



On Free Field Realization of Quantum Affine W-Algebras

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Abstract: We find an explicit formula for the conformal vector of any quantum affine W -algebra in its free field realization.

1. Introduction

The chiral part of a (super)conformal field theory is a vertex algebra which admits a conformal vector L , for which the eigenvalues of the energy operator L_0 lie in $\frac{1}{2}\mathbb{Z}_{\geq 0}$ and the multiplicity of the 0 eigenvalue is 1. An important class of such vertex algebras are quantum affine W -algebras $W^k(\mathfrak{g}, x, f)$ [KRW], [KW] (see also [DSK]), attached to a “good” datum (\mathfrak{g}, x, f, k) , where $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a basic Lie superalgebra, i.e. a simple finite-dimensional Lie superalgebra over an algebraically closed field \mathbb{F} of characteristic 0 with reductive even part $\mathfrak{g}_{\bar{0}}$ and a fixed non-degenerate even invariant supersymmetric bilinear form $(\cdot | \cdot)$, $x \in \mathfrak{g}_{\bar{0}}$ is such that the eigenspace decomposition of \mathfrak{g} with respect to $\text{ad } x$ defines a $\frac{1}{2}\mathbb{Z}$ -grading

$$\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j, \quad (1.1)$$

$f \in \mathfrak{g}_{-1}$, and $k \in \mathbb{F}$.

A datum (\mathfrak{g}, x, f, k) is called *good* if $f \in \mathfrak{g}_{-1}$ is such that

$$\mathfrak{g}^f \subset \mathfrak{g}_{\leq 0}. \quad (1.2)$$

Hereafter \mathfrak{g}^f (resp. \mathfrak{g}_j^f) denotes the centralizer of f in \mathfrak{g} (resp. \mathfrak{g}_j), and we use notation $\mathfrak{g}_{\leq m} = \bigoplus_{j \leq m} \mathfrak{g}_j$, and similarly for $\geq m$, or $< m$, or $> m$. We also denote by $p_{>0}$,

p_j , etc., the projection of \mathfrak{g} to $\mathfrak{g}_{>0}$, \mathfrak{g}_j , etc., along (1.1). A special case of a good datum is a *Dynkin datum*, defined by x and f from an $s\ell_2$ -triple $\{e, x, f\}$ in \mathfrak{g}_0 , where $[x, e] = e$, $[e, f] = x$, $[x, f] = -f$.

Recall that a bilinear form $(\cdot | \cdot)$ on \mathfrak{g} is called even if $(\mathfrak{g}_0 | \mathfrak{g}_1) = 0$, supersymmetric (resp. superskewsymmetric) if $(a|b) = (-1)^{p(a)p(b)}(b|a)$ (resp. $-(-1)^{p(a)p(b)}(b|a)$), and invariant if $([a, b]|c) = (a|[b, c])$.

In [KRW] for an arbitrary datum (\mathfrak{g}, x, f, k) a vertex algebra homology complex

$$(V^k(\mathfrak{g}) \otimes F^{\text{ch}} \otimes F^{\text{ne}}, d_{(0)}) \tag{1.3}$$

was constructed, where $V^k(\mathfrak{g})$ is the universal affine vertex algebra of level k associated to \mathfrak{g} , and F^{ch} (resp. F^{ne}) is the vertex algebra of free charged fermions based on $\mathfrak{g}_{>0} \oplus \mathfrak{g}_{>0}^*$ with reversed parity (resp. of free neutral fermions based on $\mathfrak{g}_{1/2}$), and $d_{(0)}$ is an explicitly constructed odd derivation of the vertex algebra $\mathcal{C}^k(\mathfrak{g}, x, f) := V^k(\mathfrak{g}) \otimes F^{\text{ch}} \otimes F^{\text{ne}}$.

Recall [K2] that for the construction of the vertex algebra of free fermions based on a vector superspace A , one needs a superskewsymmetric bilinear form on A . In the case of F^{ch} this bilinear form is defined via the pairing of $\mathfrak{g}_{>0}$ and its dual $\mathfrak{g}_{>0}^*$, which is identified with $\mathfrak{g}_{<0}$, using the bilinear form $(\cdot | \cdot)$; the former is non-degenerate since the latter is. In the case of F^{ne} this bilinear form is defined by the formula

$$\langle a, b \rangle^{\text{ne}} = (f|[a, b]), \quad a, b \in \mathfrak{g}_{1/2}. \tag{1.4}$$

The bilinear form (1.4) is non-degenerate if and only if

$$\text{ad } f : \mathfrak{g}_{1/2} \rightarrow \mathfrak{g}_{-1/2} \text{ is a vector superspace isomorphism.} \tag{1.5}$$

Since property (1.2) is equivalent to $[f, \mathfrak{g}_j] = \mathfrak{g}_{j-1}$ if $j \leq 1/2$ [KW], property (1.5) holds for any good datum.

The \mathbb{Z} -grading of the complex (1.3) is defined by

$$\text{deg } V^k(\mathfrak{g}) = \text{deg } F^{\text{ne}} = 0, \quad \text{deg } \mathfrak{g}_{>0} = -\text{deg } \mathfrak{g}_{>0}^* = 1.$$

The homology of the complex (1.3) is called the *quantum affine W-algebra*, attached to the datum (\mathfrak{g}, x, f, k) , and is denoted by $W^k(\mathfrak{g}, x, f)$.

For a good datum, $[\mathfrak{g}_0, f] = \mathfrak{g}_{-1}$, hence the orbit $G_0(f)$ is Zariski open in \mathfrak{g}_{-1} , and therefore the vertex algebra $W^k(\mathfrak{g}, x, f)$ is independent, up to isomorphism, of the choice of $f \in \mathfrak{g}_{-1}$, satisfying (1.2).

The main result of [KW] on the structure of the vertex algebra $W^k(\mathfrak{g}, x, f)$ is Theorem 4.1, which states that for a good datum the j^{th} homology of the complex (1.3) is zero if $j \neq 0$, and the 0-th homology is the vertex algebra $W^k(\mathfrak{g}, x, f)$, which is a subalgebra of the vertex algebra $\mathcal{C}^k(\mathfrak{g}, x, f)$ freely generated by $d_{(0)}$ -closed elements $J^{\{a_i\}}$, where a_1, \dots, a_s is a basis of \mathfrak{g}^f consisting of eigenvectors of $\text{ad } x$. The elements $J^{\{a_i\}}$ can be recursively computed, using equations (4.11) and (4.12) from [KW]. The ‘‘building blocks’’ for construction of elements $J^{\{a_i\}}$ are the elements $J^{(a)}$, $a \in \mathfrak{g}^f$, defined in [KRW], see formula (2.7) in Sect. 2 of the present paper. Theorem 4.1(a) from [KW] states that for each $a \in \mathfrak{g}_{-j}^f (= \mathfrak{g}^f \cap \mathfrak{g}_{-j})$ the element $J^{(a)} - J^{(a)}$ lies in the subalgebra of the vertex algebra $\mathcal{C}^k(\mathfrak{g}, x, f)$, generated by elements $J^{(b)}$, where $b \in \mathfrak{g}_{-s}$ with $0 \leq s < j$ (recall that $\mathfrak{g}_j^f \neq 0$ only for $j \leq 0$ by (1.2)), and by the neutral fermions.

Consider the subalgebra $\overline{\mathcal{C}}^k(\mathfrak{g}, x, f)$ of the vertex algebra $\mathcal{C}^k(\mathfrak{g}, x, f)$ generated by the elements $J^{(v)}$ with $v \in \mathfrak{g}_{\leq 0}$, and by the neutral fermions. It follows from the above

discussion that, for a good datum, all elements $J^{(v)}$, $v \in \mathfrak{g}^f$, lie in $\overline{\mathcal{C}}^k(\mathfrak{g}, x, f)$. It is easy to see that the elements $J^{(v)}$ with $v \in \mathfrak{g}_0$ and neutral fermions generate a subalgebra $\overline{\mathcal{C}}_0^k(\mathfrak{g}, x, f)$ of the vertex algebra $\mathcal{C}(\mathfrak{g}, x, f)$, and that the $J^{(v)}$ with $v \in \mathfrak{g}_{<0}$ generate an ideal $U_{<0}$ of $\overline{\mathcal{C}}^k(\mathfrak{g}, x, f)$, such that $\overline{\mathcal{C}}_0^k(\mathfrak{g}, x, f) \cap U_{<0} = 0$. Hence, the canonical map $\overline{\mathcal{C}}^k(\mathfrak{g}, x, f) \rightarrow \overline{\mathcal{C}}_0^k(\mathfrak{g}, x, f)$ induces a vertex algebra homomorphism

$$W^k(\mathfrak{g}, x, f) \rightarrow \overline{\mathcal{C}}_0^k(\mathfrak{g}, x, f). \tag{1.6}$$

Since the vertex algebra $\overline{\mathcal{C}}_0^k(\mathfrak{g}, x, f)$ is isomorphic to the tensor product of the universal affine vertex algebra $V^{k'}(\mathfrak{g}_0)$ of "shifted" level k' ([KW], formula (2.5)), and the vertex algebra F^{ne} , the map (1.6) may be viewed as a free field realization (FFR) of the W-algebra $W^k(\mathfrak{g}, x, f)$.

In the case of a good datum, for the $a_i \in \mathfrak{g}_j^f$ with $j = 0$ or $-1/2$ the elements $J^{(a_i)}$ are uniquely determined by the a_i and they are explicitly constructed in [KW], Sect. 2. The construction of these elements is still valid for an arbitrary datum, satisfying property (1.5). Furthermore, provided that $k \neq -h^\vee$ (i.e. k is not the critical level), we also constructed there an energy-momentum element L , with respect to which the elements $J^{(a_i)}$ have conformal weight $1 - m_i$, where $[x, a_i] = m_i a_i$, and this construction is again valid for an arbitrary datum satisfying property (1.5).

In [KW], Theorem 5.1(c), we found an explicit expression for L in terms of the elements $J^{(a_i)}$ and the neutral fermions in the case of minimal W-algebras, which allowed us to compute the FFR (1.6) for these W-algebras explicitly (see [KW], Theorem 5.2).

The main results of the present paper, valid for an arbitrary datum (\mathfrak{g}, x, f, k) satisfying property (1.5), are Theorem 3.1, which gives an explicit expression of the element $J^{(f)}$ in terms of the elements $J^{(f)}$, $J^{(a)}$ with $a \in \mathfrak{g}_0$ and $\mathfrak{g}_{-1/2}$, and of neutral free fermions, and Theorem 3.2, which states that $L = -\frac{1}{k+h^\vee} J^{(f)}$, provided that $k \neq -h^\vee$. This leads to an explicit formula for the image of L under the FFR (1.6) for an arbitrary quantum affine W-algebra, attached to a good datum.

Our Theorem 3.2 assumes that $k \neq -h^\vee$, i.e. k is not critical, but the construction of the W-algebra and Theorem 3.1 hold for an arbitrary k . However the structure of the W-algebra $W^{-h^\vee}(\mathfrak{g}, x, f)$ is dramatically different [FF1]. This W-algebra for $f = f_{pr}$ plays an important role in the geometric Langlands correspondence.

Throughout the paper the base field \mathbb{F} is an algebraically closed field of characteristic 0.

2. The Complex $(\mathcal{C}^k(\mathfrak{g}, x, f), d_{(0)})$ and the W-Algebra $W^k(\mathfrak{g}, x, f)$

First, recall the construction of vertex algebras $V^k(\mathfrak{g})$, F^{ch} and F^{ne} . We shall use the very convenient language of non-linear Lie conformal superalgebras and λ -brackets [DSK].

Given a Lie superalgebra \mathfrak{g} with an invariant supersymmetric bilinear form B , consider the $\mathbb{F}[\partial]$ -module $\mathbb{F}[\partial] \otimes \mathfrak{g}$ with the following non-linear λ -bracket

$$[a_\lambda b] = [a, b] + \lambda B(a, b)1, \quad a, b \in \mathfrak{g}, \tag{2.1}$$

and the universal enveloping vertex algebra $V^B(\mathfrak{g})$ of this non-linear Lie conformal superalgebra. One often fixes such a bilinear form (\cdot, \cdot) , lets $B(a, b) = k(a|b)$, $k \in \mathbb{F}$, and uses the notation $V^k(\mathfrak{g}) = V^B(\mathfrak{g})$. Then $V^k(\mathfrak{g})$ is called the universal affine vertex algebra for \mathfrak{g} of level k .

The vertex algebra $F(A)$ of fermions based on the vector superspace A with a skew-supersymmetric bilinear form $\langle \cdot, \cdot \rangle$ is defined as the universal enveloping vertex algebra of the $\mathbb{F}[\partial]$ -module $\mathbb{F}[\partial] \otimes \mathfrak{g}$ with the non-linear λ -bracket

$$[a_\lambda b] = \langle a, b \rangle 1, \quad a, b \in A. \tag{2.2}$$

Given a datum (\mathfrak{g}, x, f, k) as described in the introduction, the associated homology complex $(\mathcal{C}^k(\mathfrak{g}, x, f), d_{(0)})$ is constructed as follows. Let $A^{\text{ch}} = \Pi(\mathfrak{g}_{>0} \oplus \mathfrak{g}_{>0}^*)$, where Π stands for the reversal of parity, and define on it a skewsupersymmetric bilinear form $\langle \cdot, \cdot \rangle^{\text{ch}}$ by the pairing of the vector superspace $\Pi\mathfrak{g}_{>0}$ and its dual $\Pi\mathfrak{g}_{>0}^*$, and let $A^{\text{ne}} = \mathfrak{g}_{1/2}$ with the bilinear form $\langle a, b \rangle^{\text{ne}}$ defined by (1.4). Then $\mathcal{C}^k(\mathfrak{g}, x, f)$ is the universal enveloping vertex algebra of the non-linear Lie conformal superalgebra $\mathbb{F}[\partial](\mathfrak{g} \oplus A^{\text{ch}} \oplus A^{\text{ne}})$ with the λ -brackets defined by (2.1) and (2.2) on the summands and zero between the distinct summands. The vertex algebra $\mathcal{C}^k(\mathfrak{g}, x, f)$ is isomorphic to $V^k(\mathfrak{g}) \otimes F^{\text{ch}} \otimes F^{\text{ne}}$, where $F^{\text{ch}} = F(A^{\text{ch}})$ and $F^{\text{ne}} = F(A^{\text{ne}})$. Letting

$$\deg V^k(\mathfrak{g}) = \deg F^{\text{ne}} = 0, \quad \deg \mathfrak{g}_{>0} = -\deg \mathfrak{g}_{>0}^* = 1 \text{ on } F^{\text{ch}}, \tag{2.3}$$

defines a \mathbb{Z} -grading of this vertex algebra:

$$\mathcal{C}^k(\mathfrak{g}, x, f) = \bigoplus_{j \in \mathbb{Z}} \mathcal{C}_j^k. \tag{2.4}$$

In order to define the differential $d_{(0)}$ choose a basis $\{u_i\}_{i \in S}$ of \mathfrak{g} , compatible with parity and the $\frac{1}{2}\mathbb{Z}$ -grading (1.1), let $\{u^i\}_{i \in S}$ be its dual basis of \mathfrak{g} with respect to the bilinear form $(\cdot | \cdot)$, i.e. $(u_i | u^j) = \delta_{i,j}$, and denote by $\{u_i\}_{i \in S_{>0}}$ (resp. $\{u_i\}_{i \in S_j}$) the part of $\{u_i\}_{i \in S}$, which is a basis of $\mathfrak{g}_{>0}$ (resp. \mathfrak{g}_j). Let $\{\varphi_i\}_{i \in S_{>0}}$ be the corresponding to $\{u_i\}_{i \in S_{>0}}$ basis of $\Pi\mathfrak{g}_{>0}$, and let $\{\varphi^i\}_{i \in S_{>0}}$ be the dual basis of $\Pi\mathfrak{g}_{>0}^*$. Let $\{\Phi_i\}_{i \in S_{1/2}}$ be the corresponding to $\{u_i\}_{i \in S_{1/2}}$ basis of A^{ne} . For $u \in \mathfrak{g}$ let $\Phi_u = \sum_{i \in S_{1/2}} \gamma_i \Phi_i$ (resp. $\varphi_u = \sum_{i \in S_{>0}} \gamma_i \varphi_i$) if $p_{1/2}u$ (resp. $p_{>0}u$) $\gamma_i u_i$, where $\gamma_i \in \mathbb{F}$.

Introduce the following element of the vertex algebra $\mathcal{C}^k(\mathfrak{g}, x, f)$:

$$\begin{aligned} d &= \sum_{i \in S_{>0}} ((-1)^{p(i)} u_i \otimes \varphi^i \otimes 1 + (f | u_i) \otimes \varphi^i \otimes 1) + \sum_{i \in S_{1/2}} 1 \otimes \varphi^i \otimes \Phi_i \\ &+ \frac{1}{2} \sum_{i,j \in S_{>0}} (-1)^{p(i)} 1 \otimes : \varphi^i \varphi^j \varphi_{[u_j, u_i]} : \otimes 1, \end{aligned} \tag{2.5}$$

where $p(i)$ stands for the parity $p(u_i)$ in the Lie superalgebra \mathfrak{g} . The element d is independent of the choice of the basis of \mathfrak{g} . One checks that $[d_\lambda d] = 0$ ([KRW], Theorem 2.1), therefore $[d_{(0)}, d_{(0)}] = 0$ and $d_{(0)}^2 = 0$. Thus, $d_{(0)}$ is a homology differential of the vertex algebra $\mathcal{C}^k(\mathfrak{g}, x, f)$. The homology of the complex $(\mathcal{C}^k(\mathfrak{g}, x, f), d_{(0)})$ is the quantum affine W-algebra $W^k(\mathfrak{g}, x, f)$.

One has the following formulas for the action of $d_{(0)}$ of the generators of the vertex algebra $\mathcal{C}^k(\mathfrak{g}, x, f)$ (cf. [KRW], formula(2.4)), where $a \in \mathfrak{g}$, and thereafter we skip the

tensor product signs:

$$\begin{aligned}
 d_{(0)}a &= \sum_{j \in S_{>0}} ((-1)^{p(j)} : \varphi^j [u_j, a] : + k(a|u_j) \partial \varphi^j); \\
 d_{(0)}\varphi_a &= p_{>0}(a) + (a|f) + (-1)^{p(a)} \Phi_a + \sum_{j \in S_{>0}} : \varphi^j \varphi_{[u_j, p_{>0}a]} :; \\
 d_{(0)}\varphi^i &= -\frac{1}{2} \sum_{j,s \in S_{>0}} (-1)^{p(i)p(j)} c_{j,s}^i : \varphi^j \varphi^s :; \text{ where } [u_j, u_s] = \sum_i c_{j,s}^i u_i; \\
 d_{(0)}\Phi_a &= \sum_{j \in S_{1/2}} (u_j|[a, f])\varphi^j. \tag{2.6}
 \end{aligned}$$

Recall that the “building blocks” for elements of the W-algebra $W^k(\mathfrak{g}, x, f)$ are the following elements of $\mathcal{C}^k(\mathfrak{g}, x, f)$ for $v \in \mathfrak{g}$:

$$J^{(v)} = v + \sum_{j \in S_{>0}} (-1)^{p(v)+p(j)} : \varphi_{[v, u_j]} \varphi^j :. \tag{2.7}$$

Denote by $\mathcal{C}_-^k(\mathfrak{g}, x, f)$ the subalgebra of the vertex algebra $\mathcal{C}^k(\mathfrak{g}, x, f)$, generated by the elements $J^{(v)}$ ($v \in S_{\leq 0}$), φ^i ($i \in S_{>0}$), and Φ_i ($i \in S_{1/2}$). By (2.6), this subalgebra is $d_{(0)}$ -invariant. Let, as above, $\overline{\mathcal{C}}^k(\mathfrak{g}, x, f)$ be the subalgebra of $\mathcal{C}_-^k(\mathfrak{g}, x, f)$, generated by the $J^{(v)}$ ($v \in S_{\leq 0}$), and the Φ_i ($i \in S_{1/2}$). Then, by (2.6), we have

$$\overline{\mathcal{C}}^k(\mathfrak{g}, x, f) \cap d_{(0)}\mathcal{C}_-^k(\mathfrak{g}, x, f) = 0. \tag{2.8}$$

Let $\kappa(a, b) = \text{str}_{\mathfrak{g}}(\text{ad } a)(\text{ad } b)$ be the Killing form on \mathfrak{g} . Recall that

$$\kappa(a, b) = 2h^\vee(a|b). \tag{2.9}$$

For the projection $p_j : \mathfrak{g} \rightarrow \mathfrak{g}_j$ (resp $p_{>0} : \mathfrak{g} \rightarrow \mathfrak{g}_{>0}$) along the grading (1.1), define the "partial" Killing forms

$$\kappa_j(a, b) = \text{str}_{\mathfrak{g}}(p_j(\text{ad } a)(\text{ad } b)) \text{ (resp. } \kappa_{>0}(a, b) = \text{str}_{\mathfrak{g}}(p_{>0}(\text{ad } a)(\text{ad } b))). \tag{2.10}$$

Elements $J^{(v)}$ for $v \in \mathfrak{g}_0$ obey λ -brackets of the universal affine vertex algebra $V^{B_0}(\mathfrak{g}_0)$ [KRW, Theorem 2.4(c)]:

$$[J^{(a)}_\lambda J^{(b)}] = J^{(a,b)} + \lambda B_0(a, b), \quad a, b \in \mathfrak{g}_0, \tag{2.11}$$

where

$$B_0(a, b) = k(a|b) + \frac{1}{2}(\kappa(a, b) - \kappa_0(a, b)). \tag{2.12}$$

In fact, (2.11) holds for $a \in \mathfrak{g}_i, b \in \mathfrak{g}_j$ with $ij \geq 0$ (of course $B_0(a, b) = 0$ if $ij \geq 0$ and i or j is non-zero). Hence, we have the following

Corollary 2.1. *The factor algebra of the vertex algebra $V^{B_0}(\mathfrak{g}_{\leq 0})$ by the ideal, generated by $\mathfrak{g}_{<0}$, is isomorphic to the vertex algebra $V^{B_0}(\mathfrak{g}_0)$.*

The proof of the following formula from [KW], formula (2.6), uses formulas (2.6):

$$\begin{aligned}
 d_{(0)}(J^{(v)}) &= \sum_{j \in S_{>0}} ([f, v]|u_j)\varphi^j + \sum_{j \in S_{>0}} (-1)^{p(v)(p(j)+1)} : \varphi^j \Phi_{p_{1/2}[v, u_j]} : \\
 &\quad - \sum_{\substack{j \in S_{>0} \\ [v, u_j] \in \mathfrak{g}_{\leq 0}}} (-1)^{p(j)(p(v)+1)} : \varphi^j J^{([v, u_j])} : + \sum_{j \in S_{>0}} (k(v|u_j) + \kappa_{>0}(v, u_j))\partial\varphi^j.
 \end{aligned}
 \tag{2.13}$$

From now on we shall assume that condition (1.5) holds, so that we can define the basis $\{\Phi^i\}_{i \in S_{1/2}}$ of A^{ne} , dual to $\{\Phi_i\}_{i \in S_{1/2}}$ with respect to the bilinear form (1.4). Then we have: $d_{(0)}\Phi^i = \varphi^i$.

As has been mentioned in the introduction, for a good grading the $d_{(0)}$ -closed elements $J^{[a]}$ are uniquely determined for $a \in \mathfrak{g}_j^f$ for $j = 0$ or $-1/2$. The $d_{(0)}$ -closed elements $J^{[a]}$ for $a \in \mathfrak{g}_0^f$ can be constructed, provided that (1.5) holds, and they are as follows (see [KRW], Theorem 2.4(a)):

$$J^{[a]} = J^{(a)} + \frac{(-1)^{p(a)}}{2} \sum_{j \in S_{1/2}} : \Phi^j \Phi_{[u_j, a]} : .
 \tag{2.14}$$

These elements obey λ -brackets of the universal affine vertex algebra $V^{B_{1/2}}(\mathfrak{g}_0^f)$:

$$[J^{[a]}]_{\lambda} J^{[b]} = J^{[[a, b]]} + \lambda B_{1/2}(a, b),
 \tag{2.15}$$

where

$$B_{1/2}(a, b) = k(a|b) + \kappa_{>0}(a, b) - \frac{1}{2}\kappa_{1/2}(a, b).
 \tag{2.16}$$

The $d_{(0)}$ -closed elements $J^{[v]}$ for $v \in \mathfrak{g}_{-1/2}^f$ are as follows (see [KW], Theorem 2.1(d)):

$$\begin{aligned}
 J^{[v]} &= J^{(v)} - \frac{(-1)^{p(v)}}{3} \sum_{i, j \in S_{1/2}} : \Phi^i \Phi^j \Phi_{[u_j, [u_i, v]]} : \\
 &\quad + \sum_{i \in S_{1/2}} (: J^{([v, u_i])} \Phi^i : - (k(v|u_i) + \kappa_{>0}(v, u_i))\partial\Phi^i),
 \end{aligned}
 \tag{2.17}$$

and one has ([KW], Theorem 2.1(e)):

$$[J^{[a]}]_{\lambda} J^{[v]} = J^{[[a, v]]} \text{ if } a \in \mathfrak{g}_0^f, v \in \mathfrak{g}_{-1/2}^f.
 \tag{2.18}$$

Remark 2.1. The elements φ^i coincide with the elements, denoted by φ_i^* in [KRW] and [KW], but are different from the elements, denoted by φ^i in [DSK]. The advantage of this less natural choice is that then the construction of the W -algebra $W^k(\mathfrak{g}, x, f)$ works for an arbitrary finite-dimensional Lie superalgebra \mathfrak{g} with an arbitrary supersymmetric (possibly degenerate) invariant bilinear form (\cdot, \cdot) . (The simplicity of \mathfrak{g} and the non-degeneracy of (\cdot, \cdot) are needed in the next sections.)

3. A Formula for $J^{\{f\}}$ and the Energy-Momentum Element L of $W^k(\mathfrak{g}, x, f)$

Choose a Cartan subalgebra \mathfrak{h} of \mathfrak{g}_0 , containing a Cartan subalgebra of \mathfrak{g}_0^f . It is a Cartan subalgebra of \mathfrak{g} . Choose a set of positive roots of \mathfrak{g} , compatible with the grading (1.1). Recall that the dual Coxeter number h^\vee of the simple Lie superalgebra \mathfrak{g} with the given invariant bilinear form $(\cdot | \cdot)$ is the half of the eigenvalue of the Casimir operator $\sum_{j \in S} u^j u_j$ on \mathfrak{g} , and it is given by the formula

$$h^\vee = (\rho | \theta) + \frac{1}{2} (\theta | \theta), \tag{3.1}$$

where θ is the highest root and ρ is the half of the difference between sums of positive even roots and positive odd roots. Provided that $k \neq -h^\vee$, the energy-momentum (or Virasoro) element of the vertex algebra $\mathcal{C}^k(\mathfrak{g}, x, f)$ is defined by [KRW] for an arbitrary datum, satisfying (1.5):

$$L = L^{\mathfrak{g}} + \partial x + L^{\text{ch}} + L^{\text{ne}}, \tag{3.2}$$

where

$$\begin{aligned} L^{\mathfrak{g}} &= \frac{1}{2(k + h^\vee)} \sum_{j \in S} : u^j u_j :, \\ L^{\text{ch}} &= \sum_{j \in S_{>0}} (1 - m_j) : (\partial \varphi^j) \varphi_j : - m_j : \varphi^j \partial \varphi_j :, \\ L^{\text{ne}} &= \frac{1}{2} \sum_{j \in S_{1/2}} : (\partial \Phi^j) \Phi_j :, \end{aligned}$$

and the m_j are defined by $[x, u_j] = m_j u_j$.

The central charge of this Virasoro element is equal to (see [KRW], Remark 2.2)

$$c(\mathfrak{g}, x, k) = \text{sdim } \mathfrak{g}_0 - \frac{1}{2} \text{sdim } \mathfrak{g}_{1/2} - \frac{12}{k + h^\vee} |\rho - (k + h^\vee)x|^2. \tag{3.3}$$

With respect to this L the elements φ_j (resp. φ^j) are primary of conformal weight $1 - m_j$ (resp. m_j), the Φ_j are primary of conformal weight $1/2$, and $a \in \mathfrak{g}_j$ has conformal weight $1 - j$ and is primary, unless $j = 0$ and $(a|x) \neq 0$. Actually one has:

$$[L_\lambda a] = (\partial + \lambda)a - \lambda^2 k(a|x) \text{ for } a \in \mathfrak{g}_0. \tag{3.4}$$

Furthermore, it was shown in [KRW] that the element d , defined by (2.5) is primary of conformal weight 1, hence $[d_\lambda L] = \lambda d$ and $d_{(0)}L = [d_\lambda L]|_{\lambda=0} = 0$. Hence, the homology class of L defines an energy-momentum element of the vertex algebra $W^k(\mathfrak{g}, x, f)$, which is denoted again by L . Note that though, for a good datum, the W -algebra $W^k(\mathfrak{g}, x, f)$ is independent, up to isomorphism, of the choice of x with given f [AKM], the element L does depend on x .

As has been mentioned in the introduction, the explicit expressions of the elements $J^{(a)}$ which generate the subalgebra $W^k(\mathfrak{g}, x, f)$ of the vertex algebra $\overline{\mathcal{C}}^k(\mathfrak{g}, x, f)$, associated to a good datum, are known only for $a \in \mathfrak{g}_{-j}$, where $j = 0$ and $\frac{1}{2}$. In view of (1.6) it is important to find an explicit expression for $J^{\{f\}}$. This is the first main result of

the paper. The second main result is the formula $L = -\frac{1}{k+h^\vee} J^{\{f\}}$ in $W^k(\mathfrak{g}, x, f)$. Both results hold for an arbitrary datum (\mathfrak{g}, x, f, k) , satisfying (1.5).

Let

$$\Omega_0 = \sum_{j \in S_0} (\text{ad } u^j)(\text{ad } u_j).$$

Proposition 3.1. *The operator Ω_0 is diagonalizable on \mathfrak{g}_j for each $j > 0$.*

Proof. Choose a Cartan subalgebra \mathfrak{h} of \mathfrak{g}_0 ; it is a Cartan subalgebra of \mathfrak{g} , containing x . Choose a set of positive roots in \mathfrak{h}^* , compatible with the $\frac{1}{2}\mathbb{Z}$ -grading (1.1), and let e_i, f_i be the Chevalley generators of \mathfrak{g} . Then for each $j > 0$, the \mathfrak{g} -module \mathfrak{g}_j is the sum of lowest weight modules with the lowest weight vectors that are commutators of the e_i , such that $e_i \in \mathfrak{g}_{>0}$. Since the restriction of Ω_0 to each of these summands is diagonalizable, proposition follows. \square

Let $\rho_{>0}$ (resp. ρ_j) $\in \mathfrak{h}^* = \mathfrak{h}$ be the half of the difference between the sums of positive even and positive odd roots of \mathfrak{h} in $\mathfrak{g}_{>0}$ (resp. \mathfrak{g}_j). (We identify \mathfrak{g} with \mathfrak{g}^* using (\cdot, \cdot)).

Proposition 3.2. *The element $\rho_{>0}$ lies in the center of \mathfrak{g}_0 if (1.5) holds.*

Proof. Since $[L^{\text{ch}}_\lambda J^{(a)}] = (\partial + \lambda)(J^{(a)} - a) + \lambda^2(\rho_{>0} - h^\vee x|a)$ for $a \in \mathfrak{g}_0$, the Jacobi identity

$$[J^{(a)}_\lambda [J^{(b)}_\mu L]] - (-1)^{p(a)p(b)} [J^{(b)}_\mu [J^{(a)}_\lambda L]] = [[J^{(a)}_\lambda J^{(b)}]_{\lambda+\mu} L], \quad a, b \in \mathfrak{g}_0$$

is equivalent to the equation $0 = -(\lambda + \mu)^2(\rho_{>0}|[a, b])$. Hence $([\rho_{>0}, a]|b) = 0$ for all $b \in \mathfrak{g}_0$. It follows that $[\rho_{>0}, \mathfrak{g}_0] = 0$. \square

Theorem 3.1. *For the datum (\mathfrak{g}, x, f, k) , satisfying (1.5), the following element of $\mathcal{C}^k(\mathfrak{g}, x, f)$ is $d_{(0)}$ -closed:*

$$\begin{aligned} J^{\{f\}} = & J^{(f)} + \sum_{j \in S_{1/2}} (-1)^{p(j)} : \Phi^j J^{\{f, u_j\}} : - \frac{1}{2} \sum_{j \in S_0} : J^{(u^j)} J^{(u_j)} : \\ & - (k + h^\vee) \partial J^{(x)} + \partial J^{(\rho_{>0})} + \frac{k + h^\vee}{2} \sum_{j \in S_{1/2}} : \Phi^j \partial \Phi_j : . \end{aligned}$$

Theorem 3.2. *Assuming that the datum (\mathfrak{g}, x, f, k) satisfies (1.5), and that $k \neq -h^\vee$, the element $L + \frac{1}{k+h^\vee} J^{\{f\}}$ of $\mathcal{C}(\mathfrak{g}, x, f)$ is $d_{(0)}$ -exact. Consequently $L = -\frac{1}{k+h^\vee} J^{\{f\}}$ in $W^k(\mathfrak{g}, x, f)$.*

As an immediate consequence of (2.11), (2.12), (2.14)–(2.17) and Theorems 3.1 and 3.2, we obtain the following corollary.

Corollary 3.1. *Provided that the datum (\mathfrak{g}, x, f, k) is good and $k \neq -h^\vee$, one has a homomorphism of the vertex algebra $W^k(\mathfrak{g}, x, f)$ to the vertex algebra $V^{B_0}(\mathfrak{g}_0) \otimes$*

$F^{\text{nc}}(\mathfrak{g}_{1/2})$, such that

$$\begin{aligned}
 J^{\{a\}} &\mapsto a + \frac{(-1)^{p(a)}}{2} \sum_{j \in S_{1/2}} : \Phi^j \Phi_{[u_j, a]} : \text{ if } a \in \mathfrak{g}_0^f, \\
 J^{\{v\}} &\mapsto \sum_{i \in S_{1/2}} : [v, u_i] \Phi^i : - \frac{(-1)^{p(v)}}{3} \sum_{i, j \in S_{1/2}} : \Phi^i \Phi^j \Phi_{[u_j, [u_i, v]]} : \\
 &\quad - \sum_{i \in S_{1/2}} (k(v|u_i) + \kappa_{>0}(v, u_i)) \partial \Phi^i \text{ if } v \in \mathfrak{g}_{-1/2}^f, \\
 L &\mapsto \frac{1}{k+h^\vee} \left(\frac{1}{2} \sum_{j \in S_0} : u^j u_j : + \partial((k+h^\vee)x - \rho_{>0}) \right) - \frac{1}{2} \sum_{j \in S_{1/2}} : \Phi^j \partial \Phi_j : .
 \end{aligned}$$

Corollary 3.2. *If $k = -h^\vee$, then $J^{\{f\}}$ is a central element of $W^k(\mathfrak{g}, x, f)$.*

4. Proof of Theorem 3.1

Let U and V be finite-dimensional vector spaces over \mathbb{F} with a non-degenerate even pairing $\langle \cdot, \cdot \rangle : U \times V \rightarrow \mathbb{F}$. Choose dual bases $\{u_i\}_{i \in I}$ and $\{u^i\}_{i \in I}$ of U and V respectively, i.e. $\langle u_i, u^j \rangle = \delta_{i,j}$. Then for any $A \in \text{End } U$ and $B \in \text{End } V$ we have:

$$\text{str}_U A = \sum_{i \in I} (-1)^{p(i)} \langle Au_i, u^i \rangle, \quad \text{str}_V B = \sum_{i \in I} (-1)^{p(i)} \langle u_i, Bu^i \rangle, \quad (4.1)$$

where, as before, $p(i)$ stands for $p(u_i)$ ($= p(u^i)$). This will be used to prove the following lemma.

Lemma 4.1. *For $u, v \in \mathfrak{g}$ we have*

$$\kappa_{<0}(u, v) = \kappa_{>0}(u, v) - \text{str}_{\mathfrak{g}_{>0}} p_{>0} \text{ ad } [u, v], \quad (4.2)$$

$$\kappa_{>0}(u, v) = \frac{1}{2} (\kappa(u, v) - \kappa_0(u, v) + \text{str}_{\mathfrak{g}_{>0}} p_{>0} \text{ ad } [u, v]), \quad (4.3)$$

$$\kappa_0(u, v) = (\Omega_0 u | v) = (u | \Omega_0 v). \quad (4.4)$$

Proof. We may assume that $(\text{ad } u)(\text{ad } v)$ preserves the $\frac{1}{2}$ - Z -grading (1.1) and that $p(u) = p(v)$. In order to prove (4.2), we use (4.1) with

$$U = \mathfrak{g}_{>0}, \quad V = \mathfrak{g}_{<0}, \quad \langle \cdot, \cdot \rangle = (\cdot | \cdot), \quad A = p_{>0}(\text{ad } v)(\text{ad } u), \quad B = p_{>0}(\text{ad } u)(\text{ad } v).$$

We have by the second and then the first formula in (4.1):

$$\begin{aligned}
 \kappa_{<0}(u, v) &= \text{str}_{\mathfrak{g}_{<0}} B = \sum_{i \in S_{>0}} (-1)^{p(i)} (u_i | [u, [v, u^i]]) = \sum_{i \in S_{>0}} (-1)^{p(i)} ([u_i, u] | v) u^i \\
 &= (-1)^{p(u)p(v)} \sum_{i \in S_{>0}} (-1)^{p(i)} ([v, [u, u_i]] | u^i) = (-1)^{p(u)p(v)} \text{str}_{\mathfrak{g}_{>0}} A \\
 &= (-1)^{p(u)p(v)} \text{str}_{\mathfrak{g}_{>0}} \text{ ad } [v, u] + \text{str}_{\mathfrak{g}_{>0}} (\text{ad } u)(\text{ad } v) = -\text{str}_{\mathfrak{g}_{>0}} \text{ ad } [u, v] + \kappa_{>0}(u, v).
 \end{aligned}$$

Formula (4.3) follows from (4.2) since $\kappa(u, v) = \kappa_{>0}(u, v) + \kappa_0(u, v) + \kappa_{<0}(u, v)$. The proof of (4.4) is similar, by letting $U = V = \mathfrak{g}_0$, $A = (\text{ad } u)(\text{ad } v)$. \square

Lemma 4.2. *Let, as before, $\{u_i\}_{i \in S_{>0}}$ be a basis of $\mathfrak{g}_{>0}$ and $\{u^i\}_{i \in S_{>0}}$ the dual basis of $\mathfrak{g}_{<0}$, i.e. $(u_i|u^j) = \delta_{i,j}$, and let $v \in \mathfrak{g}_0$. Then*

$$\sum_{i \in S_{>0}} (-1)^{p(i)} [u_i, u^i] = 2\rho_{>0}, \tag{4.5}$$

$$\text{str}_{\mathfrak{g}_{>0}} \text{ad } v = 2(\rho_{>0}|v). \tag{4.6}$$

Proof. Since the LHS is independent on the choice of dual bases, we may take for $\{u_i\}_{i \in S_{>0}}$ the basis $\{e_\alpha\}_{\alpha \in \Delta_{>0}}$ of $\mathfrak{g}_{>0}$, so that the dual basis of $\mathfrak{g}_{<0}$ is $\{e_{-\alpha}\}_{\alpha \in \Delta_{>0}}$ with $(e_\alpha|e_{-\alpha}) = 1$, and hence $[e_\alpha, e_{-\alpha}] = \alpha$. Then (4.5) follows.

For (4.6) we have:

$$\text{str}_{\mathfrak{g}_{>0}} \text{ad } v = \sum_{i \in S_{>0}} (-1)^{p(i)} ([v, u_i]|u^i) = \sum_{i \in S_{>0}} (-1)^{p(i)} (v|[u_i, u^i]) = 2(v|\rho_{>0})$$

by (4.5). □

Denote by I, II, ..., IV the operator $d_{(0)}$, applied to each of the six terms in the RHS of the formula for $J^{(f)}$ in Theorem 3.1. We have to prove that the sum of these six elements of $\mathcal{C}^k(\mathfrak{g}, x, f)$ is equal to 0.

By formula (2.13) for $v = f$, element I is equal to the sum of the following four elements:

$$I_A = \sum_{j \in S_{3/2}} : \varphi^j \Phi_{[f, u_j]} : , \quad I_B = - \sum_{j \in S_1} (-1)^{p(j)} : \varphi^j J^{([f, u_j])} : , \tag{4.7}$$

$$I_C = - \sum_{j \in S_{1/2}} (-1)^{p(j)} : \varphi^j J^{([f, u_j])} : , \quad I_D = \sum_{j \in S_1} (k(f|u_j) + \kappa_{>0}(f, u_j)) \partial \varphi^j. \tag{4.8}$$

By (2.13) for $v = f$ and the last formula in (2.6), using that $d_{(0)}$ is an odd derivation of the vertex algebra $\mathcal{C}^k(\mathfrak{g}, x, f)$, one obtains that element II is equal to the sum of the following five elements:

$$II_A = \sum_{j \in S_{1/2}} (-1)^{p(j)} : \varphi^j J^{([f, u_j])} : , \quad II_B = \sum_{j \in S_{1/2}} \sum_{i \in S_{3/2}} (f|[[f, u_j], u_i]) : \Phi^j \varphi^i : , \tag{4.9}$$

$$II_C = \sum_{j \in S_{1/2}} \sum_{i \in S_1} (-1)^{p(j)(p(i)+1)} : \Phi^j \varphi^i \Phi_{[[f, u_j], u_i]} : , \tag{4.10}$$

$$II_D = - \sum_{i, j \in S_{1/2}} (-1)^{p(i)(p(j)+1)} : \Phi^j \varphi^i J^{([[f, u_j], u_i]) } : , \tag{4.11}$$

$$II_E = \sum_{i, j \in S_{1/2}} (k([[f, u_j], u_i]) + \kappa_{>0}([[f, u_j], u_i])) : \Phi^j \partial \varphi^i : . \tag{4.12}$$

It is easy to see that $I_A + II_B = 0$, and since also $I_C + II_A = 0$, we obtain

$$I + II = I_B + I_D + II_C + II_D + II_E. \tag{4.13}$$

Lemma 4.3. *One has*

(a) $II_C = 0$.

(b) $II_D = \sum_{i \in S_{1/2}} \sum_{k \in S_0} : \Phi_{[u^k, u_i]} \varphi^i J^{(u_k)} : .$

Proof. By (4.10), using that

$$\Phi_{[[f, u_j], u_k]} = \sum_{i \in S_{1/2}} \langle u_i, [[f, u_j], u_k] \rangle^{ne} \Phi^i,$$

we obtain:

$$\begin{aligned} II_C &= \sum_{i, j \in S_{1/2}} \sum_{k \in S_1} (-1)^{p(j)(p(k)+1)} \langle u_i, [[f, u_j], u_k] \rangle^{ne} : \Phi^j \varphi^k \Phi^i : \\ &= \sum_{i, j \in S_{1/2}} \sum_{k \in S_1} (-1)^{(p(i)+p(j))(p(k)+1)} \langle u_i, [[f, u_j], u_k] \rangle^{ne} : \Phi^j \Phi^i \varphi^k : \\ &= \sum_{i, j \in S_{1/2}} \sum_{k \in S_1} ([[f, u_i], [f, u_j]] | u_k) : \Phi^j \Phi^i \varphi^k : . \end{aligned}$$

If one exchanges i and j in the summation of the last expression, II_C doesn't change. On the other hand, looking at each summand in this expression, we see that it changes the sign, hence $II_C = -II_C$, proving (a).

By (4.11), using that, for $i, j \in S_{1/2}$,

$$[[f, u_j], u_i] = \sum_{k \in S_0} ([[f, u_j], u_i] | u^k) u_k,$$

we obtain:

$$II_D = - \sum_{i \in S_{1/2}} \sum_{k \in S_0} (-1)^{p(i)p(k)} : \Phi_{[u_i, u^k]} \varphi^i J^{(u_k)} :,$$

proving (b). □

Next, we treat the term III. For that introduce structure constants c_{ij}^k and $c_j^k(v)$ for $i, j, k \in S_{>0}$ and $v \in \mathfrak{g}_0$ by

$$[u_i, u_j] = \sum_k c_{ij}^k u_k, \quad [v, u_j] = \sum_k c_j^k(v) u_k.$$

Lemma 4.4. (a) *For $v \in \mathfrak{g}_0$ and $k \in S_{>0}$ one has:*

$$[\varphi^k_\lambda J^{(v)}] = \sum_{j \in S_{>0}} c_j^k(v) \varphi^j, \tag{4.14}$$

$$: \varphi^k J^{(v)} : - (-1)^{p(v)(p(k)+1)} : J^{(v)} \varphi^k : = \sum_{j \in S_{>0}} c_j^k(v) \partial \varphi^j. \tag{4.15}$$

(b) For $u \in \mathfrak{g}_0, v \in \mathfrak{g}_{1/2}$ and $j \in S_{>0}$ one has

$$::\Phi_v \varphi^j : J^{(u)}:: =: \Phi_v : \varphi^j J^{(u)}:: + \sum_{k \in S_{>0}} c_k^j(u) : (\partial \Phi_v) \varphi^k :, \tag{4.16}$$

$$::\varphi^j \Phi_v : J^{(u)}:: =: \varphi^j : \Phi_v J^{(u)}:: + \sum_{k \in S_{>0}} (-1)^{p(u)p(v)} c_k^j(u) : \varphi^k \partial \Phi_v : . \tag{4.17}$$

Proof. It uses the λ -bracket calculus, see [K2], [DSK]. Formula (4.14) follows by the non-commutative Wick formula, (4.15) by quasicommutativity, and (4.16) by quasiassociativity of a vertex algebra. As an example, we prove here (4.16). By quasiassociativity we have

$$\begin{aligned} &::\Phi_v \varphi^j : J^{(u)} : - : \Phi_v : \varphi^j J^{(u)}:: \\ &= (-1)^{p(u)(p(j)+1)} \left(\int_0^{-\partial} : \Phi_v [J^{(u)}_\lambda \varphi^j] d\lambda : - : \Phi_v \int_0^{-\partial} [J^{(u)}_\lambda \varphi^j] d\lambda : \right). \end{aligned}$$

Using (4.14), we obtain that the RHS is equal to

$$\begin{aligned} &- \sum_{k \in S_{>0}} c_k^j(u) \left(\int_0^{-\partial} : \Phi_v \varphi^k : d\lambda - : \Phi_v \int_0^{-\partial} \varphi^k d\lambda : \right) \\ &= \sum_{k \in S_{>0}} c_k^j(u) (\partial : \Phi_v \varphi^k : - : \Phi_v \partial \varphi^k :) = \sum_{k \in S_{>0}} c_k^j(u) : (\partial \Phi_v) \varphi^k :, \end{aligned}$$

proving (4.16) □

We have, by formula (2.13), for $i \in S_0$:

$$d_{(0)} J^{(u_i)} = \sum_{j \in S_1} (f|[u_i, u_j]) \varphi^j + \sum_{j \in S_{1/2}} (-1)^{p(i)(p(j)+1)} : \varphi^j \Phi_{[u_i, u_j]} : .$$

It follows that

$$d_{(0)} \sum_{i \in S_0} : J^{(u_i)} J^{(u_i)} := A_1 + A_2 + A_3 + A_4, \tag{4.18}$$

where

$$A_1 = \sum_{i \in S_0} \sum_{j \in S_1} (f|[u^i, u_j]) : \varphi^j J^{(u_i)} :, \tag{4.19}$$

$$A_2 = \sum_{i \in S_0} \sum_{j \in S_{1/2}} (-1)^{p(i)(p(j)+1)} : \varphi^j \Phi_{[u^i, u_j]} : J^{(u_i)} :, \tag{4.20}$$

$$A_3 = \sum_{i \in S_0} \sum_{j \in S_1} (-1)^{p(i)} (f|[u_i, u_j]) : J^{(u_i)} \varphi^j :, \tag{4.21}$$

$$A_4 = \sum_{i \in S_0} \sum_{j \in S_{1/2}} (-1)^{p(i)} (-1)^{p(i)(p(j)+1)} : J^{(u_i)} \varphi^j \Phi_{[u_i, u_j]} : . \tag{4.22}$$

In order to simplify expressions for those elements, recall the operator Ω_0 , defined by (3.4). By Proposition 3.1, this operator is diagonalizable in \mathfrak{g}_j . Hence we can choose

$u_i \in \mathfrak{g}_{1/2}$ (resp. \mathfrak{g}_1) to be eigenvectors of Ω_0 ; denote by a_i (resp. b_i) the corresponding eigenvalues.

We have by (4.16):

$$A_2 = \sum_{i \in S_0} \sum_{j \in S_{1/2}} (-1)^{p(i)(p(j)+1)} : \varphi^j \Phi_{[u^i, u_j]} J^{(u_i)} : + \sum_{k \in S_{1/2}} \sum_{i \in S_0} \sum_{j \in S_{1/2}} c_{i,k}^j : \varphi^k \partial \Phi_{[u^i, u_j]} : .$$

The first sum in this expression is equal to

$$A'_2 = \sum_{i \in S_0} \sum_{j \in S_{1/2}} : \Phi_{[u^i, u_j]} \varphi^j J^{(u_i)} : , \tag{4.23}$$

while the second sum is equal to

$$\sum_{k \in S_{1/2}} : \varphi^k \sum_{i \in S_0} \partial \Phi_{[u^i, [u_i, u_k]]} : .$$

Hence we obtain

$$A_2 = A'_2 + \sum_{k \in S_{1/2}} a_k : \varphi^k \partial \Phi_k : . \tag{4.24}$$

Next, we obtain, using (2.6), (4.14) and (4.21),

$$A_3 = A_1 - \sum_{k \in S_1} \sum_{i \in S_0} \sum_{j \in S_1} c_{i,k}^j (f|[u^i, u_j]) \partial \varphi^k ,$$

hence

$$A_3 = A_1 - \sum_{k \in S_1} b_k (f|u_k) \partial \varphi^k , \tag{4.25}$$

since

$$\sum_{j \in S_1} c_{i,k}^j (f|[u^i, u_j]) = (f|\Omega_0(u_k)) .$$

Finally, for A_4 , given by (4.22), we have, using (4.15) and (4.23):

$$A_4 = A'_2 - \sum_{k \in S_{1/2}} a_k : (\partial \varphi^k) \Phi_k : . \tag{4.26}$$

From (4.18)–(4.26) we obtain that the element III is equal to the sum of the four elements

$$III_A = - \sum_{i \in S_0} \sum_{j \in S_1} (f|[u^i, u_j]) : \varphi^j J^{(u_i)} : , \tag{4.27}$$

$$III_B = - \sum_{i \in S_0} \sum_{j \in S_{1/2}} : \Phi_{[u^i, u_j]} \varphi^j J^{(u_i)} : , \tag{4.28}$$

$$III_C = \frac{1}{2} \sum_{j \in S_1} b_j (f|u_j) \partial \varphi^j , \tag{4.29}$$

$$III_D = \frac{1}{2} \sum_{j \in S_{1/2}} a_j (: (\partial \varphi^j) \Phi_j : - : \varphi^j \partial \Phi_j :) . \tag{4.30}$$

We have:

$$III_A = \sum_{j \in S_1} (-1)^{p(j)} : \varphi^j \sum_{i \in S_0} ([f, u_j | u^i]) J^{(u_i)} :,$$

hence, by (4.7),

$$III_A = -I_B. \tag{4.31}$$

Using Lemma 4.3(b), we obtain

$$II_D = -III_B. \tag{4.32}$$

Hence, by (4.13) and Lemma 4.3, we have

$$I + II + III = I_D + II_E + III_C + III_D. \tag{4.33}$$

Next, we have

$$I_D + III_C = \sum_{j \in S_1} ((k + h^\vee)(f|u_j) + (\rho_{>0} |[f, u_j])) \partial \varphi^j, \tag{4.34}$$

$$\begin{aligned} II_E + III_D &= (k + h^\vee) \sum_{i \in S_{1/2}} : \Phi_i \partial \varphi^i : - \frac{1}{2} \sum_{i \in S_{1/2}} : (\partial \Phi_{\Omega_0 u_i}) \varphi^i : \\ &+ \sum_{i \in S_{1/2}} : \Phi_{[u_i, \rho_{>0}]} \partial \varphi^i :. \end{aligned} \tag{4.35}$$

Indeed, by (4.8) and (4.29), we have, using (4.3) and (2.9):

$$I_D + III_C = \sum_{j \in S_1} ((k + \frac{1}{2} b_j)(f|u_j) + h^\vee(f|u_j) - \frac{1}{2} \kappa_0(f, u_j) + \frac{1}{2} \text{str}_{\mathfrak{g}_{>0}} \text{ad} [f, u_j]) \partial \varphi^j.$$

Applying (4.4) and (4.6) to the RHS, we obtain (4.34).

In order to prove (4.35), we rewrite (4.30) as follows:

$$\begin{aligned} III_D &= \frac{1}{2} \sum_{k \in S_{1/2}} (: (\partial \varphi^k) \Phi_{\Omega_0 u_k} : - : \varphi^k \partial \Phi_{\Omega_0 u_k} :) \\ &= \frac{1}{2} \sum_{k \in S_{1/2}} (: \Phi_{\Omega_0 u_k} \partial \varphi^k : - : (\partial \Phi_{\Omega_0 u_k}) \varphi^k :). \end{aligned}$$

We also rewrite (4.12), using (4.3), (4.4) and (4.6), as follows:

$$II_E = (k + h^\vee) \sum_{i \in S_{1/2}} : \Phi_i \partial \varphi^i : - \frac{1}{2} \sum_{i \in S_{1/2}} : \Phi_{\Omega_0 u_i} \partial \varphi^i : + \sum_{i \in S_{1/2}} : \Phi_{[u_i, \rho_{>0}]} \partial \varphi^i :.$$

Adding up these two expressions, we get (4.35).

Lemma 4.5. *One has*

$$\Omega_0(u) = 2[\rho_{>0}, u] \text{ for } u \in \mathfrak{g}_{1/2}.$$

Proof. Since elements I, II, III lie in the image of $d_{(0)}$ and $d_{(0)}^2 = 0$, we obtain, using (4.33):

$$0 = d_{(0)}(I_D + III_C) + d_{(0)}(III_E + III_D).$$

Substituting here (4.34) and (4.35) and using formulas (2.6) for the action of $d_{(0)}$, we obtain:

$$0 = \sum_{i,j \in S_{1/2}} (-1)^{p(i)} \langle u_j, \frac{1}{2} \Omega_0(u_i) - [\rho_{>0}, u_i] \rangle^{ne} : \varphi^i \partial \varphi^j : .$$

Due to the non-degeneracy of the bilinear form $\langle \cdot, \cdot \rangle^{ne}$, the lemma follows. □

Finally, we treat the remaining three elements IV, V, and VI. Using (2.13), we obtain:

$$IV = -(k + h^\vee) \sum_{j \in S_1} (f|u_j) \partial \varphi^j - \frac{k + h^\vee}{2} \sum_{j \in S_{1/2}} \partial : \Phi_j \varphi^j : , \tag{4.36}$$

$$V = - \sum_{j \in S_1} (\rho_{>0} | [f, u_j]) \partial \varphi^j + \sum_{j \in S_{1/2}} \partial : \Phi_{[\rho_{>0}, u_j]} \varphi^j : . \tag{4.37}$$

Using (2.6), we obtain

$$VI = \frac{k + h^\vee}{2} \sum_{j \in S_{1/2}} (: \varphi^j \partial \Phi_j : - : \Phi_j \partial \varphi^j :). \tag{4.38}$$

Adding up (4.36)–(4.38), we obtain

$$\begin{aligned} IV + V + VI &= -(k + h^\vee) \sum_{j \in S_1} (f|u_j) \partial \varphi^j - \sum_{j \in S_1} (\rho_{>0} | [f, u_j]) \partial \varphi^j \\ &\quad - (k + h^\vee) \sum_{j \in S_{1/2}} : \Phi_j \partial \varphi^j : + \sum_{j \in S_{1/2}} \partial : \Phi_{[\rho_{>0}, u_j]} \varphi^j : . \end{aligned} \tag{4.39}$$

Adding up (4.33) and (4.39), and using (4.34), (4.35) and Lemma 4.5, we conclude that $d_{(0)} J^{\{j\}} = 0$, completing the proof of Theorem 3.1.

5. Proof of Theorem 3.2

First, introduce the following convenient notation. Let \mathfrak{a} (resp. \mathfrak{a}') be the sum of some \mathfrak{g}_j 's (resp. the remaining \mathfrak{g}_j 's) in (1.1). Then we let $\delta_{u, \mathfrak{a}} = 1$ (resp. 0) if $u \in \mathfrak{a}$ (resp. \mathfrak{a}'). Then we have for $u, v \in \mathfrak{g}$:

$$\sum_{i \in S_{>0}} (u_i | v) u^i = \delta_{v, \mathfrak{g}_{<0}} v; \quad \sum_{i \in S_{>0}} (v | u^i) u_i = \delta_{v, \mathfrak{g}_{>0}} v; \tag{5.1}$$

$$\sum_{i \in S_{>0}} (u | u^i) (u_i | v) = \delta_{u, \mathfrak{g}_{>0}} (u | v) = \delta_{v, \mathfrak{g}_{<0}} (u | v); \tag{5.2}$$

$$\sum_{i \in S_{>0}} (u | u^i) (v | u_i) = \delta_{u, \mathfrak{g}_{>0}} (v | u) = \delta_{v, \mathfrak{g}_{<0}} (v | u). \tag{5.3}$$

Similar formulas hold if we replace $S_{>0}$ by S_0 , and $\mathfrak{g}_{>0}$ and $\mathfrak{g}_{<0}$ by \mathfrak{g}_0 ; these formulas will be denoted by (5.1)', (5.2)' and (5.3)'.

Next, let v^{ch} denote the second summand on the right in (2.7). Then

$$v^{\text{ch}} = \sum_{i,j \in S_{>0}} (-1)^{p(i)} ([v, u_j] | u^i) : \varphi_i \varphi^j : . \tag{5.4}$$

Next, by condition (1.5), we have

$$u^i = [u^{(i)}, f], \quad i \in S_{1/2}, \tag{5.5}$$

where the $\{u^{(i)}\}_{i \in S_{1/2}}$ is a basis of $\mathfrak{g}_{1/2}$, dual to $\{u_i\}_{i \in S_{1/2}}$ with respect to the bilinear form (1.4).

Next, by the quasiassociativity of the normally ordered product, we have for $i, j, k, l \in S_{>0}$

$$:: \varphi_i \varphi^j : : \varphi_k : = : \varphi_i \varphi^j \varphi_k : + (-1)^{p(j)} \delta_{j,k} \partial \varphi_i ; \tag{5.6}$$

$$:: \varphi_i \varphi^j : : \varphi_k = : \varphi_i \varphi^j \varphi^k : + (-1)^{(p(i)+1)(p(j)+1)} \delta_{i,k} \partial \varphi^j ; \tag{5.7}$$

$$\begin{aligned} :: \varphi_i \varphi^j : : \varphi_k \varphi^l : &= : \varphi_i \varphi^j \varphi_k \varphi^l : + (-1)^{p(k)} \delta_{j,k} : (\partial \varphi_i) \varphi^l : \\ &- (-1)^{p(j)p(k)} (-1)^{p(i)(p(j)+p(k))} \delta_{i,l} : \varphi_k \partial \varphi^j : . \end{aligned} \tag{5.8}$$

Lemma 5.1. *We have, using (5.4):*

$$\sum_{i \in S_0} : u^i (u_i)^{\text{ch}} := \sum_{i,j \in S_{>0}} (-1)^{p(i)} : p_0([u_j, u^i]) \varphi_i \varphi^j := \sum_{i \in S_0} : (u^i)^{\text{ch}} u_i : ; \tag{5.9}$$

$$\begin{aligned} \sum_{i \in S_0} : (u^i)^{\text{ch}} (u_i)^{\text{ch}} &:= \sum_{i,j,k,l \in S_{>0}} (-1)^{p(i)+p(k)} ([u_i, u^k] | p_0[u_j, u^l]) : \varphi_i \varphi^j \varphi_k \varphi^l : \\ &+ \sum_{\substack{i,j,k \in S_{>0} \\ [u_k, u^i] \in \mathfrak{g}_0}} (-1)^{p(i)} (u_j | [u^k, [u_k, u^i]]) : (\partial \varphi_i) \varphi^j : - : \varphi_i \partial \varphi^j : . \end{aligned} \tag{5.10}$$

Proof. Using the invariance of the bilinear form $(\cdot | \cdot)$ and (5.1)', we obtain

$$\sum_{i \in S_0} : u^i (u_i)^{\text{ch}} := \sum_{j,k \in S_{>0}} (-1)^{p(j)} : p_0([u_k, u^j]) \varphi_j \varphi^k : ,$$

which is the first equality in (5.9) after replacing indices j, k by i, j . The proof of the second equality in (5.9) is the same.

Using the invariance of the bilinear form $(\cdot | \cdot)$ and (5.2)', we obtain

$$\sum_{i \in S_0} : (u^i)^{\text{ch}} (u_i)^{\text{ch}} := \sum_{j,k,r,s \in S_{>0}} (-1)^{p(j)+p(r)} ([u_s, u^r] | p_0[u_k, u^j]) : \varphi_j \varphi^k : : \varphi_r \varphi^s : .$$

Using (5.8), we see that this is equal to

$$\begin{aligned} &= \sum_{j,k,r,s \in S_{>0}} (-1)^{p(j)+p(r)} ([u_s, u^r] | p_0[u_k, u^j]) : \varphi_j \varphi^k \varphi_r \varphi^s : \\ &+ \sum_{j,k,s \in S_{>0}} (-1)^{p(j)} ([u_s, u^k] | p_0[u_k, u^j]) : (\partial \varphi_j) \varphi^s : \\ &- \sum_{j,k,s \in S_{>0}} (-1)^{p(j)} (u_s | [u^k, p_0[u_k, u^j]]) : \varphi_j \partial \varphi^s : . \end{aligned}$$

In the last term we used the invariance of $(\cdot|\cdot)$ and relabeling of indices; we also used that $(a|b) \neq 0$ implies that $p(a) = p(b)$ in order to simplify the sign. Now (5.10) easily follows. \square

Lemma 5.2. *Recalling that $[u_i, u_j] = \sum_k c_{ij}^k u_k$ for $i, j, k \in S_{>0}$, we have*

$$\sum_{i,j,k \in S_{>0}} (-1)^{p(i)+p(k)} c_{ij}^k : \varphi_k \varphi^j u^i := \sum_{i,j \in S_{>0}} (-1)^{p(i)} : p_{<0}([u_j, u^i]) \varphi_i \varphi^j : , \tag{5.11}$$

$$\sum_{i,k \in S_{>0}} (-1)^{p(i)} : [u_k, u^i] \varphi_i \varphi^k : - \sum_{i,j,k \in S_{>0}} (-1)^{p(i)+p(k)} c_{ij}^k : \varphi_k \varphi^j u^i : \tag{5.12}$$

$$= \sum_{i,j \in S_{>0}} (-1)^{p(i)} : p_{\geq 0}([u_j, u^i]) \varphi_i \varphi^j : .$$

Proof. Using that $c_{ij}^k = ([u_i, u_j]|u^k)$, that the bilinear form $(\cdot|\cdot)$ is invariant, equation (5.1), and that $p(i) + p(k) = p(j)$ if $c_{ij}^k \neq 0$, we obtain :

$$\sum_{i,j,k \in S_{>0}} (-1)^{p(i)+p(k)} c_{ij}^k : \varphi_k \varphi^j u^i := \sum_{j,k \in S_{>0}} (-1)^{p(j)} : \varphi_k \varphi^j p_{<0}[u_j, u^k] : ,$$

from which (5.11) follows.

By (5.11), the LHS of (5.12) is equal to

$$\left(\sum_{i,j \in S_{>0}} - \sum_{\substack{i,j \in S_{>0} \\ [u_j, u^i] \in \mathfrak{g}_{<0}}} \right) (-1)^{p(i)} : [u_j, u^i] \varphi_i \varphi^j : .$$

Formula (5.12) follows. \square

Lemma 5.3. *The expression*

$$A_{<0} = \sum_{i,j,k,l \in S_{>0}} (-1)^{p(i)+p(k)} ([u_l, u^k]|p_{<0}[u_j, u^i]) : \varphi_i \varphi^j \varphi_k \varphi^l :$$

is equal to $\frac{1}{2} A_{\neq 0}$, where $A_{\neq 0}$ is obtained from $A_{<0}$ by replacing $p_{<0}$ by $p_{\neq 0}$.

Proof. Exchanging i with k and j with l in $A_{<0}$, we obtain

$$A_{<0} = \sum_{i,j,k,l} (-1)^{p(i)+p(k)} ([u_l, u^k]|p_{>0}[u_j, u^i]) : \varphi_i \varphi^j \varphi_k \varphi^l : .$$

Adding the two expressions for $A_{<0}$, we obtain $A_{\neq 0}$. \square

From (5.4) we obtain

$$f^{\text{ch}} = \sum_{i,j \in S_{>0}} (-1)^{p(i)} (f|[u_j, u^i]) : \varphi_i \varphi^j : . \tag{5.13}$$

Lemma 5.4. *We have*

$$\sum_{i \in S_{1/2}} (-1)^{p(i)} : \Phi^i [f, u_i]^{\text{ch}} := \sum_{i, j \in S_{>0}} (-1)^{p(j)} : \Phi_{[u_j, u^i]} \varphi_i \varphi^j : .$$

Proof. Substituting in the LHS the expression (5.4) for $v = [f, u_i]$, we obtain, by invariance of (\cdot, \cdot) and (1.4),

$$\begin{aligned} & \sum_{i \in S_{1/2}} \sum_{j, k \in S_{>0}} (-1)^{p(i)+p(k)} \langle u_i, [u_j, u^k] \rangle^{\text{ne}} : \Phi^i \varphi_k \varphi^j : \\ &= \sum_{j, k \in S_{>0}} (-1)^{p(j)} \sum_{i \in S_{1/2}} \langle u_i, [u_j, u^k] \rangle^{\text{ne}} : \Phi^i \varphi_k \varphi^j : \\ &= \sum_{j, k \in S_{>0}} (-1)^{p(j)} : \Phi_{[u_j, u^k]} \varphi_k \varphi^j : , \end{aligned}$$

proving the lemma. □

Lemma 5.5. *Let*

$$\begin{aligned} P_0 &= -f - \sum_{i \in S_{1/2}} (-1)^{p(i)} : \Phi^i [f, u_i] : + 1/2 \sum_{i \in S_0} : u^i u_i : \\ &\quad - \partial \rho_{>0} + (k + h^\vee) \partial J^{(x)} - \frac{k + h^\vee}{2} \sum_{i \in S_{1/2}} : \Phi^i \partial \Phi_i : \\ &\quad - h^\vee \sum_{i \in S_{>0}} (-1)^{p(i)} : \varphi_i \partial \varphi^i : - \sum_{i, j, k \in S_{>0}} (-1)^{p(i)+p(k)} c_{ij}^k : \varphi_k \varphi^j u^i : \\ &\quad + \sum_{i, k \in S_{>0}} (-1)^{p(i)} : [u_k, u^i] \varphi_i \varphi^k : . \end{aligned}$$

Then

$$(k + h^\vee)L = d_{(0)} \left(\sum_{i \in S_{>0}} (-1)^{p(i)} : \varphi_i u^i : \right) + P_0. \tag{5.14}$$

Proof. By (3.2) we have

$$(k + h^\vee)L = \frac{1}{2} \sum_{j \in S} : u^j u_j : + (k + h^\vee) \partial x + (k + h^\vee)L^{\text{ch}} + (k + h^\vee)L^{\text{ne}}. \tag{5.15}$$

Choosing, as usual, dual bases $\{h_i\}$ and $\{h^i\}$, $i = 1, \dots, l$, of \mathfrak{h} and root vectors $\{e_\alpha\}_{\alpha \in \Delta_+}$, $\{e_{-\alpha}\}_{\alpha \in \Delta_+}$ of \mathfrak{g} , where $(e_\alpha | e_{-\alpha}) = 1$, we obtain, using quasicommutativity of the normally ordered product, that the first term in the RHS of (5.15) is

$$\begin{aligned} \sum_{\alpha \in \Delta_+} (-1)^{p(\alpha)} : e_\alpha e_{-\alpha} : + \frac{1}{2} \sum_{i=1}^l h^i h_i - \partial \rho &= \sum_{i \in S_{>0}} (-1)^{p(i)} : u_i u^i : \\ &\quad + \frac{1}{2} \sum_{i \in S_0} : u^i u_i : - \partial \rho_{>0}. \end{aligned} \tag{5.16}$$

We also have

$$\partial x^{\text{ch}} = \sum_{i \in S_{>0}} (-1)^{p(i)} m_i \partial (: \varphi_i \varphi^i :). \tag{5.17}$$

Using (5.16) and (5.17), equation (5.15) can be rewritten as follows:

$$\begin{aligned} (k + h^\vee)L &= \sum_{i \in S_{>0}} (-1)^{p(i)} : u_i u^i : + \frac{1}{2} \sum_{i \in S_0} : u^i u_i : - \partial \rho_{>0} \\ &+ (k + h^\vee) \partial J^{(x)} - (k + h^\vee) \sum_{i \in S_{>0}} (-1)^{p(i)} : \varphi_i \partial \varphi^i : + \frac{k + h^\vee}{2} \sum_{i \in S_{1/2}} : (\partial \Phi^i) \Phi_i : . \end{aligned} \tag{5.18}$$

Next, we compute $d_{(0)} (: \varphi_i u^i :)$, $i \in S_{>0}$, using (2.6) and that $d_{(0)}$ is an odd derivation of the normally ordered product:

$$\begin{aligned} d_{(0)} (: \varphi_i u^i :) &=: u_i u^i : + \sum_{j, k \in S_{>0}} (-1)^{p(k)} c_{ij}^k : \varphi_k \varphi^j u^i : + (f|u_i) u^i \\ &+ (-1)^{p(i)} : \Phi_{u_i} [u^{(i)}, f] : - k : \varphi_i \partial \varphi^i : - \sum_{k \in S_{>0}} : [u_k, u^i] \varphi_i \varphi^k : . \end{aligned}$$

We have used for the 3-rd term in the RHS that $(f|u_i) = 0$ if $p(i) \neq 0$, and formula (5.5) for the 4-th term. It follows that

$$\begin{aligned} \sum_{i \in S_{>0}} (-1)^{p(i)} d_{(0)} (: \varphi_i u^i :) &= \sum_{i \in S_{>0}} (-1)^{p(i)} : u_i u^i : + \sum_{i, j, k \in S_{>0}} (-1)^{p(i)+p(k)} c_{i,j}^k : \varphi_k \varphi^j u^i : \\ &+ f - \sum_{i \in S_{1/2}} (-1)^{p(i)} : \Phi^i [u_i, f] : \\ &- \sum_{i, k \in S_{>0}} (-1)^{p(i)} : [u_k, u^i] \varphi_i \varphi^k : - k \sum_{i \in S_{>0}} (-1)^{p(i)} : \varphi_i \partial \varphi^i : . \end{aligned} \tag{5.19}$$

We have used for the 3-rd term in the RHS that $f = \sum_{i \in S_{>0}} (-1)^{p(i)} (f|u_i) u^i$, and for the 4-th term that

$$- \sum_{i \in S_{1/2}} (-1)^{p(i)} : \Phi^i [u_i, f] : = \sum_{i \in S_{1/2}} (-1)^{p(i)} d_{(0)} (: \varphi_i u^i :).$$

Therefore $(k + h^\vee)L - \sum_{i \in S_{>0}} (-1)^{p(i)} d_{(0)} (: \varphi_i u^i :)$ is the difference of the right hand sides of equations (5.18) and (5.19), which is P_0 . □

Lemma 5.6. *We have*

$$P_0 = -J^{\{f\}} + P_1 ,$$

where

$$\begin{aligned}
 P_1 = & \sum_{i,j \in S_{>0}} (-1)^{p(i)} (f|[u_j, u^i]) : \varphi_i \varphi^j : + \sum_{i,j \in S_{>0}} (-1)^{p(j)} : \Phi_{[u_j, u^i]} \varphi_i \varphi^j : \\
 & - \frac{1}{2} \sum_{i,j,k,\ell \in S_{>0}} (-1)^{p(i)+p(k)} ([u_\ell, u^k]|p_0[u_j, u^i]) : \varphi_i \varphi^j \varphi_k \varphi^\ell : \\
 & - \frac{1}{2} \sum_{i,j,k \in S_{>0}} (-1)^{p(i)} (u_j|[u^k, p_0[u_k, u^i]]) (: \partial \varphi_i \varphi^j : - : \varphi_i \partial \varphi^j :) \\
 & + \sum_{i,j \in S_{>0}} (-1)^{p(i)} (\rho_{>0}|[u_j, u^i]) \partial : \varphi_i \varphi^j : \\
 & - h^\vee \sum_{i \in S_{>0}} (-1)^{p(i)} : \varphi_i \partial \varphi^i + \sum_{i,j \in S_{>0}} (-1)^{p(i)} : p_{>0}([u_j, u^i]) \varphi_i \varphi^j : .
 \end{aligned}$$

Consequently, by (5.14), we have

$$(k + h^\vee)L + J^{\{f\}} \equiv P_1 \pmod{\text{Im } d_{(0)}}. \tag{5.20}$$

Proof. First, we compute, using Lemma 5.1,

$$\begin{aligned}
 \sum_{i \in S_0} : J^{(u^i)} J^{(u_i)} : & = \sum_{i \in S_0} : u^i u_i : + 2 \sum_{i,j \in S_{>0}} (-1)^{p(i)} : p_0([u_j, u^i]) \varphi_i \varphi^j : \\
 & + \sum_{i,j,k,\ell \in S_{>0}} (-1)^{p(i)+p(k)} ([u_\ell, u^k]|p_0[u_j, u^i]) : \varphi_i \varphi^j \varphi_k \varphi^\ell : \\
 & + \sum_{i,j,k \in S_{>0}} (-1)^{p(i)} (u_j|[u^k, p_0[u_k, u^i]]) (: \partial \varphi_i \varphi^j : - : \varphi_i \partial \varphi^j :). \tag{5.21}
 \end{aligned}$$

Hence, for P_0 , defined in Lemma (5.5), and $J^{\{f\}}$, defined in Theorem 3.1, we have

$$P_0 + J^{\{f\}} = A, \tag{5.22}$$

where

$$\begin{aligned}
 A = & (f^{\text{ch}} + \sum_{i \in S_{1/2}} (-1)^{p(i)} : \Phi^i[f, u_i]^{\text{ch}} :) + \frac{1}{2} \sum_{i \in S_0} (: u^i u_i : - : J^{(u^i)} J^{(u_i)} :) \\
 & + \sum_{i,j \in S_{>0}} (-1)^{p(i)} (\rho_{>0}|[u_j, u^i]) \partial : \varphi_i \varphi^j : - h^\vee \sum_{i \in S_{>0}} (-1)^{p(i)} : \varphi_i \partial \varphi^i : \\
 & + \sum_{i,j \in S_{>0}} (-1)^{p(i)} : p_{\geq 0}([u_j, u^i]) \varphi_i \varphi^j : . \tag{5.23}
 \end{aligned}$$

Here we used Lemma 5.4 for the first term, formula (5.21) for the second term and formula (5.12) for the last term.

From (5.23) it is straightforward to deduce that $A = P_1$. This completes the proof of Lemma 5.6. □

Lemma 5.7. *Let*

$$\begin{aligned}
 P_2 = & -\frac{1}{2} \sum_{i,j,k \in S_{>0}} (-1)^{p(i)} (u_j [[u^k, p_0[u_k, u^i]]]) (\partial \varphi_i) \varphi^j : - : \varphi_i \partial \varphi^j : \\
 & + \sum_{i,j \in S_{>0}} (-1)^{p(i)} (\rho_{>0} [[u_j, u^i]]) \partial (\varphi_i \varphi^j) - h^\vee \sum_{i \in S_{>0}} (-1)^{p(i)} : \varphi_i \partial \varphi^i : \\
 & - \frac{1}{2} \sum_{i,j,k \in S_{>0}} (-1)^{p(i)} (u_j [[u^k, p_{<0}[u_k, u^i]]]) (\partial \varphi_i) \varphi^j : \\
 & + \sum_{i,j,k \in S_{>0}} (-1)^{p(i)} (u_j [[u^k, p_{>0}[u_k, u^i]]]) : \varphi_i \partial \varphi^j : .
 \end{aligned}$$

Then

$$P_2 = P_1 - \frac{1}{2} d_{(0)} \sum_{i,j \in S_{>0}} (-1)^{p(j)} : \varphi_{[u_i, u^j]} \varphi_j \varphi^i : .$$

Consequently, by (5.20), we have

$$(k + h^\vee)L + J^{\{f\}} \equiv P_2 \pmod{\text{Im } d_{(0)}}.$$

Proof. It is similar to that of Lemma 5.6, and therefore is omitted. □

Lemma 5.8. *Let*

$$\varphi = \sum_{i,j \in S_{>0}} (a_{ij} : (\partial \varphi_i) \varphi^j : + b_{ij} : \varphi_i \partial \varphi^j :),$$

where $a_{ij}, b_{ij} \in \mathbb{F}$.

Then $d_{(0)}\varphi = 0$ implies that $\varphi = 0$.

Proof. It is clear from (2.6). □

Now it is easy to complete the proof of Theorem 3.2. By Lemma 5.7, $(k + h^\vee)L + J^{\{f\}} \equiv P_2 \pmod{\text{Im } d_{(0)}}$, where $d_{(0)}P_2 = 0$ since $d_{(0)}L = 0 = d_{(0)}J^{\{f\}}$. But P_2 has the form of φ in Lemma 5.8, hence $P_2 = 0$.

6. Examples

6.1. Minimal W-algebras. Let $\theta \in \mathfrak{h}^* = \mathfrak{h}$ be the highest root for some ordering of roots of the Lie superalgebra \mathfrak{g} . The W-algebra $W^k(\mathfrak{g}, \theta/2, e_{-\theta})$ is called a *minimal W-algebra* [KRW], [KW] if the $\frac{1}{2}\mathbb{Z}$ -grading (1.1) has the form

$$\mathfrak{g} = \bigoplus_{j=-1}^1 \mathfrak{g}_j, \text{ where } \mathfrak{g}_{-1} = \mathbb{F}e_{-\theta}.$$

In this case $f = e_{-\theta}$ lies in the non-zero nilpotent orbit of minimal dimension in one of the simple components of $\mathfrak{g}_{\bar{0}}$. Conversely, if f lies in the non-zero orbit of minimal dimension in a simple component of $\mathfrak{g}_{\bar{0}}$, then the corresponding W-algebra is a minimal W-algebra in all cases, except when $\mathfrak{g} = osp(3|n)$ and the simple component of $\mathfrak{g}_{\bar{0}}$ is so_3 . Minimal W-algebras were studied in detail in [KRW] and [KW].

Obviously, for a minimal W -algebra, $\rho_1 = x$, and it follows from [KW], formulas (5.6), (5.10), that $\rho_{1/2} = (h^\vee - 2)x$. Hence,

$$\rho_{>0} = (h^\vee - 1)x. \tag{6.1}$$

Therefore, $\rho_{>0} - (k+h^\vee)x = -(k+1)x$, and the FFR, given by Corollary 3.1, coincides with that, given by [KW], Theorem 5.2.

6.2. *Principal W-algebras.* Let $\{e_*, \rho^\vee, f_*\}$ be a principal sl_2 -triple, where $x = \rho^\vee$ is the half of the sum of positive coroots of \mathfrak{g}_0 . Then the datum $(\mathfrak{g}, \rho^\vee, f_*, k)$ is a Dynkin datum. The corresponding W -algebra $W^k(\mathfrak{g}, \rho^\vee, f_*)$ is called the *principal W-algebra*, associated to \mathfrak{g} .

If \mathfrak{g} is a Lie algebra, then $\mathfrak{g}_{\pm 1/2} = 0$ and $\mathfrak{g}_0 = \mathfrak{h}$, and therefore

$$\rho_{>0} = \rho (\in \mathfrak{h}^* = \mathfrak{h}), \text{ and } \mathfrak{g}_0^{f_*} = 0, \tag{6.2}$$

where ρ is the half of the sum of positive roots of \mathfrak{g} . Hence the FFR in this case is a homomorphism $W^k(\mathfrak{g}, \rho^\vee, f_*) \rightarrow V^{B_0}(\mathfrak{h})$, for which

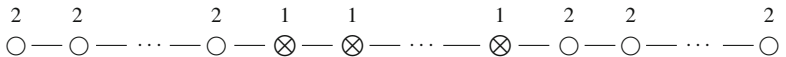
$$L \mapsto \frac{1}{2(k+h^\vee)} \sum_{j \in S_0} : u^j u_j : + \partial \rho^\vee - \frac{1}{k+h^\vee} \partial \rho. \tag{6.3}$$

The principal W -algebras for arbitrary simple Lie algebras were first constructed in [FF].

The element $2x$ is determined by its Dynkin labels $2\alpha_i(x)$, $i = 1, \dots, \text{rank } \mathfrak{g}$, which are known to take values 0, 1, and 2. In the case when \mathfrak{g} is a simple Lie algebra all the Dynkin labels of $2\rho^\vee$ are equal to 2.

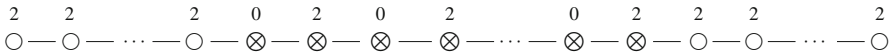
Let now \mathfrak{g} be a basic Lie superalgebra, which is not a Lie algebra. Then \mathfrak{g} may have several non-isomorphic sets of simple roots, and the Dynkin diagrams, corresponding to the choices of positive roots, compatible with the grading (1.1) may be different. Below we list the Dynkin labels $2\alpha_i(\rho^\vee)$, $i = 1, \dots, \text{rank } \mathfrak{g}$, for all basic Lie superalgebras \mathfrak{g} , which are not Lie algebras. For exceptional Lie superalgebras \mathfrak{g} they can be found in [H2]. We use notation for basic Lie superalgebras and their Dynkin diagrams from [K1].

I. $A(m, n)$, $m > n \geq 0$, $m - n = 2k + 1$, $k \in \mathbb{Z}_{\geq 0}$:



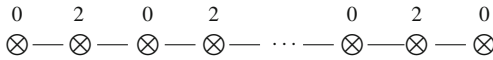
where the number of white nodes at the beginning and the end is equal to k , and the number of grey nodes is $2(n + 1)$.

II. $A(m, n)$, $m > n \geq 0$, $m - n = 2k$, $k \in \mathbb{Z}_{\geq 1}$:



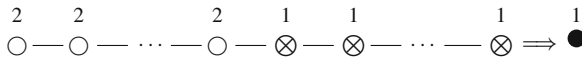
where the number of white nodes at the beginning (resp. end) is equal to k (resp. $k - 1$), and the number of grey nodes is $2(n + 1)$.

III. $A(m, m)$, $m \geq 1$:



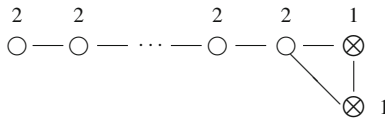
In all cases I–III the total number of nodes is $m + n + 1$.

IV. $B(m, n)$, $m \geq 0$, $n \geq 1$:



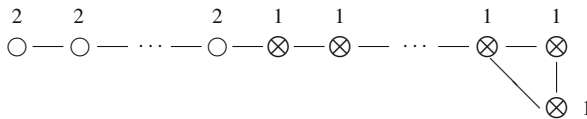
where the number of white nodes is $m - n$ if $m \geq n$, and is $n - m - 1$ if $m \leq n - 1$; the total number of nodes is $m + n$.

V. $C(n)$, $n \geq 3$:



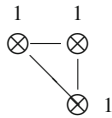
where the number of white nodes is $n - 2$.

VI. $D(m, n)$, $m \geq 2$, $n \geq 1$:



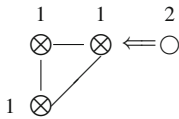
where the number of white nodes is $m - n - 1$, if $m \geq n + 1$, and is $n - m$ if $m \leq n$; the total number of nodes is $m + n$

VII. $D(2, 1; a)$:



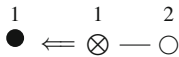
where the simple roots are $\{\epsilon_1 + \epsilon_2 + \epsilon_3, \epsilon_1 - \epsilon_2 - \epsilon_3, -\epsilon_1 - \epsilon_2 + \epsilon_3\}$.

VIII. $F(4)$:



where the simple roots are $\{\frac{1}{2}(\delta + \epsilon_1 - \epsilon_2 - \epsilon_3), \frac{1}{2}(\delta - \epsilon_1 + \epsilon_2 + \epsilon_3), \frac{1}{2}(-\delta + \epsilon_1 - \epsilon_2 + \epsilon_3), \epsilon_2 - \epsilon_3\}$.

IX. $G(3)$:



where the simple roots are $\{\delta, -\delta + \epsilon_1, \epsilon_2 - \epsilon_1\}$.

Looking at these diagrams, we see that for all basic Lie superalgebras \mathfrak{g} , except for $A(m, n)$ with $m - n$ even, which we shall exclude from consideration, the $\frac{1}{2}\mathbb{Z}$ -grading (1.1), corresponding to the principal nilpotent element, is defined by

$$\alpha_i(\rho^\vee) = 1 \text{ (resp } = \frac{1}{2}\text{) if } \alpha_i \text{ is even (resp. odd),}$$

where $\alpha_i, i = 1, \dots, \text{rank } \mathfrak{g}$, are simple roots. Hence this grading is compatible with the parity and $\mathfrak{g}_0 = \mathfrak{h}$. It follows that (6.2) still holds. Furthermore, $\mathfrak{g}_{1/2}$ (resp. $\mathfrak{g}_{-1/2}$) is a purely odd space, spanned by the e_{α_i} (resp. $e_{-\alpha_i}$), where the α_i are all odd simple roots. The element $f \in \mathfrak{g}_{-1}$ can be chosen as follows. Let f^0 (resp. f^1) be the sum of all $e_{-\alpha_i}$ with α_i even (resp. odd); then

$$f = f^0 + [f^1, f^1].$$

Remark 6.1. It is probably impossible to write down a complete FFR of an arbitrary W -algebra, with explicit expressions for all elements beyond those of conformal weight 1, $\frac{3}{2}$, and L . However, in many cases (including the minimal one) the W -algebra $W^k(\mathfrak{g}, x, f)$ is generated by elements of conformal weight 1 and $\frac{3}{2}$ (this happens, for example, when $\mathfrak{g}_{-1/2}^f$ generates $\mathfrak{g}_{<0}^f$). In such cases formulas (2.14) and (2.17) can be extended to the complete FFR of this W -algebra.

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7. Appendix A. A more Conceptual Proof of Theorem 3.2 for Dynkin Datum

The proof uses the following properties of the Dynkin datum:

- (i) the restriction of the bilinear form $(\cdot | \cdot)$ to \mathfrak{g}_0^f is non-degenerate;
- (ii) (even part of $\mathfrak{g}_{-1}) \cap (\text{center of } \mathfrak{g}^f) = \mathbb{F}f$;
- (iii) \mathfrak{g}_0^f is a direct sum of an abelian Lie algebra, basic Lie superalgebras, and $gl(n|n)$ with $n \geq 1$;
- (iv) $(\mathfrak{g}_0^f | x) = 0$.

Properties (i) and (iv) obviously hold. Property (ii) holds by the Brylinski-Kostant Theorem [BK], [P] in the Lie algebra case, and its analogue in the Lie superalgebra case, which follows from [H1] and [H2]. Property (iii) for classical (resp. exceptional) basic Lie superalgebras holds by [Ho] (resp. [H1]).

Lemma 7.1. *Let L be the element of $C^k(\mathfrak{g}, x, f)$, given by (3.1), and let $L' = -\frac{1}{k+h\nabla} J^{\{f\}}$, where $J^{\{f\}}$ is an element of $C^k(\mathfrak{g}, x, f)$ given by Theorem 3.1. Let $a \in \mathfrak{g}_0^f$ and $J^{\{a\}} \in C^k(\mathfrak{g}, x, f)$ be defined by (2.14). Then*

- (a) $[L_\lambda J^{\{a\}}] = (\partial + \lambda)J^{\{a\}} + \lambda^2(\rho_{>0}|a)$.
- (b) $[L'_\lambda J^{\{a\}}] = (\partial + \lambda)J^{\{a\}} + \lambda^2(\rho_{>0}|a)$.

Proof. From (2.11) and (2.14) we deduce

$$[J^{\{a\}}_\lambda J^{\{v\}}] = J^{\{[a,v]\}} + \lambda B_0(a, v), \quad a \in \mathfrak{g}_0^f, v \in \mathfrak{g}_{\leq 0}. \tag{7.1}$$

Using (2.14) we obtain

$$[J^{\{a\}}_\lambda \Phi_u] = (-1)^{p(a)} \Phi_{[a,u]}, \quad a \in \mathfrak{g}_0^f, u \in \mathfrak{g}_{1/2}. \tag{7.2}$$

From (7.1) and (7.2) we deduce, by making use of the non-commutative Wick formula, for $a \in \mathfrak{g}_0^f, i \in S_0, j \in S_{1/2}$:

$$\begin{aligned} [J^{\{a\}}_\lambda : J^{\{u^i\}} J^{\{u_i\}} :] &= J^{\{[a,u^i]\}} J^{\{u_i\}} : + (-1)^{p(a)p(i)} : J^{\{u^i\}} J^{\{[a,u_i]\}} : \\ &+ (-1)^{p(i)} \lambda J^{\{[u_i, [u^i, a]]\}} + \lambda(B_0(a, u^i)J^{\{u_i\}} + (-1)^{p(i)} B_0(a, u_i)J^{\{u^i\}}) + \frac{1}{2} \lambda^2 B_0(a, [u^i, u_i]), \\ [J^{\{a\}}_\lambda : \Phi^j \partial \Phi_j :] &= (-1)^{p(a)(p(j)+1)} \lambda : \Phi^j \Phi_{[a,u_j]} : + \frac{1}{2} (-1)^{p(a)} \lambda^2 \langle [a, u^j], u_j \rangle^{\text{ne}} \end{aligned}$$

$$\begin{aligned}
 &+(-1)^{p(a)} \sum_{i \in S_{1/2}} (\langle u_i, [a, u^{(j)}] \rangle >^{ne} : \Phi^i \partial \Phi_j : - \langle u_j, [a, u^{(i)}] \rangle >^{ne} : \Phi^j \partial \Phi_i :), \\
 &[J^{[a]}_\lambda : \Phi^j J^{([f, u_j])} :] = (-1)^{p(a)} : \Phi_{[a, u_j]} J^{([f, u_j])} : + (-1)^{p(a)p(j)} : \Phi^j J^{([a, [f, u_j]])} : .
 \end{aligned}$$

Summing up both sides of the first formula over $i \in S_0$, the second and third formula over $j \in S_{1/2}$, we obtain the following three formulas for $a \in \mathfrak{g}_0^f$:

$$\sum_{i \in S_0} [J^{[a]}_\lambda : J^{(u^i)} J^{(u_i)} :] = 2(k + h^\vee)\lambda J^{(a)}, \tag{7.3}$$

$$\sum_{j \in S_{1/2}} [a^{ne} : \Phi^j \partial \Phi_j :] = -2\lambda a^{ne}, \tag{7.4}$$

where a^{ne} is the second term on the right in (2.14),

$$\sum_{j \in S_{1/2}} [J^{[a]}_\lambda (-1)^{p(j)} : \Phi^j J^{([f, u_j])} :] = 0. \tag{7.5}$$

Using (7.1), we obtain

$$[J^{[a]}_\lambda J^{(f)}] = 0 = [J^{[a]}_\lambda J^{(x)}] \text{ for } a \in \mathfrak{g}_0^f. \tag{7.6}$$

Using Proposition 3.2, we obtain from (7.1):

$$[J^{[a]}_\lambda J^{(\rho_{>0})}] = \lambda(k + h^\vee)(a|_{\rho_{>0}}). \tag{7.7}$$

Now we can complete the proof of the lemma. Formula (a) is straightforward by the discussion in Sect. 3, cf. (3.4). Below we shall prove (b). We have for $a \in \mathfrak{g}_0^f$:

$$\begin{aligned}
 [J^{[a]}_\lambda J^{[f]}] &= [J^{[a]}_\lambda J^{(f)}] + \sum_{i \in S_{1/2}} (-1)^{p(i)} [J^{[a]}_\lambda : \Phi^i J^{([f, u_i])} :] \\
 &- \frac{1}{2} \sum_{i \in S_0} [J^{[a]}_\lambda : J^{(u^i)} J^{(u_i)} :] - (k + h^\vee) [J^{[a]}_\lambda \partial J^{(x)}] \\
 &+ [J^{[a]}_\lambda \partial J^{(\rho_{>0})}] + \frac{1}{2}(k + h^\vee) \sum_{i \in S_{1/2}} [J^{[a]}_\lambda : \Phi^i \partial \Phi_i :].
 \end{aligned}$$

The first and the fourth terms on the right are equal to 0 by (7.6), and the second term equals 0 by (7.5). The third term equals $-(k + h^\vee)\lambda J^{(a)}$ by (7.3). The fifth term equals $\lambda^2(k + h^\vee)(a|_{\rho_{>0}})$ by (7.7). The sixth term equals $-(k + h^\vee)\lambda a^{ne}$ by (7.4). Thus, we have :

$$\begin{aligned}
 [J^{[f]}_\lambda J^{[a]}] &= -[J^{[a]}_{-\partial-\lambda} J^{[f]}] = -(k + h^\vee)(\partial + \lambda)J^{[a]} - \lambda^2(k + h^\vee)(a|_{\rho_{>0}}) \\
 &= -(k + h^\vee)((\partial + \lambda)J^{[a]} + \lambda^2(a|_{\rho_{>0}})),
 \end{aligned}$$

proving formula (b). Here we have used property (iv) of a Dynkin data. □

Remark 7.1. The same proof shows that Lemma 7.1 holds for arbitrary grading (1.1), satisfying (1.5), if we replace the coefficient of λ^2 by $(\rho_{>0}|a) - (k + h^\vee)(x|a)$.

Lemma 7.2. *Let M be an element of conformal weight 2 (with respect to L), which lies in the subalgebra of $\mathcal{C}^k(\mathfrak{g}, x, f)$, generated by the elements $J^{\{a\}}$, $a \in \mathfrak{g}_0^f$. Suppose that*

$$[J^{\{a\}}_\lambda M] = 0 \text{ for all } a \in \mathfrak{g}_0^f. \tag{7.8}$$

Then $M = 0$.

Proof. By property (iii) of a Dynkin datum, the subalgebra of $\mathcal{C}^k(\mathfrak{g}, x, f)$ generated by the elements $J^{\{a\}}$, $a \in \mathfrak{g}_0^f$, is the affine vertex algebra $V^{B_{1/2}}(\mathfrak{g}_0^f)$ (see (2.15), (2.16)), associated to the Lie superalgebra \mathfrak{g}_0^f , which is a direct sum of its center, some basic Lie superalgebras and Lie superalgebras $gl(n|n)$, $n \geq 1$, and a non-degenerative bilinear form on it, for all, but finitely many values of k . Therefore, for all, but finitely many values of k , this vertex algebra carries the Sugawara element, with respect to which all elements $J^{\{a\}}$, $a \in \mathfrak{g}_0^f$, have conformal weight 1 (see e.g. [K2]); for $gl(n|n)$ we use the modification of the Sugawara construction, discussed in Appendix B. Hence, the center of $V^{B_{1/2}}(\mathfrak{g}_0^f)$ consists of the multiples of the vacuum vector. Since condition (7.8) implies that M is a central element of $V^{B_{1/2}}(\mathfrak{g}_0^f)$, we conclude that $M = 0$ for all, but finitely many k , hence for all k . \square

Now we can complete the proof of Theorem 3.2. Since L has conformed weight 2 (hereafter conformal weight is meant with respect to L), by [KW], Theorem 4.1(a), L is $d_{(0)}$ -equivalent to an element of the form

$$J^{\{f'\}} = J^{\{f'\}} + M_1, \tag{7.9}$$

where $f' \in \mathfrak{g}_{-1}^f$ and M_1 lies in the subalgebra of $\mathcal{C}^k(\mathfrak{g}, x, f)$ generated by the $J^{\{a\}}$ with $a \in \mathfrak{g}_0 + \mathfrak{g}_{-1/2}$ and the Φ_i , $i \in S_{1/2}$. Let $v \in \mathfrak{g}^f$ and let $J^{\{v\}} \in \overline{\mathcal{C}}^k(\mathfrak{g}, x, f)$ be an element, given by [KW], Theorem 4.1(a). Since $J^{\{f'\}}$ is $d_{(0)}$ -equivalent to L , we have in $W^k(\mathfrak{g}, x, f)$

$$[J^{\{f'\}}_\lambda J^{\{v\}}] = \partial J^{\{v\}} + O(\lambda). \tag{7.10}$$

But, due to (2.8), equation (7.10) is an equality in $\overline{\mathcal{C}}^k(\mathfrak{g}, x, f)$. It is easy to conclude from (7.9), using (2.11) (which can be used since $f' \in \mathfrak{g}_{-1}$ and $v \in \mathfrak{g}_{\leq 0}$), that we have the following equality in $\overline{\mathcal{C}}^k(\mathfrak{g}, x, f)$:

$$[J^{\{f'\}}_\lambda J^{\{v\}}] = J^{\{[f', v]\}} + A + B + O(\lambda), \tag{7.11}$$

where A is a linear combination of elements of the form $\sum : \partial^{i_1} J^{\{v_1\}} \dots \partial^{i_n} J^{\{v_n\}} :$ with $n \geq 2$ or $i_1 + \dots + i_n \geq 1$, and B is a linear combination of normally ordered products, involving the Φ_i . Comparing (7.10) and (7.11), we conclude that

$$[f', \mathfrak{g}^f] = 0.$$

Hence, by condition (ii), we have

$$f' = \gamma f \text{ for some } \gamma \in \mathbb{F}.$$

Therefore, $L - \gamma J^{(f)}$ is a sum of normally ordered products of elements of $\overline{C}^k(\mathfrak{g}, x, f)$ of conformal weight < 2 . It follows that, for $J^{(f)}$ from Theorem 3.1, we have

$$L - \gamma J^{(f)} = M, \tag{7.12}$$

where M is a $d_{(0)}$ -closed element of conformal weight 1. Hence, by [KW], Theorem 4.1(b), M is a linear combination of elements of the form $: J^{(a)} J^{(b)} :$ and $\partial J^{(c)}$, where $a, b, c \in \mathfrak{g}_0^f$. By Lemma 7.1, M satisfies (7.8), hence, by Lemma 7.2, $M = 0$, and we have

$$L = \gamma J^{(f)}. \tag{7.13}$$

In order to complete the proof of Theorem 3.2, it remains to show that

$$\gamma = -\frac{1}{k + h^\vee}. \tag{7.14}$$

For that we use the following formula, which can be deduced from the discussion of properties of L in Sect. 3 :

$$[L_\lambda J^{(v)}] = (\partial + \Delta_v \lambda) J^{(v)} + \lambda^2 (\rho_{>0} | v), \quad v \in \mathfrak{g}, \tag{7.15}$$

where Δ_v is the conformal weight of $J^{(v)}$. Using formula (7.15), we obtain

$$[L_\lambda \gamma J^{(f)}] = (\partial + 2\lambda) \gamma J^{(f)} + \lambda^3 \left(-\gamma (k + h^\vee) \frac{c(\mathfrak{g}, x, k)}{12} + \gamma \beta \right). \tag{7.16}$$

where

$$\beta = (\rho_{>0} | \rho_{>0}) - (\rho | \rho) + \frac{1}{6} (\rho_{1/2} | \rho_{>0}) + \frac{1}{24} \text{str}_{\mathfrak{g}_0 \oplus \mathfrak{g}_{1/2}} \Omega_0.$$

Using (7.13) and comparing (7.16) with

$$[L_\lambda L] = (\partial + 2\lambda) L + \frac{\lambda^3}{12} c(\mathfrak{g}, x, k),$$

where $c(\mathfrak{g}, x, k)$, given by (3.3), is non-zero for generic k , we obtain (7.14) (and also that $\beta = 0$). This completes the proof of Theorem 3.2.

8. Appendix B. Theorem 3.2 for $\mathfrak{g} = \mathfrak{gl}(n|n)$

Let \mathfrak{g} be a finite-dimensional Lie superalgebra over \mathbb{F} with an even invariant supersymmetric bilinear form $(\cdot | \cdot)$. In order to apply the Sugawara construction, we need two properties:

- (i) the bilinear form $(\cdot | \cdot)$ is non-degenerate, so that we can choose dual bases $\{u_i\}$ and $\{u^i\}$ of \mathfrak{g} with respect to this form and construct the Casimir operator

$$\Omega = \sum_i u^i u_i \in U(\mathfrak{g});$$

- (ii) the Casimir operator Ω acts on \mathfrak{g} as a scalar (which we denoted by $2h^\vee$).

In this Appendix we consider $\mathfrak{g} = gl(n|n)$ with the bilinear form $(a|b) = \text{str } ab$. The property (i) holds, but (ii) fails. However, we will show that the Sugawara operator $L^{\mathfrak{g}}$, appearing in the formula (3.2) for L can be modified, so that the resulting modified L is a Virasoro vector, which satisfies a modified Theorem 3.2 with the modified $J^{\{f\}}$.

Let I be the identity matrix in \mathfrak{g} and let

$$\omega = \sum_i : u^i u_i : \in V^k(\mathfrak{g}). \tag{8.1}$$

The following formulas are obtained by straightforward computations, where $a \in \mathfrak{g}$:

$$\Omega(a) = -2(a|I)I; \tag{8.2}$$

$$[a_\lambda \omega] = 2\lambda k a - 2\lambda(a|I)I; \tag{8.3}$$

$$[a_\lambda I] = \lambda k(a|I); \tag{8.4}$$

$$[a_\lambda : I^2 :] = 2\lambda k(a|I)I. \tag{8.5}$$

Introduce the modified Sugawara operator

$$L^{\mathfrak{g}} = \frac{1}{2k}\omega + \frac{1}{2k^2} : I^2 :. \tag{8.6}$$

It is straightforward to deduce from (8.3)–(8.5) the following two formulas:

$$[a_\lambda L^{\mathfrak{g}}] = \lambda a, \text{ hence } [L^{\mathfrak{g}} a_\lambda] = (\partial + \lambda)a; \tag{8.7}$$

$$[L^{\mathfrak{g}} a_\lambda L^{\mathfrak{g}}] = (\partial + 2\lambda)L^{\mathfrak{g}}. \tag{8.8}$$

Hence, we have the following proposition.

Proposition 8.1. *The element $L^{\mathfrak{g}}$ defined by (8.6) is a Virasoro vector of $V^k(\mathfrak{g})$ with central charge 0, for which $a \in \mathfrak{g}$ have conformal weight 1. Hence, $V^k(\mathfrak{g})$ is a conformal vertex algebra of CFT type for all $k \neq 0$.*

Finally, we have the following version of Theorem 3.2 for $\mathfrak{g} = gl(n|n)$.

Theorem 8.1. *Let $L^{\mathfrak{g}}$ be defined by (8.6), let L be defined by (3.2) with this $L^{\mathfrak{g}}$, and let $\tilde{J}^{\{f\}}$ be the element, defined in Theorem 3.1 for $h^\vee = 0$. Then the element*

$$J^{\{f\}} = \tilde{J}^{\{f\}} - \frac{1}{2k} : I^2 :$$

is $d_{(0)}$ -closed and $L = -\frac{1}{k}J^{\{f\}}$ is an energy-momentum vector of $W^k(\mathfrak{g}, x, f)$ with central charge given by formula (3.3) with $h^\vee = 0$.

Proof. By the same proof as that of Theorem 3.1, the element $\tilde{J}^{\{f\}}$ is $d_{(0)}$ -closed, and also, by (2.6), the element I is $d_{(0)}$ -closed, hence the same holds for $: I^2 :$. Hence the element $J^{\{f\}}$ is $d_{(0)}$ -closed.

Let $\tilde{L} = \frac{1}{2k}\omega + \partial x + L^{ch} + L^{ne}$ (cf. (3.2)). By the same proof as that of Theorem 3.2, we have: $\tilde{L} + \frac{1}{k}\tilde{J}^{\{f\}}$ is $d_{(0)}$ -exact. Since this element coincides with $L + \frac{1}{k}J^{\{f\}}$, the theorem is proved. □

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