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# A Stationary Black Hole Must be Axisymmetric in Effective Field Theory

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**Abstract:** The black hole rigidity theorem asserts that a rotating stationary black hole must be axisymmetric. This theorem holds for General Relativity with suitable matter fields, in four or more dimensions. We show that the theorem can be extended to any diffeomorphism invariant theory of vacuum gravity, assuming that this is interpreted in the sense of effective field theory, with coupling constants determined in terms of a "UV scale", and that the black hole solution can locally be expanded as a power series in this scale.

## 1. Introduction

Consider a stationary black hole spacetime of dimension $d \geq 4$. The orbits of the asymptotically timelike Killing vector field (KVF) $t^a$ must leave the horizon invariant and are either everywhere tangent to the null-generators of the horizon, or not. It is known in standard Einstein gravity coupled to a wide range of standard matter models that in the first case, the metric and matter fields are actually static [7,15–17,27,28,45–47]. In the other case, the black hole horizon is said to be rotating. In such a case, it is known, again for a fairly general class of standard matter models but assuming that the spacetime is real analytic and non-degenerate, that there necessarily exists another KVF $\chi^a$ that is tangent to the null-generators of the horizon. Furthermore, it is possible to show that $t^a = \chi^a + \sum_j \Omega_j \psi_j^a$, where the $\psi_j^a$ are commuting KVFs each of which has closed orbits with period $2\pi$. Thus, the black hole is necessarily stationary *and* axi-symmetric. This theorem is originally due to Hawking [22,23], who considered $d = 4$ dimensions. Later improvements include [13,40–42] (partially eliminating the analyticity assumption) as well as [1] (eliminating the analyticity- but under a smallness assumption). For higher dimensions $d \geq 4$ see [20,21,35].[1] These results are often called "rigidity theorem", because they imply among other things that the black hole must be rotating rigidly with respect to infinity with angular velocities $\Omega_j$.

[1] See also [25,36,37] for closely related work on Cauchy horizons with closed generators.

In $d = 4$, the rigidity theorem is an important stepping stone for the proof of the uniqueness—or "no hair"—theorems [6,8,9,34,44] for Kerr-Newman black holes. Even though no uniqueness theorems of comparable strength are known in $d > 4$, or in a number of Einstein-matter theories even in $d = 4$, the rigidity theorem is of major structural importance not least because it is a prerequisite for the zeroth- and first laws of black hole mechanics. It is therefore natural to ask whether the rigidity theorem remains valid for example in the presence of higher-derivative terms in the action as expected from an effective field theory (EFT) perspective.

In this paper, we will prove an extension of the rigidity theorem in dimensions $d \geq 4$ for general, local, covariant purely gravitational EFTs extending standard Einstein general relativity. Our approach is to order the terms in the EFT action/equations of motion by the numbers of derivatives that they contain and to study, so to speak, the effects of these terms to increasing accuracy. Each term in the action is multiplied by a suitable power of some length scale $\ell$ that one may think of as a cutoff scale for UV physics if desired. For example, the standard Einstein-Hilbert Lagrangian $R$ has two derivatives; the next possible terms would be linear combinations of quadratic curvature invariants with four derivatives, $R^2$, $R_{ab}R^{ab}$ and $R_{abcd}R^{abcd}$, each multiplied by $\ell^2$, and so on. Roughly speaking, the EFT approach is to restrict attention to solutions varying over a typical length scale $L$ such that $\ell/L \ll 1$, and this means, roughly speaking, that for such solutions the higher curvature terms are always smaller than the leading Einstein Hilbert term in the action. More precisely, one may say (see def. 4.1 of [24]) that the EFT condition is valid near the horizon, $\mathcal{H}$, if considering a 1-parameter family of solutions to the theory with parameter $L$ (in our case thought of roughly as the size of the black hole), any quantity of dimension $n$ built from coordinate components of the metric in a suitable class of coordinate systems will remain bounded by $C_n/L^n$ in absolute value. Then, if $\ell \ll L$, a higher derivative term of dimension $D$ will, intuitively, locally make a very small correction of order $(\ell/L)^D$ to the solution. Thinking of the length scale $L$ as fixed and making $\ell$ small instead, this motivates that we should ask whether the expansion coefficients in $\ell$ of a family of solutions $g_{ab}(\ell, x)$ labelled by the UV length scale $\ell$ locally satisfy the rigidity theorem order by order. This is what we shall actually do in this paper.

The result of this analysis in the rotating case is Theorem 3 (and its local version Theorem 1) and it assumes, just as the rigidity theorem in ordinary Einstein gravity, that the family of metrics is real analytic in the domain of outer communication and non-degenerate. Furthermore, it appears that we additionally need to assume a certain genericity requirement in $d > 4$ which however does not impose a major restriction physically. Even though Theorem 3 is an order-by-order statement for the expansion in $\ell$—and thus already a good approximation in view of the EFT hypothesis near any horizon cross section—if the metric was known to be jointly analytic in $\ell$ and $x$, it is plausible that the rigidity theorem, i.e. existence of the further KVFs $\chi^a$, $\psi_j^a$ would actually hold for finite $\ell$ sufficiently close to zero.

The proof of Theorems 1 and 3 is inductive, with the induction ascending in the power of $\ell$. For vanishing $\ell$, the usual rigidity theorems in standard Einstein gravity of course apply. To make the induction step, we follow the ideas used in standard Einstein gravity [20] up to a point. However, we cannot at higher orders in $\ell$ parallel a key step employed in [20] which is using the Raychaudhuri equation, because it gives insufficient information in the presence of higher derivative terms in the action. As is well known, the Raychaudhuri equation plays a key role in the proof of the area- and singularity—and also the rigidity—theorems in ordinary Einstein gravity (see e.g. [23,49]). It is useful because the stress energy tensor ordinarily comes with a definite sign. By contrast, in the EFTs that we consider, the higher order derivative terms in the equation of motion produce terms in the Raychaudhuri equation which are not sign-definite in general. However, we are able, within our inductive scheme, to replace the arguments normally based on the Raychaudhuri equation and the horizon area with an argument involving an entropy current-density in EFTs recently analyzed in [24] (for previous works on such entropy current-densities see [3–5,29,30,50]); for an example see e.g. (83). Furthermore, for higher derivative theories, the treatment of the rotational Killing fields seems more subtle than for Einstein gravity.

A corollary of our proof is that the surface gravity, $\kappa$, which can be defined thanks to the existence of the additional KVF $\chi^a$ tangent and normal to the horizon, is constant, i.e. that the zeroth law of black hole mechanics holds. It is interesting to note that if one *assumes* the existence of $\chi^a$, the zeroth law can be demonstrated in the EFTs that we are considering by an independent argument [2,18].

This paper is organized as follows. In Sect. 2 we present in detail our assumptions and recall Gaussian null coordinates and related constructions required in the proof of the local rigidity theorem Theorem 1, which is presented in Sect. 3. In Sect. 4 we analyze for completeness the situation regarding KVFs for non-rotating horizons summarized in Theorem 2, and in Sect. 5, we present our main result Theorem 3. Some technical material is relegated to various appendices. Our conventions and notations are the same as in [49]. Lower case Roman indices $a, b, c, \ldots$ are abstract spacetime indices whereas Greek indices $\mu, \nu, \sigma, \ldots$ refer to specific spacetime coordinates clear from the context. Upper case Roman indices $A, B, C, \ldots$ refer to coordinates on the horizon cross section, $\mathcal{C}$. We work in units such that $16\pi G = 1$. $\ell$ is a length scale.

## 2. Setup

*2.1. Standing assumptions.* We consider a generic covariant parity even[2] gravitational theory describing corrections to Einstein gravity. Such a theory has an action of the form

$$I[g] = \int_{\mathcal{M}} (R + \ell^2 L_4 + \ell^4 L_6 + \cdots + \ell^{2D-2} L_{2D})\sqrt{-g}\mathrm{d}^d x \tag{1}$$

where the $L_{2j}$ are covariant local functionals of the metric containing $2j$ derivatives[3]. By the Thomas replacement theorem [29], each term in $L_{2j}$ is therefore a contraction of the (inverse) metric with a tensor product of $\nabla_{a_1} \ldots \nabla_{a_r} R_{abcd}$ and in each tensor product, $2j$ is equal to the total number of covariant derivatives plus twice the number of Riemann tensors. We could add a cosmological constant term $L_0 = -2\Lambda$, which would result in an obvious change in the asymptotic conditions on the metric, but not in a major change in our proofs. We will briefly comment on this in remark 1) below Theorem 3.

The Euler Lagrange (Einstein-) equations for (1) can be written in the schematic form as

$$G_{ab} = \ell^2 H_{4\,ab} + \cdots + \ell^{2D-2} H_{2D\,ab} \tag{2}$$

with local covariant tensors $H_{2j\,ab}$ containing $2j$ derivatives of the metric. The standing assumptions on the solutions to (2) considered in this paper are:

[2] This assumption is made only for simplicity and there is no difficulty in principle to generalize our proofs to theories with parity odd terms.

[3] If we drop the parity even requirement, we can use the volume element $\epsilon_{a_1 \ldots a_d}$ to obtain terms with an odd number of derivatives in odd $d$.

1. We have a 1-parameter family of $d$-dimensional ($d \geq 4$), stationary, asymptotically Minkowskian solutions $(\mathcal{M}, g_{ab})$ to (2) with asymptotically timelike KVF $t^a$ with complete orbits, see defs. 2.1, 2.2 of [10] for the precise asymptotic and causality conditions. Both $t^a$, $g_{ab}$ are functions of the parameter $\ell$ and $(\mathcal{M}, g_{ab})$ contains a black hole. We require that the manifold structure of $\mathcal{M}$ is independent of $\ell$ and, as a gauge condition, we require that the location of the (future) horizon,[4] $\mathcal{H}$, is independent of $\ell$. We also require that $\mathcal{H}$ is smooth.
2. $g_{ab}(\ell, x)$ and $t^a(\ell, x)$ are jointly smooth in $(\ell, x)$, meaning that we have an asymptotic expansion of the form

$$g_{ab}(\ell, x) = \sum_{n=0}^{2M} \ell^n g^{(n)}_{ab}(x) + O(\ell^{2M+2})(x) \tag{3}$$

   where each $g^{(n)}_{ab}$ is smooth on $\mathcal{M}$ and where $M$ can be as large as we like, and similarly for $t^a$. We assume w.l.o.g. that only even powers of $\ell$ appear, the same goes for all similar expansions below. Here and in the following, $O(\ell^n)$ denotes a term such that for each coordinate neighborhood $\mathcal{U}$ of $\mathcal{H}$ with compact closure and each $k \geq 0$, there exists a sufficiently small $\ell_0 = \ell_0(\mathcal{U}, k)$ and a constant $c_n = c_n(\mathcal{U}, k)$ such that $|\partial_{\mu_1} \dots \partial_{\mu_k} O(\ell^n)(x)| \leq c_n \ell^n$ for all $|\ell| \leq \ell_0$, all $x \in \mathcal{U}$.
3. We assume that $\mathcal{H}$ has topology $\mathcal{H} = \mathbb{R} \times \mathcal{C}$, where $\mathcal{C}$ is compact and that $\mathcal{H}$ is non-degenerate for $\ell = 0$ [for the precise definition see below Eq. (15)].

We note that the asymptotic expansion in $\ell$ postulated in item 2) is not required to be uniform in $x$, e.g. we allow that the metric for finite $\ell$ could deviate from the $\ell = 0$ solution in standard Einstein gravity by an ever increasing amount as time goes to infinity, no matter how small $\ell$. In other words, we allow the corrections from the higher derivative terms, while locally small, to pile up in an unbounded manner over asymptotically large times. In a sense, we are therefore allowing secular effects.

While in the case of $d = 4$ dimensions, the above requirements will be sufficient for the proof of Theorem 1, our method of analysis appears to necessitate a further "genericity" assumption in higher dimensions $d > 4$, unless the horizon is non-rotating. Since the analysis of both cases is rather different anyhow, we shall distinguish them in the following:

(I) Rotating Case: $t^a$ is *not* tangent to the null generators of $\mathcal{H}$ for sufficiently small $|\ell|$.
(II) Nonrotating Case: $t^a$ is tangent to the null generators up to arbitrary order in $\ell$.

There is of course also the possibility that $t^a$ is tangent to the null generators of $\mathcal{H}$ only up to a finite order in $\ell$. The treatment of this case would require a combination of the methods in cases (I) and (II), depending on the order in $\ell$ in the induction procedure. Since it is only case (I) that should be considered generic anyhow, and since the analysis would be rather repetitive, we will not give it here. In case (II), it could also in principle happen that $t^a$ is not tangent to the null generators for a sequence $\{\ell_n\}$ tending to zero if $t^a$, $g_{ab}$ and/or the manifold structure of $\mathcal{M}$ is not analytic, in which case the terminology

[4] It is defined as the future boundary of the domain of outer communication. More precisely, as in [10], we assume that $\mathcal{M}$ contains an acausal hypersurface $\Sigma$ with possibly several asymptotic ends $\Sigma_1, \Sigma_2, \dots$ each $\cong \mathbb{R}^{d-1} \setminus B_R$. The slice must satisfy either def. 2.2(a,b) of [10], which precludes it from "not reaching the event horizon". Then the domain of outer communication $\mathcal{D}$ with respect to the end $\Sigma_1$ is defined as the causal completion of $\cup_{\tau \in \mathbb{R}} \phi_\tau[\Sigma_1]$ [see Eq. (63)] where $\phi_\tau$ is the flow of $t^a$ and the future/past horizon is defined as $\mathcal{H}^\pm = \partial\mathcal{D} \cap I^\pm(\mathcal{D})$.

'nonrotating' is misleading. However, this case will not be relevant for our analysis since we will only obtain results order by order in $\ell$ for Theorem 1 or assume analyticity for Theorem 3. The rotating case is treated in Sect. 3 whereas the non-rotating case is treated in Sect. 4.

In order to state the "genericity" assumption in the rotating case, we need to recall—and will demonstrate again below—that for $\ell = 0$, i.e. Einstein gravity, already the above assumptions (1)–(3) imply that the projection of the flow generated by $t^a|_{\ell=0}$ to any cross section $\mathcal{C}$ of $\mathcal{H}$ is a Killing vector field, $S^a$, of the metric $g_{ab}|_{\ell=0}$ restricted to $\mathcal{C}$. Let $\{\hat{\phi}_\tau \;\; \tau \in \mathbb{R}\}$ be the flow of $S^a$ on $\mathcal{C}$. It is an abelian subgroup, $\mathcal{A}$, of the isometry group of the compact Riemannian manifold $\mathcal{C}$ (with the metric induced from $g_{ab}|_{\ell=0}$). Its closure, $\mathcal{G} = \overline{\mathcal{A}}$, therefore is an abelian compact Lie-group, hence isomorphic to a torus $\mathcal{G} = \mathbb{T}^N$ for some $N \geq 1$. The $N$ generators of this torus correspond to $N$ KVFs of $\mathcal{C}$, $\psi_1^a, \ldots, \psi_N^a$, each generating a flow of isometries with period $2\pi$, and we have, on $\mathcal{C}$,

$$S^a = \sum_{j=1}^{N} \Omega_j \psi_j^a. \tag{4}$$

4. (Genericity) We assume that the flow of $S^a$ generates the *full* isometry group of the cross sections $\mathcal{C}$ of $\mathcal{H}$ for the restriction of the metric $g_{ab}|_{\ell=0}$. In particular, the isometry group must be the abelian group $\mathbb{T}^N$.

Note that the genericity property is trivially fulfilled in $d = 4$ dimensions since in that case, the stationary black holes in question are provided by the Kerr-family, which has only one rotational KVF. The Myers–Perry black holes [38] in $d$ dimensions have isometry group $\mathbb{R} \times \mathbb{T}^N$, where $N = \lfloor (d-1)/2 \rfloor$, unless some of the spin parameters $a_j$ happen to vanish. The genericity requirement imposes that the orbit of $S^a$ on a horizon cross section is dense in $\mathbb{T}^N$, and this will be the case if the $a_j$ are such that all non-trivial ratios $\Omega_i/\Omega_j$ are irrational numbers. Thus, 4) amounts to a genericity requirement on the values of the spin parameters $a_j$ which is satisfied for almost all values of these parameters because the rational numbers have measure zero in the real numbers.

*2.2. Gaussian null coordinates (GNCs).* We begin by picking an arbitrary compact cross section $\mathcal{C}$ of $\mathcal{H}$ and flow it with the 1-parameter group $\phi_\tau$ of isometries generated by the KVF $t^a$ by an amount $v$ and set $\mathcal{C}(v) := \phi_v[\mathcal{C}]$. Using the global structure of spacetime expressed in assumption 1) and prop. 4.1 of [10], we may assume without loss of generality that $\mathcal{C}$ has been chosen so that each orbit of $t^a$ on $\mathcal{H}$ intersects $\mathcal{C}$ precisely once, so $t^a$ is everywhere transverse to each $\mathcal{C}(v)$. On each $\mathcal{C}(v)$ we can therefore decompose

$$t^a = k^a + s^a \tag{5}$$

where $s^a$ is tangent to each $\mathcal{C}(v)$, not identically zero on $\mathcal{C}(v)$ for rotating horizons, and $k^a$ is tangent and normal to $\mathcal{H}$ and nowhere vanishing, see Fig. 1.

By construction, we have

$$\mathscr{L}_t k^a = \mathscr{L}_t s^a = \mathscr{L}_k s^a = 0, \quad \mathscr{L}_t v = 1 \tag{6}$$

on $\mathcal{H}$, where $\mathscr{L}$ is the Lie derivative. The family of cross section $\mathcal{C}(v)$ defines a foliation of $\mathcal{H}$ which we now use to set up an adapted Gaussian null coordinate (GNC) system in an open neighborhood of $\mathcal{H}$; see Appendix A for further explanations about GNCs.

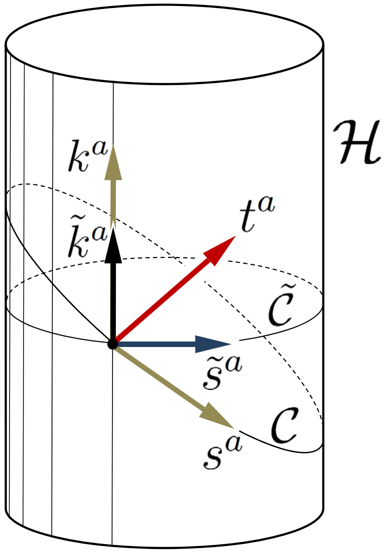


**Fig. 1.** Illustration of $t^a = k^a + s^a$ for a given cut $\mathcal{C}$. If a different cut $\tilde{\mathcal{C}}$ is chosen, we get a correspondingly different $\tilde{k}^a$ and $\tilde{s}^a$

To this end, we consider at each point of $\mathcal{H}$ a second null vector $l^a$ normalized such that $l^a k_a = 1$ and such that $l^a$ is perpendicular to the corresponding cut $\mathcal{C}(v)$. We extend $l^a$ off of $\mathcal{H}$ imposing the geodesic equation $l^a \nabla_a l^b = 0$ and let $r$ be an affine parameter on each such geodesic such that $r = 0$ on $\mathcal{H}$. Finally, we may locally pick a coordinate system $(x^A)$ on $\mathcal{C}$, which is transported off of $\mathcal{C}$ demanding that $\mathscr{L}_k x^A = \mathscr{L}_l x^A = 0$ where defined. Then, in the coordinates $(v, r, x^A)$, the metric takes the Gaussian Null Form:

$$g = 2\mathrm{d}v(\mathrm{d}r - r\alpha \mathrm{d}v - r\beta_A \mathrm{d}x^A) + \gamma_{AB}\mathrm{d}x^A \mathrm{d}x^B \tag{7}$$

By construction we have

$$k^a = \left(\frac{\partial}{\partial v}\right)^a, \quad l^a = \left(\frac{\partial}{\partial r}\right)^a, \quad s^a = s^A \left(\frac{\partial}{\partial x^A}\right)^a, \tag{8}$$

and we set

$$\gamma_{ab} = \gamma_{AB}(\mathrm{d}x^A)_a(\mathrm{d}x^B)_b, \quad \beta_a = \beta_A(\mathrm{d}x^A)_a \tag{9}$$

Even though we will of course need more than one coordinate chart $(x^A)$ to cover $\mathcal{C}$, these coordinate charts can be patched together so that the above tensor fields are defined globally and invariantly in an open neighborhood of $\mathcal{H}$, depending only on the initial choice of $\mathcal{C}$. By construction, we have

$$\mathscr{L}_t \gamma_{ab} = \mathscr{L}_t \beta_a = \mathscr{L}_t \alpha = \mathscr{L}_t r = 0. \tag{10}$$

In particular, since $[k, s]^a = 0$ on $\mathcal{H}$ by (6), it follows that the GNC components of $s^a$ are independent of $v$ on $\mathcal{H}$. Also, by construction, we have

$$k^a \nabla_a k^b = \alpha k^b \quad \text{on } \mathcal{H}. \tag{11}$$

We can think of the tensors $\alpha, \beta_a, \gamma_{ab}$ as living on the foliation $\mathcal{C}(v, r)$ of surfaces of constant $r, v$, and it will be useful to define a corresponding intrinsic covariant derivative operator and projections. For this purpose, we set $p^a{}_b = (\partial_A)^a (\mathrm{d}x^A)_b$, and $q^{ab} = (\gamma^{-1})^{AB} (\partial_A)^a (\partial_B)^b$. Then $q^a{}_b$ is the orthogonal (with respect to $g_{ab}$) projector onto $T\mathcal{C}(v, r)$, and we have $\gamma_{ab} q^{bc} = p^c{}_a$. $p^a{}_b$ is another projection onto $T\mathcal{C}(v, r)$ characterized by $p^a{}_b l^b = 0 = p^a{}_b k^b$. Note that $p^a{}_b$ is not an *orthogonal* projection where $r\beta_a$ is non-vanishing and therefore not equal to $q^a{}_b$ nor to $\gamma^a{}_b$ at such points. But on $\mathcal{H}$, i.e. for $r = 0$, the quantities $q^a{}_b, \gamma^a{}_b, p^a{}_b$ all coincide. By construction, we have

$$p^a{}_b \beta_a = \beta_b, \quad p^a{}_b p^c{}_d \gamma_{ac} = \gamma_{bd}, \quad p^a{}_b s^b = s^a \tag{12}$$

where these quantities are defined, and on $\mathcal{H}$ we could replace $p^a{}_b$ in these expressions by $q^a{}_b$.

For a covariant tensor field $T_{a_1 \dots a_r}{}^{b_1 \dots b_s}$ intrinsic to the foliation $\mathcal{C}(v, r)$, i.e. such that it coincides with its projection by $q_a{}^b$ for any index and any $(v, r)$, we define

$$D_b T_{a_1 \dots a_r}{}^{b_1 \dots b_s} := q_b{}^c q_{c_1}{}^{b_1} \cdots q_{c_s}{}^{b_s} q_{a_1}{}^{d_1} \cdots q_{a_r}{}^{d_r} \nabla_c T_{d_1 \dots d_r}{}^{c_1 \dots c_s}, \tag{13}$$

and we will denote by $R[\gamma]_{abc}{}^d$ the curvature of $D_c$ which is an intrinsically defined tensor field on each $\mathcal{C}(v, r)$. The contractions of $R_{ab}$ into $l^a, k^a, p^b{}_a$ as expressed in terms of $D_a, \gamma_{ab}, \beta_a, \alpha$ etc. are given in Appendix A.

## 3. Rotating Case

Assume that we are in the rotating case (I) of Sect. 2.1. We will give in this section a proof that there exists a KVF $\chi^a$ tangent to the null generators of $\mathcal{H}$ in the sense that $\chi^a$ Lie derives $g_{ab}$ modulo terms of order $O(\ell^n)$ where $n$ can be chosen as large as we like, and modulo terms that vanish to arbitrarily high order in any coordinate transverse to $\mathcal{H}$. $\chi^a$ is constructed such that it commutes with $t^a$ and on $\mathcal{H}$ satisfies $\chi^a \nabla_a \chi^b = \kappa \chi^b$, where $\kappa > 0$ is constant on $\mathcal{H}$. If the solution is jointly real analytic in $(x, \ell)$, we will argue in the next section that $\chi^a$ has an analytic continuation to the entire domain of outer communication which Lie-derives $g_{ab}$ exactly, i.e. without any error terms.

The basic idea is to define $\chi^a := k^a$ where $k^a$ is the vector field (VF) tangent to the horizon generators defined by fixing a cross section $\mathcal{C}$ of $\mathcal{H}$ in our construction of GNCs, see (8). The cross section $\mathcal{C}$ is arbitrary in our construction of GNCs and different choices will lead to different $k^a$. We are going to find the right $\mathcal{C}$ by demanding that for the $k^a$ defined by that $\mathcal{C}$, we have $\alpha = \kappa =$ constant on $\mathcal{H}$. A generic $\mathcal{C}$ will not do for this purpose, so we will have to pass to a new $\tilde{\mathcal{C}}$, to be determined. The determination of this $\tilde{\mathcal{C}}$ will be made order by order in $\ell$. To organize the powers of $\ell$, we define the expansion coefficients in

$$\begin{aligned} \alpha(x, \ell) &\sim \sum_{n=0}^{\infty} \ell^n \alpha^{(n)}(x) \\ \beta_a(x, \ell) &\sim \sum_{n=0}^{\infty} \ell^n \beta_a^{(n)}(x) \\ \gamma_{ab}(x, \ell) &\sim \sum_{n=0}^{\infty} \ell^n \gamma_{ab}^{(n)}(x) \end{aligned} \tag{14}$$

where "$\sim$" means an asymptotic expansion in the sense described below (3). Similar expansions are made for other tensor fields on $\mathcal{M}$, and we note that due to the structure of the action (1), only even powers of $\ell$ can appear in such expansions. The conditions $\mathscr{L}_t \gamma_{ab}(\ell, x) = \mathscr{L}_t \beta_a(\ell, x) = \mathscr{L}_t \alpha(\ell, x) = 0$ may then be expanded out in powers of $\ell$ to obtain conditions on the expansion coefficients. We define $\kappa$ to be the average of $\alpha$ over $\mathcal{C}$,

$$\kappa = \frac{1}{A[\mathcal{C}]} \int_{\mathcal{C}} \alpha \sqrt{\gamma} \mathrm{d}^{d-2} x \tag{15}$$

where $A[\mathcal{C}]$ is the area defined w.r.t. $\gamma_{ab}$. Then we have $\kappa \sim \sum_{n=0}^{\infty} \ell^n \kappa^{(n)}$, and the precise form of assumption 3) is that $\kappa^{(0)}$ so defined is $> 0$. We now make the following

**Inductive hypothesis at order** $\ell^n$: There exists a cross section $\mathcal{C}$ with corresponding GNC system $(v, r, x^A)$ such that

$$\begin{aligned} \mathscr{L}_l^m \Big( \mathscr{L}_k \gamma_{ab} \Big)_{r=0} &= O(\ell^{n+2}) \\ \mathscr{L}_l^m \Big( \mathscr{L}_k \beta_a \Big)_{r=0} &= O(\ell^{n+2}) \\ \mathscr{L}_l^m \Big( \mathscr{L}_k \alpha \Big)_{r=0} &= O(\ell^{n+2}) \end{aligned} \tag{16}$$

for any $m \in \mathbb{N}_0$. Furthermore, $\alpha^{(j)} = \kappa^{(j)} =$ constant on $\mathcal{H}$ for all $j \leq n$ in that GNC system, and the expansion coefficients of $s^a$ satisfy

$$s^{(j)a} \quad \text{are KVFs of } \gamma_{ab}|_{\ell=0} \text{ for all } j \leq n \text{ on } \mathcal{C}. \tag{17}$$

**Remarks:** (1) By the genericity assumption 4) of Sect. 2.1, the $s^{(j)a}$, $j = 0, \dots, n$ are *commuting* KVFs of the zeroth order in $\ell$ horizon metric $\gamma_{ab}|_{\ell=0}$, and we can write, on $\mathcal{C}$,

$$s^a(\ell, x) = \sum_{i=1}^{N} \Omega_i(\ell) \psi_i^a(x) + O(\ell^{n+2}). \tag{18}$$

(2) The facts that $t^a$ Lie-derives $\alpha, \beta_a, \gamma_{ab}$ for all $\ell$ together with $t^a = k^a + s^a$ and (16) give

$$\begin{aligned} \mathscr{L}_l^m \Big( \mathscr{L}_s \gamma_{ab} \Big)_{r=0} &= O(\ell^{n+2}) \\ \mathscr{L}_l^m \Big( \mathscr{L}_s \beta_a \Big)_{r=0} &= O(\ell^{n+2}) \\ \mathscr{L}_l^m \Big( \mathscr{L}_s \alpha \Big)_{r=0} &= O(\ell^{n+2}) \end{aligned} \tag{19}$$

for any $m \in \mathbb{N}_0$.

*3.1. Induction start $n = 0$.* The argument in this section has been presented in [20] but we go through some of its steps as a preparation for the induction step to familiarize the reader with the basic logic of our argument. Let $\lambda$ be an affine parameter for the null geodesic generators of $\mathcal{H}$ whose future directed tangent we denote by $n^a$. Then we have the corresponding expansion and shear $\theta, \sigma_{ab}$ on $\mathcal{H}$, given by

$$\theta = \gamma^{ab} \nabla_{(a} n_{b)}, \quad \sigma_{ab} = \gamma_a{}^c \gamma_b{}^d (\nabla_{(c} n_{d)} - \tfrac{1}{d-2} g_{cd} \nabla_e n^e). \tag{20}$$

We consider the $v$-derivative of the area (with respect to $\gamma_{ab}$):

$$\frac{\mathrm{d}}{\mathrm{d}v} A[\mathcal{C}(v)]_{v=0} = \int_{\mathcal{C}} \frac{\partial \lambda}{\partial v} \theta \sqrt{\gamma} \mathrm{d}^{d-2} x. \tag{21}$$

For convenience, we will take $\lambda = 0$ on $\mathcal{C}$. We know $\frac{\mathrm{d}}{\mathrm{d}v} A[\mathcal{C}(v)] = 0$, because the flow generated by $t^a$ is isometric by assumption 1) and because the area $A$ of a cut $\mathcal{C}$ is covariant, i.e. only dependent on the metric structure and on $\mathcal{C}$. The Raychaudhuri equation gives

$$\partial_\lambda \theta = -\sigma_{ab}\sigma^{ab} - \frac{1}{d-2}\theta^2 - R_{ab} n^a n^b \tag{22}$$

and the Einstein equation (2) gives $R_{ab} n^a n^b = O(\ell^2)$. By the same argument as for the area theorem [23] for the $\ell = 0$ solution, we cannot have $\theta^{(0)} < 0$ on $\mathcal{H}$. Combining this statement with $\frac{\mathrm{d}}{\mathrm{d}v} A[\mathcal{C}(v)] = 0$ evaluated at order $O(\ell^0)$ therefore gives $\theta^{(0)} = 0$ on $\mathcal{H}$. In view of (22) evaluated at $O(\ell^0)$, we conclude that $\sigma^{(0)}_{ab} = \theta^{(0)} = 0$ on $\mathcal{H}$.

Thus, we have the first equation in (16) for $n = m = 0$. Combining $\mathscr{L}_k \gamma^{(0)}_{ab} = 0$ on $\mathcal{H}$ with $t^a = k^a + s^a$ and the fact that $\mathscr{L}_t \gamma_{ab} = 0$, this also shows that $S^a = s^a|_{\ell=0}$ Lie derives $\gamma^{(0)}_{ab}$, i.e. is a KVF. These conclusions hold no matter how we chose the initial cut $\mathcal{C} = \mathcal{C}(0)$ to set up our GNC system.

However, the other equations in (16) for $n = 0$ are in general not satisfied for an arbitrarily chosen initial cut $\mathcal{C}$. We now wish to find the appropriate new cut $\tilde{\mathcal{C}}$ and thereby the appropriate decomposition $t^a = \tilde{k}^a + \tilde{s}^a$ (see fig. 1) and GNC system $(\tilde{v}, \tilde{r}, \tilde{x}^A)$ with associated tensors $\tilde{\alpha}$, $\tilde{\beta}_a$, $\tilde{\gamma}_{ab}$ satisfying (7). For this, we consider the $vA$-component of the Ricci tensor on $\mathcal{H}$, see (69). Using that we now know $\mathscr{L}_k \gamma_{ab} = O(\ell^2)$, this gives

$$R_{bc} k^b p^c{}_a = D_a \alpha - \frac{1}{2} \mathscr{L}_k \beta_a + O(\ell^2) \quad \text{on } \mathcal{H}. \tag{23}$$

Using the Einstein equation on $\mathcal{H}$, we have $R_{ac} k^a p^c{}_b = O(\ell^2)$, so sending $\ell \to 0$ we find that $D_a \alpha^{(0)} - \frac{1}{2} \mathscr{L}_k \beta^{(0)}_a = 0$ on $\mathcal{H}$, which implies $D_a \alpha^{(0)} + \frac{1}{2} \mathscr{L}_S \beta^{(0)}_a = 0$ on $\mathcal{C}$, where $S^a = s^a|_{\ell=0}$. We see from this equation that if we can define a new cut $\tilde{\mathcal{C}}$ with corresponding new $\tilde{\alpha}$, $\tilde{\beta}_a$, $\tilde{\gamma}_{ab}$ in such a way that $\tilde{\alpha}^{(0)}$ is constant on $\mathcal{H}$, then $\mathscr{L}_{\tilde{k}} \tilde{\beta}^{(0)}_a = 0$, and we will have all the equations in (16) for $n = 0$ for $m = 0$ for the tilde fields. Additionally, we will have learnt that $\tilde{\alpha}^{(0)} = \kappa(0) = \text{constant}$.

Let us determine the conditions that the new cut $\tilde{\mathcal{C}}$ would have to satisfy. It is clear that $\tilde{k}^a$ must be proportional to $k^a$, so it must be the case that $\tilde{k}^a = f k^a$ for some positive function $f$. Since $\mathscr{L}_t k^a = \mathscr{L}_t \tilde{k}^a = 0$, we must have $\mathscr{L}_t f = 0$ and therefore $\mathscr{L}_k f = -\mathscr{L}_s f$. Since on $\mathcal{H}$ we know that $k^a \nabla_a k^b = \alpha k^b$ and that $\tilde{k}^a \nabla_a \tilde{k}^b = \tilde{\alpha} \tilde{k}^b$. This means that $\alpha$ and $\tilde{\alpha}$ are related through $f$ by

$$\mathscr{L}_k f + \alpha f = -\mathscr{L}_s f + \alpha f = \tilde{\alpha}. \tag{24}$$

Demanding $\tilde{\alpha} = \kappa^{(0)} + O(\ell^2)$ means in view of the last relation that $f$ should be taken to be a solution of

$$-\mathscr{L}_S f + \alpha^{(0)} f = \kappa^{(0)} \tag{25}$$

using as before the notation $S^a = s^a|_{\ell=0}$. Equality (25) provides a condition that must be necessarily be satisfied on $\mathcal{C}$, cf. eq. (23) of [20]. We must additionally have $\mathscr{L}_t \tilde{v} = 1$ on $\mathcal{H}$ by (6), and since $\tilde{k}^a = (\partial/\partial\tilde{v})^a$,

$$\mathscr{L}_k \tilde{v} = \frac{1}{f} \mathscr{L}_{\tilde{k}} \tilde{v} = \frac{1}{f}. \tag{26}$$

Using $t^a = k^a + s^a$ and taking $\ell = 0$ for all the quantities shows that

$$1 - \mathscr{L}_S \tilde{v} = \frac{1}{f}. \tag{27}$$

Therefore $\tilde{v}$ must on $\mathcal{C}$ satisfy the equation

$$\mathscr{L}_S \tilde{v} = 1 + \frac{1}{\kappa^{(0)}} \left( \mathscr{L}_S \log f - \alpha^{(0)} \right), \tag{28}$$

cf. Eq. (38) of [20]. It is easy to see that (25) and (28) have a solution on $\mathcal{C}$ for $d = 4$: In that case, by the horizon topology theorem [23], $\mathcal{C} \cong \mathbb{S}^2$, so by standard results on isometric actions of spheres, the orbits of $S^a$ must close after a certain period $2\pi/\Omega$. It is then easy to see from this fact that (25) and (28) have a solution on $\mathcal{C}$. In $d > 4$, even though $S^a$ need not have closed orbits on $\mathcal{C}$, (25) and (28) have a solution on $\mathcal{C}$ by Lemmas 1 and 2 of [20] since we are assuming to be in the non-degenerate case, $\kappa^{(0)} > 0$.

This gives $\tilde{v}$ as a function on $\mathcal{C}$, and then we extend it to a function on $\mathcal{H}$ by demanding that $\mathscr{L}_t \tilde{v} = 1$, see (6). Given $\tilde{v}$, we define a new initial cut $\tilde{\mathcal{C}}$ by $\tilde{v} = 0$, and then we obtain a new GNC system $(\tilde{v}, \tilde{r}, \tilde{x}^A)$ from the foliation $\tilde{\mathcal{C}}(\tilde{v})$ with corresponding $\tilde{k}^a$, $\tilde{s}^a$ etc.

At this point we have shown $\mathscr{L}_{\tilde{k}}(\tilde{\alpha}^{(0)}, \tilde{\beta}^{(0)}_a, \tilde{\gamma}^{(0)}_{ab}) = 0$ on $\mathcal{H}$ and $\tilde{\alpha}^{(0)} = \kappa^{(0)}$ on $\mathcal{H}$. So we have all the equations in (16) for $n = 0$ and $m = 0$ for the tilde fields. Now we wish to show the same for $m = 1$ then $m = 2$, and so on. To do this, we perform an induction in $m$. First, we consider the $\tilde{v}$-derivative of the $AB$ Ricci tensor component (72) which gives, on $\mathcal{H}$

$$\mathscr{L}_{\tilde{k}}(R_{cd}\, p^c{}_a p^d{}_b) = \mathscr{L}_{\tilde{k}}[\mathscr{L}_{\tilde{k}} \mathscr{L}_{\tilde{l}} \tilde{\gamma}_{ab} + \tilde{\alpha}\, \mathscr{L}_{\tilde{l}} \tilde{\gamma}_{ab}] + O(\ell^2). \tag{29}$$

Since the tensors in this equation are Lie-derived by $t^a = \tilde{k}^a + \tilde{s}^a$, and since the Ricci tensor is $O(\ell^2)$ by the Einstein equation, this gives, on $\tilde{\mathcal{C}}$,

$$0 = \mathscr{L}_{\tilde{S}}[\mathscr{L}_{\tilde{S}} \mathscr{L}_{\tilde{l}} \tilde{\gamma}^{(0)}_{ab} - \kappa^{(0)}\, \mathscr{L}_{\tilde{l}} \tilde{\gamma}^{(0)}_{AB}] \tag{30}$$

see Eq. (56) of [20]. “Integrating” this equation along the orbits of $\tilde{S}^a$, it can be shown [20] that $\mathscr{L}_{\tilde{S}} \mathscr{L}_{\tilde{l}} \tilde{\gamma}^{(0)}_{ab} = 0$, and then $\mathscr{L}_{\tilde{l}}(\mathscr{L}_{\tilde{k}} \tilde{\gamma}^{(0)}_{ab})_{\tilde{r}=0} = 0$. This is the first equation of (16) for $n = 0$ and $m = 1$.

Let $M \geq 1$. We inductively assume that the first equation in (16) is satisfied for $n = 0$ and all $m \leq M$ whereas the second and third equations in (16) is satisfied for $n = 0$ and all $m \leq M - 1$ (we have seen that this is true when $M = 1$). Now we apply $\mathscr{L}_{\tilde{l}}^{M-1} \mathscr{L}_{\tilde{k}}$ to the $\tilde{v}\tilde{r}$ component of the Ricci tensor (68) and evaluate the result at $\tilde{r} = 0 = r$, i.e. on $\mathcal{H}$. Using the inductive hypothesis, we find

$$\mathscr{L}_{\tilde{l}}^{M-1} \Big( \mathscr{L}_{\tilde{n}} (R_{ab} \tilde{l}^a \tilde{k}^b) \Big)_{\tilde{r}=0} = \mathscr{L}_{\tilde{l}}^{M} \Big( \mathscr{L}_{\tilde{k}} \tilde{\alpha} \Big)_{\tilde{r}=0} + O(\ell^2) \tag{31}$$

and then using that the Ricci tensor is itself of order $O(\ell^2)$ by the Einstein equation, we obtain $\mathscr{L}_{\tilde{l}}^M(\mathscr{L}_{\tilde{k}}\tilde{\alpha}^{(0)})_{\tilde{r}=0} = 0$, which is the third equation in (16) for $n = 0$ and $m = M$. Now we apply $\mathscr{L}_{\tilde{l}}^{M-1}\partial_{\tilde{v}}$ to the $A\tilde{r}$ component of the Ricci tensor (71) and evaluate the result at $\tilde{r} = 0$, i.e. on $\mathcal{H}$. We find

$$\mathscr{L}_{\tilde{l}}^{M-1}\Big(\mathscr{L}_{\tilde{n}}(R_{bc}\tilde{p}^b{}_a\tilde{l}^c)\Big)_{\tilde{r}=0} = \mathscr{L}_{\tilde{l}}^{M}\Big(\mathscr{L}_{\tilde{k}}\tilde{\beta}_a\Big)_{\tilde{r}=0} + O(\ell^2) \tag{32}$$

which gives $\mathscr{L}_{\tilde{l}}^M(\mathscr{L}_{\tilde{k}}\tilde{\beta}_a^{(0)})_{\tilde{r}=0} = 0$. This is the second equation in (16) for $n = 0$ and $m = M$. Thus we see that all equation of (16) are satisfied for $n = 0$ and all $m \geq 0$ equations for the tilde GNCs. For ease of notation, we finally replace the tilde cross section $\tilde{\mathcal{C}}$ and the tilde tensor fields $\tilde{\alpha}, \tilde{\beta}_a, \tilde{\gamma}_{ab}, \tilde{k}^a, \tilde{s}^a, \tilde{l}^a$ so obtained by untilde quantities.

*3.2. Induction step $n-2 \to n$, ($n \geq 2$).* The induction step follows roughly the same path as the induction start:

1. We demonstrate that $\mathscr{L}_k\gamma_{ab} = 0$ on $\mathcal{H}$ up to order $O(\ell^{n+2})$,
2. We adjust $\mathcal{C} \to \tilde{\mathcal{C}}$ by an order $O(\ell^n)$ correction in such a way that the new tensor fields $\tilde{\alpha}, \tilde{\beta}_a, \tilde{\gamma}_{ab}$ still satisfy 1) and in addition that $\tilde{\alpha}$ is constant, and $\mathscr{L}_{\tilde{k}}\tilde{\beta}_a = 0$ on $\mathcal{H}$, up to order $O(\ell^{n+2})$.
3. We go through an additional induction loop in $m \in \mathbb{N}$ showing that $\mathscr{L}_{\tilde{l}}^m\mathscr{L}_{\tilde{k}}\{\tilde{\alpha}, \tilde{\beta}_a, \tilde{\gamma}_{ab}\} = 0$ on $\mathcal{H}$, up to order $O(\ell^{n+2})$.

The details of step 1) are different however from the induction start because the Raychaudhuri equation will have to be replaced by the "modified IWW entropy current-density" (see Appendix B). The arguments for step 1) will be broken down into Lemmas 1–3. The details of step 2) are likewise different because we need to first bring the vector field $\tilde{s}^a$ into a suitable form [see (17)] by applying a gauge transformation that needs to be found at each induction order $n$. The arguments for step 2) will be broken down into Lemmas 4 and 5. Step 3) will then not require essential innovations compared to the induction start $n = 0$.

We now go through the details. As mentioned, we would first like to show in parallel with the induction start that $\mathscr{L}_k\gamma_{ab} = O(\ell^{n+2})$ for $r = 0$, i.e. on $\mathcal{H}$. A first idea might be to consider again the $v$-derivative of the area functional, $\frac{\mathrm{d}}{\mathrm{d}v}A[\mathcal{C}(v)] = 0$, which vanishes due to stationarity. Again, let $\lambda$ be a parameter of affine null geodesics ruling $\mathcal{H}$ with tangent $n^a$. In parallel with the induction start, it seems natural that we seek to combine the Raychaudhuri equation (22) and the Einstein equation (2) as in

$$\partial_\lambda\theta = -\sigma_{ab}\sigma^{ab} - \frac{1}{d-2}\theta^2 - \ell^2 H_{4\,ab}n^a n^b - \cdots - \ell^{2D-2}H_{2D\,ab}n^a n^b \tag{33}$$

in order to gain information on $\theta^{(n)}, \sigma_{ab}^{(n)}$ at the induction order $n$. Indeed, the Raychaudhuri equation is implicitly used in the corresponding argument for the induction start $n = 0$ because it is the basis of area theorem used there [23,49]. In the present case, the induction hypothesis gives $\theta^{(j)}, \sigma_{ab}^{(j)} = 0$ for all $j \leq n-2$ on $\mathcal{H}$, so $\theta, \sigma_{ab} = O(\ell^n)$. This means that the $\sigma_{ab}\sigma^{ab}, \theta^2$ terms in the Raychaudhuri equation (33) are of order $O(\ell^{2n})$. However, the other terms on the right side of (33) are potentially only of order $O(\ell^{n+2})$! This is because it is merely known at this stage that each term $H_{j\,ab}n^a n^b$ is at least linear in positive boost weight quantities in the sense of [24], but when combined with Lemma 2, the induction hypothesis, and the explicit $\ell^{j-2}$ powers, this still leaves room for a term only of order $O(\ell^{n+j-2})$ which could e.g. be as bad as $O(\ell^{n+2})$ (for $j = 4$). So it appears that unlike for $n = 0$, we cannot get useful sign information on $\partial_\lambda \theta^{(n)}$ from the Raychaudhuri equation as the sign-definite term is no longer leading in $\ell$ in the case $n \geq 2$ considered now. As a consequence, it is not easy to see how we could conclude $\theta^{(n)}, \sigma_{ab}^{(n)} = 0$ [equivalent to the statement that $\gamma_{ab}$ is Lie derived by $k^a$ up to order $O(\ell^{n+2})$] on $\mathcal{H}$ by some sort of argument along the lines of the Raychaudhuri equation/area functional. Thus, it appears unclear how to take the first step in closing the induction.

Below in Lemma 3, we shall circumvent this problem by replacing the Raychaudhuri equation and cross section area by an equation for an entropy current-density and the corresponding generalized entropy of the cross section considered in [24]. However, before we come to this construction, we observe that the Raychaudhuri equation still gives the following preliminary result.

**Lemma 1.** *Under the inductive hypothesis, we have $\theta^{(n)} = 0$ on $\mathcal{C}$.*

*Proof.* Remember that combining the Einstein equation with the Raychaudhuri equation, we get (33). Now we expand $H_{j\,ab} n^a n^b$ in terms of "primitive monomials" with definite "boost weight" as described in [24] which is recalled in Appendix B. By the results of Sect. 2 of that paper, each summand in $H_{j\,ab} n^a n^b$ is at least linear in a positive boost weight primitive monomial. By Lemma 2 below, such a term is of order $O(\ell^n)$, and since each $H_{j\,ab} n^a n^b$ is accompanied by $\ell^{j-2}$ with $j - 2 \geq 2$ we see that the corresponding terms on the right side of the above equation are of order at least $O(\ell^{n+2})$. By the inductive hypothesis $\theta, \sigma_{ab}$ are of order $O(\ell^n)$, so since $n \geq 2$, the right side Raychaudhuri's equation (33) is of order $O(\ell^{n+2})$. This shows that $\partial_\lambda \theta^{(n)} = 0$ on $\mathcal{H}$.

Now, since $n^a = (\mathscr{L}_k \lambda)^{-1} k^a$, and since $\mathscr{L}_t[(\mathscr{L}_k \lambda)\theta] = 0$ because $[t, k]^a = 0$, it follows that

$$\mathscr{L}_k[(\mathscr{L}_k \lambda^{(0)})\theta^{(n)}] = -\mathscr{L}_S[(\mathscr{L}_k \lambda^{(0)})\theta^{(n)}] \tag{34}$$

where $S^a = s^a|_{\ell=0}$. By the usual relationship between affine- and Killing parameters at order $O(\ell^0)$, we can say that $\mathscr{L}_k \lambda^{(0)} = e^{\kappa^{(0)} v}$ (using the induction hypothesis and applying a rescaling to the affine parameter if necessary), which is constant on $\mathcal{C}$. Since we have already seen that $\mathscr{L}_k \theta^{(n)} = 0$, this gives

$$(\mathscr{L}_S + \kappa^{(0)})\theta^{(n)} = 0 \tag{35}$$

on $\mathcal{C}$. Let $\hat{\phi}_\tau$ be the flow of $S^a$ on $\mathcal{C}$. The previous equation can be rewritten as

$$\left(\frac{\mathrm{d}}{\mathrm{d}v} + \kappa^{(0)}\right) \theta^{(n)} \circ \hat{\phi}_v = 0 \tag{36}$$

Integrating this, we see that $\theta^{(n)} \circ \hat{\phi}_v = e^{-\kappa^{(0)} v} X^{(n)}$ where $X^{(n)}$ does not depend on $v$. However $\theta^{(n)} \circ \hat{\phi}_v$ is clearly bounded uniformly in $v$ because $\mathcal{C}$ is compact, so letting $v \to -\infty$ and using that $\kappa^{(0)} > 0$, we see that $X^{(n)} = 0$. We therefore conclude that $\theta^{(n)} = 0$ on $\mathcal{C}$. □

The following Lemma and its proof again refer to the notions of "primitive monomial" and "positive boost weight" from [24] which are recalled in Appendix B.

**Lemma 2.** *Let $X$ be a primitive monomial of positive boost weight. Then we have $X = O(\ell^n)$ locally near $\mathcal{C}$.*

*Proof.* Let us apply Lemma 2.2 of [24] to $X$. Then we eliminate any occurrence of the GNCs (with respect to the affine parameterization of $\mathcal{H}$) of the Ricci tensor or its covariant derivatives using the Einstein equation (2) and its covariant derivative. The new terms arising from the substitution process are decomposed into primitive monomials, and then Lemma 2.2 is applied again, and the Einstein equation is used again etc., repeating this process $n/2$ times. Thereby, $X$ is written as a sum of terms which either have an explicit pre-factor of at least $\ell^n$, or terms which are products of the monomials described in Lemma 2.2 of [24] without the occurrence of the Ricci tensor. The terms of order $O(\ell^n)$ can be ignored for the purposes of the proof, whereas each of the other terms contains at least one factor of $D_{c_1}\cdots D_{c_r}\mathscr{L}_n^N\sigma_{ab}$ or $D_{c_1}\cdots D_{c_r}\mathscr{L}_n^N\theta$ for some $N, r$. However, on $\mathcal{H}$, we can replace $\mathscr{L}_n = (\mathscr{L}_k\lambda)^{-1}\mathscr{L}_k$, so such a term clearly is of order $O(\ell^n)$ in view of the induction hypothesis. □

**Lemma 3.** *We have $\mathscr{L}_k\gamma_{ab} = O(\ell^{n+2})$ on $\mathcal{H}$.*

*Proof.* In order to get around the problem described before Lemma 1, we recall that the area is the black hole entropy for Einstein gravity $\ell = 0$, so it is natural to try an appropriate entropy functional for the higher derivative theory. We chose the "improved IWW entropy" $S[\mathcal{C}]$ defined in[5] sec. 3.3 of [24] because its properties are suitable for our purposes—in particular it is covariant. $S[\mathcal{C}]$ is defined in terms of an entropy-current-density $\sigma^a = \Sigma n^a + J^A(\partial_A)^a$ on $\mathcal{H}$ as in

$$S[\mathcal{C}] = \int_{\mathcal{C}} \Sigma\sqrt{\gamma}\mathrm{d}^{d-2}x \tag{37}$$

For further explanations regarding the definition and properties of $\Sigma$ and $J^a$, see Appendix B.

By property (1) of $\Sigma$ in Appendix B, since $t^a$ is a KVF, since $S[\mathcal{C}(v)]$ is a covariant functional that does not depend on the arbitrary choice of affine parameter, and since the flow of $t^a$ by an amount $v$ moves $\mathcal{C}$ to $\mathcal{C}(v)$, we learn that $S[\mathcal{C}(v)]$ is independent of $v$. Taking the first derivative in $v$ we get

$$0 = \frac{\mathrm{d}}{\mathrm{d}v}S[\mathcal{C}(v)]_{v=0} = \int_{\mathcal{C}} \frac{\partial\lambda}{\partial v}\,\partial_\lambda(\Sigma\sqrt{\gamma})\,\mathrm{d}^{d-2}x. \tag{38}$$

Taking the second derivative in $v$ we get

$$0 = \frac{\mathrm{d}^2}{\mathrm{d}v^2}S[\mathcal{C}(v)]_{v=0} = \int_{\mathcal{C}} \mathscr{L}_k\left[\frac{\partial\lambda}{\partial v}\,\partial_\lambda(\Sigma\sqrt{\gamma})\right]\mathrm{d}^{d-2}x. \tag{39}$$

We will now use these two identities to prove the Lemma.

First, since $k^a\nabla_a k^b = \alpha k^a$, and since $k^a = (\mathscr{L}_k\lambda)n^a$, we obtain $\alpha = \mathscr{L}_k\log\mathscr{L}_k\lambda$. Integrating this equation using $\alpha = \kappa + O(\ell^n)$ gives $\mathscr{L}_k\lambda = ae^{\kappa v}[1 + O(\ell^n)]$ where $a(x^A)$ may be chosen to be $= 1$ by a suitable choice of the affine parameter $\lambda$, giving

$$D_a\mathscr{L}_k\lambda = O(\ell^n) \tag{40}$$

on $\mathcal{C}$.

[5] The difference between the "improved IWW entropy" and its antecedents [4,50] is a specific choice for the ambiguities in that construction which are designed to render $S[\mathcal{C}]$ covariant, see prop. 1 of [24].

Next, we look at $\sqrt{\gamma}^{-1}\partial_\lambda(\Sigma\sqrt{\gamma})$. By property 3) in Appendix B, we know that this is equal to $\theta+$ terms at least linear in positive boost weight with an explicit prefactor of at least $\ell^2$. Combined with Lemmas 1, 2 and the induction hypothesis, we get that

$$\frac{1}{\sqrt{\gamma}}\partial_\lambda(\Sigma\sqrt{\gamma}) = O(\ell^{n+2}) \tag{41}$$

on $\mathcal{C}$.

Finally, we use $\mathcal{L}_k \log \mathcal{L}_k\lambda = \alpha$ which is used to write (39) as

$$\begin{aligned}0 = &\int_{\mathcal{C}} \alpha\,(\mathcal{L}_k\lambda)\,\partial_\lambda(\Sigma\sqrt{\gamma})\,\mathrm{d}^{d-2}x + \int_{\mathcal{C}}(\mathcal{L}_k\lambda)^2\partial_\lambda\left(\frac{1}{\sqrt{\gamma}}\partial_\lambda(\sqrt{\gamma}\Sigma) + D_aJ^a\right)\sqrt{\gamma}\,\mathrm{d}^{d-2}x\\ &+\int_{\mathcal{C}}(\partial_\lambda J^a)D_a(\mathcal{L}_k\lambda)^2\sqrt{\gamma}\,\mathrm{d}^{d-2}x + \int_{\mathcal{C}}\theta(\mathcal{L}_k\lambda)^2\partial_\lambda(\sqrt{\gamma}\Sigma)\,\mathrm{d}^{d-2}x\\ &-\int_{\mathcal{C}} J^a(D_a\theta)(\mathcal{L}_k\lambda)^2\sqrt{\gamma}\,\mathrm{d}^{d-2}x\end{aligned} \tag{42}$$

where we have added and subtracted $D_aJ^a$ under the integral and performed a partial integration. The terms on the right side are now treated as follows. On the first term, we use that $\alpha$ is, by induction, constant on $\mathcal{C}$ up to terms of order $O(\ell^n)$. The constant does not contribute in view of (38). The $O(\ell^n)$-terms combine with (41) to terms order $O(\ell^{2n+2})$ in total. On the second term on the right side we use the analog of the Raychaudhuri equation (79), see property 2) in Appendix B. On the third term on the right hand side we use (40) and the fact that $J^a$ is of order $O(\ell^{n+2})$: it has an explicit prefactor of $\ell^2$ and is linear in positive boost weight, see property 3) in Appendix B. By the induction hypothesis a positive boost weight term is of order $O(\ell^n)$ at least. Thus, the third term is of order $O(\ell^{2n+2})$. The fourth term on the right side is treated using that $\theta = O(\ell^{n+2})$ by Lemma 1 which is combined with (41). Thus, the fourth term is of order $O(\ell^{2n+4})$. The last term is estimated using $J^a = O(\ell^{n+2}), \theta = O(\ell^{n+2})$ and is therefore of order $O(\ell^{2n+4})$.

Combining these results, we see that (42) implies

$$\int_{\mathcal{C}}(\mathcal{L}_k\lambda)^2 F\,\sqrt{\gamma}\mathrm{d}^{d-2}x = O(\ell^{2n+2}). \tag{43}$$

At this stage, property 4) in Appendix B gives together with the induction hypothesis that

$$F = -\ell^{2n}\sigma^{(n)}_{ab}\sigma^{(n)}_{cd}\gamma^{(0)ac}\gamma^{(0)bd} + O(\ell^{2n+2}), \tag{44}$$

and so (43) yields $\sigma^{(n)}_{ab} = 0$ on $\mathcal{C}$ because $\mathcal{L}_k\lambda > 0$. Note that we already know $\theta^{(n)} = 0$ by Lemma 1, and that $\theta^{(j)} = 0 = \sigma^{(j)}_{ab}$ for $j \le n-2$ by induction. Then, using the definitions of $\theta, \sigma_{ab}$ on $\mathcal{H}$ we see that $\mathcal{L}_k\gamma_{ab} = O(\ell^{n+2})$ on $\mathcal{C}$ and since $\mathcal{L}_t\mathcal{L}_k\gamma_{ab} = 0$ on $\mathcal{H}$ it follows that $\mathcal{L}_k\gamma_{ab} = O(\ell^{n+2})$ everywhere on $\mathcal{H}$. □

Using $t^a = k^a + s^a$ and $\mathcal{L}_t\gamma_{ab} = 0$, we learn from Lemma 3 that $\mathcal{L}_s\gamma_{ab} = O(\ell^{n+2})$. We now want to use this result to establish the induction hypothesis (17) at order $n$, i.e. that $s^{(j)a}$ are KVFs of the zeroth order metric $\gamma_{ab}|_{\ell=0}$ for all $j \le n$. This is in question only for $j = n$. It does not seem possible to deduce this merely from $\mathcal{L}_s\gamma_{ab} = O(\ell^{n+2})$ and the fact that $s^{(j)a}$, $j \le n-2$ are KVFs of the zeroth order metric. But we will now show that it can be achieved if we simultaneously redefine $\tilde{t}^a = \phi^* t^a$, $\tilde{g}_{ab} = \phi^* g_{ab}$ by a suitable diffeomorphism $\phi$ preserving the horizon cross sections $\mathcal{C}(v)$. We will now construct such a diffeomorphism. The corresponding tensor fields $\tilde{\alpha}$, $\tilde{\beta}_a$, $\tilde{\gamma}_{ab}$, $\tilde{s}^a$, $\tilde{k}^a$ will clearly still satisfy $\mathscr{L}_{\tilde{k}}\tilde{\gamma}_{ab} = O(\ell^{n+2})$, $\mathscr{L}_{\tilde{k}}\tilde{\beta}_a = O(\ell^n)$, $\mathscr{L}_{\tilde{k}}\tilde{\alpha} = O(\ell^n)$. Furthermore, if $\phi$ is the identity up to order $O(\ell^2)$, then the zeroth order metric will be unchanged, $\gamma_{ab}|_{\ell=0} = \tilde{\gamma}_{ab}|_{\ell=0}$, and if $\tilde{s}^a \equiv \phi^* s^a = S^a + O(\ell^n)$ where $S^a := \sum_{j=0}^{n-2} \ell^j s^{(j)a}$, then $s^{(j)a} = \tilde{s}^{(j)a}$, $j \le n-2$, so the induction hypothesis (17) will still hold for $\tilde{s}^a$ up to order $j \le n-2$. Finally, if we even have $\phi^* s^a = S^a + \xi^a + O(\ell^{n+2})$ where $\xi^a$ is of order $O(\ell^n)$ and at the same time a KVF of the zeroth order metric $\gamma_{ab}|_{\ell=0}$, then all of the previous will still hold and in addition the induction hypothesis (17) will hold for $\tilde{s}^a$ up to order $j \le n$.

We are going to construct $\phi = \phi_1$ as the flow $\phi_\tau$ at parameter value $\tau = 1$ of a vector field $\zeta^a$ that is to be determined. First of all, the diffeomorphism should preserve the horizon cross sections $\mathcal{C}(v)$, and this will be achieved choosing a $\zeta^a$ that is tangent to $\mathcal{C}$ and such that $\mathscr{L}_k \zeta^a = 0$. In fact we will construct $\zeta^a$ initially on $\mathcal{C}$ and then define it in a neighborhood of $\mathcal{H}$ by the condition that $\mathscr{L}_k \zeta^a = 0 = \mathscr{L}_l \zeta^a$. Next, $\phi$ should be the identity up to order $O(\ell^2)$, and we will achieve this by choosing $\zeta^a = O(\ell^2)$. To analyze what requirements on $\zeta^a$ are imposed by the remaining conditions, we consider the Taylor series for $\phi_\tau^* s^a|_{\tau=1} = \tilde{s}^a$ with remainder around $\tau = 0$:

$$\begin{aligned}\tilde{s}^a &= \sum_{j=0}^{M} \frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}\tau^j}\phi_\tau^* s^a|_{\tau=0} + \frac{1}{M!}\int_0^1 (1-\tau)^M \frac{\mathrm{d}^{M+1}}{\mathrm{d}\tau^{M+1}}\phi_\tau^* s^a \,\mathrm{d}\tau \\ &= \sum_{j=0}^{M} \frac{1}{j!}\mathscr{L}_\zeta^j s^a + \frac{1}{M!}\int_0^1 (1-\tau)^M \phi_\tau^* \mathscr{L}_\zeta^{M+1} s^a \,\mathrm{d}\tau.\end{aligned} \tag{45}$$

Clearly, since $\zeta^a = O(\ell^2)$, the integral remainder term in the last line will be of order $O(\ell^{2(M+1)})$, so if we choose $2M \ge n$, then it will be of order $O(\ell^{n+2})$. Furthermore, if we knew that $\mathscr{L}_\zeta S^a = O(\ell^n)$, then it would automatically follow that each term in the sum for $j \ge 2$ would also be of order $O(\ell^{n+2})$, whereas the $j = 1$ term can be written as $-\mathscr{L}_S \zeta^a$ up to order $O(\ell^{n+2})$. Thus, we would know that

$$\tilde{s}^a = s^a - \mathscr{L}_S \zeta^a + O(\ell^{n+2}). \tag{46}$$

From this equation, we see that we would have $\ell^n \tilde{s}^{(n)a} = \ell^n s^{(n)a} - \mathscr{L}_S \zeta^a + O(\ell^{n+2})$. Since we would like to satisfy the induction hypothesis (17) at order $j \le n$, we must achieve that $\ell^n s^{(n)a} - \mathscr{L}_S \zeta^a$ is a KVF, called $\xi^a$, of order $O(\ell^n)$ of the zeroth order metric $\gamma_{ab}^{(0)}$ on $\mathcal{C}$, up to an error term of size $O(\ell^{n+2})$. That a $\zeta^a$ with all these properties exists is established in the following Lemma.

**Lemma 4.** *There exists a smooth vector field $\zeta^a = O(\ell^2)$ and a KVF $\xi^a = O(\ell^n)$ of $\gamma_{ab}^{(0)}$, both tangent to $\mathcal{C}$, such that $\ell^n s^{(n)a} = \xi^a + \mathscr{L}_S \zeta^a + O(\ell^{n+2})$, where $S^a = \sum_{k \le n-2} \ell^k s^{(k)a}$.*

*Proof.* See Appendix B □

We relabel the fields and coordinates thus obtained by the untilde ones. Then we know at this stage that $\mathscr{L}_k \gamma_{ab} = O(\ell^{n+2})$, $\mathscr{L}_k \beta_a = O(\ell^n)$, $\mathscr{L}_k \alpha = O(\ell^n)$, and we know that (17) holds up to and including order $j = n$. We now proceed to the $vA$-component of the Einstein equation (2) on $\mathcal{H}$, which using the induction hypothesis (16) and $\mathcal{L}_k \gamma_{ab} = O(\ell^{n+2})$ and the expression (69) for the corresponding Ricci-component reads

$$D_a\alpha - \frac{1}{2}\mathcal{L}_k\beta_a = \sum_{j=4}^{2D} \ell^{j-2} H_{j\,cb} k^b p^c{}_a + O(\ell^{n+2}) \quad \text{on } \mathcal{H}. \tag{47}$$

By Lemma 2 the right side is of order $O(\ell^{n+2})$. Using $t^a = k^a + s^a$, we therefore find

$$D_a\alpha + \frac{1}{2}\mathcal{L}_s\beta_a = O(\ell^{n+2}) \tag{48}$$

Following the same reasoning as in the induction start, we have to consider next a higher order version of (24), i.e. $-\mathcal{L}_s f + \alpha f = \tilde{\alpha}$, demanding now that $\tilde{\alpha} \equiv \kappa + O(\ell^{n+2})$, where $\kappa$ is the average of $\alpha$ over $\mathcal{C}$ with respect to the volume element of $\gamma_{ab}$. Since we inductively know that $\alpha = \sum_{k\leq n-2} \ell^k \kappa_{(k)} + O(\ell^n)$, it follows that $\alpha - \kappa = O(\ell^n)$. By analogy with the induction start, we next seek to define a smooth function $f$ on $\mathcal{C}$ such that

$$-\mathcal{L}_s f + \alpha f = \kappa. \tag{49}$$

Let $\hat{\phi}_\tau$ be the flow of $s^a$ on $\mathcal{C}$. We define

$$f(x) = \kappa \int_0^\infty \exp\left[-\int_0^\sigma \alpha \circ \hat{\phi}_\tau(x) \mathrm{d}\tau\right] \mathrm{d}\sigma. \tag{50}$$

Using the relation $\alpha = \sum_{k\leq n-2} \ell^k \kappa^{(k)} + O(\ell^n)$ and $\kappa^{(0)} > 0$, we see that the $\mathrm{d}\sigma$ integral converges absolutely for sufficiently small $|\ell|$, and furthermore, that $f = 1 + O(\ell^n)$. By analogy with the induction start, the new coordinate $\tilde{v}$ should now satisfy [see (28)]

$$\mathcal{L}_s\tilde{v} = 1 + \frac{1}{\kappa}(\mathcal{L}_s \log f - \alpha) + O(\ell^{n+2}). \tag{51}$$

a smooth solution for which exists by the following Lemma.

**Lemma 5.** *The equation $\mathcal{L}_s\varphi = \alpha - \kappa + O(\ell^{n+2})$ on $\mathcal{C}$ has a smooth solution $\varphi = O(\ell^n)$.*

*Proof.* See Appendix B. □

Using the solution $\varphi$ given by Lemma 5, we set

$$\tilde{v} := \frac{1}{\kappa}(-\varphi + \log f) \tag{52}$$

as a function on $\mathcal{C}$. Then it follows that $\tilde{v} = O(\ell^n)$ on $\mathcal{C}$ and then we extend $\tilde{v}$ to a function $\tilde{v} = v + O(\ell^n)$ on $\mathcal{H}$ by demanding that $\mathcal{L}_t\tilde{v} = 1$, see (6). Given $\tilde{v}$, we define a new initial cut $\tilde{\mathcal{C}}$ by $\tilde{v} = 0$, and then we obtain a new GNC system $(\tilde{v}, \tilde{r}, \tilde{x}^A) = (v, r, x^A) + O(\ell^n)$ from the foliation $\tilde{\mathcal{C}}(\tilde{v})$ with corresponding $\tilde{k}^a$ and $\tilde{s}^a$. By construction, $\tilde{\alpha}$ is constant on $\tilde{\mathcal{C}}$ modulo $O(\ell^{n+2})$, and then $\mathcal{L}_{\tilde{k}}\tilde{\beta}_a = 0$ modulo $O(\ell^{n+2})$.

So at this point we have shown $\mathcal{L}_{\tilde{k}}(\tilde{\alpha}, \tilde{\beta}_a, \tilde{\gamma}_{ab}) = O(\ell^{n+2})$ on $\mathcal{H}$ and $\tilde{\alpha} = \kappa + O(\ell^{n+2})$ on $\mathcal{H}$ and we can still assume the induction hypothesis (16) for the tilde tensors up to and including order $n-2$ when $m \geq 1$. Furthermore, we have the induction hypothesis (17) for $\tilde{s}^a$ on $\tilde{\mathcal{C}}$.

Now we wish to show that we have $\mathcal{L}_{\tilde{l}}^m(\mathcal{L}_k(\tilde{\alpha}, \tilde{\beta}_a, \tilde{\gamma}_{ab}))_{\tilde{r}=0} = O(\ell^{n+2})$ for all $m$. To do this, we again perform an induction in $m$ similar to the induction start. First, we consider the $\tilde{v}$-derivative of the $AB$ Ricci tensor component (72) which gives

$$\mathcal{L}_{\tilde{k}}(R_{cd}\,\tilde{p}^c{}_a\tilde{p}^d{}_d) = \mathcal{L}_{\tilde{k}}[\mathcal{L}_{\tilde{k}}\mathcal{L}_{\tilde{l}}\tilde{\gamma}_{ab} + \tilde{\alpha}\,\mathcal{L}_{\tilde{l}}\tilde{\gamma}_{ab}] + O(\ell^{n+2}) \tag{53}$$

on $\mathcal{H}$. Since the tensors are Lie-derived by $t^a = \tilde{k}^a + \tilde{s}^a$, we can effectively replace $\mathcal{L}_{\tilde{k}} = -\mathcal{L}_{\tilde{s}}$ and since the Ricci tensor is $O(\ell^{n+2})$ by the Einstein equation (2) and the induction hypothesis ($\mathcal{L}_{\tilde{k}}$ produces at least one primitive monomial (see Appendix B) appearing on the right side of the Einstein equation resulting in an $O(\ell^n)$ term by Lemma 2, which is multiplied at least by $\ell^2$), this gives, on $\tilde{\mathcal{C}}$,

$$\mathcal{L}_{\tilde{S}}[\mathcal{L}_{\tilde{S}}\tilde{L}_{ab} - \kappa\,\tilde{L}_{ab}] = O(\ell^{n+2}) \tag{54}$$

where $\tilde{S}^a = \sum_{j\le n}\ell^j\tilde{s}^{(j)a}$ which is a Riemannian isometry of $\tilde{\gamma}_{ab}|_{\ell=0}$ by (17), and where $\tilde{L}_{ab} := \mathcal{L}_{\tilde{l}}\tilde{\gamma}_{ab}$. "Integrating" this equation along the orbits of $\tilde{S}^a$ gives $\mathcal{L}_{\tilde{S}}\tilde{L}_{ab} = O(\ell^{n+2})$ by the same kind of argument as around eq. (56) of [20] but carrying around now the potential $O(\ell^{n+2})$ "error terms". This results in $\mathcal{L}_{\tilde{l}}(\mathcal{L}_{\tilde{k}}\tilde{\gamma}_{ab})_{\tilde{r}=0} = O(\ell^{n+2})$, where we have replaced again $\mathcal{L}_{\tilde{k}} = -\mathcal{L}_{\tilde{s}}$, so we get the first equation of (16) for our induction order $n$ and $m = 1$.

Let $M \ge 1$. We inductively assume that the first equation in (16) is satisfied for all $m \le M$ whereas the second and third equations in (16) is satisfied for all $m \le M-1$ (we have seen that this is true when $M = 1$). Now we apply $\mathcal{L}_{\tilde{l}}^{M-1}\mathcal{L}_{\tilde{k}}$ to the $\tilde{v}\tilde{r}$ component of the Ricci tensor (68) and evaluate the result at $\tilde{r} = 0$, i.e. on $\mathcal{H}$. Using the inductive hypothesis and the Einstein equation, we find

$$\mathcal{L}_{\tilde{l}}^{M-1}\Big(\mathcal{L}_{\tilde{k}}(R_{ab}\tilde{l}^a\tilde{k}^b)\Big)_{\tilde{r}=0} = \mathcal{L}_{\tilde{l}}^{M}\Big(\mathcal{L}_{\tilde{k}}\tilde{\alpha}\Big)_{\tilde{r}=0} + O(\ell^{n+2}) \tag{55}$$

and then using that a $\tilde{v}$-derivative of the Ricci tensor is itself of order $O(\ell^{n+2})$ by the Einstein equation ($\mathcal{L}_{\tilde{k}}$ hits at least one primitive monomial of non-negative boost weight appearing on the right side of the Einstein equation resulting in an $O(\ell^n)$ term which is multiplied at least by $\ell^2$), we obtain $\mathcal{L}_{\tilde{l}}^M(\mathcal{L}_{\tilde{k}}\tilde{\alpha})_{\tilde{r}=0} = O(\ell^{n+2})$, which is the third equation in (16) for our induction order $n$ and $m = M$. Now we apply $\mathcal{L}_{\tilde{l}}^{M-1}\mathcal{L}_{\tilde{k}}$ to the $A\tilde{r}$ component of the Ricci tensor (71) and evaluate the result at $\tilde{r} = 0$, i.e. on $\mathcal{H}$. We find

$$\mathcal{L}_{\tilde{l}}^{M-1}\Big(\mathcal{L}_{\tilde{k}}(R_{bc}\tilde{l}^c\,\tilde{p}^b{}_a)\Big)_{\tilde{r}=0} = \mathcal{L}_{\tilde{l}}^{M}\Big(\mathcal{L}_{\tilde{k}}\tilde{\beta}_a\Big)_{\tilde{r}=0} + O(\ell^{n+2}) \tag{56}$$

which similarly gives $\mathcal{L}_{\tilde{l}}^M(\mathcal{L}_{\tilde{k}}\tilde{\beta}_a)_{\tilde{r}=0} = O(\ell^{n+2})$. This is the second equation in (16) for $m = M$. Thus we see that all equations of (16) are satisfied for all $m \ge 0$ for the tilde GNCs. We relabel the tilde GNCs and corresponding tensors by the untilde $(v, r, x^A)$ to simplify the notation. This closes the induction loop.

Setting $\chi^a := k^a$ where $k^a$ is the vector field that is defined in a neighborhood of $\mathcal{H}$ by going through $n$ iterations of the induction step as described above, we obtain the following theorem.

**Theorem 1.** *Suppose that we have a family of spacetimes satisfying the assumptions (1)–(4) in Sect. 2.1 and let $n \in \mathbb{N}_0$. Then there exists a vector field $\chi^a$ tangent to the null generators of $\mathcal{H}$ which Lie derives $g_{ab}$ modulo terms of order $O(\ell^{n+2})$ and modulo* *terms that vanish to arbitrarily high order in any coordinate transverse to $\mathcal{H}$. In other words, if $l^a$ is a VF transverse to $\mathcal{H}$ (such as $l^a$ in the GNC above), then*

$$\mathscr{L}_l^m \mathscr{L}_\chi g_{ab}|_{\mathcal{H}} = O(\ell^{n+2}) \tag{57}$$

*for any $m \in \mathbb{N}_0$. $\chi^a$ commutes with $t^a$ and on $\mathcal{H}$ satisfies $\chi^a \nabla_a \chi^b = \kappa \chi^b$, where $\kappa$ is constant up to terms of order $O(\ell^{n+2})$. The vector field $s^a = t^a - \chi^a$ is tangent to a foliation of cross sections $\mathcal{C}(v)$ of $\mathcal{H}$ and on $\mathcal{H}$ satisfies*

$$\mathscr{L}_l^m \mathscr{L}_s g_{ab}|_{\mathcal{H}} = O(\ell^{n+2}) \tag{58}$$

*for any $m \in \mathbb{N}_0$.*

## 4. Non-rotating Case

Now we assume that we are in the non-rotating case II) of Sect. 2.1: $t^a$ is tangent to the null generators of $\mathcal{H}$ for sufficiently small $|\ell|$. We will show order by order in $\ell$ that $(\mathcal{M}, g_{ab})$ is spherically symmetric.

First, for $\ell = 0$ it follows from the staticity theorem [47] in combination with [10] and the uniqueness theorems for static vacuum black holes in Einstein gravity [7,15–17,27,28,45,46] that the metric $\bar{g}_{ab} := g_{ab}|_{\ell=0}$ is the Schwarzschild metric i.e. there is a coordinate system in which

$$\bar{g} = -f\mathrm{d}t^2 + f^{-1}\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_{d-2}^2, \quad f = 1 - \left(\frac{r_0}{r}\right)^{d-3}, \quad r_0 > 0 \tag{59}$$

with $\mathrm{d}\Omega_{d-2}^2$ the metric of the round sphere $\mathbb{S}^{d-2}$. By applying a suitable $\ell$-dependent diffeomorphism to the family $g_{ab}(\ell, x)$ which is the identity to zeroth order in $\ell$, we can ensure that the timelike KVF $t^a$ is independent of $\ell$. We assume that such a diffeomorphism has been applied and continue to call the family of metrics $g_{ab}(\ell, x)$. In particular, $\bar{g}_{ab}$ is still given by the above formula. By assumption $t^a$ is null on $\mathcal{H}$.

Let $Y^a$ be one of the KVFs of the Schwarzschild metric generating a rotation. We will now construct a formal series $Y^a(\ell, x) = \sum_{k \geq 0} \ell^k Y^{(k)a}(x)$ in $\ell$ which is tangent to $\mathcal{H}^\pm$, which is commuting with $t^a$, which Lie-derives $g_{ab}(\ell, x)$ to all orders in $\ell$, and such that $Y^a(\ell = 0, x) = Y^a(x)$. For this, we assume inductively that the terms in this expansion for $Y^a$ have been constructed up to and including order $O(\ell^{n-2})$ in such a way that $\mathscr{L}_Y g_{ab} = O(\ell^n)$ and $\mathscr{L}_t Y^a = O(\ell^n)$. If we now take $\mathscr{L}_Y$ of the Einstein equation, evaluate this at order $\ell^n$, then we see that $h_{ab}$, defined as the $O(\ell^n)$-term in $\mathscr{L}_Y g_{ab}$, satisfies the homogeneous linearized Einstein equation in Schwarzschild. Furthermore, since the linearized Einstein operator of Schwarzschild commutes with $\mathscr{L}_t$, since $\mathscr{L}_t Y^a = O(\ell^n)$ and since $\mathscr{L}_t g_{ab} = 0$, we have $\mathscr{L}_t h_{ab} = 0$. Thus, $h_{ab}$ is a stationary perturbation of Schwarzschild which is asymptotically flat, i.e. falling off roughly as $|h_{\mu\nu}| = O(1/r^{d-3})$ as $r \to \infty$ in a suitable asymptotically Cartesian coordinate system $(x^\mu)$, and regular on the horizon, see def. 2.1 of [10] for the details on such asymptotic conditions.

**Proposition 1.** *Let $h_{ab}$ be a linearized, smooth, asymptotically flat solution to the linearized Einstein equation off of Schwarzschild spacetime which is regular on $\mathcal{H}$ and Lie-derived by $t^a$. Then*

$$h_{ab} = z_{ab} + \mathscr{L}_\xi \bar{g}_{ab} \tag{60}$$

*where $\xi^a$ is a smooth vector field which is an asymptotic symmetry at null infinity and where $z_{ab}$ is a perturbation towards a Myers–Perry black hole* [38].

*Proof.* This proof is based on the analysis of master variables for gravitational perturbations given in [26,33], see Appendix B for the full argument. □

We learn from Proposition 1 that

$$\mathscr{L}_Y g_{ab} = \ell^n (z_{ab} + \mathscr{L}_\xi \bar{g}_{ab}) + O(\ell^{n+2}) \tag{61}$$

where $z_{ab}$ is an infinitesimal perturbation to a Myers–Perry black hole and where $\xi^a$ is a gauge vector field which is smooth as $r \to r_0$, and which is an asymptotic symmetry at $\mathcal{I}^+$ and $\mathcal{I}^-$, i.e. has one of the asymptotic forms IIa,b,c, III, IV of [20]. From $\mathscr{L}_t h_{ab} = 0 = \mathscr{L}_t z_{ab}$ it also follows that $\mathscr{L}_{[t,\xi]} \bar{g}_{ab} = 0$ so $\eta^a := [t, \xi]^a$ must be a linear combination of $t^a$ and a rotational KVF of Schwarzschild. By inspection, the only asymptotic forms for $\xi^a$ giving rise to a non-trivial asymptotic symmetry $\eta^a$ are type III, i.e. asymptotic boosts, in which case $\eta^a$ is an asymptotic spatial translation. This, however, is not a KVF of Schwarzschild, so we conclude that $[t, \xi]^a = 0$, in fact. It follows that $\bar{g}_{ab} t^a \xi^b$ is Lie-derived by $t^a$ on $\mathcal{H}$, and since the latter vanishes on the bifurcation surface $\mathcal{B}$, we must have $\bar{g}_{ab} t^a \xi^b = 0$ on $\mathcal{H}$ meaning that $\xi^a$ is tangent to $\mathcal{H}$ because $t^a$ is null with respect to $\bar{g}_{ab}$.

Now we redefine $Y^a \to Y^a + \ell^n \xi^a$ which is tangent to $\mathcal{H}$. Then still $[t, Y]^a = 0$, $\mathscr{L}_Y g_{ab} = O(\ell^n)$ but now $\mathscr{L}_Y g_{ab} = \ell^n z_{ab} + O(\ell^{n+2})$. Furthermore, since by assumption $s_c := g_{ab} t^a p^b{}_c|_{\mathcal{H}} = 0$ to all orders in $\ell$ by the non-rotating assumption, since $t^a$ does not depend on $\ell$, and since $\mathscr{L}_Y t^a = 0$, we get $N_c := z_{ab} t^a p^b{}_c = 0$ on $\mathcal{H}$, so $z_{ab}$ cannot be an infinitesimal perturbation towards a rotating black hole, for which $N_c$ would be the perturbed shift vector on $\mathcal{H}$ which is not zero. Thus, $z_{ab}$ must be a perturbation towards another Schwarzschild black hole. Consider now the pull-back of the equation $\mathscr{L}_Y g_{ab} = \ell^n z_{ab} + O(\ell^{n+2})$ to the bifurcation surface $\mathcal{B}$. Since $Y^a$ is tangent to $\mathcal{B}$, we get $D_a Y_b + D_b Y_a = \ell^n z_{ab} + O(\ell^{n+2})$ on $\mathcal{B}$. Taking a trace of these equation and integrating over $\mathcal{B}$ shows (in our gauge where the location of $\mathcal{B}$ is independent of $\ell$)

$$0 = 2 \int_{\mathcal{B}} D_a Y^a \sqrt{\gamma} \mathrm{d}^{d-2}x = \ell^n \int_{\mathcal{B}} z_{ab} \gamma^{ab} \sqrt{\gamma} \mathrm{d}^{d-2}x + O(\ell^{n+2}) \tag{62}$$

and this implies that $z_{ab} = 0$ because a non-trivial perturbation to another Schwarzschild black hole will result in a change of the area of $\mathcal{B}$. This closes the induction loop, showing the existence of a KVF $Y^a(\ell, x)$ commuting with $t^a$, to all orders in $\ell$.

In the above argument we can start with any rotational KVF of Schwarzschild, so we obtain from the above construction not only one, but in fact $\frac{1}{2}(d-2)(d-1)$ KVFs $Y^a_j(\ell, x)$, $j = 1, \ldots, \frac{1}{2}(d-2)(d-1)$ commuting with $t^a$, to all orders in $\ell$. Generalizing the usual argument, see e.g. [49], Appendix C, that the space of KVFs on a pseudo-Riemannian manifold is finite-dimensional to formal series of VFs and metrics (in $\ell^2$) we learn that the $Y^a_j(\ell, x)$ generate a finite dimensional Lie-algebra under the commutator of VFs. This Lie algebra must be a deformation/extension of the Lie algebra $\mathfrak{so}(d-1)$, and such extensions are classified by the cohomology ring $H^2(\mathfrak{so}(d-1), V)$ where $V$ is a finite-dimensional representation of $\mathfrak{so}(d-1)$, see e.g. [48]. As is well-known, that ring is trivial for any simple Lie-algebra and finite-dimensional representation. Thus the $Y^a_j(\ell, x)$ generate the Lie-algebra $\mathfrak{so}(d-1)$.

We therefore have shown:

**Theorem 2.** *In the non-rotating case, $g_{ab}(x, \ell = 0)$ is a Schwarzschild metric. If $Y^a(x)$ is one of its rotational KVFs, there is a formal series $Y^a(\ell, x) = \sum_{k \geq 0} \ell^k Y^{(k)a}(x)$ in $\ell$ which is tangent to $\mathcal{H}^\pm$, which is commuting with $t^a$, which Lie-derives $g_{ab}(\ell, x)$ to all* *orders in $\ell$, and such that $Y^a(\ell = 0, x) = Y^a(x)$. The Killing vector fields $Y^a(\ell, x)$ (in the sense of formal series) represent the Lie algebra of $SO(d-1)$ under the vector field commutator.*

## 5. Existence of Global Rotational Killing Field(s)

Consider the rotating case (I) of Sect. 2.1 and assume in addition to (1)–(4) in Sect. 2.1 that the manifold $\mathcal{M}$ is real analytic and the metric $g_{ab}(\ell, x)$ and stationary KVF $t^a(\ell, x)$ are jointly real analytic in $(\ell, x)$ for some atlas of analytic coordinate systems (depending possibly on $\ell$). Let us go through the proof of Theorem 1 with an eye towards analyticity of $\chi^a$ in $x$ at the various orders in $\ell$.

First, we may pick the initial cut $\mathcal{C}$ to be an analytic submanifold of the analytic manifold $\mathcal{H}$. This means that initial GNCs $(v, r, x^A)$ give analytic charts in neighborhoods of $\mathcal{M}$ covering $\mathcal{H}$. As we have described, the vector field $\chi^a = \tilde{k}^a = (\partial/\partial\tilde{v})^a$ described in Theorem 1 is constructed as a coordinate vector field for a suitable new GNC system $(\tilde{v}, \tilde{r}, \tilde{x}^A)$, and this coordinate system is analytic as we will now argue. At $n$-th order in $\ell^n$, the function $\tilde{v}$ is defined by (52) in terms of functions $f, \varphi$. The function $f$ is defined by (50), and easily checked to be analytic [20]. The function $\varphi$ is defined to be a solution to an elliptic equation (89) with analytic coefficients on $\mathcal{C}$, hence also analytic by standard results on elliptic regularity [19]. Thus $\tilde{v}$ is analytic. Likewise, the coordinates $\tilde{x}^A$ are constructed using the vector field $\zeta^A$ in Lemma 4, and this vector field is analytic because it is also defined as the solution to an elliptic equation with analytic coefficients. The coordinates $(\tilde{v}, \tilde{x}^A)$ are propagated by Lie-transport with the analytic vector field $t^a$, and so are analytic functions on the respective coordinate patches of $\mathcal{H}$. Finally, $\tilde{r}$ is defined as a parameter along affine geodesics off of $\mathcal{C}$ with analytic initial condition, hence it is also analytic. Thus, the coordinate systems $(\tilde{v}, \tilde{r}, \tilde{x}^A)$ successively determined at the various orders in $\ell$ are all analytic. Since $\chi^a$ is a coordinate vector field in this coordinate system, it is analytic, and we have $\mathscr{L}_\chi g_{ab} = O(\ell^{n+2})$ identically in an open neighborhood of $\mathcal{H}$ for the given $n$ that we fix, by Theorem 1.

At this stage, we can extend $\chi^a$ globally onto the domain of outer communication

$$\mathcal{D} := I^+\left(\bigcup_{\tau\in\mathbb{R}} \phi_\tau[\Sigma_1]\right) \cap I^-\left(\bigcup_{\tau\in\mathbb{R}} \phi_\tau[\Sigma_1]\right) \tag{63}$$

of $\mathcal{M}$ (where $\Sigma_1$ is the asymptotic region of the acausal surface and $\phi_\tau$ the flow of $t^a$, see [10] def. 2.1 and footnote 4) by the usual method of analytic extension on overlapping neighborhoods. Because that domain is simply connected $\pi_1(\mathcal{D}) = 0$ by the topological censorship theorem [12,14] the analytic continuation is single-valued. The vector field $s^a := t^a - \chi^a$ is then also globally defined on $\mathcal{D}$ and single valued. On $\mathcal{H}$, $s^a$ can be written as (18) where each $\psi_j^a$ has closed orbits and is independent of $\ell$. We Lie-drag each $\psi_j^a$ off $\mathcal{H}$ using $l^a$, which is analytic up to the order $O(\ell^n)$ considered. Then proceeding precisely in the same way[6] as in Sect. 3 of [20], one shows that each $\psi_j^a$ can be analytically continued on $\mathcal{D}$ to a single valued which Lie derives the metric up to order $O(\ell^{n+2})$. We therefore get:

**Theorem 3.** *Assume (1)–(4) as in Sect. 2.1, that $\mathcal{M}$ is real analytic and that $g_{ab} = g_{ab}(\ell, x)$, $t^a = t^a(\ell, x)$ are jointly real analytic in $(\ell, x)$ with $t^a$ not tangent to the null*

[6] This part of the argument does not use the Einstein equations, and so is applicable in our context, too.

*generators of $\mathcal{H}$. Then for any fixed $n \in \mathbb{N}_0$, there exist an analytic VF $s^a = s^a(\ell, x)$ on $\mathcal{D}$ tangent to a foliation of $\mathcal{H}$ by spacelike cross sections, and a VF $\chi^a = \chi^a(\ell, x)$ on $\mathcal{D}$ tangent and normal to $\mathcal{H}$, such that $s^a$, $\chi^a$ commute with $t^a$, such that $\mathscr{L}_s g_{ab} = O(\ell^{n+2}) = \mathscr{L}_\chi g_{ab}$ and such that we have*

$$t^a = \chi^a + s^a \tag{64}$$

*The vector field $\chi^a$ has an acceleration (surface gravity) $\kappa$ on $\mathcal{H}$ that is constant up to order $O(\ell^{n+2})$ and $s^a$ can be written as*

$$s^a = \sum_{j=1}^{N} \Omega_j \psi_j^a, \tag{65}$$

*where $\psi_j^a = \psi_j^a(\ell, x)$ are commuting VFs on $\mathcal{D}$ whose orbits are closed with period $2\pi$ such that $\mathscr{L}_{\psi_j} g_{ab} = O(\ell^{n+2})$, and where the $\Omega_j = \Omega_j(\ell)$ are constants that are defined up to order $O(\ell^{n+2})$.*

**Remarks.** (1) A similar argument will work with a cosmological constant $\Lambda < 0$, because the asymptotic structure is used in our proofs only to show that the domain of outer communication is simply connected and to show that $t^a$ does not vanish on $\mathcal{H}$. Both will work if $(\mathcal{M}, g_{ab})$ is asymptotically AdS. In the case $\Lambda > 0$, one has to make suitable assumptions on how $t^a$ behaves on $\mathcal{I}^-$ because this is spacelike now. To stay within the realm of effective field theory, the cosmological constant should in either case be so small that $|\Lambda L^2| \lesssim 1$, where $L$ is the typical scale over which the solution is varying, as explained in [24], def. 2.1.
(2) Note that the VFs in the theorem $s^a$, $\psi_j^a$, $\chi^a$ in principle depend on the order $n$ in $\ell$ up to which the Killing vector field property holds. It should also be possible to establish the existence of $s^a$, $\psi_j^a$, $\chi^a$ with the properties stated in this theorem up to arbitrary order in $\ell$. This would be tantamount to showing that the successive changes of coordinates to $(\tilde{v}, \tilde{r}, \tilde{x}^A)$ defined order by order in $\ell$ can be summed within a non-zero radius of convergence $|\ell| < \ell_0$. We expect that this should be possible, but it would be a rather tedious bookkeeping exercise. We will not carry this out here because in the effective field theory spirit, the action $I[g]$ is only valid up to a finite order in $\ell$ anyhow.
(3) It is natural to ask to what extent the (approximate) KVFs $t^a$, $\chi^a$, $\psi_j^a$ can be made independent of $\ell$ by applying a diffeomorphism to all quantities, i.e. by "changing the gauge". First we consider the rotational KVFs $\psi_j^a$. By (18) each $\psi_j^a$ is independent of $\ell$ on $\mathcal{H}$. But since $\psi_j^a$ is obtained off of $\mathcal{H}$ by Lie-dragging it along the vector field $l^a = l^a(\ell, x)$, it is in general $\ell$-dependent off of $\mathcal{H}$. Consider a point $x_0 \in \mathcal{H}$ and a sufficiently small $r > 0$. Let $x_0(r, \ell)$ be the point in $\mathcal{D}$ obtained by flowing $x_0$ along the VF $l^a(\ell, x)$. Then for sufficiently small $r > 0$, the map $x_0(r, \ell) \mapsto x_0(r, \ell = 0)$ is a diffeomorphism $\phi$ which is the identity on $\mathcal{H}$, and by construction we have $\phi^* l^a(\ell, x) = l^a(\ell = 0, x)$, so $l^a$ becomes independent of $\ell$. Consequently, by applying $\phi$ to $g_{ab}$, $\chi^a$, $t^a$, $\psi_j^a$, we have achieved that all $\psi_j^a(\ell, x)$ are actually independent of $\ell$. This of course holds only locally where the diffeomorphism $\phi$ has been defined and it would require extra non-trivial work to show that a global diffeomorphism exists satisfying $\phi^* l^a(\ell, x) = l^a(\ell = 0, x)$ throughout $\mathcal{D}$. If so, each $\psi_j^a$ would be independent globally on $\mathcal{D}$ in this gauge.

On the other hand, even if this were true, we cannot expect that the KVFs $t^a$, $\chi^a$ can be made independent of $\ell$ by a change of gauge even locally. This is because the projection of the orbits of $t^a$ to a cross section $\mathcal{C}$ of $\mathcal{H}$ will depend on $\ell$ in view of (64) and (65), and in fact their global nature will depend sensitively on the angular velocities $\Omega_j(\ell)$: If some ratios $\Omega_j(\ell)/\Omega_i(\ell)$, $i \neq j$ are rational, then the orbits of $s^a(\ell, x)$ will not be dense in $\mathcal{C}$, and otherwise they will. Obviously, this may change if we vary $\ell$ even arbitrarily close to $\ell = 0$. Whether the orbits are dense or not is a gauge invariant statement, so we cannot expect to make $t^a$ independent of $\ell$ by changing the gauge. A similar argument applies to $\chi^a$, now replacing the cut $\mathcal{C}$ on the horizon by a cut at future null infinity.

Consider next the non-rotating case (II). By Theorem 2, we have a set of KVFs in the sense of formal power series generating the Lie algebra of $SO(d-1)$ under the commutator of VFs. These KVFs are constructed order by order in $\ell^2$, and if we could show that the series converges, then it would follow that we have actual KVFs and not just formal series. Again, for such an argument to proceed we should at least know that the manifold $\mathcal{M}$ is real analytic and that the metric $g_{ab}(\ell, x)$ is jointly real analytic in $(\ell, x)$ for an analytic atlas of coordinate systems depending analytically on $\ell$. Again, one would have to go through the detailed steps of the inductive constructions, order by order in $\ell^2$.

## 6. Conclusions

For simplicity, we have considered in this paper (parity even) purely gravitational theories. However, we expect our proofs to be robust and to apply to any local covariant Einstein-gravity-matter model such that the rigidity theorem holds for the corresponding standard Einstein-gravity-matter model when $\ell = 0$. The latter applies to a broad class of models including abelian vectors coupled to scalars [20,21].

A more difficult question is whether one can remove the analyticity, non-degenerate, and genericity assumptions in our main theorem (Theorem 3). Since it is unknown how to remove non-degeneracy and analyticity even for Einstein gravity, we expect this to be highly non-trivial. Actually, in the case of higher derivative theories as considered in this paper, it is not totally clear what viewpoint to take on this problem, for the following reason. We have treated solutions in the EFT setting as (locally) small corrections to the corresponding solutions in Einstein gravity. In doing so, we are assuming in effect that the solutions are "low frequency' (relative to the EFT length scale $\ell$). If we were to try to drop the analyticity assumption, it seems plausible that we would have to study the EFT equations as an initial value problem of some sort as in [1], which requires the study of solutions of arbitrary frequency. However, for a general higher derivative theory, there is at this point no general understanding when it will possess a well-posed initial value problem, although this has been established for certain special theories [31,32,39].

Let us assume that we *have* a higher derivative theory with a well-posed initial value problem, say one of the theories with second order equations of motion studied in [31,32]. If we take such a theory seriously even for arbitrarily short wavelengths, it is not natural to consider the lightcone as defined by the metric $g_{ab}$—as we have done in this paper—but instead one wants to study an intrinsically defined propagation cone defined by the highest derivative part in the Einstein equation. This leads in general to a propagation cone different from the lightcone and correspondingly a different notion of event horizon of a black hole [43]. One might ask in such special theories whether the rigidity theorem still holds for the new notion of event horizon, and whether, in the case of stationary solutions, the different notions of event horizons may actually even coincide. We leave this interesting issue for future work.

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## A Ricci Tensor in GNCs [20]

In this Appendix, we provide expressions for the Ricci tensor in our GNC system for the convenience of the reader [20]. As in the main text, the horizon, $\mathcal{H}$, corresponds to the surface $r = 0$. Associated with the foliation $\mathcal{C}(v, r)$ by $(d-2)$-dimensional compact cross sections there are two natural projectors. The orthogonal projector $q^a{}_b$, as well as the non-orthogonal projector $p^a{}_b$ characterized by $p^a{}_b k^b = p^a{}_b l^b = 0$. When $r\beta_a \neq 0$, these do not coincide. In terms of the Gaussian null coordinate components of $\gamma_{ab}$, we have $q^{ab} = (\gamma^{-1})^{AB}(\partial x^A)^a(\partial/\partial x^B)^b$, whereas $p^a{}_b = (\partial/\partial x^A)^a(\mathrm{d}x^A)_b$. The relationship between $p^a{}_b$ and $\gamma^a{}_b$ is given by

$$p^a{}_b = -rl^a\beta_b + \gamma^a{}_b\,. \tag{66}$$

Since in terms of Gaussian null coordinates, we have $p^a{}_b = (\partial_A)^a(\mathrm{d}x^A)_b$ and $l^a = (\partial_r)^a$, $k^a = (\partial_v)^a$, and since coordinate vector fields commute, it follows that $\mathscr{L}_k p^a{}_b = 0 = \mathscr{L}_l p^a{}_b$. It also is easily seen that $q^{ac}\gamma_{cb} = p^a{}_b$ and that $p^a{}_b q^b{}_c = q^a{}_c$. We finally recall that a definition $D_a$ of an intrinsic derivative operator associated with $q_{ab}$ was given in (13), and we denote the Riemann and Ricci tensors associated with $q_{ab}$ as $R[\gamma]_{abc}{}^d$ and $R[\gamma]_{ab}$. Then we have:

$$\begin{aligned}
k^a k^b R_{ab} = &-\frac{1}{2}q^{ab}\mathscr{L}_k\mathscr{L}_k\gamma_{ab} + \frac{1}{4}q^{ca}q^{db}(\mathscr{L}_k\gamma_{ab})\mathscr{L}_k\gamma_{cd} + \frac{1}{2}\alpha\, q^{ab}\mathscr{L}_k\gamma_{ab}\\
&+\frac{r}{2}\cdot\Big[\,4\alpha\mathscr{L}_l\mathscr{L}_l\alpha + 8\alpha\mathscr{L}_l\alpha + (\mathscr{L}_l\alpha)q^{ab}\mathscr{L}_k\gamma_{ab}\\
&\qquad +q^{ab}\mathscr{L}_l\gamma_{ab}\cdot\Big\{-\mathscr{L}_k\alpha - rq^{cd}\beta_c\mathscr{L}_k\beta_d\\
&\qquad\qquad +(rq^{cd}\beta_c\beta_d + 2\alpha)\mathscr{L}_l(r\alpha) + rq^{cd}\beta_c D_d\alpha\Big\}\\
&\qquad +2q^{ab}D_a\left\{\beta_b\mathscr{L}_l(r\alpha) + D_b\alpha - \mathscr{L}_k\beta_b\right\}\\
&\qquad +q^{bc}\mathscr{L}_l(r\beta_c)\cdot\Big\{(rq^{ef}\beta_e\beta_f + 2\alpha)\mathscr{L}_l(r\beta_b)
\end{aligned}$$

$$
\begin{aligned}
&\qquad -4D_b\alpha + 2\mathscr{L}_k\beta_b + 4rq^{ae}\beta_e D_{[a}\beta_{b]}\Big\} \\
&+2(\mathscr{L}_l\alpha)\mathscr{L}_l(r^2q^{ab}\beta_a\beta_b) + 4rq^{ab}\beta_a\beta_b\mathscr{L}_l\alpha + 2rq^{ab}\beta_a\beta_b\mathscr{L}_l\mathscr{L}_l\alpha \\
&+2q^{ab}\beta_a\mathscr{L}_l(r\beta_b)\cdot\left\{2\mathscr{L}_l(r\alpha) - \frac{1}{2}rq^{cd}\beta_c\mathscr{L}_l(r\beta_d)\right\} \\
&+2r^{-1}\mathscr{L}_l\left\{r^2q^{ab}\beta_a(D_b\alpha - \mathscr{L}_k\beta_b)\right\} + 2r^{-1}\alpha\mathscr{L}_l(r^2q^{ab}\beta_a\beta_b)\Big],
\end{aligned} \tag{67}
$$

$$
\begin{aligned}
k^al^bR_{ab} = &-2\mathscr{L}_l\alpha + \frac{1}{4}q^{ca}q^{db}(\mathscr{L}_k\gamma_{cd})\mathscr{L}_l\gamma_{ab} - \frac{1}{2}q^{ab}\mathscr{L}_l\mathscr{L}_k\gamma_{ab} \\
&-\frac{1}{2}\alpha\, q^{ab}\mathscr{L}_l\gamma_{ab} - \frac{1}{2}q^{ab}\beta_a\beta_b \\
&+\frac{r}{2}\cdot\Big[-2\mathscr{L}_l\mathscr{L}_l\alpha - \frac{1}{2}q^{ab}\mathscr{L}_l\gamma_{ab}\cdot\left\{2\mathscr{L}_l\alpha + q^{cd}\beta_c\mathscr{L}_l(r\beta_d)\right\} \\
&\qquad -q^{ab}\beta_a\mathscr{L}_l\beta_b - \mathscr{L}_l\{q^{ab}\beta_a\mathscr{L}_l(r\beta_b)\} - q^{ab}D_a(\mathscr{L}_l\beta_b)\Big],
\end{aligned} \tag{68}
$$

$$
\begin{aligned}
k^bp^c{}_aR_{bc} = &-p^b{}_aD_b\alpha + \frac{1}{2}\mathscr{L}_k\beta_a + \frac{1}{4}\beta_aq^{bc}\mathscr{L}_k\gamma_{bc} - p^d{}_{[a}p^e{}_{b]}D_d(q^{bc}\mathscr{L}_k\gamma_{ce}) \\
&+\frac{r}{2}\cdot\Big[\frac{1}{2}(q^{bc}\mathscr{L}_k\gamma_{bc})\mathscr{L}_l\beta_a + \mathscr{L}_k\mathscr{L}_l\beta_a + 2\alpha\mathscr{L}_l\beta_a \\
&\quad +\mathscr{L}_l(r\beta_a)\cdot\left\{r^{-1}\mathscr{L}_l(r^2q^{bc}\beta_b\beta_c) + 2\mathscr{L}_l\alpha\right\} \\
&\quad -2p^b{}_aD_b(\mathscr{L}_l\alpha) + \mathscr{L}_l(q^{bc}\beta_b\mathscr{L}_k\gamma_{ca}) - 2r^{-1}\mathscr{L}_l\left(r^2q^{cd}\beta_cp^b{}_aD_{[b}\beta_{d]}\right) \\
&\quad -\frac{1}{2}q^{bc}\mathscr{L}_l\gamma_{bc}\cdot\Big\{-(rq^{ef}\beta_e\beta_f + 2\alpha)\mathscr{L}_l(r\beta_a) \\
&\qquad\qquad +2p^d{}_aD_d\alpha - q^{bc}\beta_b\mathscr{L}_k\gamma_{ca} + 2rq^{ef}\beta_ep^d{}_aD_{[d}\beta_{f]}\Big\} \\
&\quad -2\mathscr{L}_l(\alpha\beta_a) - 2r(\mathscr{L}_l\alpha)\mathscr{L}_l\beta_a + p^d{}_aD_b\left\{q^{bc}\beta_c\mathscr{L}_l(r\beta_d)\right\} \\
&\quad -2p^b{}_aq^{cd}D_dD_{[b}\beta_{c]} - q^{bc}(\mathscr{L}_l\beta_b)\mathscr{L}_k\gamma_{ca} \\
&\quad -q^{bc}\mathscr{L}_l(r\beta_b)\cdot\Big\{(rq^{ef}\beta_e\beta_f + 2\alpha)\mathscr{L}_l\gamma_{ca} + p^d{}_aD_c\beta_d \\
&\qquad\qquad +\beta_c\mathscr{L}_l(r\beta_a) - rq^{ef}\beta_c\beta_f\mathscr{L}_l\gamma_{ea}\Big\} \\
&\quad +q^{bc}(\mathscr{L}_l\gamma_{ca})\cdot\left\{2\beta_b\mathscr{L}_l(r\alpha) + 2D_b\alpha - \mathscr{L}_k\beta_b + 2rq^{de}\beta_eD_{[b}\beta_{d]}\right\}\Big],
\end{aligned} \tag{69}
$$

$$
l^al^bR_{ab} = -\frac{1}{2}q^{ab}\mathscr{L}_l\mathscr{L}_l\gamma_{ab} + \frac{1}{4}q^{ca}q^{db}(\mathscr{L}_l\gamma_{ab})\mathscr{L}_l\gamma_{cd}, \tag{70}
$$

$$
l^bp^c{}_aR_{bc} = -\frac{1}{4}\beta_aq^{bc}\mathscr{L}_l\gamma_{bc} - \mathscr{L}_l\beta_a + \frac{1}{2}q^{bc}\beta_c\mathscr{L}_l\gamma_{ab} - p^d{}_{[a}p^e{}_{b]}D_d\left(q^{bc}\mathscr{L}_l\gamma_{ce}\right)
$$

$$+ \frac{r}{2} \cdot \Big[ - \mathcal{L}_l \mathcal{L}_l \beta_a + \mathcal{L}_l \left( q^{bc} \beta_c \mathcal{L}_l \gamma_{ab} \right) + \frac{1}{2} (q^{cd} \mathcal{L}_l \gamma_{cd}) \left( -\mathcal{L}_l \beta_a + q^{be} \beta_e \mathcal{L}_l \gamma_{ab} \right) \Big], \quad (71)$$

$$\begin{aligned} p^c{}_a p^d{}_b R_{cd} = & -\mathcal{L}_l \mathcal{L}_k \gamma_{ab} - \alpha \mathcal{L}_l \gamma_{ab} + p^c{}_a p^d{}_b R[\gamma]_{cd} - p^c{}_{(a} p^d{}_{b)} D_c \beta_d - \frac{1}{2} \beta_a \beta_b \\ & + q^{cd} \left( \mathcal{L}_l \gamma_{d(a} \right) \mathcal{L}_k \gamma_{b)c} - \frac{1}{4} \left\{ (q^{cd} \mathcal{L}_k \gamma_{cd}) \mathcal{L}_l \gamma_{ab} + (q^{cd} \mathcal{L}_l \gamma_{cd}) \mathcal{L}_k \gamma_{ab} \right\} \\ & + \frac{r}{2} \cdot \Big[ -2\alpha \mathcal{L}_l \mathcal{L}_l \gamma_{ab} - p^e{}_a p^f{}_b D_c (q^{cd} \beta_d \mathcal{L}_l \gamma_{ef}) \\ & - \frac{1}{2} (q^{cd} \mathcal{L}_l \gamma_{cd}) \left\{ (r q^{ef} \beta_e \beta_f + 2\alpha) \mathcal{L}_l \gamma_{ab} + 2 p^e{}_{(a} p^f{}_{b)} D_e \beta_f \right\} \\ & - 2 (\mathcal{L}_l \alpha) \mathcal{L}_l \gamma_{ab} - r^{-1} \{ \mathcal{L}_l (r^2 q^{ef} \beta_e \beta_f) \} \mathcal{L}_l \gamma_{ab} \\ & - r q^{ef} \beta_e \beta_f \mathcal{L}_l \mathcal{L}_l \gamma_{ab} - 2 \mathcal{L}_l \{ p^c{}_{(a} p^d{}_{b)} D_c \beta_d \} \\ & - 2 \beta_{(a} \mathcal{L}_l \beta_{b)} - r (\mathcal{L}_l \beta_a) \mathcal{L}_l \beta_b - r q^{ce} q^{df} \beta_c \beta_d (\mathcal{L}_l \gamma_{ae}) \mathcal{L}_l \gamma_{bf} \\ & + 2 q^{cd} \beta_d \left\{ \mathcal{L}_l (r \beta_{(a}) \right\} \mathcal{L}_l \gamma_{b)c} + 2 p^e{}_{(a} p^f{}_{b)} q^{cd} (D_d \beta_e) \mathcal{L}_l \gamma_{fc} \\ & + q^{cd} (r q^{ef} \beta_e \beta_f + 2\alpha) (\mathcal{L}_l \gamma_{ca}) \mathcal{L}_l \gamma_{db} \Big]. \end{aligned} \quad (72)$$

## B Improved IWW Entropy Current-Density [24]

In order to define the "improved IWW" entropy current-density, one sets up a GNC system on $\mathcal{H}$ adapted to some choice of affine parameter, $\lambda$, along the null geodesic generators of $\mathcal{H}$. In such a GNC system, the metric takes the form

$$2\mathrm{d}\lambda(\mathrm{d}\rho - \frac{1}{2}\rho^2 f \mathrm{d}\lambda - \rho\omega_A \mathrm{d}x^A) + h_{AB}\mathrm{d}x^A \mathrm{d}x^B. \quad (73)$$

The definition of this coordinate system involves a choice of cross section, $\mathcal{C}$ and is quite similar to (7). In particular, $\rho$ is a parameter along affine ingoing null geodesics. However, the construction differs in that (7) was based on the Killing parameter, $v$, along the null geodesic generators of $\mathcal{H}$, whereas (73) is based on an affine parameter, $\lambda$. The usual ambiguities in the choice of the affine parameter $\lambda$ may be partially fixed e.g. by imposing that $\lambda = 0$ on the given cross section $\mathcal{C}$. This still leaves the possibility to rescale $\lambda$ by an arbitrary positive smooth function $a$ on $\mathcal{C}$, but the "improved IWW" entropy current-density is shown to be independent of such a change.

One sets

$$n^a = \left(\frac{\partial}{\partial\lambda}\right)^a, \quad \bar{n}^a = \left(\frac{\partial}{\partial\rho}\right)^a, \quad (74)$$

which are tangent to ingoing ($\rho$) respectively outgoing ($\lambda$) affine null geodesics. The corresponding extrinsic curvatures in the directions of $n^a$ respectively $\bar{n}^a$ are denoted by $K_{ab}$ respectively $\bar{K}_{ab}$. The corresponding expansions and shears are denoted by $\theta, \sigma_{ab}$ respectively $\bar{\theta}, \bar{\sigma}_{ab}$, where

$$\theta = g^{ab} K_{ab}, \quad \bar{\theta} = g^{ab} \bar{K}_{ab}, \quad (75)$$

and where $\sigma_{ab}, \bar{\sigma}_{ab}$ are the trace free parts of $K_{ab}, \bar{K}_{ab}$. On $\mathcal{C}$, the expansion and shear may be written in a form analogous to (20),

$$\bar{\theta} = \gamma^{ab}\nabla_{(a}\bar{n}_{b)}, \quad \bar{\sigma}_{ab} = \gamma_a{}^c\gamma_b{}^d(\nabla_{(c}\bar{n}_{d)} - \tfrac{1}{d-2}g_{cd}\nabla_e\bar{n}^e). \tag{76}$$

where we note that $h_{ab} = \gamma_{ab}$ on $\mathcal{C}$.

A useful principle to classify the terms that arise when making a decomposition in GNCs (73) of a covariant derivative $\nabla_{e_1}\dots\nabla_{e_r}R_{abcd}$ of the Riemann tensor is the concept of "boost weight". To define this, one first argues that on $\mathcal{H}$, each term in such a decomposition must be a tensor product of factors of $n_a, \bar{n}_a$ and the following factors called "*primitive monomials*":

$$(\mathscr{L}_n)^k(\mathscr{L}_{\bar{n}})^{\bar{k}}D_{c_1}\cdots D_{c_i}\{f, \omega_a, h_{ab}\}, \quad D_{e_1}\dots D_{e_j}R[h]_{abcd} \tag{77}$$

where $f, \omega_a = \omega_A(\mathrm{d}x^A)_a, h_{ab} = h_{AB}(\mathrm{d}x^A)_a(\mathrm{d}x^B)_b$ are as in (73) and where $R[h]_{abcd}$ is the Riemann tensor of $h_{ab}$. The "*boost weight*" of each factor is then defined to be $\bar{k} - k$ plus the number of $\bar{n}_a$ factors minus the number of $n_a$ factors.

Extending previous works by [3–5,29,30,50], reference [24] constructed the so-called "improved IWW" entropy current-density,

$$\sigma^a = \Sigma n^a + J^a. \tag{78}$$

It consists of an entropy density, $\Sigma$, an entropy current, $J^a = J^A(\partial_A)^a$ (referring to the GNC system (73)), and is characterized by the following properties, see prop. 1 of [24].

1. $\Sigma$ is a scalar local functional of boost weight 0 that for any cross section $\mathcal{C}$ is a contraction of the factors in the following list:
   - $\mathcal{D}_{(a_1}\cdots\mathcal{D}_{a_j}\sigma_{a)b}, \quad \mathcal{D}_{(a_1}\cdots\mathcal{D}_{a_j)}\theta, \quad \mathcal{D}_{(a_1}\cdots\mathcal{D}_{a_j}\bar{\sigma}_{a)b}, \quad \mathcal{D}_{(a_1}\cdots\mathcal{D}_{a_j)}\bar{\theta}$
     where $\mathcal{D}_a = D_a + \frac{1}{2}b\omega_a$ is a suitable covariantized derivative defined using the Hajicek 1-form $\omega_a$ (73) and the boost weight $b$ it is acting on,
   - $R^c{}_{a_1a_2d}$ or $\nabla_{(a_1}\cdots\nabla_{a_j}R^c{}_{a_{j+1}a_{j+2})b}$ where $j > 0$,
   - $n^a, \bar{n}^a$ or $g^{ab}$.

   It follows that $S[\mathcal{C}]$ (37) is independent of the remaining freedom to choose the affine parameter $\lambda$, and hence a fully covariant functional of $\mathcal{C}$ and the metric $g_{ab}$, in the sense that $S[\psi^{-1}(\mathcal{C}), \psi^*g] = S[\mathcal{C}, g]$ for any time-orientation preserving diffeomorphism $\psi$ preserving $\mathcal{H}$.
2. The analog of the Raychaudhuri equation which uses the Einstein equation (2):

$$\left(\mathscr{L}_n\nabla_a\sigma^a \equiv\right) \quad \partial_\lambda\left(\frac{1}{\sqrt{\gamma}}\partial_\lambda(\sqrt{\gamma}\Sigma) + D_aJ^a\right) = F \tag{79}$$

   on $\mathcal{H}$, where $F$ is at least quadratic in positive boost weight quantities, and where $J_a = J_A(\mathrm{d}x^A)_a$ is an entropy current. [The quantity in parenthesis is just $\nabla_a\sigma^a$.]
3. $J^a$, which is a boost weight 1 quantity, contains an explicit power $\ell^2$ and is at least linear in positive boost weight terms. $\Sigma = 1+$ zero boost weight terms containing an explicit power $\ell^2$.
4. $F = -\sigma_{ab}\sigma^{ab} - \frac{1}{d-2}\theta^2+$ terms depending explicitly on $\ell^2$.

*Example 1.* Consider the theory described by the action

$$I[g] = \int_{\mathcal{M}} (R + c_1 \ell^2 R_{ab} R^{ab} + c_2 \ell^2 R^2) \sqrt{-g} \mathrm{d}^d x. \tag{80}$$

In this case, the improved IWW entropy density is [3,4]

$$\Sigma = 1 + c_1 \ell^2 (R_{ab} n^a \bar{n}^b - \theta \bar{\theta}) + 2c_2 \ell^2 R. \tag{81}$$

Note that the first term "1" corresponds to the area contribution to $S[\mathcal{C}]$ so $S[\mathcal{C}] = A[\mathcal{C}] + O(\ell^2)$. Note that up to a cosmological constant term (which does not contribute to the entropy current-density), $I[g]$ is the most general local covariant action including up to a total number of four derivatives in four dimensions because the term $R_{abcd} R^{ab}{}_{ef} \epsilon^{cdef}$ is topological and the term $R_{abcd} R^{abcd}$ can be expressed in terms of those already present and another topological term.

*Example 2.* In $d = 4$ dimensions, let

$$I[g] = \int_{\mathcal{M}} (R + c\ell^4 R_{abcd} R^{cd}{}_{ef} R^{abef}) \sqrt{-g} \mathrm{d}^4 x. \tag{82}$$

In this case, the improved IWW entropy density is [11]

$$\begin{aligned} \Sigma = 1 &+ c\ell^4 R_{abcd} R_{efgh} (-6 n^a \bar{n}^b n^e \bar{n}^f h^{cg} h^{eh} - 24 \bar{n}^a n^b \bar{n}^c n^f \bar{n}^e n^g h^{dh} \\ &+ 12 n^a \bar{n}^b n^c \bar{n}^d n^e \bar{n}^f n^g \bar{n}^h) + c\ell^4 (-24 R_{abcd} K^b{}_e \bar{K}^{de} \bar{n}^a n^c \\ &+ 12 K^{ab} K^{cd} \bar{K}_{ac} \bar{K}_{ef} - 36 K^{ab} K_a{}^c \bar{K}_b{}^d \bar{K}_{cd}), \end{aligned} \tag{83}$$

where $h^{ab} = g^{ab} - 2n^{(a} \bar{n}^{b)}$. Modulo the freedom to add a cosmological constant term (which would not change $\Sigma$), this action can be shown to be the most general parity even action up to and including a total number of six derivatives in the action and up to perturbative field redefinitions.

## C Proof of Lemma 4

In this proof, quantities obtained by setting $\ell = 0$ are denoted by an overbar such as in $\bar{\gamma}_{ab} = \gamma^{(0)}_{ab}$ etc. Consider the equation

$$\bar{D}^a \bar{D}_{(a} \zeta_{b)} = -\frac{1}{2} \bar{D}^a \gamma_{ab}, \tag{84}$$

where indices have been raised/lowered with the Riemannian metric $\bar{\gamma}_{ab}$ on $\mathcal{C}$. The right side is $L^2$-orthogonal to the KVFs of $\bar{\gamma}_{ab}$, and since $\gamma_{ab} = \bar{\gamma}_{ab} + O(\ell^2)$, it is clearly of order $O(\ell^2)$. Now consider the usual weak formulation of this equation for $\zeta_a$ in the Sobolev space $W^{1,2}(\mathcal{C}, T^*\mathcal{C})$ obtained by contracting the equation into $\phi^a \in C^\infty(\mathcal{C}, T\mathcal{C})$, integrating over $\mathcal{C}$, and formally performing an integration by parts to move one derivative onto $\phi^a$. Then the bilinear form so obtained from the left side of (84) has a coercivity property expressed by the Poincaré type inequality

$$\| \bar{D}_{(a} X_{b)} \|_{L^2} \geq \text{const.} \| X_a - \bar{P} X_a \|_{L^2} \tag{85}$$

where $\bar{P}$ is the $L^2$-projector onto the span of the KVFs of $\bar{\gamma}_{ab}$. By standard arguments, there is hence a weak, and a forteriori smooth, solution $\zeta^a$ which is unique modulo the addition of a KVF of $\bar{\gamma}_{ab}$. In particular, since the source in the equation for $\zeta^a$ is of order $O(\ell^2)$, we can choose $\zeta^a = O(\ell^2)$ itself. Using (84), we get

$$\bar{D}^a \bar{D}_{(a}(\mathscr{L}_S \zeta_{b)} - \ell^n s^{(n)}_{b)}) = -\frac{1}{2} \bar{D}^a \mathscr{L}_s \gamma_{ab} + O(\ell^{n+2}) = O(\ell^{n+2}), \tag{86}$$

using that $[\bar{D}_a, \mathscr{L}_S] = 0$ (since $S^a$ is a KVF of $\bar{\gamma}_{ab}$ by the inductive assumption), and the fact that $s^a = s^{(0)a} + \ell^2 s^{(2)a} + \ldots \ell^n s^{(n)a} + O(\ell^{n+2})$. The Poincaré inequality implies that the kernel of the operator $\bar{D}^a \bar{D}_{(a} X_{b)}$ consists precisely of the KVFs of $\bar{\gamma}_{ab}$. It follows that $\bar{\gamma}^{ab}(\mathscr{L}_S \zeta_b - \ell^n s^{(n)}_b)$ is a KVF, which we call $\xi^a$, of $\bar{\gamma}_{ab}$ modulo $O(\ell^{n+2})$. Thus, $\ell^n s^{(n)a} = \xi^a + \mathscr{L}_S \zeta^a + O(\ell^{n+2})$, as desired.

We would finally like to show that $\xi^a = O(\ell^n)$. Let $\hat{\phi}_\tau$ be the flow of $S^a$ on $\mathcal{C}$. Since $S^a$ is a KVF of $\bar{\gamma}_{ab}$, this flow is isometric and in particular area preserving. Therefore, by basic theorems in ergodic theory (see e.g. [51]), if $T_{a_1 \ldots a_r}$ is a smooth tensor field on $\mathcal{C}$, then the limit (orbit average)

$$\langle T_{a_1 \ldots a_r}(x) \rangle := \lim_{T \to \infty} \frac{1}{T} \int_0^T \hat{\phi}^*_\tau T_{a_1 \ldots a_r}(x) \, \mathrm{d}\tau \tag{87}$$

will exist in the sense of $L^p(\mathcal{C})$ for any $p \geq 1$ and any component in an orthonormal tetrad. Since $\hat{\phi}_\tau$ is isometric, so the same will hold for $\bar{D}_a$-derivatives of $T_{a_1 \ldots a_r}$, so convergence even occurs in any Sobolev space $W^{p,q}(\mathcal{C})$, $q \geq 0$, hence in the topology of any $C^\alpha(\mathcal{C})$ for any $\alpha \geq 0$, by an appropriate Sobolev embedding theorem. As a consequence, if $T_{a_1 \ldots a_r} = O(\ell^M)$ for some $M$, then also $\langle T_{a_1 \ldots a_r} \rangle = O(\ell^M)$. Furthermore, we have

$$\begin{aligned} \langle \mathscr{L}_S T_{a_1 \ldots a_r}(x) \rangle &= \lim_{T \to \infty} \frac{1}{T} \int_0^T \hat{\phi}^*_\tau \mathscr{L}_S T_{a_1 \ldots a_r}(x) \, \mathrm{d}\tau \\ &= \lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{\mathrm{d}}{\mathrm{d}\tau} \hat{\phi}^*_\tau T_{a_1 \ldots a_r}(x) \, \mathrm{d}\tau \\ &= \lim_{T \to \infty} \frac{1}{T} \hat{\phi}^*_\tau T_{a_1 \ldots a_r}(x) \Big|_0^T = 0, \end{aligned} \tag{88}$$

where the limit exists pointwise and uniformly, together with that of all of its derivatives. Now we take the orbit average of the equation $\ell^n s^{(n)a} = \xi^a + \mathscr{L}_S \zeta^a + O(\ell^{n+2})$ and obtain $\langle \xi^a \rangle = O(\ell^n)$. Since $\xi^a$ is already known to be a KVF of $\bar{\gamma}_{ab}$ and since, by our genericity assumption 4) all such KVFs commute, it follows that $\hat{\phi}^*_\tau \xi^a = \xi^a$ so $\langle \xi^a \rangle = \xi^a$, and this gives $\xi^a = O(\ell^n)$. □

## D Proof of Lemma 5

In this proof, quantities obtained by setting $\ell = 0$ are denoted by an overbar such as in $\bar{\gamma}_{ab} = \gamma_{ab}|_{\ell=0}$ etc. We take $\varphi$ as the unique solution to

$$\bar{D}^a \bar{D}_a \varphi = -\frac{1}{2} \bar{D}^a \beta_a \tag{89}$$

that is $L^2$-orthogonal to the constant functions on $\mathcal{C}$ which exists because the right side is $L^2$-orthogonal to the constant functions. By the usual elliptic regularity results, it follows that $\varphi(x, \ell)$ is jointly smooth in $(x, \ell)$. Let $S^a := \sum_{j \leq n} \ell^j s^{(j)a}$. By (17), known at this stage up to and including order $n$, we have $[\bar{D}_a, \mathscr{L}_S] = 0 = \mathscr{L}_S \bar{\gamma}_{ab}$, from which it follows that

$$\begin{aligned} \bar{D}^a \bar{D}_a(\mathscr{L}_S\varphi - \alpha) &= \mathscr{L}_S \bar{D}^a \bar{D}_a \varphi - \bar{D}^a \bar{D}_a \alpha \\ &= -\frac{1}{2}\mathscr{L}_S \bar{D}^a \beta_a - \bar{D}^a \bar{D}_a \alpha \\ &= -\frac{1}{2}\bar{D}^a \mathscr{L}_S \beta_a - \bar{D}^a \bar{D}_a \alpha \\ &= O(\ell^{n+2}). \end{aligned} \tag{90}$$

In the last line we used $D_a\alpha + \frac{1}{2}\mathscr{L}_s\beta_a = O(\ell^{n+2})$ and that $s^a = S^a + O(\ell^{n+2})$. It follows that $\mathscr{L}_s\varphi - \alpha$ is equal to a constant function on $\mathcal{C}$ plus a function of order $O(\ell^{n+2})$. Since $s^a$ is a KVF of $\gamma_{ab}$ modulo $O(\ell^{n+2})$, it follows that

$$\begin{aligned} \mathscr{L}_s\varphi - \alpha &= O(\ell^{n+2}) + \frac{1}{A[\mathcal{C}]}\int_{\mathcal{C}} (\mathscr{L}_s\varphi - \alpha)\sqrt{\gamma}\mathrm{d}^{d-2}x \\ &= O(\ell^{n+2}) - \frac{1}{A[\mathcal{C}]}\int_{\mathcal{C}} \alpha\sqrt{\gamma}\mathrm{d}^{d-2}x = O(\ell^{n+2}) - \kappa, \end{aligned} \tag{91}$$

as desired. From the induction hypothesis, we know $\alpha - \kappa = O(\ell^n)$, so $\mathscr{L}_s\varphi = O(\ell^n)$. Using this and $\mathscr{L}_S\bar{\gamma}_{ab} = 0$ and $s^a = S^a + O(\ell^{n+2})$ we now show that we modify $\varphi$ if necessary so that $\varphi = O(\ell^n)$ by analyzing the consequences of the relations

$$s^{(0)a} D_a \varphi^{(j)} + \cdots + s^{(j)a} D_a \varphi^{(0)} = 0 \tag{92}$$

for ascending $j$ from 0 to $n-2$: We first get $s^{(0)a} D_a \varphi^{(0)} = 0$. Since $s^{(0)a}$ is an irrational linear combination KVFs $\psi_1^a, \ldots, \psi_N^a$ generating the $2\pi$-periodic flows of isometries of $\bar{\gamma}_{ab}$, it follows that $\psi_i^a D_a \varphi^{(0)} = 0$ for all $i = 1, \ldots, N$. As a consequence of the genericity assumption $s^{(j)a}$ is also a linear combination KVFs $\psi_1^a, \ldots, \psi_N^a$ and therefore $s^{(j)a} D_a \varphi^{(0)} = 0$ for $j \le n$. Then we move on to $j = 2$ and so on in (92) and continue in a similar way establishing that $\psi_i^a D_a \varphi^{(j)} = 0$ for $j \le n-2$. Therefore, $\varphi - \sum_{j \le n-2} \ell^j \varphi^{(j)} = O(\ell^n)$ still fulfills (91). □

## E Proof of Proposition 1

In this proof, we write the Schwarzschild metric in terms of coordinates $(x^A)$ on the $(d-2)$-dimensional symmetry round spheres and coordinates $(x^j) = (t, r)$ parameterizing the directions orthogonal to these round spheres. We write $(x^\mu) = (x^j, x^A)$ and $h_{ab} = h_{\mu\nu}(\mathrm{d}x^\mu)_a(\mathrm{d}x^\nu)_b$. The Schwarzschild metric is written as

$$\bar{g} = -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{d-2}^2 \equiv \bar{g}_{ij}\mathrm{d}x^i\mathrm{d}x^j + r^2\Omega_{AB}\mathrm{d}x^A\mathrm{d}x^B. \tag{93}$$

$\bar{g}_{ij}\mathrm{d}x^i\mathrm{d}x^j$ is the metric of the space of $SO(d-1)$-orbits, with derivative $D_i$, and is used to raise and lower indices $i, j, k, \ldots$. Following [33], metric perturbations are classified into scalar-, vector- and tensor type as follows.

- Scalar perturbations are of the form $h_{ij} = f_{ij}\mathbb{S}$, $h_{iA} = rf_i\mathbb{S}_A$, $h_{AB} = 2r^2(H_L\Omega_{AB}\mathbb{S} + H_T\mathbb{S}_{AB})$, where $\mathbb{S}$ denotes a scalar spherical harmonic with eigenvalue $-k_S^2$, where $\mathbb{S}_A, \mathbb{S}_{AB}$ are defined as in [33], and where $f_i, f_{ij}, H_L, H_T$ do not depend on $x^A$.

- Vector perturbations are of the form $h_{ij} = 0, h_{iA} = rf_i \mathbb{V}_A, h_{AB} = -2r^2 k_V^{-1} H_T \hat{D}_{(A}\mathbb{V}_{B)}$, where $\mathbb{V}_A$ denotes a vector spherical harmonic with eigenvalue $-k_V^2$.
- Tensor perturbations are of the form $h_{ij} = 0, h_{iA} = 0, h_{AB} = 2r^2 H_T \mathbb{T}_{AB}$, where $\mathbb{T}_{AB}$. denotes a tensor spherical harmonic with eigenvalue $-k_T^2$.

The eigenvalues of the spherical harmonics are labelled by $l \in \mathbb{N}_0$, where $l \geq 0$ for scalar-, $l \geq 1$ for vector-, and $l \geq 2$ for tensor harmonics.

For Proposition 1, we consider static, asymptotically flat perturbations, i.e. $\partial_t h_{\mu\nu} = 0$, with the usual fall-off conditions for $|h_{\mu\nu}|$ as $r \to \infty$, see def. 2.1 of [10] for details. The perturbation is additionally required to be regular at the horizon, $r = r_0$. [26,33] have constructed gauge invariant master variables for the scalar-, vector- and tensor-type perturbations built from $f_i$, $f_{ij}$, $H_T$, $H_L$ and the eigenvalues for each spherical harmonic mode number $l$. In order for static solutions $h_{ab}$ of the described type to exist, one has to find a solution to the appropriate master equation with appropriately regular behavior as $r \to r_0$ and $r \to \infty$. In fact, [26] have obtained explicit solutions to these equations for the static case for scalar type perturbations in terms of hypergeometric functions of a variable directly related to $r$. They have shown using known asymptotic formulas for hypergeometric functions that no non-zero solution with the prerequisite regularity as $r \to r_0$ and $r \to \infty$ exists, except for a 1-parameter family of perturbations of Schwarzschild corresponding to an infinitesimal change in the Schwarzschild radius $r_0$. By a similar method, one can show that there are no regular, asymptotically flat vector- or tensor type perturbations except for perturbations corresponding to an infinitesimal set of rotation parameters in the $d$-dimensional Myers–Perry family of solutions. We now go through these arguments in detail. Throughout this appendix, it is convenient to define

$$n := d - 2, \quad x := \left(\frac{r_0}{r}\right)^{n-1}. \tag{94}$$

Note that $n$ is not to be confused with the induction order as used in the main text.

*E.1 Tensor perturbations.* For tensor perturbations, the gauge invariant master variable $\Phi$ is defined by $H_T = r^{-n/2}\Phi$. For static perturbations $\Phi = \Phi(x)$ and the master equation is [33]

$$\begin{aligned}&(n-1)^2 x^2 (1-x)\, \Phi''(x) + (n-1)(n+(1-2n)x)x\, \Phi'(x)\\ &-\left((l-1)(l+n) + \frac{n(n+2)(1-x)}{4} + \frac{n(n+1)}{2x}\right)\, \Phi(x) = 0\end{aligned} \tag{95}$$

The general solution is given in terms of Legendre- and hypergeometric functions by

$$\begin{aligned}\Phi(x) &= C_1 x^{-\frac{n}{2(n-1)}} P\left(\frac{-l-n+1}{n-1}, \frac{x-2}{x}\right) + C_2 x^{-\frac{n}{2(n-1)}} Q\left(\frac{-l-n+1}{n-1}, \frac{x-2}{x}\right)\\ &= C_3 x^{-\frac{n}{2(n-1)}} F\left(-\frac{l}{n-1}, \frac{l+n-1}{n-1}; 1, \frac{1}{x}\right) +\\ &\quad C_4 x^{-\frac{n}{2(n-1)}} \left(\frac{x-2}{x}\right)^{\frac{l}{n-1}} \frac{2^{\frac{l}{n-1}}\sqrt{\pi}\,\Gamma(-\frac{l}{n-1})}{\Gamma(\frac{n-1-2l}{2n-2})}\\ &\quad \times F\left(-\frac{l}{2n-2}, \frac{n-1-l}{2n-2}; \frac{n-1-2l}{2n-2}, \frac{x^2}{(x-2)^2}\right)\end{aligned} \tag{96}$$

For asymptotic flatness, we need $h_{AB} = O(r)$ as $r \to \infty$ at a minimum which implies that we should have $\Phi(x) = O(x^{-(n-2)/2(n-1)})$ when $x \to 0$. Due to the overall factor, the second term clearly does not satisfy this so we must have $C_4 = 0$. When $\frac{l}{(n-1)}$ is not a positive integer, we may apply a standard linear transformation formula for hypergeometric functions to the first term, resulting in

$$\begin{aligned}\Phi(x) = \ & C_3 x^{-\frac{n}{2(n-1)}} \frac{1}{\Gamma(-\frac{l}{n-1})\Gamma(\frac{l+n-1}{n-1})} \sum_{k=0}^{\infty} \frac{\left(-\frac{l}{n-1}\right)_k \left(\frac{l+n-1}{n-1}\right)_k}{(k!)^2}\\ & \times \left[2\psi(k+1) - \psi\left(-\frac{l}{n-1}+k\right) - \psi\left(-\frac{l+n-1}{n-1}+k\right) - \log\left(\frac{x-1}{x}\right)\right]\left(\frac{1-x}{x}\right)^k\end{aligned} \tag{97}$$

This is singular at the horizon $x = 1$ and so is $h_{AB}$. Therefore, we must have $C_3 = 0$ and thus $\Phi(x) = 0$. When $\frac{l}{(n-1)} = m$ is a positive integer, the hypergeometric function multiplied by $C_3$ above becomes a polynomial in $1/x$ of degree at most $m$. Then even for the best possible case $m = 1$, $\Phi$ is singular at the horizon and so is $h_{AB}$. Thus, $\Phi(x) = 0$ in all cases and we conclude that there cannot exist static, regular, asymptotically flat tensor perturbations.

*E.2 Vector perturbations.* For vector perturbations, the gauge invariant master variable $\Phi$ is defined in terms of the tensor $F_j$ (eq. 5.10 of [33]) by $F_i = r^{n-1}\epsilon_{ij} D^j(r^{n/2}\Phi)$, except for the $l = 1$ mode. In that case we should instead consider the variable $\Phi = 2r\bar{\epsilon}^{ij} D_i(r^{-1} f_j)$ as the basic gauge invariant potential (eq. 5.16 of [33]). For static perturbations, $\Phi = \Phi(x)$ in either case.

$\underline{l > 1}$: The master equation for a static vector perturbation is

$$\begin{aligned}&(n-1)^2 x^2 (1-x)\ \Phi''(x) + (n-1)(n+(1-2n)x)x\ \Phi'(x) -\\ &- \left((l-1)(l+n) + \frac{n(n+2)}{4} - \frac{3n^2 x}{4}\right)\ \Phi(x) = 0\end{aligned} \tag{98}$$

The general solution is

$$\begin{aligned}\Phi(x) = C_1 x^{-\frac{2l+n}{2(n-1)}} F\left(\frac{-n-l}{n-1}, \frac{n-l}{n-1}; -\frac{2l}{n-1}, x\right) +\\ C_2 x^{\frac{2l+n-2}{2(n-1)}} F\left(\frac{l-1}{n-1}, \frac{l+2n-1}{n-1}; \frac{2l+2n-2}{n-1}, x\right)\end{aligned} \tag{99}$$

At large distances, $x \to 0$, the terms behave as $\sim C_1 x^{-\frac{2l+n}{2(n-1)}}$ respectively as $\sim C_2 x^{\frac{2l+n-2}{2(n-1)}}$. One sees from this that for the perturbation to be asymptotically flat we must require $C_1 = 0$. By applying a linear transformation formula for hypergeometric functions to the second linearly independent solution multiplied by $C_2$, we see that

$$\Phi(x) = C_2 x^{\frac{2l+n-2}{2(n-1)}} \frac{\Gamma(\frac{2l+2n-2}{n-1})}{\Gamma(\frac{l-1}{n-1})\Gamma(\frac{l+2n-1}{n-1})} \sum_{k=0}^{\infty} \frac{\left(\frac{l-1}{n-1}\right)_k \left(\frac{l+2n-1}{n-1}\right)_k}{(k!)^2} \times \left[2\psi(k+1) - \psi\left(\frac{l-1}{n-1}+k\right) - \psi\left(\frac{l+2n-1}{n-1}+k\right) - \log(1-x)\right](1-x)^k \quad (100)$$

This has a logarithmic divergence for $x \to 1$, i.e. at the horizon.
$\underline{l=1}$ : For static perturbations, the linearized Einstein equation yields $\Phi(r) = L/r^{n+1}$ for some real constant $L$, from which we learn that the only nonzero component of $h_{\mu\nu}$ is $h_{At} = r f_t \mathbb{V}_A = \frac{L}{(n+1)r^{n-1}} \mathbb{V}_A$, where $\mathbb{V}^A$ is a Killing vector field of the $n$-sphere. Such a perturbation corresponds to turning on an infinitesimal rotation parameter in the Myers–Perry solution [38].

To summarize, the only static, regular, asymptotically flat vector perturbations correspond to adding perturbatively a rotation to the Schwarzschild black hole towards a Myers–Perry black hole.

*E.3 Scalar perturbations.* For scalar perturbations, we take as our gauge invariant master variable, $Y$, the combination[7] $Y = r^{n-2}(f^{-1}F^{ij}(D_i r)D_j r - 2F)$, where $F_{ij}, F$ are defined in eqs. 2.7a,b of [33]. For static scalar perturbations, $Y = Y(x)$, which is a solution to the master equation

$$(n-1)^2 x^2 (1-x)\, Y''(x) - 2(n-1)(1+(n-2)x)x\, Y'(x) + \left((n-2)x - (l-1)(n+l)\right) Y(x) = 0 \quad (101)$$

The general solution is

$$Y(x) = C_1 x^{-\frac{l-1}{n-1}} F\left(\frac{n-l-1}{n-1}, -\frac{l}{n-1}; -\frac{2l}{n-1}, x\right) + C_2 x^{\frac{l+n}{n-1}} F\left(\frac{n+l-1}{n-1}, \frac{2(n-1)+l}{n-1}; \frac{2(l+n-1)}{n-1}, x\right) \quad (102)$$

To see which of these solutions correspond to a regular, asymptotically flat pertrubation, we apply a gauge transformation to the perturbation to bring it into Regge-Wheeler gauge in which $f_i = H_T = 0$. The transformed perturbation is still static, regular, and asymptotically flat and takes the form

$$h_{\mu\nu} dx^\mu dx^\nu = -F^t_t f \mathbb{S}\, dt^2 + f^{-1} F^r_r \mathbb{S}\, dr^2 + 2Fr^2 \mathbb{S}\, \Omega_{AB} dx^A dx^B \quad (103)$$

[7] Its relation to the master variable $\tilde{Y}$ obeying the Schrödinger-type equation 3.6 of [26] is $\tilde{Y} = \sqrt{\frac{f}{r}} \frac{r}{f'} Y$.

where in the coordinate $x$ the metric function $f(x) = 1 - x$ and the horizon is at $x = 1$. The components of the gauge invariant variables $F_i^j$, $F$ are given by the master variable $Y$ and $X = Y + 2fY'/f'$ through the relations

$$F_t^t = \frac{(n-1)X - Y}{nr^{n-2}}, \quad F_r^r = \frac{-X + (n-1)Y}{nr^{n-2}}, \quad F_t^r = 0, \quad F = -\frac{X+Y}{2nr^{n-2}}, \tag{104}$$

noting that $Z = 0$, see eqs. 4.2 of [26].
$\underline{l > 0}$ : Using this together with the asymptotic forms of the hypergeometric functions, we see that the first solution multiplied by $C_1$ would give rise to $F_t^t$, $F_r^r$, $F$ diverging as $r^l$ for $r \to \infty$. Therefore, asymptotic flatness requires $C_1 = 0$. By applying an appropriate linear transformation formula to the hypergeometric function multiplied by $C_2$, we obtain

$$\begin{aligned} Y(x) = C_2 x^{\frac{n+l}{n-1}} \Bigg[ & \frac{\Gamma(\gamma_2)}{\Gamma(\alpha_2)\Gamma(\beta_2)}(1-x)^{-1} + \\ & + \frac{\Gamma(\gamma_2)}{\Gamma(\alpha_2 - 1)\Gamma(\beta_2 - 1)} \sum_{k=0}^{\infty} \frac{(\alpha_2)_k\,(\beta_2)_k}{k!(k+1)!} \{\log(1-x) - \psi(k+1) - \psi(k+2) + \\ & \psi(\alpha_2 + k) + \psi(\beta_2 + k)\}(1-x)^k \Bigg] \end{aligned} \tag{105}$$

where $\alpha_2 = \frac{n+l-1}{n-1}$, $\beta_2 = \frac{2(n-1)+l}{n-1}$, $\gamma_2 = \alpha_2 + \beta_2 - 1$. Note that $\alpha_2 > 1$, $\beta_2 > 1$. Near the horizon we therefore find that for $x \to 1$ we have the asymptotic behavior

$$X \sim -Y \sim -C_2 \frac{\Gamma(\gamma_2)}{\Gamma(\alpha_2)\Gamma(\beta_2)}(1-x)^{-1} \tag{106}$$

and therefore $F_t^t$ and $F_r^r$ both diverge as $f(r)^{-1}$ at the horizon $r \to r_0$. Thus, there are no static, regular, asymptotically flat scalar type perturbations when $l > 0$.
$\underline{l = 0}$ : In this case, we have a spherically symmetric, static perturbation. It can be shown by analyzing the perturbation equations that these correspond up to gauge precisely to a perturbative change in the Schwarzschild radius, $r_0$, hence to a perturbation towards another Schwarzschild black hole.

Combining the results for scalar-, vector- and tensor perturbations and taking into account the gauge transformations implicitly considered in the above arguments, we have shown that $h_{\mu\nu} = z_{\mu\nu} + \mathscr{L}_\xi \bar{g}_{\mu\nu}$, where $z_{\mu\nu}$ corresponds to an infinitesimal variation of the parameters in the Myers–Perry solution [38]. By assumption the coordinate components $h_{\mu\nu}$ in an asymptotically Cartesian coordinate system built from $(t, r, x^A)$ fulfill the fall-off conditions detailed in [10], def. 2.1, for $r \to \infty$. Since furthermore $\partial_t h_{\mu\nu} = 0 = \mathscr{L}_t z_{\mu\nu}$, it follows that $\mathscr{L}_\xi \bar{g}_{\mu\nu}$ is asymptotically flat at $\mathcal{I}^\pm$, as we can see e.g. by analyzing its behavior in the coordinates $(t \pm r, r, x^A)$. Thus, $\xi^a$ is an asymptotic symmetry at $\mathcal{I}^\pm$. $\square$

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