2. The Fock representation. In [1] a Fock representation $\mathrm{d}\rho$ depending on a parameter $\lambda \in \{1, \alpha\}$ was constructed on the so called polynomial Fock space F_λ. We will briefly reconstruct it here and refer to [1, Section 4.3] for further details. Note that from now on we will exclude $\alpha = 0$ and $\alpha = 1$ since for that case the picture becomes quite different. Remark that $\alpha = 1$ correspond to the non-deformed case $D(2,1;1) = \mathfrak{osp}(4|2)$, while for $\alpha = 0$, the algebra $D(2,1;0)$ is not simple. See [1, Section 4.2] for a more detailed explanation.

Let z_1, z_2 and z_3, z_4 be the even resp. odd representatives of the coordinates on $\mathcal{P}(\mathbb{C}^{2|2})$. We define

$$V_\alpha := \{a(2z_1z_2 + z_3z_4) + bz_2^2 + cz_2z_3 + dz_2z_4 \mid a,b,c,d \in \mathbb{K}\} \subset \mathcal{P}_2 \qquad \text{if } \lambda = \alpha \text{ and}$$
$$V_1 := \{a(2\alpha z_1z_2 + z_3z_4) + bz_1^2 + cz_1z_3 + dz_1z_4 \mid a,b,c,d \in \mathbb{K}\} \subset \mathcal{P}_2 \quad \text{if } \lambda = 1.$$

Definition 2.1. Suppose $\lambda \in \{1, \alpha\}$, then the **polynomial Fock space** is defined as the superspace

$$F_\lambda := \mathcal{P}(\mathbb{C}^{2|2})/\mathcal{I}_\lambda,$$

with $\mathcal{I}_\lambda := \mathcal{P}(\mathbb{C}^{2|2})V_\lambda$.

As shown in [1, Section 5.1], if $p \in F_\alpha$, then there exists $p_{i,k} \in \mathbb{C}$ such that

$$p = p_{1,0} + \sum_{k=1}^{\infty} z_1^{k-1}(p_{1,k}z_1 + p_{2,k}z_2 + p_{3,k}z_3 + p_{4,k}z_4).$$

The explicit expression for the Fock representation $\mathrm{d}\rho$ on F_λ is as follows. For $\mathfrak{g}_-$ we obtain

$$\begin{aligned}
\mathrm{d}\rho(F_2) &= -\frac{\imath}{2}(z_1 + \mathcal{B}_\lambda(z_1)) - \frac{\imath}{2}(-\lambda + 2z_1\partial_{z_1} + z_3\partial_{z_3} + z_4\partial_{z_4}),\\
\mathrm{d}\rho(F_3) &= -\frac{\imath}{2}(z_2 + \mathcal{B}_\lambda(z_2)) - \frac{\imath}{2}(-\frac{\lambda}{\alpha} + 2z_2\partial_{z_2} + z_3\partial_{z_3} + z_4\partial_{z_4}),\\
\mathrm{d}\rho(u_{---}) &= \frac{\imath}{2}(z_3 + \mathcal{B}_\lambda(z_3)) + \frac{\imath}{2}(z_3\partial_{z_1} + 2\alpha z_2\partial_{z_4} + z_3\partial_{z_2} + 2z_1\partial_{z_4}),\\
\mathrm{d}\rho(u_{+--}) &= \frac{\imath}{4}(z_4 + \mathcal{B}_\lambda(z_4)) + \frac{\imath}{4}(z_4\partial_{z_1} - 2\alpha z_2\partial_{z_3} + z_4\partial_{z_2} - 2z_1\partial_{z_3}).
\end{aligned}$$

For $\mathfrak{g}_+$ we have

$$\begin{aligned}
\mathrm{d}\rho(E_2) &= -\frac{\imath}{2}(z_1 + \mathcal{B}_\lambda(z_1)) + \frac{\imath}{2}(-\lambda + 2z_1\partial_{z_1} + z_3\partial_{z_3} + z_4\partial_{z_4}),\\
\mathrm{d}\rho(E_3) &= -\frac{\imath}{2}(z_2 + \mathcal{B}_\lambda(z_2)) + \frac{\imath}{2}(-\frac{\lambda}{\alpha} + 2z_2\partial_{z_2} + z_3\partial_{z_3} + z_4\partial_{z_4}),\\
\mathrm{d}\rho(u_{-++}) &= -\frac{\imath}{2}(z_3 + \mathcal{B}_\lambda(z_3)) + \frac{\imath}{2}(z_3\partial_{z_1} + 2\alpha z_2\partial_{z_4} + z_3\partial_{z_2} + 2z_1\partial_{z_4}),\\
\mathrm{d}\rho(u_{+++}) &= -\frac{\imath}{4}(z_4 + \mathcal{B}_\lambda(z_4)) + \frac{\imath}{4}(z_4\partial_{z_1} - 2\alpha z_2\partial_{z_3} + z_4\partial_{z_2} - 2z_1\partial_{z_3}).
\end{aligned}$$

For $\mathfrak{g}_0$ we have

$$\begin{aligned}
&\mathrm{d}\rho(F_1) = 2z_3\partial_{z_4}, \quad \mathrm{d}\rho(E_1) = 2^{-1}z_4\partial_{z_3}, \quad \mathrm{d}\rho(H_1) = z_4\partial_{z_4} - z_3\partial_{z_3},\\
&\mathrm{d}\rho(H_2) = z_1 - \mathcal{B}_\lambda(z_1), \quad \mathrm{d}\rho(H_3) = z_2 - \mathcal{B}_\lambda(z_2),\\
&\mathrm{d}\rho(u_{--+} + u_{-+-}) = -(z_3 - \mathcal{B}_\lambda(z_3)), \quad \mathrm{d}\rho(u_{+-+} + u_{++-}) = -2^{-1}(z_4 - \mathcal{B}_\lambda(z_4)),\\
&\mathrm{d}\rho(u_{--+} - u_{-+-}) = -z_3\partial_{z_1} - 2\alpha z_2\partial_{z_4} + z_3\partial_{z_2} + 2z_1\partial_{z_4},\\
&\mathrm{d}\rho(u_{+-+} - u_{++-}) = 2^{-1}(-z_4\partial_{z_1} + 2\alpha z_2\partial_{z_3} + z_4\partial_{z_2} - 2z_1\partial_{z_3}).
\end{aligned}$$

Here $\mathcal{B}_\lambda(z_i)$ denotes the Bessel operator of z_i and are expicitly given by

$$\begin{aligned}
\mathcal{B}_\lambda(z_1) &= (-\lambda + z_1\partial_{z_1} + z_3\partial_{z_3} + z_4\partial_{z_4})\partial_{z_1} - 2\alpha z_2\partial_{z_3}\partial_{z_4},\\
\mathcal{B}_\lambda(z_2) &= (-\frac{\lambda}{\alpha} + z_2\partial_{z_2} + z_3\partial_{z_3} + z_4\partial_{z_4})\partial_{z_2} - 2z_1\partial_{z_3}\partial_{z_4},\\
\mathcal{B}_\lambda(z_3) &= (-2\lambda + 2z_1\partial_{z_1} + 2\alpha z_2\partial_{z_2} + 2(1+\alpha)z_3\partial_{z_3})\partial_{z_4} + z_3\partial_{z_1}\partial_{z_2},\\
\mathcal{B}_\lambda(z_4) &= (2\lambda - 2z_1\partial_{z_1} - \alpha z_2\partial_{z_2} - 2(1+\alpha)z_4\partial_{z_4})\partial_{z_3} + z_4\partial_{z_1}\partial_{z_2}.
\end{aligned}$$

2.3. Additional representations. As mentioned in Sect. 2.1, we have the isomorphisms $D(2,1;\alpha) \cong D(2,1;\beta)$ if and only if β is in the same orbit as α under the transformations $\alpha \mapsto \alpha^{-1}$ and $\alpha \mapsto -1-\alpha$, i.e.,

$$\beta \in \{\alpha, -1-\alpha, -1-\alpha^{-1}, \alpha^{-1}, (-1-\alpha)^{-1}, (-1-\alpha^{-1})^{-1}\}$$

These isomorphisms give rise to additional representations of $D(2,1;\alpha)$. A straightforward verification shows the isomorphism between $D(2,1;\alpha)$ and $D(2,1;\alpha^{-1})$ respects the three grading introduced in Eq. 2.2, while the isomorphism between $D(2,1;\alpha)$ and $D(2,1;-1-\alpha)$, in general, does not.

Let $\mathrm{d}\rho^{\alpha}_{\lambda}$ denote the Fock representation corresponding to $D(2,1;\alpha)$ with parameter $\lambda \in \{1,\alpha\}$. The isomorphism between $D(2,1;\alpha)$ and $D(2,1;\alpha^{-1})$ induces an equivalence between $\mathrm{d}\rho^{\alpha}_{\lambda}$ with parameter $\lambda = \alpha$ and $\mathrm{d}\rho^{\alpha^{-1}}_{\lambda}$ with $\lambda = 1$. Therefore, without loss of generality, we may choose $\lambda = \alpha$ and set $\mathrm{d}\rho^{\alpha} := \mathrm{d}\rho^{\alpha}_{\alpha}$.

For an arbitrary α we now find that precomposing $\mathrm{d}\rho^{\beta}$ with the isomorphism $D(2,1;\alpha) \to D(2,1;\beta)$ for all possible values of β gives us

$$\mathrm{d}\rho^{\alpha}, \quad \mathrm{d}\rho^{-1-\alpha}, \quad \mathrm{d}\rho^{-1-\alpha^{-1}} \quad \mathrm{d}\rho^{\alpha^{-1}}, \quad \mathrm{d}\rho^{(-1-\alpha)^{-1}}, \quad \mathrm{d}\rho^{(-1-\alpha^{-1})^{-1}},$$

which are all possibly distinct representations of $D(2,1;\alpha)$.

3. The Lie supergroup $\mathbb{D}(2,1;\alpha)$

In this section we define the supergroup $\mathbb{D}(2,1;\alpha)$ which has $D(2,1;\alpha)$ as its Lie superalgebra. We also give some basic results of $SL(V)$, which we will need later on. Note that in this section we will work over the field $\mathbb{R}$ of real numbers.

3.1. Definition of $\mathbb{D}(2,1;\alpha)$. We will use the characterisation of Lie supergroups based on pairs, see for example [22, Chapter 7] for more details.

Definition 3.1. A Lie supergroup G is a pair $(G_0, \mathfrak{g})$ together with a morphism $\sigma : G_0 \to \mathrm{End}(\mathfrak{g})$ where G_0 is a Lie group and $\mathfrak{g}$ is a Lie superalgebra for which

- $\mathrm{Lie}(G_0) \cong \mathfrak{g}_{\bar{0}}$.
- For all $g \in G_0$ we have $\sigma(g)|_{\mathfrak{g}_{\bar{0}}} = \mathrm{Ad}(g)$, where Ad is the adjoint representation of G_0 on $\mathfrak{g}_{\bar{0}}$.
- For all $X \in \mathfrak{g}_{\bar{0}}$ and $Y \in \mathfrak{g}$ we have

$$d\sigma(X)Y = \frac{d}{dt}\sigma(\exp(tX))Y\bigg|_{t=0} = [X,Y].$$

Since σ extends the adjoint representation of G_0 on $\mathfrak{g}_{\bar{0}}$ we call it the **adjoint representation** of G_0 on $\mathfrak{g}$ and denote it by Ad.

Note that these Lie supergroups are called super Harish-Chandra pairs in [2]. The term Lie supergroup is then used for a supermanifold endowed with a group structure for which the multiplication is a smooth map. However, as is mentioned in [2] these two structures are categorically equivalent.

Recall $\mathfrak{g} = D(2,1;\alpha)$ and define $G_0 := SL(V_1) \times SL(V_2) \times SL(V_3)$, where V_i is a copy of the two dimensional vector space V with basis u_+ and u_-. Then $\mathbb{D}(2,1;\alpha) :=$

$(G_0, \mathfrak{g})$ is a Lie supergroup if we extend the adjoint representation as follows. For $A_i \in SL(V_i)$ and $x \otimes y \otimes z \in \mathfrak{g}_{\bar{1}} = V_1 \otimes V_2 \otimes V_3$ we define

$$\mathrm{Ad}(A_1, A_2, A_3)x \otimes y \otimes z := A_1(x) \otimes A_2(y) \otimes A_3(z)$$

and for $X_i \in \{H_i, E_i, F_i\}$ we define

$$\mathrm{Ad}(A_1, A_2, A_3)X_i := A_i X_i A_i^{-1}$$

and extend it linearly.

3.2. Properties of $SL(V)$. Define the following one-dimensional subgroups of $SL(V_i)$ for $i \in \{1, 2, 3\}$

$$K_i := \left\{ K_i(k_i) := \begin{pmatrix} \cos(k_i) & -\sin(k_i) \\ \sin(k_i) & \cos(k_i) \end{pmatrix} \mid k_i \in \mathbb{R} \right\},$$
$$A_i := \left\{ A_i(a_i) := \begin{pmatrix} \exp(a_i) & 0 \\ 0 & \exp(-a_i) \end{pmatrix} \mid a_i \in \mathbb{R} \right\}.$$

On the one hand we have the Cartan decomposition of $SL(V_i)$.

Theorem 3.2 (Cartan decomposition). *We have a decomposition $SL(V) = KAK$, i.e., every $g \in \mathfrak{sl}(V)$ can be written as $g = kak'$ with $k, k' \in K$ and $a \in A$.*

This decomposition implies that a representation of $SL(V_i)$ is fully determined by its restriction to K_i and A_i. On the other hand we have an explicit integration of $\mathfrak{sl}(V_i)$ to $SL(V_i)$.

Lemma 3.3. *Suppose A, B and C are three anticommuting variables. Then*

$$(A + B + C)^{2j} = \sum_{a+b+c=j} \binom{j}{a, b, c} A^{2a} B^{2b} C^{2c},$$

for all $j \in \mathbb{N}$.

Proof. This follows immediately from the multinomial theorem and the fact that $(A + B + C)^2 = A^2 + B^2 + C^2$ is a sum of three commuting variables. □

Theorem 3.4. *Suppose $g \in SL(V_i)$. There exists an $X \in \mathrm{Lie}(SL(V_i))$ such that $g = \exp(X)$ if and only if g is the identity or*

$$g = \begin{pmatrix} \cosh(\rho) + a\rho^{-1}\sinh(\rho) & (l - k)\rho^{-1}\sinh(\rho) \\ (l + k)\rho^{-1}\sinh(\rho) & \cosh(\rho) - a\rho^{-1}\sinh(\rho) \end{pmatrix},$$

for some $a, k, l \in \mathbb{R}$ such that $\rho := \sqrt{a^2 + l^2 - k^2} \neq 0$. In this case we have $X = k(F_i - E_i) + aH_i + l(F_i + E_i)$.

Proof. Any element $X \in \mathfrak{sl}(V_i)$ can be written as $X = k(F_i - E_i) + aH_i + l(F_i + E_i)$ for $a, k, l \in \mathbb{R}$. We will calculate $\exp(X)$ explicitly. Note that $F_i - E_i$, H_i and $F_i + E_i$ anticommute with each other and $(\imath(F_i - E_i))^2 = H_i^2 = (F_i + E_i)^2 = I$. Using Lemma 3.3 we find

$$\begin{aligned}
\exp(X) &= \sum_{j=0}^{\infty} \frac{1}{j!}(k(F_i - E_i) + aH_i + l(F_i + E_i))^j \\
&= \sum_{j=0}^{\infty} \frac{1}{(2j)!}(k(F_i - E_i) + aH_i + l(F_i + E_i))^{2j} \\
&\quad + (k(F_i - E_i) + aH_i + l(F_i + E_i)) \\
&\quad \times \sum_{j=0}^{\infty} \frac{1}{(2j+1)!}(k(F_i - E_i) + aH_i + l(F_i + E_i))^{2j} \\
&= \sum_{j=0}^{\infty} \sum_{u+v+w=j} \frac{k^{2u}a^{2v}l^{2w}j!}{(2j)!u!v!w!}(F_i - E_i)^{2u}H_i^{2v}(F_i + E_i)^{2w} \\
&\quad + (k(F_i - E_i) + aH_i + l(F_i + E_i)) \\
&\quad \times \sum_{j=0}^{\infty} \sum_{u+v+w=j} \frac{k^{2u}a^{2v}l^{2w}j!}{(2j+1)!u!v!w!}(F_i - E_i)^{2u}H_i^{2v}(F_i + E_i)^{2w} \\
&= I \sum_{j=0}^{\infty} \sum_{u+v+w=j} \frac{(\imath k)^{2u}a^{2v}l^{2w}j!}{(2j)!u!v!w!} \\
&\quad + (k(F_i - E_i) + aH_i + l(F_i + E_i)) \sum_{j=0}^{\infty} \sum_{u+v+w=j} \frac{(\imath k)^{2u}a^{2v}l^{2w}j!}{(2j+1)!u!v!w!} \\
&= \cosh(\rho)I + (k(F_i - E_i) + aH_i + l(F_i + E_i))\rho^{-1}\sinh(\rho),
\end{aligned}$$

for $\rho \neq 0$. For $\rho = 0$ this calculation gives us $\exp(X) = I$. □

Note that in particular, we have

$$K_i = \{\exp(k_i(F_i - E_i)) \mid k_i \in \mathbb{R}\}, \qquad A_i = \{\exp(a_i H_i) \mid a_i \in \mathbb{R}\}.$$

This implies that from an explicit representation of $\mathfrak{sl}(V_i)$ we can obtain an explicit action of elements in K_i and A_i when integrated to the group level. Because of the Cartan decomposition this then defines an action of $SL(V_i)$.

Since we can write every element of $SL(V_i)$ as a finite product of exponentials of elements of $\mathfrak{sl}(V_i)$, we obtain the following corollary for $\mathbb{D}(2, 1; \alpha)$.

Corollary 3.5. *Every element of $G_0 = SL(V_1) \times SL(V_2) \times SL(V_3)$ can be written as a finite product of exponentials of elements of $\mathfrak{g}_{\overline{0}}$, i.e., for all $g \in G_0$ we have*

$$g = \prod_{i=1}^{n} \exp(X_i),$$

for some $X_i \in \mathfrak{g}_{\overline{0}}$ and $n \in \mathbb{N}$.

4. The Fock space $\mathcal{F}$

In this section we introduce the notion of a Hilbert superspace as defined in [2]. We also extend the polynomial Fock space F_λ to the Fock space $\mathcal{F}$ and show it is such a Hilbert superspace when combined with the Bessel-Fischer product.

From now on we will restrict ourselves to the case $\alpha \in \mathbb{R} \setminus \mathbb{N}$ since only then the Bessel-Fischer product will be non-degenerate. Furthermore, we also choose $\lambda = \alpha$ and denote the polynomial Fock space F_λ by F. Recall from Sect. 2.3 that the case $\lambda = 1$ is always equivalent to a representation with $\lambda = \alpha$.

4.1. The Bessel-Fischer product. In [1, Section 5], a non-degenerate, sesquilinear, superhermitian form on F was introduced. This product is a generalization of the Bessel-Fischer inner product on the polynomial space $\mathcal{P}(\mathbb{C}^m)$ considered in [23, Section 2.3].

Definition 4.1. For $p, q \in$ F we define the **Bessel-Fischer product** of p and q as

$$\langle p, q\rangle_{\mathcal{B}} := p(\mathcal{B}_\lambda)\bar{q}(z)|_{z=0},$$

where $\bar{q}(z) = \overline{q(\bar{z})}$ is obtained by conjugating the coefficients of the polynomial q and $p(\mathcal{B}_\lambda)$ is obtained by replacing z_i by $\mathcal{B}_\lambda(z_i)$.

From [1, Proposition 5.6.] we obtain the following explicit form of the Bessel-Fischer product.

Proposition 4.2. *Suppose $p, q \in \{z_1^k, z_1^k z_2, z_1^k z_3, z_1^k z_4\}$, with $k \in \mathbb{N}$. Then the only non-zero evaluations of $\langle p, q\rangle_{\mathcal{B}}$ are*

$$\left\langle z_1^k, z_1^k\right\rangle_{\mathcal{B}} = -\left\langle z_1^k z_2, z_1^k z_2\right\rangle_{\mathcal{B}} = k!(-\alpha)_k,$$
$$\left\langle z_1^k z_3, z_1^k z_4\right\rangle_{\mathcal{B}} = -\left\langle z_1^k z_4, z_1^k z_3\right\rangle_{\mathcal{B}} = 2k!(-\alpha)_{k+1},$$

where we used the Pochhammer symbol $(a)_k = a(a+1)(a+2)\cdots(a+k-1)$.

From this explicit form we can easily see that the Bessel-Fischer product is degenerate if and only if $\alpha \in \mathbb{N}$, which is why we assume $\alpha \in \mathbb{R} \setminus \mathbb{N}$.

4.2. Definitions.

Definition 4.3. A **Hermitian superspace** $(\mathcal{H}, \langle\cdot\,,\cdot\rangle)$ is a supervector space $\mathcal{H} = \mathcal{H}_{\bar{0}} \oplus \mathcal{H}_{\bar{1}}$ endowed with a non-degenerate, superhermitian, sesquilinear form $\langle\cdot, \cdot\rangle$. If the inner product is a homogeneous form of degree $\sigma(\mathcal{H}) \in \mathbb{Z}/2\mathbb{Z}$, then $\mathcal{H}$ is called a Hermitian superspace of **parity** $\sigma(\mathcal{H})$.

According to the propositions in [1, Section 5], the polynomial Fock space F endowed with the Bessel-Fischer product $\langle\cdot\,,\cdot\rangle_{\mathcal{B}}$ is such a Hermitian superspace.

Definition 4.4. A **fundamental symmetry** of a Hermitian superspace $(\mathcal{H}, \langle\cdot\,,\cdot\rangle)$ is an endomorphism J of $\mathcal{H}$ such that $J^4 = 1$, $\langle J(x), J(y)\rangle = \langle x, y\rangle$ and $(\cdot\,,\cdot)_J$ defined by

$$(x, y)_J := \langle x, J(y)\rangle,$$

for all $x, y \in \mathcal{H}$ is an inner product on $\mathcal{H}$.

For F we find the following condition on its fundamental symmetries with respect to the Bessel-Fischer product.

Proposition 4.5. *For all fundamental symmetries of* F *we must have*

$$J(z_1^k)_{z_1^k} = \epsilon_{1,k}\,\mathrm{sgn}((-\alpha)_k), \qquad J(z_1^k z_2)_{z_1^k z_2} = -\epsilon_{2,k}\,\mathrm{sgn}((-\alpha)_k),$$
$$J(z_1^k z_3)_{z_1^k z_4} = \epsilon_{3,k}\,\mathrm{sgn}((-\alpha)_{k+1}), \qquad J(z_1^k z_4)_{z_1^k z_3} = -\epsilon_{4,k}\,\mathrm{sgn}((-\alpha)_{k+1}),$$

for all $k \in \mathbb{N}$. Here $J(a)_b$ denotes the coefficient of b in $J(a)$ and $\epsilon_{i,k} > 0$ for all $i \in \{1, 2, 3, 4\}$.

Proof. Suppose J is an arbitrary fundamental symmetry of F, then we have

$$\left(z_1^k, z_1^k\right)_J = \left\langle z_1^k, J(z_1^k)\right\rangle_{\mathcal{B}} = J(z_1^k)_{z_1^k}\left\langle z_1^k, z_1^k\right\rangle_{\mathcal{B}} = J(z_1^k)_{z_1^k}k!(-\alpha)_k > 0,$$

for all $k \in \mathbb{N}$. Therefore, $J(z_1^k)_{z_1^k} = \epsilon\,\mathrm{sgn}((-\alpha)_k)$ for an $\epsilon > 0$. The other three cases are similar. □

Based on this condition, we define the endomorphism S of F by the linear extension of

$$S(z_1^k) := \mathrm{sgn}((-\alpha)_k) z_1^k, \qquad S(z_1^k z_2) := -\,\mathrm{sgn}((-\alpha)_k) z_1^k z_2,$$
$$S(z_1^k z_3) := \mathrm{sgn}((-\alpha)_{k+1}) z_1^k z_4, \qquad S(z_1^k z_4) := -\,\mathrm{sgn}((-\alpha)_{k+1}) z_1^k z_3,$$

for all $k \in \mathbb{N}$. Then, one can easily verify that S is a fundamental symmetry of F with respect to the Bessel-Fischer product.

Proposition 4.6. *Suppose $p, q \in \{z_1^k, z_1^k z_2, z_1^k z_3, z_1^k z_4\}$, with $k \in \mathbb{N}$. Then the only non-zero evaluations of $(p, q)_S$ are*

$$(z_1^k, z_1^k)_S = (z_1^k z_2, z_1^k z_2)_S = k!|(-\alpha)_k|,$$
$$(z_1^k z_3, z_1^k z_3)_S = (z_1^k z_4, z_1^k z_4)_S = 2k!|(-\alpha)_{k+1}|,$$

where we used the Pochhammer symbol $(a)_k = a(a+1)(a+2)\cdots(a+k-1)$.

Proof. This follows immediately from Proposition 4.2. □

Definition 4.7. A Hermitian superspace $(\mathcal{H}, \langle\cdot\,,\cdot\rangle)$ is a **Hilbert superspace** if there exists a fundamental symmetry J such that $(\mathcal{H}, (\cdot\,,\cdot)_J)$ is a Hilbert space.

Note that the choice of a fundamental symmetry does not matter for the topology, thanks to [2, Theorem 3.4].

Denote by $\mathcal{F}$ the completion of F with respect to $(\cdot\,,\cdot)_S$, then $(\mathcal{F}, \langle\cdot\,,\cdot\rangle_{\mathcal{B}})$ is a Hilbert superspace, which we call the Fock space. Define $\|f\|_S := \sqrt{(f, f)_S}$, then we have

$$\mathcal{F} = \left\{ f = f_{1,0} + \sum_{k=1}^{\infty} z_1^{k-1}(f_{1,k}z_1 + f_{2,k}z_2 + f_{3,k}z_3 + f_{4,k}z_4) : \|f\|_S < \infty,\ f_{i,k} \in \mathbb{C} \right\}.$$

The condition $\|f\|_S < \infty$ on f is equivalent to the condition that the sums

$$\sum_{k=0}^{\infty} k!|(-\alpha)_k||f_{1,k}|^2, \qquad \sum_{k=1}^{\infty}(k-1)!|(-\alpha)_{k-1}||f_{2,k}|^2,$$
$$\sum_{k=1}^{\infty}(k-1)!|(-\alpha)_k||f_{3,k}|^2, \qquad \sum_{k=1}^{\infty}(k-1)!|(-\alpha)_k||f_{4,k}|^2$$

converge.

5. The superunitary representation ρ_0

In this section we explicitly integrate the differential action $\mathrm{d}\rho$ of $D(2, 1; \alpha)$ on F to an action ρ_0 of $\mathbb{D}(2, 1; \alpha)$ on $\mathcal{F}$. We also introduce the notion of superunitary representations as defined in [2]. Then, we prove that our action defines a superunitary representation on $\mathcal{F}$ for $\alpha < 0$.

Recall from Sect. 4 that we assume $\alpha \in \mathbb{R} \setminus \mathbb{N}$.

5.1. Definition and explicit form. We define $\rho_0(\exp(X)) := \exp(\mathrm{d}\rho(X))$ for all $X \in \mathfrak{g}_{\bar{0}}$. Because of Corollary 3.5 this defines a representation of all of G_0. We will now describe this representation more explicitly. Note that we omit the action of $A_2(a_2)$ from our explicit representation. This case will be discussed in Sect. 5.2.

Theorem 5.1. *The representation ρ_0 acting on $f = f(z_1, z_2, z_3, z_4) \in \mathcal{F}$ is given by*

$$\rho_0(K_1(k_1))f = f(z_1, z_2, \cos(k_1)z_3 - 2^{-1}\sin(k_1)z_4, 2\sin(k_1)z_3 + \cos(k_1)z_4), \quad (5.1)$$

$$\rho_0(K_2(k_2))f = \exp(\imath\alpha k_2) f(\exp(-2\imath k_2)z_1, z_2, \exp(-\imath k_2)z_3, \exp(-\imath k_2)z_4), \quad (5.2)$$

$$\rho_0(K_3(k_3))f = \exp(\imath k_3) f(z_1, \exp(-2\imath k_3)z_2, \exp(-\imath k_3)z_3, \exp(-\imath k_3)z_4), \quad (5.3)$$

$$\rho_0(A_1(a_1))f = f(z_1, z_2, \exp(-a_1)z_3, \exp(a_1)z_4), \quad (5.4)$$

$$\begin{aligned}\rho_0(A_3(a_3))f = {} & (\cosh(a_3) + \sinh(a_3)z_2) \\ & \times f(z_1, \tanh(a_3) + \cosh(a_3)^{-2}z_2, \cosh(a_3)^{-1}z_3, \cosh(a_3)^{-1}z_4),\end{aligned} \quad (5.5)$$

Proof.

(5.1) We have

$$\begin{aligned}\rho_0(K_1(k_1)) &= \exp(\mathrm{d}\rho(k_1(F_1 - E_1))) = \exp(k_1(2z_3\partial_{z_4} - \frac{1}{2}z_4\partial_{z_3})) \\ &= \sum_{i=0}^{\infty} \frac{k_1^i}{i!}(2z_3\partial_{z_4} - \frac{1}{2}z_4\partial_{z_3})^i,\end{aligned}$$

with

$$\begin{aligned}(2z_3\partial_{z_4} - \frac{1}{2}z_4\partial_{z_3})^2 &= -(z_3\partial_{z_3} + z_4\partial_{z_4}), \\ (2z_3\partial_{z_4} - \frac{1}{2}z_4\partial_{z_3})^3 &= -(2z_3\partial_{z_4} - \frac{1}{2}z_4\partial_{z_3}),\end{aligned}$$

and therefore

$$\begin{aligned}\rho_0(K_1(k_1)) &= 1 - (z_3\partial_{z_3} + z_4\partial_{z_4}) + \sum_{i=0}^{\infty}(-1)^i \frac{k_1^{2i}}{(2i)!}(z_3\partial_{z_3} + z_4\partial_{z_4}) \\ &\quad + \sum_{i=0}^{\infty}(-1)^i \frac{k_1^{2i+1}}{(2i+1)!}(2z_3\partial_{z_4} - \frac{1}{2}z_4\partial_{z_3}) \\ &= 1 - z_3\partial_{z_3} - z_4\partial_{z_4} + \cos(k_1)(z_3\partial_{z_3} + z_4\partial_{z_4}) \\ &\quad + \sin(k_1)(2z_3\partial_{z_4} - \frac{1}{2}z_4\partial_{z_3}).\end{aligned}$$

This gives us

$$\rho_0(K_1(k_1))f(z_1,z_2,z_3,z_4) = f(z_1,z_2,\begin{pmatrix}\cos(k_1) & -2^{-1}\sin(k_1)\\ 2\sin(k_1) & \cos(k_1)\end{pmatrix}\begin{pmatrix}z_3\\ z_4\end{pmatrix}).$$

(5.2) and (5.3) We have

$$\begin{aligned}\rho_0(K_2(k_2)) &= \exp(\mathrm{d}\rho(k_2(F_2-E_2)))\\ &= \exp(\alpha\imath k_2 - 2\imath k_2 z_1\partial_{z_1} - \imath k_2 z_3\partial_{z_3} - \imath k_2 z_4\partial_{z_4})\\ &= \exp(\alpha\imath k_2)\exp(-2\imath k_2 z_1\partial_{z_1})\exp(-\imath k_2 z_3\partial_{z_3})\exp(-\imath k_2 z_4\partial_{z_4}),\end{aligned}$$

and

$$\begin{aligned}\rho_0(K_3(k_3)) &= \exp(\mathrm{d}\rho(k_3(F_3-E_3)))\\ &= \exp(\imath k_3 - 2\imath k_3 z_2\partial_{z_2} - \imath k_3 z_3\partial_{z_3} - \imath k_3 z_4\partial_{z_4})\\ &= \exp(\imath k_3)\exp(-2\imath k_3 z_2\partial_{z_2})\exp(-\imath k_3 z_3\partial_{z_3})\exp(-\imath k_3 z_4\partial_{z_4}).\end{aligned}$$

Since $\exp(az_i\partial_{z_i})f(z_i) = f(\exp(a)z_i)$ for all $a \in \mathbb{C}$ we get

$$\rho_0(K_2(k_2))f(z_1,z_2,z_3,z_4) = e^{\imath\alpha k_2}f(e^{-2\imath k_2}z_1, z_2, e^{-\imath k_2}z_3, e^{-\imath k_2}z_4),$$

and

$$\rho_0(K_3(k_3))f(z_1,z_2,z_3,z_4) = e^{\imath k_3}f(z_1, e^{-2\imath k_3}z_2, e^{-\imath k_3}z_3, e^{-\imath k_3}z_4),$$

respectively.
(5.4) We have

$$\begin{aligned}\rho_0(A_1(a_1))f &= \exp(a_1\mathrm{d}\rho(H_1))f = \exp(a_1(z_4\partial_{z_4} - z_3\partial_{z_3}))f\\ &= f(z_1,z_2,\exp(-a_1)z_3,\exp(a_1)z_4)\end{aligned}$$

(5.5) We have

$$\rho_0(A_3(a_3))f = \exp(a_3\mathrm{d}\rho(H_3))f = \exp(a_3(z_2+\partial_{z_2}))f = \sum_{i=0}^{\infty}\frac{a_3^i}{i!}(z_2+\partial_{z_2})^i f$$

with

$$\begin{aligned}(z_2+\partial_{z_2})^{2i}f &= \sum_{k=0}^{\infty} z_1^{k-1}(f_{1,k}z_1 + f_{2,k}z_2),\\ (z_2+\partial_{z_2})^{2i-1}f &= \sum_{k=0}^{\infty} z_1^k f_{2,k+1} + \sum_{k=0}^{\infty} z_1^{k-1}z_2 f_{1,k-1}\\ &= \sum_{k=0}^{\infty} z_1^{k-1}(f_{2,k+1}z_1 + f_{1,k-1}z_2),\end{aligned}$$

for $i \geq 1$. Therefore

$$\begin{aligned}
\rho_0(A_3(a_3))f &= f + (\cosh(a_3) - 1)\sum_{k=0}^{\infty} z_1^{k-1}(f_{1,k}z_1 + f_{2,k}z_2) \\
&\quad + \sinh(a_3)\sum_{k=0}^{\infty} z_1^{k-1}(f_{2,k+1}z_1 + f_{1,k-1}z_2) \\
&= \sum_{k=0}^{\infty} z_1^{k-1}((\cosh(a_3)f_{1,k} + \sinh(a_3)f_{2,k+1})z_1 \\
&\quad + (\cosh(a_3)f_{2,k} + \sinh(a_3)f_{1,k-1})z_2 + f_{3,k}z_3 + f_{4,k}z_4) \\
&= (\cosh(a_3) + \sinh(a_3)z_2) \\
&\quad \times f(z_1, \tanh(a_3) + \cosh(a_3)^{-2}z_2, \cosh(a_3)^{-1}z_3, \cosh(a_3)^{-1}z_4)
\end{aligned}$$

□

We have two alternative ways to present this representation. The first one is as follows. Suppose $f \in \mathcal{F}$ and define

$$f_1(z_1) := \sum_{k=0}^{\infty} z_1^k f_{i,k} \quad \text{and} \quad f_i(z_1) := \sum_{k=0}^{\infty} z_1^k f_{i,k+1},$$

for $i \in \{2, 3, 4\}$. Then we have $f = f_1(z_1) + f_2(z_1)z_2 + f_3(z_1)z_3 + f_4(z_1)z_4$ and we can view f as the vector

$$f = \begin{pmatrix} f_1(z_1) \\ f_2(z_1) \\ f_3(z_1) \\ f_4(z_1) \end{pmatrix}.$$

The representation ρ_0 on $\mathcal{F}$ can now be given by matrices acting on $f \in \mathcal{F}$.

Corollary 5.2. *The representation ρ_0 acting on $f \in \mathcal{F}$ given by*

$$\rho_0(K_1(k_1))f = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(k_1) & 2\sin(k_1) \\ 0 & 0 & -2^{-1}\sin(k_1) & \cos(k_1) \end{pmatrix} f,$$

$$\rho_0(K_2(k_2))f = \begin{pmatrix} e^{\imath k_2(\alpha-2\mathbb{E})} & 0 & 0 & 0 \\ 0 & e^{\imath k_2(\alpha-2\mathbb{E})} & 0 & 0 \\ 0 & 0 & e^{\imath k_2(\alpha-1-2\mathbb{E})} & 0 \\ 0 & 0 & 0 & e^{\imath k_2(\alpha-1-2\mathbb{E})} \end{pmatrix} f,$$

$$\rho_0(K_3(k_3))f = \begin{pmatrix} e^{\imath k_3} & 0 & 0 & 0 \\ 0 & e^{-\imath k_3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} f,$$

$$\rho_0(A_1(a_1))f = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{-a_1} & 0 \\ 0 & 0 & 0 & e^{a_1} \end{pmatrix} f,$$

$$\rho_0(A_3(a_3))f = \begin{pmatrix} \cosh(a_3) & \sinh(a_3) & 0 & 0 \\ \sinh(a_3) & \cosh(a_3) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} f,$$

where $\mathbb{E} := z_1\partial_{z_1}$ *denotes the Euler operator on* $f_i(z_1)$, $i \in \{1, 2, 3, 4\}$.

The second method is as follows. Denote by $\mathcal{P}_k(\mathbb{C}^{m|n})$ the space of homogeneous superpolynomials of degree k in m even variables and n odd variables. Then

$$\begin{aligned} \phi : \quad & F_\lambda \quad \to \mathcal{P}_{\text{even}}(\mathbb{C}^{1|2}) := \bigoplus_{k=0}^{\infty} \mathcal{P}_{2k}(\mathbb{C}^{1|2}) \\ & (z_1, z_2, z_3, z_4) \mapsto \quad (2^{-1}\ell_1^2, \ell_2\ell_3, \ell_1\ell_3, \ell_1\ell_2), \end{aligned}$$

defines an isomorphism between F_λ and the space of even degree superpolynomials in the even variable ℓ_1 and the two odd variables ℓ_2, ℓ_3. Here the "even" in $\mathcal{P}_{\text{even}}(\mathbb{C}^{1|2})$ refers to the degree and not the parity of the superpolynomial terms.

The representation ρ_0 on $\mathcal{F}$ can now be given as an action on $f(\ell_1, \ell_2, \ell_3) \in \phi(\mathcal{F})$.

Corollary 5.3. *The representation* ρ_0 *acting on* $f = f(\ell_1, \ell_2, \ell_3) \in \phi(\mathcal{F})$ *is given by*

$$\begin{aligned} \rho_0(K_1(k_1))f &= f(\ell_1, \cos(k_1)\ell_2 + 2\sin(k_1)\ell_3, -2^{-1}\sin(k_1)\ell_2 + \cos(k_1)\ell_3), \\ \rho_0(K_2(k_2))f &= \exp(\imath\alpha k_2)f(\exp(-\imath k_2)\ell_1, \ell_2, \ell_3), \\ \rho_0(K_3(k_3))f &= \exp(\imath k_3)f(\ell_1, \exp(-\imath k_3)\ell_3, \exp(-\imath k_3)\ell_4), \\ \rho_0(A_1(a_1))f &= f(\ell_1, \exp(a_1)\ell_2, \exp(-a_1)\ell_3), \\ \rho_0(A_3(a_3))f &= (\cosh(a_3) + \sinh(a_3)\ell_2\ell_3)f(\ell_1, \cosh(a_3)^{-1}\ell_2, \cosh(a_3)^{-1}\ell_3) \\ &\quad + \sinh(a_3)(f(\ell_1, 1, 1) - f(\ell_1, 1, 0) - f(\ell_1, 0, 1) + f(\ell_1, 0, 0) \\ &\quad + \tanh(a_3)(f(\ell_1, \ell_2, \ell_3) - f(\ell_1, \ell_2, 0) - f(\ell_1, 0, \ell_3) + f(\ell_1, 0, 0))). \end{aligned}$$

Note that the symbolic change of odd variables ℓ_2 *and* ℓ_3 *to the constant* 1 *is only well defined if we use the convention that every instance of* $\ell_3\ell_2$ *in* f *is first rewritten as* $-\ell_2\ell_3$.

5.2. The action of $A_2(a_2)$. For the action $\rho_0(A_2(a_2))$ we were unable to find an explicit form if $\alpha < 0$.

For $\alpha > 0$ we can write $\mathcal{F}$ in terms of a Generalised Laguerre polynomial basis,

$$\begin{aligned} \mathcal{F} = \Bigg\{ g = \exp(-z_1) \Bigg(& \sum_{k=0}^{\infty} (g_{1,k} + g_{2,k}z_2) L_k^{(-1-\alpha)}(2z_1) \\ & + \sum_{k=1}^{\infty} (g_{3,k}z_3 + g_{4,k}z_4) L_{k-1}^{(-\alpha)}(2z_1) \Bigg) : \|g\|_S < \infty,\ g_{i,k} \in \mathbb{C} \Bigg\}, \end{aligned}$$

Here

$$L_k^{(a)}(2x) = \frac{(-1)^k}{k!} \mathrm{U}(-k, a+1, 2x) = \frac{(-1)^k}{k!} \sum_{i=0}^{k} \frac{(-1)^i}{i!} (-a-k)_i (-k)_i (2x)^{k-i}$$

are the generalised Laguerre polynomials and $\mathrm{U}(a, b, c)$ is the confluent hypergeometric function of the second kind. Note that this does not define a basis of $\mathcal{F}$ if $\alpha < 0$, since then $\|\exp(-z_1)\|_S \not< \infty$. We can now give the actions of $A_2(a_2)$ with respect to this basis.

Proposition 5.4. *For* $\alpha > 0$ *we have*

$$\rho_0(A_2(a_2))g(z) = \exp(-z_1)\left(\sum_{k=0}^{\infty}\exp(a_2(2k-\alpha))(g_{1,k}+g_{2,k}z_2)L_k^{(-1-\alpha)}(2z_1)) + \sum_{k=1}^{\infty}\exp(a_2(2k-\alpha-1))(g_{3,k}z_3+g_{4,k}z_4)L_{k-1}^{(-\alpha)}(2z_1))\right).$$

Proof. We have

$$\begin{aligned}\rho_0(A_2(a_2)) &= \exp(a_2 \mathrm{d}\rho(H_2)) = \exp(a_2(z_1 - \mathcal{B}_\lambda(z_1)))\\ &= \exp(a_2(z_1 + (\alpha - z_1\partial_{z_1} - z_3\partial_{z_3} - z_4\partial_{z_4})\partial_{z_1}))\\ &= \sum_{i=0}^{\infty}\frac{a_2^i}{i!}D^i,\end{aligned}$$

with

$$D = z_1 + (\alpha - z_1\partial_{z_1} - z_3\partial_{z_3} - z_4\partial_{z_4})\partial_{z_1}.$$

Since

$$D(\exp(-z_1)L_k^{(-1-\alpha)}(2z_1)) = (2k-\alpha)\exp(-z_1)L_k^{(-1-\alpha)}(2z_1)$$

and

$$D(\exp(-z_1)z_jL_{k-1}^{(-\alpha)}(2z_1)) = (2k-\alpha-1)\exp(-z_1)z_jL_{k-1}^{(-\alpha)}(2z_1),$$

for $j \in \{3, 4\}$, we obtain the desired result. □

Despite not having an explicit form $\alpha < 0$, we can show that this action is unitary if and only if $\alpha < 0$.

Proposition 5.5. *The action* $\rho_0(A_2(a_2))$ *is a unitary operator on* $(\mathcal{F}, \langle\cdot\,,\cdot\rangle_{\mathcal{B}})$ *for all* $a_2 \in \mathbb{R}$ *if and only if* $\alpha < 0$.

Proof. First assume $\alpha > 0$. From Proposition 5.4 we see that the eigenvalues of $\rho_0(A_2(a_2))$ are of the form $\exp(a)$, with $a \in \mathbb{R}$. Since these eigenvalues are not roots of unity, $\rho_0(A_2(a_2))$ can not be a unitary operator on $(\mathcal{F}, \langle\cdot\,,\cdot\rangle_{\mathcal{B}})$.

Now assume $\alpha < 0$. In this case, we can easily see that the Fundamental symmetry S commutes with $\rho_0(A_2(a_2))$. Because of [1, Proposition 6.3] we have

$$\langle \mathrm{d}\rho(H_2)p, q\rangle_{\mathcal{B}} = -\langle p, \mathrm{d}\rho(H_2)q\rangle_{\mathcal{B}},$$

for $p, q \in \mathrm{F}$. This implies

$$\begin{aligned}(\rho_0(A_2(a_2))p, q)_S &= \langle\rho_0(A_2(a_2))p, S(q)\rangle_{\mathcal{B}} = \langle\exp(a_2\mathrm{d}\rho(H_2))p, S(q)\rangle_{\mathcal{B}}\\ &= \langle p, \exp(-a_2\mathrm{d}\rho(H_2))S(q)\rangle_{\mathcal{B}} = \langle p, \rho_0(A_2(-a_2))S(q)\rangle_{\mathcal{B}}\\ &= \langle p, S(\rho_0(A_2(-a_2))q)\rangle_{\mathcal{B}} = (p, \rho_0(A_2(-a_2))q)_S,\end{aligned}$$

for $p, q \in \mathrm{F}$, i.e., $\rho_0(A_2(a_2))$ acts as a unitary operator when acting on F. Since F is dense in $\mathcal{F}$, we are finished. □

5.3. Superunitary representations. The following definitions can be found in [2].

Definition 5.6. Let $(\mathcal{H}_1, \langle \cdot\,, \cdot \rangle_1)$ and $(\mathcal{H}_2, \langle \cdot\,, \cdot \rangle_2)$ be Hilbert superspaces and suppose $T : \mathcal{H}_1 \to \mathcal{H}_2$ is a linear operator. We call T a **bounded operator** between $\mathcal{H}_1$ and $\mathcal{H}_2$ if it is continuous with respect to their Hilbert topologies. The set of bounded operators is denoted by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathcal{B}(\mathcal{H}_1) := \mathcal{B}(\mathcal{H}_1, \mathcal{H}_1)$.

Definition 5.7. Let $(\mathcal{H}_1, \langle \cdot\,, \cdot \rangle_1)$ and $(\mathcal{H}_2, \langle \cdot\,, \cdot \rangle_2)$ be Hilbert superspaces and suppose $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. The **superadjoint** of T is the operator $T^\dagger \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ such that

$$\left\langle T^\dagger(x), y \right\rangle_1 = (-1)^{|T||x|} \langle x, T(y) \rangle_2 ,$$

for all $x \in \mathcal{H}_2$, $y \in \mathcal{H}_1$.

Definition 5.8. Let $(\mathcal{H}_1, \langle \cdot\,, \cdot \rangle_1)$ and $(\mathcal{H}_2, \langle \cdot\,, \cdot \rangle_2)$. A **superunitary operator** between $\mathcal{H}_1$ and $\mathcal{H}_2$ is a homogeneous operator $\psi \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ of degree 0 satisfying $\psi^\dagger \psi = \psi \psi^\dagger = \mathbb{1}$. The set of superunitary operators is denoted by $\mathcal{U}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathcal{U}(\mathcal{H}_1) := \mathcal{U}(\mathcal{H}_1, \mathcal{H}_1)$.

Definition 5.9. A **superunitary representation** of a Lie supergroup $G = (G_0, \mathfrak{g})$ is a triple $(\mathcal{H}, \pi_0, d\pi)$ such that

- $\mathcal{H}$ is a Hilbert superspace.
- $\pi_0 : G_0 \to \mathcal{U}(\mathcal{H})$ is a group morphism.
- For all $v \in \mathcal{H}$, the maps $\pi_0^v : g \mapsto \pi_0(g)v$ are continuous on G_0.
- $d\pi : \mathfrak{g} \to \mathrm{End}(\mathcal{H}^\infty)$ is a $\mathbb{R}$-Lie superalgebra morphism such that $d\pi = d\pi_0$ on $\mathfrak{g}_{\bar{0}}$, $d\pi$ is skew-supersymmetric with respect to $\langle \cdot\,, \cdot \rangle$ and

$$\pi_0(g) d\pi(X) \pi_0(g)^{-1} = d\pi(\mathrm{Ad}(g)(X)), \quad \text{for all } g \in G_0 \text{ and } X \in \mathfrak{g}_{\bar{1}}.$$

  Here $\mathcal{H}^\infty$ is the space of smooth vectors of the representation π_0 and Ad is the adjoint representation of G_0 on $\mathfrak{g}$.

Using this definition of a superunitary representation we can now prove the following result.

Theorem 5.10. *Assume $\alpha < 0$. The triple $((\mathcal{F}, \langle \cdot\,, \cdot \rangle_{\mathcal{B}}), \rho_0, \mathrm{d}\rho)$ is a superunitary representation of* $\mathbb{D}(2, 1; \alpha)$.

Proof. Thanks to Corollary 3.5, we only need to consider the representation ρ_0 on elements of the form $g = \exp(X_1) \cdots \exp(X_n) \in G_0$, with $X_i \in \mathfrak{g}_{\bar{0}}$ and $n \in \mathbb{N}$. We now prove the different conditions of Definition 5.9.

- $(\mathcal{F}, \langle \cdot, \cdot \rangle_{\mathcal{B}})$ is a Hilbert superspace:

This follows from the definitions.

- $\rho_0 : G_0 \to \mathcal{U}(\mathcal{F})$ is a group morphism:

We wish to prove that $\rho_0(\exp(X_1) \cdots \exp(X_n))$ is a superunitary operator of $\mathcal{F}$. Because of [1, Proposition 6.3] we have

$$\langle \rho_0(\exp(X)) p, q \rangle_{\mathcal{B}} = (-1)^{|X||p|} \langle p, \rho_0(\exp(-X)) q \rangle_{\mathcal{B}} ,$$

which implies that the superadjoint of $\rho_0(\exp(X_1) \cdots \exp(X_n))$ is given by $\rho_0(\exp(-X_n) \cdots \exp(-X_1))$ and therefore it is a superunitary operator of $\mathcal{F}$.

- For all $f \in \mathcal{F}$, the maps $\rho_0^f : g \mapsto \rho_0(g)f$ are continuous on G_0:
We need to prove the following

$$(\forall g \in G_0)(\forall \epsilon > 0)(\exists U \text{ neighborhood of } g)(h \in U \implies ||\rho_0^f(g) - \rho_0^f(h)||_S < \epsilon).$$

Since

$$U_r := \left\{ \prod_{i=1}^{3} K_i(k_i) A_i(a_i) K_i(k_i') g : \sum_{i=1}^{3} |k_i| + |a_i| + |k_i'| < r \right\}$$

is a neighbourhood of g for all $r > 0$, it suffices to prove

$$(\forall g \in SL(V_i))(\forall \epsilon > 0)(\exists \delta > 0)(||\rho_0(X_i(\delta))f - f||_S < \epsilon),$$

for $i \in \{1, 2, 3\}$ and $X_i \in \{K_i, A_i\}$.
For A_2 we know from Proposition 5.5 that the actions is unitary if $\alpha < 0$. Since unitarity implies continuity, we are done.
For K_3 we have

$$\begin{aligned} \rho_0(K_3(\delta))f &= e^{\imath\delta} \sum_{k=0}^{\infty} z_1^{k-1}(f_{1,k}z_1 + e^{-2\imath\delta} f_{2,k}z_2 + e^{-\imath\delta}(f_{3,k}z_3 + f_{4,k}z_4)) \\ &= \sum_{k=0}^{\infty} z_1^{k-1}(e^{\imath\delta} f_{1,k}z_1 + e^{-\imath\delta} f_{2,k}z_2 + f_{3,k}z_3 + f_{4,k}z_4) \end{aligned}$$

and therefore

$$\begin{aligned} \|\rho_0(K_3(\delta))f - f\|_S^2 &= \left\| \sum_{k=0}^{\infty} z_1^{k-1}((e^{\imath\delta} - 1) f_{1,k}z_1 + (e^{-\imath\delta} - 1) f_{2,k}z_2) \right\|_S^2 \\ &= (2 - e^{\imath\delta} - e^{-\imath\delta}) \\ &\quad \times \sum_{k=0}^{\infty} k!|(-\alpha)_k||f_{1,k}|^2 + (k-1)!|(-\alpha)_{k-1}||f_{2,k}|^2, \end{aligned}$$

which goes to 0 as δ goes to 0.
For A_3 we have

$$\begin{aligned} \rho_0(A_3(\delta))f - f = \sum_{k=0}^{\infty} z_1^{k-1}&(((\cosh(\delta) - 1) f_{1,k} + \sinh(\delta) f_{2,k+1})z_1 \\ &+ ((\cosh(\delta) - 1) f_{2,k} + \sinh(\delta) f_{1,k-1})z_2) \end{aligned}$$

and therefore

$$\begin{aligned}\|\rho_0(A_3(\delta))f - f\|_S^2 &= \sum_{k=0}^{\infty} |(\cosh(\delta)-1)f_{1,k} + \sinh(\delta)f_{2,k+1}|^2|(-\alpha)_k|k! \\ &\quad + |(\cosh(\delta)-1)f_{2,k} + \sinh(\delta)f_{1,k-1}|^2|(-\alpha)_{k-1}|(k-1)! \\ &\leq 2(\cosh(\delta)-1)^2 \sum_{k=0}^{\infty} |f_{1,k}|^2|(-\alpha)_k|k! \\ &\quad + 2\sinh(\delta)^2 \sum_{k=0}^{\infty} |f_{2,k+1}|^2|(-\alpha)_k|k! \\ &\quad + 2(\cosh(\delta)-1)^2 \sum_{k=1}^{\infty} |f_{2,k}|^2|(-\alpha)_{k-1}|(k-1)! \\ &\quad + 2\sinh(\delta)^2 \sum_{k=1}^{\infty} |f_{1,k-1}|^2|(-\alpha)_{k-1}|(k-1)!\end{aligned}$$

which goes to 0 as δ goes to 0.
For K_2 we have

$$\begin{aligned}\rho_0(K_2(\delta))f - f &= \sum_{k=0}^{\infty}(e^{\imath\delta(\alpha-2k)}-1)(z_1^k f_{1,k} + z_2 z_1^k f_{2,k+1}) \\ &\quad + \sum_{k=0}^{\infty}(e^{\imath\delta(\alpha-2k-1)}-1)(z_3 z_1^k f_{3,k+1} + z_4 z_1^k f_{4,k+1})\end{aligned}$$

and therefore

$$\begin{aligned}&\|\rho_0(K_2(\delta))f - f\|_S^2 \\ &\quad = \sum_{k=0}^{\infty} 2(1-\cos(\delta(\alpha-2k)))(|f_{1,k}|^2 + |f_{2,k+1}|^2)k!|(-\alpha)_k| \\ &\qquad + \sum_{k=0}^{\infty} 4(1-\cos(\delta(\alpha-2k-1)))(|f_{3,k+1}|^2 + |f_{4,k+1}|^2)k!|(-\alpha)_{k+1}| \\ &\quad \leq 4\sum_{k=0}^{\infty}(|f_{1,k}|^2 + |f_{2,k+1}|^2)k!|(-\alpha)_k| + 8\sum_{k=0}^{\infty}(|f_{3,k+1}|^2 + |f_{4,k+1}|^2)k!|(-\alpha)_{k+1}| \\ &\quad = 4\|f\|_S^2\end{aligned}$$

Using Lebesgue's dominated convergence theorem we now find

$$\begin{aligned}&\lim_{\delta\to 0}\|\rho_0(K_2(\delta))f - f\|_S^2 \\ &\quad = \sum_{k=0}^{\infty} 2\lim_{\delta\to 0}(1-\cos(\delta(\alpha-2k)))(|f_{1,k}|^2 + |f_{2,k+1}|^2)k!|(-\alpha)_k|\end{aligned}$$

$$+\sum_{k=0}^{\infty} 4 \lim_{\delta\to 0}(1-\cos(\delta(\alpha-2k-1)))(|f_{3,k+1}|^2+|f_{4,k+1}|^2)k!|(-\alpha)_{k+1}|$$
$$= 0,$$

as desired.

Lastly, the K_1 and A_1 cases are analogous to the K_3 and A_3 cases.

- $\mathrm{d}\rho : \mathfrak{g} \to \mathrm{End}(\mathrm{F})$ is a $\mathbb{R}$-Lie superalgebra morphism such that
  - (i) $\mathrm{d}\rho = \mathrm{d}\rho_0$ on $\mathfrak{g}_{\bar{0}}$,
  - (ii) $\mathrm{d}\rho$ is skew-supersymmetric with respect to $\langle \cdot\, , \cdot \rangle$ and
  - (iii) $\rho_0(g)\mathrm{d}\rho(X)\rho_0(g)^{-1} = \mathrm{d}\rho(\mathrm{Ad}(g)(X))$, for all $g \in G_0$ and $X \in \mathfrak{g}_{\bar{1}}$:

Item (i) follows from

$$\mathrm{d}\rho_0(X)p = \frac{d}{dt}\rho_0(\exp(tX))p\bigg|_{t=0} = \frac{d}{dt}\exp(t\mathrm{d}\rho(X))p\bigg|_{t=0} = \mathrm{d}\rho(X)p,$$

for all $p \in \mathrm{F}$ and $X \in \mathfrak{g}_{\bar{0}}$. Item (ii) follows directly from [1, Proposition 6.3] and item (iii) follows from

$$\begin{aligned}\rho_0(\exp(Y))\mathrm{d}\rho(X)\rho_0(\exp(Y))^{-1} &= \rho_0(\exp(Y))\mathrm{d}\rho(X)\rho_0(\exp(-Y))\\ &= \exp(\mathrm{d}\rho(Y))\mathrm{d}\rho(X)\exp(\mathrm{d}\rho(-Y))\\ &= \mathrm{d}\rho(\exp(Y)X\exp(-Y))\\ &= \mathrm{d}\rho(\mathrm{Ad}(\exp(Y))(X)),\end{aligned}$$

for all $X \in \mathfrak{g}_{\bar{1}}$ and $Y \in \mathfrak{g}_{\bar{0}}$.

□

The assumption $\alpha < 0$ is only used to prove the continuity of $\rho_0(A_2(\delta))$. Note that Proposition 5.5 only implies that the actions are not unitary if $\alpha > 0$. It tells us nothing about the continuity in this case. It is possible that Theorem 5.10 holds even without the assumption $\alpha < 0$.

From the discussion in Sect. 2.3, we can at least conclude that for every α there always exists a superunitary representation of $\mathbb{D}(2, 1; \alpha)$. Indeed, if $\alpha > 0$, we can look at the Fock representation of $\mathbb{D}(2, 1; -1-\alpha)$ instead of the Fock representation of $\mathbb{D}(2, 1; \alpha)$.

5.4. Strong superunitary representation. In [2, Section 4.4] the notion of a strong superunitary representation is also defined. However, it is easy to prove that our superunitary representation is not a strong superunitary representation.

Definition 5.11. A **strong superunitary representation** of a Lie supergroup $G = (G_0, \mathfrak{g})$ is a superunitary representation $(\mathcal{H}, \pi_0, d\pi)$ such that

- $(\mathcal{H}, \pi_0)$ is unitarizable,
- $(\mathcal{H}, \pi_0, d\pi)$ admits a restriction to $(D(G_0), D(\mathfrak{g}_\mathbb{R}))$.

Here $D(G_0)$ is the connected Lie subgroup of G_0 with Lie algebra $[(\mathfrak{g}_\mathbb{R})_{\bar{1}}, (\mathfrak{g}_\mathbb{R})_{\bar{1}}]$ and $D(\mathfrak{g}_\mathbb{R}) := [(\mathfrak{g}_\mathbb{R})_{\bar{1}}, (\mathfrak{g}_\mathbb{R})_{\bar{1}}] \oplus (\mathfrak{g}_\mathbb{R})_{\bar{1}}$.

Theorem 5.12. *There does not exist a fundamental symmetry on* F *such that* $((\mathcal{F}, \langle\cdot, \cdot\rangle_\mathcal{B}), \rho_0)$ *is unitarizable. As a consequence* $((\mathcal{F}, \langle \cdot\, , \cdot\rangle_\mathcal{B}), \rho_0, \mathrm{d}\rho)$ *is not a strong superunitary representation.*

Proof. Let J be an arbitrary fundamental symmetry on F. Thanks to Proposition 4.5 we may assume

$$J(z_1^k)_{z_1^k} = \epsilon_{1,k}\operatorname{sgn}((-\alpha)_k), \qquad J(z_1^k z_2)_{z_1^k z_2} = -\epsilon_{2,k}\operatorname{sgn}((-\alpha)_k),$$
$$J(z_1^k z_3)_{z_1^k z_4} = \epsilon_{3,k}\operatorname{sgn}((-\alpha)_{k+1}), \qquad J(z_1^k z_4)_{z_1^k z_3} = -\epsilon_{4,k}\operatorname{sgn}((-\alpha)_{k+1}),$$

with $\epsilon_{i,k} > 0$ for all $k \in \mathbb{N}$ and $i \in \{1, 2, 3, 4\}$. Suppose $((\mathcal{F}, \langle\cdot,\cdot\rangle_{\mathcal{B}}), \rho_0)$ is unitarizable, then the inner product on F should be invariant under the derived action of ρ_0, i.e.,

$$(d\rho(X)p, q)_J = -(p, d\rho(X)q)_J \tag{5.6}$$

for all $X \in \mathbb{D}(2, 1; \alpha)_{\overline{0}}$, $p, q \in$ F. Set $X = E_3 + F_3$, $p = z_2$ and $q = 1$. Then $d\rho(X) = -\imath(z_2 - \partial_{z_2})$ and the left hand side of equation (5.6) becomes

$$(d\rho(X)p, q)_J = \langle \imath, J(1)\rangle_{\mathcal{B}} = \imath\epsilon_{1,0}$$

while the right hand side becomes

$$-(p, d\rho(X)q)_J = -\langle z_2, -\imath J(z_2)\rangle_{\mathcal{B}} = -(\overline{-\imath})\epsilon_{2,0} = -\imath\epsilon_{2,0},$$

which implies equation (5.6) holds only if

$$\epsilon_{1,0} + \epsilon_{2,0} = 0.$$

Since both $\epsilon_{1,0}$ and $\epsilon_{2,0}$ are greater than zero, this gives us a contradiction. □

5.5. Harish-Chandra supermodules. We will end this paper by giving an alternative, non-explicit, way to integrate the algebra representation of $D(2, 1; \alpha)$ to group level. We do this by using the framework of Harish-Chandra supermodules developed in [24]. It would be interesting to know if this abstract integration gives the same representation as the explicit integration of Theorem 5.1, but we were unable to verify this.

Definition 5.13. [24, Definition 4.1] Let V be a complex super-vector space, $G = (G_0, \mathfrak{g})$ a Lie supergroup and K a maximal compact subgroup of G_0. Then V is a $(\mathfrak{g}, K)$-**module** if it is a locally finite K-representation that has also a compatible $\mathfrak{g}$-module structure, that is, the derived action of K agrees with the Lie(K)-module structure:

$$d\pi_0(X)(v) = \frac{d}{dt}\pi_0(\exp(tX))(v)\Big|_{t=0} = d\pi(X)(v) \text{ for all } X \in \mathrm{Lie}(K),\ v \in V$$

and

$$\pi_0(k)(d\pi(X)(v)) = d\pi(\mathrm{Ad}(k)(X))(\pi_0(k)(v)), \text{ for all } k \in K, X \in \mathfrak{g}, v \in V,$$

where π_0 is the K-representation and $d\pi$ the $\mathfrak{g}$-representation. A $(\mathfrak{g}, K)$-module is a **Harish–Chandra supermodule** if it is finitely generated over $U(\mathfrak{g})$ and is K-multiplicity finite.

The maximal compact subgroup of $SL(V_i)$ is K_i. The maximal compact subgroup of G_0 is therefore the 3-Torus $K := K_1 \times K_2 \times K_3$.

Proposition 5.14. *The module* F *is a Harish–Chandra supermodule.*

Proof. That F is a $(\mathfrak{g}, K)$-module follows from

$$\mathrm{d}\rho_0(X)(p) = \frac{d}{dt}\rho_0(\exp(tX))(p)\Big|_{t=0} = \frac{d}{dt}\exp(t\mathrm{d}\rho(X))(p)\Big|_{t=0} = \mathrm{d}\rho(X)p,$$

and

$$\begin{aligned}\rho_0(\exp(Y))(\mathrm{d}\rho(X)p) &= \rho_0(\exp(Y))\mathrm{d}\rho(X)\rho_0(\exp(-Y))\rho_0(\exp(Y))p\\ &= (\exp(\mathrm{d}\rho(Y))\mathrm{d}\rho(X)\exp(-\mathrm{d}\rho(Y)))\rho_0(\exp(Y))p\\ &= \mathrm{d}\rho((\exp(Y)X\exp(-Y))\rho_0(\exp(Y))p\\ &= \mathrm{d}\rho((\mathrm{Ad}(\exp(Y))X)\rho_0(\exp(Y))p,\end{aligned}$$

for all $p \in$ F and $X, Y \in \mathfrak{g}$. From the decomposition in [1, Theorem 6.4] it immediately follows that F is locally K-finite. Using Proposition 5.1 we also see that F is also K-multiplicity finite. □

Corollary 5.15. *The* $(\mathfrak{g}, K)$*-module* F *integrates to a unique smooth Fréchet representation of moderate growth for the Lie supergroup* $\mathbb{D}(2, 1; \alpha)$.

Proof. This follows immediately from [24, Theorem 4.6]. □

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