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Correction

Correction: A Convex Analysis Approach to Entropy Functions, Variational Principles and Equilibrium States

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Abstract: In Biś et al. (Commun Math Phys 394:215–256, 2022) it was stated that the entropy-like map provided by the variational principle established in Biś et al. (2022, Theorem 1) is always affine. In this note we present an example which shows that this claim is incorrect, and establish necessary and sufficient conditions for the affinity to hold.

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1. Introduction

Let (X, d) be a metric space and $\mathfrak{B}$ its σ-algebra of Borel subsets of X. Denote by $\mathbf{B}$ a Banach space over the field $\mathbb{R}$ equal to either

$$\begin{aligned} B_m(X) &= \{\varphi : X \to \mathbb{R} \mid \varphi \text{ is Borel measurable and bounded}\} \\ \text{or} \quad C_b(X) &= \{\varphi \in B_m(X) \mid \varphi \text{ is continuous}\} \\ \text{or else} \quad C_c(X) &= \{\varphi \in C_b(X) \mid \varphi \text{ has compact support}\} \end{aligned}$$

endowed with the norm $\|\varphi\|_\infty = \sup_{x \in X} |\varphi(x)|$. In what follows, $\mathcal{P}_a(X)$ stands for the set of Borel finitely additive probability measures endowed with the total variation distance, $\mathcal{P}(X)$ denotes the set of Borel σ-additive probability measures on X with the weak* topology and $C(X)$ is the space of real valued continuous maps whose domain is X.

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Definition 1.1. A function $\Gamma\colon \mathbf{B} \to \mathbb{R}$ is called a *pressure function* if it satisfies the following conditions:

(C_1) *Monotonicity*: $\varphi \leqslant \psi \;\Rightarrow\; \Gamma(\varphi) \leqslant \Gamma(\psi) \quad \forall\, \varphi,\, \psi \in \mathbf{B}$.
(C_2) *Translation invariance*: $\Gamma(\varphi + c) = \Gamma(\varphi) + c \quad \forall\, \varphi \in \mathbf{B} \quad \forall\, c \in \mathbb{R}$.
(C_3) *Convexity*: $\Gamma(t\,\varphi + (1-t)\,\psi) \leqslant t\,\Gamma(\varphi) + (1-t)\,\Gamma(\psi) \quad \forall\, \varphi,\, \psi \in \mathbf{B} \quad \forall\, t \in [0, 1]$.

The first result of [4] established the following abstract variational principle for pressure functions.

Theorem 1. [4] Let $\Gamma : \mathbf{B} \to \mathbb{R}$ be a pressure function. Then

$$\Gamma(\varphi) \;=\; \max_{\mu \in \mathcal{P}_a(X)} \left\{\mathfrak{h}(\mu) + \int \varphi\, d\mu\right\} \qquad \forall\, \varphi \in \mathbf{B} \tag{1.2}$$

where, for every $\mu \in \mathcal{P}_a(X)$,

$$\mathfrak{h}(\mu) \;=\; \inf_{\varphi \in \mathcal{A}_\Gamma} \left\{\int \varphi\, d\mu\right\} \qquad \text{and} \qquad \mathcal{A}_\Gamma \;=\; \big\{\varphi \in \mathbf{B} :\, \Gamma(-\varphi) \leqslant 0\big\}. \tag{1.3}$$

The map $\mathfrak{h}$ is concave, upper semi-continuous and

$$\mathfrak{h}(\mu) \;=\; \inf_{\varphi \in \mathbf{B}} \left\{\Gamma(\varphi) - \int \varphi\, d\mu\right\} \qquad \forall\, \mu \in \mathcal{P}_a(X).$$

Moreover, if $\alpha\colon \mathcal{P}_a(X) \to \mathbb{R} \cup \{-\infty\}$ is another function taking the role of $\mathfrak{h}$ in (1.2), then $\alpha \leqslant \mathfrak{h}$. If, in addition, X is locally compact and $\mathbf{B} = C_c(X)$, then the maximum referred to in (1.2) is attained in $\mathcal{P}(X)$.

It is immediate from (1.2) that $\mathfrak{h}$ is upper bounded by $\Gamma(0)$. Since the pointwise infimum of concave functions is concave, and affine maps are themselves concave, we get from (1.3) that the map $\mathfrak{h}$ is concave. In [4, Theorem 1] we asserted incorrectly that $\mathfrak{h}$ is affine. The following example shows that this is not true in general.

Example 1.4. Consider a two point compact metric space $X = \{a, b\}$ with the discrete metric. The space $\mathcal{P}(X)$ of Borel probability measures on X is precisely the set $\big\{t\delta_a + (1-t)\delta_b :\, t \in [0, 1]\big\}$ of convex combinations of the Dirac measures δ_a and δ_b supported on $\{a\}$ and $\{b\}$, respectively. Let $\Gamma\colon C(X) \to \mathbb{R}$ be given by

$$\varphi \in C(X) \quad \mapsto \quad \log\left(e^{\varphi(a)} + e^{\varphi(b)}\right).$$

This is a pressure function on the Banach space $C(X)$ with the norm $\|\varphi\|_\infty = \max_{x \in X} |\varphi(x)|$. Actually, the formula for Γ gives the topological pressure of the potentials on the shift space $\{0, 1\}^{\mathbb{N}}$ that are locally constant on cylinders of size 1 (cf. [6]). Thus, Theorem 1 provides the variational principle

$$\Gamma(\varphi) \;=\; \max_{\mu \in \mathcal{P}(X)} \left\{\mathfrak{h}(\mu) + \int \varphi\, d\mu\right\}$$

where

$$\mathfrak{h}(\mu) \;=\; \inf_{\psi \in \mathcal{A}_\Gamma} \int \psi\, d\mu \qquad \forall\, \mu \in \mathcal{P}(X) \tag{1.5}$$

and

$$\mathcal{A}_\Gamma = \left\{\psi \in C(X) \colon \Gamma(-\psi) \leqslant 0\right\} = \left\{\psi \in C(X) \colon e^{-\psi(a)} + e^{-\psi(b)} \leqslant 1\right\}. \quad (1.6)$$

Therefore, there is $\mu_0 \in \mathcal{P}(X)$ such that

$$\Gamma(0) = \log 2 = \mathfrak{h}(\mu_0).$$

Yet, a straightforward computation using (1.5) and (1.6) yields

$$\mathfrak{h}(\delta_a) = \inf_{\psi \in \mathcal{A}_\Gamma} \psi(a) = 0 \quad \text{and} \quad \mathfrak{h}(\delta_b) = \inf_{\psi \in \mathcal{A}_\Gamma} \psi(b) = 0.$$

Thus, $\mathfrak{h}$ is not affine.

2. Necessary and Sufficient Conditions for the Affinity of $\mathfrak{h}$

The wrong statement that $\mathfrak{h}$ is always affine is the content of Lemma 3.2 in [4]. Although this error has no impact on the other results of [4], since the possible affinity of $\mathfrak{h}$ is not needed elsewhere in that paper, it is relevant to identify where the proof of Lemma 3.2 fails and to clarify under what assumptions it becomes correct.

Let X be a locally compact metric space. The convex set $\mathcal{P}(X)$ is compact and metrizable with the weak* topology, as a consequence of Banach–Alaoglu Theorem (cf. [7, Theorem 2, V.4.2]). So, by the Krein–Milman Theorem (cf. [8, pages 187–188]), it is the closed convex hull of its extreme points. Moreover, we can use the Choquet Representation Theorem (cf. [3, Theorem 6.6]) to express each member of $\mathcal{P}(X)$ as a generalized convex combination of the extreme elements of $\mathcal{P}(X)$. More precisely, if $E(X)$ denotes the set of extreme points of $\mathcal{P}(X)$ and μ belongs to $\mathcal{P}(X)$, then there is a unique measure $\mathbb{P}_\mu$ on the Borel σ-algebra of $\mathcal{P}(X)$ such that $\mathbb{P}_\mu(E(X)) = 1$ and $\mu = \int_{E(X)} m \, d\mathbb{P}_\mu(m)$. We call the last equality the decomposition in extremes of μ.

Given a pressure function Γ, take the map $\mathfrak{h}$ provided by Theorem 1 and consider $\mu \in \mathcal{P}(X)$. In the Banach space $\mathbf{B}$, define the following binary relation

$$\varphi \preceq_\mu \psi \quad \Leftrightarrow \quad \int \varphi \, d\mu \leqslant \int \psi \, d\mu.$$

Then $\preceq_\mu$ is well defined, total (since $(\mathbb{R}, \leqslant)$ is completely ordered), reflexive and transitive, but depends on the fixed probability measure μ.

Definition 2.1. A non-empty set A with a pre-order $\lhd$ is said to be downward directed if, given $a, b \in A$, there is $c \in A$ such that $c \lhd a$ and $c \lhd b$.

Lemma 2.2. *Given $\mu \in \mathcal{P}(X)$, if $\mathfrak{h}(\mu) > -\infty$ then $\mathcal{A}_\Gamma$ with the pre-order $\preceq_\mu$ is a lower bounded downward directed set in* $\mathbf{B}$.

Proof. Given $\varphi, \psi \in \mathcal{A}_\Gamma$, one has either $\int \varphi \, d\mu \leqslant \int \psi \, d\mu$ or $\int \psi \, d\mu \leqslant \int \varphi \, d\mu$. In the former case, the map $\Psi = \varphi$ is in $\mathcal{A}_\Gamma$ and satisfies both $\Psi \preceq_\mu \varphi$ and $\Psi \preceq_\mu \psi$. In the latter case, one chooses $\Psi = \psi$. Thus, $(\mathcal{A}_\Gamma, \preceq_\mu)$ is a downward directed set.

As $\mathfrak{h}(\mu) > -\infty$, we may take the constant map $H \colon X \to \mathbb{R}$ defined by $H(x) = \mathfrak{h}(\mu)$, which is a lower bound of $(\mathcal{A}_\Gamma, \preceq_\mu)$ in $\mathbf{B}$ since

$$\int H(x) \, d\mu(x) = \int \mathfrak{h}(\mu) \, d\mu(x) = \mathfrak{h}(\mu) = \inf_{\psi \in \mathcal{A}_\Gamma} \int \psi \, d\mu \leqslant \int \psi \, d\mu \qquad \forall \psi \in \mathcal{A}_\Gamma.$$

□

Therefore, $\left(\int \psi \, d\mu\right)_{\psi \in \mathcal{A}_\Gamma}$ is a lower bounded decreasing net. So, as X is locally compact, if $\mathbf{B} = C_c(X)$ and $\mu \in \mathcal{P}(X)$, then we may apply the Monotone Convergence Theorem for nets (cf. [9, Theorem IV.15] or [5, Theorem 1, Chapter IV, §1]), thus concluding that the infimum of the net with respect to $\preceq_\mu$, say $\inf^{(\mu)} \mathcal{A}_\Gamma$, belongs to $L^1(X, \mathfrak{B}, \mu)$ and

$$\inf_{\psi \in \mathcal{A}_\Gamma} \int \psi \, d\mu \; = \; \int \inf{}^{(\mu)} \mathcal{A}_\Gamma \, d\mu.$$

Proposition 2.3. *Assume that X is locally compact and* $\inf_{\mu \in \mathcal{P}(X)} \mathfrak{h}(\mu) > -\infty$. *Then* $\mathfrak{h}$ *is affine if and only if*

$$\int \inf{}^{(\mu)} \mathcal{A}_\Gamma \, d\mu = \int_{E(X)} \left(\int \inf{}^{(m)} \mathcal{A}_\Gamma \, (x) \, dm(x) \right) d\mathbb{P}_\mu(m) \quad \forall \, \mu \in \mathcal{P}(X) \tag{2.4}$$

where $\mu = \int_{E(X)} m \; d\mathbb{P}_\mu(m)$ *is the decomposition in extremes of* μ.

Proof. Assume that $\inf_{\mu \in \mathcal{P}(X)} \mathfrak{h}(\mu) > -\infty$ and the condition (2.4) is valid. Then, for every $\mu \in \mathcal{P}(X)$,

$$\begin{aligned} \mathfrak{h}(\mu) = & \inf_{\psi \in \mathcal{A}_\Gamma} \int \psi \, d\mu \; = \; \int \inf{}^{(\mu)} \mathcal{A}_\Gamma \, d\mu \\ = & \int_{E(X)} \left(\int \inf{}^{(m)} \mathcal{A}_\Gamma \, (x) \, dm(x) \right) d\mathbb{P}_\mu(m) \\ = & \int_{E(X)} \left(\inf_{\psi \in \mathcal{A}_\Gamma} \int \psi(x) \, dm(x) \right) d\mathbb{P}_\mu(m) \\ = & \int_{E(X)} \mathfrak{h}(m) \, d\mathbb{P}_\mu(m). \end{aligned} \tag{2.5}$$

Remark 2.6. We note that, to obtain the previous estimates, which yield the equality

$$\mathfrak{h}(\mu) \; = \; \int_{E(X)} \mathfrak{h}(m) \, d\mathbb{P}_\mu(m) \qquad \forall \, \mu \in \mathcal{P}(X) \tag{2.7}$$

we only needed to assume that $\mathfrak{h}(\mu) > -\infty$ for every $\mu \in \mathcal{P}(X)$.

We are left to conclude from (2.5) that $\mathfrak{h}$ is affine. Given $\mu_1, \mu_2 \in \mathcal{P}(X)$ whose decompositions in extremes are $\mu_i = \int_{E(X)} m \; d\mathbb{P}_{\mu_i}(m)$, for $i \in \{1, 2\}$, then

$$\forall \, 0 < t < 1 \qquad t \, \mu_1 + (1-t) \, \mu_2 \; = \; \int_{E(X)} m \left[t \, d\mathbb{P}_{\mu_1} + (1-t) \, d\mathbb{P}_{\mu_2} \right](m)$$

and so, using (2.5), we get for every $0 < t < 1$

$$\mathfrak{h}\big(t \, \mu_1 + (1-t) \, \mu_2\big) \; = \; \int_{E(X)} \mathfrak{h}(m) \; d[t \, \mathbb{P}_{\mu_1} + (1-t) \, \mathbb{P}_{\mu_2}](m) \; = \; t \, \mathfrak{h}(\mu_1) + (1-t) \, \mathfrak{h}(\mu_2). \tag{2.8}$$

Remark 2.9. It may happen that there is $\nu_0 \in \mathcal{P}(X)$ such that $\mathfrak{h}(\nu_0) = -\infty$, so the first assumption in Proposition 2.3 fails, whereas the map $\mathfrak{h}$ is affine. This is precisely the case illustrated in Example 2.14, for which, despite its infinite values, the map $\mathfrak{h}$ is affine. Actually, in this example the equality (2.7) is valid, and this equality is just what one uses in the computation (2.8) to show that $\mathfrak{h}$ is affine.

Remark 2.10. If we may find a pre-order so that the function $\inf \mathcal{A}_\Gamma$ does not depend on the probability measure μ, then the request (2.4) in the statement of Proposition 2.3 is trivially fulfilled. See, for instance, Example 2.15, for which the set $\mathcal{A}_\Gamma$ is downward directed and lower bounded in $C(X)$ with respect to the usual order, defined by $\varphi \leqslant \psi \Leftrightarrow \varphi(x) \leqslant \psi(x) \, \forall x \in X$.

Regarding the converse assertion in Proposition 2.3, assume now that $\inf_{\mu \in \mathcal{P}(X)} \mathfrak{h}(\mu) > -\infty$ and $\mathfrak{h}$ is affine. We start by recalling that:

Lemma 2.11. [10, page 186] *If $F: \mathcal{P}(X) \to \mathbb{R}$ is affine, upper semi-continuous and lower bounded by a continuous map $G: \mathcal{P}(X) \to \mathbb{R}$, then*

$$F(\mu) = \int_{E(X)} F(m)\, d\mathbb{P}_\mu(m) \qquad \forall\, \mu \in \mathcal{P}(X) \tag{2.12}$$

where $\mu = \int_{E(X)} m \, d\mathbb{P}_\mu(m)$ is the decomposition in extremes of μ.

Proof. Firstly, suppose that F is continuous. As F is affine, then the equality (2.12) is an immediate consequence of the Choquet Representation Theorem.

Suppose now that F is only upper semi-continuous. Then F is the pointwise limit of a decreasing (with respect to the usual order in $C(X)$) net $(F_\alpha)_\alpha$ of continuous affine maps $F_\alpha : \mathcal{P}(X) \to \mathbb{R}$ (cf. [2, Corollary I.1.4]). As $G \leqslant F = \inf_\alpha F_\alpha$, then $(F_\alpha)_\alpha$ is bounded from below in the space of continuous real valued maps whose domain is $\mathcal{P}(X)$. Therefore, for every $\mu \in \mathcal{P}(X)$, whose decomposition in extremes is $\mu = \int_{E(X)} m \, d\mathbb{P}_\mu(m)$, one has

$$\begin{aligned} F(\mu) &= \lim_\alpha F_\alpha(\mu) = \lim_\alpha \int_{E(X)} F_\alpha(m)\, d\mathbb{P}_\mu(m) \\ &= \int_{E(X)} \lim_\alpha \, F_\alpha(m)\, d\mathbb{P}_\mu(m) = \int_{E(X)} F(m)\, d\mathbb{P}_\mu(m) \end{aligned}$$

where the last but one equality is due to the Monotone Convergence Lemma for nets (cf. [1, Lemma 19.36]), which may be applied since $\mathbb{P}_\mu$ is σ-additive on the Borel σ-algebra of the compact metric space $\mathcal{P}(X)$. □

Since $\mathfrak{h}$ is upper semi-continuous and satisfies $\inf_{\mu \in \mathcal{P}(X)} \mathfrak{h}(\mu) > -\infty$, then, if in addition it is affine, one may apply Lemma 2.11 using $F = \mathfrak{h}$ and $G = \inf_{\mu \in \mathcal{P}(X)} \mathfrak{h}(\mu)$. Therefore,

$$\mathfrak{h}(\mu) = \int_{E(X)} \mathfrak{h}(m)\, d\mathbb{P}_\mu(m) \qquad \forall\, \mu \in \mathcal{P}(X).$$

That is,

$$\int \inf{}^{(\mu)} \mathcal{A}_\Gamma \, d\mu = \int_{E(X)} \left(\int \inf{}^{(m)} \mathcal{A}_\Gamma \,(x)\, dm(x) \right) d\mathbb{P}_\mu(m) \qquad \forall\, \mu \in \mathcal{P}(X).$$

The proof of Proposition 2.3 is completed. □

Remark 2.13. In Example 1.4, one has $\inf_{\mu \in \mathcal{P}(X)} \mathfrak{h}(\mu) > -\infty$. Indeed, $\mathfrak{h}(\delta_a) = \mathfrak{h}(\delta_b) = 0$ and

$$\mathfrak{h}\big(t\,\delta_a + (1-t)\,\delta_b\big) = -t\log(t) - (1-t)\log(1-t) \qquad \forall\, 0 < t < 1.$$

So $\inf_{\mu \in \mathcal{P}(X)} \mathfrak{h}(\mu) = \min_{\mu \in \mathcal{P}(X)} \mathfrak{h}(\mu) = 0$. However, if $\mu_0 = \frac{1}{2}\,\delta_a + \frac{1}{2}\,\delta_b$, then

$$\log 2 = \int \inf{}^{(\mu_0)} \mathcal{A}_\Gamma \, d\mu_0 \quad \neq \quad \int_{E(X)=\{\delta_a,\,\delta_b\}} \left(\int \inf{}^{(m)} \mathcal{A}_\Gamma\,(x)\, dm(x) \right) d\mathbb{P}_{\mu_0}(m) = 0.$$

Example 2.14. Consider, as in Example 1.4, the set $X = \{a, b\}$ with the discrete metric and its space of Borel probability measures $\mathcal{P}(X) = \big\{t\delta_a + (1-t)\delta_b \,:\, t \in [0,1]\big\}$. Let now $\Gamma\colon C(X) \to \mathbb{R}$ be given by

$$\Gamma(\varphi) = \varphi(a).$$

It is easy to check that Γ is a pressure function on the Banach space $C(X)$. Therefore, by Theorem 1,

$$\Gamma(\varphi) = \max_{\mu \in \mathcal{P}(X)} \left\{ \mathfrak{h}(\mu) + \int \varphi\, d\mu \right\}$$

where

$$\mathfrak{h}(\mu) = \inf_{\psi \in \mathcal{A}_\Gamma} \int \psi\, d\mu \qquad \forall\, \mu \in \mathcal{P}(X)$$

and

$$\begin{aligned} \mathcal{A}_\Gamma &= \Big\{\psi \in C(X)\colon\ \Gamma(-\psi) \leqslant 0\Big\} = \Big\{\psi \in C(X)\colon\ -\psi(a) \leqslant 0\Big\} \\ &= \Big\{\psi \in C(X)\colon\ \psi(a) \geqslant 0\Big\}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathfrak{h}(\delta_a) &= \inf_{\psi \in \mathcal{A}_\Gamma} \int \psi\, d\delta_a = \inf_{\psi \in \mathcal{A}_\Gamma} \psi(a) = \inf_{\{\psi\colon\ \psi(a) \geqslant 0\}} \psi(a) = 0; \\ \mathfrak{h}(\delta_b) &= \inf_{\psi \in \mathcal{A}_\Gamma} \int \psi\, d\delta_b = \inf_{\psi \in \mathcal{A}_\Gamma} \psi(b) = \inf_{\{\psi\colon\ \psi(a) \geqslant 0\}} \psi(b) = -\infty; \end{aligned}$$

and, for every $0 < t < 1$,

$$\begin{aligned} \mathfrak{h}\big(t\,\delta_a + (1-t)\,\delta_b\big) &= \inf_{\psi \in \mathcal{A}_\Gamma} \int \psi\, d(t\,\delta_a + (1-t)\,\delta_b) \\ &= \inf_{\psi \in \mathcal{A}_\Gamma} \big(t\,\psi(a) + (1-t)\,\psi(b)\big) \\ &= \inf_{\{\psi\colon\ \psi(a) \geqslant 0\}} \big(t\,\psi(a) + (1-t)\,\psi(b)\big) \\ &= -\infty. \end{aligned}$$

Thus, $\mathfrak{h}$ is affine, although for some probability measures one has $\mathfrak{h} = -\infty$.

Apart from the known cases where $\mathfrak{h}$ is affine when restricted to an interesting convex subset of $\mathcal{P}(X)$ (see, for instance, [4, Remark 2.4]), there are examples for which a better choice of the pre-order in $\mathcal{A}_\Gamma$ is available.

Example 2.15. Consider a compact metric space (X, d) and the function $\Gamma\colon C(X) \to \mathbb{R}$ given by

$$\Gamma(\varphi) = \max_{x \in X} \varphi(x).$$

Then Γ is a pressure function, since

1. $\varphi \leqslant \psi \Rightarrow \max_{x \in X} \varphi(x) \leqslant \max_{x \in X} \psi(x)$;
2. $\max_{x \in X} (\varphi + c)(x) = \big(\max_{x \in X} \varphi(x)\big) + c \quad \forall \varphi \in C(X) \quad \forall c \in \mathbb{R}$;
3. $\forall \varphi, \psi \in C(X) \quad \forall t \in [0, 1]$,

$$\max_{x \in X} \big(t\,\varphi + (1-t)\,\psi\big)(x) \leqslant t \max_{x \in X} \varphi(x) + (1-t) \max_{x \in X} \psi(x).$$

Moreover,

$$\mathcal{A}_\Gamma = \Big\{\psi \in C(X) \,:\, \Gamma(-\psi) \leqslant 0\Big\} = \Big\{\psi \in C(X) \,:\, \max_{x \in X} -\psi(x) \leqslant 0\Big\}$$
$$= \Big\{\psi \in C(X) \,:\, \psi \geqslant 0\Big\}.$$

So

$$\mathfrak{h}(\mu) = \inf \Big\{ \int \psi \, d\mu \,:\, \psi \in C(X) \text{ and } \psi \geqslant 0\Big\} = 0 \qquad \forall \mu \in \mathcal{P}(X).$$

Hence $\mathfrak{h}$ is affine. Furthermore, given $\varphi \in C(X)$ whose maximum is attained at a point $x_0 \in X$, one has

$$\Gamma(\varphi) = \varphi(x_0) = \mathfrak{h}(\delta_{x_0}) + \int \varphi \, d\delta_{x_0}.$$

In this case, instead of the pre-order $\preceq_\mu$, we may consider in $\mathcal{A}_\Gamma$ the usual order in $C(X)$ and still conclude that $(\mathcal{A}_\Gamma, \leqslant)$ is a lower bounded downward directed set. Indeed, given $\varphi, \psi \in \mathcal{A}_\Gamma$, the map $\min\{\varphi, \psi\}$ is continuous and non-negative, hence belongs to $\mathcal{A}_\Gamma$; besides, one has $\min\{\varphi, \psi\} \leqslant \varphi$ and $\min\{\varphi, \psi\} \leqslant \psi$; and the map 0 is a lower bound for $\mathcal{A}_\Gamma$. Therefore, the function $\inf \mathcal{A}_\Gamma$ does not depend on any probability measure μ.

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