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Well-Posedness of the Ambient Metric Equations and Stability of Even Dimensional Asymptotically de Sitter Spacetimes

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Abstract: The vanishing of the Fefferman–Graham obstruction tensor was used by Anderson and Chruściel to show stability of the asymptotically de Sitter spaces in even dimensions. However, the existing proofs of hyperbolicity of this equation contain gaps. We show in this paper that it is indeed a well-posed hyperbolic system with unique up to diffeomorphism and conformal transformations smooth development for smooth Cauchy data. Our method applies also to equations defined by various versions of the Graham–Jenne–Mason–Sparling operators. In particular, we use one of these operators to propagate Gover's condition of being almost Einstein (basically conformal to Einsteinian metric). This allows us to study initial data also for Cauchy surfaces which cross the conformal boundary. As a by-product we show that on globally hyperbolic manifolds one can always choose a conformal factor such that Branson Q-curvature vanishes.

1. Anderson–Fefferman–Graham Equation

An important issue in General Relativity is the long time, asymptotic behaviour of solutions to Einstein's equations. In the case of positive cosmological constant the problem was solved by Friedrich [1,2]. He showed that there exists in four dimensions a hyperbolic system of equations for a metric and some derived variables which is satisfied if a metric is conformal to a solution of Einstein equation with a cosmological constant. This allows to study compactified versions of the solutions via conformal Penrose compactification and replace difficult long time analysis by a simpler finite time problem. Future asymptotically simple solutions are those satisfying the following condition: there exists a smooth conformal compactification in which the future boundary of physical space $\overline{\Sigma}_+$ is a Cauchy surface. An important example of such a spacetime is de Sitter universe. For this reason, these spacetimes are often called asymptotically de Sitter. In fact, we need to assume positive cosmological constant in order for the conformal boundary surface to be spacelike. From hyperbolicity of the new system one obtains immediately stability in this class of spacetimes. Moreover, the method gives explicit description of the initial data on the conformal boundary $\overline{\Sigma}_+$. However, Friedrich's method does not extend easily to higher dimension. Another important drawback is that no Lagrangean formulation for it exists. Such formulations are important for analysis of the initial data and conserved charges.

The alternative method proposed by Anderson in [3] and futher developed by Anderson and Chruściel in [4] is using the Fefferman–Graham obstruction tensor $H_{\mu\nu}$ which is defined for even dimensions $d \geq 4$ [5]. We will describe the original definition of [5] (see [6]) in Sect. 4. This tensor can be defined as the variation of the Lagrangean

$$c_d \int Q(h)\sqrt{h}\mathrm{d}^d x, \tag{1}$$

over the metric $h_{\mu\nu}$, where Q is the Branson curvature [7], a covariant object with nice conformal transformations, and c_d is a constant depending on the dimension. This functional is invariant (up to boundary terms) under both diffeomorphisms and conformal transformations. Consequently, the $H_{\mu\nu}$ tensor has interesting properties

1. It is a covariant object built out of the metric and its derivative,
2. It is conformally covariant, namely for $h^1_{\mu\nu} = e^{2\sigma} h^2_{\mu\nu}$, for σ a smooth function

$$e^{(d-2)\sigma} H_{\mu\nu}(h^1) = H_{\mu\nu}(h^2). \tag{2}$$

3. It is divergenceless $\nabla^\mu H_{\mu\nu} = 0$ and traceless $H^\mu_\mu = 0$.

Even more remarkably,

4 if $h_{\mu\nu}$ satisfies vacuum Einstein equations with a cosmological constant Λ, $G_{\mu\nu}(h) = \Lambda h_{\mu\nu}$, then $H_{\mu\nu} = 0$ [6].

The method proposed in [3,4] is to consider conformally invariant Anderson–Fefferman–Graham (AFG) equations

$$H_{\mu\nu} = 0, \tag{3}$$

instead of the Einstein equations and impose the later as a constraint at the initial surface $\overline{\Sigma}_+$. If we want to use this method, it is necessary to prove that the system (3) is well-posed after fixing diffeomorphism and conformal gauge.

However, this is a tricky problem. It is shown in [4] that one can impose gauge $\Box_{\overline{h}}(x^\mu) = 0$ and $R_{\overline{h}} = 0$ (where $\Box_{\overline{h}}$ is a scalar d'Alembert operator for the metric $\overline{h}$, x^μ are coordinates and $R_{\overline{h}}$ is the Ricci scalar). In this specific gauge the equations take the form

$$\Box_0^{d/2}\overline{h}_{\mu\nu} + F^{\overline{h}}_{\mu\nu}(D^{d-1}\overline{h}_{\mu\nu}) = 0, \tag{4}$$

where $\Box_0 := \overline{h}^{\mu\nu}\partial_\mu\partial_\nu$ and $D^m\overline{h}_{\mu\nu}$ denote m-th jets of the metric i.e. all derivatives $\partial^k\overline{h}_{\mu\nu}$ for $k \leq m$. The principal symbol is hyperbolic, but the roots have multiplicities, thus the system is not strictly hyperbolic. Such systems are complicated as we can see in the following example:

Example 1. (Not well-posed) Consider an equation on $\mathbb{R} \times S^1$ with x^1 being time coordinate

$$(\partial_1^2 - \partial_2^2)^3\phi + \partial_2(\partial_1 + \partial_2)^3\phi = 0. \tag{5}$$

The principal symbol is hyperbolic (it is a power of d'Alembert operator for a flat metric), but it has multiple characteristics. Functions $\phi_k(x^1, x^2) = e^{i(\omega(k)x^1 + kx^2)}$ are solutions for $\omega(k) = k + \frac{-1-i\sqrt{3}}{2}k^{1/3}$ with property $|\phi_k| = e^{\frac{\sqrt{3}}{2}k^{1/3}x^1}$. The smooth initial data on $\Sigma = \{x^1 = 0\}$

$$\partial_1^n \phi|_\Sigma = \sum_{k=0}^{\infty} i^n \omega(k)^n e^{-k^{1/4}} e^{ikx^2}, \quad n = 0 \ldots 5, \tag{6}$$

does not admit a Cauchy development because for every $k \geq 0$ the mode function should behave like $e^{-k^{1/4}}\phi_k(x^1, \cdot)$ for $x^1 > 0$ but the series $\sum_{k\geq 0} e^{-k^{1/4}}\phi_k(x^1, x^2)$ does not converge even in $L^2(S^1)$.

We see that in order to establish hyperbolicity of the equations with a non-strictly hyperbolic principal part, one needs to control a few lower order derivatives of the equation (by so called Levi conditions, see [8–10]). Unfortunately, the Fefferman–Graham obstruction tensor is quite complicated and one would need to control more and more of these terms the higher dimension we consider. The conditions on the lower order symbols are necessary for the case of multiple characteristics. This is the reason, why proofs of well-posedness of the Anderson–Fefferman–Graham equation in [3] and [4] cannot be correct, and our example shows that indeed it is not the case. We will provide in this paper a proof of well-posedness of AFG equations for smooth data. However, our proof will not be based on Levi conditions but on certain structural property of the obstruction tensor and the standard results for quasi-linear wave equations. Our method is in fact a modification of the approach from [4], but the special form of the system needs to be taken into account.

The problem is less complicated in lower dimensions. It is worth to mention that the well-posedness in dimension 4 was proven in [11].[1] Our approach can be regarded as a generalization, which put also [11] in a proper context.

2. Summary of the Results

The Cauchy problem of the Anderson–Fefferman–Graham (AFG) equation (3) is similar to that of the Einstein equations. The metric itself is an object of equations, which leads to additional geometric complications as the meaning of the Cauchy development depends itself on the solution (see [12]). The initial data on the surface Σ will be a set of $d-1$ jets of symmetric tensors fields in $\mathbb{R} \times \Sigma$ at Σ

$$D^{d-1}h_{\mu\nu}|_\Sigma \in C^\infty(\Sigma), \tag{7}$$

where $h_{\mu\nu}$ is a Lorentzian metric. We introduce a normal $\vec{N}$ to Σ with respect to this metric and we assume that it is a timelike vector. Assume that (7) satisfy constraints (well-defined because we know sufficiently many derivatives)

$$H(\vec{N}, \cdot)|_\Sigma = 0, \tag{8}$$

We will consider a specific (local) coordinate system and a choice of conformal factor given by conditions introduced in [4]

$$R = 0, \ \forall_\mu \ \Box(x^\mu) = 0, \tag{9}$$

[1] In dimension 4 the obstruction tensor is proportional to the celebrated Bach tensor and the analysis is less complicated.

where R is the Ricci scalar. As it is shown in [4] we can always locally transform the metric by diffeomorphism and rescaling, such that these conditions are satisfied. We then show that equations (3) are hyperbolic in this gauge. The standard analysis [12] thus shows:

Theorem 1. *The AFG equation* (3) *with initial data* (7) *and subject to constraints* (8) *forms a C^∞ well-posed system (in the Anderson–Chruściel gauge* (9)*). Every two local solutions differ by diffeomorphisms and conformal transformations.*

We will now describe our approach to the problem. Fefferman–Graham obstruction tensor is obtained by ambient metric construction. We consider expansion coefficients of the ambient metric $\tilde{g}^{[k]}_{\mu\nu}$ for $k = 0, \ldots \frac{d}{2} - 1$. This is a family of symmetric tensors on M, such that $\tilde{g}^{[0]}_{\mu\nu} = h_{\mu\nu}$. The vanishing of the Fefferman–Graham tensor is equivalent to

$$\tilde{S}^{[k]}_{\mu\nu} = 0, \quad k = 0, \ldots, \frac{d}{2} - 1, \tag{10}$$

where $\tilde{S}^{[k]}_{\mu\nu}$ is a part of the expansion of the Ricci tensor for the ambient metric. These are also symmetric tensors on the spacetime. Every tensor $\tilde{S}^{[k]}_{\mu\nu}$ depends on the second jets of the tensors $\tilde{g}^{[n]}_{\mu\nu}$, $n = 0, \ldots k$ in a way reminiscent of the Ricci tensor on the metric. We will briefly describe the ambient construction in Sect. 4. The AFG equation is obtained by recursive determination of $\tilde{g}^{[k]}_{\mu\nu}$ for $k = 1, \ldots \frac{d}{2} - 1$ in terms of $\tilde{g}^{[0]}_{\mu\nu} = h_{\mu\nu}$.

Instead of solving recursively, we will consider the equations (10) as a dynamical system for $\tilde{g}^{[k]}_{\mu\nu}$ for $k = 0, \ldots \frac{d}{2} - 1$. Ricci tensor depends on the second jets of the metric, thus every equation in (10) is of second order. However, the system is not of hyperbolic type. It is not surprising because we need gauge fixing. Following standard Choquet–Bruhat method we write

$$\tilde{S}^{[k]}_{\mu\nu} = \tilde{E}^{[k]}_{\mu\nu} + \partial_\mu \tilde{G}^{[k]}_\nu + \partial_\nu \tilde{G}^{[k]}_\mu + \ldots, \tag{11}$$

where $\tilde{E}^{[k]}_{\mu\nu}$ is of hyperbolic type and $\tilde{G}^{[k]}_\mu$ are gauge fixing functions. The addition $\ldots$ comes from an additional conformal gauge fixing term that is basically of the form $\tilde{g}^{[0]}_{\mu\nu}\tilde{\gamma}^{[k]}$, for some additional (scale) gauge fixing functions $\tilde{\gamma}^{[k]}$. As in the standard method we will solve system $\tilde{E}^{[k]}_{\mu\nu} = 0$. However, the system is still not strictly hyperbolic. Fortunately, it is some generalized type of hyperbolic equation for which we provide a proof of well-posedness in Sect. 3. The standard method now use Bianchi identity to show that gauge fixing functions $\tilde{G}^{[k]}_\mu$ and $\tilde{\gamma}^{[k]}$ propagate by a linear hyperbolic system.

Here the next problem appears. We use only part of the Ricci tensor from the ambient metric. The Bianchi identities are already used to deduce that the remaining parts of this tensor, $\tilde{S}^{[k]}_{\mu\infty}$ and $\tilde{S}^{[k]}_{\infty\infty}$ vanish. We denote by ∞ the ambient direction. In some way, the ambient metric is already in the partially gauge fixed form and additional gauge fixing is excessive. We circumvent this problem by building $\tilde{G}^{[k]}_\mu$ and $\tilde{\gamma}^{[k]}$ from these remaining parts of the ambient Ricci tensor in such a way that Bianchi identities provide generalized hyperbolic system for the gauge fixing functions (see Sect. 4.1.1).

Our method of treating the ambient metric equations is more general and allows to prove well-posedness of various equations constructed with aid of the ambient construction. In particular, it is true for Graham–Jenne Mason–Sparling (GJMS) [13] equation and its various generalizations. It is a linear system of the similar type as gauge fixed

Anderson–Fefferman–Graham equation thus it has a unique development with a given initial data, which is global on globally hyperbolic spacetimes. Using this result, we show that there always exists a metric in the conformal class with vanishing Branson Q-curvature [14] for a given globally hyperbolic spacetime. Another application is propagation of covariantly constant tractor [15,16] from the initial Cauchy surface. Existence of the covariantly constant tractor is equivalent for metric to be conformal to Einsteinian metric, except when certain nondegeneracy condition is not satisfied what corresponds to conformal boundary (see [17]). In this way one can show that the condition of being Einstein propagates to the whole development of AFG equation in a uniform way. The initial Cauchy surface can now also cross the conformal boundary (see Proposition 23).

The organization of the paper is as follows: In Sect. 3 we introduce a generalization of quasilinear wave systems which we call generalized hyperbolic (Definition 1). We prove that they are well-posed in the smooth category in Proposition 3. We provide a natural way of obtaining such systems by an ambient space construction in Sect. 3.1. Subject to the additional condition of being recursive (Definition 3), the system is equivalent to a higher order so-called derived equation (Definition 4). This equivalence and the relation between the initial data of the system and its derived equation is shown in Lemmas 4 and 5. Section 4 is devoted to well-posedness of the AFG equation in the Anderson–Chruściel gauge. We introduce a gauge-fixed system $\tilde{E}_{\mu\nu}$ in Sect. 4.1 and prove that it is well-posed in Lemma 6. In Sect. 4.1.1 we show that the gauge fixing functions vanish if they vanish on the intial surface. We translate this condition into properties of the initial data for the AFG equation in Sect. 4.1.2 (see Lemma 9). We finish the proof of well-posedness in Sect. 4.2. In the rest of the paper we consider general GJMS operators. We prove well-posedness of the homogeneous (Corollary 15) and inhomogeneous (Proposition 17) equations for GJMS-type operators. This proves existence of a metric in the conformal class with vanishing Branson Q-curvature [14] for a given globally hyperbolic spacetime (Corollary 18). In Sect. 5.4 we consider the condition of being almost Einstein (existence of a covariantly constant tractor). We prove that the Einstein scale satisfies a certain supercritical GJMS equation (Lemma 21). We use this equation to propagate covariantly constant tractor from the initial Cauchy surface provided the obstruction tensor vanishes (Lemma 23). Some applications to the stability problem are given in Sect. 5.6.

3. Generalized Hyperbolic Systems

We denote by M a d-dimensional Lorentzian manifold. We use abstract index notation and Einstein summation convention in the paper. We denote indices of tensors in M by Greek letters. We use symbol $D^m u$ to denote m jets on M of the field u on M. We assume that x^1 is a time coordinate and in what follows $\Sigma = \{x^1 = 0\} \subset M$.

We will consider a bit more general situation then a standard second order hyperbolic system. We consider a family of fields $u^{(k)}$ for $k = 0, \ldots N$, where each field is a section of some vector bundle over M. These vector bundles might be different for each field. We call such family a <u>multifield</u>.

Definition 1. Consider a system involving a multifield $u^{(k)}$ for $k = 0, \ldots N$ and a system of equations on M for fields $u^{(k)}$, $k = 0, \cdots N$

$$K^{(k)} = -\frac{1}{2} g^{\mu\nu}(x, u^{(0)}) \partial_\mu \partial_\nu u^{(k)} + F^{(k)} \left(x, u^{(l)}, \partial_\mu u^{(l)}, \partial_\mu \partial_\nu u^{(l)} \right) = 0, \quad k = 0, \ldots, N, \tag{12}$$

where functions $F^{(k)}(x, u^{(l)}, v^{(l)}_\mu, w^{(l)}_{\mu\nu})$ depends smoothly on coordinates x and $u^{(l)}$ for $l \le \max(k+1, N)$, $v^{(l)}_\mu$ for $l \le k$, $w^{(l)}_{\mu\nu}$ for $l \le k-1$. Function $g_{\mu\nu}(x, u^{(0)})$ is a lorentzian metric smoothly depending on x and $u^{(0)}$.[2] We will call such a system $K^{(n)} = 0, n = 0, \dots N$ generalized hyperbolic for $u^{(k)}, k = 0, \dots N$.

Remark 1. Equation (12) is not a system of quasi-linear wave equations because of the non-trivial dependence of functions $F^{(k)}$ on second derivatives of the fields.

We assume that $\Sigma \subset M$ is spacelike and compact. The assumption of being spacelike is a condition for $u|_\Sigma$ if the metric is u dependent. We consider smooth initial data defined by functions $f^{(l)}_0$ and $f^{(l)}_1$

$$u^{(l)}|_\Sigma = f^{(l)}_0, \quad \partial_1 u^{(l)}|_\Sigma = f^{(l)}_1, \quad l = 0, \dots N. \tag{13}$$

We assumed here some local coordinate system x^μ on M such that $\Sigma = \{x^1 = 0\}$. We denote $\partial_\mu = \frac{\partial}{\partial x^\mu}$. Locally around Σ, x^1 is a time function. We are interested in well-posedness in the smooth category. It means a series of important properties (see [18]):

1. **Existence of a unique local solution**: There exists $I = (-T_-, T_+)$, $T_\pm > 0$ such that on $M = I \times \Sigma$ we have a unique smooth solution with the given initial data at $\{0\} \times \Sigma$. Moreover, every surface $\{s\} \times \Sigma$ is a Cauchy surface with respect to the metric $g_{\mu\nu}(x, u^{(0)})$, thus M is globally hyperbolic. We can choose $T_\pm$ maximal with this property.
2. **The speed of propagation is equal to the speed of light**: Namely, if two initial data u, u' are equal on some open set $U \subset \Sigma$ then $u(x) = u'(x)$ at all points such that their future and past developments $J^\pm_{g(u)}(x)$ satisfy $[J^+_{g(u)}(x) \cup J^-_{g(u)}(x)] \cap \{0\} \times \Sigma \subset U$. From this we can deduce some version of well-posedness also for arbitrary non-compact Cauchy surfaces.
3. **Smooth dependence on the initial data**: For arbitrary $T'_\pm < T_\pm$ the solution is defined for an open neighbourhood of a given initial data. The solution depends smoothly on the initial data (as a map from a Fréchet space of smooth sections on Σ to a Fréchet space of smooth sections on M). In particular, the derivative of the family of solutions $\left.\frac{du^{(l)}_\lambda}{d\lambda}\right|_{\lambda=0}$ satisfies a system obtained by linearization at $\{u^{(l)}_0\}$.

Let us first notice that the system is non-characteristic on every Cauchy surface. Suppose that $\Sigma = \{x^1 = 0\}$, and x^1 is a time function.

Lemma 2. *There exist smooth functions $L^{(k)}\left(\bar{D}^2 u^{(l)}_{l\le k}, \bar{D}\partial_1 u^{(l)}_{l\le k}, u^{(k+1)}\right)$, $k = 0, \dots N$, valued in the same field spaces as $u^{(k)}$ (with dependence on $u^{(k+1)}$ only if $k < N$), such that the following conditions are equivalent*

1. $K^{(k)}|_\Sigma = 0$ *for* $k = 0, \dots N$,
2. $\partial_1^2 u^{(k)}|_\Sigma = L^{(k)}(\bar{D}^2 u^{(l)}_{l\le k}|_\Sigma, \bar{D}\partial_1 u^{(l)}_{l\le k}|_\Sigma, u^{(k+1)}|_\Sigma)$ *for* $k = 0, \dots N$,

where $\bar{D}^n$ denotes n jets on Σ.

[2] These conditions should hold for some open neighbourhood of values of u (for some open set in the standard Fréchet topology on smooth sections on Σ). We will take this condition as obvious in what follows.

Proof. We will prove by induction in k_0 the statement:

$$\begin{aligned}&\left(\forall k < k_0\colon K^{(k)}|_\Sigma = 0\right)\\ &\quad\Longleftrightarrow \left(\forall k < k_0\colon \partial_1^2 u^{(k)}|_\Sigma = L^{(k)}(\bar{D}^2 u^{(l)}_{l\leq k}|_\Sigma, \bar{D}\partial_1 u^{(l)}_{l\leq k}|_\Sigma, u^{(k+1)}|_\Sigma)\right).\end{aligned} \tag{14}$$

The statement for $k_0 = 0$ is tautological. Suppose that it is true for some $k_0 \geq 0$. Using $g^{11} \neq 0$ we can write

$$-\frac{2}{g^{11}} K^{(k_0)}|_\Sigma = \partial_1^2 u^{(k_0)}|_\Sigma + \ldots, \tag{15}$$

where $\ldots$ is a smooth function of $\bar{D}^1\partial_1 u^{(k)}$ for $k \leq k_0$, $\bar{D}^2 u^{(k)}$ for $k \leq k_0$, $\partial_1^2 u^{(k)}$ for $k < k_0$ and $u^{(k_0+1)}$ if $k_0 < N$. We can express $\partial_1^2 u^{(k)}$ for $k < k_0$ by $L^{(k)}$ by induction hypothesis, thus we obtain desired function finishing the induction proof. □

Proposition 3. *The generalized hyperbolic system $K^{(n)} = 0$, $n = 0, \ldots N$ (12) for the multifield $u^{(k)}$, $k = 0, \ldots N$ is well-posed in the smooth category. If the system is linear (or linear with a source term) then the solution is defined on the whole globally hyperbolic spacetime.*

Proof. We will prove Proposition 3 by induction with respect to the order N. For $N = 0$ it is a known result (see for example [19] Chapter 16.1-16.3 and [12] Appendix III). The following system is well-posed:

$$-\frac{1}{2} g^{\mu\nu}(x, u)\partial_\mu\partial_\nu u + F(x, D^1 u) = 0, \tag{16}$$

where $g_{\mu\nu}(x, u)$ is a lorentzian metric smoothly depending on coordinates x and u and F is a smooth function of coordinates and $D^1 u$ (first jets of u).

We assume now that we proved the statement for all $0 \leq N < N_0$. Consider generalized hyperbolic system with fields $u^{(k)}$ for $k = 0, \ldots N_0$ (i.e. order N_0)

$$-\frac{1}{2} g^{\mu\nu}(x, u^{(0)})\partial_\mu\partial_\nu u^{(k)} + F^{(k)}\left(x, u^{(l)}, \partial_\mu u^{(l)}, \partial_\mu\partial_\nu u^{(l)}\right) = 0, \quad k = 0, \ldots N_0, \tag{17}$$

for some functions $F^{(k)}(x, u^{(l)}, v_\mu^{l)}, w_{\mu\nu}^{(l)})$ depending on variables described in the definition of generalized hyperbolic system. Differentiating equation for $u^{(k)}$ with respect to ∂_ρ we get

$$\begin{aligned}&-\frac{1}{2} g^{\mu\nu}\partial_\mu\partial_\nu\partial_\rho u^{(k)} - \frac{1}{2}\frac{\partial g^{\mu\nu}}{\partial u^{(0)}}\partial_\rho u^{(0)}\partial_\mu\partial_\nu u^{(k)} - \frac{1}{2}\frac{\partial g^{\mu\nu}}{\partial x^\rho}\partial_\mu\partial_\nu u^{(k)}\\ &\quad+\frac{\partial F^{(k)}}{\partial u^{(l)}}\partial_\rho u^{(l)} + \frac{\partial F^{(k)}}{\partial v_\mu^{(l)}}\partial_\mu\partial_\rho u^{(l)} + \frac{\partial F^{(k)}}{\partial w_{\mu\nu}^{(l)}}\partial_\mu\partial_\nu\partial_\rho u^{(l)} + \frac{\partial F^{(k)}}{\partial x^\rho} = 0,\end{aligned} \tag{18}$$

where summation over l is implicitely assumed (as well as standard Einstein summation convention). Introducing $p_\mu^{(k)} = \partial_\mu u^{(k)}$ for $k \leq N_0 - 1$ we can write it as

$$-\frac{1}{2} g^{\mu\nu}\partial_\mu\partial_\nu p_\rho^{(k)} + G_\rho^{(k)}\left(x, u^{(l)}, \partial_\mu u^{(l)}, \partial_\mu\partial_\nu u^{(l)}, p_\rho^{(l)}, \partial_\mu p_\rho^{(l)}, \partial_\mu\partial_\nu p_\rho^{(l)}\right) = 0, \tag{19}$$

where

$$G_\rho^{(k)}(x, u^{(l)}, v_\mu^{(l)}, w_{\mu\nu}^{(l)}, p_\rho^{(l)}, q_{\mu\rho}^{(l)}, s_{\mu\nu\rho}^{(l)})$$
$$= -\frac{1}{2}\frac{\partial g^{\mu\nu}}{\partial u^{(0)}} p_\rho^{(0)} q_{\mu\nu}^{(k)} + \frac{\partial F^{(k)}}{\partial u^{(l)}} p_\rho^{(l)} + \frac{\partial F^{(k)}}{\partial v_\mu^{(l)}} q_{\mu\rho}^{(l)} + \frac{\partial F^{(k)}}{\partial w_{\mu\nu}^{(l)}} s_{\mu\nu\rho}^{(l)} - \frac{1}{2}\frac{\partial g^{\mu\nu}}{\partial x^\rho} q_{\mu\nu}^{(k)} + \frac{\partial F^{(k)}}{\partial x^\rho}, \tag{20}$$

where all derivatives of $F^{(k)}$ and $g_{\mu\nu}$ retain their original variables. We introduce new multifields

$$u'^{(k)} = \{u^{(k)}, p_\mu^{(k)}\} \text{ for } k < N_0 - 1, \tag{21}$$
$$u'^{(N_0-1)} = \{u^{(N_0-1)}, p_\mu^{(N_0-1)}, u^{(N_0)}\}, \tag{22}$$

and the system of equations

$$-\frac{1}{2}g^{\mu\nu}(x, u^{(0)})\partial_\mu\partial_\nu p_\rho^{(k)}$$
$$+ G_\rho^{(k)}\left(x, u^{(l)}, \partial_\mu u^{(l)}, \partial_\mu\partial_\nu u^{(l)}, p_\rho^{(l)}, \partial_\mu p_\rho^{(l)}, \partial_\mu\partial_\nu p_\rho^{(l)}\right) = 0, \quad k \le N_0 - 1, \tag{23}$$
$$-\frac{1}{2}g^{\mu\nu}(x, u^{(0)})\partial_\mu\partial_\nu u^{(k)} + F^{(k)}\left(x, u^{(l)}, \partial_\mu u^{(l)}, \partial_\mu\partial_\nu u^{(l)}\right) = 0, \quad k \le N_0 - 1, \tag{24}$$
$$-\frac{1}{2}g^{\mu\nu}(x, u^{(0)})\partial_\mu\partial_\nu u^{(N_0)} + F^{(N_0)}\left(x, u^{(l)}, \partial_\mu u^{(l)}, \partial_\mu p_\nu^{(l)}\right) = 0. \tag{25}$$

It is of the generalized form but the order is now $N_0 - 1$.

From solution of original system we can form solution of this system by taking

$$p_\mu^{(k)} = \partial_\mu u^{(k)}. \tag{26}$$

We proved uniqueness by induction hypothesis.

In order to prove existence we construct initial data

$$p_\mu^{(k)}|_\Sigma = \partial_\mu u^{(k)}|_\Sigma, \quad \partial_1 p_\mu^{(k)}|_\Sigma = \partial_1\partial_\mu u^{(k)}|_\Sigma, \tag{27}$$

with $\partial_1\partial_\mu u^{(k)}|_\Sigma$ obtained by Lemma 2 (the original system is non-characteristic with respect to a Cauchy surface, thus it is possible).

Now we notice that the difference ∂_ρ (24) – (23) is equal to

$$-\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu w_\mu^{(k)} - \frac{1}{2}\frac{\partial g^{\mu\nu}}{\partial u^{(0)}} w_\rho^{(0)}\partial_\mu\partial_\nu u_\nu^{(k)} - \frac{1}{2}\frac{\partial g^{\mu\nu}}{\partial u^{(0)}} p_\rho^{(0)}\partial_\mu w_\nu^{(k)} - \frac{1}{2}\frac{\partial g^{\mu\nu}}{\partial x^\rho}\partial_\mu w_\nu^{(k)}$$
$$+ \frac{\partial F^{(k)}}{\partial u^{(l)}} w_\rho^{(l)} + \frac{\partial F^{(k)}}{\partial v_\mu^{(l)}}\partial_\mu w_\rho^{(l)} + \frac{\partial F^{(k)}}{\partial w_{\mu\nu}^{(l)}}\partial_\mu\partial_\nu w_\rho^{(l)} = 0, \quad k \le N_0 - 1, \tag{28}$$

where $w_\mu^{(k)} = \partial_\mu u^{(k)} - p_\mu^{(k)}, k = 0, \ldots N_0 - 1$. These equations form linear generalized hyperbolic system for $w_\mu^{(k)}$ and as the initial data

$$w_\mu^{(k)}|_\Sigma = \partial_\mu u^{(k)} - p_\mu^{(k)}|_\Sigma = 0, \quad \partial_1 w_\mu^{(k)}|_\Sigma = \partial_1\partial_\mu u^{(k)} - \partial_1 p_\mu^{(k)}|_\Sigma = 0, \tag{29}$$

we have by uniqueness of solution (due to induction hypothesis)

$$p_\mu^{(k)} = \partial_\mu u^{(k)}, \tag{30}$$

and the solution of the lower order system gives the solution of the original one.

The solution of u' system depends smoothly on the initial data, so it is also true for the original system. The induction is complete. □

Remark 2. In fact one can show that it is well-posed in Sobolev spaces, but of different order for every $u^{(k)}$. We leave the details for further investigations. Such hyperbolic systems with shifted weights were first considered by Leray. The theory for first order hyperbolic systems is described for example in his book [20].

3.1. The ambient construction. We consider an ambient space[3]

$$\tilde{M} = M \times \mathbb{R}. \tag{31}$$

with coordinates (x^μ, ρ) where x^μ are coordinates on M. We will denote fields on $\tilde{M}$ with $\tilde{}$. We regard them as a formal series in ρ. We denote differentiation with respect to ρ by ∂_∞ or $'$. As before, we denote indices in M by greek letters. Coordinate x^1 is a time coordinate and $\Sigma = \{x^1 = 0\} \subset M$. We use $\tilde{D}^m \tilde{u}$ to denote m jets on $\tilde{M}$ of the field $\tilde{u}$, whereas $D^m u$ is the m jets on M of the field u on M.

Let us consider a ρ-dependent field on M, $\tilde{v}$. It can be also regarded as a field on $\tilde{M}$. We can write an expansion

$$\tilde{v} = \sum_{m=0} \tilde{v}^{[m]}(x^\mu)\rho^m + O(\rho^\infty), \tag{32}$$

where $\tilde{v}^{[m]}$ are rescaled Taylor expansion coefficients and $O(\rho^\infty)$ means a term that vanishes to infinite order at $\rho = 0$ surface. In what follows we will be interested in such formal series.

Remark 3. Every term $\tilde{v}^{[m]}$ in the expansion is a field on M. Thus, we have a family of fields $\tilde{v}^{[k]}$, $k = 0, \ldots$. Moreover, every differential equation $\tilde{K}(\tilde{v}) = 0$ on $\tilde{M}$, induces system of equations $\tilde{K}^{[k]} = 0$ on M. We will now apply results of Sect. 3 to systems obtained in this way.

Definition 2. Let $\tilde{u}$ be a field on $\tilde{M}$. We say that a formal series $\tilde{F}$ in ρ is <u>of order N</u> in $\tilde{u}$ if for each $n = 0, \ldots$, $\tilde{F}^{[n]}$ is a function of x and

$$\left\{ D^m \tilde{u}^{[l]} \right\}, \quad m = \min\{n + N - l, 2\},\ l \le n + N. \tag{33}$$

For example for $\tilde{F}$ of order 2, we have dependences

$$\tilde{F}^{[0]}(x, D^2\tilde{u}^{[0]}, D^1\tilde{u}^{[1]}, \tilde{u}^{[2]}),\ \ \tilde{F}^{[1]}(x, D^2\tilde{u}^{[0]}, D^2\tilde{u}^{[1]}, D^1\tilde{u}^{[2]}, \tilde{u}^{[3]}),\ \ \tilde{F}^{[2]}(x, D^2\tilde{u}^{[0]}, \ldots D^2\tilde{u}^{[2]}, D^1\tilde{u}^{[3]}, \tilde{u}^{[4]}), \ldots$$

[3] We will consider later another ambient space $\mathbf{M} = \mathbb{R} \times \tilde{M}$ which was introduced by Fefferman and Graham [6].

and so on. Remind, that $\tilde{D}^m \tilde{u}$ denotes m jets on $\tilde{M}$ of the field $\tilde{u}$. If $\tilde{F}(x, \tilde{D}^2 \tilde{u})$ is a smooth function then it is of order 2. Let $\tilde{f}^{\mu\nu}(x, \tilde{u})$ be a smooth tensor function then

$$\tilde{f}^{\mu\nu} \partial_\mu \partial_\nu \tilde{u} = [\tilde{f}^{\mu\nu}]^{[0]} \partial_\mu \partial_\nu \tilde{u} + \tilde{F}, \tag{34}$$

where $\tilde{F}$ is of order 1.

Important example of generalized hyperbolic systems can be obtained from $\tilde{K}^{[n]} = 0$, $n = 0, \ldots, N$ for a multifield $\tilde{u}^{[n]}$, $n = 0, \ldots, N$ if

$$\tilde{K} = -\frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \partial_\nu \tilde{u} + \tilde{F}, \tag{35}$$

and $\tilde{F}$ is of order 1 and $\tilde{F}^{[n]}$ for $n \leq N$ decouple in the sense that they do not depend on $O(\rho^{N+1})$ part of the expansion.

3.2. The derived equation. The equations of interest have also another important property:

Definition 3. We say that a system $K^{(n)}(x, D^2 u) = 0$, $n = 0, \ldots, N$ for $u^{(k)}$, $k = 0, \ldots, N$ is recursive (or recursive till order N) if

1. For every $n < N$, $K^{(n)}$ is a function of $D^2 u^{(k)}$ for $k \leq n$ and $u^{(n+1)}$,
2. For every $n < N$, $K^{(n)}$ depends linearly on $u^{(n+1)}$ and we can determine $u^{(n+1)}$ from equation $K^{(n)} = 0$ in terms of other variables.

We will consider in this paper the generalized hyperbolic systems given by

$$\tilde{K}^{[k]} = -\frac{1}{2} g^{\mu\nu}(\tilde{u}^{[0]}) \partial_\mu \partial_\nu \tilde{u}^{[k]} + [\tilde{F}]^{[k]}, \tag{36}$$

which are recursive till order N. In order to determine this property it is enough to study $\tilde{F}$.

The property of being recursive allows us to determine higher order variables from sufficiently high jets of the lowest order $u^{(0)}$. In our application we need a local version of this procedure that is described by a following lemma:

Lemma 4. *Let $K^{(n)}$ be recursive in $u^{(k)}$ for $0 \leq k \leq N$ till order N. There exist smooth functions $H_K^{(n)}$ for $0 < n \leq N$ depending on $x \in M$ and on variables $D^{2n} u^{(0)}$, such that for any point $x \in M$ and an integer $N' > 0$ the following conditions are equivalent*

1. $D^{N'-2k-2} K^{(k)}(x) = 0$ *for* $0 \leq k \leq N-1$, $2k+2 \leq N'$,
2. $D^{N'-2k} \left(u^{(k)} - H_K^{(k)} \left(x, D^{2k} u^{(0)} \right) \right)(x) = 0$ *for* $1 \leq k \leq N$, $2k \leq N'$.

where D^m denotes m-th jets. In the case of linear system the functions $H_K^{(n)}$ are also linear. If the system does not directly depend on x then the same is true for $H_K^{(n)}$.

Remark 4. We will use subscript for $H^{(k)}$ to indicate which system is used to determine recursive functions.

Proof. We proceed by induction in k_0. Suppose that

$$D^{N'-2k}u^{(k)}(x) = D^{N'-2k}H_K^{(k)}\left(x, D^{2k}u^{(0)}\right)|_x, \quad 1 \le k \le k_0 - 1 \tag{37}$$

is equivalent to

$$D^{N'-2k-2}K^{(k)}(x) = 0, \quad 0 \le k \le k_0 - 2. \tag{38}$$

Solving equation $K^{(k_0-1)} = 0$ for $u^{(k_0)}$ we introduce functions $G^{(k_0)}$

$$u^{(k_0)} = G^{(k_0)}\left(x, \left\{D^2u^{(l)}\right\}_{l\le k_0-1}\right) \tag{39}$$

Let us notice that for $k_0 \le N$ and $2k_0 \le N'$

$$D^{N'-2k_0}K^{(k_0-1)}(x) = 0, \Longleftrightarrow D^{N'-2k_0}(u^{(k_0)} - G^{(k_0)})|_x = 0, \tag{40}$$

due to linear dependence. By inserting recursively variables from the lower orders we show the result. Induction starts with $k_0 = 0$ where it is a trivial statement. □

This lemma allows us to determine initial data for the generalized hyperbolic system from the sufficienty high jets of the lowest order field $u^{(0)}$ on the Cauchy surface. Important is that the evolved generalized system will have $u^{(0)}$ in agreement with this data. This property is guaranteed by the following fact:

Lemma 5. *Let $K^{(n)} = 0$ be the generalized hyperbolic system, recursive in the multifield $u^{(k)}$, $k = 0, \dots N$. Consider initial data*

$$u^{(k)}|_\Sigma = H_K^{(k)}\left(\cdot, D^{2k}v|_\Sigma\right), \quad \partial_1 u^{(k)}|_\Sigma = \partial_1 H_K^{(k)}\left(\cdot, D^{2k}v|_\Sigma\right), \tag{41}$$

defined by jets $D^{2N+1}v|_\Sigma$ of the field v. Then a solution with these initial data satisfies on the Cauchy surface

$$D^{2N+1}u^{(0)}|_\Sigma = D^{2N+1}v|_\Sigma. \tag{42}$$

Proof. Let $u^{(k)}$ be a development. We denote

$$\partial_1^m v^{(k)}|_\Sigma = \partial_1^m H_K^{(k)}\left(\left\{D^m v|_\Sigma\right\}_{m\le 2k}\right), \quad m + 2k \le 2N + 1. \tag{43}$$

Consider a set

$$A = \{(m, k) \in \mathbb{Z}\times\mathbb{Z} \colon m \ge 0,\ k \ge 0,\ m+2k \le 2N+1,\ \partial_1^m v^{(k)}|_\Sigma \ne \partial_1^m u^{(k)}|_\Sigma\}. \tag{44}$$

We should show that this set is empty. By contradiction assume otherwise and define

$$m_0 = \min(m \colon \exists k,\ (m, k) \in A), \quad k_0 = \min(k \colon (m_0, k) \in A). \tag{45}$$

We notice that $m_0 \ge 2$ because of the way $u^{(k)}|_\Sigma$ and $\partial_1 u^{(k)}|_\Sigma$ are defined. Consider $\partial_1^{m_0-2}K^{(k_0)}$ in terms of $u^{(k_0)}$

$$\partial_1^{m_0-2}K^{(k_0)}|_\Sigma = -\frac{1}{2}g^{11}\partial_1^{m_0}u^{(k_0)}|_\Sigma + \dots, \tag{46}$$

where $\dots$ is a function of the terms which do not belong to A by definition. Similarly

$$\partial_1^{m_0-2}K^{(k_0)}|_\Sigma = -\frac{1}{2}g^{11}\partial_1^{m_0}v^{(k_0)}|_\Sigma + \dots, \tag{47}$$

and as ... of the same property. From the definition of (m_0, k_0) the remainders ... are equal thus

$$-\frac{1}{2}g^{11}\partial_1^{m_0}u^{(k_0)}|_\Sigma = -\frac{1}{2}g^{11}\partial_1^{m_0}v^{(k_0)}|_\Sigma, \tag{48}$$

and as $g^{11}|_\Sigma$ is nonvanishing we obtain a contradiction. □

If the equation system $K^{(n)} = 0$ is recursive till order N and it decouples at this order, then the equation for $n = N$ gives us by Lemma 4 the equation of higher order for $u^{(0)}$

$$0 = K^{(N)}\left(x, \left\{D^2 u^{(l)}\right\}_{l\leq N}\right). \tag{49}$$

This object will be of some importance, thus we introduce a definition.

Definition 4. Consider a generalized hyperbolic system $K^{(n)} = 0, n = 0, \dots N$ for a multifield $u^{(k)}$, $k = 0, \dots N$, which is recursive till order N and decouples at this order. We will call

$$\hat{H}_K\left(x, D^{2N+2}v\right) := K^{(N)}\left(x, \left\{D^2 u^{(l)}\right\}_{l\leq N}\right), \quad u^{(l)} := H_K^{(k)}\left(x, D^{2k}v\right) \tag{50}$$

the derived operator for the system $K^{(n)}$ and the equation $\hat{H}_K = 0$ will be called the derived equation for this system.

In case of a linear system the derived operator is also linear.

If the system $K^{(n)} = 0$, $n = 0, \dots, N$ is satisfied, then the equation $\hat{H}_K = 0$ for $u^{(0)}$ also holds. From solution of the derived equation we can obtain solution to the system. Initial conditions for this equation provide also initial conditions for the system, if we know sufficiently high jets on the initial surface.

4. The Fefferman–Graham Ambient Metric Construction

We are working in even d dimensions. Moreover we assume that $d \geq 4$. Let us introduce an ambient space $\mathbf{M}$ for the spacetime M

$$\mathbf{M} = \mathbb{R}_+ \times \tilde{M}, \quad \tilde{M} = M \times \mathbb{R}, \tag{51}$$

with coordinates (t, x^μ, ρ) and (x^μ, ρ) respectively, where x^μ are coordinates on M and the metric on $\mathbf{M}$ takes the form

$$\mathbf{g}_{IJ}dx^I dx^J = 2\rho dt^2 + 2t dt d\rho + t^2 \tilde{g}_{\mu\nu}(x^\mu, \rho)dx^\mu dx^\nu. \tag{52}$$

In the following we will denote objects on $\tilde{M}$ with $\tilde{}$ and objects on $\mathbf{M}$ bold. We denote by $\mathbf{g}_{IJ}$, $\boldsymbol{\nabla}_I$, $\mathbf{S}_{IJ}$ metric covariant derivative and Ricci tensor respectively on $\mathbf{M}$. Indices $I = 0, \infty$ or μ in the case of index on M. We use $\mathbf{g}_{IJ}$ to raise or lower indices. The metric $\tilde{g}_{\mu\nu}$, connection $\tilde{\nabla}_\mu$ and Ricci tensor $\tilde{R}_{\mu\nu}$ are ρ-dependent objects on M. We use $\tilde{g}_{\mu\nu}$ to raise and lower indices for such objects.

Let $h_{\mu\nu}$ be a given metric on M. The ambient metric on $\mathbf{M}$ is a metric that satisfies $\tilde{g}^{[0]}_{\mu\nu} = h_{\mu\nu}$ and

$$\mathbf{S}_{IJ} = O(\rho^{d/2-1}), \quad \mathbf{S} = O(\rho^{d/2}), \tag{53}$$

where $\mathbf{S}_{IJ}$ and $\mathbf{S}$ are Ricci tensor and Ricci scalar of the metric on $\mathbf{M}$. Symbol $\mathbf{F} = O(\rho^n)$ means that $\lim_{\rho\to 0}\rho^{-n}\mathbf{F}$ exists.

One can show that $\mathbf{S}_{0I} = 0$ and that $\mathbf{S}_{IJ}$ is t independent. Essentially, it is a function on $\tilde{M}$ (see [6])

$$\mathbf{S}_{\mu\nu} := \tilde{S}_{\mu\nu}, \quad \mathbf{S}_{\mu\infty} := \tilde{S}_{\mu\infty}, \quad \mathbf{S}_{\infty\infty} := \tilde{S}_{\infty\infty}. \tag{54}$$

We have (eq. 3.17 in [6])

$$\tilde{S}_{\mu\nu} = \rho\tilde{g}''_{\mu\nu} - \rho\tilde{g}^{\xi\chi}\tilde{g}'_{\xi\mu}\tilde{g}'_{\chi\nu} - \left(\frac{d}{2}-1\right)\tilde{g}'_{\mu\nu} - \frac{1}{2}\tilde{g}^{\xi\chi}\tilde{g}'_{\xi\chi}\tilde{g}_{\mu\nu} + \tilde{R}_{\mu\nu}, \tag{55}$$

$$\tilde{S}_{\mu\infty} = \frac{1}{2}\tilde{g}^{\xi\chi}\left(\tilde{\nabla}_\xi\tilde{g}'_{\mu\chi} - \tilde{\nabla}_\mu\tilde{g}'_{\xi\chi}\right), \tag{56}$$

$$\tilde{S}_{\infty\infty} = -\frac{1}{2}\tilde{g}^{\xi\chi}\tilde{g}''_{\xi\chi} + \frac{1}{4}\tilde{g}^{\xi\chi}\tilde{g}^{\mu\nu}\tilde{g}'_{\mu\xi}\tilde{g}'_{\nu\chi}, \tag{57}$$

where $\tilde{R}_{\mu\nu}$ denotes the Ricci tensor in the metric $\tilde{g}_{\mu\nu}$ depending on ρ. The equations (53) are equivalent to

$$\tilde{S}^{[n]}_{\mu\nu} = 0, \quad n = 0, \ldots d/2-2, \tag{58}$$

$$(\tilde{g}^{[0]})^{\mu\nu}\tilde{S}^{[d/2-1]}_{\mu\nu} = 0, \tag{59}$$

and then other components automatically vanish. Namely, (see [6] and compare with Proposition 26),

$$\tilde{S}_{\mu\infty} = O(\rho^{d/2-1}), \quad \tilde{S}_{\infty\infty} = O(\rho^{d/2-1}). \tag{60}$$

One can check that $\tilde{S}^{[n]}_{\mu\nu}$ is recursive till order $d/2-1$. Thus, we obtain[4]

$$\tilde{g}^{[n]}_{\mu\nu} = H^{(n)}_{\tilde{S},\,\mu\nu}(D^{2n}\tilde{g}^{[0]}), \quad n \leq d/2-1, \tag{61}$$

so higher orders of the metric are determined through $\tilde{g}^{[0]}_{\mu\nu} = h_{\mu\nu}$. The last equation $(\tilde{g}^{[0]})^{\mu\nu}\tilde{S}^{[d/2-1]}_{\mu\nu} = 0$ allows to compute the trace $\operatorname{tr}\tilde{g}^{[d/2]} = (\tilde{g}^{[0]})^{\mu\nu}\tilde{g}^{[d/2]}_{\mu\nu}$. The formula for $\tilde{S}^{[d/2-1]}_{\mu\nu}$ depends on $\tilde{g}^{[d/2]}$ only through the trace. It does not depend on the choice of the ambient metric.

The Fefferman–Graham obstruction tensor for $h_{\mu\nu}$ is defined by

$$H_{\mu\nu} = \tilde{S}^{[d/2-1]}_{\mu\nu}. \tag{62}$$

The constraints $H_{\mu\nu} = 0$ are equivalent to

$$\tilde{S}^{[n]}_{\mu\nu} = 0, \quad n = 0, \ldots d/2-1. \tag{63}$$

Let us notice, that the specific combination

$$\tilde{S}^{[d/2-1]}_{\mu\nu} - \frac{1}{d/2-1}\tilde{g}^{[0]}_{\mu\nu}\tilde{S}^{[d/2-2]}_{\infty\infty} \tag{64}$$

depends only on $\tilde{g}^{[k]}$ for $k \leq d/2-1$ and its derivatives. Importantly,

$$H_{\mu\nu} = \tilde{S}^{[d/2-1]}_{\mu\nu} - \frac{1}{d/2-1}\tilde{g}^{[0]}_{\mu\nu}\tilde{S}^{[d/2-2]}_{\infty\infty}, \tag{65}$$

because $\tilde{S}^{[d/2-2]}_{\infty\infty} = 0$ by (60). This form of the obstruction tensor does not involve $\operatorname{tr}\tilde{g}^{[d/2]}$.

[4] This recurrence procedure breaks down for $n = d/2-1$ and this is the source of the obstruction tensor.

4.1. Hyperbolicity of the Anderson–Fefferman–Graham equation. The system of (10) is not hyperbolic for the same reason as Einstein's gravity, because of the gauge transformations. The first step is to introduce a hyperbolic system in a specific gauge. We will use a natural gauge introduced in [3]. Then we show that this gauge is preserved in the evolution and as a result the obtained solution is also a solution of the Anderson–Fefferman–Graham equations. It is a standard treatment in gravity (see [12] for application to Einstein's equations).

Let us remind the following known identity (see [12])

$$\tilde{R}_{\mu\nu} = -\frac{1}{2}\tilde{g}^{\xi\chi}\partial_\xi\partial_\chi\tilde{g}_{\mu\nu} + \frac{1}{2}\left(\partial_\mu\tilde{F}_\nu + \partial_\nu\tilde{F}_\mu\right) + \ldots, \tag{66}$$

where ... means terms of order 1 (see Definition 2) and

$$\tilde{F}_\mu = \tilde{g}^{\xi\chi}\left(\partial_\xi\tilde{g}_{\chi\mu} - \frac{1}{2}\partial_\mu\tilde{g}_{\xi\chi}\right). \tag{67}$$

We notice that

$$\partial_\infty^2\tilde{F}_\mu = \tilde{g}^{\xi\chi}\left(\partial_\xi\tilde{g}''_{\chi\mu} - \frac{1}{2}\partial_\mu\tilde{g}''_{\xi\chi}\right) + \ldots, \tag{68}$$

where ... denotes terms of order at most 2. Comparing it with

$$\partial_\infty\tilde{S}_{\mu\infty} = \frac{1}{2}\tilde{g}^{\xi\chi}\left(\partial_\xi\tilde{g}''_{\mu\chi} - \partial_\mu\tilde{g}''_{\xi\chi}\right) + \ldots, \tag{69}$$

$$\partial_\mu\tilde{S}_{\infty\infty} = -\frac{1}{2}\tilde{g}^{\xi\chi}\partial_\mu\tilde{g}''_{\xi\chi} + \ldots, \tag{70}$$

we obtain the formula

$$\partial_\infty^2\tilde{F}_\mu = 2\partial_\infty\tilde{S}_{\mu\infty} - \partial_\mu\tilde{S}_{\infty\infty} + \ldots, \tag{71}$$

where ... denotes terms of order at most 2.

In order to write a slightly modified $\tilde{F}_\mu$ in terms of $\tilde{S}_{\mu\infty}$ and $\tilde{S}_{\infty\infty}$ we extend the notion of derivatives with respect to ρ. For $n > 0$ we introduce n-times integration of a multifield $\tilde{u}$ (a collection of fields on $\tilde{M}$)

$$\partial_\infty^{-n}\tilde{u}(x^\mu,\rho) = \int_0^\rho d\rho' \, \frac{(\rho-\rho')^{n-1}}{(n-1)!}\tilde{u}(x^\mu,\rho'), \tag{72}$$

that is $\partial_\infty^{-n}\sum_{k=0}u^{[k]}\rho^k = \sum_{k=0}\frac{1}{(k+1)\cdots(k+n)}u^{[k]}\rho^{k+n}$. Suppose that $\tilde{F}$ is of order N then $\partial_\infty^{-n}\tilde{F}$ is of order $N-n$.

We introduce additional tensors

$$\tilde{\gamma} = -\frac{1}{2}\tilde{g}^{[0]\xi\chi}\tilde{g}^{[1]}_{\xi\chi} + \partial_\infty^{-1}\tilde{S}_{\infty\infty}, \tag{73}$$

$$\tilde{G}_\mu = \tilde{F}^{[0]}_\mu + 2\partial_\infty^{-1}\tilde{S}_{\mu\infty} - \partial_\mu\partial_\infty^{-1}\tilde{\gamma}, \tag{74}$$

$$\tilde{E}_{\mu\nu} = \tilde{S}_{\mu\nu} - \frac{1}{2}(\tilde{\nabla}_\mu\tilde{G}_\nu + \tilde{\nabla}_\nu\tilde{G}_\mu) - \tilde{g}_{\mu\nu}\tilde{\gamma}, \tag{75}$$

These tensors will be used in our analysis of the AFG equations. The reason for occurence of additional term $\tilde{g}_{\mu\nu}\tilde{\gamma}$ is explained in the proof below.

Lemma 6. *The equation system*

$$\tilde{E}_{\mu\nu} = O(\rho^{d/2}) \tag{76}$$

is generalized hyperbolic and recursive in $\tilde{g}^{[n]}_{\mu\nu}$, $n = 0, \ldots, d/2 - 1$ *till order* $d/2 - 1$ *and it decouples at this order. Thus, it is well-posed.*

Proof. We will first prove that it is recursive till order $d/2 - 1$ and that it decouples at this order. Functions $\tilde{G}_\mu$ and $\tilde{\gamma}$ are of order 1. Hence, $\tilde{E}_{\mu\nu}$ is of order 2. Moreover, $\tilde{G}^{[n]}_\mu$ does not depend on $\tilde{g}^{[k]}_{\mu\nu}$, $k > n$ (nor its derivatives). The dependence of $\tilde{S}^{[n]}_{\mu\nu}$ and $\tilde{\gamma}^{[n]}$ on $D^m \tilde{g}^{[k]}_{\mu\nu}$, $k > n$ is only by linear terms in $\tilde{g}^{[n+1]}_{\mu\nu}$. In fact, we can compute

$$[\tilde{S}_{\mu\nu} - \tilde{g}_{\mu\nu}\tilde{\gamma}]^{[n]} = \left(n - \frac{d}{2} + 1\right)(n+1)\tilde{g}^{[n+1]}_{\mu\nu} + \ldots, \tag{77}$$

where $\ldots$ are terms depending on $D^2\tilde{g}^{[l]}_{\mu\nu}$ for $l \leq n$. For $n < d/2 - 1$ we can uniquely determined $\tilde{g}^{[n+1]}_{\mu\nu}$. Additionally, $[\tilde{S}_{\mu\nu} - \tilde{g}_{\mu\nu}\tilde{\gamma}]^{[d/2-1]}$ depends only on $\tilde{g}^{[k]}_{\mu\nu}$ for $k \leq d/2-1$ and their derivatives (see also (64)). Thus, the system is recursive and it decouples.

We need to show that $\tilde{E}_{\mu\nu}$ is of the form (35). As a preliminary step we prove that

$$\tilde{F}_\nu = \tilde{G}_\nu + \ldots, \tag{78}$$

where $\ldots$ denotes term of order 0. Indeed, $\tilde{F}^{[0]}_\nu = \tilde{G}^{[0]}_\nu$. Direct computation gives

$$\tilde{G}^{[1]}_\mu = [\partial_\infty \tilde{G}_\mu]^{[0]} = 2\tilde{S}^{[0]}_{\mu\infty} - \partial_\mu \tilde{\gamma}^{[0]}, \tag{79}$$

which can be compared to

$$\tilde{F}^{[1]}_\mu = 2\tilde{S}^{[0]}_{\mu\infty} + \frac{1}{2}\partial_\mu(\tilde{g}^{[0]\xi\chi}\tilde{g}^{[1]}_{\xi\chi}) + \ldots, \tag{80}$$

where $\ldots$ denotes terms depending on x and $D^m\tilde{g}^{[k]}_{\mu\nu}$ for $m + k \leq 1$. Finally, by (71) we obtain

$$\partial^2_\infty \tilde{F}_\nu = \partial^2_\infty \tilde{G}_\nu + \ldots, \tag{81}$$

where $\ldots$ denotes term of order 2. This shows (78).

We thus have

$$\tilde{\nabla}_\mu \tilde{F}_\nu = \tilde{\nabla}_\mu \tilde{G}_\nu + \ldots, \tag{82}$$

where $\ldots$ denotes term of order 1 (both $\tilde{F}_\mu$ and $\tilde{G}_\mu$ depends only on up to first derivatives of the metric). Taking this and (66) into account, the following yields

$$\begin{aligned}\tilde{E}_{\mu\nu} &= \tilde{S}_{\mu\nu} - \frac{1}{2}(\tilde{\nabla}_\mu \tilde{G}_\nu + \tilde{\nabla}_\nu \tilde{G}_\mu) - \tilde{g}_{\mu\nu}\tilde{\gamma} \\ &= \tilde{R}_{\mu\nu} - \frac{1}{2}(\tilde{\nabla}_\mu \tilde{F}_\nu + \tilde{\nabla}_\nu \tilde{F}_\mu) + \ldots = -\frac{1}{2}\tilde{g}^{\xi\chi}\partial_\xi\partial_\chi\tilde{g}_{\mu\nu} + \ldots,\end{aligned} \tag{83}$$

where $\ldots$ is of order 1. Expanding first term the form described in (35) is obtained. Well-posedness follows from Proposition 3. □

4.1.1. Propagation of the gauge In this section we will explain that $\tilde{G}_\mu = O(\rho^{d/2})$ and $\tilde{\gamma} = O(\rho^{d/2-1})$ provided that these functions vanish on the initial surface together with their time derivatives and secondly $\tilde{E} = O(\rho^{d/2})$. As usual this is achieved by showing that these variables obey a system of linear hyperbolic equations.

Let us introduce two tensors

$$\tilde{B}^1_\mu = -\frac{1}{2}\tilde{\nabla}^\xi\tilde{\nabla}_\xi\tilde{G}_\mu - \frac{1}{2}\tilde{R}^\nu_\mu\tilde{G}_\nu - \left(\frac{d}{2} - 1 - \rho\partial_\infty\right)\partial_\infty\tilde{G}_\mu + \frac{1}{2}\tilde{g}^{\xi\chi}\tilde{g}'_{\xi\chi}\rho\partial_\infty\tilde{G}_\mu + \frac{1}{2}\rho\tilde{g}^{\xi\chi}\tilde{g}'_{\xi\chi}\partial_\mu\tilde{\gamma}, \tag{84}$$

$$\tilde{B}^2 = -\frac{1}{2}\tilde{\nabla}^\mu\partial_\mu\tilde{\gamma} - \left(\frac{d}{2} - 2 - \rho\partial_\infty\right)\partial_\infty\tilde{\gamma} + \tilde{g}^{\mu\nu}\tilde{g}'_{\mu\nu}\rho\partial_\infty\tilde{\gamma} + \frac{1}{2}\tilde{Q}^\mu\tilde{G}_\mu + \frac{1}{2}\tilde{g}'_{\mu\nu}\tilde{\nabla}^\mu\tilde{G}^\nu + \frac{1}{2}\tilde{g}'_{\mu\nu}\tilde{g}^{\mu\nu}\tilde{\gamma}, \tag{85}$$

where $\tilde{Q}^\mu = \partial_\infty(\tilde{g}^{\nu\xi}\tilde{\Gamma}^\mu_{\nu\xi})$ and $\tilde{\Gamma}^\mu_{\xi\chi}$ is ρ-dependent Christoffel symbol.

We will prove in Proposition 30 of Appendix A that if $\tilde{E} = O(\rho^{d/2})$ then $\tilde{B}^1_\mu = O(\rho^{d/2})$ and $\tilde{B}^2 = O(\rho^{d/2-1})$. The next two lemmas will show that the system of equations $\tilde{B}^1_\mu = O(\rho^{d/2})$ and $\tilde{B}^2 = O(\rho^{d/2-1})$ allows us to deduce that the gauge functions vanish ($\tilde{G}_\mu = O(\rho^{d/2})$ and $\tilde{\gamma} = O(\rho^{d/2-1})$) if their initial data vanish on the Cauchy surface.

Lemma 7. *The equation system*

$$\tilde{B}^1_\mu = O(\rho^{d/2}),\ \tilde{B}^2 = O(\rho^{d/2-1}) \tag{86}$$

is linear generalized hyperbolic for $\tilde{G}^{[n]}_\mu$, $n = 0, \ldots d/2-1$ *and* $\tilde{\gamma}^{[n]}$, $n = 0, \ldots d/2-2$. *Moreover, if* (86) *is satisfied and at a point* $x \in M$ *the following equations are true*

$$D^{d-2}\tilde{G}^{[0]}_\mu(x) = 0, \quad D^{d-3}\tilde{\gamma}^{[0]}(x) = 0, \tag{87}$$

then

$$D^{d-2k-2}\tilde{G}^{[k]}_\mu(x) = 0,\ k = 0, \ldots d/2-1, \quad D^{d-2k-3}\tilde{\gamma}^{[k]}(x) = 0,\ k = 0, \ldots d/2-2. \tag{88}$$

Proof. Inspection of the equations shows that

$$\tilde{B}^1_\mu = -\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\partial_\nu\tilde{G}_\mu + \tilde{F}^1_\mu, \quad \tilde{B}^2 = -\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\partial_\nu\tilde{G}_\mu + \tilde{F}^2, \tag{89}$$

where $\tilde{F}^1_\mu$ and $\tilde{F}^2$ are of order 1. Generalized hyperbolicity follows from two facts which ensure that the system decouples:

1. The dependence of $[\tilde{B}^1_\mu]^{[n]}$ on $D^m\tilde{G}^{[k]}_\nu$ for $k > n$ is by

$$\left[\left(\frac{d}{2} - 1 - \rho\partial_\infty\right)\partial_\infty\tilde{G}_\mu\right]^{[n]} = \left(\frac{d}{2} - 1 - n\right)(n+1)\left[\tilde{G}_\mu\right]^{[n+1]}, \tag{90}$$

   and it does not depend on $D^m\tilde{\gamma}^{[k]}$ for $k \geq n$.

2. The dependence of $[\tilde{B}^2]^{[n]}$ on $D^m \tilde{\gamma}^{[k]}$ for $k > n$ is by

$$\left[\left(\frac{d}{2} - 2 - \rho\partial_\infty\right)\partial_\infty\tilde{\gamma}\right]^{[n]} = \left(\frac{d}{2} - 2 - n\right)(n+1)\left[\tilde{\gamma}\right]^{[n+1]}, \tag{91}$$

and it does not depend on $D^m \tilde{G}_\mu^{[k]}$ for $k > n$.

Let us now prove the second statement of the lemma by induction on k_0. Suppose that for all $0 \leq k < k_0$

$$D^{d-2k-2}\tilde{G}_\mu^{[k]} = 0, \quad D^{d-2k-3}\tilde{\gamma}^{[k]}(x) = 0, \tag{92}$$

then taking up to $d - 2k_0 - 2$ derivatives of $[\tilde{B}_\mu^1]^{[k_0-1]} = 0$ and up to $d - 2k_0 - 3$ derivatives of $[\tilde{B}^2]^{[k_0-1]} = 0$ we get due to (92)

$$D^{d-2k_0-2}\tilde{G}_\mu^{[k_0]} = 0, \quad D^{d-2k_0-3}\tilde{\gamma}^{[k_0]}(x) = 0, \tag{93}$$

which shows the induction together with a trivial statement for $k_0 = 1$. □

As a result, we obtain:

Lemma 8. *Suppose that $\tilde{E}_{\mu\nu} = O(\rho^{d/2})$ and on the initial surface*

$$\tilde{G}_\mu|_\Sigma = \partial_1\tilde{G}_\mu|_\Sigma = O(\rho^{d/2}), \ \tilde{\gamma}|_\Sigma = \partial_1\tilde{\gamma}|_\Sigma = O(\rho^{d/2-1}), \tag{94}$$

then $\tilde{G}_\mu = O(\rho^{d/2})$ and $\tilde{\gamma} = O(\rho^{d/2-1})$.

Proof. It follows from Proposition 30 that $\tilde{B}_\mu^1 = O(\rho^{d/2})$ and $\tilde{B}^2 = O(\rho^{d/2-1})$ and by generalized hyperbolicity the solution is unique. From linearity, it is just zero. □

Additionally, we have

$$\tilde{\gamma}^{[d/2-1]} = -\frac{1}{2(d/2-1)} g^{[0]\mu\nu}\tilde{g}_{\mu\nu}^{[d/2]} + \ldots, \tag{95}$$

where … denote terms depending only on $\tilde{g}_{\mu\nu}^{[n]}$ for $n \leq d/2 - 1$. By modifying $\tilde{g}_{\mu\nu}^{[d/2]}$ we can assume that $\tilde{\gamma}^{[d/2-1]} = 0$.

4.1.2. Gauge fixing conditions We assume that on the initial surface

$$D^{d-2}\tilde{G}_\mu^{[0]}|_\Sigma = 0, \quad D^{d-3}\tilde{\gamma}^{[0]}|_\Sigma = 0, \tag{96}$$

and prove that in such a case

$$\tilde{G}_\mu|_\Sigma = \partial_1\tilde{G}_\mu|_\Sigma = O(\rho^{d/2}), \ \tilde{\gamma}|_\Sigma = \partial_1\tilde{\gamma}|_\Sigma = O(\rho^{d/2-1}). \tag{97}$$

We show it by noticing that the equations $\tilde{B}_\mu^1 = O(\rho^{d/2})$, $\tilde{B}^2 = O(\rho^{d/2-1})$ hold. We can invoke Lemma 7 to show that

$$D^{d-2k-2}\tilde{G}_\mu^{[k]}|_\Sigma = 0, \ k \leq \frac{d}{2} - 1, \quad D^{d-2k-3}\tilde{\gamma}^{[k]}|_\Sigma = 0, \ k \leq \frac{d}{2} - 2. \tag{98}$$

Comparing with (97) we see that the missing condition is $\partial_1\tilde{G}_\mu^{[d/2-1]}|_\Sigma = 0$.

Lemma 9. *Suppose that $\tilde{E}_{\mu\nu} = O(\rho^{d/2})$ and on the Cauchy surface Σ*

$$H(\vec{N}, \cdot)|_\Sigma = 0, \quad \partial_1^k \tilde{G}_\mu^{[n]}|_\Sigma = 0, \ \partial_1^k \tilde{\gamma}^{[n]}|_\Sigma = 0, \ k + 2n \leq d - 2, \tag{99}$$

then $\partial_1 \tilde{G}_\mu^{[d/2-1]}|_\Sigma = 0$ and $D^{d-2k-1}\tilde{g}_{\mu\nu}^{[k]}|_\Sigma = D^{d-2k-1} H_{\tilde{S},\ \mu\nu}^{(k)}(\tilde{g}_{\xi\chi}^{[0]})|_\Sigma$.

Proof. We will assume for simplicity that $\vec{N} = \partial_1$. The modification of the proof for the general case is minor. We have the identity for jets

$$D^{d-2n-3}\tilde{S}_{\mu\nu}^{[n]}|_\Sigma = D^{d-2n-3}\left[\tilde{E}_{\mu\nu} + \frac{1}{2}(\tilde{\nabla}_\mu \tilde{G}_\nu + \tilde{\nabla}_\nu \tilde{G}_\mu) + \tilde{g}_{\mu\nu}\tilde{\gamma}\right]^{[n]}|_\Sigma = 0 \tag{100}$$

From Lemma 4 the jets of the expansion of the ambient metric

$$D^{d-2k-1}\tilde{g}_{\mu\nu}^{[k]}|_\Sigma, \tag{101}$$

agree with the Fefferman–Graham ambient metric construction. As $\tilde{S}_{1\nu}^{[d/2-1]}$ depends only on first derivatives of $\tilde{g}_{\mu\nu}^{[d/2-1]}$ in x^1 direction, we deduce

$$0 = H_{1\nu}(h)|_\Sigma = \tilde{S}_{1\nu}^{[d/2-1]}|_\Sigma - \frac{1}{d/2-1}\tilde{g}_{1\nu}^{[0]}\tilde{S}_{\infty\infty}^{[d/2-2]}|_\Sigma = \tilde{S}_{1\nu}^{[d/2-1]}|_\Sigma, \tag{102}$$

by using (65) and $\tilde{S}_{\infty\infty}^{[d/2-2]}|_\Sigma = 0$ (due to $\tilde{\gamma}^{[d/2-1]}|_\Sigma = 0$). Additionally, from $\tilde{\gamma}|_\Sigma = O(\rho^{d/2})$ and $\tilde{G}_\mu|_\Sigma = O(\rho^{d/2})$ it follows that

$$\begin{aligned}0 &= \tilde{S}_{1\nu}^{[d/2-1]}|_\Sigma = \tilde{E}_{1\nu}^{[d/2-1]}|_\Sigma + \frac{1}{2}([\tilde{\nabla}_1\tilde{G}_\nu]^{[d/2-1]}|_\Sigma + [\tilde{\nabla}_\nu\tilde{G}_1]^{[d/2-1]}|_\Sigma) + [\tilde{g}_{1\nu}\tilde{\gamma}]|_\Sigma^{[d/2-1]} \\ &= \frac{1}{2}\partial_1\tilde{G}_\nu^{[d/2-1]}|_\Sigma,\end{aligned} \tag{103}$$

for $\nu \neq 1$. Similarly, for $\nu = 1$ we get

$$\begin{aligned}0 &= \tilde{S}_{11}^{[d/2-1]}|_\Sigma = \tilde{E}_{11}^{[d/2-1]}|_\Sigma + \frac{1}{2}([\tilde{\nabla}_1\tilde{G}_1]^{[d/2-1]}|_\Sigma + [\tilde{\nabla}_1\tilde{G}_1]^{[d/2-1]}|_\Sigma) + [\tilde{g}_{11}\tilde{\gamma}]|_\Sigma^{[d/2-1]} \\ &= \partial_1\tilde{G}_1^{[d/2-1]}|_\Sigma,\end{aligned} \tag{104}$$

showing the statement of the lemma. □

We can combine the results obtained so far to prove a proposition about existence of the solution in the Anderson–Chruściel gauge. We consider initial data $\partial_1^k h_{\mu\nu}|_\Sigma$, $k = 0, \ldots d-1$ subject to the conditions on the initial surface Σ:

$$D^{d-2}(\Box x^\nu)|_\Sigma = 0, \quad D^{d-3}(R)|_\Sigma = 0, \ H(\vec{N}, \cdot)|_\Sigma = 0. \tag{105}$$

The conditions are well-defined because $\Box x^\mu$ depends on jets of the metric up to the first order, R depends on jets of the metric up to the second order and $H(\vec{N}, \cdot)$ depends on $d-1$ order jets of the metric on Σ (see [4] p. 564-565 for a thorough discussion).

Proposition 10. *We consider initial data $\partial_1^k h_{\mu\nu}|_\Sigma$, $k = 0, \ldots d-1$. Suppose that on the initial surface Σ conditions (105) are satisfied. Then there exists a unique solution to AFG system $H_{\mu\nu} = 0$ with the given initial data and which satisfies $\Box x^\nu = 0$, $R = 0$.*

Proof. As $\tilde{E}_{\mu\nu}$ is recursive till order $d/2-1$ we compute by Lemma 4 the initial value data for the system by

$$\tilde{h}^{[k]}_{\mu\nu}|_\Sigma = H^{(k)}_{\tilde{E}\;\mu\nu}(D^{2k}h_{\mu\nu}|_\Sigma), \quad \partial_1\tilde{h}^{[k]}_{\mu\nu}|_\Sigma = \partial_1 H^{(k)}_{\tilde{E}\;\mu\nu}(D^{2k}h_{\mu\nu}|_\Sigma), \quad k \le d/2-1. \tag{106}$$

This allows us to determine the initial data for the equation $\tilde{E} = O(\rho^{d/2})$. We consider a unique solution $\tilde{g}_{\mu\nu}$ of this equation. Since the system is generalized hyperbolic, we get

$$\partial_1^n \tilde{g}^{[0]}_{\mu\nu}|_\Sigma = \partial_1^n h_{\mu\nu}|_\Sigma, \quad n \le d-1, \tag{107}$$

by Lemma 5. Thus, the solution has prescribed initial data.

In particular, it is true that

$$D^{d-2}\tilde{G}^{[0]}_\mu|_\Sigma = D^{d-2}\left(h_{\mu\nu}h^{\xi\chi}\Gamma^\nu_{\xi\chi}\right) = -D^{d-2}(h_{\mu\nu}\Box x^\nu) = 0, \tag{108}$$

and by (55), (75) and (73)

$$D^{d-3}\left[R_{\mu\nu} - \left(\frac{d}{2}-1\right)\tilde{g}^{[1]}_{\mu\nu}\right]\Bigg|_\Sigma = D^{d-3}\left[\tilde{E}_{\mu\nu} + \frac{1}{2}(\tilde{\nabla}_\mu\tilde{G}_\nu + \tilde{\nabla}_\nu\tilde{G}_\mu)\right]^{[0]}\Bigg|_\Sigma = 0. \tag{109}$$

We take the trace of this equality to derive

$$D^{d-3}\tilde{\gamma}^{[0]}|_\Sigma = D^{d-3}\left[-\frac{1}{2}\tilde{g}^{[0]\xi\chi}\tilde{g}^{[1]}_{\xi\chi}\right]\Bigg|_\Sigma = -D^{d-3}\left.\frac{R}{d-2}\right|_\Sigma = 0, \tag{110}$$

This means that the condition (96) is satisfied. From Lemmas 8 and 9 we conclude

$$\tilde{G}_\mu = O(\rho^{d/2}), \;\; \tilde{\gamma} = O(\rho^{d/2-1}). \tag{111}$$

We can always assume $\tilde{\gamma}^{[d/2-1]} = 0$ (see (95) and (65)). Taking this into account, we obtain

$$\tilde{S}^{[n]}_{\mu\nu} = \tilde{E}^{[n]}_{\mu\nu} + \frac{1}{2}([\tilde{\nabla}_\mu\tilde{G}_\nu]^{[n]} + [\tilde{\nabla}_\nu\tilde{G}_\mu]^{[n]}) + [\tilde{g}_{\mu\nu}\tilde{\gamma}]^{[n]} = 0, \tag{112}$$

for $n \le d/2-1$. We have a solution with $\tilde{S}_{\mu\nu} = O(\rho^{d/2})$. $\square$

4.2. The AFG equation in the Anderson–Chruściel gauge. In this section, the correspondence of our gauge functions to $R = 0$ and $\Box x^\mu = 0$ gauge will be investigated. This gauge was proposed in [3] and [4].

4.2.1. Uniqueness of the solution of the AFG equation Assume that we have a solution $H_{\mu\nu}(h) = 0$ with the given initial conditions at Σ.

Lemma 11. [3,4] *Suppose that we have a local coordinate system $y^2, \ldots y^d$ on Σ. Locally there exists a coordinate system x^μ and a conformal factor σ such that for $h'_{\mu\nu} = e^{2\sigma}h_{\mu\nu}$,*

1. $\Box' x^\mu = 0$, $R' = 0$,
2. $x^\xi|_\Sigma = y^\xi$ *for* $\xi = 2, \ldots d$, $x^1|_\Sigma = 0$,
3. ∂_1 *is a unit normal vector to* Σ

Here $\Box'$ is a scalar d'Alembert operator with respect to the metric h'. Function σ is freely specified by initial data $\sigma|_\Sigma$ and $\partial_1\sigma|_\Sigma$.

Proof. First we find σ as a solution of Yamabe problem (that is $R[e^{2\sigma}h] = 0$), which is a nonlinear hyperbolic system for σ (see [4] for discussion). Define x^ξ for $\xi \geq 2$ as a unique solution to $\Box'\phi = 0$ with initial conditions (local development)

$$\phi|_U = y^\xi, \quad N^\mu \partial_\mu \phi|_U = 0, \tag{113}$$

where N^μ is a unit normal to Σ. Finally, we define x^1 as a unique solution with initial value

$$\phi|_U = 0, \quad N^\mu \partial_\mu \phi|_U = 1. \tag{114}$$

One can check that on $U' \subset U$ this new coordinates are independent, so it is also true in the small neighborhood of the Cauchy surface. $\square$

We can thus work in this gauge. From the solution to $H_{\mu\nu} = 0$ we now construct iteratively

$$\tilde{g}^{[n+1]}_{\mu\nu} \quad n \leq d/2 - 2. \tag{115}$$

by (61). Let us notice that $\tilde{g}^{[0]}_{\mu\nu} = h_{\mu\nu}$ and $\tilde{g}^{[1]}_{\mu\nu} = 2P_{\mu\nu}$

where $P_{\mu\nu} = \frac{1}{d-2}\left(R_{\mu\nu} - \frac{1}{2(d-1)}Rh_{\mu\nu}\right)$ is the Schouten tensor [6]. Due to the gauge condition, one obtains

$$\tilde{G}^{[0]}_\mu = h_{\mu\nu}h^{\xi\chi}\Gamma^\nu_{\xi\chi} = -h_{\mu\nu}\Box x^\nu = 0, \quad \tilde{\gamma}^{[0]} = -h^{\mu\nu}P_{\mu\nu} = -\frac{1}{2(d-1)}R = 0. \tag{116}$$

Moreover, from $\tilde{S}_{\mu\infty} = O(\rho^{d/2-1})$ and $\tilde{S}_{\infty\infty} = O(\rho^{d/2-1})$ we have[5]

$$\tilde{G}_\mu = O(\rho^{d/2}), \quad \tilde{\gamma} = O(\rho^{d/2}). \tag{117}$$

Additionally, $\tilde{S}_{\mu\nu} = O(\rho^{d/2})$ and so

$$\tilde{E}_{\mu\nu} = O(\rho^{d/2}). \tag{118}$$

From the uniqueness of the solution of (118) we obtain the uniqueness of the solution of AFG equation (in the given gauge).

4.2.2. Existence of the solutions of the AFG equation Let us now assume that the initial data is given by

$$\partial_1^n h_{\mu\nu}|_\Sigma, \quad n = 0, \dots d-1, \tag{119}$$

which satisfies the constraints $H(\vec{N}, \cdot)|_\Sigma = 0$ and the gauge is satisfied:

$$\partial_1^n \Box x^\mu|_\Sigma = 0, \quad n = 0, \dots d-2, \tag{120}$$

$$\partial_1^n R|_\Sigma = 0, \quad n = 0, \dots d-3. \tag{121}$$

The conditions are well-defined because $\Box x^\mu$ depends on the jets of the metric up to first order and R depends on the jets of the metric up to the second order.

[5] Notice that the additional normalization in the Fefferman–Graham ambient metric allows us to obtain vanishing of $\tilde{\gamma}$ to one order higher than from propagation equation. This should be compare with (95).

By a formal coordinate change on $\Sigma \subset M$ we can always assume these conditions together with $\vec{N} = \partial_1$ where N is a normal vector. In this way, the constraints take the form

$$H_{1\mu}|_{\Sigma} = 0. \tag{122}$$

We also compute

$$D^{d-2}\tilde{G}^{[0]}_{\mu}|_{\Sigma} = -D^{d-2}(h_{\mu\nu}\Box x^{\nu})|_{\Sigma} = 0, \quad D^{d-3}\tilde{\gamma}^{[0]}|_{\Sigma} = \frac{1}{2(d-2)}D^{d-3}R|_{\Sigma} = 0, \tag{123}$$

by (120) and (121). The existence of the solution of AFG equation would follow from Proposition 10 if we were able to use this gauge globally on a compact Σ. However, it is not possible (see [4] Remark 4.4 for discussion), so we need to apply some version of a gluing argument.

4.2.3. Proof of the Theorem 1 The harmonic gauge is well-suited for $\Sigma = \mathbb{R}^{d-1}$. If we want to apply our result, we need to extend the notion of this gauge to compact Cauchy surfaces. This can be done in the case of Σ being a torus, where x^{μ} for $\mu = 2, \ldots d$ are defined modulo 2π. Due to the finite speed of propagation, the method provides an existence and uniqueness result also for open subsets of the torus. Uniqueness of the development allows to apply the standard gluing argument [12] to obtain Theorem 1.

4.3. Infinite order extension of the ambient metric. Suppose that the obstruction tensor vanishes. The results of [6] show that Taylor expansions of the metrics, which are Ricci flat of the order $O(\rho^{\infty})$, are in one-to-one correspondence with the traceless symmetric tensors $k_{\mu\nu}$ satisfying

$$\nabla^{\mu}k_{\mu\nu} = D_{\nu}, \tag{124}$$

where D_{ν} is a certain 1-form (defined in eq. 3.36 in [6]). The tensors $k_{\mu\nu}$ define trace-free part of $\tilde{g}^{[d/2]}_{\mu\nu}$ since the trace is already determined. In the case of Euclidean signature manifolds, it is not obvious that such a tensor exists. We will prove that this is the case for any globally hyperbolic spacetime.

Proposition 12. *There exists $k_{\mu\nu}$ satisfying (124) on a AFG globally hyperbolic spacetime.*

Proof. We will look for the tensor given in a special form

$$k_{\mu\nu} = \nabla_{\mu}u_{\nu} + \nabla_{\nu}u_{\mu} - \frac{2}{d}h_{\mu\nu}\nabla^{\rho}u_{\rho}, \tag{125}$$

for some covector field u_{μ}. It is already symmetric, traceless and the equation (124) takes a form

$$D_{\nu} = \nabla^{\mu}\nabla_{\mu}u_{\nu} + R^{\rho}_{\nu}u_{\rho} + \left(1 - \frac{2}{d}\right)\nabla_{\nu}(\nabla^{\rho}u_{\rho}). \tag{126}$$

Taking the divergence, we obtain an additional equation

$$\nabla^{\nu}D_{\nu} = \nabla^{\mu}\nabla_{\mu}(\nabla^{\nu}u_{\nu}) + 2\nabla^{\mu}(R^{\nu}_{\mu}u_{\nu}) + \left(1 - \frac{2}{d}\right)\nabla^{\mu}\nabla_{\mu}(\nabla^{\nu}u_{\nu}), \tag{127}$$

where we used $\nabla^\nu\nabla^\mu\nabla_\mu u_\nu = \nabla^\mu\nabla_\mu(\nabla^\nu u_\nu) + \nabla^\mu(R_\mu^\nu u_\nu)$. We now introduce a new variable $A = \frac{1}{d}\nabla^\mu u_\mu$ and a system of equations (equivalent to (126) and (127), respectively)

$$\nabla^\mu\nabla_\mu u_\nu + R_\nu^\rho u_\rho + (d-2)\,\nabla_\nu A = D_\nu, \tag{128}$$
$$2(d-1)\nabla^\mu\nabla_\mu A + 2\nabla^\mu(R_\mu^\nu u_\nu) = \nabla^\nu D_\nu. \tag{129}$$

It is indeed a hyperbolic second-order linear system if we regard variables A and u_ν as independent. Thus, with any given initial data on a Cauchy surface, it has a solution. We need to show that A and u_ν obtained by evolution satisfies $dA = \nabla^\nu u_\nu$ if the initial data is chosen properly. Substracting (129) from the divergence of (128) we obtain an equation:

$$\nabla^\mu\nabla_\mu(\nabla^\rho u_\rho - dA) = 0, \tag{130}$$

which is satisfied for every solution of the system (128), (129). If we choose the initial data such that $A|_\Sigma = d^{-1}\nabla^\rho u_\rho|_\Sigma$ and $\partial_1 A|_\Sigma = d^{-1}\partial_1\nabla^\rho u_\rho|_\Sigma$ (computed by the Cauchy–Kovalevskaya algorithm), then $A = \frac{1}{d}\nabla^\rho u_\rho$ in the whole spacetime and $\nabla^\mu k_{\mu\nu} = D_\nu$. □

We can thus always assume that the metric is Ricci flat to an infinite order, but it is not uniquely defined except terms $\tilde{g}_{\mu\nu}^{[n]}$ for $n \leq d/2 - 1$ and $\operatorname{tr}\tilde{g}^{[d/2]}$.

5. GJMS Type Operators for Tractor Bundles

We will now concentrate on various linear systems, which arise by the ambient metric construction in a recursive fashion simlar to the way the gauge fixed AFG equations are obtained from $\tilde{E}_{\mu\nu}$ system. They share the common property with the gauged fixed AFG equation, that the principal symbol is a power of the d'Alembert operator. In this part of the paper, we will also shortly describe Graham–Jenne–Mason–Sparling (GJMS) type systems. A quite general method of introducing this type of operators is by tractor calculus [21,22]. We will only focus on the essential parts of this theory in terms of the ambient metric construction (see [23]). For the short review of the tractor calculus in application to general relativity, we refer the reader to [17]. We only use tractors with lowered indices.

5.1. Tractors from the ambient metrics. We work on manifold $\mathbf{M} = \mathbb{R}\times\tilde{M}$, $\mathbf{T} = t\partial_t$ is a conformal Killing vector with property that for every vector field $\mathbf{F}^I$ it satisfies

$$\nabla_{\mathbf{F}}\mathbf{T} = \mathbf{F}, \quad \nabla_I\mathbf{T}_J = \mathbf{g}_{IJ}. \tag{131}$$

We also introduce $\mathbf{\Omega} = \frac{1}{2}\mathbf{T}^I\mathbf{T}_I = \rho t^2$ with the properties:

$$\nabla_I\mathbf{\Omega} = \mathbf{T}^J\nabla_I\mathbf{T}_J = \mathbf{T}_I, \quad \nabla^I\nabla_I\mathbf{\Omega} = \nabla^I\mathbf{T}_I = (d+2), \quad \nabla_I\mathbf{\Omega}\nabla^I\mathbf{\Omega} = 2\mathbf{\Omega}. \tag{132}$$

An important submanifold $\mathbf{N} = \{\mathbf{\Omega} = 0\} = \{\rho = 0\}$ is preserved by $\mathbf{T}$. It can be identified with a tautological bundle of conformal scales over M [6]. Results of [23] allows us to define tractors in the following way.

Definition 5. An n-tractor of weight $w \in \mathbb{R}, \mathcal{U}_{I_1 \dots I_n}$ is a section of a bundle $(T^*\mathbf{M}|_{\mathbf{N}})^{\otimes n}$ over $\mathbf{N} \subset \mathbf{M}$ with the property

$$\mathcal{L}_{\mathbf{T}} \mathcal{U}_{I_1 \dots I_n} = (w + n) \mathcal{U}_{I_1 \dots I_n}. \tag{133}$$

Simplifying the notation, we will often skip tractor indices if it does not cause confusion. By evaluating $\mathcal{U}_{I_1 \dots I_n}$ at $t = 1$ we can identify tractors with sections of the bundle $\mathcal{T}^{\otimes n}$

$$U_{I_1 \dots I_n} = \mathcal{U}_{I_1 \dots I_n}|_{t=1} \in \Gamma(\mathcal{T}^{\otimes n}), \quad \text{where } \mathcal{T} = T^*\mathbf{M}|_{t=1,\rho=0} = \mathbb{R} \oplus T^*M \oplus \mathbb{R}, \tag{134}$$

We used symbol $\mathbb{R}$ to denote the trivial 1-dimensional vector bundle over M. The fiber basis of $\mathcal{T}$ is $dt, dx^{\mu}, d\rho$. We will call both $\mathcal{U}_{I_1 \dots I_n}$ and $U_{I_1 \dots I_n}$ tractors, but to avoid confusion we denote the tractors as functions over $\mathbf{N}$ by caligraphic letters. We will work mostly with the restrictions to $t = 1$. Importance of the tractors is based on their simple transformation law under conformal change of the metric. Let us remind that every conformal transformation $h_{\mu\nu} \longrightarrow e^{2\sigma} h_{\mu\nu}$ induces a (formal) diffeomorphism of the ambient space [6].[6] This diffeomorphism Ψ is uniquely determined by conditions (see [6] Theorem 2.3 and Proposition 2.6):

1. It preserves submanifold $\mathbf{N} = \{\rho = 0\}$ and

$$\Psi|_{\mathbf{N}}(t, x^{\mu}, 0) = (e^{\sigma} t, x^{\mu}, 0). \tag{135}$$

2. It preserves the form of the ambient metric described in (52).

This diffeomorphism preserves not only submanifold $\mathbf{N}$, but also the function $\mathbf{\Omega}$ and the vector field $\mathbf{T}$. As a consequence, it induces also transformation of the tractor. It is easy to check that this transformation is linear (see [17] for direct derivation by the Cartan method).

There is a natural way to obtain tractors. For every n-covector $\mathbf{U}$ in the ambient space satisfying

$$\mathcal{L}_{\mathbf{T}} \mathbf{U}_{I_1 \dots I_n} = (w + n) \mathbf{U}_{I_1 \dots I_n}, \tag{136}$$

its restriction to $\mathbf{N} = \{\rho = 0\}$ is an n-tractor of weight w, namely

$$\mathcal{U} := \mathbf{U}|_{\rho=0} \text{ or in our identification } U := \mathbf{U}|_{t=1,\rho=0}. \tag{137}$$

Using our identification we can regard $\tilde{U} = \mathbf{U}|_{t=1}$ as a ρ-dependent family of tractors. It should be stress that this identification is not conformally covariant except for $\rho = 0$.

A choice of homogeneity $w + n$ in the definition of the weight will simplify certain computations as it is explained by the following lemma:

Lemma 13. *Suppose that* $\mathcal{L}_{\mathbf{T}} \mathbf{U}_{I_1 \dots I_n} = (w + n) \mathbf{U}_{I_1 \dots I_n}$ *then* $\nabla_{\mathbf{T}} \mathbf{U} = w \mathbf{U}$*, where n is a valency of the field.*

[6] It is a formal diffeomorphism defined by a series in ρ. In the even dimension the ambient metric is determined uniquely only till the order $O(\rho^{d/2})$. The extension to the infinite order is an additional data, which also need to transform accordingly under conformal transformations.

Proof. We use induction on n. For $n = 0$ (that means $\mathbf{U}$ is a functions) Lie derivative and covariant derivative agrees. Suppose that the result is true for $n < n_0$. Consider a vector field $\mathbf{F}^I$ with homogeneity 0 ($\mathcal{L}_{\mathbf{T}}\mathbf{F} = 0$). We have

$$\nabla_{\mathbf{T}}\mathbf{F} = \mathcal{L}_{\mathbf{T}}\mathbf{F} + \nabla_{\mathbf{F}}\mathbf{T} = \mathbf{F}. \tag{138}$$

For any $\mathbf{U}_{I_1\ldots I_{n_0}}$, n_0 covector with weight w, we denote by $\mathbf{F}\llcorner\mathbf{U}$

$$\mathbf{U}_{I_1\ldots I_{n_0}}\mathbf{F}^{I_1}. \tag{139}$$

It is a $n_0 - 1$ covector with weight $w + 1$, thus

$$(w+1)\mathbf{F}\llcorner\mathbf{U} = \nabla_{\mathbf{T}}(\mathbf{F}\llcorner\mathbf{U}) = \mathbf{F}\llcorner(\nabla_{\mathbf{T}}\mathbf{U}) + (\nabla_{\mathbf{T}}\mathbf{F})\llcorner\mathbf{U} = \mathbf{F}\llcorner(\nabla_{\mathbf{T}}\mathbf{U}) + \mathbf{F}\llcorner\mathbf{U}. \tag{140}$$

So $\mathbf{F}\llcorner[w\mathbf{U} - \nabla_{\mathbf{T}}\mathbf{U}] = 0$. As the restriction to $t = 1$, $\tilde{F}^I$ is arbitrary, we show the induction. □

5.2. GJMS operators. The GJMS operators were introduced in [13] (initially for scalar functions) as conformal powers of the Laplacian. They are constructed with the help of the d'Alembert operator in the ambient space $\mathbf{M}$. We consider an operator $\boxdot$ defined on n-covectors:

$$\boxdot\mathbf{U} = \nabla^I\nabla_I\mathbf{U}. \tag{141}$$

The result has the weight $w - 2$. We can thus define

$$\tilde{\boxdot}_w\tilde{U} = [\boxdot\mathbf{U}]|_{t=1} \text{ for } \tilde{U} = \mathbf{U}|_{t=1},\ \mathcal{L}_{\mathbf{T}}\mathbf{U} = (w+n)\mathbf{U}. \tag{142}$$

Proposition 14. *For any n, the operator $\tilde{\boxdot}_w$ on ρ-dependent n-tractors $\tilde{U}$ of weight w has the property*

$$\tilde{\boxdot}_w\tilde{U} = \tilde{g}^{\mu\nu}\partial_\mu\partial_\nu\tilde{U} + (d - 2 + 2w - 2\rho\partial_\infty)\,\partial_\infty\tilde{U} + \tilde{F}, \tag{143}$$

where $\tilde{F}^{[m]}$ depends only on $D^1\tilde{U}^{[k]}$ for $k \leq m$.

In other words

$$[\tilde{\boxdot}_w\tilde{U}]^{[m]} = [\tilde{g}^{\mu\nu}\partial_\mu\partial_\nu\tilde{U}]^{[m]} + (d - 2 + 2w - 2m)\,(m+1)\tilde{U}^{[m+1]} + \tilde{F}^{[m]}. \tag{144}$$

Proof. Clearly, we can write

$$\tilde{\boxdot}_w\tilde{U} = \tilde{g}^{\mu\nu}\partial_\mu\partial_\nu\tilde{U} + \tilde{G}, \tag{145}$$

where $\tilde{G}^{[m]}$ depends on $D^1\tilde{U}^{[k]}$ for $k \leq m+1$ and $\tilde{U}^{[m+2]}$.

We need to determine the dependence of $[\tilde{\boxdot}\tilde{U}]^{[m]}$ on $D^l\tilde{U}^{[k]}$ for $k > m$. This can be done by considering fields that depend only on the Taylor expansion coefficients for $k > m$. The most convenient way is to use $\mathbf{\Omega} = \rho t^2$ in the expansion instead of ρ, because $\mathbf{\Omega}$ is covariantly defined. We consider $\mathbf{U} = \mathbf{\Omega}^{m+1}\mathbf{F} = O(\rho^{m+1})$ where $\mathcal{L}_{\mathbf{T}}\mathbf{F} = (w + n - 2(m+1))\mathbf{F}$ (in order that $\mathbf{U}$ has the proper weight) and thus

$$\nabla_{\mathbf{T}}\mathbf{F} = (w - 2(m+1))\mathbf{F}. \tag{146}$$

We substitute the special form of $\mathbf{U}$ to obtain the following formula

$$\boldsymbol{\nabla}^I\boldsymbol{\nabla}_I\mathbf{U} = (m+1)\boldsymbol{\Omega}^{m-1}\left[\mathbf{F}\left(m\boldsymbol{\nabla}^I\boldsymbol{\Omega}\boldsymbol{\nabla}_I\boldsymbol{\Omega} + \boldsymbol{\Omega}(\boldsymbol{\nabla}^I\boldsymbol{\nabla}_I\boldsymbol{\Omega})\right) + 2\boldsymbol{\Omega}\nabla^I\boldsymbol{\Omega}\nabla_I\mathbf{F}\right] + \boldsymbol{\Omega}^{m+1}\boldsymbol{\nabla}^I\boldsymbol{\nabla}_I\mathbf{F}. \tag{147}$$

We use the known form of derivatives of $\boldsymbol{\Omega}$ to get the nice expression:

$$\begin{aligned}\boldsymbol{\nabla}^I\boldsymbol{\nabla}_I\mathbf{U} &= (m+1)\boldsymbol{\Omega}^m\left((2m+d+2)\mathbf{F} + 2\boldsymbol{\nabla}_T\mathbf{F}\right) + O(\rho^{m+1}) = \\ &= (m+1)\boldsymbol{\Omega}^m\mathbf{F}\left(d-2+2w-2m\right) + O(\rho^{m+1}).\end{aligned} \tag{148}$$

This shows that

$$\tilde{\boxdot}_w\tilde{U} = (d-2+2w-2\rho\partial_\infty)\,\partial_\infty\tilde{U} + \tilde{H} \tag{149}$$

where $\tilde{H}^{[m]}$ depends on $D^2\tilde{U}^{[k]}$ for $k \leq m$. Together with the previous expansion (145) it proves the lemma. □

Together with Proposition 3 this result leads immediately to a corollary:

Corollary 15. *Let $n \in \mathbb{Z}_+ \cup \{0\}$ and $w \in \mathbb{Z}$ such that $N = d/2 - 1 + w \in \mathbb{Z}_+ \cup \{0\}$, then the system*

$$\tilde{\boxdot}_w\tilde{U} + \tilde{D}(\tilde{U}) = O(\rho^{N+1})\; \textit{for}\; \tilde{U}^{[k]},\; k = 0, \ldots, N \tag{150}$$

for $\tilde{U}^{[k]}$ (n-tractors of weight w) is generalized hyperbolic, recursive and linear. Here $\tilde{D}$ is linear transformation on the space of n-tractors. In particular, the Cauchy problem with smooth initial data is well-posed.

Proof. The assumption about the metric is sufficient to satisfy the requirements for the linear generalized hyperbolic system to be well-posed in Proposition 3. □

Remark 5. The ambient metric construction determines $\tilde{g}^{[k]}_{\mu\nu}$ for $k = 0, \ldots, d/2-1$ and $\operatorname{tr}\tilde{g}^{[d/2]}$. Consequently, only equations depending on this part of the metric expansion are defined uniquely. They provide conformal equations on the spacetime M. This holds if $0 > w > 1 - d/2$. Interestingly, it is also true for $\tilde{\boxdot}_0$ on scalars of weight 0, where the operator depends on the aforementioned trace. If $n \geq 1$ and $w \geq 0$, then the system of equations explicitly depends on the choice of the extension of the ambient metric.

The initial data for the derived equation $\hat{H}_{\tilde{\boxdot}_w} = 0$ is given by $\partial_1^k\tilde{U}^{[0]}|_\Sigma$, $k \leq 2N+1$ and we get from Lemma 5 and Proposition 3 the unique global development. Let us remind that [13][7]

$$\hat{H}_{\tilde{\boxdot}_w}(U) = c_d\Box^{N+1}U + \ldots, \quad N = \frac{d}{2} - 1 + w \in \mathbb{Z}_+ \cup \{0\}, \quad c_d \in \mathbb{R} \tag{151}$$

In particular, the scalar GJMS operators P_{2k} are (up to a nonzero constant depending only on dimension and k) derived operators for $\tilde{\boxdot}_w$ acting on scalar tractors of weight $w = k - \frac{d}{2}$,

$$P_{2k}f := c_k\hat{H}_{\tilde{\boxdot}_w}(f), \quad c_k \in \mathbb{R}. \tag{152}$$

We thus obtain a corollary (due to Lemma 5 and Proposition 3):

Corollary 16. *The equation $P_{2k}f = 0$ involving scalar GJMS operators for $k > 0$ is well-posed in the smooth category.*

Let us remark that operators P_{2k} are also well-defined for $k > \frac{d}{2}$ (the so called supercritical case), but they depend on the choice of the ambient metric extension.

[7] In [13] only the scalar case was considered, but the method applied in [13] extends almost verbatim to the general tractor case.

5.3. Spacetimes with vanishing Q curvature. We can apply the Corollary 15 to the problem of finding a conformal factor, which yields the vanishing Branson Q curvature [7]. This is an important and quite mysterious object in conformal geometry (see [24,25] for an introduction to the application and meaning of the Q curvature). In particular Q curvature has an interesting affine type of conformal transformation [7]

$$e^{d\sigma} Q[e^{2\sigma} h] = Q[h] + P_d \sigma, \tag{153}$$

where the critical GJMS operator P_d is defined with respect to the metric h (in fact, P_d for metric $e^{2\sigma} h$ differs only by a factor of $e^{d\sigma}$). In order to find metric in the conformal class with vanishing Q curvature one needs to solve inhomogeneous equation

$$P_d \sigma = -Q. \tag{154}$$

The existence (and classification) of solutions will follow from the following proposition:

Proposition 17. *Consider an equation $\hat{H}_{\tilde{\boxdot}_w}(U) = V$ for U, where V is given. Here U and V are n-tractors of weight w. This system is well-posed in the smooth category.*

Proof. The equation $\hat{H}_{\tilde{\boxdot}_w}(U) - V = 0$ is the derived equation for the system for $\tilde{U}^{[k]}$, $k = 0, \dots N$

$$\tilde{\boxdot}_w \tilde{U} - \rho^N V = O(\rho^{N+1}), \quad N = \frac{d}{2} - 1 + w. \tag{155}$$

It is a recursive, decoupled and generalized hyperbolic equation by Corollary 15. The well-posedness of the derived equation follows from Lemma 5 and Proposition 3. □

We can apply Proposition 17 to equation (154).

Corollary 18. *On every globally hyperbolic spacetime, there exists a function σ such that $Q[e^{2\sigma} h] = 0$.*

The proof raises a question whether other forms of source terms than considered in (155) are interesting. In fact, we can show that all of them can be reduced to the case used in the proof of Proposition 17. Suppose that we would like to solve ($N = \frac{d}{2} - 1 + w$)

$$\tilde{\boxdot}_w \tilde{U} = \tilde{V} + O(\rho^{N+1}), \tag{156}$$

for a fixed ρ-dependent n-tractor $\tilde{V}$ of weight w. By recursive property of the equation, there exists $\tilde{F}$ such that

$$\tilde{\boxdot}_w \tilde{F} = \tilde{V} + O(\rho^N), \quad \tilde{F}^{[0]} = 0, \tag{157}$$

and it is locally determined. If we write $W = [\tilde{V} - \tilde{\boxdot}_w \tilde{F}]^{[N]}$ then

$$\tilde{\boxdot}_w (\tilde{U} - \tilde{F}) = \rho^N W + O(\rho^{N+1}), \tag{158}$$

which is the form considered in the proof of Proposition 17.

5.4. Almost Einstein condition. It is a standard computation [22,26] that the conformally rescaled metric $f^{-2}h_{\mu\nu}$ for a smooth function $f \in C^\infty(M)$ satisfies Einstein's equations with a cosmological constant if and only if $f \neq 0$ and

$$\mathrm{tf}(\nabla_\mu \nabla_\nu f + P_{\mu\nu} f) = 0, \tag{159}$$

where tf denotes the trace-free part. It is natural to skip the non-degeneracy condition $f \neq 0$. Following [27] we call a spacetime almost Einstein if there exists a non-trivial (i.e. not everywhere zero) solution of (159). We will say that such solution f defines an almost Einstein structure. The condition of being almost Einstein describes in a uniform way physical space and its conformal boundary $\{f = 0\}$. It turns out that this condition has also a very nice interpretation in terms of 1-tractors of weight 0, introduced in [22] (see [26] for an alternative derivation).[8]

Tractors of weight 0 have especially nice properties. In particular, we can introduce a connection on the tractor bundle. The tractor derivative for 1-tractor U_I of weight 0 (see tractor derivative, for example, in [16] and [17] with a version for covectors) is defined by the formula:

$$\nabla^{\mathcal{T}}_\mu \begin{pmatrix} U_0 \\ U_\nu \\ U_\infty \end{pmatrix} = \begin{pmatrix} \nabla_\mu U_0 - U_\mu \\ \nabla_\mu U_\nu + h_{\mu\nu} U_\infty + P_{\mu\nu} U_0 \\ \nabla_\mu U_\infty - P^{\nu}_{\mu} U_\nu \end{pmatrix}. \tag{160}$$

It is a conformally covariant object. It is shown in [22] that the metric is almost Einstein if there exists a non-trivial 1-tractor of weight 0, U_I which is covariantly constant

$$\nabla^{\mathcal{T}}_\mu U_I = 0. \tag{161}$$

Indeed, this condition is equivalent to

$$U_0 = f,\ U_\mu = \partial_\mu f,\ U_\infty = -\frac{1}{d}(\nabla^\mu \nabla_\mu U + P^{\mu}_{\mu} f), \tag{162}$$

for some function f satisfying (159).

The conformal factor rescaling the metric to Einsteinian is given by $e^\sigma = U_0$ and the cosmological constant is equal $\Lambda = c_d U^I U_I$ where c_d is a dimension dependent constant [26]. The conformally invariant scalar product for weight 0 tractors is given by

$$U_I U^I := 2U_0 U_\infty + U_\mu U^\mu. \tag{163}$$

Conformal boundary corresponds to $U_0 = 0$. This set is a hypersurface (apart from possibly some isolated point in the case of $\Lambda = 0$) with vanishing extrinsic curvature and it enjoys very special properties [27]. In almost Einstein spaces, the Fefferman–Graham obstruction tensor vanishes.

It turns out that the tractor derivative has a natural interpretation from the ambient point of view. Namely, let $\mathbf{V}_I$ be a covector field on $\mathbf{M}$ such that (weight 0)

$$\mathcal{L}_{\mathbf{T}} \mathbf{V}_I = \mathbf{V}_I, \quad \mathbf{V}_I|_{t=1,\rho=0} = V_I, \tag{164}$$

then (see [23])

$$\nabla^{\mathcal{T}}_\mu V_I = \left[\boldsymbol{\nabla}_\mu \mathbf{V}_I\right]_{t=1,\rho=0}. \tag{165}$$

The identity extends to the case of tensors of higher valency.

[8] We are working in a specific metric [26,27] and not in a framework of conformal densities [22].

This point of view raises a natural question about ambient metric version of the condition (161). It is shown in [16] that one can prolong the covariant tractor from $\rho = 0$ surface. We will need a detailed statement of this result. We remind the following fact (proven in [16], see also [27])

Proposition 19. *Suppose that* $\nabla^{\mathcal{T}}_{\mu} U_I = 0$ *for a tractor of weight* 0*, then there exists* $\mathbf{U}_I$ *on the ambient space such that* $\mathbf{U}_I|_{t=1,\rho=0} = U_I$ *and*

$$\boldsymbol{\nabla}_I \mathbf{U}_J = O(\rho^{d/2-1}), \quad \mathcal{L}_{\mathbf{T}} \mathbf{U}_I = \mathbf{U}_I. \tag{166}$$

In general, we cannot ensure vanishing of the error term $O(\rho^{d/2-1})$ (see [16], Proposition 4.5) unless the metric is so-called even with respect to the conformal boundary.

Our goal is to find a propagation equation for $\mathbf{U}_I$. It is useful to remind first the structure of $\mathbf{U}_I$ in the case of the odd dimension, which was analyzed in details in [27] (see also [28,29] for the case of Einstein metrics). In this case we can assume $\boldsymbol{\nabla}_I \mathbf{U}_J = O(\rho^\infty)$, where $\mathbf{U}_I = \partial_I \mathbf{f}$ for some function $\mathbf{f}$, such that

$$\mathbf{f}|_{t=1,\rho=0} = f. \tag{167}$$

This function satisfies $\boxdot \mathbf{f} = O(\rho^\infty)$. In fact, this harmonicity condition together with (167) determines $\mathbf{f}$ uniquely due to recursive property of the equation.

We will look for a similar description in the case of even dimension. In the even dimension d the recursive relation breaks at order $\frac{d}{2}$, thus in general we can only assume $\boxdot \mathbf{f} = O(\rho^{d/2})$. This allows us to determine $\mathbf{f}$ up to order $O(\rho^{d/2+1})$ and then $P_{d+2} f = c_k [\boxdot \mathbf{f}|_{t=1}]^{[d/2]}$. Although, the supercritical GJMS operator P_{d+2} depends on a choice of the extension of the ambient metric, one can check that $\mathbf{f} + O(\rho^{d/2+1})$ does not depend on this choice (see Appendix B).

We will first need an auxiliary result allowing us to determine vanishing of $\boldsymbol{\nabla}_I \mathbf{U}_J|_{t=1,\rho=0}$ from properties of function $\mathbf{f}$. For future applications, we will state it in a local form (a symbol D^n denote jets on M):

Lemma 20. *Let* $x \in M$*. Suppose that* f *satisfies* $D^2 \left[\mathrm{tf}(\nabla_\mu \nabla_\nu f + P_{\mu\nu} f)\right](x) = 0$ *and function* $\mathbf{f}$ *is such that*

$$\boxdot \mathbf{f}|_{t=1} = O(\rho^{d/2}), \quad \mathcal{L}_{\mathbf{T}} \mathbf{f} = \mathbf{f}, \quad \mathbf{f}|_{t=1,\rho=0} = f. \tag{168}$$

Then $\boldsymbol{\nabla}_I \boldsymbol{\nabla}_J \mathbf{f}|_{t=1,\rho=0}(x) = 0$.

Proof. We compute using $[\boldsymbol{\nabla}_I \boldsymbol{\nabla}^I \mathbf{f}]^{[0]} = 0$

$$k := [\mathbf{f}]^{[1]}|_{t=1} = -\frac{1}{d}(\nabla^\mu \nabla_\mu f + P^\mu_\mu f). \tag{169}$$

Let us now define $\mathbf{U}_I = \partial_I \mathbf{f}$, $U_I = [\mathbf{U}_I]^{[0]}|_{t=1}$. We notice that

$$U_0 = f, \; U_\mu = \partial_\mu f, \; U_\infty = k. \tag{170}$$

Due to $\tilde{S}_{IJ} = O(\rho^{d/2-1})$ we obtain

$$\boxdot \mathbf{U}_J = \boldsymbol{\nabla}_I \boldsymbol{\nabla}^I \boldsymbol{\nabla}_J \mathbf{f} = \boldsymbol{\nabla}_J (\boldsymbol{\nabla}_I \boldsymbol{\nabla}^I \mathbf{f}) + O(\rho^{d/2-1}) = O(\rho^{d/2-1}). \tag{171}$$

It is useful to introduce Thomas $\hat{\mathrm{D}}$-operator [22], which in such case is proportional to covariant derivative (see [23] for the ambient metric definition of this operator for tractors of weight 0)

$$\hat{\mathrm{D}}_I U_J := [(d-2)\boldsymbol{\nabla}_I \mathbf{U}_J + \mathbf{T}_I \boxdot \mathbf{U}_J]\,|_{t=1}^{[0]} = (d-2)\,[\boldsymbol{\nabla}_I \mathbf{U}_J]\,|_{t=1}^{[0]}. \tag{172}$$

The advantage is that this operator depends only on the value of $\mathbf{U}_I$ on $\mathbf{N}$. The formula for $\hat{\mathrm{D}}$ operator acting on weight 0 tractors is given by (see [17])

$$\hat{\mathrm{D}}_0 U_J = 0 \tag{173}$$

$$\hat{\mathrm{D}}_\mu U_J = (d-2)\nabla_\mu^{\mathcal{T}} U_J \tag{174}$$

$$\hat{\mathrm{D}}_\infty U_J = -\Box^{\mathcal{T}} U_J, \tag{175}$$

where $\Box^{\mathcal{T}} U_J = h^{\mu\nu}\nabla_\mu^{T^*M\otimes\mathcal{T}}\nabla_\nu^{\mathcal{T}} U_J$ is the d'Alembert operator on the tractor bundle. We need to show that $\hat{\mathrm{D}}_I U_J(x) = 0$.

The condition $D^2\left[\mathrm{tf}(\nabla_\mu\nabla_\nu f + P_{\mu\nu} f)\right](x) = 0$ gives by (160)

$$D^1\nabla_\mu^{\mathcal{T}} U_J(x) = 0. \tag{176}$$

The nontrivial condition for $\nabla_\mu^{\mathcal{T}} U_\infty$ is shown by divergence of (159) (see [17]). For this reason, we only get the condition for D^1 jets.

We can now show by (176)

$$\hat{\mathrm{D}}_\infty U_J(x) = -\Box^{\mathcal{T}} U_J(x) = -h^{\mu\nu}\nabla_\mu^{T^*M\otimes\mathcal{T}}\nabla_\nu^{\mathcal{T}} U_J(x) = 0. \tag{177}$$

Finally we see that $\boldsymbol{\nabla}_I \mathbf{U}_J|_{t=1}^{[0]}(x) = \frac{1}{d-2}\hat{\mathrm{D}}_I U_J(x) = 0.$ □

We remind that on almost Einstein spacetimes, the obstruction tensor vanishes (AFG equations are satisfied) and by Proposition 12 we have a distinguished class of extensions that are Ricci flat to infinite order in ρ. We can now state our enhancement of Proposition 19.

Proposition 21. *Let f be a non-trivial solution of* (159)*. Then there exists $\mathbf{f}$ satisfying the following conditions*

1. *$\mathcal{L}_{\mathbf{T}}\mathbf{f} = \mathbf{f}$, $\mathbf{f}|_{t=1}^{[0]} = f$.*
2. *In an arbitrary extension of the ambient metric that is Ricci flat to infinite order in ρ*

$$\boxdot\mathbf{f} = O(\rho^{d/2+1}). \tag{178}$$

In particular $P_{d+2} f = 0$.

3. *The covector $\mathbf{U}_I := \partial_I \mathbf{f}$ satisfies* (166)*, that is*

$$\boldsymbol{\nabla}_I \mathbf{U}_J = O(\rho^{d/2-1}). \tag{179}$$

The proof of the proposition will be based on some relation which we will use also for propagation of the almost Einstein condition. Let us record an identity for an arbitrary function $\mathbf{F}$

$$\boxplus(\boldsymbol{\nabla}_I\boldsymbol{\nabla}_J\mathbf{F}) = \boldsymbol{\nabla}_I\boldsymbol{\nabla}_J(\boxdot\mathbf{F}) + (\boldsymbol{\nabla}_J\mathbf{S}_I^K + \boldsymbol{\nabla}_I\mathbf{S}_J^K - \boldsymbol{\nabla}^K\mathbf{S}_{IJ})\boldsymbol{\nabla}_K\mathbf{F}, \tag{180}$$

where $\mathbf{S}^{K\,L}_{I\,J}$, $\mathbf{S}^{K}_{I}$ are the Riemann tensor and Ricci tensor on $\mathbf{M}$ and we introduced an operator $\boxplus$ on 2- covectors $\mathbf{K}_{IJ}$ on $\mathbf{M}$

$$\boxplus(\mathbf{K}_{IJ}) = \boldsymbol{\nabla}^L\boldsymbol{\nabla}_L\mathbf{K}_{IJ} + 2\mathbf{S}^{K\,L}_{I\,J}\mathbf{K}_{KL} - \mathbf{S}^K_I\mathbf{K}_{KJ} - \mathbf{S}^K_J\mathbf{K}_{KI}. \tag{181}$$

Similarly as in the case of wave operator we define

$$\tilde{\boxplus}_w\tilde{K}_{IJ} = [\boxplus(\mathbf{K}_{IJ})]_{t=1}, \quad \mathbf{K}_{IJ}|_{t=1} = \tilde{K}_{IJ}, \quad \mathcal{L}_{\mathbf{T}}\mathbf{K}_{IJ} = (w+2)\mathbf{K}_{IJ}. \tag{182}$$

We will be mainly interested in $\tilde{\boxplus}_{-1}$. By Corollary 15, $\tilde{\boxplus}_{-1}$ is recursive till order $\frac{d}{2}-2$.

Remark 6. Let us notice that both $\tilde{\boxdot}_1$ and $\tilde{\boxplus}_{-1}$ depend on the choice of the extension of the ambient metric. Once such an extension is chosen, they are well-defined operators. Nonetheless, in their definition we need more structure than just the conformal class of the metric.

Proof of Proposition 21. We define $\mathbf{f}$ up to $O(\rho^{d/2+1})$ order by

$$\boxdot\mathbf{f} = O(\rho^{d/2}), \quad \mathcal{L}_{\mathbf{T}}\mathbf{f} = \mathbf{f}, \quad \mathbf{f}_{t=1,\rho=0} = f. \tag{183}$$

As we said earlier, $\mathbf{f}$ does not depend on the choice of Ricci flat extension of the ambient metric. Let us choose such an extension. Then from (180) and $\mathbf{S}_{IJ} = O(\rho^\infty)$ it follows that

$$\boxplus(\mathbf{D}_{IJ}) = \boldsymbol{\nabla}_I\boldsymbol{\nabla}_J(\boxdot\mathbf{f}) + O(\rho^\infty) = O(\rho^{d/2-2}), \quad \mathbf{D}_{IJ} = \boldsymbol{\nabla}_I\boldsymbol{\nabla}_J\mathbf{f}. \tag{184}$$

Thus, $[\boxplus(\mathbf{D}_{IJ})|_{t=1}]^{[d/2-2]} = \hat{H}_{\tilde{\boxplus}_{-1}}(D_{IJ})$ (derived operator) where $D_{IJ} = \boldsymbol{\nabla}_I\boldsymbol{\nabla}_J\mathbf{f}|_{t=1,\rho=0}$. However, by Lemma 20, $D_{IJ} = 0$ and by recursive property of $\tilde{\boxplus}_{-1}$

$$\mathbf{D}_{IJ} = O(\rho^{d/2-1}) \Longrightarrow \boldsymbol{\nabla}_I\mathbf{U}_J = O(\rho^{d/2-1}). \tag{185}$$

Additionally $\hat{H}_{\tilde{\boxplus}_{-1}}(D_{IJ}) = 0$ and hence $\boxplus(\mathbf{D}_{IJ}) = O(\rho^{d/2-1})$. We obtain by equation (180) and $\mathbf{S}_{IJ} = O(\rho^\infty)$

$$\boldsymbol{\nabla}_I\boldsymbol{\nabla}_J(\boxdot\mathbf{f}) = \boxplus(\mathbf{D}_{IJ}) + O(\rho^\infty) = O(\rho^{d/2-1}) \Longrightarrow \boxdot\mathbf{f} = O(\rho^{d/2+1}). \tag{186}$$

This property holds independently of the chosen ambient metric extension provided it is Ricci flat to infinite order. It means that $P_{d+2}f = 0$. □

It is a bit surprising that although P_{d+2} depends on a choice of the Ricci flat extension, $P_{d+2}f = 0$ independently of this choice. It is explained by the following fact proven in Appendix B.

Proposition 22. *Let $\tilde{g}^{\pm}_{\mu\nu}$ be two extensions of the ambient metric of $h_{\mu\nu}$ which are Ricci flat to infinite order in ρ. Denote*

$$k_{\mu\nu} = [\tilde{g}^+_{\mu\nu}]^{[\frac{d}{2}]} - [\tilde{g}^-_{\mu\nu}]^{[\frac{d}{2}]}. \tag{187}$$

It is a symmetric, traceless and divergence-free tensor. Let $P^{\pm}_{d+2}$ denote the supercritical GJMS operators defined by extensions $\tilde{g}^{\pm}_{\mu\nu}$. Then

$$P^+_{d+2}\phi - P^-_{d+2}\phi = -c_k k^{\mu\nu}\left(\nabla_\mu\nabla_\nu\phi + P_{\mu\nu}\phi\right), \tag{188}$$

where constant c_k is defined in (152). In particular, for functions ϕ satisfying (159), the right-hand side vanishes.

5.5. Propagation of the almost Einstein structure. Proposition 21 is an important observation. As $\mathbf{f}$ is a scalar of weight 1, (178) is a generalized hyperbolic system for $\tilde{f} = \mathbf{f}|_{t=1}$. We will use it to evolve f from the Cauchy surface to the whole globally hyperbolic development. We will prove that this f defines an almost Einstein structure. Indeed, identity (180) shows that if (178) holds, then $\boldsymbol{\nabla}_I \mathbf{U}_J$ satisfies a linear hyperbolic equation too and then what remains is to show that the initial data for this system vanish.[9] In order to define evolution of f we need to specify the choice of extension of the ambient metric. We will choose an arbitrary extension that is Ricci flat to all orders. In principle, the solution may depend on this choice, but we will see that this particular one does not.

Proposition 23. *Let us consider a metric satisfying the AFG equation. We also choose an extension of the ambient metric which is Ricci flat to infinite order in ρ. Suppose that we have initial data $D^{d+1} f|_\Sigma$ such that*

$$D^{d-1}\,\mathrm{tf}(\nabla_\mu \nabla_\nu f + P_{\mu\nu} f)|_\Sigma = 0. \tag{189}$$

Then the metric is almost Einstein with a covariant tractor given by

$$U_0 = f,\ U_\mu = \partial_\mu f,\ U_\infty = -\frac{1}{d}(\nabla^\mu \nabla_\mu f + P^\mu_\mu f), \tag{190}$$

where f is the solution of the scalar supercritical GJMS equation:

$$P_{d+2} f = 0, \tag{191}$$

with the given initial data. The solution f does not depend on the choice of the Ricci flat extension.

Proof. We define now $\mathbf{f}$ as a solution to generalized hyperbolic system

$$\boxdot \mathbf{f} = O(\rho^{d/2+1}), \quad \mathcal{L}_{\mathbf{T}} \mathbf{f} = \mathbf{f}, \tag{192}$$

that is $\tilde{\boxdot}_1 \tilde{f} = O(\rho^{d/2+1})$ for $\tilde{f} = \mathbf{f}|_{t=1}$. The initial data $D^1 \tilde{f}|_\Sigma$ are prescribed by $D^{d+1} f|_\Sigma$ according to Lemma 5. Equation (180) together with $\mathbf{S}_{IJ} = O(\rho^\infty)$ and (192) gives the following system for $\tilde{D}_{IJ} = \boldsymbol{\nabla}_I \boldsymbol{\nabla}_J \mathbf{f}|_{t=1}$

$$\tilde{\boxplus}_{-1} \tilde{D}_{IJ} = O(\rho^{d/2-1}). \tag{193}$$

By proposition 14, this is a generalized hyperbolic and recursive equation for 2-tractors of weight -1, $\tilde{D}^{[k]}_{IJ}$, $k = 0, \ldots d/2-2$. We thus need to show that

$$\tilde{D}_{IJ}|_\Sigma = O(\rho^{d/2-1}),\ \partial_1 \tilde{D}_{IJ}|_\Sigma = O(\rho^{d/2-1}). \tag{194}$$

From recursive property (see Lemma 5), this is equivalent to showing that

$$D^{d-3}[\boldsymbol{\nabla}_I \boldsymbol{\nabla}_J \mathbf{f}]|^{[0]}_{t=1,\Sigma} = 0. \tag{195}$$

A symbol D^n denote jets in directions of M. We can now apply Lemma 20 to show that (195) holds if (189) is satisfied.

Finally, although we used a specific ambient extension to obtain f, the result does not depend on this choice. In fact, we can now check that $P_{d+2} f = 0$ for every Ricci flat extension by Proposition 21. □

[9] Such method of propagation equations is widely used in General Relativity (see [30]).

Remark 7. In the case $f \neq 0$ on the Cauchy surface, we can change conformally the metric such that it satisfies Einstein constraints on the initial surface. Therefore, we can evolve Einstein equations. The result needs to agree with the metric evolved with AFG equation up to conformal rescaling and diffeomorphism. In this way we obtain propagation of the Einsteinian condition up to the conformal boundary (see [3]). Prescribing initial data on a surface with $f = 0$ is more delicate. Our method has the advantage that it allows to treat all cases simultaneously. For example, the initial Cauchy surface can cross the conformal boundary.

Remark 8. Appearance of non-conformally invariant operators may seem a bit puzzling in propagation of a conformally invariant condition. However, these are only auxiliary systems used for this purpose. As in the case of Killing initial data, there may exist other equations which can be used in this context. However, it is not obvious if one can find conformally invariant equations of this kind. We expect it to be unlikely, at least in the class of generalized hyperbolic systems, in light of non-existence of conformal operators of the form $\Box^{\frac{d}{2}+1} + \ldots$ [31].

5.6. Application to stability of asymptotically de Sitter spacetimes. Theorem 1 together with Proposition 23 allow us to prove various results about asymptotic properties of solutions to Einstein's equations with a positive cosmological constant. This was the initial motivation for studying the AFG equations. We will state some of these results, refering readers to [3,4] for various extensions. Let us notice that we proved well-posedness in the smooth category. The case of Sobolev spaces would demand various shifting with respect to what is stated in [3,4]. We are working in even dimension d. Well-posedness of the AFG equation together with propagation of the almost Einstein condition proves:

Theorem 24 (cf. [4] Thm. 6.1). *Let Σ be a compact Cauchy surface. The initial data on Σ that correspond to the future (or past) asymptotically simple solutions of the Einstein's equations form an open set in C^∞ topology.*

In particular, initial data close to de Sitter spacetime develop a complete future and past asymptotically simple solution. We can also define initial data at a conformal boundary. It is known (see for example [6] Theorem 4.8) that the initial data on conformal boundary $\overline{\Sigma}$ for the Einstein equation is given by a smooth pair $(\gamma_{ij}, \kappa_{ij})$ where γ is an Euclidean metric and κ is a symmetric two form satisfying

$$\nabla^i_\gamma \kappa_{ij} = 0, \quad \kappa^i_i = 0. \tag{196}$$

We use latin indices for tensors on $\overline{\Sigma}$. Such data define a formal solution of the Einstein equations with a positive cosmological constant.[10] Namely, there exists a smooth function f, vanishing at $\overline{\Sigma}$ and a metric $h_{\mu\nu}$ such that $g_{\mu\nu} = f^{-2} h_{\mu\nu}$ satisfies Einstein's equations up to infinite jets at $\overline{\Sigma}$. The metric γ is a restriction of $h_{\mu\nu}$ to $\overline{\Sigma}$, whereas κ is equal to the so-called holographic stress-energy tensor which is an object obtained from high derivatives of $h_{\mu\nu}$ and f at $\overline{\Sigma}$ (see [6,32]). Let us notice that existence of a formal solution to the Einstein equations allow us to compute from (196) the initial data for AFG equations. The data satisfy AFG constraints. Similarly we can compute initial

[10] The value of the cosmological constant is fixed, but arbitrary.

data for the scale f and this initial data satisfy assumptions of Proposition 23. We can thus replace the formal expansion by an analytic result.

If two pairs $(h^{\pm}_{\mu\nu}, f^{\pm})$ are related by a diffeomorphism Ψ then corresponding initial data transform by a restriction of Ψ to $\overline{\Sigma}$. The other gauge tranformation is given by conformal rescaling

$$h^{+}_{\mu\nu} = e^{2\sigma} h^{-}_{\mu\nu}, \quad f^{+} = e^{\sigma} f^{-}. \tag{197}$$

This induces the following tranformation of the initial data (see [32])

$$\gamma^{+}_{ij} = e^{2\sigma_0}\gamma^{-}_{ij}, \quad \kappa^{+}_{ij} = e^{-(d-3)\sigma_0}\kappa^{-}_{ij}. \tag{198}$$

where $\sigma_0 = \sigma|_{\overline{\Sigma}}$. We see that two initial data $(\gamma^{\pm}_{ij}, \kappa^{\pm}_{ij})$ give equivalent solutions if and only if there exist diffeomorphism Φ on $\overline{\Sigma}$ and a smooth function σ_0 such that

$$\gamma^{+} = e^{2\sigma_0}\Phi_{*}\gamma^{-}, \quad \kappa^{+} = e^{-(d-3)\sigma_0}\Phi_{*}\kappa^{-}. \tag{199}$$

Lemma 11, Proposition 23 and Theorem 1 now show ($\Lambda > 0$ is chosen):

Theorem 25. *Let $(\gamma_{ij}, \kappa_{ij})$ on a compact surface $\overline{\Sigma}$ satisfy constraints (196). There exist (on a neighbourhood of $\overline{\Sigma} \subset M$) a metric $h_{\mu\nu}$ and a function f such that*

1. *$\overline{\Sigma} = \{f = 0\}$,*
2. *γ is the metric h restricted to $\overline{\Sigma}$ and κ is the holographic stress energy tensor,*
3. *$f^{-2}h_{\mu\nu}$ satisfies Einstein's equations with cosmological constant Λ.*

Two initial data $(\gamma^{\pm}_{ij}, \kappa^{\pm}_{ij})$ give locally equivalent solutions if and only if there exist diffeomorphism Φ on $\overline{\Sigma}$ and a smooth function σ_0 such that (199) is satisfied. Solutions are unique up to diffeomorphism and conformal transformation.

6. Summary

The Fefferman–Graham obstruction tensor and GJMS operators are very special objects. One additional nice property is related to their behavior as evolution systems. Both GJMS as well as Fefferman–Graham tensor in the suitable gauge have multiple characteristics, thus they are in principle only weakly hyperbolic. However, we proved that still they enjoy well-posed Cauchy problem. In addition, the property of being almost Einstein propagates from the initial surface. We proved it in the smooth category, but with an arbitrary Cauchy surface. Namely, the Cauchy surface can cross or partially coincide with the conformal boundary of the spacetime. This allows to use AFG equation for proving the stability of asymptotically simple solutions (as was done in [3,4]). We should notice that this is not the most effective proof of stability as the metric needs to be of high regularity. However, it provides some advantages: it is a Lagrangean theory, which allows to apply various techniques like Noether charge definition, Hamiltonian formulations on the level of conformally compactified spacetime. The meaning of such defined charges for Einsteinian solutions is still unclear. The relation to GR charges should be investigated in future.

Declarations

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A Bianchi identities

For convenience of the reader, we provide here a proof of the Bianchi identities in the ambient space. Our first goal is to show that

Proposition 26. *The following holds*

$$\tilde{\nabla}^\mu \left(\tilde{S}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{S}\right) + \rho\partial_\mu \tilde{S}_{\infty\infty} + (d-2-2\rho\partial_\infty)\tilde{S}_{\mu\infty} - \rho\tilde{g}^{\xi\chi}\tilde{g}'_{\xi\chi}\tilde{S}_{\mu\infty} = 0, \tag{200}$$

$$\tilde{\nabla}^\mu \tilde{S}_{\mu\infty} + (d-2-\rho\partial_\infty)\tilde{S}_{\infty\infty} - \rho\tilde{g}^{\mu\nu}\tilde{g}'_{\mu\nu}\tilde{S}_{\infty\infty} - \frac{1}{2}\tilde{g}'_{\mu\nu}\tilde{S}^{\mu\nu} - \frac{1}{2}\partial_\infty\tilde{S} = 0, \tag{201}$$

where $\tilde{S} = \tilde{S}^\mu_\mu$.

The proof will be divided in a series of lemmas. Let us denote by $\mathbf{g}_{IJ}$, $\boldsymbol{\nabla}_I$, $\mathbf{S}_{IJ}$ metric covariant derivative and Ricci tensor respectively in the ambient space $\mathbf{M}$. Indices are $I = 0, \infty$ or μ in the case of index on M.

Lemma 27. *Let* $\tilde{F}_\mu$ *be a* ρ*-dependent one-form and* $\tilde{F}_\infty$ *a function on* $\tilde{M}$. *Define a one-form* $\mathbf{F}_I$ *on* $\mathbf{M}$ *by*

$$\mathbf{F}_\mu = \tilde{F}_\mu, \quad \mathbf{F}_\infty = \tilde{F}_\infty, \quad \mathbf{F}_0 = 0. \tag{202}$$

Then

$$\boldsymbol{\nabla}^I\mathbf{F}_I = t^{-2}\left(\tilde{\nabla}^\mu\tilde{F}_\mu + (d-2-2\rho\partial_\infty)\tilde{F}_\infty - \rho\tilde{g}^{\mu\nu}\tilde{g}'_{\mu\nu}\tilde{F}_\infty\right). \tag{203}$$

Proof. Let us notice the identity

$$\mathbf{g}^{IJ}\boldsymbol{\nabla}_I\mathbf{F}_J = \frac{1}{\sqrt{\mathbf{g}}}\partial_I(\sqrt{\mathbf{g}}\mathbf{g}^{IJ}\mathbf{F}_J). \tag{204}$$

Now $\mathbf{g}^{0\infty} = \mathbf{g}^{\infty 0} = t^{-1}$, $\mathbf{g}^{\infty\infty} = -2\rho t^{-2}$ and $\mathbf{g}^{\mu\nu} = \tilde{g}^{\mu\nu}$ the rest of the components vanishes. Moreover, $\sqrt{\mathbf{g}} = t^{d+1}\sqrt{\tilde{g}}$. Thus, remembering that $\mathbf{F}_0 = 0$ it follows that

$$\begin{aligned}\mathbf{g}^{IJ}\nabla_I \mathbf{F}_J &= t^{-2}\frac{1}{\sqrt{\tilde{g}}}\partial_\mu(\sqrt{\tilde{g}}\tilde{g}^{\mu\nu}\tilde{F}_\nu)\\ &+ \frac{1}{t^{d+1}\sqrt{\tilde{g}}}\partial_0(t^{d+1}\sqrt{\tilde{g}}t^{-1}\tilde{F}_\infty) + \frac{1}{\sqrt{\tilde{g}}}\partial_\infty(\sqrt{\tilde{g}}(-2\rho t^{-2})\tilde{F}_\infty)\\ &= t^{-2}\left(\tilde{\nabla}^\mu \tilde{F}_\mu + (d-2-2\rho\partial_\infty)\tilde{F}_\infty - \rho\tilde{g}^{\mu\nu}\tilde{g}'_{\mu\nu}\tilde{F}_\infty\right),\end{aligned} \tag{205}$$

where we used $\frac{1}{\sqrt{\tilde{g}}}\partial_\mu(\sqrt{\tilde{g}}\tilde{g}^{\mu\nu}\tilde{F}_\nu) = \tilde{\nabla}^\mu \tilde{F}_\mu$. □

Lemma 28. *Let $\mathbf{D}_{IJ}$ be a symetric tensor and $\mathbf{U}$ a vector field, then*

$$(\nabla^I \mathbf{D}_{IJ})\mathbf{U}^J = \nabla^I(\mathbf{D}_{IJ}\mathbf{U}^J) - \frac{1}{2}\mathbf{D}^{IJ}\mathcal{L}_{\mathbf{U}}\mathbf{g}_{IJ}, \tag{206}$$

where $\mathcal{L}_{\mathbf{U}}$ is a Lie derivative

Proof. Follows from $\mathcal{L}_{\mathbf{U}}\mathbf{g}_{IJ} = \nabla_I \mathbf{U}_J + \nabla_J \mathbf{U}_I$ and symmetry of $\mathbf{D}_{IJ}$. □

Lemma 29. *We have*

$$\nabla^I \mathbf{S}_{I\mu} = t^{-2}\left(\tilde{\nabla}^\nu \tilde{S}_{\nu\mu} + (d-2-2\rho\partial_\infty)\tilde{S}_{\mu\infty} - \rho\tilde{g}^{\mu\nu}\tilde{g}'_{\mu\nu}\tilde{S}_{\mu\infty}\right), \tag{207}$$

$$\nabla^I \mathbf{S}_{I\infty} = t^{-2}\left(\tilde{\nabla}^\nu \tilde{S}_{\nu\infty} + (d-2-2\rho\partial_\infty)\tilde{S}_{\infty\infty} - \rho\tilde{g}^{\mu\nu}\tilde{g}'_{\mu\nu}\tilde{S}_{\infty\infty} - \frac{1}{2}\tilde{S}^{\nu\mu}\tilde{g}'_{\mu\nu} - \tilde{S}_{\infty\infty}\right). \tag{208}$$

Proof. Let us choose first $\mathbf{U}^\mu = U^\mu$, $\mathbf{U}^0 = 0$ and $\mathbf{U}^\infty = 0$ for some U^μ, vector field on M. The form $\mathbf{S}_{IJ}\mathbf{U}^J$ satisfies assumptions of Lemma 27, thus

$$\begin{aligned}\nabla^I(\mathbf{S}_{IJ}\mathbf{U}^J) &= t^{-2}\left(\tilde{\nabla}^\xi(\tilde{S}_{\xi\nu}U^\nu) + (d-2-2\rho\partial_\infty)\tilde{S}_{\infty\mu}U^\mu - \rho\tilde{g}^{\mu\nu}\tilde{g}'_{\mu\nu}\tilde{S}_{\infty\xi}U^\xi\right) =\\ &= t^{-2}\left(U^\mu\tilde{\nabla}^\xi\tilde{S}_{\xi\mu} + \tilde{S}_{\xi\nu}(\tilde{\nabla}^\xi U^\nu) + (d-2-2\rho\partial_\infty)\tilde{S}_{\infty\mu}U^\mu - \rho\tilde{g}^{\mu\nu}\tilde{g}'_{\mu\nu}\tilde{S}_{\infty\xi}U^\xi\right).\end{aligned} \tag{209}$$

Moreover, we have

$$(\mathcal{L}_{\mathbf{U}}\mathbf{g}_{IJ})dx^I dx^J = t^2(\mathcal{L}_U \tilde{g}_{\xi\nu})dx^\xi dx^\nu = t^2 2(\tilde{\nabla}_\xi U_\nu)dx^\xi dx^\nu, \tag{210}$$

By Lemma 28 we derive

$$\mathbf{U}^J\nabla^I \mathbf{S}_{IJ} = t^{-2}U^\mu\left(\tilde{\nabla}^\xi\tilde{S}_{\xi\mu} + (d-2-2\rho\partial_\infty)\tilde{S}_{\infty\mu} - \rho\tilde{g}^{\xi\nu}\tilde{g}'_{\xi\nu}\tilde{S}_{\infty\mu}\right), \tag{211}$$

which shows (207). Similarly choosing $\mathbf{U}^I = \delta^I_\infty$ we have

$$(\mathcal{L}_{\mathbf{U}}\mathbf{g}_{IJ})dx^I dx^J = 2dt^2 + t^2\tilde{g}'_{\mu\nu}dx^\mu dx^\nu, \tag{212}$$

where $\tilde{g}'_{\mu\nu}$ is the derivative in ρ. Finally, we obtain

$$\nabla^I(\mathbf{S}_{IJ}\mathbf{U}^J) = t^{-2}\left(\tilde{\nabla}^\xi\tilde{S}_{\xi\infty} + (d-2-2\rho\partial_\infty)\tilde{S}_{\infty\infty} - \rho\tilde{g}^{\mu\nu}\tilde{g}'_{\mu\nu}\tilde{S}_{\infty\infty}\right). \tag{213}$$

Thus we conclude

$$\boldsymbol{\nabla}^I \mathbf{S}_{I\infty} = t^{-2}\Big(\tilde{\nabla}^\nu \tilde{S}_{\nu\infty} + (d-2-2\rho\partial_\infty)\tilde{S}_{\infty\infty} - \rho\tilde{g}^{\mu\nu}\tilde{g}'_{\mu\nu}\tilde{S}_{\infty\infty} - \frac{1}{2}\tilde{S}^{\mu\nu}\tilde{g}'_{\mu\nu} - \tilde{S}_{\infty\infty}\Big), \tag{214}$$

which shows the result. □

Proof of Proposition 26. Let us now notice that

$$\mathbf{g}^{IJ}\mathbf{S}_{IJ} = t^{-2}(\tilde{g}^{\mu\nu}\tilde{S}_{\mu\nu} - 2\rho\tilde{S}_{\infty\infty}). \tag{215}$$

Now we use identity

$$\boldsymbol{\nabla}^I \mathbf{S}_{IJ} - \frac{1}{2}\partial_J(\mathbf{g}^{KL}\mathbf{S}_{KL}) = 0, \tag{216}$$

to get

$$\tilde{\nabla}^\mu \left(\tilde{S}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{S}\right) + \rho\partial_\mu\tilde{S}_{\infty\infty} + (d-2-2\rho\partial_\infty)\tilde{S}_{\mu\infty} - \rho\tilde{g}^{\xi\chi}\tilde{g}'_{\xi\chi}\tilde{S}_{\mu\infty} = 0, \tag{217}$$

$$\tilde{\nabla}^\mu\tilde{S}_{\mu\infty} + (d-2-\rho\partial_\infty)\tilde{S}_{\infty\infty} - \rho\tilde{g}^{\mu\nu}\tilde{g}'_{\mu\nu}\tilde{S}_{\infty\infty} - \frac{1}{2}\tilde{g}'_{\mu\nu}\tilde{S}^{\mu\nu} - \frac{1}{2}\partial_\infty\tilde{S} = 0, \tag{218}$$

which is the result. □

Let us remind

$$\begin{aligned}\tilde{B}^1_\mu &= -\frac{1}{2}\tilde{\nabla}^\xi\tilde{\nabla}_\xi\tilde{G}_\mu - \frac{1}{2}\tilde{R}^\nu_\mu\tilde{G}_\nu - \left(\frac{d}{2} - 1 - \rho\partial_\infty\right)\partial_\infty\tilde{G}_\mu \\ &\quad + \frac{1}{2}\tilde{g}^{\xi\chi}\tilde{g}'_{\xi\chi}\rho\partial_\infty\tilde{G}_\mu + \frac{1}{2}\rho\tilde{g}^{\xi\chi}\tilde{g}'_{\xi\chi}\partial_\mu\tilde{\gamma},\end{aligned} \tag{219}$$

$$\begin{aligned}\tilde{B}^2 &= -\frac{1}{2}\tilde{\nabla}^\mu\partial_\mu\tilde{\gamma} - \left(\frac{d}{2} - 2 - \rho\partial_\infty\right)\partial_\infty\tilde{\gamma} + \tilde{g}^{\mu\nu}\tilde{g}'_{\mu\nu}\rho\partial_\infty\tilde{\gamma} \\ &\quad + \frac{1}{2}\tilde{Q}^\mu\tilde{G}_\mu + \frac{1}{2}\tilde{g}'_{\mu\nu}\tilde{\nabla}^\mu\tilde{G}^\nu + \frac{1}{2}\tilde{g}'_{\mu\nu}\tilde{g}^{\mu\nu}\tilde{\gamma},\end{aligned} \tag{220}$$

where $\tilde{Q}^\mu = \partial_\infty(\tilde{g}^{\nu\xi}\tilde{\Gamma}^\mu_{\nu\xi})$. We will prove the following important properties of these objects.

Proposition 30. *Suppose that $\tilde{E}_{\mu\nu}{=}O(\rho^{d/2})$ then $\tilde{B}^1_\mu{=}O(\rho^{d/2})$ and $\tilde{B}^2{=}O(\rho^{d/2-1})$.*

Proof. We will use the Bianchi identity (200) and (201). Let us now also notice

$$\tilde{\nabla}^\mu\left(\tilde{\nabla}_\mu\tilde{G}_\nu + \tilde{\nabla}_\nu\tilde{G}_\mu - \tilde{g}_{\mu\nu}\tilde{\nabla}^\xi\tilde{G}_\xi\right) = \tilde{\nabla}^\xi\tilde{\nabla}_\xi\tilde{G}_\nu + \tilde{R}^\mu_\nu\tilde{G}_\mu. \tag{221}$$

Then we compute

$$\tilde{\nabla}^\mu\left(\tilde{S}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{S}\right) = \tilde{\nabla}^\mu\left(\tilde{E}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{E}\right) + \frac{1}{2}(\tilde{\nabla}^\xi\tilde{\nabla}_\xi\tilde{G}_\nu + \tilde{R}^\mu_\nu\tilde{G}_\mu) - \left(\frac{d}{2} - 1\right)\partial_\mu\tilde{\gamma}. \tag{222}$$

Moreover, differentiating (73) and (74) with respect to ρ we get

$$\tilde{S}_{\infty\infty} = \partial_\infty \tilde{\gamma}, \quad \tilde{S}_{\mu\infty} = \frac{1}{2}(\partial_\infty \tilde{G}_\mu + \partial_\mu \tilde{\gamma}). \tag{223}$$

Remembering that $\tilde{E} = O(\rho^{d/2})$ and combining (223) with (200) and (222) we obtain

$$\begin{aligned}&\frac{1}{2}\tilde{\nabla}^\xi \tilde{\nabla}_\xi \tilde{G}_\mu + \frac{1}{2}\tilde{R}^\nu_\mu \tilde{G}_\nu \\ &+ \left(\frac{d}{2} - 1 - \rho\partial_\infty\right)\partial_\infty \tilde{G}_\mu - \frac{1}{2}\tilde{g}^{\xi\chi}\tilde{g}'_{\xi\chi}\rho\partial_\infty \tilde{G}_\mu - \frac{1}{2}\rho\tilde{g}^{\xi\chi}\tilde{g}'_{\xi\chi}\partial_\mu\tilde{\gamma} = O(\rho^{d/2}).\end{aligned} \tag{224}$$

Similar computation with

$$\tilde{S} = \tilde{E} + \tilde{\nabla}^\mu \tilde{G}_\mu + d\tilde{\gamma} = \tilde{\nabla}^\mu \tilde{G}_\mu + d\tilde{\gamma} + O(\rho^{d/2}), \tag{225}$$

$$\tilde{g}'_{\mu\nu}\tilde{S}^{\mu\nu} = \tilde{g}'_{\mu\nu}\tilde{E}^{\mu\nu} + \tilde{g}'_{\mu\nu}\tilde{\nabla}^\mu\tilde{G}^\nu + \tilde{g}'_{\mu\nu}\tilde{g}^{\mu\nu}\tilde{\gamma} = \tilde{g}'_{\mu\nu}\tilde{\nabla}^\mu\tilde{G}^\nu + \tilde{g}'_{\mu\nu}\tilde{g}^{\mu\nu}\tilde{\gamma} + O(\rho^{d/2}), \tag{226}$$

reveals after inserting it into (201)

$$\begin{aligned}&\frac{1}{2}\tilde{\nabla}^\mu(\partial_\mu\tilde{\gamma} + \partial_\infty\tilde{G}_\mu) + (d - 2 - \rho\partial_\infty)\partial_\infty\tilde{\gamma} - \tilde{g}^{\mu\nu}\tilde{g}'_{\mu\nu}\rho\partial_\infty\tilde{\gamma} \\ &\quad + -\frac{1}{2}\partial_\infty\tilde{\nabla}^\mu\tilde{G}_\mu - \frac{d}{2}\partial_\infty\tilde{\gamma} - \frac{1}{2}\tilde{g}'_{\mu\nu}\tilde{\nabla}^\mu\tilde{G}^\nu - \frac{1}{2}\tilde{g}'_{\mu\nu}\tilde{g}^{\mu\nu}\tilde{\gamma} = O(\rho^{d/2-1}).\end{aligned} \tag{227}$$

Taking into account that

$$\partial_\infty\tilde{\nabla}^\mu\tilde{G}_\mu - \tilde{\nabla}^\mu\partial_\infty\tilde{G}_\mu = \tilde{Q}^\mu\tilde{G}_\mu, \quad \tilde{Q}^\mu = \partial_\infty\tilde{g}^{\nu\xi}\tilde{\Gamma}^\mu_{\nu\xi}, \tag{228}$$

we conclude

$$\begin{aligned}&\frac{1}{2}\tilde{\nabla}^\mu\partial_\mu\tilde{\gamma} + \left(\frac{d}{2} - 2 - \rho\partial_\infty\right)\partial_\infty\tilde{\gamma} - \tilde{g}^{\mu\nu}\tilde{g}'_{\mu\nu}\rho\partial_\infty\tilde{\gamma} \\ &\quad - \frac{1}{2}\tilde{Q}^\mu\tilde{G}_\mu - \frac{1}{2}\tilde{g}'_{\mu\nu}\tilde{\nabla}^\mu\tilde{G}^\nu - \frac{1}{2}\tilde{g}'_{\mu\nu}\tilde{g}^{\mu\nu}\tilde{\gamma} = O(\rho^{d/2-1}),\end{aligned} \tag{229}$$

and the proposition is proven. □

B Scalar GJMS Operators

We will prove some basic properties of supercritical scalar GJMS operator P_{d+2} which were recalled in the main part of the paper. By direct computation we obtain

$$\tilde{\boxdot}_1\tilde{\phi} = \tilde{\Box}\tilde{\phi} + (d - 2\rho\partial_\infty)\tilde{\phi}' + (\partial_\infty \ln\sqrt{\tilde{g}})(-2\rho\tilde{\phi}' + \tilde{\phi}). \tag{230}$$

This shows that $[\tilde{\boxdot}_1\tilde{\phi}]^{[n]}$ depends only on the $\tilde{g}^{[k]}_{\mu\nu}$ for $k \leq n$ and tr $\tilde{g}^{[n+1]}$. In particular, recursive determination of $\tilde{\phi} + O(\rho^{d/2+1})$ involves beside $\tilde{\phi}^{[0]}$ only $\tilde{g}^{[k]}_{\mu\nu}$ for $k \leq \frac{d}{2} - 1$ and tr $\tilde{g}^{[\frac{d}{2}]}$. This part of the metric is determined by the Fefferman–Graham construction.

Proof of Proposition 22. Let us now analyze $[\tilde{\boxdot}_1\tilde{\phi}]^{[\frac{d}{2}]}$. We consider the metric $h_{\mu\nu}$ with vanishing obstruction tensor and let $\tilde{g}_{\mu\nu}$ be an extension of the ambient metric for $h_{\mu\nu}$ which is Ricci flat to infinite order in ρ. For any tensor $k_{\mu\nu}$ satisfying

$$\nabla^{\mu} k_{\mu\nu} = 0, \quad \operatorname{tr} k = 0, \tag{231}$$

we can consider a Ricci flat extension $\tilde{g}_{\mu\nu}(s)$ defined by the property that

$$[\tilde{g}_{\mu\nu}(s)]^{[d/2]} = [\tilde{g}_{\mu\nu}]^{[d/2]} + s k_{\mu\nu} \tag{232}$$

The higher order expansion is uniquely determined by this property [6]. Our goal is to compute $\frac{d}{ds}[\tilde{\boxdot}_1\tilde{\phi}]^{[\frac{d}{2}]}$.

Lemma 31. *The following holds*

$$\frac{d}{ds}\tilde{g}^{\mu\nu}(s)|_{s=0} = -\rho^{\frac{d}{2}} k^{\mu\nu} + O(\rho^{d/2+1}) \tag{233}$$

$$\frac{d}{ds}\ln\sqrt{\tilde{g}(s)}|_{s=0} = -\frac{2}{d+2}\rho^{\frac{d}{2}+1} k^{\mu}_{\nu} P^{\nu}_{\mu} + O(\rho^{d/2+2}) \tag{234}$$

where $P_{\mu\nu}$ is the Schouten tensor.

Proof. The first equation is shown by direct computation

$$\begin{aligned}\frac{d}{ds}\tilde{g}^{\mu\nu}(s)|_{s=0} &= -\tilde{g}^{\mu\xi}\frac{d}{ds}\tilde{g}_{\xi\sigma}(s)|_{s=0}\tilde{g}^{\sigma\nu} = -\tilde{g}^{\mu\xi}\left(\rho^{\frac{d}{2}} k_{\xi\sigma} + O(\rho^{d/2+1})\right)\tilde{g}^{\sigma\nu} \\ &= -\rho^{\frac{d}{2}} k^{\mu\nu} + O(\rho^{d/2+1}).\end{aligned} \tag{235}$$

Let us introduce $\tilde{A}^{\mu}_{\nu}(s) := \tilde{g}^{\mu\xi}(s)\tilde{g}'_{\xi\nu}(s)$. We remark that $\tilde{A}^{\mu}_{\mu}(s) = 2\partial_{\infty}\ln\sqrt{\tilde{g}(s)}$. The condition $\tilde{S}_{\infty\infty}(s) = O(\rho^{\infty})$ gives

$$-\frac{1}{2}\partial_{\infty}\tilde{A}^{\mu}_{\mu}(s) - \frac{1}{4}\tilde{A}^{\mu}_{\nu}(s)\tilde{A}^{\nu}_{\mu}(s) = O(\rho^{\infty}), \tag{236}$$

by (57). We now differentiate (236) to obtain

$$-\partial^2_{\infty}\left(\frac{d}{ds}\ln\sqrt{\tilde{g}(s)}\right) - \frac{1}{2}\tilde{A}^{\mu}_{\nu}(s)\frac{d}{ds}\tilde{A}^{\nu}_{\mu}(s) = O(\rho^{\infty}). \tag{237}$$

We now notice that

$$\tilde{A}^{\nu}_{\mu}(s) = 2P^{\nu}_{\mu} + O(\rho), \quad \frac{d}{ds}\tilde{A}^{\nu}_{\mu}(s) = \frac{d}{2}\rho^{\frac{d}{2}-1} k^{\nu}_{\mu} + O(\rho^{d/2}), \tag{238}$$

thus by (237)

$$\partial^2_{\infty}\left(\frac{d}{ds}\ln\sqrt{\tilde{g}(s)}\right) = -\frac{d}{2}\rho^{\frac{d}{2}-1} P^{\nu}_{\mu} k^{\mu}_{\nu} + O(\rho^{d/2}). \tag{239}$$

As $\left[\frac{d}{ds}\ln\sqrt{\tilde{g}(s)}\right]^{[0]} = \frac{d}{ds}\ln\sqrt{\tilde{g}} = 0$ and $\left[\frac{d}{ds}\ln\sqrt{\tilde{g}(s)}\right]^{[1]} = \frac{d}{ds}P^{\mu}_{\mu} = 0$ we conclude (234) by integrating twice (239). $\square$

From (230)

$$\frac{d}{ds}\tilde{\boxdot}_1\tilde{\phi}(s)\bigg|_{s=0} = \left(\frac{d}{ds}\tilde{\Box}\right)\tilde{\phi} + \tilde{\Box}\frac{d\tilde{\phi}}{ds} + (d - 2\rho\partial_\infty)\frac{d\tilde{\phi}'}{ds}$$
$$+ \left(\partial_\infty \frac{d}{ds}\ln\sqrt{\tilde{g}}\right)(-2\rho\tilde{\phi}' + \tilde{\phi}) + \left(\partial_\infty \ln\sqrt{\tilde{g}}\right)\frac{d}{ds}(-2\rho\tilde{\phi}' + \tilde{\phi}). \tag{240}$$

Let us remind that $\frac{d}{ds}\tilde{\phi}(s)\Big|_{s=0} = O(\rho^{d/2+1})$, thus $\tilde{\Box}\frac{d\tilde{\phi}}{ds} = O(\rho^{d/2+1})$. Moreover,

$$\left[(d - 2\rho\partial_\infty)\frac{d\tilde{\phi}'}{ds}\right]^{[\frac{d}{2}]} = 0, \quad \left[\frac{d}{ds}(-2\rho\tilde{\phi}' + \tilde{\phi})\right]^{[\frac{d}{2}]} = 0. \tag{241}$$

The two remaining terms in (240) can be computed:

$$\frac{d}{ds}\tilde{\Box} = \rho^{d/2}\frac{1}{\sqrt{g}}\partial_\mu\left(-k^{\mu\nu}\sqrt{g}\partial_\nu\right) + O(\rho^{d/2+1}) = \rho^{d/2}\nabla_\mu\left(-k^{\mu\nu}\nabla_\nu\right) + O(\rho^{d/2+1}) \tag{242}$$

However, $\nabla_\mu k^{\mu\nu} = 0$ thus

$$\frac{d}{ds}\tilde{\Box} = -\rho^{d/2}k^{\mu\nu}\nabla_\mu\nabla_\nu + O(\rho^{d/2+1}). \tag{243}$$

Similarly,

$$\left(\partial_\infty \frac{d}{ds}\ln\sqrt{\tilde{g}}\right) = -\rho^{\frac{d}{2}}k^\mu_\nu P^\nu_\mu + O(\rho^{d/2+1}). \tag{244}$$

Finally,

$$\frac{d}{ds}[\tilde{\boxdot}_1\tilde{\phi}(s)]^{[\frac{d}{2}]}\bigg|_{s=0} = \left[\frac{d}{ds}\tilde{\Box}\right]^{[\frac{d}{2}]}\tilde{\phi}^{[0]} + \left[\partial_\infty\frac{d}{ds}\ln\sqrt{\tilde{g}}\right]^{[\frac{d}{2}]}[-2\rho\tilde{\phi}' + \tilde{\phi}]^{[0]}$$
$$= -k^{\mu\nu}\left(\nabla_\mu\nabla_\nu\tilde{\phi}^{[0]} + P_{\mu\nu}\tilde{\phi}^{[0]}\right). \tag{245}$$

We can now integrate the result over s from 0 to 1 assuming $\tilde{g}_{\mu\nu} = \tilde{g}^-_{\mu\nu}$ and $k_{\mu\nu} = [\tilde{g}^+_{\mu\nu}]^{[d/2]} - [\tilde{g}^-_{\mu\nu}]^{[d/2]}$ to obtain Proposition 22. □

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