



An Index Theorem for Quarter-Plane Toeplitz Operators via Extended Symbols and Gapped Invariants Related to Corner States

Shin Hayashi^{1,2,3} 

¹ Advanced Institute for Materials Research, Tohoku University, 2-1-1 Katahira, Aoba, Sendai 980-8577, Japan.

E-mail: shin.hayashi.e2@tohoku.ac.jp

² PRESTO, Japan Science and Technology Agency, 4-1-8 Honcho, Kawauchi, Saitama 332-0012, Japan

³ Mathematics for Advanced Materials - Open Innovation Laboratory,
National Institute of Advanced Industrial Science and Technology, 2-1-1 Katahira, Aoba, Sendai 980-8577, Japan

Received: 9 April 2022 / Accepted: 28 November 2022
Published online: 17 December 2022 – © The Author(s) 2022

Abstract: In this paper, we discuss index theory for Toeplitz operators on a discrete quarter-plane of two-variable rational matrix function symbols. By using Gohberg–Kreĭn theory for matrix factorizations, we extend the symbols defined originally on a two-dimensional torus to some three-dimensional sphere and derive a formula to express their Fredholm indices through extended symbols. Variants for families of (self-adjoint) Fredholm quarter-plane Toeplitz operators and those preserving real structures are also included. For some bulk-edge gapped single-particle Hamiltonians of finite hopping range on a discrete lattice with a codimension-two right angle corner, topological invariants related to corner states are provided through extensions of bulk Hamiltonians.

1. Introduction

Topological corner states have attracted much interest in condensed matter physics as a characteristic of higher-order topological insulators. Aimed at studies of topological corner states, we discuss index theory for some Toeplitz operators on a discrete quarter-plane. Index theory for quarter-plane Toeplitz operators has been investigated by Simonenko, Douglas–Howe [16, 43], and a necessary and sufficient condition for these operators to be Fredholm is obtained in terms of the invertibility of two associated half-plane Toeplitz operators. Index formulas for Fredholm quarter-plane Toeplitz operators are obtained by Coburn–Douglas–Singer, Dudučava, Park [15, 17, 36]. Coburn–Douglas–Singer derived their formula by showing that there is a deformation to some quarter-plane Toeplitz operators of a standard form preserving Fredholm indices [15]. Dudučava employed Gohberg–Kreĭn theory for the factorization of some matrix functions on a circle [14, 21, 46] and obtained a formula by using a construction of parametrix [17]. Park obtained an index formula by a construction of a cyclic cocycle and using a pairing between K -theory and cyclic cohomology [36].

A characteristic feature of topological insulators is the existence of topological edge states. Although the bulk is gapped (insulating), edge states exist that account for the

metallic properties of the boundary of the system. This appearance of edge states is known to originate from a topological invariant for the gapped bulk, called the *bulk-edge correspondence* [22,23]. A typical example is the integer quantum Hall system in which the bulk topological invariant is known to be the first Chern number of a complex vector bundle (Bloch bundle) over a two-dimensional torus (Brillouin torus). Bellissard investigated the quantum Hall effect through noncommutative geometry [9], and Kellendonk–Richter–Schulz–Baldes gave a proof of the bulk-edge correspondence based on index theory for Toeplitz operators [29]. K -theory was employed for the classification of topological insulators [18,32] (see also [39] and the references therein). We note that matrix factorizations are also used in recent physical studies of (first-order) topological insulators [1].

For *higher-order topological insulators* [10,24,31,41], an actively studied topic in condensed matter physics, the bulk-edge correspondence is greatly generalized to include corner states. For two-dimensional second-order topological insulators, for example, the bulk and two edges whose intersection form a codimension-two corner are gapped, though there exist topological corner states. There are many physical studies of higher-order topological insulators in various directions: classifications of higher-order topological insulators [19,30,35], topological invariants characterizing them as nested Wilson loops [10], symmetry indicators [38], multipole chiral numbers [11], relations between some gapped topology and corner/hinge states [3,47,49]. Experimental realizations are also reported [37].

In [25], a mathematical approach to topological corner states is proposed based on index theory for quarter-plane Toeplitz operators, where topological invariants for bulk-edge gapped Hamiltonians are defined as elements of a K -group of some C^* -algebra and its relation with hinge states is proved. Although this shows a relation between some gapped topology and corner states, gapped invariants are defined abstractly and much more geometric understanding is required, in order both for investigation from the physical point of view and its computation. For this purpose, we investigate further index theory for quarter-plane Toeplitz operators, especially Dudučava's idea of using matrix factorizations [17] from a topological point of view. We consider Fredholm quarter-plane Toeplitz operators of two-variable rational matrix function symbols. For each of them, there are two associated invertible half-plane Toeplitz operators having the same symbol. In Sect. 3, we investigate the geometric implications of this invertibility condition. Through the Fourier transform in a direction parallel to the boundary, a half-plane Toeplitz operator corresponds to a one-parameter family of Toeplitz operators, and the problem reduces to a study of invertible Toeplitz operators. For an invertible Toeplitz operator of a rational matrix function symbol, Gohberg–Kreĭn theory states that there is a decomposition of the symbol as a product of two matrix-valued functions such that each factor of the decomposition can be analytically continued to a disk. By using analytic continuation, we see that the symbol of the quarter-plane Toeplitz operator defined originally on a two-dimensional torus can be extended as a continuous nonsingular matrix-valued function over some three sphere. This extension is shown to be independent of the choice of the factorization, therefore is canonically associated with our operator. We then show in Sect. 4 that the Fredholm index of the quarter-plane Toeplitz operator is given through the three-dimensional winding number of the extended symbol (Corollary 4.10). Note that a part of its proof is based on Coburn–Douglas–Singer's idea [15]. Our formula can be extended to Fredholm quarter-plane Toeplitz operators which are self-adjoint, preserving real structures and families of them, and these variants are proved in a parallel way. In this paper, we mainly discuss

families of (self-adjoint) Fredholm quarter-plane Toeplitz operators for which we use complex K -theory (Theorem 4.1), and the results for those operators preserving real structures are contained in Sect. 5. Necessary results about quarter-plane Toeplitz operators and Gohberg–Kreĭn theory for matrix factorizations used in this paper are collected in Sect. 2.

Applications to topological corner states are discussed in Sect. 6. We consider translation invariant single-particle Hamiltonians of finite hopping range on the lattice \mathbb{Z}^d in each of the ten Altland–Zirnbauer classes [2], and discuss its restrictions onto $(\mathbb{Z}_{\geq 0})^2 \times \mathbb{Z}^{d-2}$ assuming the Dirichlet boundary condition. When the bulk Hamiltonian and its compressions onto two half-spaces $\mathbb{Z} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}^{d-2}$ and $\mathbb{Z}_{\geq 0} \times \mathbb{Z} \times \mathbb{Z}^{d-2}$ are gapped, we associate a nonsingular matrix function (over some three sphere for two-dimensional systems) through the matrix factorization which is an extension of the bulk Hamiltonian. We define a topological invariant for such a bulk-edge gapped Hamiltonian as a K -class in some topological K -theory group of this extended bulk Hamiltonian (Definition 6.1). A relation between this gapped topological invariant and corner/hinge states is given in Theorem 6.2, which provides a geometric formulation for the relation between the abstractly defined gapped topological invariant and corner states in [25]. In order to construct extensions of bulk Hamiltonians, we need to take matrix factorizations. For matrix factorizations of rational matrix functions on the unit circle in the complex plane, an algorithm is known [14, 20, 21] and the finite hopping range condition is assumed correspondingly. In [27], a classification of topological invariants related to corner states in each of the Altland–Zirnbauer classes is proposed based on index theory, where Boersema–Loring’s formulation of KO -theory for real C^* -algebras [12] is employed. Since topological corner states are one motivation of this work, some parts of the discussions in this paper are organized in this framework. For example, the real symmetries discussed in Sect. 5 are taken from Boersema–Loring’s picture. Note that the integration formula for our gapped topological invariants, like the integration of the Berry curvature for the first Chern number, is still missing since our three sphere is not smooth, although our formulation provides a way to understand gapped topological invariants related to corner states in a geometric way. For example, for a two-dimensional class AIII system, the gapped topological invariant is given by the three-dimensional winding number of the extension of the bulk Hamiltonian (Example 6.3). For a three-dimensional class A system, it is provided as a topological invariant for an extension of the Bloch bundle (Example 6.4).

Topological corner states provide contact with index theory for quarter-plane Toeplitz operators and higher-order topological insulators [26, 34]. Through it, interesting questions are posed for our index theory such as relations with invariants introduced in studies of higher-order topological insulators, developments of the theory to support robustness of (topological) corner states against disorder. For example, for two-dimensional class AIII systems with codimension-two corners, integer-valued multipole chiral numbers are defined in [11] in real space. For this class, we investigate integer-valued invariants in [26] in momentum space, and their relationship is not clear. In this paper, we add a geometric picture for the gapped K -class invariants in [25–27] restricted to bulk periodic systems with codimension-two right angle corners or hinges, as the Bloch bundle for the first Chern number in first-order topological insulators, for further investigations.

2. Preliminaries

In this section, we collect the necessary results and notations used in this paper.

2.1. *Quarter-plane Toeplitz operators.* Let \mathbb{T} be the unit circle in the complex plane equipped with the normalized Haar measure. For $f \in C(\mathbb{T}^n)$, we write M_f for the bounded linear operator on $l^2(\mathbb{Z}^n)$ corresponding to the multiplication operator on $L^2(\mathbb{T}^n)$ generated by f through the Fourier transform $L^2(\mathbb{T}^n) \cong l^2(\mathbb{Z}^n)$. For an integer k , we write $\mathbb{Z}_{\geq k}$ for the set of integers greater than or equal to k . Let δ_n be the characteristic function of a point $n \in \mathbb{Z}$ and let $l^2(\mathbb{Z}_{\geq 0})$ be the closed subspace of $l^2(\mathbb{Z})$ spanned by $\{\delta_n \mid n \geq 0\}$. Let P be the orthogonal projection of $l^2(\mathbb{Z})$ onto $l^2(\mathbb{Z}_{\geq 0})$. For $f \in C(\mathbb{T})$, the operator on $l^2(\mathbb{Z}_{\geq 0})$ defined by $T_f \varphi = P M_f \varphi$ for $\varphi \in l^2(\mathbb{Z}_{\geq 0})$ is called the *Toeplitz operator* of continuous symbol f . Let \mathcal{T} be the C^* -algebra generated by those Toeplitz operators. We have the following Toeplitz extension:

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{T} \xrightarrow{\sigma} C(\mathbb{T}) \rightarrow 0, \tag{1}$$

where \mathcal{K} is the compact operator algebra and σ is a $*$ -homomorphism that maps T_f to its symbol f .

Let $\delta_{m,n}$ be the characteristic function of the point (m, n) in \mathbb{Z}^2 , and let \mathcal{H}^0 , \mathcal{H}^∞ and $\mathcal{H}^{0,\infty}$ be closed subspaces of $l^2(\mathbb{Z}^2)$ spanned by $\{\delta_{m,n} \mid n \geq 0\}$, $\{\delta_{m,n} \mid m \geq 0\}$, and $\{\delta_{m,n} \mid m \geq 0 \text{ and } n \geq 0\}$, respectively. Let P^0 , P^∞ and $P^{0,\infty}$ be the orthogonal projection of $l^2(\mathbb{Z}^2)$ onto \mathcal{H}^0 , \mathcal{H}^∞ and $\mathcal{H}^{0,\infty}$, respectively. Note that $\mathcal{H}^{0,\infty} = \mathcal{H}^0 \cap \mathcal{H}^\infty$ and $P^{0,\infty} = P^0 P^\infty = P^\infty P^0$. For $f \in C(\mathbb{T}^2)$, the operators T_f^0 on \mathcal{H}^0 and T_f^∞ on \mathcal{H}^∞ defined by $T_f^0 \varphi = P^0 M_f \varphi$ for $\varphi \in \mathcal{H}^0$ and $T_f^\infty \varphi = P^\infty M_f \varphi$ for $\varphi \in \mathcal{H}^\infty$, respectively, are called *half-plane Toeplitz operators*. The operator on $\mathcal{H}^{0,\infty}$ defined by $T_f^{0,\infty} \varphi = P^{0,\infty} M_f \varphi$ for $\varphi \in \mathcal{H}^{0,\infty}$ is called the *quarter-plane Toeplitz operator*. Let \mathcal{T}^0 and \mathcal{T}^∞ be the C^* -algebras generated by half-plane Toeplitz operators of the form T_f^0 and T_f^∞ , respectively, and let $\mathcal{T}^{0,\infty}$ be the C^* -algebra generated by quarter-plane Toeplitz operators. Note that $\mathcal{T}^0 \cong C(\mathbb{T}) \otimes \mathcal{T}$ and $\mathcal{T}^\infty \cong \mathcal{T} \otimes C(\mathbb{T})$ by the Fourier transform in a direction parallel to the boundary of half-planes. Corresponding to these isomorphisms, let $\sigma^0 = 1_{C(\mathbb{T})} \otimes \sigma$ and $\sigma^\infty = \sigma \otimes 1_{C(\mathbb{T})}$ which are $*$ -homomorphisms from \mathcal{T}^0 and \mathcal{T}^∞ to $C(\mathbb{T}^2)$. Let $\mathcal{S}^{0,\infty}$ be the pullback C^* -algebra of these two $*$ -homomorphisms,

$$\begin{array}{ccc} \mathcal{S}^{0,\infty} & \xrightarrow{p^\infty} & \mathcal{T}^\infty \\ p^0 \downarrow & & \downarrow \sigma^\infty \\ \mathcal{T}^0 & \xrightarrow{\sigma^0} & C(\mathbb{T}^2) \end{array}$$

therefore, the C^* -algebra $\mathcal{S}^{0,\infty}$ consists of pairs $(T^0, T^\infty) \in \mathcal{T}^0 \oplus \mathcal{T}^\infty$ satisfying $\sigma^0(T^0) = \sigma^\infty(T^\infty)$. Maps p^0 and p^∞ are $*$ -homomorphisms sending (T^0, T^∞) to T^0 and T^∞ , respectively. Let $\sigma_S = \sigma^0 \circ p^0 = \sigma^\infty \circ p^\infty$. The following short exact sequence of C^* -algebras is known [36]:

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{T}^{0,\infty} \xrightarrow{\gamma} \mathcal{S}^{0,\infty} \rightarrow 0, \tag{2}$$

where the map $\mathcal{K} \rightarrow \mathcal{T}^{0,\infty}$ is the inclusion and γ is a $*$ -homomorphism that maps $T_f^{0,\infty}$ to (T_f^0, T_f^∞) for $f \in C(\mathbb{T}^2)$.

2.2. Matrix factorizations. In this subsection, we collect necessary results about the Gohberg–Kreĭn theory for factorizations of rational matrix functions in cases on the unit circle in the complex plane. For details, we refer the reader to [14,20,21].

Let us consider the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The unit circle \mathbb{T} is contained in $\hat{\mathbb{C}}$ and $\hat{\mathbb{C}} \setminus \mathbb{T}$ consists of two connected components. Let $D_+ = \{z \in \mathbb{C} \mid |z| < 1\}$ and $D_- = \{z \in \mathbb{C} \mid |z| > 1\} \cup \{\infty\}$ which are open disks. We write \mathbb{D}^2 for the closed unit disk $\mathbb{T} \cup D_+$ in the complex plane. Let $f: \mathbb{T} \rightarrow GL_n(\mathbb{C})$ be a nonsingular rational matrix function, that is a nonsingular matrix function of entries consisting of rational functions, with poles off \mathbb{T} . We have the following decomposition, called a (*right*) *matrix factorization*:

$$f = f_- \Lambda f_+, \tag{3}$$

where f_- , Λ and f_+ are rational matrix functions on \mathbb{T} satisfying the conditions below.

- f_+ (resp. f_-) admits a continuous extension onto $\mathbb{T} \cup D_+ = \mathbb{D}^2$ (resp. $\mathbb{T} \cup D_-$) as a nonsingular matrix function and is analytic on D_+ (resp. D_-). We write f_+^e (resp. f_-^e) for the extension.
- Λ is the diagonal matrix function of the form $\Lambda(z) = \text{diag}(z^{\kappa_1}, \dots, z^{\kappa_n})$ with nonincreasing sequence of integers $\kappa_1 \geq \dots \geq \kappa_n$ called *partial indices*.

Among the many results known for matrix factorizations, we note the following:

- The partial indices $\kappa_1, \dots, \kappa_n$ are uniquely determined by f .
- The Toeplitz operator T_f is invertible if and only if all of the partial indices are zero. In this case, factorization (3) is called a *canonical factorization*.
- If we have two canonical factorizations $f = f_- f_+ = g_- g_+$, there exists an invertible matrix $B \in GL_n(\mathbb{C})$, considered a constant matrix function, satisfying $f_+ = B g_+$ and $f_- = g_- B^{-1}$.

Remark 2.1. There is known a general class of Banach algebras of continuous functions on the circle that admits matrix factorizations (inverse closed decomposing Banach algebras containing rational functions [14,20]). One example is the Wiener algebra over the circle consisting of all complex-valued functions on the circle admitting an absolutely convergent Fourier series. The results in this paper are also valid for such continuous functions admitting matrix factorizations. Note that the algebra $C(\mathbb{T})$ of continuous functions is not decomposing. In this paper, we mainly discuss rational functions on the unit circle in view of our applications discussed in Sect 6.

3. Extension of Symbols Through Matrix Factorizations

Let $f: \mathbb{T}^2 \rightarrow GL_n(\mathbb{C})$ be a continuous map. The associated quarter-plane Toeplitz operator $T_f^{0,\infty}$ is Fredholm if and only if the half-plane Toeplitz operators T_f^0 and T_f^∞ are invertible [16, Corollary in p.209]. In this section, we discuss the geometric implication of this condition when the symbol f is a rational matrix function for both of the two variables of \mathbb{T}^2 . By using matrix factorizations, we provide a way to extend f to a nonsingular matrix-valued continuous function on a three sphere (Sect. 3.1). We also discuss its variants for families of them (Sect. 3.2) and the cases when matrix functions take value in (skew-)hermitian matrices (Sect. 3.3).

3.1. Extension of nonsingular matrix functions of trivial partial indices. Let f be a nonsingular rational matrix function on the circle \mathbb{T} with poles off \mathbb{T} , and assume that the associated Toeplitz operator T_f is invertible. Let $f = f_- f_+$ be a canonical factorization of f . Let f_+^e and f_-^e be the associated extensions of f_+ and f_- onto $\mathbb{T} \cup D_+$ and $\mathbb{T} \cup D_-$, respectively. For two disks $\mathbb{D}^2 = \mathbb{T} \cup D_+$ and $\mathbb{T} \cup D_-$, we consider the following identification:

$$I: \mathbb{T} \cup D_+ \rightarrow \mathbb{T} \cup D_-, \quad I(z) = \bar{z}^{-1},$$

where we set $I(0) = \infty$. By using this identification, we associate the following nonsingular matrix-valued continuous map for a canonical factorization:

$$f^e = (I^* f_-^e) \cdot f_+^e: \mathbb{D}^2 \rightarrow GL_n(\mathbb{C}). \tag{4}$$

Explicitly, $f^e(z) = f_-^e(\bar{z}^{-1})f_+^e(z)$ for $z \in \mathbb{D}^2$. Since $\bar{z}^{-1} = z$ for $z \in \mathbb{T}$, f^e coincides with f on \mathbb{T} and is its continuous extension onto \mathbb{D}^2 .

Lemma 3.1. *f^e is independent of the choice of canonical factorization.*

Proof. For two canonical factorizations $f = f_- f_+ = g_- g_+$, there exists $B \in GL_n(\mathbb{C})$ satisfying $f_+ = Bg_+$ and $f_- = g_- B^{-1}$. Note that $f_+^e = (Bg_+)^e = Bg_+^e$ and $f_-^e = g_-^e B^{-1}$, which follow from the uniqueness of analytic continuation. Therefore, for $z \in \mathbb{D}^2$,

$$f_-^e(\bar{z}^{-1}) \cdot f_+^e(z) = g_-^e(\bar{z}^{-1})B^{-1} \cdot Bg_+^e(z) = g_-^e(\bar{z}^{-1}) \cdot g_+^e(z).$$

□

By Lemma 3.1, for a nonsingular rational matrix function with poles off \mathbb{T} , $f: \mathbb{T} \rightarrow GL_n(\mathbb{C})$, whose associated Toeplitz operator T_f is invertible, there is a canonically associated extension f^e onto the disk \mathbb{D}^2 . For a non-negative real number t , we write $\mathbb{T}_t = \{z \in \mathbb{C} \mid |z| = t\}$. For $0 \leq t \leq 1$, let $m_t: \mathbb{T} \rightarrow \mathbb{T}_t$ be the map defined by $m_t(z) = tz$. We take the pullback $m_t^*(f^e|_{\mathbb{T}_t})$ of $f^e|_{\mathbb{T}_t}$ onto \mathbb{T} by m_t . Let us consider the Toeplitz operator $T_t := T_{m_t^*(f^e|_{\mathbb{T}_t})}$ associated with this matrix function. In other words, we consider \mathbb{D}^2 to be a family of circles of radius $0 \leq t \leq 1$ and consider the associated family of Toeplitz operators. For these operators, the following holds:

Lemma 3.2. *Let f be a nonsingular rational matrix function on \mathbb{T} with poles off \mathbb{T} of trivial partial indices, and consider its extension $f^e: \mathbb{D}^2 \rightarrow GL_n(\mathbb{C})$ in (4). For $0 \leq t \leq 1$, the Toeplitz operator T_t is invertible.*

Proof. The invertibility of $T_1 = T_f$ is included in our assumption. When $t = 0$, the Toeplitz operator T_0 is associated with nonsingular constant matrix function and is invertible. Let us consider the case in which $0 < t < 1$. For $z \in \mathbb{T}$, we have

$$m_t^*(f^e|_{\mathbb{T}_t})(z) = f^e(tz) = f_-^e((\bar{t}z)^{-1})f_+^e(tz) = f_-^e(t^{-1}z)f_+^e(tz).$$

Therefore, we have the following decomposition for the symbol of T_t :

$$m_t^*(f^e|_{\mathbb{T}_t}) = m_{t^{-1}}^*(f_-^e|_{\mathbb{T}_{t^{-1}}}) \cdot m_t^*(f_+^e|_{\mathbb{T}_t}). \tag{5}$$

Each component of equation (5) is a rational matrix function on \mathbb{T} . Let $D_{+,t} = \{z \in \mathbb{C} \mid |z| < t\}$ and $D_{-,t^{-1}} = \{z \in \mathbb{C} \mid |z| > t^{-1}\} \cup \{\infty\}$. Let us consider the maps $m_{+,t}: D_+ \rightarrow D_{+,t}$, $z \mapsto tz$, and $m_{-,t^{-1}}: D_- \rightarrow D_{-,t^{-1}}$, $z \mapsto t^{-1}z$. Since f_-^e and f_+^e are extensions of f_- and f_+ that are analytic on D_- and D_+ , pullbacks of their restrictions $m_{-,t^{-1}}^*(f_-^e|_{D_{-,t^{-1}}})$ and $m_{+,t}^*(f_+^e|_{D_{+,t}})$ onto D_- and D_+ also provide such extensions of $m_{t^{-1}}^*(f_-^e|_{\mathbb{T}_{t^{-1}}})$ and $m_t^*(f_+^e|_{\mathbb{T}_t})$. Therefore, equation (5) is a canonical factorization of $m_t^*(f^e|_{\mathbb{T}_t})$, and the associated Toeplitz operator T_t is invertible. □

3.2. *Extension of families of matrix functions.* We next extend the discussions in Sect. 3.1 to families of rational matrix functions of trivial partial indices. Matrix factorizations for such families are studied by Šubin [46].

Lemma 3.3. *Let X be a topological space. Let $f : \mathbb{T} \times X \rightarrow GL_n(\mathbb{C})$ be a continuous map such that for each $x \in X$, $f(x)$ is a rational matrix function on \mathbb{T} of trivial partial indices. Through the matrix factorization, there is a canonically associated continuous map $f^e : \mathbb{D}^2 \times X \rightarrow GL_n(\mathbb{C})$ that extends f .*

Proof. Following [46], for each $x_0 \in X$, there exists an open neighborhood $U \subset X$ of x_0 and continuous matrix functions f_+ and f_- on $\mathbb{T} \times U$ such that for $x \in U$, $f(x) = f_-(x)f_+(x)$ is a canonical factorization. By using this factorization, we obtain a continuous extension $f^e : \mathbb{D}^2 \times U \rightarrow GL_n(\mathbb{C})$ of f as in equation (4). As in Lemma 3.1, this f^e is independent of the choice of canonical factorization. We cover X by such open sets $\{U_\alpha\}_{\alpha \in J}$. When $U_\alpha \cap U_\beta \neq \emptyset$, we may consider two extensions of $f|_{\mathbb{T} \times (U_\alpha \cap U_\beta)}$ onto $\mathbb{D}^2 \times (U_\alpha \cap U_\beta)$ corresponding to extensions onto $\mathbb{D}^2 \times U_\alpha$ and $\mathbb{D}^2 \times U_\beta$, and they coincide by Lemma 3.1. Therefore, we obtain the desired extension onto $\mathbb{D}^2 \times X$. \square

We next consider families of two-variable rational matrix functions whose associated quarter-plane Toeplitz operators are Fredholm. Let

$$\tilde{\mathbb{S}}^3 := \partial(\mathbb{D}^2 \times \mathbb{D}^2) = \mathbb{T} \times \mathbb{D}^2 \cup_{\mathbb{T}^2} \mathbb{D}^2 \times \mathbb{T},$$

which is topologically a three-dimensional sphere. Since $\mathbb{D}^2 \subset \mathbb{C}$, we consider $\tilde{\mathbb{S}}^3$ as a subspace of \mathbb{C}^2 and use complex variables $(z, w) \in \mathbb{C}^2$ to parametrize $\tilde{\mathbb{S}}^3$.

Proposition 3.4. *Let X be a topological space. Let $f : \mathbb{T}^2 \times X \rightarrow GL_n(\mathbb{C})$ be a continuous map such that for each $x \in X$, $f(x)$ is a two-variable rational matrix function for which the associated quarter-plane Toeplitz operator $T_{f(x)}^{0,\infty}$ is Fredholm. Through matrix factorization, there is a canonically associated continuous map $f^E : \tilde{\mathbb{S}}^3 \times X \rightarrow GL_n(\mathbb{C})$ that extends f .*

Proof. For each $x \in X$, since $T_{f(x)}^{0,\infty}$ is Fredholm, both of the associated half-plane Toeplitz operators $T_{f(x)}^0$ and $T_{f(x)}^\infty$ are invertible [16]. Through a Fourier transform in a direction parallel to the boundary, the invertible half-plane Toeplitz operator $T_{f(x)}^0$ corresponds to a family of invertible Toeplitz operators $\{T_{f(z,\cdot,x)}\}_{z \in \mathbb{T}}$ parametrized by the circle. Therefore, by Lemma 3.3, there is a canonically associated continuous extension $f^e : \mathbb{T} \times \mathbb{D}^2 \times X \rightarrow GL_n(\mathbb{C})$ of f through matrix factorization. By the invertibility of $T_{f(x)}^\infty$, we also obtain an extension of f onto $\mathbb{D}^2 \times \mathbb{T} \times X$ through matrix factorization. Combined with them, we obtain an extension f^E of f as a nonsingular matrix-valued continuous function on $\tilde{\mathbb{S}}^3 \times X$. \square

3.3. *Hermitian and Skew-Hermitian matrix functions.* In this subsection, we discuss the case in which the nonsingular rational matrix functions in Sect. 3.1 and 3.2 take values in hermitian or skew-hermitian matrices. Let $GL_n(\mathbb{C})^{\text{sa}}$ (resp. $GL_n(\mathbb{C})^{\text{sk}}$) be the space of n -by- n hermitian (resp. skew-hermitian) invertible matrices.

Lemma 3.5. *Let \mathfrak{J} be $GL_n(\mathbb{C})^{\text{sa}}$ or $GL_n(\mathbb{C})^{\text{sk}}$. Let X be a topological space and $f : \mathbb{T} \times X \rightarrow \mathfrak{J}$ be a continuous map such that, for each $x \in X$, $f(x)$ is a rational matrix function on \mathbb{T} with trivial partial indices. Then the extension f^e of f in Lemma 3.3 is also a hermitian or skew-hermitian matrix function; that is, $f^e : \mathbb{D}^2 \times X \rightarrow \mathfrak{J}$.*

Proof. We consider hermitian matrix functions, that is, the case in which $\mathfrak{J} = GL_n(\mathbb{C})^{\text{sa}}$. By Lemma 3.3, it is sufficient to show that $f^e(x)$ for each $x \in X$ is a hermitian matrix-valued function. Therefore, it is sufficient to consider the case when X is one point set and we assume this condition. Let $f = f_- f_+$ be a canonical factorization of f . Since f is hermitian, we have $f_- f_+ = f = f^* = f_+^* f_-^*$. The equation $f = f_+^* f_-^*$ is also a canonical factorization since $(f_+^e)^* \circ I^{-1}$ (resp. $(f_-^e)^* \circ I$) provides a continuous extension of f_+^* onto $\mathbb{T} \cup D_-$ (resp. f_-^* onto $\mathbb{T} \cup D_+$), which is analytic on D_- (resp. D_+). Therefore, there exists $B \in GL_n(\mathbb{C})$ such that $f_+^* = f_- B^{-1}$ and $f_-^* = B f_+$. The uniqueness of analytic continuation leads to the relations $(f_+^e)^* \circ I^{-1} = (f_+^*)^e = f_-^e B^{-1}$ and $(f_-^e)^* \circ I = (f_-^*)^e = B f_+^e$. Therefore, for $z \in \mathbb{D}^2$,

$$(f^e(z))^* = (f_-^e(\bar{z}^{-1}) \cdot f_+^e(z))^* = (f_+^e(z))^* (f_-^e(\bar{z}^{-1}))^* = f_-^e(\bar{z}^{-1}) B^{-1} B f_+^e(z) = f^e(z).$$

The result for skew-hermitian matrix functions is proved in a similar manner. □

By Proposition 3.4 and Lemma 3.5, we obtain the following result:

Proposition 3.6. *Let \mathfrak{J} be $GL_n(\mathbb{C})^{\text{sa}}$ or $GL_n(\mathbb{C})^{\text{sk}}$. Let X be a topological space. Let $f : \mathbb{T}^2 \times X \rightarrow \mathfrak{J}$ be a continuous map such that, for each $x \in X$, $f(x)$ is a two-variable rational matrix function for which the associated quarter-plane Toeplitz operator $T_{f(x)}^{0, \infty}$ is Fredholm. Through matrix factorization, there is a canonically associated continuous map $f^E : \mathbb{S}^3 \times X \rightarrow \mathfrak{J}$ that extends f .*

4. Index Theorem for Quarter-Plane Toeplitz Operators via Extended Symbols

In this section, we give a formula to express family indices for (self-adjoint) Fredholm quarter-plane Toeplitz operators of two-variable rational matrix function symbols by using extended symbols obtained through the matrix factorizations in Sect. 3. The main theorem in this section is Theorem 4.1 which is formulated by using complex K -theory and we start from the preliminaries of K -theory.

Let $n_0 = 2$ and $n_1 = 1$. Let $M_n^{(0)}(\mathbb{C}) = M_{2n}(\mathbb{C})^{\text{sa}}$, that is, the set of $2n$ -by- $2n$ hermitian matrices, and $M_n^{(1)}(\mathbb{C}) = M_n(\mathbb{C})$. For $i = 0, 1$, let $GL_n^{(i)}(\mathbb{C})$ be the subspace of $M_n^{(i)}(\mathbb{C})$ consisting of invertible matrices, therefore, $GL_n^{(0)}(\mathbb{C}) = GL_{2n}(\mathbb{C})^{\text{sa}}$ and $GL_n^{(1)}(\mathbb{C}) = GL_n(\mathbb{C})$. Let $I^{(0)} = \text{diag}(1, -1)$ and $I^{(1)} = 1$. We write $I_m^{(i)}$ for the $n_i m$ -by- $n_i m$ diagonal matrix $\text{diag}(I^{(i)}, \dots, I^{(i)})$. For a unital C^* -algebra A , let $GL_n^{(0)}(A)$ be the set of self-adjoint invertible elements in $M_{2n}(A)$, and let $GL_n^{(1)}(A)$ be the set of invertible elements in $M_n(A)$. For $i = 0, 1$, let $U_n^{(i)}(A)$ be the subspace of $GL_n^{(i)}(A)$ consisting of unitary elements. The K -group $K_i(A)$ for $i = 0, 1$ is defined as $K_i(A) = \bigcup_{n=1}^{\infty} GL_n^{(i)}(A) / \sim_i$, where the equivalence relation \sim_i is generated by homotopy and stabilization by $I^{(i)}$. For a compact Hausdorff space X , complex topological K -groups are defined as $K^{-i}(X) = K_i(C(X))$. For a locally compact Hausdorff space Y , we denote $K_{\text{cpt}}^{-i}(Y)$ for the compactly supported K -group of Y . For these K -groups, we

have Bott periodicity isomorphisms $K_{\text{cpt}}^{-i}(Y) \cong K_{\text{cpt}}^{-i}(\mathbb{R}^2 \times Y)$, which is given by the composite of the following maps:

$$K_{\text{cpt}}^0(\mathbb{R}^2 \times Y) \cong K_{\text{cpt}}^{-1}(\mathbb{R} \times Y) \hookrightarrow K_{\text{cpt}}^{-1}(\mathbb{T} \times Y) \xrightarrow{\partial^T} K_{\text{cpt}}^0(Y),$$

where, the first isomorphism is the suspension isomorphism, the second inclusion is given by identifying \mathbb{T} with the one-point compactification of \mathbb{R} and the third map ∂^T is the boundary map associated with the Toeplitz extension (1). That for other degrees follow by using suspension isomorphisms. Therefore, the push-forward map through Toeplitz operators provides the Bott periodicity isomorphism (see [6, Section 7]). In this paper, we reduce the problem to a study of families of Toeplitz operators and use its relation with the Bott periodicity in the discussion. For the basics of K -theory used in this paper, we refer the reader to [5, 6, 8, 12, 28, 40].

Let $i = 0$ or 1 , and let X be a compact Hausdorff space. Let $f : \mathbb{T}^2 \times X \rightarrow GL_N^{(i)}(\mathbb{C})$ be a continuous map, such that for each $x \in X$, $f(x)$ is a two-variable rational matrix function, and the associated quarter-plane Toeplitz operator $T_{f(x)}^{0,\infty}$ is Fredholm. When $i = 1$, $\{T_{f(x)}^{0,\infty}\}_{x \in X}$ is a family of Fredholm operators which defines an element of the even complex K -group $K^0(X)$. When $i = 0$, $\{T_{f(x)}^{0,\infty}\}_{x \in X}$ is a family of self-adjoint Fredholm operators. In this case, we may stabilize f if necessary (i.e., take a direct sum with $I^{(0)}$) and assume that the essential spectrum of $T_{f(x)}^{0,\infty}$ is not contained in the set of positive real numbers $\mathbb{R}_{>0}$ or the set of negative real numbers $\mathbb{R}_{<0}$ for any $x \in X$. Under this assumption, $\{T_{f(x)}^{0,\infty}\}_{x \in X}$ defines an element of the odd complex K -group $K^1(X)$. In both of these cases, we write $[T_f^{0,\infty}]$ for the K -class of the family of (self-adjoint) Fredholm¹ quarter-plane Toeplitz operators. By using matrix factorization, there is an associated continuous extension f^E of f onto $\tilde{\mathbb{S}}^3 \times X$ that takes values in hermitian invertible matrices (when $i = 0$; see Proposition 3.6) or invertible matrices (when $i = 1$; see Proposition 3.4). This matrix function f^E defines an element $[f^E]$ of the K -group $K^0(\tilde{\mathbb{S}}^3 \times X)$ when $i = 0$, or $K^{-1}(\tilde{\mathbb{S}}^3 \times X)$ when $i = 1$. Let us consider the following isomorphism,

$$K^{-i}(\tilde{\mathbb{S}}^3 \times X) \cong K^{-i}(X) \oplus K_{\text{cpt}}^{-i}(\mathbb{R}^3 \times X) \cong K^{-i}(X) \oplus K^{-i-3}(X). \tag{6}$$

We write $\beta : K^{-i}(\tilde{\mathbb{S}}^3 \times X) \rightarrow K^{-i+1}(X)$ for the composite of the projection $K^{-i}(\tilde{\mathbb{S}}^3 \times X) \rightarrow K^{-i-3}(X)$ through the above decomposition and the square of the Bott periodicity isomorphism $K^{-i-3}(X) \cong K^{-i+1}(X)$. The following is the main theorem in this paper.

Theorem 4.1. *Let X be a compact Hausdorff space. Let $f : \mathbb{T}^2 \times X \rightarrow GL_N^{(i)}(\mathbb{C})$ be a continuous map such that for each $x \in X$, $f(x)$ is a two-variable rational matrix function, and the associated quarter-plane Toeplitz operator $T_{f(x)}^{0,\infty}$ is Fredholm. Let $f^E : \tilde{\mathbb{S}}^3 \times X \rightarrow GL_N^{(i)}(\mathbb{C})$ be the extension of f through matrix factorization in the hermitian case of Proposition 3.6 (when $i = 0$) or Proposition 3.4 (when $i = 1$). Then, $[T_f^{0,\infty}] = \beta([f^E])$ in the K -group $K^{-i+1}(X)$.*

¹ Self-adjoint Fredholm operators when $i = 0$ and Fredholm operators when $i = 1$. In this section, the term (self-adjoint) should be read when $i = 0$.

For the f in Theorem 4.1, the associated family of half-plane Toeplitz operators $\{T_{f(x)}^0\}_{x \in X}$ and $\{T_{f(x)}^\infty\}_{x \in X}$ are (self-adjoint) invertible and define an element $[(T_f^0, T_f^\infty)]$ of the K -group $K_i(\mathcal{S}^{0,\infty} \otimes C(X))$. Let

$$\partial^{qT}: K_i(\mathcal{S}^{0,\infty} \otimes C(X)) \rightarrow K_{i-1}(C(X)) \cong K^{-i+1}(X)$$

be the boundary map of the six-term exact sequence for K -theory of C^* -algebras associated with the following extension obtained by taking a tensor product of the sequence (2) with $C(X)$,

$$0 \rightarrow \mathcal{K} \otimes C(X) \rightarrow \mathcal{T}^{0,\infty} \otimes C(X) \xrightarrow{\gamma \otimes 1} \mathcal{S}^{0,\infty} \otimes C(X) \rightarrow 0. \tag{7}$$

Note that $\partial^{qT}([(T_f^0, T_f^\infty)]) = [T_f^{0,\infty}]$, since $T_f^{0,\infty}$ is a (self-adjoint) lift of (T_f^0, T_f^∞) (see [12, 40], Theorem 3 of [26], for example). Let us consider the following diagram:

$$\begin{array}{ccc} K^{-i}(\mathbb{T} \times X) \oplus K^{-i}(\mathbb{T} \times X) & & \\ \partial^{\text{pair}} \downarrow & \searrow \partial^T \oplus -\partial^T & \\ K^{-i+1}([-2, 2] \times \mathbb{T} \times X, \{\pm 2\} \times \mathbb{T} \times X) & \xrightarrow{\alpha} & K^{-i+1}(X) \end{array} \tag{8}$$

where ∂^T is the boundary map associated with the Toeplitz extension (1) and ∂^{pair} is the boundary map of the six-term exact sequence of topological K -theory associated with the pair $([-2, 2] \times \mathbb{T} \times X, \{\pm 2\} \times \mathbb{T} \times X)$. Maps $\partial^T \oplus -\partial^T$ and ∂^{pair} are surjective, and the kernel of ∂^{pair} is the diagonals that are contained in the kernel of $\partial^T \oplus -\partial^T$. Therefore, the horizontal map α making the diagram commutative is induced. To show Theorem 4.1, we will construct the following key diagram:

$$\begin{array}{ccc} K_i(\mathcal{S}^{0,\infty} \otimes C(X)) & & \\ \psi \downarrow & \searrow \partial^{qT} & \\ K^{-i+1}([-2, 2] \times \mathbb{T} \times X, \{\pm 2\} \times \mathbb{T} \times X) / \text{Ker}(\alpha) & \xrightarrow{\bar{\alpha}} & K^{-i+1}(X) \\ \phi \uparrow & \nearrow \beta & \\ K^{-i}(\tilde{\mathbb{S}}^3 \times X) & & \end{array} \tag{9}$$

The map α is surjective and the induced homomorphism $\bar{\alpha}$ in the above diagram is an isomorphism. Note that there is the following isomorphism,

$$K^{-i+1}([-2, 2] \times \mathbb{T} \times X, \{\pm 2\} \times \mathbb{T} \times X) \cong K^{-i}(\mathbb{T} \times X) \cong K^{-i}(X) \oplus K^{-i-1}(X).$$

Through this decomposition, $\text{Ker}(\alpha) \cong K^{-i}(X)$ and we have the isomorphism,

$$K^{-i+1}([-2, 2] \times \mathbb{T} \times X, \{\pm 2\} \times \mathbb{T} \times X) / \text{Ker}(\alpha) \cong K^{-i-1}(X). \tag{10}$$

Through this isomorphism, the map $\bar{\alpha}$ corresponds to the Bott periodicity isomorphism $K^{-i-1}(X) \cong K^{-i+1}(X)$.

4.1. *Construction of the map ψ .* In this subsection, we construct the homomorphism ψ in the diagram (9). For its construction, Coburn–Douglas–Singer’s idea to use homotopy lifting property for some fibrations plays a key role, and we introduce them first.

Let $U_n(\mathcal{T})$ and $U_n(C(\mathbb{T}))$ be subspaces of $M_n(\mathcal{T})$ and $M_n(C(\mathbb{T}))$, respectively, consisting of unitary elements, let $U_n(\mathcal{T})^{\text{sa}}$ and $U_n(C(\mathbb{T}))^{\text{sa}}$ be subspaces of self-adjoint unitaries. The map σ in (1) induces the following maps, which we also denote as σ :

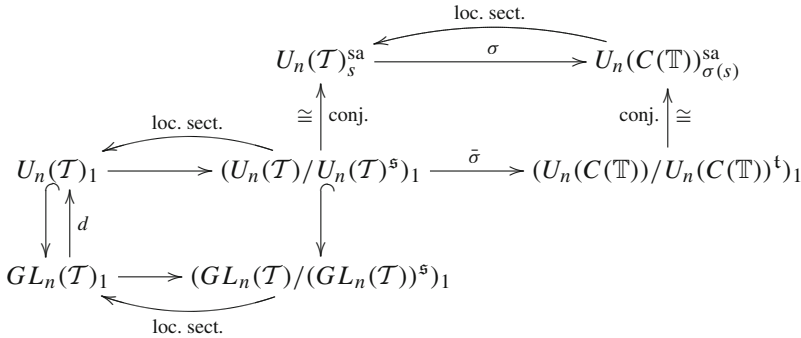
$$\sigma : U_n(\mathcal{T}) \rightarrow U_n(C(\mathbb{T})). \tag{11}$$

$$\sigma : U_n(\mathcal{T})^{\text{sa}} \rightarrow U_n(C(\mathbb{T}))^{\text{sa}}. \tag{12}$$

The map (11) is a Hurewicz fibration, which is used in [15]. We also have its variants (Lemma 4.2 and Proposition 5.5) which will be well-known [8, 50], though we briefly contain its proof since, as in [15], they will play a key role in our discussion. The proof of Lemma 4.2 is simply an application of discussions around Proposition 4.1 of [8] to the Toeplitz extension. For a space Y and its element y , we write Y_y for the connected component of Y containing y . For a Banach algebra A , its subset $S \subset A$ and an \mathbb{R} -linear operator \mathfrak{s} on A of order two, we write $S^\mathfrak{s}$ and $S^{-\mathfrak{s}}$ for the subsets of S that are pointwise fixed by \mathfrak{s} and $-\mathfrak{s}$, respectively.

Lemma 4.2. *The map (12) is a Hurewicz fibration.*

Proof. Since $U_n(C(\mathbb{T}))^{\text{sa}}$ is a paracompact Hausdorff space, it is sufficient to show that (12) is a fiber bundle [44, Corollary 14 in p.96]. Let u be a self-adjoint unitary in $M_n(C(\mathbb{T}))$. There exists a self-adjoint invertible lift $x \in M_n(\mathcal{T})$ of u since a self-adjoint Fredholm operator can be perturbed to a self-adjoint invertible operator by a self-adjoint compact operator. Then, the self-adjoint unitary $x|x|^{-1} \in M_n(\mathcal{T})$ satisfies $\sigma(x|x|^{-1}) = u$, and the map $\sigma : U_n(\mathcal{T})^{\text{sa}} \rightarrow U_n(C(\mathbb{T}))^{\text{sa}}$ is surjective. It is now sufficient to show that, for any $s \in U_n(\mathcal{T})^{\text{sa}}$, the map $\sigma : U_n(\mathcal{T})_s^{\text{sa}} \rightarrow U_n(C(\mathbb{T}))_{\sigma(s)}^{\text{sa}}$ is a fiber bundle. Note that the map $\sigma : M_n(\mathcal{T}) \rightarrow M_n(C(\mathbb{T}))$ has a continuous linear section given by compression, that is, mapping f to T_f . Let $k = s - T_{\sigma(s)}$ which is a self-adjoint compact operator, and let $l : M_n(C(\mathbb{T})) \rightarrow M_n(\mathcal{T})$ be a map given by $l(f) = T_f + k$. This map l preserves self-adjoint elements and satisfies $l(\sigma(s)) = s$. Combined with the map $d : GL_n(\mathcal{T}) \rightarrow U_n(\mathcal{T})$, $d(T) = T|T|^{-1}$, we obtain a continuous local section of the map $\sigma : U_n(\mathcal{T})_s^{\text{sa}} \rightarrow U_n(C(\mathbb{T}))_{\sigma(s)}^{\text{sa}}$ in a neighborhood of $\sigma(s)$ that maps $\sigma(s)$ to s . Let \mathfrak{s} be an operator on $M_n(\mathcal{T})$ given by the conjugation of s . Then, as in [50], the quotient map $GL_n(\mathcal{T})_1 \rightarrow (GL_n(\mathcal{T})/(GL_n(\mathcal{T}))^\mathfrak{s})_1$ has a continuous local section which follows from Corollary 7.3 of [33]. Combined with the map d , we obtain a continuous local section of the quotient map $U_n(\mathcal{T})_1 \rightarrow (U_n(\mathcal{T})/U_n(\mathcal{T})^\mathfrak{s})_1$. Let us consider the map $U_n(\mathcal{T}) \rightarrow U_n(\mathcal{T})^{\text{sa}}$ given by $T \mapsto TsT^*$, that comes from the action of unitaries to self-adjoint unitaries by conjugation. Its stabilizer subgroup at s is $U_n(\mathcal{T})^\mathfrak{s}$. As in Lemma 4.1 of [50], each orbit of this action is open and induces a homeomorphism $(U_n(\mathcal{T})/U_n(\mathcal{T})^\mathfrak{s})_1 \xrightarrow{\cong} U_n(\mathcal{T})_s^{\text{sa}}$. Let t be an operator on $M_n(C(\mathbb{T}))$ given by the conjugation of $\sigma(s)$. We consider similar discussion for the algebra $C(\mathbb{T})$ and obtain the following diagram,



where $\bar{\sigma}$ is the map induced by σ . Therefore, $U_n(\mathcal{T})_1 \rightarrow (U_n(C(\mathbb{T}))/U_n(C(\mathbb{T}))^t)_1$ has a continuous local section and the result follows (see Sect. 7.4 of [45]). \square

Let $i = 0$ or 1 , and let (T^0, T^∞) be an element in $GL_N^{(i)}(\mathcal{S}^{0,\infty} \otimes C(X))$ for some $N \in \mathbb{N}$, and let $[(T^0, T^\infty)] \in K_i(\mathcal{S}^{0,\infty} \otimes C(X))$ be its K -class. Let $f = (\sigma^0 \otimes 1_{C(X)})(T^0) = (\sigma^\infty \otimes 1_{C(X)})(T^\infty)$, which is an element in $C(\mathbb{T}^2 \times X, GL_N^{(i)}(\mathbb{C}))$. Let $\pi : \mathbb{T}^2 \times X \rightarrow X$ be the projection.

Lemma 4.3. *For a sufficiently large integer $N' \geq N$, there exists a continuous map $g : X \rightarrow GL_{N'}^{(i)}(\mathbb{C})$ such that $f \oplus I_{N'-N}^{(i)}$ is homotopic to π^*g in $C(\mathbb{T}^2 \times X, GL_{N'}^{(i)}(\mathbb{C}))$. We take a path $\{g_t\}_{0 \leq t \leq 1}$ from $g_0 = f \oplus I_{N'-N}^{(i)}$ to $g_1 = \pi^*g$ in $C(\mathbb{T}^2 \times X, GL_{N'}^{(i)}(\mathbb{C}))$. Then, there also exists a path $\{(T_t^0, T_t^\infty)\}_{0 \leq t \leq 1}$ in $GL_{N'}^{(i)}(\mathcal{S}^{0,\infty} \otimes C(X))$ satisfying the following conditions:*

- (i) $(T_0^0, T_0^\infty) = (T^0 \oplus I_{N'-N}^{(i)}, T^\infty \oplus I_{N'-N}^{(i)})$.
- (ii) $(\sigma^0 \otimes 1_{C(X)})(T_t^0) = (\sigma^\infty \otimes 1_{C(X)})(T_t^\infty) = g_t$ for $0 \leq t \leq 1$.
- (iii) There exist $K \in \mathbb{N}$ and a continuous map $t^0 : \mathbb{T} \times X \rightarrow GL_{KN'}^{(i)}(\mathbb{C})$ such that for $(z, x) \in \mathbb{T} \times X$, $T_1^0(z, x) = t^0(z, x) \oplus g(x)$ as an operator on $l^2(\mathbb{Z}_{\geq 0}; \mathbb{C}^{n_i N'}) \cong \mathbb{C}^{n_i KN'} \oplus l^2(\mathbb{Z}_{\geq K}; \mathbb{C}^{n_i N'})$.
- (iv) There exist $L \in \mathbb{N}$ and a continuous map $t^\infty : \mathbb{T} \times X \rightarrow GL_{LN'}^{(i)}(\mathbb{C})$ such that for $(w, x) \in \mathbb{T} \times X$, $T_1^\infty(w, x) = t^\infty(w, x) \oplus g(x)$ as an operator on $l^2(\mathbb{Z}_{\geq 0}; \mathbb{C}^{n_i N'}) \cong \mathbb{C}^{n_i LN'} \oplus l^2(\mathbb{Z}_{\geq L}; \mathbb{C}^{n_i N'})$.

In (iii), we consider $T_1^0 \in GL_{N'}^{(i)}(T^0 \otimes C(X)) \cong GL_{N'}^{(i)}(C(\mathbb{T}) \otimes T \otimes C(X))$ as a family of Toeplitz operators parametrized by $\mathbb{T} \times X$, and similarly for (iv).

Proof. We first take the deformation retraction of invertible elements T^0 and T^∞ to unitaries; that is, for $0 \leq t \leq \frac{1}{3}$ and $j \in \{0, \infty\}$, let $Q_t^j = (1 - 3t)T^j + 3tT^j|T^j|^{-1}$ and $f_t = (\sigma^0 \otimes 1_{C(X)})(Q_t^0) = (\sigma^\infty \otimes 1_{C(X)})(Q_t^\infty)$. We next consider the following two homomorphisms,

$$\begin{aligned}
 (\sigma_S \otimes 1_{C(X)})_* : K_i(\mathcal{S}^{0,\infty} \otimes C(X)) &\rightarrow K_i(C(\mathbb{T}^2 \times X)) \cong K^{-i}(\mathbb{T}^2 \times X), \\
 \pi^* : K^{-i}(X) &\rightarrow K^{-i}(\mathbb{T}^2 \times X).
 \end{aligned}
 \tag{13}$$

Note that $K_i(\mathcal{S}^{0,\infty}) = \mathbb{Z}$ for $i = 0, 1$, and that the map $(\sigma_{\mathcal{S}} \otimes 1_{C(X)})_*$ is computed by using the Künneth theorem [36,42]. The images of the above two maps coincide. Since

$$[f] = (\sigma_{\mathcal{S}} \otimes 1_{C(X)})_*([T^0, T^\infty]) \in \text{Im}((\sigma_{\mathcal{S}} \otimes 1_{C(X)})_*) = \text{Im}(\pi^*),$$

there exist $g: X \rightarrow U_{N'}^{(i)}(\mathbb{C})$ for some N' satisfying $[f] = [f_{\frac{1}{3}}] = \pi^*[g]$ in $K^{-i}(\mathbb{T}^2 \times X)$. Note that π^* is injective, and the K -class $[g]$ of g is uniquely determined. Since the equivalence relation used to define the K -group $K^{-i}(\mathbb{T}^2 \times X)$ is generated by homotopy and stabilization, if we take N' to be sufficiently large, there also exists a homotopy in $C(\mathbb{T}^2 \times X, U_{N'}^{(i)}(\mathbb{C}))$ from $f_{\frac{1}{3}} \oplus I_{N'-N}^{(i)}$ to π^*g . Let $T_t^0 = Q_t^0 \oplus I_{N'-N}^{(i)}$, $T_t^\infty = Q_t^\infty \oplus I_{N'-N}^{(i)}$ and $g_t = f_t \oplus I_{N'-N}^{(i)}$ for $0 \leq t \leq \frac{1}{3}$, and for $\frac{1}{3} \leq t \leq \frac{2}{3}$, let g_t be this homotopy from $f_{\frac{1}{3}} \oplus I_{N'-N}^{(i)}$ to π^*g . By the homotopy lifting property of fibration (11), there exists a homotopy $T_t^0: \mathbb{T} \times X \rightarrow U_{N'}^{(i)}(\mathcal{T})$ for $\frac{1}{3} \leq t \leq \frac{2}{3}$ that lifts g_t starting from $T_{\frac{1}{3}}^0 = Q_{\frac{1}{3}}^0 \oplus I_{N'-N}^{(i)}$. Note that $\sigma^0(T_{\frac{2}{3}}^0(x))(z) = g_{\frac{2}{3}}(z, x) = g(x)$ is independent of $z \in \mathbb{T}$. By a similar discussion for T^∞ , we obtain a homotopy $T_t^\infty: \mathbb{T} \times X \rightarrow U_{N'}^{(i)}(\mathcal{T})$ for $\frac{1}{3} \leq t \leq \frac{2}{3}$ starting from $T_{\frac{1}{3}}^\infty = Q_{\frac{1}{3}}^\infty \oplus I_{N'-N}^{(i)}$ and lifts g_t . Since $\sigma^0(T_t^0(x)) = \sigma^\infty(T_t^\infty(x))$ and $\sigma^0(T_1^0(x))(z) = \sigma^\infty(T_1^\infty(x))(w) = g(x)$, we obtain the following:

- $(T_t^0, T_t^\infty) \in GL_{N'}^{(i)}(\mathcal{S}^{0,\infty} \otimes C(X))$ for $0 \leq t \leq \frac{2}{3}$,
- $T_{\frac{2}{3}}^0(z, x) = g(x) + k^0(z, x)$, where $k^0(z, x)$ is a compact operator,
- $T_{\frac{2}{3}}^\infty(w, x) = g(x) + k^\infty(w, x)$, where $k^\infty(w, x)$ is a compact operator.

For $\frac{2}{3} \leq t \leq 1$, let $g_t = g_{\frac{2}{3}}$. We perturb k^0 and k^∞ to families of (self-adjoint) finite rank operators to obtain homotopies T_t^0 and T_t^∞ for $\frac{2}{3} \leq t \leq 1$ in the space of (self-adjoint) invertible half-plane Toeplitz operators and obtain the desired result. □

For simplicity, in what follows, we assume that our representative (T^0, T^∞) of the K -class $[(T^0, T^\infty)]$ in $K_i(\mathcal{S}^{0,\infty} \otimes C(X))$ is sufficiently stabilized to take $N = N'$ in Lemma 4.3. Under this assumption, we take a continuous map g , a homotopy $\{g_t\}_{0 \leq t \leq 1}$ from f to π^*g in $C(\mathbb{T}^2 \times X, GL_N^{(i)}(\mathbb{C}))$, a homotopy $\{(T_t^0, T_t^\infty)\}_{0 \leq t \leq 1}$ and continuous maps t^0 and t^∞ that satisfy the conditions in Lemma 4.3. t^0 and t^∞ provide elements $[t^0]$ and $[t^\infty]$ in $K^{-i}(\mathbb{T} \times X)$. Let $r: \mathbb{Z} \times \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0} \times \mathbb{Z}$ be the map defined by $r(m, n) = (n, -m)$ and $r^*: l^2(\mathbb{Z}_{\geq 0} \times \mathbb{Z}, \mathbb{C}^{n_i N}) \rightarrow l^2(\mathbb{Z} \times \mathbb{Z}_{\geq 0}, \mathbb{C}^{n_i N})$ be the induced Hilbert space isomorphism. For an operator $T \in M_{n_i N}(T^\infty)$, we write

$$\tilde{T} = r^* \circ T \circ (r^*)^{-1} \in M_{n_i N}(T^0). \tag{14}$$

By taking this transformation for $T_1^\infty(w, x) = t^\infty(w, x) \oplus g(x)$ in Lemma 4.3, we obtain $\tilde{T}_1^\infty(z, x) = \tilde{t}^\infty(z, x) \oplus g(x)$ where $\tilde{t}^\infty: \mathbb{T} \times X \rightarrow GL_{L_{N'}}^{(i)}(\mathbb{C})$. Then, we have $\partial^T([\tilde{t}^\infty]) = -\partial^T([t^\infty])$ in $K^{-i+1}(X)$ since the orientation of \mathbb{T} is reversed through (14). For these elements, the following holds [15]:

Proposition 4.4. $\partial^{\text{qT}}([(T^0, T^\infty)]) = (\partial^T \oplus -\partial^T)([t^0], [\tilde{t}^\infty])$.

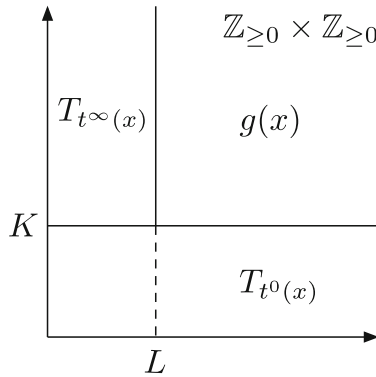


Fig. 1. Schematic picture for the quarter-plane Toeplitz operator $A(x)$

Proof. We decompose the Hilbert space $l^2(\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}, \mathbb{C}^{n_i N})$ into a direct sum of three closed subspaces,

$$l^2(\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}, \mathbb{C}^{n_i N}) = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3,$$

where $\mathcal{H}_1 = l^2(\mathbb{Z}_{\geq 0} \times \{0, \dots, K - 1\}, \mathbb{C}^{n_i N})$, $\mathcal{H}_2 = l^2(\{0, \dots, L - 1\} \times \mathbb{Z}_{\geq K}, \mathbb{C}^{n_i N})$ and $\mathcal{H}_3 = l^2(\mathbb{Z}_{\geq K} \times \mathbb{Z}_{\geq L}, \mathbb{C}^{n_i N})$. We consider the following family of quarter-plane Toeplitz operators:

$$A(x) = \begin{cases} T_{t^0(x)} & \text{on } \mathcal{H}_1 \cong l^2(\mathbb{Z}_{\geq 0}, \mathbb{C}^{n_i KN}), \\ T_{t^\infty(x)} & \text{on } \mathcal{H}_2 \cong l^2(\mathbb{Z}_{\geq 0}, \mathbb{C}^{n_i LN}), \\ g(x) & \text{on } \mathcal{H}_3, \end{cases}$$

where $x \in X$ (see Fig. 1). A is a family of (self-adjoint) Fredholm operators that satisfies $\gamma(A) = (T_1^0, T_1^\infty)$. Since $g(x)$ on \mathcal{H}_3 is (self-adjoint) invertible, the following equality holds in the K -group $K^{-i+1}(X)$.

$$\begin{aligned} (\partial^T \oplus -\partial^T)([t^0], [\tilde{t}^\infty]) &= \partial^T([t^0]) + \partial^T([\tilde{t}^\infty]) = [T_{t^0}] + [T_{t^\infty}] = [T_{t^0}] + [T_{t^\infty}] + [g] \\ &= [A] = \partial^{\text{qT}}([(T_1^0, T_1^\infty)]) = \partial^{\text{qT}}([(T^0, T^\infty)]). \end{aligned}$$

□

Let $\{a_t\}_{0 \leq t \leq 1}$ (resp. $\{b_t\}_{0 \leq t \leq 1}$) be a path from t^0 (resp. \tilde{t}^∞) to $\overbrace{g \oplus \dots \oplus g}^{K \text{ (resp. } L)}$ in $C(\mathbb{T} \times X, M_{n_i KN}^{(i)}(\mathbb{C}))$ (resp. $C(\mathbb{T} \times X, M_{n_i LN}^{(i)}(\mathbb{C}))$). These paths exist since $M_n^{(i)}(\mathbb{C})$ is contractible. We now consider the family of Toeplitz operators $F: [-2, 2] \times \mathbb{T} \times X \rightarrow M_{n_i N}(T)$ defined by

$$F(s, z, x) = \begin{cases} T_{s+2}^0(z, x), & -2 \leq s \leq -1, \\ a_{s+1}(z, x) \oplus g(x), & -1 \leq s \leq 0, \\ b_{1-s}(z, x) \oplus g(x), & 0 \leq s \leq 1, \\ \tilde{T}_{2-s}^\infty(z, x), & 1 \leq s \leq 2. \end{cases}$$

Note that $F(0, z, x) = g(x)$ is (self-adjoint) invertible and independent of $z \in \mathbb{T}$. F is a family of (self-adjoint) Fredholm operators on $[-2, 2] \times \mathbb{T} \times X$, which is invertible on

$([-2, -1] \sqcup [1, 2]) \times \mathbb{T} \times X$, therefore, it defines the following element of the relative K -group,

$$[F] \in K^{-i+1}([-2, 2] \times \mathbb{T} \times X, ([-2, -1] \sqcup [1, 2]) \times \mathbb{T} \times X).$$

We write

$$[[F]] \in K^{-i+1}([-2, 2] \times \mathbb{T} \times X, ([-2, -1] \sqcup [1, 2]) \times \mathbb{T} \times X) / \text{Ker}(\alpha)$$

for the class of $[F]$ in the quotient group, and we define

$$\psi: K_i(S^{0,\infty} \otimes C(X)) \rightarrow K^{-i+1}([-2, 2] \times \mathbb{T} \times X, ([-2, -1] \sqcup [1, 2]) \times \mathbb{T} \times X) / \text{Ker}(\alpha)$$

by $\psi([(T^0, T^\infty)]) = [[F]]$.

Proposition 4.5. *ψ is a well-defined group homomorphism.*

Proof. We show that the element $[[F]]$ depends only on the K -class $[(T^0, T^\infty)]$ in $K_i(S^{0,\infty} \otimes C(X))$. By the excision property of topological K -theory, we can remove components of F corresponding to subspaces $[-2, -1)$ and $(1, 2]$ of $[-2, 2]$. That is, the inclusion $i: [-1, 1] \hookrightarrow [-2, 2]$ induces an isomorphism

$$\begin{aligned} (i \times \text{id}_{\mathbb{T} \times X})^*: K^{-i+1}([-2, 2] \times \mathbb{T} \times X, ([-2, -1] \sqcup [1, 2]) \times \mathbb{T} \times X) \\ \xrightarrow{\cong} K^{-i+1}([-1, 1] \times \mathbb{T} \times X, \{\pm 1\} \times \mathbb{T} \times X), \end{aligned}$$

and satisfies $(i \times \text{id}_{\mathbb{T} \times X})^*[F] = [F|_{[-1,1] \times \mathbb{T} \times X}]$. We identify these two K -groups by this map. For simplicity of description, we take K and L in Lemma 4.3 sufficiently large and assume $K = L$. Let us consider the map $\tilde{F}: [-1, 1] \times \mathbb{T} \times X \rightarrow M_{n_i KN}^{(i)}(\mathbb{C})$ given by

$$\tilde{F}(s, z, x) = \begin{cases} a_{s+1}(z, x), & -1 \leq s \leq 0, \\ b_{1-s}(z, x), & 0 \leq s \leq 1. \end{cases}$$

Then, $F|_{[-1,1] \times \mathbb{T} \times X} = \tilde{F} \oplus g$, and $[F|_{[-1,1] \times \mathbb{T} \times X}] = [\tilde{F} \oplus g] = [\tilde{F} \oplus I_N^{(i)}]$ in the group $K^{-i+1}([-1, 1], \{\pm 1\}) \times \mathbb{T} \times X$ by Kuiper's theorem. Since $\tilde{F}(s = -1) = t^0$ and $\tilde{F}(s = 1) = \tilde{t}^\infty$, \tilde{F} is a (self-adjoint) lift of $t^0 \sqcup \tilde{t}^\infty: \{\pm 1\} \times \mathbb{T} \times X \rightarrow GL_{KN}^{(i)}(\mathbb{C})$ through the restriction map,

$$\text{res}: M_{n_i KN}(C([-1, 1] \times \mathbb{T} \times X)) \rightarrow M_{n_i KN}(C(\{\pm 1\} \times \mathbb{T} \times X)).$$

Therefore, we have $\partial^{\text{pair}}([t^0], [\tilde{t}^\infty]) = [F|_{[-1,1] \times \mathbb{T} \times X}]$. By the definition of the map $\bar{\alpha}$ and Proposition 4.4, we have

$$\bar{\alpha}([[F]]) = (\partial^T \oplus -\partial^T)([t^0], [\tilde{t}^\infty]) = \partial^{\text{qT}}([(T^0, T^\infty)]).$$

Since $\partial^{\text{qT}}([(T^0, T^\infty)])$ depends only on the K -class $[(T^0, T^\infty)]$ and $\bar{\alpha}$ is an isomorphism, the element $[[F]]$ is independent of the choice made to construct F , and depends only on the K -class $[(T^0, T^\infty)]$. \square

4.2. *Construction of the map ϕ .* In this subsection, we construct the following group homomorphism ϕ in the diagram (9).

Let $f: \tilde{\mathbb{S}}^3 \times X \rightarrow GL_N^{(i)}(\mathbb{C})$ be a continuous map that represents $[f] \in K^{-i}(\tilde{\mathbb{S}}^3 \times X)$, and let $f_r = f|_{\mathbb{T}^2 \times X}$. The inclusion $\mathbb{T}^2 \times X \hookrightarrow \tilde{\mathbb{S}}^3 \times X$ induces the homomorphism $K^{-i}(\tilde{\mathbb{S}}^3 \times X) \rightarrow K^{-i}(\mathbb{T}^2 \times X)$, and the image of this map is the same as that of π^* in (13). Therefore, as in the proof of Lemma 4.3, there exists $N' \geq N$, a continuous map $g: X \rightarrow GL_{N'}^{(i)}(\mathbb{C})$ and a homotopy $\{g_t\}_{0 \leq t \leq 1}$ from $g_0 = f_r \oplus I_{N'-N}^{(i)}$ to $g_1 = \pi^*g$ in $C(\mathbb{T}^2 \times X, GL_{N'}^{(i)}(\mathbb{C}))$. As in Sect. 4.1, we assume that the representative f of the K -class $[f]$ is sufficiently stabilized to take $N' = N$ for simplicity.

Let $R: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be an orientation-preserving homeomorphism given by $R(z, w) = (w, \bar{z})$. Corresponding to the notation (14) for half-plane Toeplitz operators, we write $\tilde{k} = (R \times \text{id}_X)^*k$ for a matrix-valued continuous function k on $\mathbb{T}^2 \times X$; that is $\tilde{k}(z, w, x) = k(w, \bar{z}, x)$. By using this notation, let $G: [-2, 2] \times \mathbb{T}^2 \times X \rightarrow GL_N^{(i)}(\mathbb{C})$ be a continuous map defined as follows:

$$G(s, z, w, x) = \begin{cases} f(z, (2+s)w, x), & -2 \leq s \leq -1, \\ g_{1+s}(z, w, x), & -1 \leq s \leq 0, \\ \tilde{g}_{1-s}(z, w, x) = g_{1-s}(w, \bar{z}, x), & 0 \leq s \leq 1, \\ \tilde{f}(z, (2-s)w, x) = f((2-s)w, \bar{z}, x), & 1 \leq s \leq 2. \end{cases} \tag{15}$$

Since $g_1(z, w, x) = g(x)$ does not depend on z and w , this family G defines a continuous map on $[-2, 2] \times \mathbb{T}^2 \times X$. We consider a family of Toeplitz operators T_G by using second \mathbb{T} parametrized by w for the symbols of Toeplitz operators; that is, for $(s, z, x) \in [-2, 2] \times \mathbb{T} \times X$, we set

$$T_G(s, z, x) = T_{G(s,z,\cdot,x)}.$$

T_G is a family of Toeplitz operators of invertible symbols and therefore a family of Fredholm operators parametrized by $[-2, 2] \times \mathbb{T} \times X$. When $s = \pm 2$, these Toeplitz operators have constant invertible symbols and are invertible. This family T_G provides the element $[T_G]$ of the K -group $K^{-i+1}([-2, 2], \{\pm 2\}) \times \mathbb{T} \times X$. We define

$$\phi: K^{-i}(\tilde{\mathbb{S}}^3 \times X) \rightarrow K^{-i+1}([-2, 2] \times \mathbb{T} \times X, \{\pm 2\} \times \mathbb{T} \times X) / \text{Ker}(\alpha)$$

by $\phi([f]) = [[T_G]]$.

Proposition 4.6. *ϕ is a well-defined group homomorphism.*

Proof. We show that $[[T_G]]$ depends only on the K -class $[f]$ in $K^{-i}(\tilde{\mathbb{S}}^3 \times X)$. The proof is divided into three steps.

(i) We first fix f and g and show that $[[T_G]]$ is independent of the choice of homotopy g_t . For this purpose, we take another homotopy $\{h_t\}_{0 \leq t \leq 1}$ from $h_0 = f_r$ to $h_1 = \pi^*g$ in $C(\mathbb{T}^2 \times X, GL_N^{(i)}(\mathbb{C}))$. As in (15), we define $H: [-2, 2] \times \mathbb{T}^2 \times X \rightarrow GL_N^{(i)}(\mathbb{C})$ by using h_t in place of g_t , and we consider a family of Toeplitz operators T_H defined as in T_G . In what follows, we show that

$$[[T_G]] = [[T_H]] \in K^{-i+1}([-2, 2] \times \mathbb{T} \times X, \{\pm 2\} \times \mathbb{T} \times X) / \text{Ker}(\alpha). \tag{16}$$

Let $q: [-2, 2] \rightarrow [-2, 2]$ be the map given by $q(s) = -s$, and let $H' = (q \times \text{id}_{\mathbb{T}^2 \times X})^*(H)$. Note that $(q \times \text{id}_{\mathbb{T} \times X})^*(T_H) = T_{H'}$ and the isomorphism $(q \times \text{id}_{\mathbb{T} \times X})^*$ on

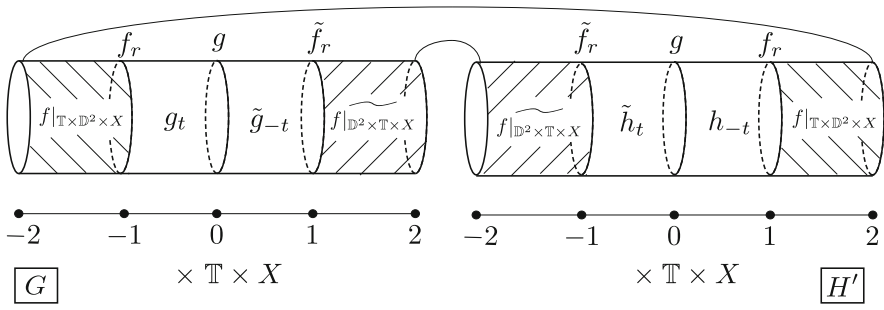


Fig. 2. Schematic picture for matrix functions G and H' . The signs of the subscripts of homotopies (e.g. g_t or h_{-t}) indicate that the orientation of the interval $[-2, 2]$ is consistent with that for each homotopy (e.g., g_t) or not (e.g., h_{-t})

$K^{-i+1}([-2, 2], \{\pm 2\}) \times \mathbb{T} \times X$ is a multiplication by -1 . Therefore, $[T_{H'}] = -[T_H]$ in $K^{-i+1}([-2, 2], \{\pm 2\}) \times \mathbb{T} \times X$. Let us consider the following map,

$$K^{-i+1}([-2, 2], \{\pm 2\}) \times \mathbb{T} \times X \times K^{-i+1}([-2, 2], \{\pm 2\}) \times \mathbb{T} \times X \xrightarrow{\mathcal{G}} K^{-i+1}(\mathbb{T}^2 \times X),$$

which is constructed as follows:

- For spaces, two endpoints of two intervals $[-2, 2]$ are connected (connect ± 2 with ∓ 2) to construct the circle \mathbb{T} .
- For two families of Fredholm operators that are invertible on $\{\pm 2\} \times \mathbb{T} \times X$, we connect invertible operators on two endpoints correspondingly by a continuous path of (self-adjoint) invertible families. This is possible and unique up to homotopy by Kuiper’s theorem.
- We then obtain a family of Fredholm operators on $\mathbb{T}^2 \times X$, which defines an element of $K^{-i+1}(\mathbb{T}^2 \times X)$.

Note that we have the following isomorphism:

$$K^{-i+1}(\mathbb{T}^2 \times X) \cong K^{-i+1}(\mathbb{T} \times X) \oplus K^{-i+1}([-2, 2] \times \mathbb{T} \times X, \{\pm 2\} \times \mathbb{T} \times X) \cong K^{-i+1}(X) \oplus K^{-i}(X)^{\oplus 2} \oplus K^{-i-1}(X). \tag{17}$$

Let $\iota: K^{-i+1}([-2, 2] \times \mathbb{T} \times X, \{\pm 2\} \times \mathbb{T} \times X) \hookrightarrow K^{-i+1}(\mathbb{T}^2 \times X)$ be the inclusion corresponding to the direct sum decomposition in (17). The map \mathcal{G} is the same as the composite of the addition of the group $K^{-i+1}([-2, 2], \{\pm 2\}) \times \mathbb{T} \times X$ followed by ι . Let us consider the projection $\varpi: K^{-i+1}(\mathbb{T}^2 \times X) \rightarrow K^{-i-1}(X)$ corresponding to the decomposition in (17). Up to the identification (10), $[[T_G]]$ and $\varpi \circ \iota([T_G])$ are equal. Therefore, in order to show (16), it is sufficient to show that

$$\varpi \circ \iota([T_G]) = \varpi \circ \iota([T_H]) \in K^{-i-1}(X). \tag{18}$$

Let us discuss the element $\mathcal{G}([T_G], [T_{H'}])$. The families of Toeplitz operators T_G and $T_{H'}$ have symbols G and H' , respectively, which are shown in Fig. 2. Since these two have the same matrix functions at the boundaries, we can glue G and H' as indicated in Fig. 2, and obtain a continuous map $\mathbb{T}^3 \times X \rightarrow GL_N^{(i)}(\mathbb{C})$ that defines an element of the

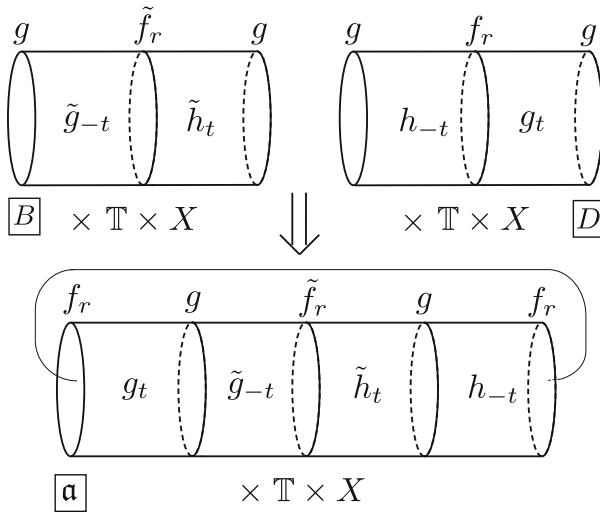


Fig. 3. Schematic picture for α , B and D

group $K^{-i}(\mathbb{T}^3 \times X)$. Let $\partial^T : K^{-i}(\mathbb{T}^3 \times X) \rightarrow K^{-i+1}(\mathbb{T}^2 \times X)$ be the push-forward map through Toeplitz operators with respect to the last variable of \mathbb{T}^3 . Then, this K -group element maps to $\mathcal{G}([T_G], [T_{H'}])$ through ∂^T . On the two pairs of dashed areas in Fig. 2, there are the same matrix functions with opposite parameter directions; therefore these dashed areas can be cancelled by a homotopy, and this K -class is the same as the K -class of the continuous map $\alpha : [-2, 2]/\{\pm 2\} \times \mathbb{T}^2 \times X \rightarrow GL_N^{(i)}(\mathbb{C})$ given by,

$$\alpha(s, z, w, x) = \begin{cases} g_{2+s}(z, w, x), & -2 \leq s \leq -1, \\ \tilde{g}_{-s}(z, w, x) = g_{-s}(w, \bar{z}, x), & -1 \leq s \leq 0, \\ \tilde{h}_s(z, w, x) = h_s(w, \bar{z}, x), & 0 \leq s \leq 1, \\ h_{2-s}(z, w, x), & 1 \leq s \leq 2, \end{cases}$$

as indicated in Fig. 3. In summary, we have

$$\partial^T([\alpha]) = [T_\alpha] = \mathcal{G}([T_G], [T_{H'}]) = \iota([T_G]) + \iota([T_{H'}]) = \iota([T_G]) - \iota([T_H]).$$

Let $B, D : [-1, 1] \times \mathbb{T}^2 \times X \rightarrow GL_N^{(i)}(\mathbb{C})$ be given as $B = \alpha|_{-1 \leq s \leq 1}$ and

$$D(s, z, w, x) = \begin{cases} h_{-s}(z, w, x), & -1 \leq s \leq 0, \\ g_s(z, w, x), & 0 \leq s \leq 1. \end{cases}$$

As shown in Fig. 3, B and D correspond to two pieces obtained by cut α along $s = \pm 1$. Both B and D have the same value at boundaries $s = \pm 1$ and reduce to the map on $[-1, 1]/\{\pm 1\} \times \mathbb{T}^2 \times X$, for which we write \mathfrak{b} and \mathfrak{d} , respectively.

$$\mathfrak{b}, \mathfrak{d} : [-1, 1]/\{\pm 1\} \times \mathbb{T} \times \mathbb{T} \times X \rightarrow GL_N^{(i)}(\mathbb{C}).$$

For B, D, \mathfrak{b} and \mathfrak{d} , we associate families of Toeplitz operators $T_B, T_D, T_{\mathfrak{b}}$ and $T_{\mathfrak{d}}$ by taking right \mathbb{T} for the symbols. These are families of (self-adjoint) Fredholm operators, and since

$B(s = \pm 1)$ and $D(s = \pm 1)$ are given by (the pull-back of) g , both $T_B(s = \pm 1)$ and $T_D(s = \pm 1)$ are invertible. Therefore, they define the following elements of K -groups:

$$\begin{aligned} [T_B], [T_D] &\in K^{-i+1}([-1, 1] \times \mathbb{T} \times X, \{\pm 1\} \times \mathbb{T} \times X), \\ [T_b], [T_{\bar{b}}] &\in K^{-i+1}([-1, 1]/\{\pm 1\} \times \mathbb{T} \times X). \end{aligned}$$

Since a is obtained by gluing B and D , we have

$$[T_a] = \mathcal{G}([T_B], [T_D]) = \iota([T_B]) + \iota([T_D]) = [T_b] + [T_{\bar{b}}].$$

Let us consider the homeomorphism $q \times R$ on $[-1, 1]/\{\pm 1\} \times \mathbb{T}^2$ given by $(q \times R)(s, z, w) = (-s, w, \bar{z})$, and consider the map $(q \times R \times \text{id}_X)^*$ on the K -group $K^{-i}(\mathbb{T}^3 \times X)$. Then $(q \times R \times \text{id}_X)^*[\bar{b}] = [b]$ holds. Furthermore, corresponding to the direct sum decomposition

$$K^{-i}(\mathbb{T}^3 \times X) \cong K^{-i}(X) \oplus K^{-i-1}(X)^{\oplus 3} \oplus K^{-i-2}(X)^{\oplus 3} \oplus K^{-i-3}(X), \tag{19}$$

the map $(q \times R \times \text{id}_X)^*$ acts on the direct summand $K^{-i-3}(X)$ as multiplication by -1 . By the map $\partial^{\mathbb{T}}$, this direct summand $K^{-i-3}(X)$ in (19) maps isomorphically to the direct summand $K^{-i-1}(X)$ in (17). Therefore, we have $\varpi([T_b]) = -\varpi([T_{\bar{b}}])$, which leads to equation (18); that is,

$$\varpi \circ \iota([T_G]) - \varpi \circ \iota([T_H]) = \varpi \circ \mathcal{G}([T_G], [T_{H'}]) = \varpi([T_a]) = \varpi([T_b]) + \varpi([T_{\bar{b}}]) = 0.$$

(ii) We next fix f and show that $[[T_G]]$ is independent of the choice of a representative g of the K -class $[g]$ in $K^{-i}(X)$ satisfying $[f_r] = \pi^*[g]$. Note that the K -class $[g]$ is uniquely determined since π^* is injective. Let $g' : X \rightarrow GL_N^{(i)}(\mathbb{C})$ be another representative; that is, $[g] = [g']$. By taking N and N' sufficiently large, we assume that $N = N'$ and that there exists a homotopy $\{k_t\}_{0 \leq t \leq 1}$ from $k_0 = g$ to $k_1 = g'$ in $C(X, GL_N^{(i)}(\mathbb{C}))$. Combined with the homotopies $\{g_t\}_{0 \leq t \leq 1}$ (from f_r to π^*g) and $\{\pi^*k_t\}_{0 \leq t \leq 1}$, we obtain a homotopy from f_r to π^*g' in $C(\mathbb{T}^2 \times X, GL_N^{(i)}(\mathbb{C}))$. The associated family of Toeplitz operators is expressed as T_K , where $K : [-2, 2] \times \mathbb{T}^2 \times X \rightarrow GL_N^{(i)}(\mathbb{C})$ is a continuous map given by

$$K(s, z, w, x) = \begin{cases} f(z, (2+s)w, x), & -2 \leq s \leq -1, \\ g_{2+2s}(z, w, x), & -1 \leq s \leq -\frac{1}{2}, \\ \pi^*k_{1+2s}(z, w, x) = k_{1+2s}(x), & -\frac{1}{2} \leq s \leq 0, \\ \widetilde{\pi^*k_{1-2s}}(z, w, x) = k_{1-2s}(x), & 0 \leq s \leq \frac{1}{2}, \\ \tilde{g}_{2-2s}(z, w, x), & \frac{1}{2} \leq s \leq 1, \\ \tilde{f}(z, (2-s)w, x), & 1 \leq s \leq 2. \end{cases}$$

Within $-\frac{1}{2} \leq s \leq \frac{1}{2}$, this family K has k_t of the opposite parameter direction, and it can be collapsed by a homotopy to obtain G . This homotopy preserves $s = \pm 2$ and provides a homotopy from T_G to T_K in the space of families of (self-adjoint) Fredholm operators that are invertible at $s = \pm 2$; therefore, $[T_G] = [T_K]$.

(iii) Finally, we show that $[[T_G]]$ is independent of the choice of a representative f of the K -class $[f]$ in $K^{-i}(\tilde{\mathbb{S}}^3 \times X)$. Let $f': \tilde{\mathbb{S}}^3 \times X \rightarrow GL_N^{(i)}(\mathbb{C})$ be another representative; that is, $[f'] = [f]$. As in (2), we assume $N = N'$ and that there exists a homotopy $\{f_t\}_{0 \leq t \leq 1}$ from $f_0 = f$ to $f_1 = f'$ in $C(\tilde{\mathbb{S}}^3 \times X, GL_N^{(i)}(\mathbb{C}))$. In this case, $\{(f_t)_r\}_{0 \leq t \leq 1}$ provides a path from f_r to f'_r in $C(\mathbb{T}^2 \times X, GL_N^{(i)}(\mathbb{C}))$. Combined with the homotopy $\{g_t\}_{0 \leq t \leq 1}$, we obtain a homotopy from f_r to π^*g , which provides the family of Toeplitz operators $T_{G'}$ and a homotopy from T_G to $T_{G'}$. Explicitly, for $0 \leq u \leq 1$, let $G_u: [-2, 2] \times \mathbb{T}^2 \times X \rightarrow GL_N^{(i)}(\mathbb{C})$ be a continuous map defined as follows:

$$G_u(s, z, w, x) = \begin{cases} f_u(z, (2+s)w, x), & -2 \leq s \leq -1, \\ f_{-2s+u-2}(z, w, x), & -1 \leq s \leq -1 + \frac{u}{2}, \\ g_{\frac{2-u}{2}s+1}(z, w, x), & -1 + \frac{u}{2} \leq s \leq 0, \\ \tilde{g}_{\frac{-2}{2-u}s+1}(z, w, x), & 0 \leq s \leq 1 - \frac{u}{2}, \\ \tilde{f}_{2s+u-2}(z, w, x), & 1 - \frac{u}{2} \leq s \leq 1, \\ \tilde{f}_u(z, (2-s)w, x), & 1 \leq s \leq 2. \end{cases}$$

Then, the associated family of Toeplitz operators $\{T_{G_u}\}_{u \in [0,1]}$ provides a path of families of (self-adjoint) Fredholm operators invertible at $s = \pm 2$ satisfying $T_{G_0} = T_G$ and $T_{G_1} = T_{G'}$; therefore, $[T_G] = [T_{G'}]$. \square

The following lemma states that the diagram (9) commutes:

Lemma 4.7. $\tilde{\alpha} \circ \phi = \beta$.

Proof. The map β is given through the decomposition (6) and the square of the Bott periodicity isomorphism. The map $\tilde{\alpha}$ corresponds to the Bott periodicity isomorphism, and we see a relation between the map ϕ and the Bott periodicity.

Let $\pi': \tilde{\mathbb{S}}^3 \times X \rightarrow X$ be the projection. Through the decomposition (6), the direct summand $K^{-i}(X)$ in $K^{-i}(\tilde{\mathbb{S}}^3 \times X)$ is represented by $[\pi'^*g]$, where $g: X \rightarrow GL_N^{(i)}(\mathbb{C})$ for some N . π'^*g is a continuous map on $\tilde{\mathbb{S}}^3 \times X$, which is constant with respect to $\tilde{\mathbb{S}}^3$. In this case, a homotopy $\{g_t\}_{0 \leq t \leq 1}$ to construct G in (15) can be taken to be constant $g_t = \pi'^*g$, and the corresponding family T_G of Toeplitz operators representing $\phi([\pi'^*g])$ to be a family of (self-adjoint) invertible operators. Therefore, $\phi([\pi'^*g]) = 0$, and the direct summand $K^{-i}(X)$ maps to zero through ϕ .

We next consider the direct summand of $K^{-i}(\tilde{\mathbb{S}}^3 \times X)$ corresponding to $K_{\text{cpt}}^{-i}(\mathbb{R}^3 \times X)$ in (6). We consider $\tilde{\mathbb{S}}^3$ as the one-point compactification of \mathbb{R}^3 , where the point at infinity corresponds to the base point $s_0 = (1, 1) \in \tilde{\mathbb{S}}^3$. K -classes in this component are represented by a continuous maps $f: \tilde{\mathbb{S}}^3 \times X \rightarrow GL_N^{(i)}(\mathbb{C})$ satisfying $f(s_0, x) = I_N^{(i)}$ for any $x \in X$. Since $\tilde{\mathbb{S}}^3 \setminus \{s_0\} \cong \mathbb{R}^3$ is contractible, we can take a small three-dimensional open ball \mathbb{B} in $\tilde{\mathbb{S}}^3$ and deform f to a continuous map which is $I_N^{(i)}$ on $(\tilde{\mathbb{S}}^3 \setminus \mathbb{B}) \times X$. Therefore, the K -class $[f]$ is represented by a continuous map $f': \tilde{\mathbb{S}}^3 \times X \rightarrow GL_N^{(i)}(\mathbb{C})$, which is $I_N^{(i)}$ on $\mathbb{D}^2 \times \mathbb{T} \times X$ and $\mathbb{T} \times \{0\} \times X$, where 0 is the origin of $\mathbb{D}^2 \subset \mathbb{C}$. Since $f'_r = I_N^{(i)}$, we can take a homotopy $\{g_t\}_{0 \leq t \leq 1}$ to construct G in (15) as $g_t = I_N^{(i)}$. Then,

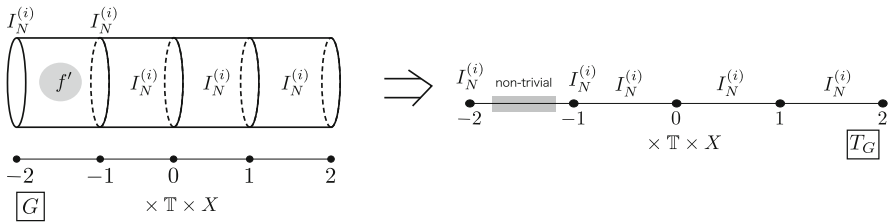
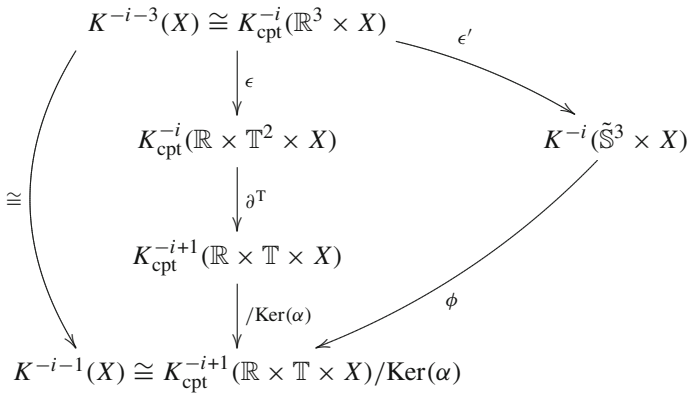


Fig. 4. Schematic picture for G and T_G in the proof of Lemma 4.7. A gray area in the left figure corresponds to the small open ball \mathbb{B} in \mathbb{S}^3 , and that in the right indicates its image

the map $G: [-2, 2] \times \mathbb{T}^2 \times X \rightarrow GL_N^{(i)}(\mathbb{C})$ is $I_N^{(i)}$ when $s = -2$ and $-1 \leq s \leq 2$, and corresponding family of Toeplitz operators T_G is also $I_N^{(i)}$ on $s = -2$ and $-1 \leq s \leq 2$. Therefore, ϕ on $K_{\text{cpt}}^{-i}(\mathbb{R}^3 \times X)$ is simply the push-forward map through the Toeplitz operators (see Fig. 4). Explicitly, the following diagram commutes:



In the above, the map ϵ is constructed as follows: we embed \mathbb{R}^3 into $\mathbb{R} \times \mathbb{T}^2$ as a small open ball corresponding to \mathbb{B} . A K -class in $K_{\text{cpt}}^{-i}(\mathbb{R}^3 \times X)$ is represented by a continuous map g from $\mathbb{R}^3 \times X$ to $GL_N^{(i)}(\mathbb{C})$ for some N , which is $I_N^{(i)}$ outside of a compact set. The image of the K -class $[g]$ in $K_{\text{cpt}}^{-i}(\mathbb{R} \times \mathbb{T}^2 \times X)$ is represented by the extension of g to $\mathbb{R} \times \mathbb{T}^2 \times X$ by $I_N^{(i)}$. In the same way, the map ϵ' is constructed by identifying \mathbb{R}^3 with the open ball \mathbb{B} in $\tilde{\mathbb{S}}^3$. The composite of vertical arrows is the Bott periodicity isomorphism, and the commutativity of the right hand side of the diagram follows from the construction of ϕ (see Fig. 4). Therefore, ϕ on $K_{\text{cpt}}^{-i}(\mathbb{R}^3 \times X)$ is given by the Bott periodicity isomorphism.

In summary, the composite $\bar{\alpha} \circ \phi$ is a projection onto $K_{\text{cpt}}^{-i}(\mathbb{R}^3 \times X)$ followed by the square of the Bott periodicity isomorphism, which is β . □

4.3. Proof of Theorem 4.1. Theorem 4.1 follows from the proposition below and the commutativity of the diagram (9).

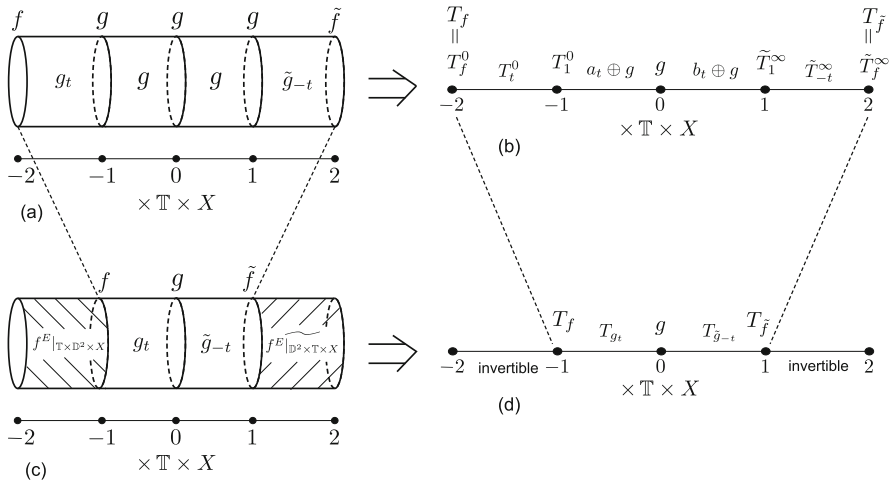


Fig. 5. Families of Toeplitz operators and their symbols used to define $\psi([(T_f^0, T_f^\infty)])$ **(b, a)** and $\phi([f^E])$ **(d, c)**: **a** symbols of F , **b** family of Toeplitz operators F , **c** symbols G of T_G , **d** family of Toeplitz operators T_G

Proposition 4.8. *Let $f : \mathbb{T}^2 \times X \rightarrow GL_N^{(i)}(\mathbb{C})$ be a continuous map such that for each $x \in X$, $f(x)$ is a two-variable rational matrix function such that the associated quarter-plane Toeplitz operator $T_{f(x)}^{0,\infty}$ is Fredholm. Let $f^E : \mathbb{S}^3 \times X \rightarrow GL_N^{(i)}(\mathbb{C})$ be the extension of f through matrix factorization in the hermitian case of Proposition 3.6 (when $i = 0$) or Proposition 3.4 (when $i = 1$). We then have $\psi([(T_f^0, T_f^\infty)]) = \phi([f^E])$.*

Proof. For the definition of $\psi([(T_f^0, T_f^\infty)])$, we take a path $\{(T_t^0, T_t^\infty)\}_{0 \leq t \leq 1}$ in $GL_N^{(i)}(\mathcal{S}^{0,\infty} \otimes C(X))$ in Lemma 4.3 and construct a family F of (self-adjoint) Fredholm operators on $[-2, 2] \times \mathbb{T}^2 \times X$ representing the element $\psi([(T_f^0, T_f^\infty)])$ ((b) in Fig. 5). Note that T_f^0 is identified with T_f through the isomorphism $\mathcal{T}^0 \cong C(\mathbb{T}) \otimes \mathcal{T}$ and that \tilde{T}_f^∞ is identified with $T_{\tilde{f}}$. Associated with the family $\{(T_t^0, T_t^\infty)\}_{0 \leq t \leq 1}$, there is a path² $g_t = \sigma^0(T_t^0) = \sigma^\infty(T_t^\infty)$ from f to π^*g in $C(\mathbb{T}^2 \times X, GL_N^{(i)}(\mathbb{C}))$. We use this path $\{g_t\}_{0 \leq t \leq 1}$ to define $\phi([f^E])$. Families of (self-adjoint) Fredholm Toeplitz operators T_G representing the K -group element $\phi([f^E])$ and their symbols G are indicated in (d) and (c) of Fig. 5. Note that $f_r^E = f^E|_{\mathbb{T}^2 \times X} = f$ in our setup. Since f^E in Propositions 3.4 and 3.6 is the extension of f constructed through matrix factorization, the families of Toeplitz operators T_G are (self-adjoint) invertible on $([-2, -1] \sqcup [1, 2]) \times \mathbb{T} \times X$ by Lemma 3.2.

Let us compare F and T_G . Let $p : [-2, 2] \rightarrow [-1, 1]$ be a map defined as follows:

$$p(s) = \begin{cases} s + 1, & -2 \leq s \leq -1, \\ 0, & -1 \leq s \leq 1, \\ s - 1, & 1 \leq s \leq 2, \end{cases}$$

² We assume that f is sufficiently stabilized to take $N' = N$ for simplicity.

that is, collapsing the subinterval $[-1, 1]$ in $[-2, 2]$ to zero. We take a restriction of T_G onto $-1 \leq s \leq 1$ and extend $\{s = 0\}$ to $-1 \leq s \leq 1$; that is, we consider

$$T'_G = (p \times \text{id}_{\mathbb{T} \times X})^*(T_G|_{[-1,1] \times \mathbb{T} \times X}).$$

Since T_G is invertible when $-2 \leq s \leq -1$ and $1 \leq s \leq 2$, T'_G is also invertible when $s = \pm 2$, and the K -class of T'_G in $K^{-i+1}([-2, 2] \times \mathbb{T} \times X, \{\pm 2\} \times \mathbb{T} \times X)$ is the same as that of T_G . Families F and T'_G coincide at boundaries $s = \pm 2$, and on the interior, these two are families of (self-adjoint) Fredholm Toeplitz operators of the same symbol. Therefore, they differ by (self-adjoint) compact operators (see Fig. 5), and the linear path $(1 - t)F + tT'_G$ for $0 \leq t \leq 1$ provides a homotopy between F and T'_G in the space of (self-adjoint) Fredholm operators on $[-2, 2] \times \mathbb{T} \times X$, which is invertible on $\{\pm 2\} \times \mathbb{T} \times X$. We thus have,

$$[F] = [T'_G] = [T_G] \in K^{-i+1}([-2, 2] \times \mathbb{T} \times X, \{\pm 2\} \times \mathbb{T} \times X).$$

By taking a quotient by $\text{Ker}(\alpha)$, we obtain $\psi([(T_f^0, T_f^\infty)]) = \phi([f^E])$. □

Proof of Theorem 4.1. By Proposition 4.8 and the commutativity of the diagram (9), we have,

$$[T^{0,\infty}] = \partial^{\text{qT}}([(T_f^0, T_f^\infty)]) = \bar{\alpha} \circ \psi([(T_f^0, T_f^\infty)]) = \bar{\alpha} \circ \phi([f^E]) = \beta([f^E]).$$

□

The following is an example of Theorem 4.1 when $i = 1$ and $X = \{\text{pt}\}$.

Example 4.9. Let $f : \mathbb{T}^2 \rightarrow GL(2, \mathbb{C})$ be the two-variable rational matrix function given by $f(z, w) = \begin{pmatrix} z & -w^{-1} \\ w & z^{-1} \end{pmatrix}$. We first consider f as a family of rational matrix functions on the circle \mathbb{T} of variable z parametrized by $w \in \mathbb{T}$. By applying an algorithm from [20], we obtain a right factorization of the following form:

$$f(z, w) = f_-(z, w) \cdot f_+(z, w) = \begin{pmatrix} 1 & 0 \\ -wz^{-1} & 1 \end{pmatrix} \begin{pmatrix} z & -w^{-1} \\ 2w & 0 \end{pmatrix}.$$

Since this is a canonical factorization, the half-plane Toeplitz operator T_f^∞ is invertible. Our extended symbol f^E in Proposition 3.4 is given as,

$$f^E(z, w) = f_-(\bar{z}^{-1}, w) \cdot f_+(z, w) = \begin{pmatrix} z & -w^{-1} \\ -w|z|^2 + 2w & \bar{z} \end{pmatrix}, \tag{20}$$

for $(z, w) \in \mathbb{D}^2 \times \mathbb{T}$. We next consider $z \in \mathbb{T}$ as a parameter, and by a similar discussion for rational matrix functions f with respect to $w \in \mathbb{T}$, we also obtain a canonical factorization. Therefore, the half-plane Toeplitz operator T_f^0 is invertible and f^E for $(z, w) \in \mathbb{T} \times \mathbb{D}^2$ is given as

$$f^E(z, w) = \begin{pmatrix} -z|w|^2 + 2z & -\bar{w} \\ w & z^{-1} \end{pmatrix}. \tag{21}$$

For $0 \leq t \leq 1$, let $g_t : \tilde{\mathbb{S}}^3 \rightarrow GL(2, \mathbb{C})$ be a continuous map given as follows: On $\mathbb{D}^2 \times \mathbb{T}$, we set g_t as the replacement of the $(2, 1)$ component of f^E in (20) with

$tw(-|z|^2+1) + w$. On $\mathbb{T} \times \mathbb{D}^2$ we set g_t as the replacement of the $(1, 1)$ component of f^E in (21) with $tz(-|w|^2+1) + z$. This g_t provides a homotopy between $g_1 = f^E$ and $g_0(z, w) = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix}$ in $C(\tilde{\mathbb{S}}^3, GL(2, \mathbb{C}))$. Therefore, $[f^E] = [g_0]$ in $K^{-1}(\tilde{\mathbb{S}}^3)$.

Note that the map $\beta: K^{-1}(\tilde{\mathbb{S}}^3) \rightarrow K^0(\{\text{pt}\}) \cong \mathbb{Z}$ is the square of the Bott periodicity isomorphism. Since g_0 is the square of a Bott element, which is a generator of the K -group $K^{-1}(\tilde{\mathbb{S}}^3) \cong \mathbb{Z}$, we have $\beta([f^E]) = \beta([g_0]) = 1$. Since T_f^0 and T_f^∞ are invertible, the quarter-plane Toeplitz operator $T_f^{0,\infty}$ is Fredholm, and its Fredholm index is 1 by Theorem 4.1, which is also computed in [15, 17] by different methods.

Let us consider the special case of Theorem 4.1 when $i = 1$ and $X = \{\text{pt}\}$. Our extended symbol defines an element $[f^E] \in K^{-1}(\tilde{\mathbb{S}}^3)$ and, through the isomorphism $\beta: K^{-1}(\tilde{\mathbb{S}}^3) \xrightarrow{\cong} \mathbb{Z}$, we obtain an integer $W_3(f^E) := \beta([f^E])$. This integer corresponds to $\pi_3(GL_n(\mathbb{C})) \cong \mathbb{Z}$ for $n \geq 2$ and is the three-dimensional winding number. Therefore, we obtain the following result.

Corollary 4.10. *Let $f: \mathbb{T}^2 \rightarrow GL_N(\mathbb{C})$ be a nonsingular two-variable rational matrix function with poles off \mathbb{T}^2 such that the associated quarter-plane Toeplitz operator $T_f^{0,\infty}$ is Fredholm. Let $f^E: \tilde{\mathbb{S}}^3 \rightarrow GL_N(\mathbb{C})$ be the extension of f through matrix factorization in Proposition 3.4. Then the Fredholm index of $T_f^{0,\infty}$ is given by,*

$$\text{index}(T_f^{0,\infty}) = W_3(f^E).$$

Remark 4.11. In Example 4.9, the matrix function f^E is smooth both on $\mathbb{D}^2 \times \mathbb{T}$ and $\mathbb{T} \times \mathbb{D}^2$. If we apply the integration formula for three-dimensional winding numbers to f^E on $\mathbb{D}^2 \times \mathbb{T}$ in (20), we obtain $\frac{1}{24\pi^2} \int_{\mathbb{D}^2 \times \mathbb{T}} \text{Tr}((f^E)^{-1}df^E)^3 = \frac{1}{2}$. Similar integration for f^E on $\mathbb{T} \times \mathbb{D}^2$ in (21) is also $\frac{1}{2}$. We see that their sum is 1, which is the same as the three-dimensional winding number computed in Example 4.9, though we do not have a general proof that the sum coincides with the three-dimensional winding number since our three sphere $\tilde{\mathbb{S}}^3$ is not smooth.

5. Quarter-Plane Toeplitz Operators Preserving Real Structures

In this section, we discuss a variant of Theorem 4.1 in real K -theory. For this purpose, we use Atiyah’s KR -theory for spaces equipped with involutions [4] and Boersema–Loring’s formulation for the KO -theory of real C^* -algebras [12]. Index theory for quarter-plane Toeplitz operators preserving some real structures and application to topological corner states are discussed in [27], which we mainly follow.

5.1. Matrix functions preserving real or quaternionic structures. Let \mathcal{C} be a real or a quaternionic structure on \mathbb{C}^n , that is, an antiunitary operator on \mathbb{C}^n whose square is $+1$ or -1 . Note that, when we consider a quaternionic structure, the positive integer n must be even. We write $\text{Ad}_{\mathcal{C}}$ for a real linear automorphism of order two on $M_n(\mathbb{C})$ given by $\text{Ad}_{\mathcal{C}}(x) = \mathcal{C}x\mathcal{C}^*$. We also write $*$ for the operation on $M_n(\mathbb{C})$ taking the hermitian conjugate of matrices. We write c for complex conjugation on \mathbb{C} , that is, $c(z) = \bar{z}$. Then, (\mathbb{T}, c) is a \mathbb{Z}_2 -space, which is a \mathbb{Z}_2 -subspace of (\mathbb{D}^2, c) .

Lemma 5.1. *Let \mathfrak{J} be a \mathbb{Z}_2 -space (i) $(GL_n(\mathbb{C}), \text{Ad}_C)$ or (ii) $(GL_n(\mathbb{C}), \text{Ad}_C \circ *)$. Let (X, ζ) be a \mathbb{Z}_2 -space and $f: (\mathbb{T} \times X, c \times \zeta) \rightarrow \mathfrak{J}$ be a \mathbb{Z}_2 -equivariant continuous map such that for each $x \in X$, $f(x)$ is a rational matrix function of trivial partial indices. Then, its extension f^e in Lemma 3.3 is a \mathbb{Z}_2 -map $f^e: (\mathbb{D}^2 \times X, c \times \zeta) \rightarrow \mathfrak{J}$.*

Proof. By Lemma 3.3, the map f^e is continuous. We now show its \mathbb{Z}_2 -equivalence. For each $x \in X$, we take a canonical factorization $f(x) = f_-(x)f_+(x)$.

(i) We first consider the case in which $\mathfrak{J} = (GL_n(\mathbb{C}), \text{Ad}_C)$. By assumption, the following equation holds for $(z, x) \in \mathbb{T} \times X$:

$$f_-(z, x)f_+(z, x) = f(z, x) = Cf(\bar{z}, \zeta(x))C^* = Cf_-(\bar{z}, \zeta(x))C^* \cdot Cf_+(\bar{z}, \zeta(x))C^*. \tag{22}$$

Let $f_+^e(x)$ and $f_-^e(x)$ be continuous extensions of $f_+(x)$ and $f_-(x)$ onto $\mathbb{T} \cup D_+$ and $\mathbb{T} \cup D_-$, which are analytic on D_+ and D_- , respectively. The function $z \mapsto Cf_+(\bar{z}, \zeta(x))C^*$ is a rational matrix function on \mathbb{T} and $Cf_+^e(\bar{z}, \zeta(x))C^*$ provides its continuous extension onto $\mathbb{T} \cup D_+$ which is analytic on D_+ as a nonsingular matrix function. A similar observation holds for $Cf_-(\bar{z}, \zeta(x))C^*$ and the equation (22) provides two canonical factorizations of $f(x)$. Therefore, for each $x \in X$, there exists $B \in GL_n(\mathbb{C})$ such that $Bf_+(z, x) = Cf_+(\bar{z}, \zeta(x))C^*$ and $f_-(z, x)B^{-1} = Cf_-(\bar{z}, \zeta(x))C^*$ for $z \in \mathbb{T}$. By the uniqueness of analytic continuation, we obtain

$$\begin{aligned} Bf_+^e(z, x) &= Cf_+^e(\bar{z}, \zeta(x))C^* \text{ for } z \in \mathbb{T} \cup D_+, \\ f_-^e(z, x)B^{-1} &= Cf_-^e(\bar{z}, \zeta(x))C^* \text{ for } z \in \mathbb{T} \cup D_-. \end{aligned}$$

Therefore, for $z \in \mathbb{T} \cup D_+ = \mathbb{D}^2$,

$$\begin{aligned} f^e(z, x) &= f_-^e(\bar{z}^{-1}, x)f_+^e(z, x) = f_-^e(\bar{z}^{-1}, x)B^{-1} \cdot Bf_+^e(z, x) \\ &= Cf_-^e(\bar{z}^{-1}, \zeta(x))C^* \cdot Cf_+^e(\bar{z}, \zeta(x))C^* = Cf^e(\bar{z}, \zeta(x))C^*. \end{aligned}$$

(ii) We next consider the case of $\mathfrak{J} = (GL_n(\mathbb{C}), \text{Ad}_C \circ *)$. By assumption, we have

$$f_-(z, x)f_+(z, x) = f(z, x) = Cf_+(\bar{z}, \zeta(x))^*C^* \cdot Cf_-(\bar{z}, \zeta(x))^*C^* \tag{23}$$

for $(z, x) \in \mathbb{T} \times X$. The matrix function $Cf_+^e(\bar{z}^{-1}, \zeta(x))^*C^*$ (resp. $Cf_-^e(\bar{z}^{-1}, \zeta(x))^*C^*$) provides a continuous extension of $Cf_+(\bar{z}, \zeta(x))^*C^*$ (resp. $Cf_-(\bar{z}, \zeta(x))^*C^*$) onto $\mathbb{T} \cup D_-$ (resp. $\mathbb{T} \cup D_+$), which is analytic on D_- (resp. D_+), and the right hand side of equation (23) is also a canonical factorization of f . Therefore, as in the proof of (i), there exists $B \in GL_n(\mathbb{C})$ satisfying,

$$\begin{aligned} Bf_+^e(z, x) &= Cf_-^e(\bar{z}^{-1}, \zeta(x))^*C^* \text{ for } z \in \mathbb{T} \cup D_+, \\ f_-^e(z, x)B^{-1} &= Cf_+^e(\bar{z}^{-1}, \zeta(x))^*C^* \text{ for } z \in \mathbb{T} \cup D_-. \end{aligned}$$

By these equations, for $z \in \mathbb{T} \cup D_+ = \mathbb{D}^2$,

$$\begin{aligned} f^e(z, x) &= f_-^e(\bar{z}^{-1}, x)B^{-1} \cdot Bf_+^e(z, x) = Cf_+^e(\bar{z}, \zeta(x))^*C^* \cdot Cf_-^e(\bar{z}^{-1}, \zeta(x))^*C^* \\ &= C(f_-^e(\bar{z}^{-1}, \zeta(x))f_+^e(\bar{z}, \zeta(x)))^*C^* = Cf^e(\bar{z}, \zeta(x))^*C^*. \end{aligned}$$

□

Lemma 5.2. *Let \mathfrak{J} be a \mathbb{Z}_2 -space (i) $(GL_n(\mathbb{C})^{\text{sa}}, \text{Ad}_{\mathcal{C}})$ or (ii) $(GL_n(\mathbb{C})^{\text{sa}}, -\text{Ad}_{\mathcal{C}})$. Let (X, ζ) be a \mathbb{Z}_2 -space and $f : (\mathbb{T} \times X, c \times \zeta) \rightarrow \mathfrak{J}$ be a \mathbb{Z}_2 -equivariant continuous map such that for each $x \in X$, $f(x)$ is a rational matrix function of trivial partial indices. Then, its extension f^e in Lemma 3.5 is a \mathbb{Z}_2 -map $f^e : (\mathbb{D}^2 \times X, c \times \zeta) \rightarrow \mathfrak{J}$.*

Proof. (i) follows from the hermitian case of Lemma 3.5 and (i) of Lemma 5.1. For (ii), note that a \mathbb{Z}_2 -map $f : (\mathbb{T} \times X, c \times \zeta) \rightarrow (GL_n(\mathbb{C})^{\text{sa}}, -\text{Ad}_{\mathcal{C}})$ provides, by multiplication by the imaginary unit, a \mathbb{Z}_2 -map $\sqrt{-1}f : (\mathbb{T} \times X, c \times \zeta) \rightarrow (GL_n(\mathbb{C})^{\text{sk}}, \text{Ad}_{\mathcal{C}})$. Therefore, the result follows from the skew-hermitian case of Lemma 3.5 and (i) of Lemma 5.1. \square

Let ν be an involution on $\tilde{\mathfrak{S}}^3$ given by the restriction of $c^2 = c \times c$ on \mathbb{C}^2 onto $\tilde{\mathfrak{S}}^3$. By Proposition 3.4 and Lemmas 5.1 and 5.2, we obtain the following result.

Proposition 5.3. *Let (X, ζ) be a \mathbb{Z}_2 -space, and let \mathfrak{J} be a \mathbb{Z}_2 -space $(GL_n(\mathbb{C}), \text{Ad}_{\mathcal{C}})$, $(GL_n(\mathbb{C}), \text{Ad}_{\mathcal{C}} \circ *)$, $(GL_n(\mathbb{C})^{\text{sa}}, \text{Ad}_{\mathcal{C}})$ or $(GL_n(\mathbb{C})^{\text{sa}}, -\text{Ad}_{\mathcal{C}})$. Let $f : (\mathbb{T}^2 \times X, c^2 \times \zeta) \rightarrow \mathfrak{J}$ be a \mathbb{Z}_2 -continuous map such that for each $x \in X$, $f(x)$ is a two-variable rational matrix function for which the associated quarter-plane Toeplitz operator $T_{f(x)}^{0, \infty}$ is Fredholm. Then, through matrix factorization, there is a canonically associated \mathbb{Z}_2 -continuous map $f^E : (\tilde{\mathfrak{S}}^3 \times X, \nu \times \zeta) \rightarrow \mathfrak{J}$ that extends f .*

5.2. Index theorem: real cases. Let \mathcal{R} be the antiunitary operator on \mathbb{C}^n given by $\mathcal{R} = \text{diag}(c, \dots, c)$, where c is the complex conjugation on \mathbb{C} . Let j be an antiunitary operator on \mathbb{C}^2 given by $j(x, y) = (-\bar{y}, \bar{x})$. When n is even, let $\mathcal{J} = \text{diag}(j, \dots, j)$ be a quaternionic structure on \mathbb{C}^n . Let A be a unital C^* -algebra equipped with a real structure³ τ . Let τ be the antiautomorphism on A of order two given by $\tau(a) = \tau(a^*)$. We call τ the *transposition* and write a^τ for $\tau(a)$. The pair (A, τ) is called $C^{*, \tau}$ -algebra in [12]. The transposition τ on A is extended to the transposition on the matrix algebra $M_n(A)$ by $(a_{ij})^\tau = (a_{ji}^\tau)$. Let $\sharp \otimes \tau$ be a transposition on $M_2(A)$ defined by

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{\sharp \otimes \tau} = \begin{pmatrix} a_{22}^\tau & -a_{12}^\tau \\ -a_{21}^\tau & a_{11}^\tau \end{pmatrix},$$

which is extended to the transposition on $M_{2n}(A)$ by $(b_{ij})^{\sharp \otimes \tau} = (b_{ji}^{\sharp \otimes \tau})$ where $b_{ij} \in M_2(A)$ and $1 \leq i, j \leq n$. For $i = -1, 0, \dots, 6$, let $n_{r,i}$ be a positive integer, $R^{(i)}$ be a relation and $I_r^{(i)}$ be a matrix as indicated in Table 1. Let $GL_n^{(i)}(A, \tau)$ be the set of all invertible elements in $M_{n_{r,i-n}}(A)$ satisfying the relation $R^{(i)}$. Following Boersema–Loring [12], we define⁴ the KO -group of (A, τ) as $KO_i(A, \tau) = \cup_{n=1}^\infty GL_n^{(i)}(A, \tau) / \sim_i$ where the equivalence relation \sim_i is generated by homotopy and stabilization by $I_r^{(i)}$. For a finite \mathbb{Z}_2 -CW complex (X, ζ) , we associate a (complex) C^* -algebra $C(X)$ with a transposition⁵ τ_ζ given by $(f^{\tau_\zeta})(x) = f(\zeta(x))$. We define $KR^{-i}(X, \zeta) = KO_i(C(X), \tau_\zeta)$ which are Atiyah’s Real K -groups [4] for the \mathbb{Z}_2 -space (X, ζ) . Note that an element

³ An antilinear $*$ -automorphism on A satisfying $\tau^2 = 1$.

⁴ In [12], KO -groups are defined via unitaries though we can also define KO -groups through invertibles preserving symmetries in [12] since deformation retraction from invertibles to unitaries preserves these symmetries. We discuss invertible elements since, in our application discussed in Sect. 6, Hamiltonians will be expressed as multivariable nonsingular rational matrix functions, though not necessarily unitaries.

⁵ Correspondingly, the real structure on $C(X)$ is given by $\tau_\zeta(f)(x) = \overline{f(\zeta(x))}$.

Table 1. KO -theory via invertible elements (Boersema–Loring [12])

i	$n_{r,i}$	$R^{(i)}$	$I_r^{(i)}$	$GL_{r,n}^{(i)}(\mathbb{C})$
-1	1	$x^\tau = x$	1	$(GL_n(\mathbb{C}), \text{Ad}_{\mathcal{R}} \circ *)$
0	2	$x = x^*, x^\tau = x^*$	$\text{diag}(1, -1)$	$(GL_{2n}(\mathbb{C})^{\text{sa}}, \text{Ad}_{\mathcal{R}})$
1	1	$x^\tau = x^*$	1	$(GL_n(\mathbb{C}), \text{Ad}_{\mathcal{R}})$
2	2	$x = x^*, x^\tau = -x$	$\begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}$	$(GL_{2n}(\mathbb{C})^{\text{sa}}, -\text{Ad}_{\mathcal{R}})$
3	2	$x^{\sharp \otimes \tau} = x$	1_2	$(GL_{2n}(\mathbb{C}), \text{Ad}_{\mathcal{J}} \circ *)$
4	4	$x = x^*, x^{\sharp \otimes \tau} = x^*$	$\text{diag}(1_2, -1_2)$	$(GL_{4n}(\mathbb{C})^{\text{sa}}, \text{Ad}_{\mathcal{J}})$
5	2	$x^{\sharp \otimes \tau} = x^*$	1_2	$(GL_{2n}(\mathbb{C}), \text{Ad}_{\mathcal{J}})$
6	2	$x = x^*, x^{\sharp \otimes \tau} = -x$	$\begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}$	$(GL_{2n}(\mathbb{C})^{\text{sa}}, -\text{Ad}_{\mathcal{J}})$

$f \in GL_n^{(i)}(C(X), \tau_\zeta)$ corresponds to a \mathbb{Z}_2 -equivariant continuous map $f : (X, \zeta) \rightarrow GL_{r,n}^{(i)}(\mathbb{C})$ where $GL_{r,n}^{(i)}(\mathbb{C})$ is the \mathbb{Z}_2 -space as indicated in Table 1. Following [4], let $\mathbb{R}^{p,q} := \mathbb{R}^q \oplus i\mathbb{R}^p$ be the \mathbb{Z}_2 -space whose involution is given by id on \mathbb{R}^q and $-i$ on \mathbb{R}^p , and let $\mathbb{S}^{p,q}$ be the unit sphere in $\mathbb{R}^{p,q}$. Note that our \mathbb{Z}_2 -space $(\tilde{\mathbb{S}}^3, \nu)$ is \mathbb{Z}_2 -equivariantly homeomorphic to $\mathbb{S}^{2,2}$. The Bott periodicity for KR -theory is given as $KR^{-i}(X, \zeta) \cong KR_{\text{cpt}}^{-i}(\mathbb{R}^{1,1} \times (X, \zeta))$ whose isomorphism is given as in the complex case.

On $l^2(\mathbb{Z}^2)$, we consider an antiunitary operator of order two given by the pointwise operation of complex conjugation, for which we simply write c . Conjugation of c provide real structures for the quarter-plane Toeplitz algebra $\mathcal{T}^{0,\infty}$ and the pull-back C^* -algebra $\mathcal{S}^{0,\infty}$. We write $\tau_{0,\infty}$ and $\tau_{\mathcal{S}}$ for corresponding transpositions on $\mathcal{T}^{0,\infty}$ and $\mathcal{S}^{0,\infty}$, respectively. For $i = -1, 0, \dots, 6$, and a positive integer N , let $GL_{r,N}^{(i)}$ be the \mathbb{Z}_2 -space indicated in Table 1. Let (X, ζ) be a finite \mathbb{Z}_2 -CW complex, and let $f : (\mathbb{T}^2 \times X, c^2 \times \zeta) \rightarrow GL_{r,N}^{(i)}(\mathbb{C})$ be a \mathbb{Z}_2 -map such that for each $x \in X$, $f(x)$ is a two-variable rational matrix function and the associated quarter-plane Toeplitz operator $T_{f(x)}^{0,\infty}$ is Fredholm. In this case, half-plane Toeplitz operators $\{T_{f(x)}^0\}_{x \in X}$ and $\{T_{f(x)}^\infty\}_{x \in X}$ and quarter-plane Toeplitz operators $\{T_{f(x)}^{0,\infty}\}_{x \in X}$ preserve some symmetry corresponding to the \mathbb{Z}_2 -equivalence of the map f . As in [26], pairs of invertible half-plane Toeplitz operators (T_f^0, T_f^∞) define an element $[(T_f^0, T_f^\infty)]$ of the KO -group $KO_i(\mathcal{S}^{0,\infty} \otimes C(X), \tau_{\mathcal{S}} \otimes \tau_\zeta)$, and (self-adjoint, when i is even) Fredholm quarter-plane Toeplitz operators define an element $[T_f^{0,\infty}]$ of the KR -group $KR^{-i+1}(X, \zeta)$. By Proposition 5.3, through the matrix factorization, there is an associated \mathbb{Z}_2 -map f^E which defines an element $[f^E]$ of the KR -group $KR^{-i}(\tilde{\mathbb{S}}^3 \times X, \nu \times \zeta)$. Let

$$\partial^{q\Gamma} : KO_i(\mathcal{S}^{0,\infty} \otimes C(X), \tau_{\mathcal{S}} \otimes \tau_\zeta) \rightarrow KO_{i-1}(C(X), \tau_\zeta) \cong KR^{-i+1}(X, \zeta),$$

be the boundary map of the 24-term exact sequence for KO -theory associated with the short exact sequence of C^*, τ -algebras,

$$\begin{aligned} 0 \rightarrow (\mathcal{K} \otimes C(X), \tau_{\mathcal{K}} \otimes \tau_\zeta) &\rightarrow (\mathcal{T}^{0,\infty} \otimes C(X), \tau_{0,\infty} \otimes \tau_\zeta) \\ &\xrightarrow{\gamma \otimes 1} (\mathcal{S}^{0,\infty} \otimes C(X), \tau_{\mathcal{S}} \otimes \tau_\zeta) \rightarrow 0. \end{aligned} \tag{24}$$

Then, we have $\partial^{\text{qT}}((T_f^0, T_f^\infty)) = [T_f^{0,\infty}]$. For $(\tilde{\mathbb{S}}^3, \nu)$, we take a \mathbb{Z}_2 -fixed point $s_0 = (1, 1)$ in $\tilde{\mathbb{S}}^3$ as its base point, and obtain the following decomposition:

$$\begin{aligned} KR^{-i}(\tilde{\mathbb{S}}^3 \times X, \nu \times \zeta) &\cong KR^{-i}(X, \zeta) \oplus KR_{\text{cpt}}^{-i}(\mathbb{R}^{2,1} \times (X, \zeta)) \\ &\cong KR^{-i}(X, \zeta) \oplus KR^{-i+1}(X, \zeta), \end{aligned}$$

where the second isomorphism is given by the square of the Bott periodicity isomorphism. Let $\beta : KR^{-i}(\tilde{\mathbb{S}}^3 \times X, \nu \times \zeta) \rightarrow KR^{-i+1}(X, \zeta)$ be the projection corresponding to this decomposition.

Theorem 5.4. *Let (X, ζ) be a finite \mathbb{Z}_2 -CW complex. Let $f : (\mathbb{T}^2 \times X, c^2 \times \zeta) \rightarrow GL_{r,N}^{(i)}(\mathbb{C})$ be a \mathbb{Z}_2 -map such that for each $x \in X$, $f(x)$ is a two-variable rational matrix function and the associated quarter-plane Toeplitz operator $T_{f(x)}^{0,\infty}$ is Fredholm. Let $f^E : (\tilde{\mathbb{S}}^3 \times X, \nu \times \zeta) \rightarrow GL_{r,N}^{(i)}(\mathbb{C})$ be the extension of f through matrix factorization in Proposition 5.3. Then, we have $[T_f^{0,\infty}] = \beta([f^E])$ in $KR^{-i+1}(X, \zeta)$.*

The proof of Theorem 4.1 concerns three parts: matrix factorizations, the homotopy lifting property and K -theory. For Theorem 5.4, matrix factorizations are discussed in Sect. 5.1. Here, we note the following result concerning the \mathbb{Z}_2 -equivariant homotopy lifting property. For $\mathcal{C} = \mathcal{R}$ or \mathcal{J} , let $c_{\mathcal{T}}$ be an involution on $M_n(\mathcal{T})$ given by $c_{\mathcal{T}}(T) = \mathcal{C}T\mathcal{C}^*$, and let $c_{\mathbb{T}}$ be an involution on $M_n(C(\mathbb{T}))$ given by $c_{\mathbb{T}}(f)(z) = \mathcal{C}f(\bar{z})\mathcal{C}^*$ for $z \in \mathbb{T}$.

Proposition 5.5. *The following are Serre \mathbb{Z}_2 -fibrations:*

- (i) $\sigma : (U_n(\mathcal{T}), c_{\mathcal{T}}) \rightarrow (U_n(C(\mathbb{T})), c_{\mathbb{T}})$,
- (ii) $\sigma : (U_n(\mathcal{T}), c_{\mathcal{T}} \circ *) \rightarrow (U_n(C(\mathbb{T})), c_{\mathbb{T}} \circ *)$,
- (iii) $\sigma : (U_n(\mathcal{T})^{\text{sa}}, c_{\mathcal{T}}) \rightarrow (U_n(C(\mathbb{T}))^{\text{sa}}, c_{\mathbb{T}})$,
- (iv) $\sigma : (U_n(\mathcal{T})^{\text{sa}}, -c_{\mathcal{T}}) \rightarrow (U_n(C(\mathbb{T}))^{\text{sa}}, -c_{\mathbb{T}})$.

Since (11) and (12) are Hurewicz fibrations and by Theorem 4.1 of [13], to show Proposition 5.5, it is sufficient to show that the restrictions on the \mathbb{Z}_2 -fixed point sets are fibrations. (see also [48]).

Lemma 5.6. *The following are Hurewicz fibrations:*

- (i) $\sigma : U_n(\mathcal{T})^{c_{\mathcal{T}}} \rightarrow U_n(C(\mathbb{T}))^{c_{\mathbb{T}}}$,
- (ii) $\sigma : U_n(\mathcal{T})^{c_{\mathcal{T}} \circ *} \rightarrow U_n(C(\mathbb{T}))^{c_{\mathbb{T}} \circ *}$,
- (iii) $\sigma : (U_n(\mathcal{T})^{\text{sa}})^{c_{\mathcal{T}}} \rightarrow (U_n(C(\mathbb{T}))^{\text{sa}})^{c_{\mathbb{T}}}$,
- (iv) $\sigma : (U_n(\mathcal{T})^{\text{sa}})^{-c_{\mathcal{T}}} \rightarrow (U_n(C(\mathbb{T}))^{\text{sa}})^{-c_{\mathbb{T}}}$.

Proof. (i) and (iii) are real analogues of (11) and (12). (iv) follows from the skew-adjoint analogue of (iii), since multiplication by $\sqrt{-1}$ provides the homeomorphism from $(U_n(\mathcal{T})^{\text{sa}})^{-c_{\mathcal{T}}}$ to $(U_n(\mathcal{T})^{\text{sk}})^{c_{\mathcal{T}}}$.

For (ii), we put unitaries preserving these real structures in the framework of Wood [50]. Let $A = M_{2n}(\mathcal{T})$ and set $U(A) = U_{2n}(\mathcal{T})$. Let

$$e = \begin{pmatrix} \sqrt{-1} \cdot 1_n & 0 \\ 0 & -\sqrt{-1} \cdot 1_n \end{pmatrix}, \quad \tilde{c} = \begin{pmatrix} 0 & -\mathcal{C} \cdot 1_n \\ \mathcal{C} \cdot 1_n & 0 \end{pmatrix},$$

and let $\tilde{c}_{\mathcal{T}} : A \rightarrow A$ be a real linear automorphism of order two given by $\tilde{c}_{\mathcal{T}}(a) = \tilde{c}a\tilde{c}^*$. Its fixed point set $A^{\tilde{c}_{\mathcal{T}}}$ is a unital real Banach $*$ -algebra containing e . Let ϵ be an operator

on $A^{\tilde{c}\mathcal{T}}$ given by the conjugation of e . Let us consider the space $(U(A^{\tilde{c}\mathcal{T}})^{\text{sk}})^{-\epsilon}$, that is, the elements in $A = M_{2n}(\mathcal{T})$ which are skew-adjoint unitary, commute with \tilde{C} and anti-commute with e . We have the following identification,

$$U_n(\mathcal{T})^{c\mathcal{T}o*} \xrightarrow{\cong} (U(A^{\tilde{c}\mathcal{T}})^{\text{sk}})^{-\epsilon}, \quad a \mapsto \begin{pmatrix} 0 & -a^* \\ a & 0 \end{pmatrix}.$$

Let $u \in U_n(\mathcal{T})^{c\mathcal{T}o*}$ and $s = \begin{pmatrix} 0 & -u^* \\ u & 0 \end{pmatrix} \in (U(A^{\tilde{c}\mathcal{T}})^{\text{sk}})^{-\epsilon}$. We consider an operator s on $U(A^{\tilde{c}\mathcal{T}})^{\epsilon}$ given by the conjugation of k . By Lemma 4.2 of [50], we have the following homeomorphism,

$$(U(A^{\tilde{c}\mathcal{T}})^{\epsilon} / (U(A^{\tilde{c}\mathcal{T}})^{\epsilon})^s)_1 \xrightarrow{\cong} (U(A^{\tilde{c}\mathcal{T}})^{\text{sk}})^{-\epsilon}_s \cong U_n(\mathcal{T})_u^{c\mathcal{T}o*},$$

given by $[T] \mapsto TsT^*$. We also have a similar homeomorphism for the algebra $C(\mathbb{T})$, and as in the proof of Lemma 4.2, we obtain that the map

$$\sigma : U_n(\mathcal{T})_u^{c\mathcal{T}o*} \rightarrow U_n(C(\mathbb{T}))_{\sigma(u)}^{c\mathbb{T}o*}$$

is a fiber bundle and the result follows. □

For the proof of Theorem 5.4, we replace $n_i, I^{(i)}$ and $GL_n^{(i)}(\mathbb{C})$ in the proof of Theorem 4.1 with $n_{r,i}, I_r^{(i)}$ and $GL_{r,n}^{(i)}(\mathbb{C})$ in Table 1 and obtain the following commutative diagram:

$$\begin{array}{ccc} KO_i(S^{0,\infty} \otimes C(X), \tau_S \otimes \tau_{\zeta}) & & \\ \downarrow \psi & \searrow \partial^{\text{qT}} & \\ KR^{-i+1}([[-2, 2], \{\pm 2\}] \times \mathbb{T} \times X, \text{id} \times c \times \zeta) / \text{Ker}(\alpha) & \xrightarrow{\bar{\alpha}} & KR^{-i+1}(X, \zeta) \\ \uparrow \phi & \nearrow \beta & \\ KR^{-i}(\mathbb{S}^3 \times X, \nu \times \zeta) & & \end{array}$$

where the map $\bar{\alpha}$ is induced from the map α , defined as the map that makes the following diagram commutative:

$$\begin{array}{ccc} KR^{-i}(\mathbb{T} \times X, c \times \zeta) \oplus KR^{-i}(\mathbb{T} \times X, c \times \zeta) & & \\ \downarrow \partial^{\text{pair}} & \searrow \partial^{\mathbb{T} \oplus -\partial^{\mathbb{T}}} & \\ KR^{-i+1}([[-2, 2], \{\pm 2\}] \times \mathbb{T} \times X, \text{id} \times c \times \zeta) & \xrightarrow{\alpha} & KR^{-i+1}(X, \zeta) \end{array}$$

Theorem 5.4 is proved in a parallel way as Theorem 4.1 by using real K -theory in place of complex K -theory.

6. Gapped Topological Invariants Related to Corner States

In this section, we discuss some bulk-edge gapped Hamiltonians on a lattice with a codimension-two corner or hinge. By using Theorem 4.1 and Theorem 5.4, we provide more geometric way to formulate a relation between topological invariants for such gapped Hamiltonians and corner/hinge states than that in [25, 27].

We consider a translation invariant Hamiltonian on the lattice \mathbb{Z}^d of the following form:

$$H: l^2(\mathbb{Z}^d, \mathbb{C}^N) \rightarrow l^2(\mathbb{Z}^d, \mathbb{C}^N), \quad H = \sum_{\text{finite}} a_{j_1 \dots j_d} S_1^{j_1} \dots S_d^{j_d},$$

where $a_{j_1 \dots j_d} \in M_N(\mathbb{C})$, S_j is the shift operator in the j -th direction and the subscript *finite* means that $a_{j_1 \dots j_d} = 0$ except for finitely many $(j_1, \dots, j_d) \in \mathbb{Z}^d$. We assume that H is self-adjoint and consider such Hamiltonians in each of the ten Altland–Zirnbauer classes. In classes A, AI and AII, we further assume that the spectrum of the bulk Hamiltonian is not contained in $\mathbb{R}_{>0}$ and $\mathbb{R}_{<0}$. Through the Fourier transform, the bulk Hamiltonian corresponds to a hermitian matrix-valued function $H: \mathbb{T}^d \rightarrow M_N(\mathbb{C})^{\text{sa}}$ on the d -dimensional Brillouin torus \mathbb{T}^d . We write (z_1, \dots, z_d) for an element in \mathbb{T}^d . Corresponding to our finite hopping range condition, each entry of this matrix consists of a d -variable Laurent polynomial; therefore, the bulk Hamiltonian H corresponds to a d -variable rational matrix function on \mathbb{T}^d . We next introduce our models for two edges and the corner. Note that, for each $\mathbf{z} = (z_3, \dots, z_d) \in \mathbb{T}^{d-2}$, the matrix function $H(\mathbf{z})$ on \mathbb{T}^2 is a two-variable rational matrix function. Let $H^0(\mathbf{z}) = T_{H(\mathbf{z})}^0$ and $H^\infty(\mathbf{z}) = T_{H(\mathbf{z})}^\infty$ be the associated half-plane Toeplitz operators, and let $H^{0,\infty}(\mathbf{z}) = T_{H(\mathbf{z})}^{0,\infty}$ be the associated quarter-plane Toeplitz operator. That is, for models of two edges, we consider the restrictions of our bulk Hamiltonian onto half-spaces $\mathbb{Z} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}^{d-2}$ and $\mathbb{Z}_{\geq 0} \times \mathbb{Z} \times \mathbb{Z}^{d-2}$, and for the model of codimension-two right angle corner, we consider the restriction onto the lattice $(\mathbb{Z}_{\geq 0})^2 \times \mathbb{Z}^{d-2}$, where we assume the Dirichlet boundary condition. We assume that, for any $\mathbf{z} \in \mathbb{T}^{d-2}$, half-plane Toeplitz operators $H^0(\mathbf{z})$ and $H^\infty(\mathbf{z})$ are invertible. Under this assumption, $H(\mathbf{z})$ is also invertible and $H^{0,\infty}(\mathbf{z})$ is Fredholm. Therefore, we assume that our model Hamiltonians for the bulk and two edges that makes the corner are gapped. Under this assumption, we discuss a relation between some gapped topological invariant and corner states. As in [27], the family of self-adjoint Fredholm operators $\{H^{0,\infty}(\mathbf{z})\}_{\mathbf{z} \in \mathbb{T}^{d-2}}$ defines an element of the complex K -group $K^{-i+1}(\mathbb{T}^{d-2})$ for classes A and AIII, or the KR -group $KR^{-i+1}(\mathbb{T}^{d-2}, c^{d-2})$ for classes AI, BDI, D, DIII, AII, CII, C and CI of some degree i corresponding to its Altland–Zirnbauer class \spadesuit as indicated in Table 2. We write $\mathcal{I}_{\text{Gapless}}^\spadesuit(H)$ for this element of the K -group. If $\mathcal{I}_{\text{Gapless}}^\spadesuit(H)$ is non-trivial, there exist topological corner/hinge states. For classes AIII, BDI, DIII, CII and CI where the Hamiltonians preserve chiral symmetry, a Hamiltonian H anti-commute pointwise with the chiral symmetry operator Π and can be represented by the off-diagonal form $H = \begin{pmatrix} 0 & h^* \\ h & 0 \end{pmatrix}$. This h is also a nonsingular d -variable rational matrix function on \mathbb{T}^d . As in [27], this H or h preserves the symmetries of Boersema–Loring’s formulation of complex or real K -theory groups. Under our assumption, by using matrix factorizations (Proposition 3.6 and Proposition 5.3), H or h on \mathbb{T}^d is extended to a nonsingular matrix-valued continuous map $H^E: \tilde{\mathbb{S}}^3 \times \mathbb{T}^{d-2} \rightarrow GL_N(\mathbb{C})^{\text{sa}}$ or $h^E: \tilde{\mathbb{S}}^3 \times \mathbb{T}^{d-2} \rightarrow GL_{\frac{N}{2}}(\mathbb{C})$. Corresponding to its Altland–Zirnbauer class, this matrix function defines the following element of the complex or real K -group which is a gapped topological invariant for our bulk-edge gapped system:

Table 2. i and \mathcal{H}^E for each of the Altland–Zirnbauer classes ♠

♠	A	AIII	AI	BDI	D	DIII	AII	CII	C	CI
i	0	1	0	1	2	3	4	5	6	-1
\mathcal{H}^E	$H^E \oplus 1_N$	h^E	$H^E \oplus 1_N$	h^E	H^E	h^E	$H^E \oplus 1_N$	h^E	H^E	h^E

Definition 6.1. In classes ♠ = A and AIII, we define,

$$\mathcal{I}_{\text{Gapped}}^\spadesuit(H) = [\mathcal{H}^E] \in K^{-i}(\tilde{\mathbb{S}}^3 \times \mathbb{T}^{d-2}).$$

In classes ♠ = AI, BDI, D, DIII, AII, CII, C and CI, we define,

$$\mathcal{I}_{\text{Gapped}}^\spadesuit(H) = [KR^{-i}(\tilde{\mathbb{S}}^3 \times \mathbb{T}^{d-2}, \nu \times c^{d-2})],$$

where i and \mathcal{H}^E are as indicated in Table 2.

As in [25, 27], from the pair of invertible half-space operators H^0 and H^∞ , we can define an element of the K -group $K_i(\mathcal{S}^{0,\infty} \otimes C(\mathbb{T}^{d-2}))$ or $KO_i(\mathcal{S}^{0,\infty} \otimes C(\mathbb{T}^{d-2}), \tau_{\mathcal{S}} \otimes \tau_{c^{d-2}})$, and $\mathcal{I}_{\text{Gapless}}^\spadesuit(H)$ is the image of this element through the boundary map ∂^{qT} of the long exact sequence of K -theory for C^* -algebras or KO -theory for $C^{*,\tau}$ -algebras associated with the extension (7) or (24). Therefore, by Theorem 4.1 and Theorem 5.4, we obtain the following result.

Theorem 6.2. In classes ♠ = A and AIII, we have

$$\beta(\mathcal{I}_{\text{Gapped}}^\spadesuit(H)) = \mathcal{I}_{\text{Gapless}}^\spadesuit(H) \in K^{-i+1}(\mathbb{T}^{d-2}),$$

In classes ♠ = AI, BDI, D, DIII, AII, CII, C and CI, we have,

$$\beta(\mathcal{I}_{\text{Gapped}}^\spadesuit(H)) = \mathcal{I}_{\text{Gapless}}^\spadesuit(H) \in KR^{-i+1}(\mathbb{T}^{d-2}, c^{d-2}),$$

where i and \mathcal{H}^E are as indicated in Table 2.

Theorem 6.2 provides a geometric formulation for a relation between a gapped topological invariant and corner states in [25, 27], though, since our three sphere $\tilde{\mathbb{S}}^3$ is not smooth as the boundary of $\mathbb{D}^2 \times \mathbb{D}^2$, an integration formula for numerical gapped invariants, like integration of the Berry curvature for the first Chern number, is still missing. At this stage, we simply note the following understanding of numerical gapped invariants for two-dimensional class AIII systems with a corner and three-dimensional class A systems with a hinge.

Example 6.3. For a two-dimensional class AIII bulk-edge gapped Hamiltonian on the lattice $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, our extension of the (off-diagonal part of the) bulk Hamiltonian defines an element $\mathcal{I}_{\text{Gapped}}^{\text{AIII}}(H) = [h^E] \in K^{-1}(\tilde{\mathbb{S}}^3)$. By Corollary 4.10, the three-dimensional winding number $W_3(h^E)$ of h^E is the same as the Fredholm index $\text{index} T_h^{0,\infty} = \text{Tr}(\Pi|_{\text{Ker} H^{0,\infty}}) = \mathcal{I}_{\text{Gapless}}^{\text{AIII}}(H)$ and accounts for topological corner states. The two-variable rational matrix function f in Example 4.9 provides an example in this class and corresponds to Benalcazar–Bernevig–Hughes’ two-dimensional model of a second-order topological insulator [10] as discussed in [26]. In [26], the topological invariant $\mathcal{I}_{\text{Gapless}}^{\text{AIII}}(H)$ is computed by expressing Benalcazar–Bernevig–Hughes’ model in a specific way, though Example 4.9 demonstrates an alternative way of computing invariants which is not restricted to the specific form.

Example 6.4. For a three-dimensional class A bulk-edge gapped Hamiltonian on the lattice $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}$, our gapped topological invariant is $\mathcal{I}_{\text{Gapped}}^A(H) = [H^E \oplus 1_N] \in K^0(\tilde{\mathbb{S}}^3 \times \mathbb{T})$. Since H^E is a continuous family of self-adjoint invertible matrices, we define a complex vector bundle E on $\tilde{\mathbb{S}}^3 \times \mathbb{T}$ whose fiber at $(z, w, t) \in \tilde{\mathbb{S}}^3 \times \mathbb{T}$ is

$$E_{(z,w,t)} = \bigcup_{\mu < 0} \text{Ker}(H^E(z, w, t) - \mu).$$

This vector bundle E is an extension of the Bloch bundle since H^E is an extension of the bulk Hamiltonian H . By Theorem 4.1 and the results in [25], the minus of the pairing of second Chern class of this extended Bloch bundle E with fundamental class of $\tilde{\mathbb{S}}^3 \times \mathbb{T}$ is the same as the spectral flow of the family of self-adjoint Fredholm operators $\{H^{0,\infty}(t)\}_{t \in \mathbb{T}}$, therefore accounts for the number of topological hinge states⁶.

Summarizing, we consider a gapped translation invariant single-particle Hamiltonian of finite hopping range on the lattice \mathbb{Z}^d in each of the ten Altland–Zirnbauer classes. We use Gohberg–Kreĭn theory to factorize the bulk Hamiltonian on the Brillouin torus about two variables z_1 and z_2 . For this purpose, there is a relevant algorithm since our bulk Hamiltonian corresponds to a multivariable rational matrix function on the torus \mathbb{T}^d [14, 20, 21]. If all of the partial indices of right matrix factorizations are trivial (equivalently, if compressions of the bulk Hamiltonian onto two half-spaces $\mathbb{Z} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}^{d-2}$ and $\mathbb{Z}_{\geq 0} \times \mathbb{Z} \times \mathbb{Z}^{d-2}$ are invertible), we define two topological invariants: One is defined through the restriction of the bulk Hamiltonian onto the lattice $(\mathbb{Z}_{\geq 0})^2 \times \mathbb{Z}^{d-2}$ assuming the Dirichlet boundary condition which provides a (family of) Fredholm operator(s) and its K -class $\mathcal{I}_{\text{Gapless}}^\blacklozenge(H)$ account for topological corner/hinge states. The other is defined as the K -class $\mathcal{I}_{\text{Gapped}}^\blacklozenge(H)$ of the extension of the bulk Hamiltonian onto $\tilde{\mathbb{S}}^3 \times \mathbb{T}^{d-2}$ obtained through matrix factorizations, which is our gapped topological invariant (Definition 6.1). There is a relation between these two topological invariants (Theorem 6.2), therefore, corresponding to the gapped topological invariant $\mathcal{I}_{\text{Gapped}}^\blacklozenge(H)$, corner states appear.

Acknowledgement. This work was supported by JSPS KAKENHI (Grant Nos. JP17H06461, JP19K14545) and JST PRESTO (Grant No. JPMJPR19L7).

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Declarations

Competing interests The author does not have any other competing interests to declare.

⁶ For its proof, we take a deformation of our three sphere $\tilde{\mathbb{S}}^3$ to the unit three sphere \mathbb{S}^3 in \mathbb{C}^2 . For two- and four-dimensional unit balls in \mathbb{C} and \mathbb{C}^2 , we consider spin^c structures they inherit as subspaces of Euclidean spaces. We equip \mathbb{T} and \mathbb{S}^3 their boundary spin^c structures and take the product spin^c structure on $\mathbb{S}^3 \times \mathbb{T}$. Then the result follows from Atiyah–Singer’s index formula for twisted spin^c Dirac operators [7].

Funding This work was supported by JSPS KAKENHI (Grant Nos. JP17H06461, JP19K14545) and JST PRESTO (Grant No. JPMJPR19L7).

Data availability This article does not have any external supporting data.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

- Alase, A.: Boundary Physics and Bulk-boundary Correspondence in Topological Phases of Matter. Springer Theses. Springer, Heidelberg ([2019] © 2019). Doctoral thesis accepted by Dartmouth University, USA
- Altland, A., Zirnbauer, M.R.: Nonstandard symmetry classes in mesoscopic normal-superconduction hybrid structures. *Phys. Rev. B* **55**(2), 1142–1161 (1997)
- Araki, H., Mizoguchi, T., Hatsugai, Y.: \mathbb{Z}_Q Berry phase for higher-order symmetry-protected topological phases. *Phys. Rev. Res.* **2**, 012009 (2020)
- Atiyah, M.F.: K -theory and reality. *Q. J. Math. Oxf. Ser.* **2**(17), 367–386 (1966)
- Atiyah, M.F.: K -Theory. Lecture Notes by D. W. Anderson. W. A. Benjamin, Inc., New York-Amsterdam (1967)
- Atiyah, M.F.: Bott periodicity and the index of elliptic operators. *Q. J. Math. Oxf. Ser.* **2**(19), 113–140 (1968)
- Atiyah, M.F., Singer, I.M.: The index of elliptic operators. III. *Ann. Math.* **2**(87), 546–604 (1968)
- Atiyah, M.F., Singer, I.M.: Index theory for skew-adjoint Fredholm operators. *Inst. Hautes Études Sci. Publ. Math.* **37**, 5–26 (1969)
- Bellissard, J., van Elst, A., Schulz-Baldes, H.: The noncommutative geometry of the quantum Hall effect. *J. Math. Phys.* **35**(10), 5373–5451 (1994)
- Benalcazar, W.A., Bernevig, B.A., Hughes, T.L.: Quantized electric multipole insulators. *Science* **357**, 61–66 (2017)
- Benalcazar, W.A., Cerjan, A.: Chiral-symmetric higher-order topological phases of matter. *Phys. Rev. Lett.* **128**, 127601 (2022)
- Boersema, J.L., Loring, T.A.: K -theory for real C^* -algebras via unitary elements with symmetries. *N. Y. J. Math.* **22**, 1139–1220 (2016)
- Bredon, G.E.: *Equivariant Cohomology Theories*. Lecture Notes in Mathematics, No. 34. Springer, Heidelberg (1967)
- Clancey, K.F., Gohberg, I.: Factorization of Matrix Functions and Singular Integral Operators. *Operator Theory: Advances and Applications*, vol. 3. Birkhäuser Verlag, Basel-Boston, Mass (1981)
- Coburn, L.A., Douglas, R.G., Singer, I.M.: An index theorem for Wiener–Hopf operators on the discrete quarter-plane. *J. Differ. Geom.* **6**, 587–593 (1972)
- Douglas, R.G., Howe, R.: On the C^* -algebra of Toeplitz operators on the quarterplane. *Trans. Am. Math. Soc.* **158**, 203–217 (1971)
- Duducava, R.V.: Discrete convolution operators on the quarter plane, and their indices. *Izv. Akad. Nauk SSSR Ser. Mat.* **41**(5), 1125–1137 (1977)
- Freed, D.S., Moore, G.W.: Twisted equivariant matter. *Ann. Henri Poincaré* **14**(8), 1927–2023 (2013)
- Geier, M., Trifunovic, L., Hoskam, M., Brouwer, P.W.: Second-order topological insulators and superconductors with an order-two crystalline symmetry. *Phys. Rev. B* **97**, 205135 (2018)
- Gohberg, I., Kaashoek, M.A., Spitkovsky, I.M.: An overview of matrix factorization theory and operator applications. In: *Factorization and Integrable Systems* (Faro, 2000). *Oper. Theory Adv. Appl.*, vol. 141, pp. 1–102. Birkhäuser, Basel (2003)
- Gohberg, I.C., Kreĭn, M.G.: Systems of integral equations on the half-line with kernels depending on the difference of the arguments. *Uspehi Mat. Nauk (N.S.)* **13**(2 (80)), 3–72 (1958)
- Graf, G.M., Porta, M.: Bulk-edge correspondence for two-dimensional topological insulators. *Commun. Math. Phys.* **324**(3), 851–895 (2013)
- Hatsugai, Y.: Chern number and edge states in the integer quantum Hall effect. *Phys. Rev. Lett.* **71**, 3697–3700 (1993)
- Hashimoto, K., Wu, X., Kimura, T.: Edge states at an intersection of edges of a topological material. *Phys. Rev. B* **95**, 165443 (2017)
- Hayashi, S.: Topological invariants and corner states for Hamiltonians on a three-dimensional lattice. *Commun. Math. Phys.* **364**(1), 343–356 (2018)
- Hayashi, S.: Toeplitz operators on concave corners and topologically protected corner states. *Lett. Math. Phys.* **109**(10), 2223–2254 (2019)

27. Hayashi, S.: Classification of topological invariants related to corner states. *Lett. Math. Phys.* **111**(5), 118 (2021)
28. Higson, N., Roe, J.: *Analytic K -homology*. Oxford Mathematical Monographs. Oxford University Press, Oxford Science Publications, Oxford (2000)
29. Kellendonk, J., Richter, T., Schulz-Baldes, H.: Edge current channels and Chern numbers in the integer quantum Hall effect. *Rev. Math. Phys.* **14**(1), 87–119 (2002)
30. Khalaf, E.: Higher-order topological insulators and superconductors protected by inversion symmetry. *Phys. Rev. B* **97**, 205136 (2018)
31. Khalaf, E., Po, H.C., Vishwanath, A., Watanabe, H.: Symmetry indicators and anomalous surface states of topological crystalline insulators. *Phys. Rev. X* **8**, 031070 (2018)
32. Kitaev, A.: Periodic table for topological insulators and superconductors. *AIP Conf. Proc.* **1134**(1), 22–30 (2009)
33. Michael, E.: Convex structures and continuous selections. *Can. J. Math.* **11**, 556–575 (1959)
34. Okugawa, R., Hayashi, S., Nakanishi, T.: Second-order topological phases protected by chiral symmetry. *Phys. Rev. B* **100**, 235302 (2019)
35. Okuma, N., Sato, M., Shiozaki, K.: Topological classification under nonmagnetic and magnetic point group symmetry: application of real-space Atiyah–Hirzebruch spectral sequence to higher-order topology. *Phys. Rev. B* **99**, 085127 (2019)
36. Park, E.: Index theory and Toeplitz algebras on certain cones in Z^2 . *J. Oper. Theory* **23**(1), 125–146 (1990)
37. Peterson, C.W., Benalcazar, W.A., Hughes, T.L., Gaurav, B.: A quantized microwave quadrupole insulator with topologically protected corner states. *Nature* **555**, 346–350 (2018)
38. Po, H.C., Vishwanath, A., Watanabe, H.: Symmetry-based indicators of band topology in the 230 space groups. *Nat. Commun.* **8**, 50 (2017)
39. Prodan, E., Schulz-Baldes, H.: *Bulk and Boundary Invariants for Complex Topological Insulators: From K -Theory to Physics*. Mathematical Physics Studies. Springer, Heidelberg (2016)
40. Rørdam, M., Larsen, F., Laustsen, N.: *An introduction to K -Theory for C^* -Algebras*. London Mathematical Society Student Texts, vol. 49. Cambridge University Press, Cambridge (2000)
41. Schindler, F., Cook, A.M., Vergniory, M.G., Wang, Z., Parkin, S.S.P., Bernevig, B.A., Neupert, T.: Higher-order topological insulators. *Sci. Adv.* **4**(6), eaat0346 (2018)
42. Schochet, C.: Topological methods for C^* -algebras. II. Geometric resolutions and the Künneth formula. *Pac. J. Math.* **98**(2), 443–458 (1982)
43. Simonenko, I.B.: Convolution type operators in cones. *Mat. Sb. (N.S.)* **74**(116), 298–313 (1967)
44. Spanier, E.H.: *Algebraic Topology*. McGraw-Hill Book Co., New York-Toronto, Ont.-London (1966)
45. Steenrod, N.: *The Topology of Fibre Bundles*. Princeton Mathematical Series, vol. 14. Princeton University Press, Princeton, NJ (1951)
46. Šubin, M.A.: Factorization of matrix functions dependent on a parameter in normed rings, and related questions in the theory of Noetherian operators. *Mat. Sb. (N.S.)* **73**(115), 610–629 (1967)
47. Takahashi, R., Tanaka, Y., Murakami, S.: Bulk-edge and bulk-hinge correspondence in inversion-symmetric insulators. *Phys. Rev. Research* **2**, 013300 (2020)
48. tom Dieck, T.: *Transformation Groups*. De Gruyter Studies in Mathematics, vol. 8. Walter de Gruyter & Co., Berlin (1987)
49. Trifunovic, L., Brouwer, P.W.: Higher-order bulk-boundary correspondence for topological crystalline phases. *Phys. Rev. X* **9**, 011012 (2019)
50. Wood, R.: Banach algebras and Bott periodicity. *Topology* **4**, 371–389 (1966)

Communicated by Y. Ogata