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Mathematical Structures of Non-perturbative Topological String Theory: From GW to DT Invariants

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Abstract: We study the Borel summation of the Gromov–Witten potential for the resolved conifold. The Stokes phenomena associated to this Borel summation are shown to encode the Donaldson–Thomas (DT) invariants of the resolved conifold, having a direct relation to the Riemann–Hilbert problem formulated by Bridgeland (Invent Math 216(1), 69–124, 2019). There exist distinguished integration contours for which the Borel summation reproduces previous proposals for the non-perturbative topological string partition functions of the resolved conifold. These partition functions are shown to have another asymptotic expansion at strong topological string coupling. We demonstrate that the Stokes phenomena of the strong-coupling expansion encode the DT invariants of the resolved conifold in a second way. Mathematically, one finds a relation to Riemann–Hilbert problems associated to DT invariants which is different from the one found at weak coupling. The Stokes phenomena of the strong-coupling expansion turn out to be closely related to the wall-crossing phenomena in the spectrum of BPS states on the resolved conifold studied in the context of supergravity by Jafferis and Moore (Wall crossing in local Calabi Yau manifolds, arXiv:0810.4909, 2008).

Contents

1. Introduction

The study of the geometric structures associated to quantum field and string theories has been extremely fruitful in revealing connections between different areas of mathematics as well as in putting forward organizing principles and relations for mathematical structures and invariants.

The focus of this work is on the connection of two types of invariants associated to families of Calabi–Yau (CY) threefolds. On the one hand, the Gromov–Witten invariants are characteristics of the enumerative geometry of maps into the CY. Their generating function is closely related to the partition function of topological string theory. The latter is a formal power series which is asymptotic in the topological string coupling constant. On the other hand, the Donaldson–Thomas or BPS invariants associated to the same geometry can be defined using the enumeration of coherent sheaves supported on holomorphic submanifolds on the same CY subject to a stability condition. Physically, the latter correspond to BPS states, which are realized by D-branes supported on subspaces of the CY geometry. The generating functions of BPS invariants are expected to correspond to physical partition functions of black holes. In physical terms, the topological string theory is obtained from a perturbative formulation of the underlying string theory, while the BPS invariants represent data representing non-perturbative effects in string theory. Relations between the two very different types of data and mathematical invariants have long been expected both from the points of view of physics [INOV08, DVV06, OSV04] as well as mathematics [MNOP06a, MNOP06b].

The link between GW and DT invariants is thus expected to be intimately related to the non-perturbative structure of topological string theory. Since the latter is defined by an asymptotic series in the topological string coupling, the most canonical path to its non-perturbative structure is to consider the theory of resurgence and Borel resummation; see [Mn14] and references therein for an overview. This has indeed been applied to topological string theory in connection with Chern–Simons theory and matrix models in [PS10] as well as for the resolved conifold in [HO15]. In particular, [HO15] used a generalization of the Borel resummation and produced via Borel resummation a partition function which matched the expectations of a proposal for the non-perturbative structure of topological string theory on non-compact CY manifolds put forward earlier in [HMnMO14,GHMn16]. A non-perturbative definition of the topological string free energy for general toric CY has been proposed in [GHMn16] in terms of the spectral determinants of the finite difference operators obtained by quantising the mirror curves. In [CSESV15], techniques of resurgence and transseries were applied to the study of topological string theory via the holomorphic anomaly equations of BCOV [BCOV94]; see also [CS14] and references therein. These techniques have been applied to the study of the proposal of [GHMn16] in [CSMnS17]. The link to BPS structures started to emerge more clearly recently [GGMn20b,GMn21] where connections between Stokes phenomena and BPS invariants have been investigated. See also [KS20,GGMn20a] for works in related directions.

On the side of DT invariants and BPS structures, exciting insights are coming from the study of wall-crossing phenomena. The wall-crossing formulas of Kontsevich and Soibelman [KS08] as well as Joyce and Song [JS12] have led to a lot of progress on wall-crossing phenomena of BPS states. In [GMN10,GMN13b,GMN13a], Gaiotto, Moore and Neitzke (GMN) provided a physical interpretation of these developments as well as new geometric constructions of hyperkähler manifolds having metrics determined by the BPS spectra; see e. g. [Nei14]. More recent developments are concerned with the analytic and integrable structures behind wall-crossing phenomena. The emerging links indicate new connections between DT invariants and GW invariants, going substantially beyond the scope of the MNOP relation [MNOP06a,MNOP06b]. Bridgeland [Bri19] formulated a Riemann–Hilbert associated to the Donaldson–Thomas invariants of a given derived category and defined an associated potential called Tau-function in [Bri19]. In simple examples including the resolved conifold [Bri20], it was shown that an asymptotic expansion of the Tau-function reproduces the full Gromov–Witten potential. In [CLT20] it was proposed that the topological string partition functions for a certain class of local CY represent local sections of certain canonical holomorphic line bundles defined by the relevant solutions to the Riemann-Hilbert problems from [Bri19].

In this paper, we will revisit the Borel summation of the resolved conifold partition function from a new perspective. We will show, on the one hand, that the Stokes jumps of the Borel summation of the expansion in powers of the topological string coupling have a close relation to the jumps defining the Riemann–Hilbert problem defined by Bridgeland using DT invariants as input data in [Bri20]. The Stokes jumps serve as certain types of potentials for the jumps of the Darboux coordinates defining the Riemann–Hilbert problem in [Bri20].

The Borel summations along different rays ρ are found to have the following structure

$$F_\rho(\lambda, t) = F_{\rm GV}(\lambda, t) + F_{\rm D}(\lambda, t; \rho), \tag{1.1}$$

where λ is the topological string coupling, and t the complexified Kähler parameter. The contribution denoted $F_{\rm GV}(\lambda, t)$ is the canonical re-organisation known from the work of Gopakumar and Vafa of the formal series in powers of λ as a series in powers of $Q = e^{2\pi i t}$ which is convergent for $\mathrm{Im}(t) > 0$. $F_{\rm GV}(\lambda, t)$ does not depend on the ray ρ. The second part, $F_{\rm D}(\lambda, t; \rho)$ strongly depends on the choice of a ray ρ. $F_{\rm D}(\lambda, t; \rho)$ can be represented as functions of the variables $Q' = e^{4\pi^2 \mathrm{i}t/\lambda}$ and $q' = e^{4\pi^2\mathrm{i}/\lambda}$, suggesting an interpretation in terms of non-perturbative effects associated to D-branes in type II string theory. It is known that there exist non-perturbative effects in string theory represented by disk amplitudes associated to stable D-branes. Closely related effects have recently been identified with non-perturbative corrections to the metric on the hypemultiplet moduli space in type II string theory on CY three-folds [ASS21]. It seems natural to interpret the jumps of $F_\rho(\lambda, t)$ across Stokes rays as the consequences of changes of the set of stable objects contributing to the non-perturbative effects in the partition functions. The explicit results for the jumps take a particularly simple form, having a direct relation to the Riemann-Hilbert problems associated to DT-invariants in [Bri20] further discussed below.

The results associated to different rays ρ interpolate between two special functions which had previously been proposed as candidates for non-perturbative definitions of the topological string partition functions: Integration along the imaginary axis yields the Gopakumar–Vafa resummation $F_{\mathrm{GV}}(\lambda, t)$ on the one hand, while choosing ρ to be the positive real axis, $\rho = \mathbb{R}_{>0}$, yields a function closely related to the triple sine function. In the case $\rho = \mathbb{R}_{>0}$, we find that the function $F_{\mathrm{D}}(\lambda, t; \rho)$ appearing in equation (1.1) can be expressed in terms of the previously known function $F_{\mathrm{NS}}(\lambda, t)$, which can be obtained from the refined version of F_{GV} introduced in [IKV09] in the limits studied by Nekrasov and Shatashvili [NS09]. The combination

$$F_{\mathrm{np}}(\lambda, t) := F_{\mathrm{GV}}(\lambda, t) + \frac{1}{2\pi \mathrm{i}} \frac{\partial}{\partial \lambda} \lambda F_{\mathrm{NS}}\big(\tfrac{4\pi^2}{\lambda}, \tfrac{2\pi}{\lambda}(t - \tfrac{1}{2})\big), \tag{1.2}$$

appearing on the right side of (1.1) in the case $\rho = \mathbb{R}_{>0}$ has been studied before as a candidate for a non-perturbative completion of the topological string partition function [HMnMO14]. Relations to previous work studying the function $F_{\mathrm{np}}(\lambda, t)$ in connection to topological string theory are further discussed in Sect. 2.4.

It turns out that there is an appealing way to encode the Stokes data geometrically, in line with the previous suggestions made in [CLT20]. It will be shown that the Stokes jumps can be interpreted as transition functions of a certain line bundle canonically associated to the solution of the Riemann–Hilbert problem considered by Bridgeland. We will show that this line bundle is closely related to the hyperholomorphic line bundles studied in relation to hyperkähler geometry in [APP11b,Nei11]. The Borel summations of the topological string partition functions represent local sections of this line bundle. In the previous work [CPT18,CLT20], the relations discovered in [GIL] had been used to demonstrate that the Fourier transforms of the topological string partition functions associated to a certain class of local CY are related to the isomonodromic tau-functions which represent local sections of this line bundle. Due to the absence of compact four-cycles, the tau-functions simply coincide with the topological string partition functions for the case at hand.

The Borel summation along the positive real axis appears to be distinguished in some ways. This function also has an asymptotic expansion for $\lambda \to \infty$, referred to as the strong-coupling expansion in the following. The Borel summations of the strong-coupling expansion along different rays ρ' are found to have the following structure:

$$F'_{\rho'}(\lambda, t) = F_{\mathrm{BPS}}(\lambda, t; \rho') + F_{\mathrm{NS}}(\lambda, t). \tag{1.3}$$

The contribution $F_{\mathrm{NS}}(\lambda, t)$ is now independent of ρ', while $F_{\mathrm{BPS}}(\lambda, t; \rho')$ exhibits jumps when ρ' crosses certain rays in the complex plane of the variable $1/\lambda$. We find that $Z_{\mathrm{BPS}}(\lambda, t; \rho') := e^{F_{\mathrm{BPS}}(\lambda, t; \rho')}$ is closely related to the counting functions for BPS states previously studied in the context of supergravity by Jafferis and Moore [JM08]. The Stokes jumps of $Z_{\mathrm{BPS}}(\lambda, t; \rho')$ display a precise correspondence to the wall-crossing behaviour of the counting functions for BPS states studied in [JM08]. In the case of $\rho' = \mathbb{R}_{>0}$ we recover $F_{\mathrm{GV}}(\lambda, t)$.

Mathematically one may again observe a close relation to a Riemann–Hilbert problem associated to DT theory. However, the jumps of $Z_{\mathrm{BPS}}(\lambda, t; \rho')$ now directly coincide with the jumps of a particular coordinate function in a close relative of Bridgeland's Riemann–Hilbert problem, as could have been expected from previous computations of $Z_{\mathrm{BPS}}(\lambda, t; \rho')$ on the basis of wall-crossing formulae [BLR19, Appendix A]. It should be stressed that both the location of jumps, and the functional form of the jumps are different for weak- and strong-coupling expansions. However, we find that both are determined by Riemann–Hilbert type problems associated to DT invariants, albeit quite remarkably in somewhat different ways.

At least in the example studied in this paper, we have identified two new ways to extract non-perturbative information on DT invariants from the GW invariants defining the topological string partition functions. Our results suggest that these data are deeply encoded in the analytic structures of non-perturbatively defined partition functions. The way this happens indicates close connections to string-theoretic S-duality conjectures, as will be briefly discussed in Sect. 6. We expect that key features of the resulting picture will be found in much larger classes of string backgrounds. Such generalisations are the subject of work in progress. A preliminary discussion can be found in [Te22].

2. Borel Summations of the Resolved Conifold Partition Function

We are going to study the formal series

$$\widetilde{F}(\lambda, t) = \frac{1}{\lambda^2}\mathrm{Li}_3(Q) + \frac{B_2}{2}\mathrm{Li}_1(Q) + \sum_{g=2}^{\infty} \lambda^{2g-2} \frac{(-1)^{g-1} B_{2g}}{2g(2g-2)!} \mathrm{Li}_{3-2g}(Q),$$

with $Q = \exp(2\pi \mathrm{i} t)$, and polylogarithms $\mathrm{Li}_s(z)$ and Bernoulli numbers B_n defined by

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}, \quad s \in \mathbb{C}, \qquad \frac{w}{e^w - 1} = \sum_{n=0}^{\infty} B_n \frac{w^n}{n!}. \tag{2.1}$$

Borel summation of this formal series will repackage the information contained in it in an interesting way, revealing non-obvious mathematical structures. Our goals in this section will be to state the results on the Stokes phenomena of the Borel sums of $\widetilde{F}(\lambda, t)$, to discuss some of its implications, and relations to previous results in the literature.

2.1. Motivation: Topological string theory on the resolved conifold. Topological string theory motivates the consideration of the topological string partition functions. One expects to be able to associate such partition functions to families of Calabi–Yau (CY) threefolds $X = X_t$, with $t = (t^1, \ldots, t^n)$ being a set of distinguished local coordinates on the CY Kähler moduli space $\mathcal{M}$ of dimension $n = h^{1,1}(X_t)$. The partition function is expected to be defined by an asymptotic series in the topological string coupling λ of the form

$$Z_{\mathrm{top}}(\lambda, t) = \exp\left(\sum_{g=0}^{\infty} \lambda^{2g-2} F^g(t) \right). \tag{2.2}$$

In order to provide a rigorous mathematical basis for the definition of topological string partition functions, one may start by defining the GW potential of a Calabi–Yau threefold X as the formal power series

$$\mathcal{F}(Q,\lambda)=\sum_{g\geq 0}\lambda^{2g-2}\mathcal{F}^g(Q)=\sum_{g\geq 0}\sum_{\beta\in H_2(X,\mathbb{Z})}\lambda^{2g-2}N^g_\beta\, Q^\beta\,, \tag{2.3}$$

where $Q^\beta=\prod_{r=1}^n Q_r^{\beta_r}$ if $\beta=\sum_{r=1}^n\beta_r\gamma_r$, with $\{\gamma_1,\dots,\gamma_n\}$ being an integral basis for $H_2(X,\mathbb{Z})$, and Q_r being formal variables for $r=1,\dots,n$. One may note that the term associated to $\beta=0$ is independent of the Kähler class β, motivating the decomposition

$$\mathcal{F}(Q,\lambda)=\mathcal{F}_0(\lambda)+\widetilde{\mathcal{F}}(Q,\lambda)\,, \tag{2.4}$$

where the contribution $\mathcal{F}_0(\lambda)$ takes the universal form [FP98]

$$\mathcal{F}_0(\lambda)=\sum_{g\geq 0}\lambda^{2g-2}F_0^g,\qquad F_0^g=\frac{\chi(X)(-1)^{g-1}\,B_{2g}\,B_{2g-2}}{4g(2g-2)\,(2g-2)!}\,,\quad g\geq 2\,, \tag{2.5}$$

with $\chi(X)$ being the Euler characteristic of X. The formal series $\widetilde{\mathcal{F}}(Q,\lambda)$ is defined as

$$\widetilde{\mathcal{F}}(Q,\lambda)=\sum_{g\geq 0}\sum_{\beta\in\Gamma}\lambda^{2g-2}[\mathrm{GW}]_{\beta,g}\,Q^\beta, \tag{2.6}$$

where $\Gamma=\{\beta\in H_2(X,\mathbb{Z});\,\beta\neq 0\}$, with $[\mathrm{GW}]_{\beta,g}$ being the Gromov–Witten invariants. In this way, one arrives at a precise definition of $\mathcal{F}(Q,\lambda)$ as a formal series.

There is a class of CY manifolds where the series $\mathcal{F}^g(Q)$ actually have finite radii of convergence, allowing us to define the functions $F^g(t)=\mathcal{F}^g(e^{2\pi\mathrm{i}t_1},\dots,e^{2\pi\mathrm{i}t_n})$, where $t=(t_1,\dots,t_n)$. The resulting power series in λ is not expected to be convergent, in general. One may hope, however, that there can exist analytic functions having the series $F(\lambda,t)=\sum_{g\geq 0}\lambda^{2g-2}F^g(t)$ as asymptotic expansion.
We are here considering a particular example of a CY manifold X called the resolved conifold. This CY threefold represents the total space of the rank two bundle over the projective line:

$$X:=\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbb{P}^1\,, \tag{2.7}$$

and corresponds to the resolution of the conifold singularity.

The GW potential for this geometry was determined in physics [GV98b,GV99], and in mathematics [FP00] with the following outcome for the non-constant maps:[1]

$$\widetilde{F}(\lambda,t)=\sum_{g=0}^{\infty}\lambda^{2g-2}\widetilde{F}^g(t)=\frac{1}{\lambda^2}\mathrm{Li}_3(Q)+\sum_{g=1}^{\infty}\lambda^{2g-2}\frac{(-1)^{g-1}B_{2g}}{2g(2g-2)!}\mathrm{Li}_{3-2g}(Q), \tag{2.8}$$

using the notation $Q=e^{2\pi\mathrm{i}t}$. The constant map constribution has the form (2.5) with $\chi(X)=2$ and $F_0^0=-\zeta(3)$. The value of F_0^1 only shifts $\mathcal{F}(Q,\lambda)$ by an overall constant, and its specific value won't be important. Our first goal will be to study the Borel summability of the series (2.8) and (2.5). This was first studied in [PS10]. The results presented below complete and clarify previous work on this subject, as will be discussed in more detail below.

[1] See also [MnM99] for the determination of F^g from a string theory duality and the explicit appearance of the polylogarithm expressions.

2.2. Statement of results for the Borel sum and its Stokes phenomena.. Here we state a theorem collecting the results that we wish to prove. The proof of each part will be presented in Sect. 3.

Before stating the theorem, we briefly recall the definition of Borel summation. Given a formal power series $a(\check{\lambda}) \in \check{\lambda}\mathbb{C}[[\check{\lambda}]]$, we consider its Borel transform $\mathcal{B}(a)(\xi)$, where

$$\mathcal{B}\colon \check{\lambda}\mathbb{C}[[\check{\lambda}]] \to \mathbb{C}[[\xi]], \quad \mathcal{B}(\check{\lambda}^{n+1}) = \frac{\xi^n}{n!}. \tag{2.9}$$

Let $\check{\lambda} \in \mathbb{C}^\times$ and let ρ be a ray from 0 to ∞ in the complex ξ-plane. If $\mathcal{B}(a)(\xi)$ defines an analytic function along ρ, we define the Borel sum of $a(\check{\lambda})$ at $\check{\lambda}$, along ρ by

$$\int_\rho \mathrm{d}\xi\, e^{-\xi/\check{\lambda}}\mathcal{B}(a)(\xi)\,. \tag{2.10}$$

If (2.10) is finite, we say $a(\check{\lambda})$ is Borel summable at $\check{\lambda}$, along ρ.

Theorem 2.1. *Consider the formal series*

$$\begin{aligned}\widetilde{F}(\lambda, t) &= \frac{1}{\lambda^2}\mathrm{Li}_3(Q) + \frac{B_2}{2}\mathrm{Li}_1(Q) + \sum_{g=2}^{\infty}\lambda^{2g-2}\frac{(-1)^{g-1}B_{2g}}{2g(2g-2)!}\mathrm{Li}_{3-2g}(Q)\\ &= \frac{1}{\lambda^2}\mathrm{Li}_3(Q) + \frac{B_2}{2}\mathrm{Li}_1(Q) + \Phi(\check{\lambda}, t)\,, \quad \check{\lambda} = \frac{\lambda}{2\pi}\,, \quad Q = e^{2\pi \mathrm{i} t}\,.\end{aligned} \tag{2.11}$$

Then we have the following:

(i) (Borel transform) For $t \in \mathbb{C}^\times$ with $|\mathrm{Re}(t)| < 1/2$, let $G(\xi, t) := \mathcal{B}(\Phi(-, t))(\xi)$ denote the Borel transform of $\Phi(\check{\lambda}, t)$. Then $G(\xi, t)$ converges for $|\xi| < 2\pi|t|$. Furthermore, $G(\xi, t)$ admits a series representation of the form

$$G(\xi, t) = \frac{1}{(2\pi)^2}\sum_{m\in\mathbb{Z}\setminus\{0\}}\frac{1}{m^3}\frac{1}{2\xi}\frac{\partial}{\partial\xi}\left(\frac{\xi^2}{1-e^{-2\pi\mathrm{i}t+\xi/m}} - \frac{\xi^2}{1-e^{-2\pi\mathrm{i}t-\xi/m}}\right). \tag{2.12}$$

We can use the above series representation to analytically continue $G(\xi, t)$ in the ξ variable to a meromorphic function with poles at $\xi = 2\pi\mathrm{i}(t+k)m$ for $k \in \mathbb{Z}$ and $m \in \mathbb{Z}\setminus\{0\}$.

(ii) (Borel sum) For $t \in \mathbb{C} - \mathbb{Z}$ and $k \in \mathbb{Z}$ let $l_k := \mathbb{R}_{<0}\cdot 2\pi\mathrm{i}(t+k)$ and $l_\infty := \mathrm{i}\mathbb{R}_{<0}$. Given any ray ρ from 0 to ∞ different from $\{\pm l_k\}_{k\in\mathbb{Z}} \cup \{\pm l_\infty\}$,

and λ in the half-plane $\mathbb{H}_\rho$ centered at ρ, we define the Borel sum of $\widetilde{F}(\lambda, t)$ along ρ as

$$F_\rho(\lambda, t) := \frac{1}{\lambda^2}\mathrm{Li}_3(Q) + \frac{B_2}{2}\mathrm{Li}_1(Q) + \int_\rho \mathrm{d}\xi\, e^{-\xi/\check{\lambda}}G(\xi, t)\,. \tag{2.13}$$

Taking $\rho = \mathbb{R}_{>0}$, and assuming that $\mathrm{Im}(t) > 0$ and $0 < \mathrm{Re}(t) < 1$, we have the following identity whenever $\mathrm{Re}(t) < \mathrm{Re}(\check{\lambda}+1)$:

$$F_{\mathbb{R}_{>0}}(\lambda, t) = -\int_{\mathbb{R}+\mathrm{i}0^+}\frac{\mathrm{d}u}{8u}\frac{e^{u(t-1/2)}}{\sinh(u/2)(\sinh(\check{\lambda}u/2))^2}\,. \tag{2.14}$$

(iii) (Stokes jumps) Let ρ_k be a ray in the sector determined by the Stokes rays l_k and l_{k-1}. Then if $\mathrm{Im}(t) > 0$*, on the overlap of their domains of definition in the* λ *variable we have*

$$\phi_{\pm l_k}(\lambda, t) := F_{\pm\rho_{k+1}}(\lambda, t) - F_{\pm\rho_k}(\lambda, t) = \frac{1}{2\pi \mathrm{i}} \partial_{\check{\lambda}} \Big(\check{\lambda}\, \mathrm{Li}_2\big(e^{\pm 2\pi \mathrm{i}(t+k)/\check{\lambda}}\big)\Big)\,. \tag{2.15}$$

If $\mathrm{Im}(t) < 0$*, then the previous jumps also hold provided ρ_{k+1} is interchanged with ρ_k in the above formula.*

(iv) (Limits to $\pm\mathrm{i}\mathbb{R}_{>0}$) Let ρ_k denote any ray between the rays l_k and l_{k-1}. Furthermore, assume that $0 < \mathrm{Re}(t) < 1$*,* $\mathrm{Im}(t) > 0$*,* $\mathrm{Re}(\lambda) > 0$*,* $\mathrm{Im}(\lambda) < 0$*, and* $\mathrm{Re}\, t < \mathrm{Re}(\check{\lambda}+1)$*. Then*

$$\lim_{k\to\infty} F_{\rho_k}(\lambda, t) = \lim_{k\to\infty} F_{-\rho_k}(-\lambda, t) = \sum_{k=1}^{\infty} \frac{e^{2\pi \mathrm{i} k t}}{k\big(2\sin\big(\frac{\lambda k}{2}\big)\big)^2}\,. \tag{2.16}$$

Furthermore, we can write the sum of the Stokes jumps along l_k for $k \geq 0$ as

$$\sum_{k=0}^{\infty} \phi_{l_k}(\lambda, t) = \frac{1}{2\pi \mathrm{i}} \partial_\lambda \Big(\lambda \sum_{l=1}^{\infty} \frac{w^l}{l^2(1-\widetilde{q}^l)}\Big), \quad w := e^{2\pi \mathrm{i} t/\check{\lambda}}, \ \ \widetilde{q} := e^{2\pi \mathrm{i}/\check{\lambda}}\,. \tag{2.17}$$

If, on the other hand, we take $0 < \mathrm{Re}(t) < 1$*,* $\mathrm{Im}(t) > 0$*,* $\mathrm{Re}(\lambda) > 0$*,* $\mathrm{Im}(\lambda) > 0$*,* $\mathrm{Re}(t) < \mathrm{Re}(\check{\lambda} + 1)$ *and furthermore assume that* $|e^{2\pi \mathrm{i} t/\check{\lambda}}| < 1$*, then we also have*

$$\lim_{k\to-\infty} F_{\rho_k}(\lambda, t) = \lim_{k\to-\infty} F_{-\rho_k}(-\lambda, t) = \sum_{k=1}^{\infty} \frac{e^{2\pi \mathrm{i} k t}}{k\big(2\sin\big(\frac{\lambda k}{2}\big)\big)^2}. \tag{2.18}$$

Let us note that under the assumptions of the first part of (iv), $\lim_{k\to\infty} F_{\rho_k}(\lambda, t)$ differs from $F_{\mathbb{R}_{>0}}(\lambda, t)$ by the sum over all jumps $\phi_{l_k}(\lambda, t)$ for $k \geq 0$, leading to the decomposition

$$F_{\mathbb{R}_{>0}}(\lambda, t) = \lim_{k\to\infty} F_{\rho_k}(\lambda, t) - \frac{1}{2\pi \mathrm{i}} \frac{\partial}{\partial \lambda} \Big(\lambda \sum_{l=1}^{\infty} \frac{w^l}{l^2(1-\widetilde{q}^l)}\Big). \tag{2.19}$$

As we will see in Proposition 3.15, this decomposition can be obtained by evaluating the integral on the right of (2.14) as a sum over residues. Part (iv) of the theorem above identifies the second term on the right of (2.19) with the sum over the Stokes jumps in the lower right quadrant of the Borel plane (Fig. 1).

We further remark that the constraint $\mathrm{Im}(t) > 0$ and $0 < \mathrm{Re}(t) < 1$ is needed to conclude that $F_{\mathbb{R}_{>0}}(\lambda, t) = F_{\mathrm{np}}(\lambda, t)$ and the statements of point (iv) of Theorem 2.1 and (2.19). These statements are nevertheless easy to generalize to more general ranges of t. For example, if $t \in \mathrm{i}\mathbb{R}_{>0}$, then we cannot perform the Borel sum along $\mathbb{R}_{>0}$, since l_0 coincides with $\mathbb{R}_{>0}$, while it is easy to check that $F_{\rho_0}(\lambda, t) = F_{\mathrm{np}}(\lambda, t)$ for ρ_0 between l_0 and l_{-1}.

It will turn out that the Borel summation $F_{0,\rho}(\lambda)$ of the formal series (2.5) is closely related to the value of the function $F_\rho(\lambda, 0)$ defined in Theorem 2.1. The relation will be found to be of the form

$$F_{0,\rho}(\lambda) = -F_\rho(\lambda, 0) - \frac{1}{12} \log \check{\lambda} + C, \tag{2.20}$$

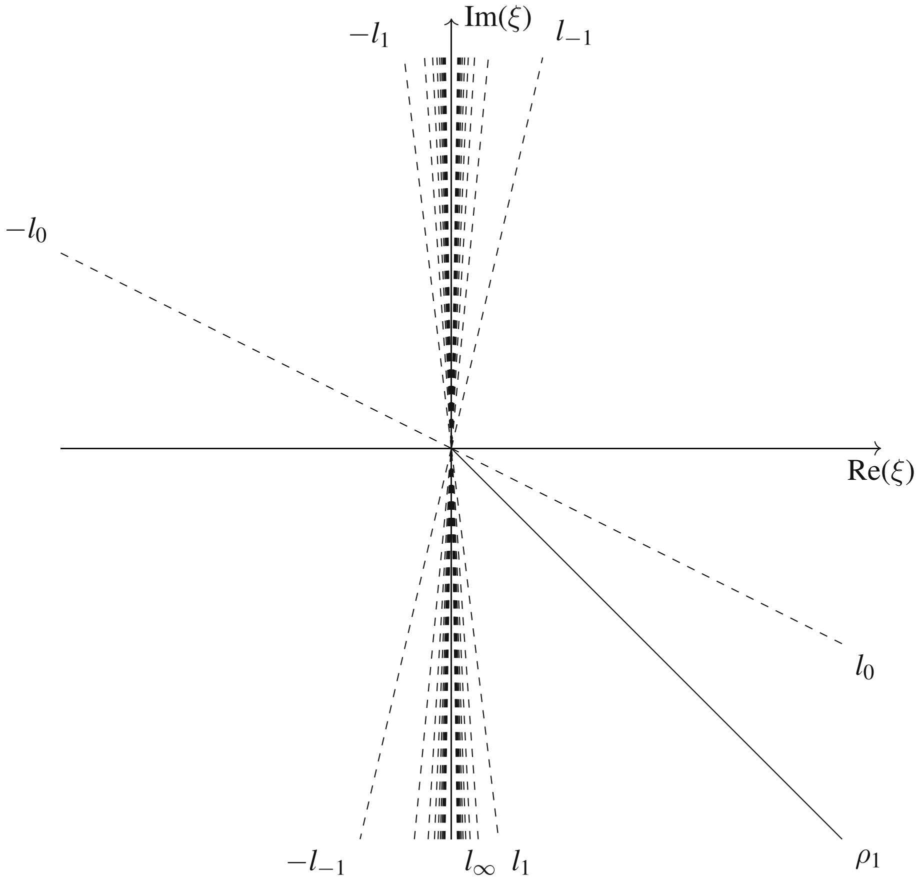

Fig. 1. Illustration of the Stokes rays $l_k = \mathbb{R}_{<0} \cdot 2\pi \mathrm{i}(t+k)$ in the Borel plane, plotted for $t = \frac{1}{\pi}\left(1 + \frac{\mathrm{i}}{2}\right)$ and $k = -10, \dots, 10$, as well as a possible integration ray ρ_1

where C is a constant independent of ρ which won't be of interest for us. Equation (2.20) finally allows us to represent the Borel summation $\hat{F}_\rho(\lambda, t)$ of the full free energy $F(\lambda, t) = F_0(\lambda) + \widetilde{F}(\lambda, t)$ of the topological string theory by the formula

$$\hat{F}_\rho(\lambda, t) = F_\rho(\lambda, t) - F_\rho(\lambda, 0) - \frac{1}{12} \log \check{\lambda} + C. \tag{2.21}$$

In the following two subsections we will first discuss the interpretation of Theorem 2.1 in the context of topological string theory. This will be followed by a discussion of the relation to previous results in this direction.

2.3. Connection to topological string theory. In the case of the resolved conifold, non-perturbative definitions of the topological string partition functions should be analytic functions of λ and t such that (2.8) gives an asymptotic series expansion for $\lambda \to 0$ of the corresponding free energy $\widetilde{F}(\lambda, t)$.

2.3.1

The Gopakumar–Vafa (GV) resummation of the GW potential [GV98a] re-organises the non-constant part $\widetilde{F}(\lambda, t)$ of the GW potential in the following form:

$$\sum_{g\geq 0}\lambda^{2g-2}\sum_{\beta\in\Gamma}[\mathrm{GW}]_{\beta,g}\,Q^{\beta}=\sum_{\beta\in\Gamma}\sum_{g\geq 0}[\mathrm{GV}]_{\beta,g}\sum_{k\geq 1}\frac{1}{k}\left(2\sin\left(\tfrac{k\lambda}{2}\right)\right)^{2g-2}Q^{k\beta}\,. \tag{2.22}$$

Equation (2.22) can be understood as an equality of formal power series in Q^{β} with coefficients being Laurent series in λ. One can thereby regard (2.22) as a definition of the GV invariants $[\mathrm{GV}]_{\beta,g}$ in terms of the Gromov–Witten invariants $[\mathrm{GW}]_{\beta,g}$.

Using the known results for the invariants $\mathrm{GW}_{\beta,g}$ of the conifold, one finds that the right-hand side of (2.22) simplifies to

$$F_{\mathrm{GV}}(\lambda,t)=\sum_{k=1}^{\infty}\frac{e^{2\pi\mathrm{i}kt}}{k\left(2\sin\left(\frac{\lambda k}{2}\right)\right)^{2}}\,. \tag{2.23}$$

This has also been derived using the topological vertex formalism [AKMnV05]. Assuming $\mathrm{Im}(t) > 0$, one may notice that the series defining $F_{\mathrm{GV}}(\lambda, t)$ in (2.23) is convergent for $\mathrm{Im}(\lambda) > 0$ or $\mathrm{Im}(\lambda) < 0$. One may regard $F_{\mathrm{GV}}(\lambda, t)$ as a minimal summation of the divergent series (2.11), in the sense that it is obtained by a rearrangement of the formal series $\widetilde{F}(\lambda, t)$ into a convergent series in powers of $Q = e^{2\pi\mathrm{i}t}$ that defines functions analytic in λ away from the real line $\mathbb{R}$.

Our results relate $F_{\mathrm{GV}}(\lambda, t)$ to the limits of the Borel summations along rays ρ_k for $k \to \pm\infty$ when the rays ρ_k approach the imaginary axis.

2.3.2

As mentioned above, the function $F_{\mathrm{GV}}(\lambda, t)$ is not well-defined for $\lambda \in \mathbb{R}$. This is one of the motivations to look for analytic functions having the same asymptotic expansion, but larger domains of definition, as candidates for non-perturbative definitions of the topological string partition functions.

A general proposal has been made in [HMnMO14] for non-perturbative definitions of topological string partition functions. This proposal was motivated by the observation [HMO13] that one can systematically add functions of $e^{(2\pi)^2\frac{\mathrm{i}}{\lambda}}$ to the function $F_{\mathrm{GV}}(\lambda, t)$ cancelling all the singularities that $F_{\mathrm{GV}}(\lambda, t)$ develops on the real λ-axis. The function of $e^{(2\pi)^2\frac{\mathrm{i}}{\lambda}}$ having this property can be interpreted as certain non-perturbative corrections in string theory.

Specialised to the conifold, the proposal made in [HMnMO14], see also [Ha15], yields

$$F_{\mathrm{np}}(\lambda,t):=F_{\mathrm{GV}}(\lambda,t)+\frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial\lambda}\lambda F_{\mathrm{NS}}\left(\tfrac{4\pi^2}{\lambda},\tfrac{2\pi}{\lambda}\left(t-\tfrac{1}{2}\right)\right), \tag{2.24}$$

using the notations

$$F_{\mathrm{GV}}(\lambda,t):=\sum_{k=1}^{\infty}\frac{e^{2\pi\mathrm{i}kt}}{k\left(2\sin\left(\frac{\lambda k}{2}\right)\right)^{2}}\,,\qquad F_{\mathrm{NS}}(g,t):=\frac{1}{2\mathrm{i}}\sum_{k=1}^{\infty}\frac{e^{2\pi\mathrm{i}kt}}{k^{2}\sin\left(\frac{gk}{2}\right)}\,. \tag{2.25}$$

It is easy to see that the right side of (2.24) coincides with the expression on the right of (2.19) (i.e. that $F_{\mathbb{R}_{>0}} = F_{\mathrm{np}}$).

2.3.3

Using Borel summation is another natural approach to finding non-perturbative definitions of the topological string partition functions, as previously investigated in [PS10] and in [HO15]. A formula for the Borel transform had been first proposed in [PS10], and in [HO15] it was conjectured that the Borel transform along the real axis is equal to (2.24). Extensive numerical studies provided convincing evidence for these proposals.

Our Theorem 2.1 offers a more complete picture. It shows that the Borel summations $F_\rho(\lambda, t)$ interpolate between $F_{\mathrm{GV}}(\lambda, t)$ and $F_{\mathrm{np}}(\lambda, t)$. All the functions $F_\rho(\lambda, t)$ defined by different choices of the ray ρ can be regarded as different re-packagings of the same information, contained in the formal series (2.8). Any of these summations can serve as a candidate for a non-perturbative definition of the topological string partition function of the resolved conifold. Additional requirements have to be imposed to distinguish a particular choice among others.

Defining the topological string partition functions by Borel summation whenever this possibility exists seems to be the most canonical way to associate actual functions to the divergent series (2.8). That is, we set $F_{\mathrm{can}}(\lambda, t) := F_{\rho_\lambda}(\lambda, t)$ where $\rho_\lambda = \mathbb{R}_{>0} \cdot \lambda$. The price to pay is that the resulting function is only piecewise analytic, having jumps across the rays $\pm l_k$. However, as will be explained in the rest of the paper, there is interesting information contained in these jumps. We are going to demonstrate that the jump functions encode information on the spectrum of BPS states on the resolved conifold in a particularly simple and transparent way by relating them to the Riemann–Hilbert problem formulated in [Bri20] which takes as input data the generalised DT invariants for the resolved conifold.

The Borel summations $F_{\rho_k}(\lambda, t)$ each have natural domains of definition, bounded by the rays l_k and l_{k-1}. It seems important to note, however, that the functions $F_{\rho_k}(\lambda, t)$ can be analytically continued in λ to larger domains of definition containing l_k and l_{k-1}, namely, the half plane $\mathbb{H}_{\rho_k}$. This suggests to regard the analytically continued functions $F_{\rho_k}(\lambda, t)$ as local sections of a line bundle defined by taking exponentials of the jumps $\phi_{l_k}(\lambda, t) = F_{\rho_{k+1}}(\lambda, t) - F_{\rho_k}(\lambda, t)$ as transition functions. This line bundle, together with the collection of distinguished local sections $F_{\rho_k}(\lambda, t)$ is a natural geometric object canonically associated to the formal series (2.8) by Borel summation. We will see that it is a natural analog of the line bundle proposed in [CLT20] for the case of the resolved conifold.

It should be noted that $F_{\mathrm{can}}(\lambda, t)$ is not well-defined for the case of $t \in \mathrm{i}\mathbb{R}_{>0}$ and $\lambda > 0$, where l_0 is on the real axis. It is clear from the results presented above that there is no canonical extension of our definitions covering such a case. Being mainly interested in the information provided by the series (2.8) alone, for the time being, we will not discuss possible prescriptions that could extend the definition to these cases in this paper.

2.3.4

Let us note that the differences $F_{\mathrm{D},k}(\lambda, t) := F_{\rho_k}(\lambda, t) - F_{\mathrm{GV}}(\lambda, t)$ can be represented as sums of terms which are all proportional to an exponential function having dependence with respect to the topological string coupling λ of the form $e^{(const.)/\lambda}$. It is therefore natural to associate the differences $F_{\mathrm{D},k}(\lambda, t)$ with non-perturbative effects in string theory. They can be represented as a sum over the Stokes jumps across the rays l_k enclosed by ρ_k and $\mathrm{i}\mathbb{R}$. We will see that these jumps are in a one-to-one correspondence with D-branes in type II string theory on the resolved conifold.

A dependence of the form of the form $e^{(const.)/\lambda}$ is characteristic for non-perturbative effects in string theory having a world-sheet description through disk amplitudes with boundaries associated to D-branes. Such disk amplitudes can represent central charge functions of D-branes in type II string theory [HIV00]. We will later see that the functions $F_{\mathrm{D},k}(\lambda, t)$ can indeed be expressed in terms of the central charge functions of the D-branes associated to the jumps. It turns out that the functions $F_{\mathrm{D},k}(\lambda, t)$ can be represented as sums over all terms which are exponentially suppressed in the wedge of the λ-plane bounded by l_k and l_{k-1}.

These observations suggest that the non-perturbative effects represented by the functions $F_{\mathrm{D},k}(\lambda, t)$ have a world-sheet description in terms of disk amplitudes with boundaries associated to stable D-branes representing states in the BPS-spectrum of the resolved conifold. The set of D-branes contributing to the non-perturbatively defined partition functions depends on the phase of λ, and jumps across the rays l_k.

2.4. Previous results. Previous work on this subject had obtained several important partial results. The first study of the Borel summability of the series $\widetilde{F}(\lambda, t)$ was performed in [PS10], where an explicit formula for the Borel transform was found. While the direct comparison of the formula derived in [PS10] with (2.12) is not completely straightforward, it is easy to see that the poles and residues agree.

Another approach to the summation of the formal series $\widetilde{F}(\lambda, t)$ has been proposed in [HO15]. The summation considered in [HO15] is an analytic function $F^{\mathrm{resum}}_{\mathrm{coni}}(\lambda, t)$ defined through an explicit integral representation. Numerical evidence has been presented for the conjecture that $F^{\mathrm{resum}}_{\mathrm{coni}}(\lambda, t)$ is equal to the Borel summation $F_{\mathbb{R}_{>0}}(\lambda, t)$ along $\rho = \mathbb{R}_{>0}$ in our notations. We will later in Sect. 3.4.1 explicitly establish the relation between $F^{\mathrm{resum}}_{\mathrm{coni}}(\lambda, t)$ and $F_{\mathbb{R}_{>0}}(\lambda, t)$ considered in our paper. It was furthermore proposed in [HO15] that the function $F^{\mathrm{resum}}_{\mathrm{coni}}(\lambda, t)$ admits the decomposition (2.24). This conjecture has been extensively checked numerically.

Interesting relations with spectral determinants of finite difference operators along the lines of [GHMn16] have been found in [BGT19]. Further exploration of the relations to our results should be illuminating.

It has been demonstrated in [Bri20] that a special function closely related to the triple sine function has (2.11) as its asymptotic expansion. The relation between the triple sine function and the formal series $\widetilde{F}(\lambda, t)$ has stimulated the work [Ali20, AS21, Ali21] studying the function defined on the right side of (2.14) as a promising candidate for a non-perturbative definition of the topological string partition function. It was identified in [AS21] as a solution with pleasant analytic properties of a difference equation [Ali20] which governs the topological string free energy. In [Ali21], the non-perturbative content of this function was extracted demonstrating that this function admits the decomposition (2.24) and matching in particular with the results of [HO15].

Further work on the function $F_{\mathrm{np}}(\lambda, t)$ in connection with the non-perturbative structure of topological strings can be found in [LV18, KM15].

3. Proofs of the Results of Sect. 2.2

In this section we prove each of the points of Theorem 2.1. Our approach is strongly inspired by the paper [GK20] which has studied the analogous problem for the non-compact quantum dilogarithm function. Each of the four subsections below corresponds to each of the four points of the Theorem.

3.1. The Borel transform. We start by proving the first part of Theorem 2.1, concerning the Borel transform of of $\widetilde{F}(\lambda, t)$. We remark that an alternative expression of the Borel transform was previously given in [PS10], which we recall in Sect. 3.1.1.

Recall the asymptotic expansion of the topological string free energy for the resolved conifold, which is given by (2.8):

$$\begin{aligned}\widetilde{F}(\lambda, t) &= \sum_{g=0}^{\infty} \lambda^{2g-2} \widetilde{F}^g(t) \\ &= \frac{1}{\lambda^2}\mathrm{Li}_3(Q) + \frac{B_2}{2}\mathrm{Li}_1(Q) + \sum_{g=2}^{\infty} \lambda^{2g-2} \frac{(-1)^{g-1} B_{2g}}{2g(2g-2)!} \mathrm{Li}_{3-2g}(Q) \\ &= \frac{1}{\lambda^2}\mathrm{Li}_3(Q) + \frac{B_2}{2}\mathrm{Li}_1(Q) + \Phi(\check{\lambda}, t), \quad \check{\lambda} = \frac{\lambda}{2\pi}, \quad Q = e^{2\pi i t}, \end{aligned} \tag{3.1}$$

We use the property

$$\theta_Q \mathrm{Li}_s(Q) = \mathrm{Li}_{s-1}(Q), \quad \theta_Q := Q \frac{d}{dQ}, \tag{3.2}$$

to write

$$\widetilde{F}^g = \frac{(-1)^{g-1} B_{2g}}{2g(2g-2)!} \theta_Q^{2g} \mathrm{Li}_3(Q), \quad g \geq 2. \tag{3.3}$$

Furthermore, using that $\theta_Q = \frac{1}{2\pi \mathrm{i}} \partial_t$ we obtain

$$\widetilde{F}^g = \frac{(-1) B_{2g}}{2g(2g-2)!(2\pi)^{2g}} \partial_t^{2g} \mathrm{Li}_3(Q), \quad g \geq 2. \tag{3.4}$$

We thus have

$$\Phi(\check{\lambda}, t) = -\frac{1}{4\pi^2} \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)!} \check{\lambda}^{2g-2} \partial_t^{2g} \mathrm{Li}_3(Q). \tag{3.5}$$

We now wish to compute the Borel transform of $\Phi(\check{\lambda}, t)$ and specify its domain of convergence. The Borel transform is defined as the formal power series $G(\xi, t) := \mathcal{B}(\Phi(-, t))(\xi)$, where

$$\mathcal{B} \colon \check{\lambda}\mathbb{C}[[\check{\lambda}]] \to \mathbb{C}[[\xi]], \quad \mathcal{B}(\check{\lambda}^{n+1}) = \frac{\xi^n}{n!}. \tag{3.6}$$

Namely, we wish to study

$$G(\xi, t) = -\frac{1}{4\pi^2} \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)!(2g-3)!} \xi^{2g-3} \partial_t^{2g} \mathrm{Li}_3(Q). \tag{3.7}$$

In order to do this, it will be convenient to first recall the Hadamard product and a certain integral representation thereof. The techniques used below follow the lines of [GK20].

Definition 3.1. Consider two formal power series $\sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} b_n z^n \in \mathbb{C}[[z]]$. Then the Hadamard product $\circledast\colon \mathbb{C}[[z]] \times \mathbb{C}[[z]] \to \mathbb{C}[[z]]$ is defined by

$$\Big(\sum_{n=0}^{\infty} a_n z^n\Big) \circledast \Big(\sum_{n=0}^{\infty} b_n z^n\Big) = \sum_{n=0}^{\infty} a_n b_n z^n \,. \tag{3.8}$$

Lemma 3.2. *Consider two holomorphic functions near $z = 0$ having series expansions*

$$f_1(z) = \sum_{n=0}^{\infty} a_n z^n, \quad f_2(z) = \sum_{n=0}^{\infty} b_n z^n \tag{3.9}$$

with radius of convergence $r_1 > 0$ and $r_2 > 0$, respectively. Then $(f_1 \circledast f_2)(z)$ converges for $|z| < r_1 r_2$, and for any $\rho \in (0, r_1)$ the following holds for $|z| < \rho r_2$:

$$(f_1 \circledast f_2)(z) = \frac{1}{2\pi i} \int_{|s|=\rho} \frac{\mathrm{d}s}{s} f_1(s) f_2\Big(\frac{z}{s}\Big) \,. \tag{3.10}$$

Proof. By using the limsup definition of the radius of convergence, one can easily check that the radius of convergence of $(f_1 \circledast f_2)(z)$ must be bigger or equal than $r_1 r_2$. On the other hand, we have that for $|z| < \rho r_2 < r_1 r_2$,

$$\begin{aligned}(f_1 \circledast f_2)(z) &= \sum_{n=0}^{\infty} a_n b_n z^n = \sum_{n=0}^{\infty} \Big(\frac{1}{2\pi \mathrm{i}} \int_{|s|=\rho} \mathrm{d}s \, \frac{f_1(s)}{s^{n+1}}\Big) b_n z^n \\ &= \frac{1}{2\pi \mathrm{i}} \int_{|s|=\rho} \frac{\mathrm{d}s}{s} f_1(s) \sum_{n=0}^{\infty} b_n \Big(\frac{z}{s}\Big)^n = \frac{1}{2\pi \mathrm{i}} \int_{|s|=\rho} \frac{\mathrm{d}s}{s} f_1(s) f_2\Big(\frac{z}{s}\Big)\end{aligned} \tag{3.11}$$

where the interchange of sum and integrals is justfied by the Fubini–Tonelli theorem and the absolute convergence of $(f_1 \circledast f_2)(z)$. □

The idea is to write

$$G(\xi, t) = (f_1 \circledast f_2(-, t))(\xi) \tag{3.12}$$

for two functions $f_1(\xi)$, $f_2(\xi, t)$ which are holomorphic near $\xi = 0$, and then use the first part of the previous lemma. We will take $f_1(\xi)$, $f_2(\xi, t)$ to be the following:

$$\begin{aligned} f_1(\xi) &= -\frac{1}{4\pi^2} \sum_{g=2}^{\infty} \frac{(2g-1)\, B_{2g}}{(2g)!} \xi^{2g-3} \\ f_2(\xi, t) &= \sum_{g=2}^{\infty} \frac{\xi^{2g-3}}{(2g-3)!} \, \partial_t^{2g} \mathrm{Li}_3(Q) = \sum_{g=2}^{\infty} \frac{\xi^{2g-3}}{(2g-3)!} \, (2\pi \mathrm{i})^{2g} \mathrm{Li}_{3-2g}(Q) \,. \end{aligned} \tag{3.13}$$

Proposition 3.3. *Let $t \in \mathbb{C}^\times$ with $|\mathrm{Re}(t)| < \frac{1}{2}$. Then $G(\xi, t)$ converges for $|\xi| < 2\pi |t|$.*

Proof. Using the fact that

$$B_{2g} \sim (-1)^{g+1} 4\sqrt{\pi g}\Big(\frac{g}{\pi e}\Big)^{2g} \quad \text{as} \quad g \to \infty, \tag{3.14}$$

we find that the radius of convergence for $f_1(\xi)$ is 2π. On the other hand, using the fact that for $|\mathrm{Re}(t)| < 1/2$, we have

$$\mathrm{Li}_{3-2g}(e^{2\pi \mathrm{i} t}) \sim \Gamma(1-3+2g)(-2\pi \mathrm{i} t)^{3-2g-1} \quad \text{as} \quad g \to \infty, \tag{3.15}$$

we find that the radius of convergence of $f_2(\xi, t)$ is $r_2(t) = |t|$.

By the use of Lemma 3.2, we find that provided $t \in \mathbb{C}^\times$ satisfies $|\mathrm{Re}(t)| < \frac{1}{2}$, we have that $G(\xi, t) = (f_1 \circledast f_2(-, t))(\xi)$ converges for $|\xi| < r_1 r_2(t) = 2\pi|t|$. □

We now wish to use the integral representation of the Hadamard product to find a more convenient representation of $G(\xi, t)$.

Proposition 3.4. *With the same hypothesis as in Proposition 3.3, we have*

$$G(\xi, t) = \frac{1}{(2\pi)^2} \sum_{m \in \mathbb{Z}\setminus\{0\}} \frac{1}{m^3}\Big(1 + \frac{\xi}{2}\frac{\partial}{\partial \xi}\Big)\Big(\frac{1}{1 - e^{-2\pi \mathrm{i} t + \xi/m}} - \frac{1}{1 - e^{-2\pi \mathrm{i} t - \xi/m}}\Big) \tag{3.16}$$

In particular, for fixed t, the expression on the right allows us to analytically continue $G(\xi, t)$ to a meromorphic function in ξ with poles at $\xi = 2\pi \mathrm{i}(t + k)m$ for $k \in \mathbb{Z}$ and $m \in \mathbb{Z} \setminus \{0\}$.

Proof. The idea is to now use the integral representation of the Hadamard product in Lemma 3.2, together with the results in the proof of Proposition 3.3. In particular, for $t \in \mathbb{C}^\times$ with $|\mathrm{Re}(t)| < 1/2$ and $\rho \in (0, 2\pi)$, we have for $|\xi| < \rho|t|$

$$G(\xi, t) = \frac{1}{2\pi \mathrm{i}} \int_{|s|=\rho} \frac{\mathrm{d}s}{s} f_1(s) f_2\Big(\frac{\xi}{s}, t\Big) \tag{3.17}$$

where $f_1(\xi)$ and $f_2(\xi, t)$ are as in Proposition 3.3.

We have, on the one hand,

$$\begin{aligned} f_1(\xi) &= -\frac{1}{4\pi^2} \sum_{g=2}^{\infty} \frac{(2g-1)\, B_{2g}}{(2g)!} \xi^{2g-3} = -\frac{1}{4\pi^2}\frac{1}{\xi}\partial_\xi \left(\frac{1}{\xi}\sum_{g=2}^{\infty}\frac{B_{2g}}{(2g)!}\xi^{2g}\right) \\ &= -\frac{1}{4\pi^2}\frac{1}{\xi}\partial_\xi \left(\frac{1}{\xi}\left(\sum_{g=0}^{\infty}\frac{B_g}{g!}\xi^g - 1 + \frac{\xi}{2} - \frac{\xi^2}{12}\right)\right) \\ &= -\frac{1}{4\pi^2}\frac{1}{\xi}\partial_\xi \left(\frac{1}{\xi}\left(\frac{\xi}{e^\xi - 1} - 1 + \frac{\xi}{2} - \frac{\xi^2}{12}\right)\right) \\ &= -\frac{1}{4\pi^2}\left(\frac{1}{\xi^3} - \frac{1}{\xi(e^{\xi/2} - e^{-\xi/2})^2} - \frac{1}{12\xi}\right), \end{aligned} \tag{3.18}$$

where we have used the expression for the generating function of the Bernoulli numbers

$$\frac{w}{e^w - 1} = \sum_{n=0}^{\infty} B_n \frac{w^n}{n!} , \tag{3.19}$$

and the fact that except $B_1 = -\frac{1}{2}$, all odd Bernoulli numbers vanish. From the final expression we see that $f_1(\xi)$ admits an analytic continuation to a meromorphic function with double poles at $\xi = 2\pi \mathrm{i}\mathbb{Z} \setminus \{0\}$.

On the other hand, for $f_2(\xi, t)$, we have

$$\begin{aligned} f_2(\xi, t) &= \sum_{g=2}^{\infty} \frac{\xi^{2g-3}}{(2g-3)!} \partial_t^{2g} \mathrm{Li}_3(Q) = \partial_\xi^3 \left(\sum_{g=1}^{\infty} \frac{\partial_t^{2g} \mathrm{Li}_3(Q)}{(2g)!} \xi^{2g} \right) \\ &= \partial_\xi^3 \left(\frac{1}{2} (\mathrm{Li}_3(e^{2\pi \mathrm{i}(t+\xi)}) + \mathrm{Li}_3(e^{2\pi \mathrm{i}(t-\xi)})) - \mathrm{Li}_3(e^{2\pi \mathrm{i} t}) \right) \\ &= \frac{(2\pi \mathrm{i})^3}{2} \left(\mathrm{Li}_0\left(e^{2\pi \mathrm{i}(t+\xi)}\right) - \mathrm{Li}_0\left(e^{2\pi \mathrm{i}(t-\xi)}\right) \right) \\ &= \frac{(2\pi \mathrm{i})^3}{2} \left(\frac{e^{2\pi \mathrm{i}(t+\xi)}}{1 - e^{2\pi \mathrm{i}(t+\xi)}} - \frac{e^{2\pi \mathrm{i}(t-\xi)}}{1 - e^{2\pi \mathrm{i}(t-\xi)}} \right), \end{aligned} \tag{3.20}$$

so that $f_2(\xi, t)$ admits an analytic continuation in ξ with simple poles at $\pm t + \mathbb{Z}$. The integral representation then becomes

$$G(\xi, t) = \frac{1}{2} \int_{|s|=\rho} \frac{\mathrm{d}s}{s} \left(\frac{1}{s^3} - \frac{e^s}{s(e^s - 1)^2} - \frac{1}{12s} \right) \left(\frac{e^{2\pi \mathrm{i}(t+\xi/s)}}{1 - e^{2\pi \mathrm{i}(t+\xi/s)}} - \frac{e^{2\pi \mathrm{i}(t-\xi/s)}}{1 - e^{2\pi \mathrm{i}(t-\xi/s)}} \right). \tag{3.21}$$

Notice that $f_2(\xi/s, t)$ as a function of s has simple poles at $s = \pm\xi/(t+k)$ for all $k \in \mathbb{Z}$. By our assumption that $|\xi| < \rho|t|$ and $|\mathrm{Re}(t)| < \frac{1}{2}$, we have

$$\left| \frac{\pm\xi}{t+k} \right| < \rho \frac{|t|}{|t+k|} \le \rho , \tag{3.22}$$

so that all the poles of $f_2(\xi/s, t)$ lie inside the contour. Furthermore, since $\rho < 2\pi$, all the poles of $f_1(s)$ lie outside the contour.

If, for each $k \in \mathbb{Z}$ with $k > 1$, we denote as γ_k the contour given by $|s| = \pi(2k+1)$, then between $|s| = \rho$ and γ_k, we have the poles at $\pm 2\pi \mathrm{i} n$ for $n = 1, \ldots, k$ due to $f_1(s)$. We can therefore write the following for any $k > 1$:

$$\begin{aligned} G(\xi, t) &= \frac{1}{2} \int_{|s|=\rho} \frac{\mathrm{d}s}{s} \left(\frac{1}{s^3} - \frac{e^s}{s(e^s - 1)^2} - \frac{1}{12s} \right) \left(\frac{e^{2\pi \mathrm{i}(t+\xi/s)}}{1 - e^{2\pi \mathrm{i}(t+\xi/s)}} - \frac{e^{2\pi \mathrm{i}(t-\xi/s)}}{1 - e^{2\pi \mathrm{i}(t-\xi/s)}} \right) \\ &= 2\pi \mathrm{i} \sum_{m \in \mathbb{Z}: -k<m<k, m \neq 0} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{e^s (s - 2\pi \mathrm{i} m)^2}{(e^s - 1)^2 s^2} \left(\frac{e^{2\pi \mathrm{i}(t+\xi/s)}}{1 - e^{2\pi \mathrm{i}(t+\xi/s)}} - \frac{e^{2\pi \mathrm{i}(t-\xi/s)}}{1 - e^{2\pi \mathrm{i}(t-\xi/s)}} \right) \right) \Bigg|_{s=2\pi \mathrm{i} m} \\ &\quad + \frac{1}{2} \int_{\gamma_k} \frac{\mathrm{d}s}{s} \left(\frac{1}{s^3} - \frac{e^s}{s(e^s - 1)^2} - \frac{1}{12s} \right) \left(\frac{e^{2\pi \mathrm{i}(t+\xi/s)}}{1 - e^{2\pi i(t+\xi/s)}} - \frac{e^{2\pi \mathrm{i}(t-\xi/s)}}{1 - e^{2\pi \mathrm{i}(t-\xi/s)}} \right), \end{aligned} \tag{3.23}$$

where the terms in the sum come from the contribution of the (clockwise) contours around the poles between the two contours.

One can check that $f_2(\xi/s,t) = \mathcal{O}(1/s)$ as $s \to \infty$, while $f_1|_{\gamma_k} = \mathcal{O}(1/k)$ as $k \to \infty$. Hence, taking the limit $k \to \infty$ in (3.23) we obtain the following expression:

$$\begin{aligned} G(\xi,t) = 2\pi \mathrm{i} \sum_{m\in\mathbb{Z}-\{0\}} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{e^s(s-2\pi \mathrm{i}m)^2}{(e^s-1)^2 s^2}\left(\frac{e^{2\pi\mathrm{i}(t+\xi/s)}}{1-e^{2\pi\mathrm{i}(t+\xi/s)}} - \frac{e^{2\pi\mathrm{i}(t-\xi/s)}}{1-e^{2\pi\mathrm{i}(t-\xi/s)}}\right)\right)\Bigg|_{s=2\pi\mathrm{i}m} \\ = -\sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{1}{(2\pi\mathrm{i})^2}\left(\frac{1}{m^3}\left(\frac{e^{2\pi\mathrm{i}t+\xi/m}}{1-e^{2\pi\mathrm{i}t+\xi/m}} - \frac{e^{2\pi\mathrm{i}t-\xi/m}}{1-e^{2\pi\mathrm{i}t-\xi/m}}\right)\right. \\ \left. + \frac{\xi}{2m^4}\left(\frac{e^{2\pi\mathrm{i}t+\xi/m}}{(1-e^{2\pi\mathrm{i}t+\xi/m})^2} + \frac{e^{2\pi\mathrm{i}t-\xi/m}}{(1-e^{2\pi\mathrm{i}t-\xi/m})^2}\right)\right). \end{aligned} \tag{3.24}$$

The result then follows by a simple rewriting of the summands. □

A direct check that $G(\xi,t)$ is the Borel transform of $\widetilde{F}(\lambda,t)$ can be found in Lemma B.1 from Appendix B.

3.1.1. Previous expression for the Borel transform. In the following we review an expression for the Borel transform of the topological string free energy for the resolved conifold obtained in [PS10], starting again with

$$\begin{aligned} \widetilde{F}(\lambda,t) &= \sum_{g=0}^{\infty} \lambda^{2g-2}\widetilde{F}^g(t) \\ &= \frac{1}{\lambda^2}\mathrm{Li}_3(q) + \frac{B_2}{2}\mathrm{Li}_1(q) + \sum_{g=2}^{\infty}\lambda^{2g-2}\frac{(-1)^{g-1}B_{2g}}{2g(2g-2)!}\mathrm{Li}_{3-2g}(q) \\ &= \frac{1}{\lambda^2}\mathrm{Li}_3(q) + \frac{B_2}{2}\mathrm{Li}_1(q) + \Phi(\check{\lambda},t), \quad \check{\lambda} = \frac{\lambda}{2\pi}. \end{aligned} \tag{3.25}$$

Using the series representation of the polylogarithm

$$\mathrm{Li}_s(e^{2\pi i t}) = \Gamma(1-s)\sum_{k\in\mathbb{Z}}(2\pi i)^{s-1}(k-t)^{s-1}, \tag{3.26}$$

valid for $\mathrm{Re}(s) < 0$ and $t \notin \mathbb{Z}$, we can write

$$\begin{aligned} \Phi(\check{\lambda},t) &= \sum_{g=2}^{\infty}\lambda^{2g-2}\frac{(-1)^{g-1}B_{2g}}{2g(2g-2)!}\mathrm{Li}_{3-2g}(q) \\ &= \sum_{g=2}^{\infty}\check{\lambda}^{2g-2}\frac{B_{2g}}{2g(2g-2)}\sum_{k\in\mathbb{Z}}(k-t)^{2-2g}. \end{aligned} \tag{3.27}$$

Taking the Borel transform of $\Phi(\check{\lambda},t)$, we find

$$\begin{aligned} G(\xi,t) &= \sum_{g=2}^{\infty}\frac{B_{2g}}{2g\,(2g-2)!}\xi^{2g-3}\sum_{k\in\mathbb{Z}}(k-t)^{2-2g} \\ &= \sum_{k\in\mathbb{Z}}\frac{1}{\xi}\left(\frac{(k-t)^2}{\xi^2} - \frac{e^{\xi/(k-t)}}{(e^{\xi/(k-t)}-1)^2} - \frac{1}{12}\right), \end{aligned} \tag{3.28}$$

where the second equality follows from

$$\frac{e^w}{(e^w-1)^2} = \frac{1}{w^2} - \frac{1}{12} - \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)!} w^{2g-2}, \tag{3.29}$$

which can be obtained by taking a derivative of the generating function of Bernoulli numbers (3.19) and rearranging the outcome. We note here that this expression for the Borel transform, which was previously obtained in [PS10] has the same set of poles at

$$\xi = 2\pi \mathrm{i} m(t+k), \quad m \in \mathbb{Z} \setminus \{0\}, \quad k \in \mathbb{Z},$$

as the one we have obtained in Theorem 2.1. We will later show that it also has the same Stokes jumps.

One advantage of the expression (3.16) compared to (3.28) is that the first gives a well-defined expression for $t \in \mathbb{Z}$. As we will see below, this will allow us to express the Borel transform of constant map contribution of (2.4) in terms of $G(\xi, 0)$ (see Corollary 3.13).

Remark 3.5. It seems one might be able to obtain (3.28) from the integral representation (3.21) by deforming the contour to 0 instead of ∞, and summing over the residues of the poles inside the contour. The only technical issue is that it seems harder to show that the contour limiting to $s = 0$ limits to a zero contribution.

3.2. The Borel sum along $\mathbb{R}_{>0}$. We now prove the second point of Theorem 2.1. Hence, we wish to study the Borel sum of $\widetilde{F}(\lambda, t)$ along $\mathbb{R}_{>0}$. More generally, we define the following:

Definition 3.6. Given $t \in \mathbb{C}$ with $\mathrm{Im}(t) \neq 0$, and a ray $\rho \subset \mathbb{C}^\times$ from 0 to ∞ avoiding the poles $2\pi\mathrm{i}(t+k)m$ of $G(\xi, t)$ and which is different from $\pm\mathrm{i}\mathbb{R}_{>0}$, we define

$$F_\rho(\lambda, t) := \frac{1}{\lambda^2}\mathrm{Li}_3(Q) + \frac{B_2}{2}\mathrm{Li}_1(Q) + \int_\rho \mathrm{d}\xi\, e^{-\xi/\check{\lambda}} G(\xi, t), \quad Q = e^{2\pi \mathrm{i} t}, \tag{3.30}$$

for λ in the half-plane $\mathbb{H}_\rho$ centered at ρ. We call $F_\rho(\lambda, t)$ the Borel sum of $\widetilde{F}(\lambda, t)$ along ρ.

The integral appearing in (2.14) corresponds to an integral representation of a certain function $G_3(z \,|\, \omega_1, \omega_2)$ related to the triple sine function. Hence, before studying the Borel sum along $\mathbb{R}_{>0}$, we recall how this function is defined and some of its properties. For a convenient review of the special functions appearing below and their properties, see for example [Bri20, Section 4] and the references cited therein.

Definition 3.7. For $z \in \mathbb{C}$ and $\omega_1, \omega_2 \in \mathbb{C}^\times$, we define

$$G_3(z \,|\, \omega_1, \omega_2) := \exp\left(\frac{\pi \mathrm{i}}{6} B_{3,3}(z + \omega_1 \,|\, \omega_1, \omega_1, \omega_2)\right) \cdot \sin_3(z + \omega_1 \,|\, \omega_1, \omega_1, \omega_2), \tag{3.31}$$

where $\sin_3(z \,|\, \omega_1, \omega_2, \omega_3)$ denotes the triple sine function, and $B_{3,3}(z \,|\, \omega_1, \omega_2, \omega_3)$ is the multiple Bernoulli polynomial.

What will be most important for us are the following properties:

Proposition 3.8. [Bri20, Prop. 4.2] [Nar04, Prop. 2] *$G_3(z\,|\,\omega_1,\omega_2)$ is a single-valued meromorphic function under the assumption $\omega_1/\omega_2 \notin \mathbb{R}_{<0}$. Furthermore, we have*

- *It is regular everywhere, and vanishes only at the points*

$$z = a\omega_1 + b\omega_2, \quad a, b \in \mathbb{Z}, \tag{3.32}$$

with $a < 0$ and $b \le 0$, or $a > 0$ and $b > 0$.

- *Let $\mathrm{Re}(\omega_i) > 0$ and $-\mathrm{Re}(\omega_1) < \mathrm{Re}(z) < \mathrm{Re}(\omega_1+\omega_2)$. Then*

$$G_3(z\,|\,\omega_1,\omega_2) = \exp\left(-\int_{\mathbb{R}+\mathrm{i}0^+}\frac{\mathrm{d}u}{8u}\frac{e^{u(z-\omega_2/2)}}{\sinh(\omega_2 u/2)(\sinh(\omega_1 u/2))^2}\right). \tag{3.33}$$

Definition 3.9. For $\mathrm{Re}\,\check{\lambda} > 0$ and $-\mathrm{Re}(\check{\lambda}) < \mathrm{Re}(t) < \mathrm{Re}(\check{\lambda}+1)$, we define

$$F_{\mathrm{np}}(\lambda, t) := \log G_3(t\,|\,\check{\lambda}, 1) = -\int_{\mathbb{R}+\mathrm{i}0^+}\frac{\mathrm{d}u}{8u}\frac{e^{u(t-1/2)}}{\sinh(u/2)(\sinh(\check{\lambda}u/2))^2}\,, \tag{3.34}$$

We now wish to relate $F_{\mathbb{R}_{>0}}(\lambda, t)$ to $F_{\mathrm{np}}(\lambda, t)$. For this, we will need the following lemma, giving a "Woronowicz form" for F_{np}. We remark that a similar form for the triple-sine function was conjectured in [AP21, Equation B.17].

Lemma 3.10. *Let $t \in \mathbb{C}$ be such that $0 < \mathrm{Re}(t) < 1$, $\mathrm{Im}(t) > 0$, and let λ be in the sector determined by $l_0 = \mathbb{R}_{<0}\cdot 2\pi\mathrm{i}t$ and $l_{-1} = \mathbb{R}_{<0}\cdot 2\pi\mathrm{i}(t-1)$. Furthermore, assume that $\mathrm{Re}(t) < \mathrm{Re}(\check{\lambda}+1)$. Then $F_{\mathrm{np}}(\lambda, t)$ admits the following Woronowicz form:*

$$F_{\mathrm{np}}(\lambda, t) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}+\mathrm{i}0^+}\mathrm{d}v\frac{v}{1-e^v}\log(1-e^{\check{\lambda}v+2\pi\mathrm{i}t})\,. \tag{3.35}$$

Proof. We will follow the method of [GK20], based on the unitarity of the Fourier transform:

$$\langle f, g\rangle = \langle Ff, Fg\rangle, \qquad \langle f, g\rangle = \int_{\mathbb{R}}\mathrm{d}x\; f(x)\overline{g(x)}, \qquad (F\psi)(x) = \int_{\mathbb{R}}\mathrm{d}y\; e^{2\pi\mathrm{i}xy}\psi(y).$$

We start by defining for sufficiently small $\epsilon > 0$,

$$\begin{aligned} f_\epsilon(x) &:= e^{-\epsilon x}\log\left(1-e^{\check{\lambda}x+2\pi\mathrm{i}t}\right), \qquad g_\epsilon(x) := e^{+\epsilon x}\frac{1}{1-e^{x+\mathrm{i}\epsilon}},\\ G_\epsilon(x) &:= e^{+\epsilon x}\frac{x}{1-e^{x+\mathrm{i}\epsilon}}. \end{aligned} \tag{3.36}$$

We then easily see that

$$\lim_{\epsilon\to 0^+}\frac{1}{(2\pi)^2}\langle f_\epsilon, G_\epsilon\rangle = \frac{1}{(2\pi)^2}\int_{\mathbb{R}+\mathrm{i}0^+}dv\frac{v}{1-e^v}\log(1-e^{\check{\lambda}v+2\pi\mathrm{i}t})\,. \tag{3.37}$$

We now compute the Fourier transform of f_ϵ, g_ϵ, and G_ϵ. Setting $\zeta = 2\pi x + \mathrm{i}\epsilon$, we find that

$$Ff_\epsilon(x) = \int_{\mathbb{R}}\mathrm{d}y\; e^{\mathrm{i}y\zeta}\log(1-e^{\check{\lambda}y+2\pi\mathrm{i}t}) = \frac{\mathrm{i}\check{\lambda}}{\zeta}\int_{\mathbb{R}}\mathrm{d}y\;\frac{e^{\mathrm{i}y\zeta}}{1-e^{-\check{\lambda}y-2\pi\mathrm{i}t}}\,, \tag{3.38}$$

where we have integrated by parts, and used that the boundary terms vanish. The last integral has simple poles at $y = 2\pi\mathrm{i}(k-t)/\check{\lambda}$, and under our assumptions for the parameters t and λ, it is easy to check that the poles on the upper half-plane correspond to $k > 0$, while those in the lower half-plane correspond to $k \leq 0$. If $\mathrm{Re}(x) > 0$, by an application of Jordan's lemma and the residue theorem, we can compute $Ff_\epsilon(x)$ by summing up the residues in the upper half-plane, obtaining

$$\begin{aligned}\frac{\mathrm{i}\check{\lambda}}{\zeta}\int_{\mathbb{R}}\mathrm{d}y\,\frac{e^{\mathrm{i}y\zeta}}{1-e^{-\check{\lambda}y-2\pi\mathrm{i}t}} &= 2\pi\mathrm{i}\frac{\mathrm{i}\check{\lambda}}{\zeta}\sum_{k=1}^{\infty}\frac{e^{\mathrm{i}y\zeta}}{\check{\lambda}}\Big|_{y=2\pi\mathrm{i}(k-t)/\check{\lambda}} = -\frac{2\pi}{\zeta}e^{2\pi t\zeta/\check{\lambda}}\sum_{k=1}^{\infty}e^{-2\pi k\zeta/\check{\lambda}}\\ &= -\frac{\pi}{\zeta}e^{\pi\zeta(2t-1)/\check{\lambda}}\Big(2e^{-\pi\zeta/\check{\lambda}}\sum_{k=0}^{\infty}e^{-2\pi\zeta k/\check{\lambda}}\Big) = -\frac{\pi}{\zeta}\frac{e^{\pi\zeta(2t-1)/\check{\lambda}}}{\sinh(\pi\zeta/\check{\lambda})}\,,\end{aligned} \tag{3.39}$$

where in the last equality we have used the Dirichlet series representation of $1/\sinh(z)$. Similarly, if $\mathrm{Re}(x) < 0$, we can compute Ff_ϵ summing up the residues in the lower half-plane, obtaining

$$\begin{aligned}\frac{\mathrm{i}\check{\lambda}}{\zeta}\int_{\mathbb{R}}\mathrm{d}y\,\frac{e^{\mathrm{i}y\zeta}}{1-e^{-\check{\lambda}y-2\pi\mathrm{i}t}} &= -2\pi\mathrm{i}\frac{\mathrm{i}\check{\lambda}}{\zeta}\sum_{k=0}^{-\infty}\frac{e^{\mathrm{i}y\zeta}}{\check{\lambda}}\Big|_{y=2\pi\mathrm{i}(k-t)/\check{\lambda}} = \frac{2\pi}{\zeta}e^{2\pi t\zeta/\check{\lambda}}\sum_{k=0}^{\infty}e^{2\pi k\zeta/\check{\lambda}}\\ &= -\frac{\pi}{\zeta}e^{\pi\zeta(2t-1)/\check{\lambda}}\Big(-2e^{\pi\zeta/\check{\lambda}}\sum_{k=0}^{\infty}e^{2\pi\zeta k/\check{\lambda}}\Big) = -\frac{\pi}{\zeta}\frac{e^{\pi\zeta(2t-1)/\check{\lambda}}}{\sinh(\pi\zeta/\check{\lambda})}\,,\end{aligned} \tag{3.40}$$

so that $Ff_\epsilon(x)$ exists for $x \neq 0$ and

$$Ff_\epsilon(x) = -\frac{\pi}{\zeta}\frac{e^{\pi\zeta(2t-1)/\check{\lambda}}}{\sinh(\pi\zeta/\check{\lambda})}\,. \tag{3.41}$$

The computation of $Fg_\epsilon(x)$ is simpler, and follows similar lines. One obtains that for $x \neq 0$,

$$\overline{Fg_\epsilon(x)} = \pi\mathrm{i}\frac{e^{(\epsilon-\pi)\zeta}}{\sinh(\pi\zeta)}\,. \tag{3.42}$$

On the other hand, since $G_\epsilon(x) = xg_\epsilon(x)$, we find

$$\overline{FG_\epsilon(x)} = -\frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial x}\overline{Fg_\epsilon(x)} = \frac{\partial}{\partial\zeta}\Big(\frac{2\pi e^{\epsilon\zeta}}{1-e^{2\pi\zeta}}\Big)\,. \tag{3.43}$$

We then have that

$$\begin{aligned}\lim_{\epsilon\to 0^+}\frac{1}{(2\pi)^2}\langle f_\epsilon, G_\epsilon\rangle &= \lim_{\epsilon\to 0^+}\frac{1}{(2\pi)^2}\langle Ff_\epsilon, FG_\epsilon\rangle\\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}+\mathrm{i}0^+}\mathrm{d}x\Big(-\frac{\pi}{2\pi x}\frac{e^{\pi(2\pi x)(2t-1)/\check{\lambda}}}{\sinh(2\pi^2 x/\check{\lambda})}\Big)\Big(\frac{(2\pi)^2}{4(\sinh(2\pi^2 x))^2}\Big)\\ &= -\int_{\check{\lambda}^{-1}\cdot(\mathbb{R}+\mathrm{i}0^+)}\frac{\mathrm{d}v}{8v}\frac{e^{v(t-1/2)}}{\sinh(v/2)(\sinh(\check{\lambda}v/2))^2}\\ &= -\int_{\mathbb{R}+\mathrm{i}0^+}\frac{\mathrm{d}v}{8v}\frac{e^{v(t-1/2)}}{\sinh(v/2)(\sinh(\check{\lambda}v/2))^2} = F_{\mathrm{np}}(\lambda, t)\,,\end{aligned} \tag{3.44}$$

where we used the fact that the range of the parameter λ allows us to deform the contour back to $\mathbb{R} + i0^+$. The result then follows. □

We are now ready to prove the second point of Theorem 2.1. Another proof is given in Appendix A.

Proposition 3.11. *Under the same assumptions as in Lemma 3.10, we have*

$$F_{\mathrm{np}}(\lambda,t)=\frac{1}{\lambda^2}\mathrm{Li}_3(Q)+\frac{B_2}{2}\mathrm{Li}_1(Q)+\int_0^\infty \mathrm{d}\xi\, e^{-\xi/\check{\lambda}}G(\xi,t)\,, \tag{3.45}$$

where $G(\xi,t)$ is the Borel transform (3.16) obtained in the previous section, $Q=e^{2\pi\mathrm{i}t}$, and $\check{\lambda}=\lambda/2\pi$.

Proof. We start by performing the change of variables $y=\lambda v/2\pi$ on (3.35), obtaining

$$\begin{aligned}F_{\mathrm{np}}(\lambda,t)&=\frac{1}{\lambda^2}\int_{\lambda(\mathbb{R}+\mathrm{i}0^+)}\mathrm{d}y\ \ \frac{y}{1-e^{2\pi y/\lambda}}\log(1-e^{y+2\pi\mathrm{i}t})\\&=\frac{1}{\lambda^2}\int_{\mathbb{R}+\mathrm{i}0^+}\mathrm{d}y\ \ \frac{y}{1-e^{2\pi y/\lambda}}\log(1-e^{y+2\pi\mathrm{i}t})\\&=\lim_{\epsilon\to0^+}\frac{1}{\lambda^2}\int_{\mathbb{R}}\mathrm{d}y\,\frac{y}{1-e^{2\pi y/\lambda-i\epsilon}}\log(1-e^{y+2\pi\mathrm{i}t})\,,\end{aligned}\tag{3.46}$$

where in the second equality we have used that the range of λ allows us to deform the contour back to $\mathbb{R}+\mathrm{i}0^+$. Now using

$$\frac{\mathrm{d}}{\mathrm{d}y}(-\log(1-e^{-2\pi y/\lambda+\mathrm{i}\epsilon}))=\frac{2\pi}{\lambda(1-e^{2\pi y/\lambda-\mathrm{i}\epsilon})}\tag{3.47}$$

and integration by parts, we find

$$\begin{aligned}F_{\mathrm{np}}(\lambda,t)=\lim_{\epsilon\to0}\Big[&-\frac{y}{2\pi\lambda}\log(1-e^{-2\pi y/\lambda+i\epsilon})\log(1-e^{y+2\pi it})\Big|_{y=-\infty}^{\infty}\\&+\frac{1}{2\pi\lambda}\int_{\mathbb{R}}\mathrm{d}y\ \ \log(1-e^{-2\pi y/\lambda+i\epsilon})\Big(\log(1-e^{y+2\pi it})+\frac{y}{1-e^{-y-2\pi it}}\Big)\Big].\end{aligned}\tag{3.48}$$

Because $\mathrm{Re}(\lambda)>0$, we obtain that the boundary terms vanish. Furthermore, splitting the integration over the left and right half intervals, one then obtains

$$\begin{aligned}F_{\mathrm{np}}(\lambda,t)&=\lim_{\epsilon\to0}\Big[\frac{1}{2\pi\lambda}\int_0^\infty\mathrm{d}y\ \ \log(1-e^{-2\pi y/\lambda+\mathrm{i}\epsilon})\Big(\log(1-e^{y+2\pi\mathrm{i}t})+\frac{y}{1-e^{-y-2\pi\mathrm{i}t}}\Big)\\&\quad+\frac{1}{2\pi\lambda}\int_0^\infty\mathrm{d}y\ \ \log(1-e^{2\pi y/\lambda+\mathrm{i}\epsilon})\Big(\log(1-e^{-y+2\pi\mathrm{i}t})-\frac{y}{1-e^{y-2\pi\mathrm{i}t}}\Big)\Big]\\&=\widetilde{H}(\lambda,t)+\lim_{\epsilon\to0}H(\lambda,t,\epsilon)\,,\end{aligned}\tag{3.49}$$

where we have defined

$$\begin{aligned}\widetilde{H}(\lambda,t)&:=\frac{1}{2\pi\lambda^2}\int_0^\infty\mathrm{d}y\ \ (2\pi y+\pi\mathrm{i}\lambda)\Big(\log(1-e^{-y+2\pi\mathrm{i}t})-\frac{y}{1-e^{y-2\pi it}}\Big)\\H(\lambda,t,\epsilon)&:=\frac{1}{2\pi\lambda}\int_0^\infty\mathrm{d}y\ \ \log(1-e^{-2\pi y/\lambda+\mathrm{i}\epsilon})\Big(\log(1-e^{y+2\pi\mathrm{i}t})+\log(1-e^{-y+2\pi\mathrm{i}t})\\&\qquad+\frac{y}{1-e^{-y-2\pi\mathrm{i}t}}-\frac{y}{1-e^{y-2\pi\mathrm{i}t}}\Big)\,.\end{aligned}\tag{3.50}$$

One can compute $\widetilde{H}(\lambda, t)$ explicitly by performing an integration by parts to get rid of the log term:

$$\begin{aligned}\widetilde{H}(\lambda, t) = &\frac{1}{2\pi\lambda^2}\left((\pi y^2 + \pi \mathrm{i}\lambda y)\log(1 - e^{-y+2\pi \mathrm{i}t})\Big|_{y=0}^{\infty} - \int_0^\infty \mathrm{d}y \ \ (\pi y^2 + \pi\mathrm{i}\lambda y)\frac{-1}{1 - e^{y-2\pi \mathrm{i}t}}\right)\\ &-\frac{1}{2\pi\lambda^2}\int_0^\infty \mathrm{d}y \ \ (2\pi y + \pi\mathrm{i}\lambda)\frac{y}{1 - e^{y-2\pi\mathrm{i}t}}\\ = &\frac{1}{2\lambda^2}\int_0^\infty \mathrm{d}y \frac{y^2}{e^{y-2\pi\mathrm{i}t} - 1}\ . \end{aligned} \tag{3.51}$$

Since $\mathrm{Im}(t) > 0$, we find that $|e^{2\pi \mathrm{i}t}| < 1$, so that the last integral in (3.51) corresponds to an integral representation of $\mathrm{Li}_3(e^{2\pi\mathrm{i}t})/\lambda^2$. Hence, we conclude that

$$\widetilde{H}(\lambda, t) = \frac{1}{\lambda^2}\mathrm{Li}_3(e^{2\pi\mathrm{i}t})\ . \tag{3.52}$$

On the other hand, by expanding the first log term of H and applying the Fubini–Tonelli theorem, we find that

$$\begin{aligned} H(\lambda, t, \epsilon) = -\sum_{n=1}^{\infty}\frac{1}{2\pi\lambda}\int_0^\infty \mathrm{d}y \ \ &\frac{\mathrm{e}^{-2\pi n y/\lambda + \mathrm{i}n\epsilon}}{n}\Big(\log(1 - e^{y+2\pi\mathrm{i}t}) + \log(1 - e^{-y+2\pi\mathrm{i}t})\\ &+\frac{y}{1 - e^{-y-2\pi\mathrm{i}t}} - \frac{y}{1 - e^{y-2\pi\mathrm{i}t}}\Big)\ . \end{aligned} \tag{3.53}$$

Performing a change of variables in each integral, and interchanging integral and summations again, we obtain

$$\begin{aligned} H(\lambda, t, \epsilon) = -\tfrac{1}{2\pi\lambda}\textstyle\int_0^\infty \mathrm{d}y \ \ \mathrm{e}^{-2\pi y/\lambda}\sum_{n=1}^{\infty}\frac{e^{in\epsilon}}{n^2}&\Big(\log(1 - e^{y/n+2\pi\mathrm{i}t}) + \log(1 - e^{-y/n+2\pi\mathrm{i}t})\\ &+\tfrac{y/n}{1-e^{-y/n-2\pi\mathrm{i}t}} - \tfrac{y/n}{1-e^{y/n-2\pi\mathrm{i}t}}\Big) \end{aligned} \tag{3.54}$$

Letting $H(\lambda, t) := \lim_{\epsilon\to 0} H(\lambda, t, \epsilon)$, we get

$$\begin{aligned} H(\lambda, t) \ = -\frac{1}{2\pi\lambda}\int_0^\infty \mathrm{d}y \ \ \mathrm{e}^{-2\pi y/\lambda}\sum_{n=1}^{\infty}\frac{1}{n^2}&\Big(\log(1 - e^{y/n+2\pi\mathrm{i}t}) + \log(1 - e^{-y/n+2\pi\mathrm{i}t})\\ &+\frac{y/n}{1 - e^{-y/n-2\pi\mathrm{i}t}} - \frac{y/n}{1 - e^{y/n-2\pi\mathrm{i}t}}\Big)\ . \end{aligned} \tag{3.55}$$

Finally, using that $-\frac{2\pi}{\lambda}e^{-2\pi y/\lambda} = \frac{\mathrm{d}}{\mathrm{d}y}e^{-2\pi y/\lambda}$ and integrating by parts yields

$$\begin{aligned} H(\lambda, t) = &\Bigg[\frac{e^{-2\pi y/\lambda}}{(2\pi)^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\Big(\log(1 - e^{y/n+2\pi\mathrm{i}t}) + \log(1 - e^{-y/n+2\pi\mathrm{i}t})\\ &+\frac{y/n}{1 - e^{-y/n-2\pi\mathrm{i}t}} - \frac{y/n}{1 - e^{y/n-2\pi\mathrm{i}t}}\Big)\Bigg]\Bigg|_{y=0}^{\infty}\\ &-\int_0^\infty dy \ \ \frac{e^{-2\pi y/\lambda}}{(2\pi)^2}\frac{\mathrm{d}}{\mathrm{d}y}\Bigg[\sum_{n=1}^{\infty}\frac{1}{n^2}\Big(\log(1 - e^{y/n+2\pi\mathrm{i}t}) + \log(1 - e^{-y/n+2\pi\mathrm{i}t})\\ &+\frac{y/n}{1 - e^{-y/n-2\pi\mathrm{i}t}} - \frac{y/n}{1 - e^{y/n-2\pi\mathrm{i}t}}\Big)\Bigg]. \end{aligned} \tag{3.56}$$

Using that the boundary term at ∞ vanishes, and interchanging the derivative with the sum, we obtain

$$\begin{aligned} H(\lambda, t) &= -\frac{2}{(2\pi)^2}\log(1-Q)\sum_{n=1}^{\infty}\frac{1}{n^2} + \int_0^{\infty} \mathrm{d}y \;\; \mathrm{e}^{-2\pi y/\lambda} G(y,t) \\ &= \frac{1}{2}\mathrm{Li}_1(Q) B_2 + \int_0^{\infty} \mathrm{d}y \;\; \mathrm{e}^{-2\pi y/\lambda} G(y,t) \, , \end{aligned} \tag{3.57}$$

where we used that $\sum_{n=1}^{\infty}\frac{1}{n^2} = \pi^2 B_2$, $\mathrm{Li}_1(Q) = -\log(1-Q)$.

Hence, putting (3.49), (3.52) and (3.57) together gives us (3.45). □

We finish this section with the following corollaries:

Corollary 3.12. *Let $l_k = \mathbb{R}_{<0} \cdot 2\pi\mathrm{i}(t+k)$, and let ρ_k be a ray between l_k and l_{k-1}. Then the following holds for $n \in \mathbb{Z}$:*

$$F_{\rho_{k-n}}(\lambda, t+n) = F_{\rho_k}(\lambda, t)\, . \tag{3.58}$$

In particular, if $0 < \mathrm{Re}(t) < 1$ and $\mathrm{Im}(t) > 0$, we have on their common domains of definition

$$F_{\rho_{-n}}(\lambda, t+n) = \log(G_3(t \,|\, \check{\lambda}, 1)) \tag{3.59}$$

Proof. Note that the labels l_k (and hence also ρ_k) depend on t. In the following, we denote $l_k(t)$ and $\rho_k(t)$ to emphasize the t dependence. In particular, we have the relations $l_k(t+n) = l_{k+n}(t)$ and $\rho_k(t+n) = \rho_{k+n}(t)$ for $n \in \mathbb{Z}$.

Using the fact that $G(\xi, t) = G(\xi, t+n)$ for any $n \in \mathbb{Z}$, we thus obtain

$$\begin{aligned} F_{\rho_{k-n}}(\lambda, t+n) &= \frac{1}{\lambda^2}\mathrm{Li}_3(e^{2\pi\mathrm{i}(t+n)}) + \frac{B_2}{2}\mathrm{Li}_1(e^{2\pi\mathrm{i}(t+n)}) + \int_{\rho_{k-n}(t+n)} \mathrm{d}\xi \; e^{-\xi/\check{\lambda}} G(\xi, t+n) \\ &= \frac{1}{\lambda^2}\mathrm{Li}_3(e^{2\pi\mathrm{i}t}) + \frac{B_2}{2}\mathrm{Li}_1(e^{2\pi\mathrm{i}t}) + \int_{\rho_{k-n}(t+n)} \mathrm{d}\xi \; e^{-\xi/\check{\lambda}} G(\xi, t) \\ &= \frac{1}{\lambda^2}\mathrm{Li}_3(e^{2\pi\mathrm{i}t}) + \frac{B_2}{2}\mathrm{Li}_1(e^{2\pi\mathrm{i}t}) + \int_{\rho_k(t)} \mathrm{d}\xi \; e^{-\xi/\check{\lambda}} G(\xi, t) \\ &= F_{\rho_k}(\lambda, t)\, . \end{aligned} \tag{3.60}$$

The final result then follows from Proposition 3.11 and 3.8. □

Corollary 3.13. *Let ρ be a ray different from $\pm i\mathbb{R}_{>0}$. The Borel-transform $F_{0,\rho}(\lambda)$ of the formal series $F_0(\lambda)$ defined in (2.5) can be represented in the form*

$$F_{0,\rho}(\lambda) = -\frac{1}{\lambda^2}\zeta(3) + F_0^1 - \int_{\rho} \mathrm{d}\xi \; e^{-\xi/\check{\lambda}} \left(G(\xi, 0) + \frac{1}{12\xi} \right). \tag{3.61}$$

It is related to the Borel transform $F_\rho(\lambda, t)$ of $\tilde{F}(\lambda, t)$ by the equation (2.20).

Proof. We will consider a limit of $F_\rho(\lambda, t)$ as $t \to 0$, where t is taken to satisfy $\mathrm{Re}(t) > 0$, $\mathrm{Im}(t) > 0$; and such that along the limit, ρ is always between l_{-1} and l_0 (resp. $-l_{-1}$ and $-l_0$) if ρ is on the right (resp. left) Borel half-plane. Let us first assume that ρ is taken on the right half plane. The limit requires some care as $G(\xi, 0)$ has a simple pole with residue $-\frac{B_2}{2} = -\frac{1}{12}$ at $\xi = 0$. We may observe, however, that

$$F_\rho(\lambda, t) = \frac{1}{\lambda^2}\mathrm{Li}_3(e^{2\pi \mathrm{i} t}) + \int_\rho \mathrm{d}\xi \, \left(e^{-\xi/\check{\lambda}}G(\xi, t) + \frac{1}{12}\frac{1}{e^{\xi - 2\pi \mathrm{i} t} - 1}\right)$$

has a well-defined limit for $t \to 0$, as the poles at $\xi = 0$ in the integrand cancel each other. This limit can be represented as

$$F_\rho(\lambda, 0) = \frac{1}{\lambda^2}\zeta(3) + \int_\rho \mathrm{d}\xi \; e^{-\xi/\check{\lambda}}\left(G(\xi, 0) + \frac{1}{12\xi}\right) + \frac{1}{12}\int_\rho \mathrm{d}\xi \, \left(\frac{1}{e^\xi - 1} - \frac{e^{-\xi/\check{\lambda}}}{\xi}\right). \tag{3.62}$$

The derivative of the second integral with respect to $\check{\lambda}$ is easily found to be equal to $-\frac{1}{12}\frac{1}{\check{\lambda}}$. It follows that this integral is equal to $-\frac{1}{12}\log\check{\lambda} + C$, with C being an undetermined constant. The integrand of the first integral in (3.62), on the other hand, is analytic at $\xi = 0$. By straightforward computation of the Taylor series one may check that $G_0(\xi) := -G(\xi, 0) - \frac{1}{12\xi}$ is the Borel transform of the formal series $F_0(\lambda) + \zeta(3)/\lambda^2 - F_0^1$ (see Lemma B.2 for the computation), so the relation (2.20) then follows. On the other hand, if ρ is on the left half plane, one can apply the same argument from before by using the relation $F_\rho(\lambda, t) = F_{-\rho}(-\lambda, t)$ (see Lemma 3.16 below). □

3.3. Stokes phenomena of the Borel sum. In the previous section, we studied $F_\rho(\lambda, t)$ for $\rho = \mathbb{R}_{>0}$. However, the ray $\mathbb{R}_{>0}$ is a choice, and any other ray ρ that avoids the poles of $G(\xi, t)$ in principle is an equally valid choice to perform the Borel sum. In this section we study the dependence on this choice.

Proposition 3.14. *Assume that* $\mathrm{Im}(t) > 0$ *and for* $k \in \mathbb{Z}$ *let* $l_k = \mathbb{R}_{<0} \cdot 2\pi\mathrm{i}(t + k)$. *Furthermore let* ρ *be a ray in the sector determined by the Stokes rays* l_{k+1} *and* l_k, *and* ρ' *a ray in the sector determined by* l_k *and* l_{k-1}. *Then for* $\lambda \in \mathbb{H}_\rho \cap \mathbb{H}_{\rho'}$ *(resp.* $\lambda \in \mathbb{H}_{-\rho} \cap \mathbb{H}_{-\rho'}$*) we have*

$$F_{\pm\rho}(\lambda, t) - F_{\pm\rho'}(\lambda, t) = \frac{1}{2\pi i}\partial_{\check{\lambda}}\left(\check{\lambda}\,\mathrm{Li}_2\left(e^{\pm 2\pi \mathrm{i}(t+k)/\check{\lambda}}\right)\right). \tag{3.63}$$

If $\mathrm{Im}(t) < 0$, *then the previous jumps also hold provided* ρ *is interchanged with* ρ' *in the above formulas.*

Proof. Notice that

$$F_\rho(\lambda, t) - F_{\rho'}(\lambda, t) = \int_{\mathcal{H}(l_k)} \mathrm{d}\xi \; e^{-\xi/\check{\lambda}}G(\xi, t)\,, \tag{3.64}$$

where $\mathcal{H}(l_k)$ is a Hankel contour around $l_k = \mathbb{R}_{<0} \cdot 2\pi\mathrm{i}(t + k)$.

To compute this, notice that for our range of parameters, $G(\xi, t)$ has double poles at $\xi = 2\pi \mathrm{i} m(t + k)$ for all $k \in \mathbb{Z}$ and $m \in \mathbb{Z} \setminus \{0\}$ with generalized residues

$$\frac{\mathrm{d}}{\mathrm{d}\xi}(e^{-\xi/\check{\lambda}}(\xi - 2\pi \mathrm{i} m(t + k))^2 G(\xi, t))\bigg|_{\xi = 2\pi \mathrm{i} m(t+k)}$$

$$= -\frac{e^{-2\pi \mathrm{i} m(t+k)/\check{\lambda}}}{(2\pi)^2 m^2}\left(1+\frac{2\pi \mathrm{i} m(t+k)}{\check{\lambda}}\right). \tag{3.65}$$

In particular, we have that

$$\begin{aligned}
\int_{\mathcal{H}(l_k)} \mathrm{d}\xi\, e^{-\xi/\check{\lambda}} G(\xi,t) &= 2\pi \mathrm{i} \sum_{m=-1}^{-\infty} \frac{\mathrm{d}}{\mathrm{d}\xi}\left(e^{-\xi/\check{\lambda}}(\xi - 2\pi \mathrm{i} m(t+k))^2 G(\xi,t)\right)\Big|_{\xi=2\pi \mathrm{i} m(t+k)} \\
&= -\frac{\mathrm{i}}{2\pi}\sum_{m=1}^{\infty}\frac{e^{2\pi \mathrm{i} m(t+k)/\check{\lambda}}}{m^2}\left(1-\frac{2\pi \mathrm{i} m(t+k)}{\check{\lambda}}\right) \\
&= -\frac{\mathrm{i}}{2\pi}\left(\mathrm{Li}_2\left(e^{2\pi \mathrm{i}(t+k)/\check{\lambda}}\right)+\frac{2\pi \mathrm{i}(t+k)}{\check{\lambda}}\log\left(1-e^{2\pi \mathrm{i}(t+k)/\check{\lambda}}\right)\right) \\
&= \frac{1}{2\pi \mathrm{i}}\partial_{\check{\lambda}}\left(\check{\lambda}\mathrm{Li}_2\left(e^{2\pi \mathrm{i}(t+k)/\check{\lambda}}\right)\right), \qquad (3.66)
\end{aligned}$$

where we have used the series representation of $\mathrm{Li}_s(z)$ for $s = 1, 2$; and the fact that for λ in a sufficiently small sector containing l_k, we have $|e^{2\pi \mathrm{i}(t+k)/\check{\lambda}}| < 1$.

A similar argument follows for the rest of the cases in the statement of the proposition. □

3.3.1. Stokes jumps of the other Borel transform. Recall the expression for the Borel transform which was obtained previously in [PS10], given in (3.28):

$$G(\xi,t) = \sum_{k\in\mathbb{Z}} \frac{1}{\xi}\left(\frac{(k-t)^2}{\xi^2} - \frac{e^{\xi/(k-t)}}{(e^{\xi/(k-t)}-1)^2} - \frac{1}{12}\right), \tag{3.67}$$

Similarly to the previous discussion, if $y_m = \xi - 2\pi \mathrm{i} m(t+k)$ near the pole given by $\xi = 2\pi \mathrm{i} m(t+k)$, then by Taylor expanding the integrand near $y_m = 0$ we obtain

$$e^{-\xi/\check{\lambda}} G(\xi,t) = e^{-2\pi \mathrm{i} m(t+k)/\check{\lambda}}\left(\frac{1}{y_m^2}\frac{\mathrm{i}(t+k)}{2\pi m} + \frac{1}{y_m}\left(-\frac{1}{4\pi^2 m^2} - \frac{\mathrm{i}}{2\pi m\check{\lambda}}(t+k)\right) + \mathcal{O}(1)\right). \tag{3.68}$$

Hence, by following the argument of Proposition 3.14, we obtain the same Stokes jumps.

3.4. The limits $\lim_{k\to\pm\infty} F_{\rho_k}(\lambda,t)$ *and* $F_{\mathrm{GV}}(\lambda,t)$. To finish the proof of Theorem 2.1, we study the limits of $F_\rho(\lambda,t)$, discussed in point (iv).

Proposition 3.15. *Let* ρ_k *denote any ray between the Stokes rays* l_k *and* l_{k-1}. *Furthermore, assume that* $0 < \mathrm{Re}(t) < 1$, $\mathrm{Im}(t) > 0$, $\mathrm{Re}(\lambda) > 0$, $\mathrm{Im}(\lambda) < 0$, *and* $\mathrm{Re}(t) < \mathrm{Re}(\check{\lambda}+1)$. *Then*

$$\lim_{k\to\infty} F_{\rho_k}(\lambda,t) = \sum_{k=1}^{\infty}\frac{e^{2\pi \mathrm{i} kt}}{k\left(2\sin\left(\frac{\lambda k}{2}\right)\right)^2} = F_{\mathrm{GV}}(\lambda,t). \tag{3.69}$$

Furthermore, we can write the sum of the Stokes jumps along l_k *for* $k \geq 0$ *as*

$$\sum_{k=0}^{\infty}\phi_{l_k}(\lambda,t) = \frac{1}{2\pi i}\partial_\lambda\left(\lambda\sum_{l=1}^{\infty}\frac{w^l}{l^2(1-\tilde{q}^l)}\right), \quad w := e^{2\pi \mathrm{i} t/\check{\lambda}},\ \ \tilde{q} := e^{2\pi \mathrm{i}/\check{\lambda}}. \tag{3.70}$$

Proof. By Proposition 3.14 we find that

$$F_{\rho_k} - F_{\rho_{k+1}} = \frac{\mathrm{i}}{2\pi}\Big(\mathrm{Li}_2\big(e^{2\pi\mathrm{i}(t+k)/\check{\lambda}}\big) + \log(e^{2\pi\mathrm{i}\frac{t+k}{\check{\lambda}}})\log\big(1-e^{2\pi\mathrm{i}(t+k)/\check{\lambda}}\big)\Big). \quad (3.71)$$

Denoting $w = e^{2\pi\mathrm{i}t/\check{\lambda}}$ and $\widetilde{q} = e^{2\pi\mathrm{i}/\check{\lambda}}$, we find

$$\begin{aligned} F_{\rho_0}(\lambda,t) - \lim_{k\to\infty} F_{\rho_k}(\lambda,t) &= \sum_{k=0}^{\infty} F_{\rho_k}(\lambda,t) - F_{\rho_{k+1}}(\lambda,t) \\ &= \frac{\mathrm{i}}{2\pi}\sum_{k=0}^{\infty}\Big(\mathrm{Li}_2\big(w\widetilde{q}^k\big) + \log\big(w\widetilde{q}^k\big)\log\big(1-w\widetilde{q}^k\big)\Big). \end{aligned} \quad (3.72)$$

We now use the following identities:

$$\sum_{k=0}^{\infty}\log(1-w\widetilde{q}^k) = -\sum_{l=1}^{\infty}\frac{1}{l}\frac{w^l}{1-\widetilde{q}^l}, \quad (3.73a)$$

$$\sum_{k=0}^{\infty}k\log(1-w\widetilde{q}^k) = -\sum_{l=1}^{\infty}\frac{\widetilde{q}^l}{l}\frac{w^l}{(1-\widetilde{q}^l)^2}, \quad (3.73b)$$

$$\sum_{k=0}^{\infty}\mathrm{Li}_2(w\widetilde{q}^k) = \sum_{l=1}^{\infty}\frac{1}{l^2}\frac{w^l}{1-\widetilde{q}^l}. \quad (3.73c)$$

The first two identities (3.73a) and (3.73b) are easily established by using the Taylor expansion of the logarithm function; using that $|\widetilde{q}| < 1$ and $|w| < 1$; and exchanging the two summations. In order to verify (3.73c), one can first act on it with $w\frac{\mathrm{d}}{\mathrm{d}w}$. The left side of the resulting equation is easily seen to be equal to

$$-\sum_{k=0}^{\infty}\log(1-w\widetilde{q}^k) = \sum_{l=1}^{\infty}\frac{1}{l}\frac{w^l}{1-\widetilde{q}^l},$$

using (3.73a). It follows that (3.73c) holds up to addition of a term which is constant with respect to w. In order to fix this freedom, it suffices to note that (3.73c) holds for $w = 0$.

Using the previous identities, we obtain

$$\begin{aligned} F_{\rho_0}(\lambda,t) - \lim_{k\to\infty} F_{\rho_k}(\lambda,t) &= \frac{\mathrm{i}}{2\pi}\sum_{l=1}^{\infty}\frac{w^l}{l(1-\widetilde{q}^l)}\left(\frac{1}{l} - \frac{\widetilde{q}^l\log\widetilde{q}}{1-\widetilde{q}^l} - \log w\right) \\ &= -\frac{\mathrm{i}}{2\pi}\sum_{l=1}^{\infty}\frac{\partial}{\partial l}\left(\frac{w^l}{l(1-\widetilde{q}^l)}\right). \end{aligned} \quad (3.74)$$

Now notice that under our assumptions on t and λ, we have $F_{\rho_0} = F_{\mathrm{np}}$ by Proposition 3.11. We now show that F_{ρ_0} admits the following representation as sum over residues:

$$F_{\rho_0}(\lambda,t) = \frac{1}{2\pi\mathrm{i}}\sum_{l=1}^{\infty}\frac{\partial}{\partial l}\left(\frac{w^l}{l(1-\widetilde{q}^l)}\right) + \sum_{k=1}^{\infty}\frac{e^{2\pi\mathrm{i}kt}}{k\big(2\sin\big(\frac{\lambda k}{2}\big)\big)^2}. \quad (3.75)$$

In order to see this, let us recall that by Proposition 3.8, we have

$$
\begin{aligned}
F_{\mathrm{np}}(\lambda, t) &= -\int_{\mathbb{R}+\mathrm{i}0^+} \frac{\mathrm{d}u}{8u} \frac{e^{u(t-\frac{1}{2})}}{\sinh(u/2)(\sinh(\lambda u/4\pi))^2} \\
&= \int_{\mathbb{R}+\mathrm{i}0^+} \frac{\mathrm{d}u}{u} \frac{e^{ut}}{1-e^u} \frac{1}{(2\sinh(\lambda u/4\pi))^2} .
\end{aligned}
$$

The integrand has two series of poles, one at $u = u_l := (2\pi)^2 \frac{\mathrm{i}}{\lambda} l$, $l \in \mathbb{Z}$ and the other at $u = \widetilde{u}_k := 2\pi \mathrm{i} k$, $k \in \mathbb{Z}$. We can compute the previous integral by closing the contour in the upper half-plane. The contributions from the poles at $u = u_l$ are calculated as

$$
\begin{aligned}
2\pi\mathrm{i}\left(\frac{4\pi}{2\lambda}\right)^2 \frac{\partial}{\partial u} \frac{e^{ut}}{(1-e^u)u}\bigg|_{u=(2\pi)^2\frac{\mathrm{i}}{\lambda}l} &= 2\pi\mathrm{i}\,\frac{(2\pi)^2}{\lambda^2}\left(\frac{\lambda}{\mathrm{i}(2\pi)^2}\right)^2 \frac{\partial}{\partial l} \frac{w^l}{(1-\widetilde{q}^l)l} \\
&= \frac{1}{2\pi\mathrm{i}} \frac{\partial}{\partial l} \frac{w^l}{(1-\widetilde{q}^l)l} ,
\end{aligned}
$$

while the contributions of the poles at $u = 2\pi \mathrm{i} k$ give the remaining term.

In particular, we conclude that

$$
\begin{aligned}
\lim_{k\to\infty} F_{\rho_k}(\lambda, t) &= \lim_{k\to\infty} (F_{\rho_k}(\lambda, t) - F_{\rho_0}(\lambda, t)) + F_{\rho_0}(\lambda, t) \\
&= -\frac{1}{2\pi\mathrm{i}} \sum_{l=1}^{\infty} \frac{\partial}{\partial l}\left(\frac{w^l}{l(1-\widetilde{q}^l)}\right) + \frac{1}{2\pi\mathrm{i}} \sum_{l=1}^{\infty} \frac{\partial}{\partial l}\left(\frac{w^l}{l(1-\widetilde{q}^l)}\right) \\
&\quad + \sum_{k=1}^{\infty} \frac{e^{2\pi\mathrm{i}kt}}{k\big(2\sin\big(\frac{\lambda k}{2}\big)\big)^2} \\
&= \sum_{k=1}^{\infty} \frac{e^{2\pi\mathrm{i}kt}}{k\big(2\sin\big(\frac{\lambda k}{2}\big)\big)^2} .
\end{aligned} \tag{3.76}
$$

The last statement follows easily by noticing that

$$
\frac{\mathrm{i}}{2\pi} \sum_{l=1}^{\infty} \frac{\partial}{\partial l}\left(\frac{w^l}{l(1-\widetilde{q}^l)}\right) = \frac{1}{2\pi\mathrm{i}} \partial_\lambda\left(\lambda \sum_{l=1}^{\infty} \frac{w^l}{l^2(1-\widetilde{q}^l)}\right). \tag{3.77}
$$

□

To study the other limits to the imaginary rays, we use the following lemma:

Lemma 3.16. *For ρ any ray not in $\{\pm l_k\} \cup \{\pm l_\infty\}$ and $\lambda \in \mathbb{H}_\rho$, we have*

$$
F_\rho(\lambda, t) = F_{-\rho}(-\lambda, t) . \tag{3.78}
$$

Proof. The main thing to notice is that

$$
G(\xi, t) = -G(-\xi, t) . \tag{3.79}
$$

Using the earlier relation, we obtain

$$
F_\rho(\lambda, t) = \frac{1}{\lambda^2}\mathrm{Li}_3(Q) + \frac{B_2}{2}\mathrm{Li}_1(Q) - \int_\rho \mathrm{d}\xi\, e^{-\xi/\check{\lambda}} G(-\xi, t)
$$

$$= \frac{1}{\lambda^2}\mathrm{Li}_3(Q) + \frac{B_2}{2}\mathrm{Li}_1(Q) + \int_{-\rho} \mathrm{d}\xi\, e^{\xi/\check{\lambda}} G(\xi, t)$$
$$= F_{-\rho}(-\lambda, t)\,. \tag{3.80}$$

□

As an immediate corollary from Proposition 3.15 and Lemma 3.16, we obtain:

Corollary 3.17. *With the same notation as in Proposition 3.15, assume that* $0 < Re(t) < 1$, $\mathrm{Im}(t) > 0$, $\mathrm{Re}(\lambda) < 0$, $\mathrm{Im}(\lambda) > 0$ *and* $\mathrm{Re}(t) < \mathrm{Re}(-\check{\lambda} + 1)$. *Then*

$$\lim_{k\to\infty} F_{-\rho_k}(\lambda, t) = F_{\mathrm{GV}}(\lambda, t)\,. \tag{3.81}$$

Proposition 3.18. *With the same notation as in Proposition 3.15, assume that* $0 < Re(t) < 1$, $Im(t) > 0$, $Re(\lambda) < 0$, $Im(\lambda) < 0$, $\mathrm{Re}\, t < \mathrm{Re}(-\check{\lambda} + 1)$ *and that* $|w^{-1}| < 1$. *Then*

$$\lim_{k\to-\infty} F_{-\rho_k}(\lambda, t) = F_{\mathrm{GV}}(\lambda, t) \tag{3.82}$$

Remark 3.19. For fixed λ satisfying $\mathrm{Re}(\lambda) < 0$, $\mathrm{Im}(\lambda) < 0$, the condition $|w^{-1}| < 1$ can be satisfied by picking t such that $0 < \mathrm{Re}(t) < 1$, $\mathrm{Im}(t) > 0$, and $\mathrm{Im}(t)$ is sufficiently large. Similarly, for fixed t with $0 < \mathrm{Re}(t) < 1$, $\mathrm{Im}(t) > 0$, $|w^{-1}| < 1$ can be satisfied by picking λ such that $\mathrm{Re}(\lambda) < 0$, $\mathrm{Im}(\lambda) < 0$ and $|\mathrm{Im}(\lambda)| << |\mathrm{Re}(\lambda)|$.

Proof. Using the jumps along the Stokes rays l_k for $k < 0$, we find that

$$\begin{aligned}
F_{-\rho_0} - \lim_{k\to-\infty} F_{-\rho_k} &= \sum_{k=-1}^{-\infty} F_{-\rho_{k+1}} - F_{-\rho_k} \\
&= -\frac{\mathrm{i}}{2\pi}\sum_{k=-1}^{-\infty}\left(\mathrm{Li}_2\big(e^{-2\pi\mathrm{i}(t+k)/\check{\lambda}}\big) + \log(e^{-2\pi\mathrm{i}\frac{t+k}{\check{\lambda}}})\log\big(1 - e^{-2\pi\mathrm{i}(t+k)/\check{\lambda}}\big)\right) \\
&= -\frac{\mathrm{i}}{2\pi}\sum_{k=0}^{\infty}\left(\mathrm{Li}_2\big(w^{-1}\widetilde{q}^k\big) + \log(w^{-1}\widetilde{q}^k)\log\big(1 - w^{-1}\widetilde{q}^k\big)\right) \\
&\quad + \frac{\mathrm{i}}{2\pi}\left(\mathrm{Li}_2\big(w^{-1}\big) + \log(w^{-1})\log\big(1 - w^{-1}\big)\right).
\end{aligned} \tag{3.83}$$

Using the constraints on t and λ, we find that $|w^{-1}| < 1$ and $|\widetilde{q}| < 1$, so that we can expand in series as in Proposition 3.15 and write

$$\begin{aligned}
&F_{-\rho_0} - \lim_{k\to-\infty} F_{-\rho_k} \\
&= -\frac{\mathrm{i}}{2\pi}\sum_{l=1}^{\infty}\frac{w^{-l}}{l(1-\widetilde{q}^l)}\left(\frac{1}{l} - \frac{\widetilde{q}^l\log\widetilde{q}}{1-\widetilde{q}^l} - \log w^{-1}\right) \\
&\quad + \frac{\mathrm{i}}{2\pi}\left(\mathrm{Li}_2\big(w^{-1}\big) + \log(w^{-1})\log\big(1 - w^{-1}\big)\right) \\
&= -\frac{1}{2\pi\mathrm{i}}\sum_{l=1}^{\infty}\frac{\partial}{\partial l}\left(\frac{w^{-l}}{l(1-\widetilde{q}^l)}\right) + \frac{\mathrm{i}}{2\pi}\left(\mathrm{Li}_2\big(w^{-1}\big) + \log(w^{-1})\log\big(1 - w^{-1}\big)\right).
\end{aligned} \tag{3.84}$$

On the other hand, under our conditions on the parameters t and λ, and by Lemma 3.16, we have that

$$F_{-\rho_0}(\lambda, t) = F_{\text{np}}(-\lambda, t)\,. \tag{3.85}$$

Following the same argument as in Proposition 3.15 using the integral representation of F_{np}, we find that

$$F_{\text{np}}(-\lambda, t) = \frac{1}{2\pi \mathrm{i}} \sum_{l=1}^{\infty} \frac{\partial}{\partial l}\left(\frac{w^{-l}}{l(1-\tilde{q}^{-l})} \right) + \sum_{k=1}^{\infty} \frac{e^{2\pi \mathrm{i} k t}}{k\big(2\sin\left(\frac{\lambda k}{2}\right)\big)^2}.$$

Hence, we find that for λ and t as in the hypothesis

$$F_{-\rho_0}(\lambda, t) = \frac{1}{2\pi \mathrm{i}} \sum_{l=1}^{\infty} \frac{\partial}{\partial l}\left(\frac{w^{-l}}{l(1-\tilde{q}^{-l})} \right) + F_{\text{GV}}(\lambda, t) \tag{3.86}$$

Joining our results together, we conclude that

$$\lim_{k\to -\infty} F_{-\rho_k}(\lambda, t) = \frac{1}{2\pi \mathrm{i}} \sum_{l=1}^{\infty} \left(\frac{\partial}{\partial l}\left(\frac{w^{-l}}{l(1-\tilde{q}^{-l})} \right) + \frac{\partial}{\partial l}\left(\frac{w^{-l}}{l(1-\tilde{q}^{l})} \right)\right) + F_{\text{GV}}(\lambda, t)$$
$$+\frac{1}{2\pi \mathrm{i}}\Big(\mathrm{Li}_2\big(w^{-1}\big) + \log(w^{-1})\log\big(1-w^{-1}\big)\Big)\,. \tag{3.87}$$

Finally, notice that

$$\begin{aligned}\sum_{l=1}^{\infty} \left(\frac{\partial}{\partial l}\left(\frac{w^{-l}}{l(1-\tilde{q}^{-l})} + \frac{w^{-l}}{l(1-\tilde{q}^{l})} \right)\right) &= \sum_{l=1}^{\infty} \left(\frac{\partial}{\partial l}\left(\frac{w^{-l}}{l} \right)\right) \\ &= \log(w^{-1}) \sum_{l=1}^{\infty} \frac{w^{-l}}{l} - \sum_{l=1}^{\infty} \frac{w^{-l}}{l^2} \\ &= -(\mathrm{Li}_2(w^{-1}) + \log(w^{-1})\log(1-w^{-1})),\end{aligned} \tag{3.88}$$

where in the last equality we used that $|w^{-1}| < 1$ under our hypotheses. Hence, we conclude that

$$\lim_{k\to -\infty} F_{-\rho_k}(\lambda, t) = F_{\text{GV}}(\lambda, t)\,. \tag{3.89}$$

□

By using Lemma 3.16 and Proposition 3.18, we get the following immediate corollary:

Corollary 3.20. *With the same notation as in Proposition 3.15, assume that* $0 < \mathrm{Re}(t) < 1$, $\mathrm{Im}(t) > 0$, $\mathrm{Re}(\lambda) > 0$, $\mathrm{Im}(\lambda) > 0$, $\mathrm{Re}(t) < \mathrm{Re}(\check{\lambda} + 1)$ *and such that* $|w| < 1$. *Then*

$$\lim_{k\to -\infty} F_{\rho_k}(\lambda, t) = F_{\text{GV}}(\lambda, t)\,. \tag{3.90}$$

The limits studied above can be informally interpreted as the relation between $F_{\text{GV}}(\lambda, t)$ and the Borel summations along the imaginary axes.

3.4.1. Relation between $F_{\mathbb{R}_{>0}}$ and $F_{\text{coni}}^{\text{resum}}$. In this subsection, we briefly explain how $F_{\mathbb{R}_{>0}} = F_{np}$ relates to $F_{\text{coni}}^{\text{resum}}$ from [HO15].

On the one hand, from part (iv) of Theorem 2.1 together with the comments of Sect. 2.3, we find that

$$F_{\mathbb{R}_{>0}}(\lambda, t) = F_{\text{GV}}(\lambda, t) + \frac{1}{2\pi i}\frac{\partial}{\partial\lambda}\lambda F_{NS}\Big(\frac{4\pi^2}{\lambda}, \frac{2\pi}{\lambda}\Big(t - \frac{1}{2}\Big)\Big). \tag{3.91}$$

On the other hand, in [HO15] the following function is considered

$$\begin{aligned} &F_{\text{coni}}^{\text{resum}}(\lambda, t) = \frac{\text{Li}_3(Q)}{\lambda^2} + \int_0^\infty dv\, \frac{v}{1 - e^{2\pi v - i0^+}}\log(1 + Q^2 - 2Q\cosh(\lambda v)), \\ &Q = e^{2\pi i t}, \end{aligned} \tag{3.92}$$

and it is conjectured that

$$F_{\text{coni}}^{\text{resum}}(\lambda, t) = F_{\text{GV}}(\lambda, t) + \frac{1}{2\pi i}\frac{\partial}{\partial\lambda}\lambda F_{NS}\Big(\frac{4\pi^2}{\lambda}, \frac{2\pi}{\lambda}\Big(t - \frac{1}{2}\Big)\Big), \tag{3.93}$$

as explained in Sect. 2.4.

We show that this in indeed the case, by the use of the Woronowicz form of $F_{\mathbb{R}_{>0}}$ of Lemma 3.10.

Proposition 3.21. *Let $t \in \mathbb{C}$ be such that $0 < \text{Re}(t) < 1$, $\text{Im}(t) > 0$, and let λ be in the sector determined by $l_0 = \mathbb{R}_{<0} \cdot 2\pi \mathrm{i} t$ and $l_{-1} = \mathbb{R}_{<0} \cdot 2\pi \mathrm{i}(t-1)$. Then $F_{\mathbb{R}_{>0}} = F_{\text{coni}}^{\text{resum}}$ on their common domains of definition.*

Proof. First notice that since

$$1 + Q^2 - 2Q\cosh(\lambda x) = (1 - e^{\lambda x}Q)(1 - e^{-\lambda x}Q), \tag{3.94}$$

we can rewrite (3.92) as follows

$$\begin{aligned} F_{\text{coni}}^{\text{resum}} &= \tfrac{\text{Li}_3(Q)}{\lambda^2} + \textstyle\int_{\mathbb{R}+\mathrm{i}0^+} \mathrm{d}v \tfrac{v}{1-e^{2\pi v}}\log(1 - e^{\lambda v}Q) \\ &- \textstyle\int_{-\infty}^0 \mathrm{d}v \tfrac{v}{1-e^{2\pi v - \mathrm{i}0^+}}\log(1 - e^{\lambda v}Q) + \int_0^\infty \mathrm{d}v \tfrac{v}{1-e^{2\pi v-\mathrm{i}0^+}}\log(1 - e^{-\lambda v}Q) \\ &= \tfrac{\text{Li}_3(Q)}{\lambda^2} + \textstyle\int_{\mathbb{R}+\mathrm{i}0^+} \mathrm{d}v \tfrac{v}{1-e^{2\pi v}}\log(1 - e^{\lambda v}Q) + \int_0^\infty \mathrm{d}v\; v\log(1 - e^{-\lambda v}Q). \end{aligned} \tag{3.95}$$

On the other hand, notice that we can rewrite the last term in the above expression as follows:

$$\begin{aligned} \int_0^\infty \mathrm{d}v\; v\log(1 - e^{-\lambda v}Q) &= \frac{1}{\lambda^2}\int_{\lambda\cdot\mathbb{R}_{>0}} \mathrm{d}v\; v\log(1 - e^{-v}Q) \\ &= \frac{1}{\lambda^2}\int_0^\infty \mathrm{d}v\; v\log(1 - e^{-v}Q) \\ &= \frac{1}{2\lambda^2}\int_0^\infty \mathrm{d}v\, \frac{v^2}{1 - e^{v}Q^{-1}} = -\frac{1}{\lambda^2}\text{Li}_3(Q) \end{aligned} \tag{3.96}$$

where in the second equality we have used that the range of λ allows us to deform back the contour to $\mathbb{R}_{>0}$; in the third equality we have integrated by parts; and in the last one we have used that $\text{Im}(t) > 0$ implies that $|e^{2\pi i t}| < 1$, and hence we can use the integral representation of Li_3.

Hence,

$$F_{\text{coni}}^{\text{resum}} = \int_{\mathbb{R}+i0^+} dv \frac{v}{1-e^{2\pi v}} \log(1-e^{\lambda v}Q)$$
$$= \frac{1}{(2\pi)^2} \int_{\mathbb{R}+i0^+} dv \frac{v}{1-e^{v}} \log(1-e^{\frac{\lambda}{2\pi}v+2\pi i t}) \tag{3.97}$$

so the result follows from Lemma 3.10 and Proposition 3.11. □

4. Relation to the Riemann–Hilbert Problem and Line Bundles

In Sects. 2 and 3 we discussed the Borel sum $F_\rho(\lambda, t)$ of $\widetilde{F}(\lambda, t)$ along the ray ρ, and its dependence on ρ in terms of the Stokes jumps. Our objectives in this section are the following:

- On one hand, in [Bri20] a Riemann–Hilbert problem is associated to the BPS spectrum of the resolved conifold. This involves finding piecewise holomorphic functions $\mathcal{X}_\gamma$ on $\mathbb{C}^\times \times M$, with M being called the space of stability structures, related by certain Stokes jumps along rays in $\mathbb{C}^\times$. Introducing a coordinate λ_B for $\mathbb{C}^\times$ called twistor variable one may interpret the family of functions $\mathcal{X}_\gamma(\lambda_B, -)$ on M as complex coordinates defining a family of complex structures on M. We will show that the jumps of $F_\rho(\lambda, t)$ serve as "potentials" for the Stokes jumps associated to the Riemann–Hilbert problem.
- On the other hand, in thinking of $F_\rho(\lambda, t)$ more geometrically, it is natural to consider the partition functions

$$Z_\rho(\lambda, t) = \exp(F_\rho(\lambda, t)) , \tag{4.1}$$

  and interpret them as defining a section of a line bundle $\mathcal{L}$, having transition functions equal to the exponentials of the Stokes jumps. This perspective follows the ideas of [CLT20], specialized to the case of the resolved conifold.
- We will furthermore demonstrate that the the line bundle $\mathcal{L}$ is related to certain hyperholomorphic line bundles previously considered in [Nei11,APP11b]. These hyperholomorphic line bundles are canonically defined by a given BPS spectrum and represented by transition functions defined from the Rogers dilogarithm function. We will show that the hyperholomorphic line bundles considered in [Nei11,APP11b] are in the case of the resolved conifold related to the line bundle $\mathcal{L}$ by performing a certain "conformal limit" previously considered in [Gai14].

In order to facilitate comparison with [Bri20], we will represent the parameters λ and t used in this paper in the following form

$$t = v/w, \qquad \lambda = 2\pi\lambda_{\text{B}}/w , \tag{4.2}$$

where λ_{B} is the notation used here for the variable denoted t in [Bri20], and consider a projectivized partition function

$$Z_\rho(\lambda_{\text{B}}, v, w) := \exp\left(F_{w^{-1}\cdot\rho}\left(\frac{2\pi\lambda_{\text{B}}}{w}, \frac{v}{w}\right)\right). \tag{4.3}$$

We will show that after appropriately normalizing the partition functions $Z_\rho \to \widehat{Z}_\rho$, the BPS spectrum of the resolved conifold will be neatly encoded in the transition functions of the line bundle defined by $\widehat{Z}_\rho$. Furthermore, we will see that the normalization reintroduces the constant map constribution 2.5, giving a partition function whose free energy has asymptotic expansion equal to 2.4.

4.1. Bridgeland's Riemann–Hilbert problem and its solution. We begin by recalling the Riemann–Hilbert problem considered in [Bri20]. The initial data for such Riemann–Hilbert problems is the following:

Definition 4.1. A variation of BPS structures is given by a tuple (M, Γ, Z, Ω), where

- M is a complex manifold.
- Charge lattice: $\Gamma \to M$ is a local system of lattices with a skew-symmetric, covariantly constant paring $\langle -, - \rangle \colon \Gamma \times \Gamma \to \mathbb{Z}$.
- Central charge: Z is a holomorphic section of $\Gamma^* \otimes \mathbb{C} \to M$.
- BPS indices: $\Omega \colon \Gamma \to \mathbb{Z}$ is a function satisfying $\Omega(\gamma) = \Omega(-\gamma)$ and the Kontsevich–Soibelman wall-crossing formula [KS08, Bri19].

The tuple (M, Γ, Z, Ω) should also satisfy the following conditions:

- Support property: Let $\mathrm{Supp}(\Omega) := \{\gamma \in \Gamma \mid \Omega(\gamma) \neq 0\}$. Given a compact set $K \subset M$ and a choice of covariantly constant norm $|\cdot|$ on $\Gamma|_K \otimes_{\mathbb{Z}} \mathbb{R}$, there is a constant $C > 0$ such that for any $\mathrm{Supp}(\Omega) \cap \Gamma|_K$:

$$|Z_\gamma| > C|\gamma|\,. \tag{4.4}$$

- Convergence property: for each $p \in M$, there is an $R > 0$ such that

$$\sum_{\gamma \in \Gamma_p} |\Omega(\gamma)| e^{-R|Z_\gamma|} < \infty\,. \tag{4.5}$$

The variation of BPS structures associated to the resolved conifold is then taken to be the tuple (M, Γ, Z, Ω), where:

- M is the complex 2-dimensional manifold

$$M := \{(v, w) \in \mathbb{C}^2 \mid w \neq 0, \quad v + nw \neq 0 \quad \text{for all} \quad n \in \mathbb{Z}\}\,. \tag{4.6}$$

- $\Gamma \to M$ is given by the trivial local system

$$\Gamma = \mathbb{Z} \cdot \delta \oplus \mathbb{Z} \cdot \beta \oplus \mathbb{Z} \cdot \delta^\vee \oplus \mathbb{Z} \cdot \beta^\vee, \tag{4.7}$$

with pairing defined such that $(\beta^\vee, \beta, \delta^\vee, \delta)$ is a Darboux frame. Namely,

$$\langle \delta^\vee, \delta \rangle = \langle \beta^\vee, \beta \rangle = 1\,, \tag{4.8}$$

with all other pairings equal to 0.
- If for $\gamma \in \Gamma$, we denote $Z_\gamma := Z(\gamma)$, Z is defined by

$$Z_{n\beta + m\delta + p\beta^\vee + q\delta^\vee} = 2\pi \mathrm{i}(nv + mw), \quad \text{for} \quad n, m, p, q \in \mathbb{Z}\,. \tag{4.9}$$

- Ω is given by the BPS spectrum of the resolved conifold [JS12], see also [BLR19]:

$$\Omega(\gamma) = \begin{cases} 1 & \text{if } \gamma = \pm\beta + n\delta \quad \text{for} \quad n \in \mathbb{Z}\,, \\ -2 & \text{if } \gamma = k\delta \quad \text{for} \quad k \in \mathbb{Z} \setminus \{0\}\,, \\ 0 & \text{otherwise.} \end{cases} \tag{4.10}$$

To this data, the following Riemann–Hilbert problem is associated[2]. First, we define $\mathcal{L}_k := \mathbb{R}_{<0} \cdot 2\pi\mathrm{i}(v+kw)$ and $\mathcal{L}_\infty := \mathbb{R}_{<0} \cdot 2\pi\mathrm{i}w$, and assume that $(v, w) \in M$ satisfies $\mathrm{Im}(v/w) > 0$ (the case $\mathrm{Im}(v/w) \leq 0$ is also considered in [Bri20], but we restrict to $\mathrm{Im}(v/w) > 0$ for simplicity). Then, for each ray ρ from 0 to ∞ not in $\{\pm\mathcal{L}_k\}_{k\in\mathbb{Z}} \cup \{\pm\mathcal{L}_\infty\}$, we should find a holomorphic function $\mathcal{X}_{\gamma,\rho}(v, w, -) \colon \mathbb{H}_\rho \to \mathbb{C}^\times$ labeled by $\gamma \in \Gamma$ such that they satisfy the following:

- Twisted homomorphism property: for $\gamma, \gamma' \in \Gamma$ we have

$$\mathcal{X}_{\gamma+\gamma',\rho}(v, w, -) = (-1)^{\langle\gamma,\gamma'\rangle}\mathcal{X}_{\gamma,\rho}(v, w, -)\mathcal{X}_{\gamma',\rho}(v, w, -)\,. \tag{4.11}$$

- Stokes jumps: we denote by ρ_k a ray between $\mathcal{L}_k$ and $\mathcal{L}_{k-1}$. We then have

$$\begin{aligned}&\mathcal{X}_{\gamma,\pm\rho_{k+1}}(v, w, \lambda_{\mathrm{B}})\\ &= \mathcal{X}_{\gamma,\pm\rho_k}(v, w, \lambda_{\mathrm{B}})(1 - \mathcal{X}_{\pm(\beta+k\delta)}(v, w, \lambda_{\mathrm{B}}))^{\langle\gamma,\pm(\beta+k\delta)\rangle\Omega(\beta+k\delta)}.\end{aligned} \tag{4.12}$$

On the other hand, consider ρ_{k_1} and ρ_{k_2} with $k_1 \neq k_2$, and let $[\rho_{k_1}, -\rho_{k_2}]$ denote the smallest of the two sectors determined by ρ_{k_1} and $-\rho_{k_2}$. In the case $\mathcal{L}_\infty \subset [\rho_{k_1}, -\rho_{k_2}]$, for λ_{B} in the corresponding common domains we have

$$\begin{aligned}\mathcal{X}_{\gamma,-\rho_{k_2}} = \mathcal{X}_{\gamma,\rho_{k_1}} \cdot \Bigg(&\prod_{k\geq k_1} (1 - \mathcal{X}_{\beta+k\delta})^{\langle\gamma,\beta+k\delta\rangle\Omega(\beta+k\delta)}\\ &\prod_{k>-k_2} (1 - \mathcal{X}_{-\beta+k\delta})^{\langle\gamma,-\beta+k\delta\rangle\Omega(\beta-k\delta)} \prod_{k\geq 1}(1 - \mathcal{X}_{k\delta})^{\langle\gamma,k\delta\rangle\Omega(k\delta)}\Bigg)\,,\end{aligned} \tag{4.13}$$

while if $-\mathcal{L}_\infty \subset [\rho_{k_1}, -\rho_{k_2}]$, we have

$$\begin{aligned}\mathcal{X}_{\gamma,\rho_{k_1}} = \mathcal{X}_{\gamma,-\rho_{k_2}} \cdot \Bigg(&\prod_{k\geq k_2} (1 - \mathcal{X}_{-\beta-k\delta})^{\langle\gamma,-\beta-k\delta\rangle\Omega(\beta+k\delta)}\\ &\prod_{k>-k_1} (1 - \mathcal{X}_{\beta-k\delta})^{\langle\gamma,\beta-k\delta\rangle\Omega(\beta-k\delta)} \prod_{k\geq 1}(1 - \mathcal{X}_{-k\delta})^{\langle\gamma,-k\delta\rangle\Omega(k\delta)}\Bigg)\,.\end{aligned} \tag{4.14}$$

- Asymptotics as $\lambda_{\mathrm{B}} \to 0$: For each ray ρ and $\gamma \in \Gamma$, we have

$$\mathcal{X}_{\gamma,\rho}(v, w, \lambda_{\mathrm{B}})e^{-Z_\gamma(z,w)/\lambda_{\mathrm{B}}} \to 1, \quad \text{as} \quad \lambda_{\mathrm{B}} \to 0, \quad \lambda_{\mathrm{B}} \in \mathbb{H}_\rho. \tag{4.15}$$

- Polynomial growth as $\lambda_{\mathrm{B}} \to \infty$: for each ray ρ and $\gamma \in \Gamma$, we have the following for some $k > 0$:

$$|\lambda_{\mathrm{B}}|^{-k} < |\mathcal{X}_{\gamma,\rho}(v, w, \lambda_{\mathrm{B}})| < |\lambda_{\mathrm{B}}|^k, \quad \text{for} \quad |\lambda_{\mathrm{B}}| \gg 0\,. \tag{4.16}$$

Such a problem is shown to admit a unique solution [Bri19, Lemma 4.9], and the solution is given as follows. By the twisted homomorphism property, it is enough to describe $\mathcal{X}_{\gamma,\rho}$ for $\gamma \in \{\beta^\vee, \beta, \delta^\vee, \delta\}$. The solutions for $\gamma = \beta$ and $\gamma = \delta$ have trivial Stokes jumps, and they are given by

$$\mathcal{X}_{\beta,\rho}(v, w, \lambda_{\mathrm{B}}) = e^{2\pi\mathrm{i}v/\lambda_{\mathrm{B}}}, \quad \mathcal{X}_{\delta,\rho}(z, w, \lambda_{\mathrm{B}}) = e^{2\pi\mathrm{i}w/\lambda_{\mathrm{B}}}, \tag{4.17}$$

---

[2] We will follow slightly different conventions from [Bri20]. In particular, what we call $\mathcal{L}_k$ corresponds in Bridgeland's convention to $-\mathcal{L}_k$.

for any ray ρ. On the other hand, for a ray ρ_k between $\mathcal{L}_k$ and $\mathcal{L}_{k-1}$, the functions $\mathcal{X}_{\beta^\vee,-\rho_k}(v, w, \lambda_{\mathrm{B}})$ and $\mathcal{X}_{\delta^\vee,-\rho_k}(v, w, \lambda_{\mathrm{B}})$ are given by (see [Bri20, Equation (67)])

$$\begin{aligned}\mathcal{X}_{\beta^\vee,-\rho_k}(v, w, \lambda_{\mathrm{B}}) &= F^*(v + kw \mid w, -\lambda_{\mathrm{B}})\\ \mathcal{X}_{\delta^\vee,-\rho_k}(v, w, \lambda_{\mathrm{B}}) &= H^*(v + kw \mid w, -\lambda_{\mathrm{B}})(F^*(v + kw \mid w, -\lambda_{\mathrm{B}}))^k ,\end{aligned} \tag{4.18}$$

where F^* is defined in terms of double sine function, and H^* in terms of the triple sine function (see [Bri20, Section 5] for more details on the definitions of F^* and H^*). On the other hand, $\mathcal{X}_{\beta^\vee,\rho_k}$ and $\mathcal{X}_{\delta^\vee,\rho_k}$ are determined by the relation

$$\mathcal{X}_{\gamma,\rho_k}(v, w, \lambda_{\mathrm{B}}) = 1/\mathcal{X}_{\gamma,-\rho_k}(v, w, -\lambda_{\mathrm{B}}) , \tag{4.19}$$

that follows from uniqueness of the solutions of the Riemann–Hilbert problem.

4.2. Relation of the partition function to the Riemann–Hilbert problem. In this section we wish to relate the Stokes jumps of $F_\rho(\lambda, t)$ with the Stokes jumps of the Riemann–Hilbert problem. More specifically, we will first consider a normalization of the partition function $\exp(F_\rho(\lambda, t))$ by a factor proportional to $\lambda^{1/12} \exp(F_\rho(\lambda, 0))$. This normalization will not only capture the required Stokes jumps at $\pm l_\infty$, but will also allow us to recover the constant map contribution of (2.4). Indeed, by Corollary 3.13, this normalization introduces back the Borel sum of the constant map constribution to the free energy. We will then relate the Stokes jumps of the normalized partition function to the jumps of the Riemann–Hilbert problem of Sect. 4.1. This will in turn show us how the BPS spectrum of the resolved conifold is encoded in the Stokes jumps of $F_\rho(\lambda, t)$. We will use the notation of Sect. 4.1 throughout.

To establish the link with Sect. 4.1 more clearly, it will be convenient to consider the projectivized parameters

$$t = v/w, \quad \check{\lambda} = \lambda_{\mathrm{B}}/w, \tag{4.20}$$

where we recall that $\check{\lambda} = \lambda/2\pi$. We think of the tuple (v, w) as a point of M.

We consider the rays $\mathcal{L}_k = \mathbb{R}_{<0} \cdot 2\pi\mathrm{i}(v + kw) = \mathbb{R}_{<0} \cdot Z_{\beta+k\delta}$ for $k \in \mathbb{Z}$ and $\mathcal{L}_\infty = \mathbb{R}_{<0} \cdot 2\pi\mathrm{i}w = \mathbb{R}_{<0} \cdot Z_{k\delta}$. The relation to the old Stokes rays is then given by $\mathcal{L}_k = w \cdot l_k$ and $\mathcal{L}_\infty = w \cdot l_\infty$.

Definition 4.2. Given a ray ρ different from $\{\pm\mathcal{L}_k\}_{k\in\mathbb{Z}} \cup \{\pm\mathcal{L}_\infty\}$ we define the for $\lambda_{\mathrm{B}} \in \mathbb{H}_\rho$ and $(v, w) \in M$ with $\mathrm{Im}(v/w) \neq 0$,

$$\mathcal{F}_\rho(\lambda_{\mathrm{B}}, v, w) := F_{w^{-1}\cdot\rho}\left(\frac{2\pi\lambda_{\mathrm{B}}}{w}, \frac{v}{w}\right) = F_{w^{-1}\cdot\rho}(\lambda, t) . \tag{4.21}$$

Notice that

$$\mathcal{F}_\rho(\lambda_{\mathrm{B}}, v, 1) = F_\rho(\lambda, t) . \tag{4.22}$$

Following the same argument as in Proposition 3.14, it is easy to check the following:

Proposition 4.3. *Let ρ_k be a ray in the sector determined by the rays $\mathcal{L}_k$ and $\mathcal{L}_{k-1}$. Then, if* $\mathrm{Im}(v/w) > 0$*, on the overlap of their domains of definition in the* λ_{B} *variable we have*

$$\begin{aligned}\Phi_{\pm\mathcal{L}_k}(\lambda_{\mathrm{B}}, v, w) &:= \mathcal{F}_{\pm\rho_{k+1}}(\lambda_{\mathrm{B}}, v, w) - \mathcal{F}_{\pm\rho_k}(\lambda_{\mathrm{B}}, v, w) \\ &= \frac{1}{2\pi \mathrm{i}} \partial_{\lambda_{\mathrm{B}}} \left(\lambda_{\mathrm{B}} \mathrm{Li}_2\left(e^{\pm 2\pi \mathrm{i}(v+kw)/\lambda_{\mathrm{B}}}\right)\right) .\end{aligned} \tag{4.23}$$

If $\mathrm{Im}(v/w) < 0$*, then the previous jumps also hold provided ρ_{k+1} is interchanged with ρ_k in the above formulas.*

Proof. After a change of integration variables, we obtain

$$\mathcal{F}_{\rho_{k+1}}(\lambda_{\mathrm{B}}, v, w) - \mathcal{F}_{\rho_k}(\lambda_{\mathrm{B}}, v, w) = \frac{1}{w} \int_{\mathcal{H}(\mathcal{L}_k)} \mathrm{d}\xi \; e^{-\xi/\lambda_{\mathrm{B}}} G(\xi/w, v/w) \, , \tag{4.24}$$

where $\mathcal{H}(\mathcal{L}_k)$ is a Hankel contour around $\mathcal{L}_k$. The result then follows by summing over residues along the poles in $\mathcal{L}_k$, as in Proposition 3.14. □

4.2.1. Normalizing the partition function and the Stokes jumps at $\pm l_\infty$. We would like to first make sense of a limit of the form

$$F_\rho(\lambda, 0) := \lim_{t \to 0} F_\rho(\lambda, t) \, , \tag{4.25}$$

where t is taken to satisfy $\mathrm{Re}(t) > 0$, $\mathrm{Im}(t) > 0$; and such that along the limit, ρ is always between l_{-1} and l_0 (resp. $-l_{-1}$ and $-l_0$) if ρ is on the right (resp. left) Borel half-plane.

Lemma 4.4. *The limit $F_\rho(\lambda, 0)$ from above exists for $\lambda \in \mathbb{H}_\rho$. In fact, if ρ is on the right (resp. left) Borel plane, is can be analytically continued to $\lambda \in \mathbb{C}^\times \setminus \mathbb{R}_{\le 0}$ (resp. $\lambda \in \mathbb{C}^\times \setminus \mathbb{R}_{\ge 0}$).*

Proof. Let us assume that ρ is on the right Borel plane. Then by Propositions 3.11 and 3.8, and our condition on t, we have

$$F_\rho(\lambda, t) = F_{\mathbb{R}_{>0}}(\lambda, t) = \log(G_3(t \,|\, \check{\lambda}, 1)) \, , \tag{4.26}$$

where we recall that G_3 is the function defined in terms of the triple sine in Definition 3.7.

By Proposition 3.8, the function $G_3(t \,|\, \check{\lambda}, 1)$ has a well defined value at $t = 0$; and $G_3(0 \,|\, \check{\lambda}, 1)$ is everywhere regular, vanishing only at the points $\check{\lambda} \in \mathbb{Q}_{\le 0}$.

In particular, for $\check{\lambda} \in \mathbb{H}_\rho$, we have

$$F_\rho(\lambda, 0) = \lim_{t \to 0} F_\rho(\lambda, t) = \log(G_3(0 \,|\, \check{\lambda}, 1)) \, , \tag{4.27}$$

and we can analytically continue $F_\rho(\lambda, 0)$ to $\lambda \in \mathbb{C}^\times \setminus \mathbb{R}_{\le 0}$.

If ρ is on the left Borel plane, the statement follows from the previous case, together with the relation $F_\rho(\lambda, t) = F_{-\rho}(-\lambda, t)$ from Lemma 3.16. □

Notice that by Lemma 4.4, we can also write

$$F_\rho(\lambda,0) = \frac{1}{\lambda^2}\mathrm{Li}_3(1) + \lim_{t\to 0}\left(\frac{B_2}{2}\mathrm{Li}_1(e^{2\pi \mathrm{i}t}) + \int_\rho \mathrm{d}\xi\, e^{-\xi/\check{\lambda}} G(\xi,t)\right), \tag{4.28}$$

where the limit in t is assumed to satisfy the constraints from above.

The following proposition suggests that one can obtain the appropriate Stokes jumps at $\pm l_\infty$ by considering a normalization involving $F_\rho(\lambda,0)$:

Proposition 4.5. *Let ρ (resp. ρ') be a ray close to $l_\infty = \mathrm{i}\mathbb{R}_{<0}$ from the left (resp. right). Then for λ in their common domain of definition*

$$F_{\pm\rho}(\lambda,0) - F_{\pm\rho'}(\lambda,0) = \frac{1}{\pi \mathrm{i}}\sum_{k\geq 1}\partial_{\check{\lambda}}\Big(\check{\lambda}\,\mathrm{Li}_2\big(e^{\pm 2\pi \mathrm{i}k/\check{\lambda}}\big)\Big) - \frac{\mathrm{i}\pi}{12}\,. \tag{4.29}$$

Furthermore $F_\rho(\lambda,0)$ only has Stokes jumps along $\pm l_\infty$.

Proof. First, notice that by our definition of the limit in t, we have

$$\begin{aligned} F_\rho(\lambda,0) - F_{\rho'}(\lambda,0) &= \lim_{t\to 0}\Big(\int_\rho \mathrm{d}\xi\, e^{-\xi/\check{\lambda}}G(\xi,t) - \int_{\rho'}\mathrm{d}\xi\, e^{-\xi/\check{\lambda}}G(\xi,t)\Big) \\ &= \lim_{t\to 0}\int_{\mathcal{H}}\mathrm{d}\xi\, e^{-\xi/\check{\lambda}}G(\xi,t)\,, \end{aligned} \tag{4.30}$$

where $\mathcal{H} = \rho - \rho'$ denotes a Hankel contour along $\mathrm{i}\mathbb{R}_{<0}$, containing l_k for $k \geq 0$ and $-l_k$ for $k < 0$. Hence, for λ close to l_∞, the Hankel contour just gives the contribution of these rays that we previously computed:

$$\begin{aligned} F_\rho(\lambda,0) - F_{\rho'}(\lambda,0) &= \lim_{t\to 0}\int_{\mathcal{H}}\mathrm{d}\xi\, e^{-\xi/\check{\lambda}}G(\xi,t) \\ &= \lim_{t\to 0}\left(\frac{1}{2\pi \mathrm{i}}\sum_{k\geq 1}\Big[\partial_{\check{\lambda}}\Big(\check{\lambda}\,\mathrm{Li}_2(e^{2\pi \mathrm{i}(t+k)/\check{\lambda}})\Big) + \partial_{\check{\lambda}}\Big(\check{\lambda}\,\mathrm{Li}_2(e^{-2\pi \mathrm{i}(t-k)/\check{\lambda}})\Big)\Big] \right.\\ &\qquad \left. + \frac{1}{2\pi \mathrm{i}}\partial_{\check{\lambda}}\Big(\mathrm{Li}_2(e^{2\pi \mathrm{i}t/\check{\lambda}})\Big)\right) \\ &= \frac{1}{\pi \mathrm{i}}\sum_{k\geq 1}\partial_{\check{\lambda}}\Big(\check{\lambda}\,\mathrm{Li}_2(e^{2\pi \mathrm{i}k/\check{\lambda}})\Big) + \frac{1}{2\pi \mathrm{i}}\mathrm{Li}_2(1) \\ &= \frac{1}{\pi \mathrm{i}}\sum_{k\geq 1}\partial_{\check{\lambda}}\Big(\check{\lambda}\,\mathrm{Li}_2(e^{2\pi \mathrm{i}k/\check{\lambda}})\Big) - \frac{\pi \mathrm{i}}{12}\,. \end{aligned} \tag{4.31}$$

A similar argument follows for $-l_\infty = \mathrm{i}\mathbb{R}_{>0}$. Furthermore, the fact that there are no other Stokes jumps follows from the way we have defined the limit $F_\rho(\lambda,0)$. For example, if ρ and ρ' are both on the right Borel plane, then along the limit ρ and ρ' are both between l_0 and l_{-1}, and hence $F_\rho(\lambda,0) = F_{\rho'}(\lambda,0)$. □

We can as before projectivize, and define

$$\mathcal{F}_\rho(\lambda_{\mathrm{B}},0,w) := \lim_{v\to 0}\mathcal{F}_\rho(\lambda_{\mathrm{B}},v,w)\,, \tag{4.32}$$

where the limit in v is such that $t = v/w$ satisfies the conditions of the previous limit in $F_\rho(\lambda,0)$.

Proposition 4.6. *Let ρ (resp. ρ') be a ray close to $\mathcal{L}_\infty = \mathrm{i}\mathbb{R}_{<0}$ from the left (resp. right). Then for λ_B in their common domain of definition*

$$\mathcal{F}_{\pm\rho}(\lambda_\mathrm{B}, 0, w) - \mathcal{F}_{\pm\rho'}(\lambda_\mathrm{B}, 0, w) = \sum_{k\geq 1} \Phi_{\pm\mathcal{L}_\infty,k} - \frac{\pi \mathrm{i}}{12}, \tag{4.33}$$

where

$$\Phi_{\pm\mathcal{L}_\infty,k}(\lambda_\mathrm{B}, v, w) := \frac{1}{\pi \mathrm{i}} \partial_{\lambda_\mathrm{B}} \Big(\lambda_\mathrm{B} \mathrm{Li}_2\big(e^{\pm 2\pi \mathrm{i} k w/\lambda_\mathrm{B}}\big)\Big) \tag{4.34}$$

Furthermore, $\mathcal{F}_{\pm\rho}(\lambda_\mathrm{B}, 0, w)$ only has Stokes jumps along $\pm\mathcal{L}_\infty$.

On the other hand, Corollary 3.13 and (2.20) suggest to not only normalize by $\mathcal{F}_\rho(\lambda_B, 0, w)$ but to also normalize by a $\frac{1}{12}\log(\lambda_B/w)$ term as follows:

Definition 4.7. Given a ray ρ different from $\{\pm\mathcal{L}_k\}_{k\in\mathbb{Z}} \cup \{\pm\mathcal{L}_\infty\}$ we define the following for $\lambda_\mathrm{B} \in \mathbb{H}_\rho$ and $(v, w) \in M$ with $\mathrm{Im}(v/w) > 0$:

$$\widehat{\mathcal{F}}_\rho(\lambda_\mathrm{B}, v, w) := \mathcal{F}_\rho(\lambda_\mathrm{B}, v, w) - \mathcal{F}_\rho(\lambda_\mathrm{B}, 0, w) - \frac{1}{12}\log\left(\frac{\lambda_B}{w}\right), \tag{4.35}$$

where we place the branch cut of the log term at $l_\infty = \mathrm{i}\mathbb{R}_{<0}$. We also define the normalized partition function as

$$\widehat{Z}_\rho(\lambda_\mathrm{B}, v, w) := \exp(\widehat{\mathcal{F}}_\rho(\lambda_\mathrm{B}, v, w)). \tag{4.36}$$

We remark that due to (2.20) and Corollary 3.13, asymptotic expansion of the the normalized free energy $\hat{\mathcal{F}}_\rho(\lambda, t, 1)$ as $\lambda \to 0$ recovers the full $\mathcal{F}(Q, \lambda)$ of (2.4) up to an overall shift by a constant, which can be absorbed in a redefinition of $\widehat{\mathcal{F}}_\rho(\lambda_\mathrm{B}, v, w)$.

We immediately obtain the following:

Corollary 4.8. *The Stokes jumps of $\hat{\mathcal{F}}(\lambda_B, v, w)$ at $\pm\mathcal{L}_k$ are given by (4.23), while (using the notation of Proposition 4.6) we find that the Stokes jumps at $\pm\mathcal{L}_\infty$ are determined by*

$$\begin{aligned} -\mathcal{F}_{\pm\rho}(\lambda_\mathrm{B}, 0, w) + \mathcal{F}_{\pm\rho'}(\lambda_\mathrm{B}, 0, w) - \lim_{\lambda_B \to \pm\mathcal{L}_\infty^+} \frac{1}{12}\log\left(\frac{\lambda_B}{w}\right) - \lim_{\lambda_B \to \pm\mathcal{L}_\infty^-} \frac{1}{12}\log\left(\frac{\lambda_B}{w}\right) \\ = -\sum_{k\geq 1} \Phi_{\pm\mathcal{L}_\infty,k} \mp \frac{\pi \mathrm{i}}{12}, \end{aligned} \tag{4.37}$$

where $\lim_{\lambda_B\to\pm\mathcal{L}_\infty^+}$ (resp. $\lim_{\lambda_B\to\pm\mathcal{L}_\infty^-}$) denotes the anti-clockwise (resp. clockwise) limit to $\pm\mathcal{L}_\infty$.

Proof. The only new thing to notice is that because of the $\log(\check{\lambda})$ term with branch cut at l_∞ of the normalization, we find that

$$-\lim_{\lambda_B \to \mathcal{L}_\infty^+} \frac{1}{12}\log\left(\frac{\lambda_B}{w}\right) - \lim_{\lambda_B \to \mathcal{L}_\infty^-} \frac{1}{12}\log\left(\frac{\lambda_B}{w}\right) = -\frac{2\pi \mathrm{i}}{12} = -\frac{\pi \mathrm{i}}{6}, \tag{4.38}$$

while

$$-\lim_{\lambda_B \to -\mathcal{L}_\infty^+} \frac{1}{12}\log\left(\frac{\lambda_B}{w}\right) - \lim_{\lambda_B \to -\mathcal{L}_\infty^-} \frac{1}{12}\log\left(\frac{\lambda_B}{w}\right) = 0. \tag{4.39}$$

Combining these jumps with (minus) the jumps of (4.33) gives the desired result. □

4.2.2. Relation to the Riemann–Hilbert problem. In this subsection, we wish to relate the Stokes jumps of $\widehat{Z}_\rho$ with the Stokes jumps of the Riemann–Hilbert problem in Sect. 4.1. We will see that the jumps of $\widehat{Z}_\rho$ serve as "potentials" for the jumps of the RH problem.

For the discussion below, it will be useful to note that the Stokes jumps can be represented in terms of the double sine function, revealing some important properties. A useful review of definition and relevant properties of the double sine function $\sin_2(z\,|\,\omega_1,\omega_2)$ can be found in [Bri20, Section 4] and references therein.

Definition 4.9. For $z \in \mathbb{C}$ and $\omega_1, \omega_2 \in \mathbb{C}^\times$, let

$$F(z\,|\,\omega_1,\omega_2) := \exp\Big(-\frac{\pi\mathrm{i}}{2}B_{2,2}(z\,|\,\omega_1,\omega_2)\Big)\cdot \sin_2(z\,|\,\omega_1,\omega_2) \tag{4.40}$$

where $B_{2,2}(z\,|\,\omega_1,\omega_2)$ is a multiple Bernoulli polynomial.

We will use the following properties of the function F.

Proposition 4.10 (See [Bri20, Proposition 4.1]). *The function $F(z\,|\,\omega_1,\omega_2)$ is a single valued meromorphic function of the variables $z \in \mathbb{C}$ and $\omega_1,\omega_2 \in \mathbb{C}^\times$ under the assumption $\omega_1/\omega_2 \notin \mathbb{R}_{<0}$. It has the following properties:*

- *The function is regular and non-vanishing except at the points*

$$z = a\omega_1 + b\omega_2, \quad a, b \in \mathbb{Z}\,. \tag{4.41}$$

- *It is invariant under simultaneous rescaling of the three arguments, and symmetric in ω_1,ω_2.*
- *It satisfies the difference equation*

$$\frac{F(z+\omega_1\,|\,\omega_1,\omega_2)}{F(z\,|\,\omega_1,\omega_2)} = (1-e^{2\pi\mathrm{i}z/\omega_2})^{-1}\,. \tag{4.42}$$

- *When* $\mathrm{Re}(\omega_i) > 0$ *and* $0 < \mathrm{Re}(z) < \mathrm{Re}(\omega_1+\omega_2)$ *there is an integral representation*

$$F(z\,|\,\omega_1,\omega_2) = \exp\Big(\int_{\mathbb{R}+\mathrm{i}0^+}\frac{\mathrm{d}u}{u}\,\frac{e^{zu}}{(e^{\omega_1 u}-1)(e^{\omega_2 u}-1)}\Big)\,. \tag{4.43}$$

Definition 4.11. Let $\Phi_{\pm\mathcal{L}_k}(\lambda_{\mathrm{B}}, v, w)$ and $\Phi_{\pm\mathcal{L}_\infty,k}(\lambda_{\mathrm{B}}, v, w)$ be as in (4.23) and (4.34), respectively. We then define[3]

$$\Xi_{\pm\mathcal{L}_k}(v,w,\lambda_{\mathrm{B}}) := e^{\Phi_{\pm\mathcal{L}_k}(\lambda_{\mathrm{B}},v,w)}, \qquad \Xi_{\pm\mathcal{L}_\infty,k}(v,w,\lambda_{\mathrm{B}}) := e^{-\Phi_{\pm\mathcal{L}_\infty,k}(\lambda_{\mathrm{B}},v,w)}\,. \tag{4.44}$$

We now give a proposition relating the exponentials of the Stokes jumps (4.44) of $\widehat{Z}_\rho$ to the function F.

Proposition 4.12. *Assuming* $\mathrm{Im}(v/w) > 0$, *we can write on their common domains of definition*

$$\begin{aligned}\Xi_{\pm\mathcal{L}_k}(v,w,\lambda_{\mathrm{B}}) &= (F(\pm(v+kw)/\lambda_{\mathrm{B}}+1\,|\,1,1))^{-1}\\ \Xi_{\pm\mathcal{L}_\infty,k}(v,w,\lambda_{\mathrm{B}}) &= (F(\pm kw/\lambda_{\mathrm{B}}+1\,|\,1,1))^{2}\,.\end{aligned} \tag{4.45}$$

Furthermore, the functions on the right of (4.45) are holomorphic and non-vanishing for $\lambda_{\mathrm{B}} \in \mathbb{H}_{\pm\mathcal{L}_k}$ and $\lambda_{\mathrm{B}} \in \mathbb{H}_{\pm\mathcal{L}_\infty}$, respectively.

[3] The minus sign in the exponent of the second expression in (4.44) is due to the normalization of the partition function.

Proof. First, notice that

$$\begin{aligned}\Phi_{\mathcal{L}_k}(\lambda_{\rm B}, v, w) &= \frac{1}{2\pi\mathrm{i}}\partial_{\lambda_{\rm B}}\Big(\lambda_{\rm B}\mathrm{Li}_2\big(e^{2\pi\mathrm{i}(v+kw)/\lambda_{\rm B}}\big)\Big)\\ &= \frac{1}{2\pi\mathrm{i}}\partial_{\check{\lambda}}\Big(\check{\lambda}\mathrm{Li}_2(e^{2\pi\mathrm{i}(t+k)/\check{\lambda}})\Big)\\ &= \phi_{l_k}(\lambda, t)\,. \end{aligned} \tag{4.46}$$

In particular, if $w = e^{2\pi\mathrm{i}t/\check{\lambda}}$ and $\widetilde{q} = e^{2\pi\mathrm{i}k/\check{\lambda}}$, then for λ near l_k, we can expand

$$\Phi_{\mathcal{L}_k}(\lambda_{\rm B}, v, w) = \frac{1}{2\pi\mathrm{i}}\sum_{l=1}^{\infty}\frac{(w\widetilde{q}^k)^l}{l}\Big(\frac{1}{l} - \log(w\widetilde{q}^k)\Big)\,. \tag{4.47}$$

On the other hand, assuming for the moment that $0 < \mathrm{Re}(t+k) < \mathrm{Re}(\check{\lambda}), \mathrm{Re}(\check{\lambda}) > 0$, and using the scaling invariance of F, we find by Proposition 4.10 that

$$\begin{aligned}F((v+kw)/\lambda_{\rm B}+1\,|\,1,1) &= F((t+k)/\check{\lambda}+1\,|\,1,1)\\ &= F(t+k+\check{\lambda}\,|\,\check{\lambda},\check{\lambda})\\ &= \exp\Big(\int_{\mathbb{R}+i0^+}\frac{\mathrm{d}u}{u}\frac{e^{u(t+k)}}{(2\sinh(\check{\lambda}u/2))^2}\Big)\,. \end{aligned} \tag{4.48}$$

Using that $\mathrm{Im}(v/w) = \mathrm{Im}(t) > 0$, we can compute the previous integral by closing the contour in the upper half-plane and sum over the residues at $2\pi\mathrm{i}l/\check{\lambda}$ for $l \geq 1$, obtaining

$$\begin{aligned}\int_{\mathbb{R}+i0^+}\frac{\mathrm{d}u}{u}\frac{e^{u(t+k)}}{(2\sinh(\check{\lambda}u/2))^2} &= 2\pi\mathrm{i}\sum_{l=1}^{\infty}\Big(\frac{1}{\check{\lambda}}\Big)^2\frac{\partial}{\partial u}\frac{e^{u(t+k)}}{u}\Big|_{u=2\pi\mathrm{i}l/\check{\lambda}}\\ &= -\frac{1}{2\pi\mathrm{i}}\sum_{l=1}^{\infty}\frac{(w\widetilde{q}^k)^l}{l}\Big(\frac{1}{l} - \log(w\widetilde{q}^k)\Big)\,. \end{aligned} \tag{4.49}$$

Comparing (4.47) with (4.49), the first result then follows, with the other cases being analogous.

To check the second statement, we use the fact from Proposition 4.10 that the function $F(z\,|\,\omega_1, \omega_2)$ is regular and non-vanishing except at the points

$$z = a\omega_1 + b\omega_2, \quad a, b \in \mathbb{Z}\,. \tag{4.50}$$

We then find that $F(\pm(v+kw)/\lambda_{\rm B}+1\,|\,1,1)$ is regular except at the points where

$$(v+kw)/\lambda_{\rm B} \in \mathbb{Z} \quad \Longleftrightarrow \lambda_{\rm B} = \frac{v+kw}{n}, \quad n \in \mathbb{Z}\,. \tag{4.51}$$

In particular, $F(\pm(v+kw)/\lambda_{\rm B}+1\,|\,1,1)$ is regular for $\lambda_{\rm B}$ in the half-plane centered at $\pm\mathcal{L}_k = \pm\mathbb{R}_{<0}\cdot 2\pi\mathrm{i}(v+nw)$. The other case follows similarly. □

From the previous proposition we obtain the following corollary:

Corollary 4.13. *Let* $(v, w) \in M$ *such that* $\mathrm{Im}(v/w) > 0$. *Then* $\Xi_{\pm\mathcal{L}_k}$ *and* $\Xi_{\pm\mathcal{L}_{\infty,k}}$ *serve as potentials for the jumps of the Riemann–Hilbert problem of Sect. 4.1, in the sense that*

$$\begin{aligned}\frac{\Xi_{\pm\mathcal{L}_k}(v+\lambda_{\mathrm{B}}, w, \lambda_{\mathrm{B}})}{\Xi_{\pm\mathcal{L}_k}(v, w, \lambda_{\mathrm{B}})} &= (1 - e^{\pm 2\pi\mathrm{i}(v+kw)/\lambda_{\mathrm{B}}})^{\pm 1} = (1 - \mathcal{X}_{\pm(\beta+k\delta)})^{\pm\langle\beta^\vee, \beta+k\delta\rangle\Omega(\beta+k\delta)},\\ \frac{\Xi_{\pm\mathcal{L}_k}(v, w+\lambda_{\mathrm{B}}, \lambda_{\mathrm{B}})}{\Xi_{\pm\mathcal{L}_k}(v, w, \lambda_{\mathrm{B}})} &= (1 - e^{\pm 2\pi\mathrm{i}(v+kw)/\lambda_{\mathrm{B}}})^{\pm k} = (1 - \mathcal{X}_{\pm(\beta+k\delta)})^{\pm\langle\delta^\vee, \beta+k\delta\rangle\Omega(\beta+k\delta)},\\ \frac{\Xi_{\pm\mathcal{L}_{\infty,k}}(0, w+\lambda_{\mathrm{B}}, \lambda_{\mathrm{B}})}{\Xi_{\pm\mathcal{L}_{\infty,k}}(0, w, \lambda_{\mathrm{B}})} &= (1 - e^{\pm 2\pi\mathrm{i}kw/\lambda_{\mathrm{B}}})^{\mp 2k} = (1 - \mathcal{X}_{\pm k\delta})^{\pm\langle\delta^\vee, k\delta\rangle\Omega(k\delta)}.\end{aligned} \tag{4.52}$$

Proof. We use the fact from Proposition 4.10 that $F(z\,|\,1,1)$ satisfies the difference equation

$$\frac{F(z+1\,|\,1,1)}{F(z\,|\,1,1)} = \frac{1}{1-e^{2\pi\mathrm{i}z}}\,. \tag{4.53}$$

Then by Proposition 4.12 we find that

$$\frac{\Xi_{\mathcal{L}_k}(v+\lambda_{\mathrm{B}}, w, \lambda_{\mathrm{B}})}{\Xi_{\mathcal{L}_k}(v, w, \lambda_{\mathrm{B}})} = \left(\frac{F((v+kw)/\lambda_{\mathrm{B}}+1+1|1,1)}{F((v+kw)/\lambda_{\mathrm{B}}+1,|1,1)}\right)^{-1} = 1 - e^{2\pi\mathrm{i}(v+kw)/\lambda_{\mathrm{B}}}\,. \tag{4.54}$$

The other identities follow similarly. □

Remark 4.14. From corollary 4.13 we see how the BPS spectrum is encoded in the jumps of the normalized partition function. Namely, the BPS spectrum of the resolved conifold appears in the expressions of the jumps of $\widehat{Z}_\rho(\lambda_B, v, w)$ as follows:

$$\begin{aligned}\Xi_{\pm\mathcal{L}_k}(v, w, \lambda_{\mathrm{B}}) &= \exp\left(\frac{\Omega(\beta+k\delta)}{2\pi\mathrm{i}}\partial_{\lambda_{\mathrm{B}}}\left(\lambda_{\mathrm{B}}\mathrm{Li}_2(e^{\pm Z_{\beta+k\delta}/\lambda_{\mathrm{B}}})\right)\right)\\ \Xi_{\pm\mathcal{L}_{\infty,k}}(v, w, \lambda_{\mathrm{B}}) &= \exp\left(\frac{\Omega(k\delta)}{2\pi\mathrm{i}}\partial_{\lambda_{\mathrm{B}}}\left(\lambda_{\mathrm{B}}\mathrm{Li}_2\left(e^{\pm Z_{k\delta}/\lambda_{\mathrm{B}}}\right)\right)\right),\end{aligned} \tag{4.55}$$

making explicit how the DT-invariant are encoded in the jumps.

4.3. The line bundle defined by the normalized partition function. We would like to discuss how to define a line bundle $\mathcal{L} \to \mathbb{C}^\times \times M$, such that the partition functions $\widehat{Z}_\rho(\lambda_{\mathrm{B}}, v, w)$ define a section of $\mathcal{L}$.

To concretely define the line bundle, we restrict for simplicity to[4]

$$M_+ := \{(v, w) \in M \mid \mathrm{Im}(v/w) > 0\}\,. \tag{4.56}$$

Furthermore, let ρ_k be a ray between $\mathcal{L}_k$ and $\mathcal{L}_{k-1}$. For definiteness, we pick ρ_k to be always in the middle of $\mathcal{L}_k$ and $\mathcal{L}_{k-1}$, and consider the open sets

$$U_k^\pm := \{(\lambda_{\mathrm{B}}, v, w) \in \mathbb{C}^\times \times M_+ \mid \lambda_{\mathrm{B}} \in \mathbb{H}_{\pm\rho_k}\}\,. \tag{4.57}$$

[4] See Remark 4.16 for the other points of M.

We remark that the condition on λ_{B} actually depends on (v, w), since the latter specifies the rays $\mathcal{L}_k$ (and hence also ρ_k). We then clearly have that $\{U_k^+\}_{k\in\mathbb{Z}} \cup \{U_k^-\}_{k\in\mathbb{Z}}$ forms an open cover of $\mathbb{C}^\times \times M_+$.

If $U_{k_1}^+ \cap U_{k_2}^+ \neq \emptyset$ for $k_1 < k_2$, we then define for $(\lambda_{\mathrm{B}}, v, w) \in U_{k_1}^+ \cap U_{k_2}^+$,

$$g_{k_1,k_2}^+(\lambda_{\mathrm{B}}, v, w) := \prod_{k_1 \leq k < k_2} \Xi_{\mathcal{L}_k}(\lambda_{\mathrm{B}}, v, w)\,. \tag{4.58}$$

Notice that if $(\lambda_{\mathrm{B}}, v, w) \in U_{k_1}^+ \cap U_{k_2}^+$, then $(\lambda_{\mathrm{B}}, v, w) \in \mathbb{H}_{\mathcal{L}_k}$ for $k_1 \leq k \leq k_2$, so by Proposition 4.12 we have that g_{k_1,k_2}^+ is $\mathbb{C}^\times$-valued:

$$g_{k_1,k_2}^+ \colon U_{k_1}^+ \cap U_{k_2}^+ \to \mathbb{C}^\times\,. \tag{4.59}$$

With the assumptions $U_{k_1}^+ \cap U_{k_2}^+ \neq \emptyset$ for $k_1 < k_2$, we also define $g_{k_2,k_1}^+ := (g_{k_1,k_2}^+)^{-1}$ and $g_{k,k}^+ := 1$ for any $k \in \mathbb{Z}$.

If $U_{k_1}^- \cap U_{k_2}^- \neq \emptyset$ for $k_1 < k_2$, then we similarly define

$$g_{k_1,k_2}^- \colon U_{k_1}^- \cap U_{k_2}^- \to \mathbb{C}^\times\,. \tag{4.60}$$

by

$$g_{k_1,k_2}^-(\lambda_{\mathrm{B}}, v, w) := \prod_{k_1 \leq k < k_2} \Xi_{-\mathcal{L}_k}(\lambda_{\mathrm{B}}, v, w)\,, \tag{4.61}$$

and $g_{k_2,k_1}^- := (g_{k_1,k_2}^-)^{-1}$, $g_{k,k}^- := 1$.

On the other hand, if for some $k_1, k_2 \in \mathbb{Z}$ we have $U_{k_1}^+ \cap U_{k_2}^- \neq \emptyset$, then $\rho_{k_1} \neq \rho_{k_2}$ and hence out of the two sectors determined by ρ_{k_1} and $-\rho_{k_2}$ there is a smallest one, which we denote by $[\rho_{k_1}, -\rho_{k_2}]$. For all $(\lambda_{\mathrm{B}}, v, w) \in U_{k_1}^+ \cap U_{k_2}^-$ we must either have that $\mathcal{L}_\infty \subset [\rho_{k_1}, -\rho_{k_2}]$ or $-\mathcal{L}_\infty \subset [\rho_{k_1}, -\rho_{k_2}]$. In the first case we define[5]

$$\begin{aligned} g_{k_1,k_2}^\infty(\lambda_{\mathrm{B}}, v, w) := e^{-\pi \mathrm{i}/12} \prod_{k \geq k_1} \Xi_{\mathcal{L}_k}(\lambda_{\mathrm{B}}, v, w) \\ \prod_{k<k_2} \Xi_{-\mathcal{L}_k}(\lambda_{\mathrm{B}}, v, w) \prod_{k \geq 1} \Xi_{\mathcal{L}_{\infty,k}}(\lambda_{\mathrm{B}}, v, w)\,, \end{aligned} \tag{4.62}$$

and $g_{k_2,k_1}^\infty := (g_{k_1,k_2}^\infty)^{-1}$.

Notice that in this first case we have $\lambda_{\mathrm{B}} \in \mathbb{H}_{\mathcal{L}_k}$ for $k \geq n_1$, $\lambda_{\mathrm{B}} \in \mathbb{H}_{-\mathcal{L}_k}$ for $k < n_2$, and $\lambda_{\mathrm{B}} \in \mathbb{H}_{\mathcal{L}_\infty}$. Hence, by Proposition 4.12 and the convergence of the above product[6], we find that

$$g_{k_1,k_2}^\infty(\lambda_{\mathrm{B}}, v, w) \colon U_{k_1}^+ \cap U_{k_2}^- \to \mathbb{C}^\times\,. \tag{4.63}$$

On the other hand, in the second case we define

[5] Recall that the $e^{\pm\pi \mathrm{i}/12}$ factors are due to the jumps (4.37) of the normalization of the partition function.

[6] Here we use that the corresponding infinite sums of $\Phi_{\mathcal{L}_k}$, $\Phi_{-\mathcal{L}_k}$ and $\Phi_{\mathcal{L}_{\infty,k}}$, converge for $(\lambda_B, v, w) \in U_{k_1}^+ \cap U_{k_2}^-$.

$$g_{k_2,k_1}^{-\infty}(\lambda_{\rm B}, v, w) := e^{\pi {\rm i}/12} \prod_{k\geq k_2} \Xi_{-\mathcal{L}_k}(\lambda_{\rm B}, v, w) \prod_{k<k_1} \Xi_{\mathcal{L}_k}(\lambda_{\rm B}, v, w) \prod_{k\geq 1} \Xi_{\mathcal{L}_{\infty},-k}(\lambda_{\rm B}, v, w)\,, \tag{4.64}$$

and $g_{k_1,k_2}^{-\infty} := (g_{k_2,k_1}^{-\infty})^{-1}$.

With the previous results, the following proposition then follows:

Proposition 4.15. *The functions* $g_{k_1,k_2}^{\pm}$, $g_{k_1,k_2}^{\pm\infty}$ *associated to the cover* $\{U_k^+\}_{k\in\mathbb{Z}}\cup\{U_k^-\}_{k\in\mathbb{Z}}$ *define a* 1*-Čech cocycle over* $\mathbb{C}^\times \times M_+$*, and hence a line bundle* $\mathcal{L} \to \mathbb{C}^\times \times M_+$*. Furthermore, assuming* $\mathrm{Im}(v/w) > 0$*, the normalized partition functions* $\widehat{Z}_\rho(\lambda, z, w)$ *glue together into a section of* $\mathcal{L}$*.*

Proof. The fact that $g_{k_1,k_2}^{\pm}$ and $g_{k_1,k_2}^{\pm\infty}$ define a 1-Čech cocycle follows directly from their definitions. Furthermore, the fact that the $\widehat{Z}_\rho(\lambda, z, w)$ glue together into a section follows from our previous discussions on the Stokes jumps of $\widehat{Z}_\rho(\lambda, z, w)$. □

Remark 4.16. Let

$$M_- := \{(v, w) \in M \mid \mathrm{Im}(v/w) < 0\}, \quad M_0 := \{(v, w) \in M \mid \mathrm{Im}(v/w) = 0\}\,, \tag{4.65}$$

so that $M = M_+ \cup M_- \cup M_0$. By using the Stokes jumps for the case $(u, v) \in M_-$ (resp. $(u, v) \in M_0$, where all the Stokes rays collapse to either $\mathcal{L}_\infty$ or $-\mathcal{L}_\infty$) we can as before define a line bundle over $\mathbb{C}^\times \times M_-$ (resp. $\mathbb{C}^\times \times M_0$) having the normalized partition function as a section. Since the Borel summations $F_\rho(\lambda_B, v, w)$ make sense on sufficiently small open subsets of $\mathbb{C}^\times \times M$, and furthermore depend holomorphically on the parameters, these line bundles glue together into a holomorphic line bundle $\mathcal{L} \to \mathbb{C}^\times \times M$ having the normalized partition function as a section.

4.4. Relation to the Rogers dilogarithm and hyperholomorphic line bundles. Notice that one can write the Stokes jumps of $\widehat{Z}_\rho(\lambda_{\rm B}, v, w)$ along $\pm\mathcal{L}_k = \pm\mathbb{R}_{<0} \cdot Z_{\beta+k\delta}$ as

$$\Phi_{\mathcal{L}_k}(\lambda_{\rm B}, v, w) = \frac{\Omega(\beta + k\delta)}{2\pi {\rm i}}\Big(\mathrm{Li}_2(\mathcal{X}_{\pm(\beta+k\delta)}) + \log(\mathcal{X}_{\pm(\beta+k\delta)}) \log(1 - \mathcal{X}_{\pm(\beta+k\delta)})\Big)\,, \tag{4.66}$$

where

$$\log(\mathcal{X}_{\pm(\beta+k\delta)}) = \pm 2\pi {\rm i}(v + kw)/\lambda_{\rm B}\,. \tag{4.67}$$

Up to a factor of $\frac{1}{2}$ in the second summand, this matches

$$\frac{\Omega(\beta + k\delta)}{2\pi {\rm i}} L(\mathcal{X}_{\pm(\beta+k\delta)})\,, \tag{4.68}$$

where $L(x)$ denotes the Rogers dilogarithm

$$L(x) := \mathrm{Li}_2(x) + \frac{1}{2}\log(x)\log(1 - x)\,. \tag{4.69}$$

In previous works [Nei11, APP11b], hyperholomorphic line bundles with transition functions having the form of the exponentials of (4.68) have been discussed in the context of instanton-corrected hyperkähler and quaternionic-Kähler geometries. Our goal in the rest of this section is then two-fold:

- We would first like to explain how (4.66) and (4.68) are related by changes of local trivialization involving the solutions of the RH problem of Sect. 4.1.
- This will then be used to relate the line bundle $\mathcal{L} \to \mathbb{C}^\times \times M_+$ constructed in Sect. 4.3 with a certain "conformal limit" of the line bundles constructed in [Nei11, APP11b].

4.4.1. Relation to the Rogers dilogarithm. In order to relate to the Rogers dilogarithm, we follow the idea suggested in [CLT20, Appendix H] (see also Lemma 4.18, below).

We start by considering the solutions of the RH problem from Sect. 4.1. Notice that since $\mathcal{X}_{\gamma,\rho}(v, w, -) : \mathbb{H}_\rho \to \mathbb{C}^\times$, then there must exist $x_{\gamma,\rho}(v, w, -) : \mathbb{H}_\rho \to \mathbb{C}$ such that

$$\mathcal{X}_{\gamma,\rho}(v, w, \lambda_{\mathrm{B}}) = \exp(x_{\gamma,\rho}(v, w, \lambda_{\mathrm{B}})). \tag{4.70}$$

We then define for $(\lambda_{\mathrm{B}}, v, w) \in U_\rho = \{(v, w, \lambda_B) \in M_+ \times \mathbb{C}^\times \mid \lambda_B \in \mathbb{H}_\rho\}$,

$$x_{\beta,\rho} := 2\pi \mathrm{i} v/\lambda_{\mathrm{B}}, \quad x_{\delta,\rho} := 2\pi \mathrm{i} w/\lambda_{\mathrm{B}}, \quad x_{\beta^\vee,\rho} := \log \mathcal{X}_{\beta^\vee,\rho}, \quad x_{\delta^\vee,\rho} := \log \mathcal{X}_{\delta^\vee,\rho}\,. \tag{4.71}$$

In taking the logs in the last two coordinates, we do as the following lemma:

Lemma 4.17. *The branches of the logs in $x_{\beta^\vee,\rho}$ and $x_{\delta^\vee,\rho}$ can be taken such that the following relations are satisfied on the common domains of definition:*

- *Along $\pm\mathcal{L}_k$:*

$$\begin{aligned} x_{\beta^\vee,\pm\rho_{k+1}} &= x_{\beta^\vee,\pm\rho_k} \pm \log(1 - \mathcal{X}_{\pm(\beta+k\delta)})\,, \\ x_{\delta^\vee,\pm\rho_{k+1}} &= x_{\delta^\vee,\pm\rho_k} \pm k\log(1 - \mathcal{X}_{\pm(\beta+k\delta)})\,, \end{aligned} \tag{4.72}$$

- *If ρ_{k_1}, and ρ_{k_2} are such that $\mathcal{L}_\infty \subset [\rho_{k_1}, -\rho_{k_2}]$:*

$$x_{\beta^\vee,-\rho_{k_2}} = x_{\beta^\vee,\rho_{k_1}} + \Big(\sum_{k\geq k_1} \log(1 - \mathcal{X}_{\beta+k\delta}) - \sum_{k>-k_2} \log(1 - \mathcal{X}_{-\beta+k\delta})\Big)\,, \tag{4.73}$$

and

$$\begin{aligned} x_{\delta^\vee,-\rho_{k_2}} &= x_{\delta^\vee,\rho_{k_1}} + \Big(\sum_{k\geq k_1} k\log(1 - \mathcal{X}_{\beta+k\delta}) + \sum_{k>-k_2} k\log(1 - \mathcal{X}_{-\beta+k\delta}) \\ &\quad - \sum_{k\geq 1} 2k\log(1 - \mathcal{X}_{k\delta})\Big)\,, \end{aligned} \tag{4.74}$$

- *If ρ_{k_1}, and ρ_{k_2} are such that $-\mathcal{L}_\infty \subset [\rho_{k_1}, -\rho_{k_2}]$:*

$$x_{\beta^\vee,\rho_1} = x_{\beta^\vee,-\rho_{k_2}} + \Big(-\sum_{k\geq k_2} \log(1 - \mathcal{X}_{-\beta-k\delta}) + \sum_{k>-k_1} \log(1 - \mathcal{X}_{\beta-k\delta})\Big)\,, \tag{4.75}$$

and

$$\begin{aligned} x_{\delta^\vee,\rho_{k_1}} &= x_{\delta^\vee,-\rho_{k_2}} + \Big(-\sum_{k\geq k_2} k\log(1 - \mathcal{X}_{-\beta-k\delta}) - \sum_{k>-k_1} k\log(1 - \mathcal{X}_{\beta-k\delta}) \\ &\quad + \sum_{k\geq 1} 2k\log(1 - \mathcal{X}_{-k\delta})\Big)\,. \end{aligned} \tag{4.76}$$

Proof. We do the argument for $\beta^\vee$, since for the $\delta^\vee$ is the same.

First we pick ρ_k for some k and fix a branch for $\log \mathcal{X}_{\beta^\vee,\rho_k}$. We then fix the branches of $\log \mathcal{X}_{\beta^\vee,\rho_n}$ with $n \in \mathbb{Z}$ by enforcing the jumps (4.72). On the other hand, we set $x_{\beta^\vee,-\rho_k}(\lambda_B) := -\log \mathcal{X}_{\beta^\vee,\rho_n}(-\lambda_B)$. This indeed gives a log of $\mathcal{X}_{\beta^\vee,-\rho_k}$ due to (4.19). It is then easy to check that the $x_{\beta^\vee,-\rho_n}$ for $n \in \mathbb{Z}$ must satisfy the corresponding jumps in (4.72).

With such choices, the jumps (4.73) and (4.75) must be satisfied up to summands of $2\pi i n_1$ and $2\pi i n_2$ respectively. Furthermore, it is easy to check that n_1 and n_2 must be independent of the rays ρ_{k_1} and ρ_{k_2} satisfying $\mathcal{L}_\infty \subset [\rho_{k_1}, -\rho_{k_2}]$ or $-\mathcal{L}_\infty \subset [\rho_{k_1}, -\rho_{k_2}]$, respectively. Furthermore, by the condition $x_{\beta^\vee,-\rho_n}(\lambda_B) = -\log \mathcal{X}_{\beta^\vee,\rho_n}(-\lambda_B)$, one finds that $n_1 = -n_2$. It is then easy to check that by setting $x_{\beta^\vee,\rho_n} := \log \mathcal{X}_{\beta^\vee,\rho_n} + 2\pi i n_1$ for all $n \in \mathbb{Z}$, the jumps (4.72), (4.73) and (4.75) are satisfied. □

Lemma 4.18. *For ρ_k between $\mathcal{L}_k$ and $\mathcal{L}_{k-1}$, consider the following holomorphic function on U_k^+ (resp. U_k^-):*

$$f_{\pm\rho_k} := x_{\beta,\pm\rho_k} \cdot x_{\beta^\vee,\pm\rho_k} + x_{\delta,\pm\rho_k} \cdot x_{\delta^\vee,\pm\rho_k} \,. \tag{4.77}$$

We then have the following relations:

- *On $U_{k_1}^\pm \cap U_{k_2}^\pm$ with $k_1 < k_2$:*

$$f_{\pm\rho_{k_2}} - f_{\pm\rho_{k_1}} = \sum_{k_1 \le k < k_2} \log(\mathcal{X}_{\pm(\beta+k\delta)}) \log(1 - \mathcal{X}_{\pm(\beta+k\delta)}) \,. \tag{4.78}$$

- *If $U_{k_1}^+ \cap U_{k_2}^- \neq \emptyset$, then recall that for all $(\lambda_{\mathrm{B}}, v, w) \in U_{k_1}^+ \cap U_{k_2}^-$ we either have $\mathcal{L}_\infty \subset [\rho_{k_1}, -\rho_{k_2}]$ or $-\mathcal{L}_\infty \subset [\rho_{k_1}, -\rho_{k_2}]$. In the first case, we have*

$$\begin{aligned} f_{-\rho_{k_2}} - f_{\rho_{k_1}} &= \sum_{k \ge k_1} \log(\mathcal{X}_{\beta+k\delta}) \log(1 - \mathcal{X}_{\beta+k\delta}) + \sum_{k > -k_2} \log(\mathcal{X}_{-\beta+k\delta}) \log(1 - \mathcal{X}_{-\beta+k\delta}) \\ &\quad -2 \sum_{k \ge 1} \log(\mathcal{X}_{k\delta}) \log(1 - \mathcal{X}_{k\delta}) \,, \end{aligned} \tag{4.79}$$

while in the second case

$$\begin{aligned} f_{\rho_{k_1}} - f_{-\rho_{k_2}} = \textstyle\sum_{k \ge k_2} \log(\mathcal{X}_{-\beta-k\delta}) \log(1 - \mathcal{X}_{-\beta-k\delta}) + \sum_{k > -k_1} \log(\mathcal{X}_{\beta-k\delta}) \log(1 - \mathcal{X}_{\beta-k\delta}) \\ -2 \textstyle\sum_{k \ge 1} \log(\mathcal{X}_{-k\delta}) \log(1 - \mathcal{X}_{-k\delta}) \,. \end{aligned} \tag{4.80}$$

Proof. From the Stokes jumps (4.72) we obtain that on $U_{k_1}^+ \cap U_{k_2}^+$, we have

$$\begin{aligned} f_{\rho_{k_2}} - f_{\rho_{k_1}} &= \sum_{k_1 \le k < k_2} (x_{\beta,\rho'} + k x_{\delta,\rho'}) \log(1 - \mathcal{X}_{\beta+k\delta}) \\ &= \sum_{k_1 \le k < k_2} \left(\frac{2\pi \mathrm{i}(v + kw)}{\check{\lambda}} \right) \log(1 - \mathcal{X}_{\beta+k\delta}) \\ &= \sum_{k_1 \le k < k_2} \log(\mathcal{X}_{\beta+k\delta}) \log(1 - \mathcal{X}_{\beta+k\delta}) \,. \end{aligned} \tag{4.81}$$

The other cases follow similarly by using the other jumping relations (4.72), (4.73), (4.74), (4.75), (4.76). For example, if $U_{k_1}^+ \cap U_{k_2}^- \neq \emptyset$ and $\mathcal{L}_\infty \subset [\rho_{k_1}, -\rho_{k_2}]$, we then have

$$f_{-\rho_{k_2}} - f_{\rho_{k_1}} = x_{\beta,\rho_{k_1}} \cdot \Big(\sum_{k \ge k_1} \log(1 - \mathcal{X}_{\beta+k\delta}) - \sum_{k > -k_2} \log(1 - \mathcal{X}_{-\beta+k\delta}) \Big)$$

$$+ x_{\delta,\rho_{k_1}} \cdot \Big(\sum_{k \geq k_1} k \log(1 - \mathcal{X}_{\beta+k\delta}) + \sum_{k > -k_2} k \log(1 - \mathcal{X}_{-\beta+k\delta}) - \sum_{k \geq 1} 2k \log(1 - \mathcal{X}_{k\delta}) \Big)$$
$$= \sum_{k \geq k_1} \log(\mathcal{X}_{\beta+k\delta}) \log(1 - \mathcal{X}_{\beta+k\delta}) + \sum_{k > -k_2} \log(\mathcal{X}_{-\beta+k\delta}) \log(1 - \mathcal{X}_{-\beta+k\delta})$$
$$- 2 \sum_{k \geq 1} \log(\mathcal{X}_{k\delta}) \log(1 - \mathcal{X}_{k\delta}) \,. \tag{4.82}$$

□

We therefore obtain the following:

Proposition 4.19. *In terms of the local trivializations of $\mathcal{L} \to \mathbb{C}^\times \times M_+$ given by*

$$\exp\Big(\frac{\mathrm{i}}{4\pi} f_{\pm\rho_k} \mp \frac{\mathrm{i}\pi}{24}\Big) \widehat{Z}_{\pm\rho_k} : U_k^\pm \to \mathcal{L} \,. \tag{4.83}$$

The transition functions of $\mathcal{L}$ are given as follows:

- *If $U_{k_1}^\pm \cap U_{k_2}^\pm \neq \emptyset$ for $k_1 < k_2$, we have*

$$\widetilde{g}^{\pm}_{k_1,k_2} = \prod_{k_1 \leq k < k_2} \exp\Big(\frac{\Omega(\beta + k\delta)}{2\pi \mathrm{i}} L(\mathcal{X}_{\pm(\beta+k\delta)})\Big) \,. \tag{4.84}$$

- *If $U_{k_1}^+ \cap U_{k_2}^- \neq \emptyset$, and $\mathcal{L}_\infty \subset [\rho_{k_1}, -\rho_{k_2}]$, then*

$$\widetilde{g}^{\infty}_{k_1,k_2}(\lambda_{\mathrm{B}}, v, w) = \prod_{k \geq k_1} \exp\Big(\frac{\Omega(\beta + k\delta)}{2\pi \mathrm{i}} L(\mathcal{X}_{\beta+k\delta})\Big) \prod_{k < k_2} \exp\Big(\frac{\Omega(\beta + k\delta)}{2\pi \mathrm{i}} L(\mathcal{X}_{-(\beta+k\delta)})\Big)$$
$$\cdot \prod_{k \geq 1} \exp\Big(\frac{\Omega(k\delta)}{2\pi \mathrm{i}} L(\mathcal{X}_{k\delta})\Big) \,, \tag{4.85}$$

and the analogous relation for the case $\mathcal{L}_{-\infty} \subset [\rho_{k_1}, -\rho_{k_2}]$. In particular, one can get rid of the $e^{\pm i\pi/12}$ factors appearing in (4.62) and (4.64).

Proof. For simplicity, we compute the new transition functions in the case $U_{k+1}^+ \cap U_k^+$, with all the others following the same type of argument using the rest of the identities in Lemma 4.18. We have

$$\widetilde{g}^{+}_{k,k+1} = \exp\Big(\frac{\mathrm{i}}{4\pi}(f_{\rho_{k+1}} - f_{\rho_k})\Big) \cdot g^{+}_{k,k+1}$$
$$= \exp\Big(-\frac{1}{4\pi \mathrm{i}} \log(\mathcal{X}_{\beta+k\delta}) \log(1 - \mathcal{X}_{\beta+k\delta})\Big) \cdot \exp\Big(\frac{1}{2\pi \mathrm{i}} \partial_{\lambda_{\mathrm{B}}}\Big(\lambda_{\mathrm{B}} \mathrm{Li}_2\big(\mathcal{X}_{\beta+k\delta}\big)\Big)\Big)$$
$$= \exp\Big(\frac{\Omega(\beta + k\delta)}{2\pi \mathrm{i}} L(\mathcal{X}_{\beta+k\delta})\Big) \,, \tag{4.86}$$

where on the second line we have used Lemma 4.18. □

4.4.2. Relation to hyperholomorphic line bundles. We briefly recall the setting of [Nei11, APP11b]. The starting point is again a certain type of variations of BPS structures (M, Γ, Z, Ω). The variation of BPS structures should be such that the data (M, Γ, Z) defines a (possibly indefinite) affine special Kähler (ASK) geometry on M (or conical ASK in the case of [APP11b]). More precisely, the pair (M, Γ, Z) should satisfy:

- The pairing $\langle -, - \rangle$ admits local Darboux frames $(\widetilde{\gamma}_i, \gamma^i)$[7], and the 1-forms dZ_{γ^i} give a local frame of T^*M.
- By using the identification $\Gamma \cong \Gamma^*$ given by $\gamma \to \langle \gamma, - \rangle$, we can induce a $\mathbb{C}$-bilinear pairing on $\Gamma^* \otimes \mathbb{C}$. With respect to this pairing, we should have

$$\langle \mathrm{d}Z \wedge \mathrm{d}Z \rangle = 0\,. \tag{4.87}$$

- The two-form

$$\omega := \frac{1}{4}\langle \mathrm{d}Z \wedge \mathrm{d}\overline{Z} \rangle \tag{4.88}$$

  is non-degenerate.

Under the above conditions, $\{Z_{\gamma^i}\}_{i=1}^{\dim_{\mathbb{C}}(M)}$ give local coordinates on M, and it is not hard to check that τ_{ij} defined by $dZ_{\widetilde{\gamma}_i} = \tau_{ij} dZ_{\gamma^j}$ must be symmetric by (4.87), and

$$\omega = \frac{1}{4}(\mathrm{d}Z_{\widetilde{\gamma}_i} \wedge \mathrm{d}\overline{Z}_{\gamma^i} - \mathrm{d}Z_{\gamma^i} \wedge \mathrm{d}\overline{Z}_{\widetilde{\gamma}_i}) = \frac{\mathrm{i}}{2}\mathrm{Im}(\tau_{ij})\mathrm{d}Z_{\gamma^i} \wedge \mathrm{d}\overline{Z}_{\gamma^j}\,. \tag{4.89}$$

In particular, ω gives a Kähler form of a (possibly indefinite) ASK geometry. The functions $\{Z_{\gamma^i}\}$ then define special holomorphic coordinates, while $\{Z_{\widetilde{\gamma}_i}\}$ define a conjugate system of special holomorphic coordinates.

By following the prescription in [GMN10, Nei14], one can then define an "instanton-corrected"[8] hyperkähler (HK) structure on the total space of a torus fibration $\pi : \mathcal{M} \to M$, where

$$\mathcal{M}|_p := \{\theta : \Gamma|_p \to \mathbb{R}/2\pi\mathbb{Z} \mid \theta_{\gamma+\gamma'} = \theta_\gamma + \theta_{\gamma'} + \pi\langle \gamma, \gamma' \rangle\}\,. \tag{4.90}$$

The hyperkähler geometry is encoded in terms of certain $\mathbb{C}^\times$-valued local functions $\mathcal{Y}_\gamma(\zeta, \theta)$ on the twistor space $\mathcal{Z} := \mathbb{C}P^1 \times \mathcal{M}$ of $\mathcal{M}$. The functions are labeled by local sections γ of Γ, and must satisfy the GMN integral equations [GMN10, Nei14].

Given such data, a certain holomorphic line bundle $\mathcal{L}_{\mathcal{Z}} \to \mathcal{Z}$ is constructed in [Nei11, APP11b], descending to a hyperholomorphic line bundle $\mathcal{L}_{\mathcal{M}} \to \mathcal{M}$ (that is, a hermitian bundle having a unitary connection with curvature of type (1, 1) in all the complex structures of the HK structure of $\mathcal{M}$). Our concern in the following will not be the hyperholomorphic structure of $\mathcal{L}_{\mathcal{M}}$ itself, but the topology of $\mathcal{L}_{\mathcal{Z}}$.

We would like to now describe $\mathcal{L}_{\mathcal{Z}}$ in the case of the resolved conifold. We need, however, to solve the following issue: the variation of BPS structures (M, Γ, Z, Ω) we

[7] One can relax this condition by allowing "flavor charges" (i.e. charges γ such that $\langle \gamma, - \rangle = 0$). For a description of this more general case see for example [Nei14] or [AST21, Section 2.2].

[8] What this means is that the data of the BPS indices is used to obtain a new HK geometry from the semi-flat HK geometry associated to the ASK geometry via the rigid c-map. In the context of 4d $\mathcal{N} = 2$ theories compactified on S^1, the modifications to the semi-flat HK geometry correspond to instanton corrections of the HK geometry associated to the corresponding low-energy effective theory.

have discussed in Sect. 4.1 does not satisfy the ASK geometry condition, since for that case $Z_{\beta^\vee} = Z_{\delta^\vee} = 0$, and hence

$$\langle \mathrm{d}Z \wedge \mathrm{d}\overline{Z}\rangle = \mathrm{d}Z_{\beta^\vee} \wedge \mathrm{d}\overline{Z}_\beta + \mathrm{d}Z_{\delta^\vee} \wedge \mathrm{d}\overline{Z}_\delta - \mathrm{d}Z_\beta \wedge \mathrm{d}\overline{Z}_{\beta^\vee} - \mathrm{d}Z_\delta \wedge \mathrm{d}\overline{Z}_{\delta^\vee} = 0\,. \tag{4.91}$$

By possibly restricting to an open set $M' \subset M$, and taking $\Gamma' := \Gamma|_{M'}$, we can assume we have a central charge $Z' \colon M' \to \Gamma' \otimes \mathbb{C}$ satisfying the ASK geometry property and such that $Z'|_{\mathbb{Z}\beta\oplus\mathbb{Z}\delta} = Z|_{\mathbb{Z}\beta\oplus\mathbb{Z}\delta}$. Setting $(\gamma^1, \gamma^2) = (\beta, \delta)$, such a central charge can be found by picking a holomorphic function $\mathfrak{F}(Z_{\gamma^i}) \colon M' \to \mathbb{C}$ such that the matrix $\mathrm{Im}(\partial^2\mathfrak{F}/\partial Z_{\gamma^i}\partial Z_{\gamma^j})$ is non-degenerate, and taking $Z'_{\beta^\vee} := \partial_{Z_\beta}\mathfrak{F}$, $Z'_{\delta^\vee} := \partial_{Z_\delta}\mathfrak{F}$ (for example, we can just pick $\mathfrak{F}(Z_\beta, Z_\delta) = \mathrm{i}(Z_\beta^2 + Z_\delta^2)/2$). For simplicity, we will assume in the following that have chosen Z' satisfying the ASK condition and such that $M' = M$. We can therefore consider the HK manifold $\mathcal{M}$ and line bundle $\mathcal{L}_{\mathcal{Z}} \to \mathcal{Z}$ associated to (M, Γ, Z', Ω).

To describe $\mathcal{L}_{\mathcal{Z}} \to \mathcal{Z}$ in this case, we do as follows: recall that a quadratic refinement for $(\Gamma|_p, \langle -, -\rangle)$ is a function $\sigma : \Gamma|_p \to \mathbb{Z}_2$ such that

$$\sigma(\gamma)\sigma(\gamma') = (-1)^{\langle\gamma,\gamma'\rangle}\sigma(\gamma + \gamma')\,. \tag{4.92}$$

In our particular case, we can make a global choice of quadratic refinement $\sigma : \Gamma \to \mathbb{Z}_2$ determined by $\sigma(\beta) = \sigma(\delta) = \sigma(\beta^\vee) = \sigma(\delta^\vee) = 1$. With such a choice, we can identify

$$\mathcal{M} \cong \{\theta \colon \Gamma \to \mathbb{R}/2\pi\mathbb{Z} \mid \theta_{\gamma+\gamma'} = \theta_\gamma + \theta_{\gamma'}\} \cong M \times (S^1)^4\,, \tag{4.93}$$

via $e^{\mathrm{i}\theta_\gamma} \to \sigma(\gamma)e^{\mathrm{i}\theta_\gamma}$.

We consider the bundle $\widetilde{\pi} \colon \widetilde{\mathcal{M}} \to M$, whose fibers are the universal cover of the fibers of $\pi : \mathcal{M} \to M$. Namely,

$$\widetilde{\mathcal{M}} := \{\theta : \Gamma \to \mathbb{R} \mid \theta_{\gamma+\gamma'} = \theta_\gamma + \theta_{\gamma'}\} \cong M \times \mathbb{R}^4\,. \tag{4.94}$$

The main reason for going to the universal cover is to avoid certain issues regarding the domains of definitions of the transition functions involving the Rogers dilogarithm expressions, as we will see below.

Since our end-goal is to compare with $\mathcal{L} \to \mathbb{C}^\times \times M_+$ from Sect. 4.2, we will restrict to $\widetilde{\mathcal{M}}_+ := \widetilde{\pi}^{-1}(M_+)$. However, a similar argument follows for the line bundle over $\mathbb{C}^\times \times M_-$ and $\mathbb{C}^\times \times M_0$ (recall Remark 4.16).

It is easy to see that the HK structure on $\mathcal{M}_+$ lifts to $\widetilde{\mathcal{M}}_+$, and we will denote the corresponding twistor space by $\widetilde{\mathcal{Z}}_+ = \mathbb{C}P^1 \times \widetilde{\mathcal{M}}_+$. We consider the rays $\mathcal{L}_k = \mathbb{R}_{<0} \cdot 2\pi\mathrm{i}(v + kw)$, and pick ρ_k between $\mathcal{L}_k$ and $\mathcal{L}_{k-1}$. We furthermore consider the cover $\{V_k^\pm\}_{k\in\mathbb{Z}}$ of $\mathbb{C}^\times \times \widetilde{\mathcal{M}}_+$ given by

$$V_k^\pm := \{(\zeta, \theta) \in \mathbb{C}^\times \times \widetilde{\mathcal{M}}_+ \mid \zeta \in \mathbb{H}_{\pm\rho_k}\}\,. \tag{4.95}$$

Notice that the condition on ζ depends on $\widetilde{\pi}(\theta) = (u, v)$, since the latter determines the rays $\mathcal{L}_k$, and hence ρ_k. Furthermore, notice that $V_k^\pm = \widetilde{\pi}^{-1}(U_k^\pm)$, where $U_k^\pm$ was defined in (4.57).

We define a line bundle $\mathcal{L}_{\widetilde{\mathcal{Z}}_+} \to \mathbb{C}^\times \times \widetilde{\mathcal{M}}_+$ via the following cocycle associated to the cover $\{V_k^\pm\}_{k\in\mathbb{Z}}$ (compare with [Nei11, Equation 4.8] or [APP11b, Equation 3.29]):

- If $V^{\pm}_{k_1} \cap V^{\pm}_{k_2} \neq \emptyset$ for $k_1 < k_2$, then

$$h^{\pm}_{k_1,k_2}(\zeta,\theta) := \prod_{k_1 \leq k < k_2} \exp\left(\frac{\Omega(\beta+k\delta)}{2\pi \mathrm{i}} L(\mathcal{Y}_{\pm(\beta+k\delta)}(\zeta,\theta))\right), \tag{4.96}$$

where[9]

$$\mathcal{Y}_{\pm(\beta+k\delta)}(\zeta,\theta) = \exp\left(\zeta^{-1} Z_{\pm(\beta+k\delta)}(\widetilde{\pi}(\theta)) + \mathrm{i}\theta_{\pm(\beta+k\delta)} + \zeta \overline{Z}_{\pm(\beta+k\delta)}(\widetilde{\pi}(\theta))\right). \tag{4.97}$$

Notice that

$$L(\mathcal{Y}_{\pm(\beta+k\delta)}) = \mathrm{Li}_2(\mathcal{Y}_{\pm(\beta+k\delta)}) + \frac{1}{2}\log(\mathcal{Y}_{\pm(\beta+k\delta)})\log(1-\mathcal{Y}_{\pm(\beta+k\delta)}) \tag{4.98}$$

with

$$\log(\mathcal{Y}_{\pm(\beta+k\delta)}) = \zeta^{-1} Z_{\pm(\beta+k\delta)}(\widetilde{\pi}(\theta)) + \theta_{\pm(\beta+k\delta)} + \zeta \overline{Z}_{\pm(\beta+k\delta)}(\widetilde{\pi}(\theta)) \tag{4.99}$$

is well defined for $\zeta \in \mathbb{H}_{\mathbb{R}_{<0}\cdot Z_{\pm(\beta+k\delta)}}$, since for such ζ we have $|\mathcal{Y}_{\pm(\beta+k\delta)}| < 1$, and hence $\mathrm{Li}_2(\mathcal{Y}_{\pm(\beta+k\delta)})$ and $\log(1-\mathcal{Y}_{\pm(\beta+k\delta)})$ make sense with their principal branches. We also set $h^{\pm}_{k_2,k_1} := (h^{\pm}_{k_1,k_2})^{-1}$.

- If $V^{+}_{k_1} \cap V^{-}_{k_2} \neq \emptyset$, and $\mathcal{L}_\infty \subset [\rho_{k_1}, -\rho_{k_2}]$, then

$$\begin{aligned} h^{\infty}_{k_1,k_2}(\zeta,\theta) \;&:= \prod_{k \geq k_1} \exp\left(\frac{\Omega(\beta+k\delta)}{2\pi \mathrm{i}} L(\mathcal{Y}_{\beta+k\delta})\right) \prod_{k<k_2} \exp\left(\frac{\Omega(\beta+k\delta)}{2\pi \mathrm{i}} L(\mathcal{Y}_{-(\beta+k\delta)})\right) \\ &\cdot \prod_{k\geq 1} \exp\left(\frac{\Omega(k\delta)}{2\pi \mathrm{i}} L(\mathcal{Y}_{k\delta})\right), \end{aligned} \tag{4.100}$$

while for the case $\mathcal{L}_{-\infty} \subset [\rho_{k_1}, -\rho_{k_2}]$

$$\begin{aligned} h^{-\infty}_{k_2,k_1}(\zeta,\theta) \;&:= \prod_{k \geq k_2} \exp\left(\frac{\Omega(\beta+k\delta)}{2\pi \mathrm{i}} L(\mathcal{Y}_{-(\beta+k\delta)})\right) \prod_{k<k_1} \exp\left(\frac{\Omega(\beta+k\delta)}{2\pi \mathrm{i}} L(\mathcal{Y}_{\beta+k\delta})\right) \\ &\cdot \prod_{k\geq 1} \exp\left(\frac{\Omega(k\delta)}{2\pi \mathrm{i}} L(\mathcal{Y}_{-k\delta})\right). \end{aligned} \tag{4.101}$$

We also set as before $h^{\infty}_{k_2,k_1} := (h^{\infty}_{k_1,k_2})^{-1}$ and $h^{-\infty}_{k_1,k_2} := (h^{-\infty}_{k_2,k_1})^{-1}$.

In [Nei11,APP11b], it is argued that such a bundle extends to a holomorphic bundle $\mathcal{L}_{\widetilde{\mathcal{Z}}_+} \to \widetilde{\mathcal{Z}}_+$, and that it descends to a hyperholomorphic line bundle $\mathcal{L}_{\widetilde{\mathcal{M}}_+} \to \widetilde{\mathcal{M}}_+$. The corresponding line bundle $\mathcal{L}_{\mathcal{M}_+} \to \mathcal{M}_+$ can then be obtained by a quotient by a certain action of $\Gamma^* \to M_+$, acting fiberwise on both $\mathcal{L}_{\widetilde{\mathcal{M}}_+} \to M_+$ and $\widetilde{\mathcal{M}}_+ \to M_+$, and equivariantly with respect to $\mathcal{L}_{\widetilde{\mathcal{M}}_+} \to \widetilde{\mathcal{M}}_+$ (see for example [Nei11, Equation 3.7]). The pullback of $\mathcal{L}_{\mathcal{M}_+}$ to the twistor space then gives $\mathcal{L}_{\mathcal{Z}} \to \mathbb{C}P^1 \times \mathcal{M}_+$.

We now wish to relate the bundle $\mathcal{L}_{\mathcal{Z}}|_{\mathbb{C}^\times \times M_+}$ with the previous bundle $\mathcal{L} \to \mathbb{C}^\times \times M_+$ defined by the normalized partition functions $\widehat{Z}_\rho$ in Sect. 4.2. We will focus on

[9] Formula (4.97) gives the so–called semi-flat coordinate labeled by $\pm(\beta+k\delta)$. In the case of the resolved conifold, only the coordinates of the form $\mathcal{Y}_{n\beta+m\delta+p\beta^\vee+q\delta^\vee}$ with $p \neq 0$ or $q \neq 0$ get "instanton corrected" away from the semi-flat form.

the complex Lagrangian submanifold $L \subset \mathcal{M}$ (with respect to one of the complex symplectic structures of the HK structure) given by

$$L := \{\theta \in \mathcal{M} \mid \theta_\beta = \theta_\delta = \theta_{\beta^\vee} = \theta_{\delta^\vee} = 0\}\,. \tag{4.102}$$

The fact that this defines a complex Lagrangian submanifold L of $\mathcal{M}$, can be seen for example from formula [CT21, equation 3.10] of the instanton corrected holomorphic symplectic form (see also [Gai14]). Since L can be identified with M as complex manifolds, we will do so in the following.

The line bundle $\mathcal{L}_{\mathcal{Z}}|_{\mathbb{C}^\times \times M_+}$ can be described by the transition functions $h^{\pm}_{k_1,k_2}|_{\mathbb{C}^\times \times M_+}$ and $h^{\pm\infty}_{k_1,k_2}|_{\mathbb{C}^\times \times M_+}$ associated to the cover $\{U^{\pm}_k\}_{k\in\mathbb{Z}}$ of $\mathbb{C}^\times \times M_+$ given in (4.57). It is now easy to see how to obtain $\mathcal{L} \to \mathbb{C}^\times \times M_+$ from $\mathcal{L}_Z|_{\mathbb{C}^\times \times M_+} \to \mathbb{C}^\times \times M_+$. Namely, one considers the following conformal limit, studied in [Gai14]:

- First, one introduces a scaling parameter $Z \to RZ$ for $R > 0$.
- One then considers the limit of the transition functions as $R \to 0$, while keeping the quotient $\lambda_{\rm B} = \zeta/R$ fixed.

After taking the conformal limit, we see that

$$\mathcal{Y}_{n\beta+m\delta}|_{\mathbb{C}^\times \times M_+} \to \mathcal{X}_{n\beta+m\delta}\,, \tag{4.103}$$

and hence

$$h^{\pm}_{k_1,k_2}|_{\mathbb{C}^\times \times M_+} \to \widetilde{g}^{\pm}_{k_1,k_2}, \qquad h^{\pm\infty}_{k_1,k_2}|_{\mathbb{C}^\times \times M_+} \to \widetilde{g}^{\pm\infty}_{k_1,k_2}\,, \tag{4.104}$$

where $\widetilde{g}^{\pm}_{k_1,k_2}$ and $\widetilde{g}^{\pm\infty}_{k_1,k_2}$ correspond to the 1-cocycle associated to the cover $\{U^{\pm}_k\}_{k\in\mathbb{Z}}$ describing $\mathcal{L} \to \mathbb{C}^\times \times M_+$ from Proposition 4.19.

From the previous discussion, we obtain the following:

Proposition 4.20. *Consider the* 1*-cocycles associated to the cover* $\{U^{\pm}_k\}$ *of* $\mathbb{C}^\times \times M_+$ *and given by* $\{\widetilde{g}^{\pm}_{k_1,k_2}, \widetilde{g}^{\pm\infty}_{k_1,k_2}\}$ *and* $\{h^{\pm}_{k_1,k_2}|_{\mathbb{C}^\times \times M_+}, h^{\pm\infty}_{k_1,k_2}|_{\mathbb{C}^\times \times M_+}\}$*, respectively. Then the* 1*-cocycles are related by the conformal limit from above.*

5. The Strong-Coupling Expansion and Its Stokes Phenomena

In this section, we will demonstrate that the topological string partition function has a Borel summable strong-coupling expansion for $\lambda \to \infty$. The Stokes jumps of the strong-coupling expansion are found to reproduce the wall-crossing behaviour of counting functions for the *framed* BPS states representing composites of D0 and D2 branes bound to a heavy D6 brane in string theory on the resolved conifold. This wall-crossing has previously been studied by Jafferis and Moore [JM08]. This work in particular gave a physical derivation of the results of Szendröi on the generating function of non-commutative DT invariants [Sze08], see also [NN] for related work in mathematics and [DG10, AOVY11] in physics.

Because the techniques required in this section are the same as in the previous sections, we will give less details of the intermediate computations.

5.1. Borel summation of the strong-coupling expansion. In order to derive the strong-coupling expansion we shall start with the Woronowicz form of $F_{\rm np}(\lambda, t)$ given in (3.35), now rewritten as

$$F_{\rm np}(\lambda, t) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}+{\rm i}0^+} {\rm d}v\, \frac{v+\alpha}{1-e^{v+\alpha}} \log\left(1-e^{\check{\lambda}v}\right). \tag{5.1}$$

using the notations $\alpha = -2\pi{\rm i}t'$ and $t' = t/\check{\lambda}$. As before, we may rewrite this in terms of a Laplace transformation,

$$\begin{aligned} F_{\rm np}(\lambda, t) &= \int_0^\infty \frac{{\rm d}v}{(2\pi)^2}\left(\frac{(v+\alpha)\log\left(1-e^{\check{\lambda}v-{\rm i}0^+}\right)}{1-e^{v+\alpha}} - \frac{(v-\alpha)\log\left(1-e^{-\check{\lambda}v-{\rm i}0^+}\right)}{1-e^{\alpha-v}}\right) \\ &= \frac{1}{(2\pi)^2}\int_0^\infty {\rm d}v\left[\frac{(v+\alpha)(\check{\lambda}v+\pi{\rm i})}{1-e^{v+\alpha}} + \left(\frac{v+\alpha}{1-e^{v+\alpha}} - \frac{v-\alpha}{1-e^{\alpha-v}}\right)\log\left(1-e^{-\check{\lambda}v-{\rm i}0^+}\right)\right] \\ &= -\frac{\lambda}{(2\pi)^3}(2{\rm Li}_3(Q') + \alpha\,{\rm Li}_2(Q')) - \frac{\rm i}{4\pi}({\rm Li}_2(Q') + \alpha\,{\rm Li}_1(Q')) \\ &\quad + \int_0^\infty {\rm d}v\, e^{-\check{\lambda}v} G_s(v, t')\,, \end{aligned} \tag{5.2}$$

using the notations $Q' = e^{2\pi{\rm i}t'}$ and

$$G_{\rm s}(v, t') = \frac{1}{(2\pi)^2}\sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{1}{n^3}\frac{v+2\pi{\rm i}\,nt'}{1-e^{-v/n-2\pi{\rm i}t'}}\,.$$

Having represented the function $F_{\rm np}(\lambda, t)$ as a Laplace transform makes it straightforward to derive an asymptotic series in inverse powers of λ for which (5.2) represents a Borel transform.

$G_{\rm s}(v, t')$ has poles at $v = v_{kn}^{\pm} := \mp 2\pi{\rm i}n(t'+k)$, $k \in \mathbb{Z}\setminus\{0\}$, $n \in \mathbb{Z}_{>0}$. In the case ${\rm Im}(t') > 0$, the poles v_{kn}^+ and v_{kn}^- are in the right and left half-planes, respectively. Assuming ${\rm Re}(t') < 1$, one finds that the strings of poles $\{v_{kn}^+ \mid n \in \mathbb{Z}_{>0}\}$ with $k \in \mathbb{Z}_{<0}$ are located in the upper half-plane.

We may decompose the complex plane representing values of the integration variable v into a union of rays $\pm l'_k := \pm\mathbb{R}_{<0}\cdot 2\pi{\rm i}(t'+k)$ and wedges $[\pm l'_k, \pm l'_{k-1}]$ bounded by $\pm l'_k$ and $\pm l'_{k-1}$. Letting $\lambda' := 1/\check{\lambda}$, for ρ_k in the wedge $[l'_k, l'_{k-1}]$ and $\lambda' \in \mathbb{H}_{\pm\rho_k}$, we define

$$\begin{aligned} F'_{\pm\rho_k}(\lambda', t') :=& -\frac{1}{(2\pi)^2\lambda'}\left(2{\rm Li}_3(Q') - 2\pi{\rm i}t'{\rm Li}_2(Q')\right) - \frac{\rm i}{4\pi}\left({\rm Li}_2(Q') - 2\pi{\rm i}t'{\rm Li}_1(Q')\right) \\ &+ \int_{\pm\rho_k} {\rm d}v\, e^{-\frac{v}{\lambda'}} G_{\rm s}(v, t')\,. \end{aligned} \tag{5.3}$$

The wedges $[\pm l'_k, \pm l'_{k-1}]$ are natural domains of definition (in the λ' variable) for the functions $F'_{\pm\rho_k}(\lambda', t')$, differing by Stokes jumps from the strings of poles $\{v_{kn}^{\pm} \mid n \in \mathbb{Z}_{>0}\}$ of $G_{\rm s}(v, t)$.

5.2. Stokes jumps. To compute the Stokes jumps, we follow the strategy from Section 3.3. The relevant residues are

$$\operatorname*{Res}_{v=v_{kn}^{\pm}} e^{-v/\lambda'} G_{\rm s}(v,t) = \frac{e^{-v_{kn}^{\pm}/\lambda'}}{(2\pi)^2 n^3}(\mp 2\pi\mathrm{i}\, nk)n = \pm\frac{1}{2\pi\mathrm{i}}\frac{k}{n}\, e^{\pm\mathrm{i}\lambda n(t'+k)}. \tag{5.4}$$

It follows that the Stokes jumps across $\pm l_k'$ are explicitly given as

$$\begin{aligned} F'_{\pm\rho_{k+1}} - F'_{\pm\rho_k} &= 2\pi\mathrm{i}\sum_{n=1}^{\infty} \pm e^{\pm\lambda\mathrm{i}n(t'+k)}\frac{k}{2\pi\mathrm{i}n} = \mp k\log\left(1-e^{\pm\lambda\mathrm{i}(t'+k)}\right) \qquad Q := e^{2\pi\mathrm{i}t},\\ &= \mp k\log\left(1-e^{\pm 2\pi\mathrm{i}(t+\check{\lambda}k)}\right) = \mp k\log(1-Q^{\pm 1}q^{\pm k}), \qquad q := e^{\mathrm{i}\lambda}. \end{aligned} \tag{5.5}$$

Note that there is no jump for $k=0$.

For the rest of the section, we will assume that $0 < \mathrm{Re}(t') < 1$ and $\mathrm{Im}(t') > 0$. Taking $\mathrm{Im}(\lambda') > 0$ ($\iff$ $\mathrm{Im}(\lambda) < 0$), we can sum the jumps in the upper half-plane, and obtain

$$\begin{aligned} \lim_{k\to-\infty} F'_{\rho_k} - F'_{\rho_0} &= \sum_{k=-1}^{-\infty}(F'_{\rho_k} - F'_{\rho_{k+1}}) = -\sum_{k=1}^{\infty} k\log(1-Qq^{-k})\\ &= \sum_{l=1}^{\infty}\frac{q^{-l}}{l}\frac{Q^l}{(1-q^{-l})^2} = -\sum_{k=1}^{\infty}\frac{1}{k}\frac{e^{2\pi\mathrm{i}kt}}{\left(2\sin\left(\frac{k\lambda}{2}\right)\right)^2}. \end{aligned} \tag{5.6}$$

On the other hand, if $\mathrm{Im}(\lambda') < 0$ ($\iff$ $\mathrm{Im}(\lambda) > 0$), we can sum the jumps in the lower half-plane of the variable, which leads to

$$\begin{aligned} \lim_{k\to\infty} F'_{\rho_k} - F'_{\rho_0} &= \sum_{k=0}^{\infty}(F'_{\rho_{k+1}} - F'_{\rho_k}) = -\sum_{k=1}^{\infty} k\log(1-Qq^{k})\\ &= \sum_{l=1}^{\infty}\frac{q^{l}}{l}\frac{Q^l}{(1-q^{l})^2} = -\sum_{k=1}^{\infty}\frac{1}{k}\frac{e^{2\pi\mathrm{i}kt}}{\left(2\sin\left(\frac{k\lambda}{2}\right)\right)^2}. \end{aligned} \tag{5.7}$$

Note that the domains of definition of $\lim_{k\to\infty} F'_{\rho_k} - F'_{\rho_0}$ and $\lim_{k\to-\infty} F'_{\rho_k} - F'_{\rho_0}$ have empty intersection.

It will be instructive to consider the normalised partition functions

$$\mathcal{Z}_{\pm\rho_k}(\lambda',t') := \frac{Z'_{\rho_0}(\lambda',t')}{Z'_{\pm\rho_k}(\lambda',t')}\left(\frac{Z'_{\rho_0}(\lambda',0)}{Z'_{\pm\rho_k}(\lambda',0)}\right)^{-1}, \qquad Z'_{\pm\rho_k}(\lambda',t') = e^{F'_{\pm\rho_k}(\lambda',t')}. \tag{5.8}$$

The jumping behaviour of the normalised partition functions can be summarised as follows. Equation (5.5) immediately implies that across l_k', we have

$$\mathcal{Z}_{\rho_{k+1}}(\lambda',t') = (1-Qq^k)^k\mathcal{Z}_{\rho_k}(\lambda',t').$$

It follows that for $k \geq 0$

$$\mathcal{Z}_{\rho_{k+1}}(\lambda',t') = \prod_{j=1}^{k}(1-Qq^j)^j,$$

where we have used that $Z'_{\rho_1} = Z'_{\rho_0}$.

Considering the functions $\mathcal{Z}_{-\rho_k}(\lambda', t')$, one needs to take into account the fact that the jumps of the normalising factor accumulate at the imaginary axis. It is then straightforward to compute

$$\lim_{k\to\infty} \mathcal{Z}_{\rho_k}(\lambda', t') = \prod_{k=1}^{\infty}(1 - q^k Q)^k, \tag{5.9}$$

$$\lim_{k\to-\infty} \mathcal{Z}_{-\rho_k}(\lambda', t') = (M(q))^2 \prod_{k=1}^{\infty}(1 - q^k Q)^k, \tag{5.10}$$

$$\mathcal{Z}_{-\rho_0}(\lambda', t') = (M(q))^2 \prod_{k=1}^{\infty}(1 - q^k Q)^k (1 - q^k Q^{-1})^k, \tag{5.11}$$

with $M(q) = \prod_{k=1}^{\infty}(1 - q^k)^{-k}$ being the MacMahon function. We note furthermore that $\mathcal{Z}_{-\rho_0}(\lambda', t')$ is the expression obtained in [Sze08] as a generating function of non-commutative DT invariants.

5.3. Relation to framed BPS states. Our findings can be compared with the known results on counting of *framed* BPS-states, representing bound states of D0- and D2-branes with a single infinitely heavy D6 in string theory on local CY manifolds. A useful characteristic of the spectrum of BPS states are the BPS indices (generalised DT invariants) $[\mathrm{DT}]^{\mathcal{C}}_{n\delta+k\beta+\delta^\vee}$ which are locally constant with respect to the Kähler parameters, but may jump along walls of marginal stability in the Kähler moduli space $\mathcal{M}_{\text{Käh}}$ and therefore depend on the choice of a chamber $\mathcal{C} \subset \mathcal{M}_{\text{Käh}}$. The BPS partition functions are generating functions for the BPS indices for the case of the conifold defined as

$$\mathcal{Z}_{\text{BPS}}(u, v; \mathcal{C}) = \sum_{k=0}^{\infty}\sum_{n=1}^{\infty} [\mathrm{DT}]^{\mathcal{C}}_{n\delta+k\beta+\delta^\vee}\, u^n v^k. \tag{5.12}$$

The pattern of chambers can be described as follows [JM08]. The processes associated to walls of marginal stability represent decay or recombination of framed BPS–states with charges $\gamma_1 = k'\delta + m'\beta + \delta^\vee$ and unframed BPS-state with charges $\gamma_2 = k\delta + m\beta$. By regarding the resolved conifold as a limit $\Lambda \to \infty$ of a family of compact CY manifolds having a complexified Kähler parameter $\Lambda e^{i\varphi}$, one may introduce a regularised central charge function $Z(\gamma_1)$, to leading order in Λ given by $(\Lambda e^{i\varphi})^3$. Unframed BPS-states with charges $\gamma_2 = k\delta + m\beta$ have central charge function $Z(\gamma_2) = mz - k$, where z is the complexified Kähler parameter associated to the compact two-cycle of the resolved conifold. The phases of $Z(\gamma_1)$ and $Z(\gamma_2) = mz - k$ align if

$$3\varphi = \arg(mz - k) + 2\pi n, \qquad n \in \mathbb{Z}.$$

Taking into account that there only exist BPS-states with $m = \pm 1$, one arrives at the pattern of walls $\mathcal{W}^m_k$ described in [JM08], decomposing the parameter space into a collection of chambers $\mathcal{C}^-_k = [\mathcal{W}^{-1}_{k-1}\mathcal{W}^{-1}_k]$ and $\mathcal{C}^+_k = [\mathcal{W}^1_k \mathcal{W}^1_{k-1}]$, respectively.

Of special interest are the core region $\mathcal{C}_0^+ \cup \mathcal{C}_1^+$, the limits $\mathcal{C}_\infty^\pm$, and the chamber $\mathcal{C}_0^-$ called non-commutative chamber following [JM08]. The partition functions are

$$\mathcal{Z}_{\rm BPS}(u,v;\mathcal{C}_\infty^+) = \prod_{k=1}^\infty (1-(-u)^k v)^k, \qquad \mathcal{Z}_{\rm BPS}(u,v;\mathcal{C}_{\rm core}) = 1, \tag{5.13}$$

$$\mathcal{Z}_{\rm DT}(u,v) := \mathcal{Z}_{\rm BPS}(u,v;\mathcal{C}_\infty^-) = (M(-u))^2 \prod_{k=1}^\infty (1-(-u)^k v)^k, \tag{5.14}$$

$$\mathcal{Z}_{\rm BPS}(u,v;\mathcal{C}_0^-) = (M(-u))^2 \prod_{k=1}^\infty (1-(-u)^k v)^k (1-(-u)^k v^{-1})^k. \tag{5.15}$$

One may identify the exponents in (5.15) with the unframed BPS indices defining the BPS Riemann–Hilbert problem for the conifold.

The GW-DT correspondence [MNOP06a, MNOP06b, MOOP11] relates the BPS partition function to the topological string partition function through the following relation[10]

$$\mathcal{Z}_{\rm DT}(-q,Q) = (M(q))^{\chi(X)} e^{F_{\rm GV}(\lambda,t)}, \qquad q = e^{{\rm i}\lambda}, \quad Q = e^{2\pi {\rm i} t}. \tag{5.16}$$

Taking into account the relation between the variables u, v and q, Q following from (5.16), and identifying $\arg(\lambda') = 3\varphi$, $z = t'$, we find a one-to-one correspondence between the chambers $\mathcal{C}_k^\pm$ and the wedges $[\pm l_k', \pm l_{k-1}']$ representing natural domains of definition for the Borel summations $F_{\pm\rho_k}'(\lambda', t')$ of the strong-coupling expansion, together with a precise match between the BPS partition functions $\mathcal{Z}_{\rm BPS}(u, v; \mathcal{C}_k^\pm)$ and the normalised partition functions $\mathcal{Z}_{\pm\rho_k}(\lambda', t')$ defined in (5.8), chamber by chamber.

6. S-duality

It seems interesting to observe that the wall-crossing behaviour of the generating functions $\mathcal{Z}_{\rm BPS}(u, v; \mathcal{C})$ for BPS indices involves jumps related to the jumps in Bridgeland's RH problem by the replacements

$$\lambda \mapsto \lambda_{\rm D} = -\frac{4\pi^2}{\lambda}, \qquad t \mapsto t_{\rm D} = \frac{2\pi}{\lambda} t. \tag{6.1}$$

This suggests that we can use the framed wall-crossing phenomena studied in [JM08] causing the jumps of the BPS partition functions $\mathcal{Z}_{\rm BPS}(u, v; \mathcal{C})$ to define a "dual" version of the RH problem studied by Bridgeland in [Bri20]. The location of walls and the explicit formulae for the jumps of the dual RH problem are obtained by replacing λ and t by $\lambda_{\rm D}$ and $t_{\rm D}$, respectively.

The dependence on the variable λ suggests that Bridgleland's RH problem describes wall-crossing phenomena in non-perturbative effects due to disk instantons in string theory, while the dual RH problem describes the wall-crossing of BPS states in supergravity. As an outlook we will now briefly indicate how weak and strong-coupling expansions can be combined to get a more global geometric picture of the space $\mathcal{M}_{\text{Käh}} \times \mathbb{C}^\times$ with coordinates (t, λ), outline connections to the S-duality conjectures in string theory, and point out a relation to the mathematical phenomenon called Langlands modular duality in the context of quantum cluster algebras [FG09].

10 Comparing with [MNOP06a, MNOP06b], one should note that the variable q used in these papers corresponds to the quantity $-q$ in our notations.

6.1. Global aspects. In the space $\mathcal{M}_{\text{Käh}} \times \mathbb{C}^\times$ with coordinates (t, λ), one may naturally consider two asymptotic regions, referred to as weak and strong-coupling regions, respectively. The weak coupling region is defined by sending $\lambda \to 0$ keeping t fixed, while the strong coupling region can be described by sending $\lambda \to \infty$ with constant t_D. The asymptotic expansions of the non-perturbative free energy $F_\text{np}(\lambda, t)$ in powers of λ and λ^{-1} are valid in the weak and strong-coupling regions, respectively.

In order to get a more global picture, it seems natural to include the rays and jumps of the strong coupling expansion into the definition of a refined version of the line bundle discussed in the previous section 4. More precisely:

- On the complex 2-dimensional parameter space $\mathcal{M}_{\text{Käh}} \times \mathbb{C}^\times$ parametrized by (t, λ) or $(t', \lambda') = (t_D, -\lambda_D/2\pi)$, one can consider the real 3-dimensional walls

$$\begin{aligned} \mathcal{W}^{\pm}_{\text{weak},k} &:= \{(t,\lambda) \mid \lambda \in \pm\mathbb{R}_{<0} 2\pi i (t+k)\}, \quad k \in \mathbb{Z}, \\ \mathcal{W}^{\pm}_{\text{strong},k} &:= \left\{(t',\lambda') \mid \lambda' \in \pm\mathbb{R}_{<0} 2\pi i (t'+k)\right\}, \quad k \in \mathbb{Z} - \{0\} \\ &= \left\{(t,\lambda) \mid \lambda = \tfrac{-2\pi t \pm i r}{k}, \quad r \in \mathbb{R}_{>0}\right\}, \quad k \in \mathbb{Z} - \{0\}\,. \end{aligned} \tag{6.2}$$

  and the chambers defined by the connected components of the complement of $\mathcal{W} := \cup_{k\in\mathbb{Z}} \mathcal{W}^{\pm}_{\text{weak},k} \cup_{k\in\mathbb{Z}-\{0\}} \mathcal{W}^{\pm}_{\text{strong},k}$ (notice that since there is no jump associated to $\pm l'_0$ it is safe to exclude the case $k = 0$ for the strong coupling walls). Intersecting $\mathcal{W}^{\pm}_{\text{weak},k}$ with a t-slice $\{t\} \times \mathbb{C}^\times$, one obtains $\pm l_k$ in the corresponding λ-plane; while intersecting $\mathcal{W}^{\pm}_{\text{strong},k}$ with a t'-slice $\{t' = t/\check{\lambda} = \text{const}\}$, one obtains $\pm l'_k$ in the corresponding λ'-plane. Furthermore, the intersection of $\mathcal{W}^{\pm}_{\text{strong},k}$ with $\{t\} \times \mathbb{C}^\times$ gives a ray starting at $-2\pi t/k$ and parallel to the imaginary axis in the corresponding λ-plane. The rays corresponding to the intersection of $\mathcal{W}^{+}_{\text{strong},k}$ and $\mathcal{W}^{-}_{\text{strong},k}$ with $\{t\} \times \mathbb{C}^\times$ combine into a line parallel to the imaginary axis, and missing the point $-2\pi t/k$. In particular, these lines accumulate near the imaginary axis of the λ-plane $\{t\} \times \mathbb{C}^\times$. Assuming $\text{Re}(t) > 0$, the ones to the right of the imaginary axis correspond to $k < 0$, while the ones on the left correspond to $k > 0$.
- Taking into account the chamber structure on $(\mathcal{M}_{\text{Käh}} \times \mathbb{C}^\times) - \mathcal{W}$, the corresponding refined line bundle would then have transition functions along the walls determined by the jumps obtained at strong and weak coupling. In particular, there is in $(\mathcal{M}_{\text{Käh}} \times \mathbb{C}^\times) - \mathcal{W}$ a distinguished chamber $\mathcal{D}$ determined by the constraint $0 < \text{Re}(t) < 1$, $\text{Im}(t) > 0$, and the condition that $\mathcal{D} \cap (\{t\} \times \mathbb{C}^\times)$ gives the region of the Stokes sector $[l_0, l_{-1}]$ to the right of the line $\mathcal{W}^{\pm}_{\text{strong},-1} \cap (\{t\} \times \mathbb{C}^\times)$. On this region $F_\text{np}(\lambda, t)$ is defined and matches $F_{\mathbb{R}_{>0}}(\lambda, t)$. We can then use the jumps along the walls to extend F_np to the other regions. In particular, if we fix t and we cross the infinite set of weak coupling walls $\mathcal{W}^{+}_{\text{weak},k}$ for $k > 0$ (or $k < 0$) while avoiding the strong coupling walls, one is left with F_GV; while if we cross the infinite set of strong coupling walls $\mathcal{W}^{+}_{\text{strong},k}$ for $k < 0$ while avoiding the weak coupling walls (i.e. while remaining in the sector $[l_0, l_{-1}]$), we are left with F_NS.

The original and dual RH problems have jumps arranged according to peacock patterns in the product of two complex planes with coordinates (λ, t) and $(\lambda_\text{D}, t_\text{D})$, respectively. Assuming that $0 < \text{Re}(t) < 1$ and $\text{Im}(t) > 0$, one finds that the positive and negative real half-axes are distinguished by the property of being self-dual in the sense that they are contained both in $\mathcal{C}_0^{\pm}$ and in the wedges between $\pm l_0$ and $\pm l_{-1}$. The self-duality of the intersection of these chambers strengthens the sense in which $F_\text{np}(\lambda, t)$ is distinguished as a non-perturbative definition of the topological string partition function.

6.2. Relation to string-theoretic S-duality. Relation (6.1) resembles the realisation of S-duality discussed in [APSV09, APP11a] on complex Darboux coordinates for the QK manifolds representing the hypermultiplet moduli spaces of type II string theory (see [Ale13, AMPP15] for reviews). This is probably no accident.

One may in particular notice that a Riemann–Hilbert problem similar to the one studied in [Bri20] is expected to be solved by twistor coordinates for the hypermultiplet moduli space in type II string theory on the resolved conifold. This Riemann–Hilbert problem should reproduce the problem studied in [Bri20] in a limit called the conformal limit. Both Riemann–Hilbert problems are defined with the help of the same BPS structure, implying that the symplectic transformations used in the definitions coincide. The main differences will concern the asymptotic conditions imposed in the formulation of the two problems. These considerations suggest that the complex structures on $M \times \mathbb{C}^{\times}$ defined by the coordinate functions solving the RH problem from [Bri20] are limits of the complex structures on the conifold hypermultiplet moduli space defined by twistor coordinates.

The QK metrics defined by mutually local D-instanton corrections have been studied intensively already [RLRS+07, AS09, AB15, CT21]. Infinite-distance limits of such QK-metrics have been studied in [BMW20] motivated by the swampland conjectures in type II string theories. Two infinite-distance limits play a basic role. The first, called the D1 limit in [BMW20], is characterised by large volume and large coupling $g_s = 1/\tau_2$. The second is called the F1 limit. It is simply described by small coupling g_s at finite values of the Kähler moduli. The two limits are related by S-duality. This implies that the D1 limit is characterised by a scaling of the form

$$\tau_2(\sigma) = e^{-\frac{3}{2}\sigma}\tau_2(0)\,, \qquad t(\sigma) = e^{\frac{3}{2}\sigma}t(0)\,,$$

taking into account the leading quantum corrections to the QK metric in this limit, as expressed most clearly in [BMW20, Equation (3.41)].

It is known that a scaling of g_s induces the same scaling of the topological string coupling λ in the conformal limit. This relates the F1 and D1 limits to the weak- and strong-coupling regions in the space $M \times \mathbb{C}^{\times}$, respectively. As the F1 and D1 limits are exchanged by S-duality, it seems natural to conjecture that the relations between the Stokes jumps of weak- and strong-coupling expansions observed above are related to the S-duality phenomenon by the conformal limit.

It has been argued in [APP11b], see also [AMPP15] for a review, that the string theoretical S-duality conjectures relating D5 and NS5 branes predict relations between BPS partition functions and NS5-brane partition functions. As discussed in [APP11b, AMPP15], the NS5-brane partition function lives precisely in the line bundle governed by Rogers dilogarithm discussed in Section 4.[11]

6.3. Langlands modular duality. It seems finally worth pointing out that the coordinate changes associated to Stokes jumps in weak- and strong coupling expansions are related by the phenomenon called Langlands modular duality in the terminology introduced by Fock and Goncharov in the context of quantum cluster algebras [FG09] following [Fad00]. An essential aspect of this phenomenon, specialized to the case at hand, is the possibility to introduce dual shift operators

$$(Tf)(t) = f(t + \lambda/2\pi), \qquad (\tilde{T}f)(t) = f(t+1),$$

[11] This was pointed out to us by S. Alexandrov.

which act on the variables $\tilde{Q} := e^{4\pi^2 \mathrm{i}\, t/\lambda} \equiv w$ and $Q = e^{2\pi \mathrm{i} t}$ as

$$TQ = qQ, \qquad \tilde{T}\tilde{Q} = \tilde{q}\tilde{Q}, \qquad \tilde{T}Q = Q, \qquad T\tilde{Q} = \tilde{Q}.$$

This implies in particular that the functions representing the cluster coordinate transformations associated to the weak coupling jumps are invariant under the shift $\tilde{T}$, while the shift T acts trivially on the cluster coordinate transformations associated to the strong coupling jumps. This simple phenomenon has a natural generalisation which is the root of some remarkable features of quantized cluster algebras [FG09]. We can't help the feeling that this manifestation of Langlands modular duality in the case of the resolved conifold partition functions can be the tip of an iceberg.

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A. Alternative Proof for the Borel Sum

In this section we present an alternative derivation of the Borel sum and transform of $\widetilde{F}(\lambda, t)$. The alternative proof uses the integral representation of the Hadamard product used in Section 3.1.

Proposition A.1. *Take* $t \in \mathbb{C}$ *with* $\mathrm{Im}(t) > 0$ *and* $0 < \mathrm{Re}(t) < 1$. *Then* $F_{\mathrm{np}}(\lambda, t)$ *equals* $F_{\mathbb{R}_{>0}}$ *on their common domain of definition. More specifically:*

$$F_{\mathrm{np}}(\lambda, t) = \frac{1}{\lambda^2}\mathrm{Li}_3(Q) + \frac{B_2}{2}\mathrm{Li}_1(Q) + \int_0^\infty \mathrm{d}\xi\, e^{-\xi/\check{\lambda}} G(\xi, t)\,. \tag{A.1}$$

Proof. We will first write down an integral representation for $F_{\mathbb{R}_{>0}}$ assuming that $t \in (0,1)$, and $\check{\lambda} > 0$ satisfies the conditions of Proposition 3.8. We will then deform t to $\mathrm{Im}(t) > 0$ and show what we want.
We recall the Hadamard product representation (see Proposition 3.4):

$$G(\xi, t) = \frac{1}{2\pi i} \int_{\gamma} \frac{\mathrm{d}s}{s} f_1(s)\, f_2\Big(\frac{\xi}{s}, t\Big), \tag{A.2}$$

where γ was an appropriate counterclockwise contour around 0, and

$$\begin{aligned} f_1(s) &= -\frac{1}{4\pi^2}\left(\frac{1}{\xi^3} - \frac{1}{\xi(e^{\xi/2} - e^{-\xi/2})^2} - \frac{1}{12\xi}\right), \\ f_2(\xi, t) &= \frac{(2\pi\mathrm{i})^3}{2}\Big(\mathrm{Li}_0\big(e^{2\pi\mathrm{i}(t+\xi)}\big) - \mathrm{Li}_0\big(e^{2\pi\mathrm{i}(t-\xi)}\big)\Big). \end{aligned} \tag{A.3}$$

Integrating $e^{-\xi/\check{\lambda}} G(\xi, t)$ along the positive real line and swapping the integral signs, we get

$$\begin{aligned} &\int_0^\infty \mathrm{d}\xi\, e^{-\xi/\check{\lambda}} G(\xi, t) \\ &= \frac{(2\pi\mathrm{i})^2}{2} \int_\gamma \frac{\mathrm{d}s}{s} \left(\int_0^\infty \mathrm{d}\xi\ f_1(s) e^{-\xi/\check{\lambda}} \mathrm{Li}_0\big(e^{2\pi i(t+\xi/s)}\big) - f_1(s) e^{-\xi/\check{\lambda}} \mathrm{Li}_0\big(e^{2\pi i(t-\xi/s)}\big)\right). \end{aligned} \tag{A.4}$$

Next we simultaneously rescale $s \mapsto \check{\lambda} s$ and $\xi \mapsto \check{\lambda} s \xi$ on the first term, while simultaneously rescaling $s \mapsto -\check{\lambda} s$ and $\xi \mapsto \check{\lambda} s \xi$ on the second term to obtain

$$\begin{aligned} &\int_0^\infty \mathrm{d}\xi\, e^{-\xi/\check{\lambda}} G(\xi, t) \\ &= \frac{(2\pi\mathrm{i})^2}{2} \int_\gamma \frac{\mathrm{d}s}{s} \left(\int_0^{s^{-1}\infty} \mathrm{d}\xi\ \check{\lambda} s \big(f_1(\check{\lambda} s) - f_1(-\check{\lambda} s)\big) e^{-s\xi} \mathrm{Li}_0\big(e^{2\pi\mathrm{i}(t+\xi)}\big)\right) \\ &= (2\pi\mathrm{i})^2 \int_\gamma \mathrm{d}s\, \check{\lambda} f_1(\check{\lambda} s) \left(\int_0^{s^{-1}\infty} \mathrm{d}\xi\ e^{-s\xi} \mathrm{Li}_0\big(e^{2\pi\mathrm{i}(t+\xi)}\big)\right). \end{aligned} \tag{A.5}$$

Let $\mathcal{C}$ and $\mathcal{C}'$ denote the contours following the real line from $-\infty$ to ∞ avoiding 0 by a small detour in the upper and lower half-planes respectively. We may in fact take them to the lines with imaginary parts ϵ and $-\epsilon$ respectively, for some small $\epsilon > 0$. Since $\mathcal{C}' - \mathcal{C} = \gamma$ up to homology, we can write

$$\begin{aligned} \int_0^\infty \mathrm{d}\xi\, e^{-\xi/\check{\lambda}} G(\xi, t) &= (2\pi\mathrm{i})^2 \left(\int_{\mathcal{C}'} - \int_{\mathcal{C}}\right) \mathrm{d}s\, \check{\lambda} f_1(\check{\lambda} s) \left(\int_0^{s^{-1}\infty} \mathrm{d}\xi\ e^{-s\xi} \mathrm{Li}_0\big(e^{2\pi\mathrm{i}(t+\xi)}\big)\right) \\ &= (2\pi\mathrm{i})^2 \int_{-\infty}^\infty \mathrm{d}s \bigg(\check{\lambda} f_1(\check{\lambda}(s - \mathrm{i}\epsilon)) \left(\int_0^{(s+\mathrm{i}\epsilon)\infty} \mathrm{d}\xi\ e^{-(s-\mathrm{i}\epsilon)\xi} \mathrm{Li}_0\big(e^{2\pi\mathrm{i}(t+\xi)}\big)\right) \\ &\quad - \check{\lambda} f_1(\check{\lambda}(s + \mathrm{i}\epsilon)) \left(\int_0^{(s-\mathrm{i}\epsilon)\infty} \mathrm{d}\xi\ e^{-(s+\mathrm{i}\epsilon)\xi} \mathrm{Li}_0\big(e^{2\pi\mathrm{i}(t+\xi)}\big)\right)\bigg). \end{aligned} \tag{A.6}$$

Now taking the limit $\epsilon \to 0^+$ then gives us

$$\int_0^\infty d\xi\, e^{-\xi/\check{\lambda}} G(\xi,t) = (2\pi \mathrm{i})^2 \int_{-\infty}^{\infty} ds\, \check{\lambda} f_1(\check{\lambda}s) \left(\int_{\mathcal{H}_s} d\xi\, e^{-s\xi} \mathrm{Li}_0\big(e^{2\pi \mathrm{i}(t+\xi)}\big)\right), \tag{A.7}$$

where $\mathcal{H}_s$ is a counterclockwise Hankel contour along the negative real axis when $s < 0$, a clockwise Hankel contour along the positive real axis when $s > 0$, and the imaginary axis from $-\mathrm{i}\infty$ to $\mathrm{i}\infty$ when $s = 0$.
The poles and residues of the inner integrand are given by

$$\mathrm{Res}_{-(t+k)}\big(e^{-s\xi}\mathrm{Li}_0(e^{2\pi \mathrm{i}(t+\xi)})\big) = -\frac{1}{2\pi \mathrm{i}} e^{s(t+k)}, \tag{A.8}$$

for all $k \in \mathbb{Z}$. We can thus deduce using Cauchy's residue theorem that the inner integral is the sum of $2\pi \mathrm{i}$ times the residues from the poles at $-(t+k)$ with $k \geq 0$ when $s < 0$ and minus the sum of $2\pi \mathrm{i}$ times the residues from the poles at $-(t+k)$ with $k > 0$ when $s > 0$:

$$\begin{aligned}
\int_{\mathcal{H}_s} d\xi\, e^{-s\xi}\mathrm{Li}_0\big(e^{2\pi \mathrm{i}(t+\xi)}\big) &= -\sum_{k=0}^{\infty} e^{s(t+k)} = -\frac{e^{st}}{1-e^s}, && \text{when } s < 0,\\
\int_{\mathcal{H}_s} d\xi\, e^{-s\xi}\mathrm{Li}_0\big(e^{2\pi \mathrm{i}(t+\xi)}\big) &= \sum_{k=-1}^{-\infty} e^{s(t+k)} = \frac{e^{st}e^{-s}}{1-e^{-s}} = -\frac{e^{st}}{1-e^s}, && \text{when } s > 0.
\end{aligned} \tag{A.9}$$

Putting everything together, we get

$$\begin{aligned}
\int_0^\infty d\xi\, e^{-\xi/\check{\lambda}} G(\xi,t) &= -\int_{-\infty}^{\infty} \frac{ds}{s}\left(\frac{e^{\check{\lambda}s}}{(e^{\check{\lambda}s}-1)^2} - \frac{1}{(\check{\lambda}s)^2} + \frac{1}{12}\right)\frac{e^{st}}{e^s-1}\\
&= -\int_{\mathcal{C}} \frac{ds}{s}\left(\frac{e^{\check{\lambda}s}}{(e^{\check{\lambda}s}-1)^2} - \frac{1}{(\check{\lambda}s)^2} + \frac{1}{12}\right)\frac{e^{st}}{e^s-1},
\end{aligned} \tag{A.10}$$

where we remark that the integrand of the integral over $\mathbb{R}$ is actually regular at $s = 0$. Both expressions in the equality (A.10) above are analytic in t and $\check{\lambda}$, so we can deform t to $\mathrm{Im}(t) > 0$ with $\mathrm{Im}(t)$ small, and λ away from $\mathbb{R}_{>0}$, so that (A.10) continues to hold in their common domain of definition. The result to be proved will follow if we can show the following for $m = 0$ and $m = 1$:

$$\int_{\mathcal{C}} \frac{ds}{s^{2m+1}} \frac{e^{ts}}{e^s-1} = \frac{1}{(2\pi \mathrm{i})^{2m}} \mathrm{Li}_{2m+1}(e^{2\pi \mathrm{i}t}). \tag{A.11}$$

The integrand has a pole of order $2m+2$ at $s = 0$ and simple poles at $s = 2\pi \mathrm{i}k$ with residues $e^{2\pi \mathrm{i}kt}/(2\pi \mathrm{i}k)^{2m+1}$ for all nonzero integers k. The contour $\mathcal{C}$ contains only the simple poles with $k > 0$. Thus, again by Cauchy's residue theorem, we have:

$$\int_{\mathcal{C}} \frac{ds}{s^{2m+1}} \frac{e^{ts}}{e^s-1} = 2\pi \mathrm{i} \sum_{k=1}^{\infty} \frac{e^{2\pi \mathrm{i}kt}}{(2\pi \mathrm{i}k)^{2m+1}} = \frac{1}{(2\pi \mathrm{i})^{2m}} \mathrm{Li}_{2m+1}(e^{2\pi \mathrm{i}t}). \tag{A.12}$$

Where in the last equality we used the fact that $\mathrm{Im}(t) > 0$ and hence $|e^{2\pi \mathrm{i}t}| < 1$, so that the series representation of $\mathrm{Li}_s(z)$ holds. This completes the proof. □

B. Asymptotic Series from Borel Transforms

Lemma B.1. *The expression*

$$G(\xi,t) = -\frac{1}{4\pi^2}\sum_{g=2}^{\infty}\frac{B_{2g}}{2g(2g-2)!(2g-3)!}\xi^{2g-3}\,\partial_t^{2g}\mathrm{Li}_3(Q)\,, \tag{B.1}$$

of the Borel transform can be obtained back from

$$\begin{aligned} G(\xi,t) = -\sum_{m\in\mathbb{Z}\backslash\{0\}}\frac{1}{(2\pi\mathrm{i})^2}\Bigg(\frac{1}{m^3}\left(\frac{e^{2\pi\mathrm{i}t+\xi/m}}{1-e^{2\pi\mathrm{i}t+\xi/m}}-\frac{e^{2\pi\mathrm{i}t-\xi/m}}{1-e^{2\pi\mathrm{i}t-\xi/m}}\right) \\ +\frac{\xi}{2m^4}\left(\frac{e^{2\pi\mathrm{i}t+\xi/m}}{(1-e^{2\pi\mathrm{i}t+\xi/m})^2}+\frac{e^{2\pi\mathrm{i}t-\xi/m}}{(1-e^{2\pi\mathrm{i}t-\xi/m})^2}\right)\Bigg)\,. \end{aligned} \tag{B.2}$$

Proof. We first write the second expression of $G(\xi,t)$ as

$$G(\xi,t) = -\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\frac{\xi^2}{(2\pi\mathrm{i})^2}\sum_{m=1}^{\infty}\left(\frac{1}{m^3}\left(\frac{e^{2\pi\mathrm{i}t+\xi/m}}{1-e^{2\pi\mathrm{i}t+\xi/m}}-\frac{e^{2\pi\mathrm{i}t-\xi/m}}{1-e^{2\pi\mathrm{i}t-\xi/m}}\right)\right)\right) \tag{B.3}$$

we next use the Taylor expansion around $\xi = 0$:

$$\frac{e^{2\pi\mathrm{i}t+\xi/m}}{1-e^{2\pi\mathrm{i}t+\xi/m}} = \mathrm{Li}_0(e^{2\pi\mathrm{i}t+\xi/m}) = \sum_{k=0}^{\infty}\frac{\xi^k}{m^k}\mathrm{Li}_{-k}(e^{2\pi\mathrm{i}t})\,, \tag{B.4}$$

which makes use of the property

$$\theta_Q\mathrm{Li}_s(Q) = \mathrm{Li}_{s-1}(Q)\,,\quad \theta_Q := Q\,\frac{\mathrm{d}}{\mathrm{d}Q}\,, \tag{B.5}$$

We thus obtain:

$$\begin{aligned} G(\xi,t) &= -\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\frac{\xi^2}{(2\pi\mathrm{i})^2}\sum_{m=1}^{\infty}\left(\frac{2}{m^3}\left(\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\left(\frac{\xi}{m}\right)^{2k+1}\mathrm{Li}_{-2k-1}(e^{2\pi\mathrm{i}t})\right)\right)\right) \\ &= -\frac{2}{\xi}\frac{\partial}{\partial\xi}\left(\frac{\xi^2}{(2\pi\mathrm{i})^2}\left(\sum_{k=0}^{\infty}\zeta(2k+4)\frac{1}{(2k+1)!}\xi^{2k+1}\mathrm{Li}_{-2k-1}(e^{2\pi\mathrm{i}t})\right)\right) \\ &= -\frac{2}{\xi}\frac{\partial}{\partial\xi}\left(\frac{\xi^2}{(2\pi\mathrm{i})^2}\left(\sum_{k=0}^{\infty}(-1)^{k+3}\frac{B_{2k+4}\,(2\pi)^{2k+4}}{2(2k+4)!}\frac{1}{(2k+1)!}\xi^{2k+1}\mathrm{Li}_{-2k-1}(e^{2\pi\mathrm{i}t})\right)\right) \\ &= -\left(\frac{1}{(2\pi\mathrm{i})^2}\sum_{k=0}^{\infty}(-1)^{k+3}\frac{B_{2k+4}\,(2\pi)^{2k+4}\,(2k+3)}{(2k+4)!(2k+1)!}\xi^{2k+1}\,\mathrm{Li}_{-2k-1}(e^{2\pi\mathrm{i}t})\right) \\ &= -\frac{1}{4\pi^2}\sum_{g=2}^{\infty}\frac{B_{2g}}{2g(2g-2)!(2g-3)!}\xi^{2g-3}\,\partial_t^{2g}\mathrm{Li}_3(Q)\,, \end{aligned} \tag{B.6}$$

where in going from the first to the second line we have used the following expression for the Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}\,,\quad \mathrm{Re}(s) > 0\,,$$

and in going from the second to the third line we have used the following identity:

$$\zeta(2n) = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!} ,$$

and where we have changed the summation variable in the fifth line to $g = k + 2$ and made use of

$$(-1)^g \, (2\pi)^{2g} \mathrm{Li}_{3-2g}(Q) = \partial_t^{2g} \mathrm{Li}_3(Q) .$$

□

Lemma B.2. *The expression* $-G(\xi, 0) - \frac{1}{12\xi}$ *gives the Borel transform of* $F_0(\lambda) + \zeta(3)/\lambda^2 - F_0^1$.

Proof. From the definition of $F_0(\lambda)$ in (2.5), we have the following (recall that we take $\chi(X) = 2$ for the resolved conifold):

$$F_0(\lambda) + \zeta(3)/\lambda^2 - F_0^1 = \sum_{g \geq 2} \lambda^{2g-2} \frac{(-1)^{g-1} B_{2g} B_{2g-2}}{2g(2g-2)(2g-2)!} . \tag{B.7}$$

The Borel transform $G_0(\xi)$ of the previous series is then given by

$$G_0(\xi) = \sum_{g \geq 2} \xi^{2g-3} \frac{(-1)^{g-1} B_{2g} B_{2g-2} (2\pi)^{2g-2}}{2g\, ((2g-2)!)^2} . \tag{B.8}$$

On the other hand, we have

$$\begin{aligned} G(\xi, 0) &= \frac{2}{(2\pi)^2} \sum_{m>0} \frac{1}{m^3} \Big(1 + \frac{\xi}{2} \frac{\partial}{\partial \xi}\Big) \Big(\frac{1}{1 - e^{\xi/m}} - \frac{1}{1 - e^{-\xi/m}}\Big) \\ &= \frac{2}{(2\pi)^2} \sum_{m>0} \frac{1}{m^3} \Big[\frac{1}{2}\Big(\frac{1}{1 - e^{\xi/m}} - \frac{1}{1 - e^{-\xi/m}}\Big) + \frac{1}{2} \frac{\partial}{\partial \xi} \Big(\frac{\xi}{1 - e^{\xi/m}} - \frac{\xi}{1 - e^{-\xi/m}}\Big)\Big] \end{aligned} \tag{B.9}$$

Using (2.1) one finds

$$\begin{aligned} G(\xi, 0) &= \frac{2}{(2\pi)^2} \sum_{m>0} \frac{1}{m^3} \Big[- \frac{m}{\xi} \sum_{k=0}^{\infty} \frac{B_{2k}}{(2k)!} \Big(\frac{\xi}{m}\Big)^{2k} - m \frac{\partial}{\partial \xi} \Big(\sum_{k=0}^{\infty} \frac{B_{2k}}{(2k)!} \Big(\frac{\xi}{m}\Big)^{2k} \Big) \Big] \\ &= - \frac{2}{(2\pi)^2} \sum_{k=0}^{\infty} (2k+1) \frac{B_{2k}}{(2k)!} \xi^{2k-1} \zeta(2k+2) \\ &= - \frac{2}{(2\pi)^2} \sum_{k=0}^{\infty} (2k+1) \frac{B_{2k}}{(2k)!} \xi^{2k-1} \Big(\frac{(-1)^k B_{2k+2} (2\pi)^{2k+2}}{(2k+2)!2} \Big) \\ &= - \sum_{g \geq 1} \xi^{2g-3} \frac{(-1)^{g-1} B_{2g} B_{2g-2} (2\pi)^{2g-2}}{2g\, ((2g-2)!)^2} \\ &= -G_0(\xi) - \frac{1}{12\xi} \end{aligned} \tag{B.10}$$

and the result follows. □

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