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Spinors and mass on weighted manifolds

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Abstract: This paper generalizes classical spin geometry to the setting of weighted manifolds (manifolds with density) and provides applications to the Ricci flow. Spectral properties of the naturally associated weighted Dirac operator, introduced by Perelman, and its relationship with the weighted scalar curvature are investigated. Further, a generalization of the ADM mass for weighted asymptotically Euclidean (AE) manifolds is defined; on manifolds with nonnegative weighted scalar curvature, it satisfies a weighted Witten formula and thereby a positive weighted mass theorem. Finally, on such manifolds, Ricci flow is the gradient flow of said weighted ADM mass, for a natural choice of weight function. This yields a monotonicity formula for the weighted spinorial Dirichlet energy of a weighted Witten spinor along Ricci flow.

0. Introduction

Manifolds *with density*, or *weighted* manifolds, have long appeared in mathematics. A weighted manifold is a Riemannian manifold (M, g) endowed with a function $f : M \to \mathbb{R}$, defining the measure $e^{-f} dV_g$. After being introduced by Lichnerowicz in [Lic1,Lic2], more recent attention has been given to the differential geometry of weighted manifolds, including a generalization of Ricci curvature. A central idea of Perelman's spectacular proofs [P] required considering manifolds with density and their evolution. This led him to introduce a notion of weighted scalar curvature which is *not* the trace of the weighted Ricci curvature of Bakry-Émery. Sometimes called the P-scalar curvature, this weighted scalar curvature has only been moderately studied; see for instance [Fa,AC,LM,D,BH].

This paper shows that the intimate relationship between scalar curvature and the Dirac operator generalizes naturally to the weighted scalar curvature and an associated weighted Dirac operator, defined below. Well-known theorems relating scalar curvature and the Dirac operator include Friedrich's eigenvalue estimate [Fr1], Witten's proof

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Table 1. Classical vs. weighted quantities

	Riemannian	with density
Volume form	dV	$e^{-f}\,dV$
Ricci curvature	Ric	$\mathrm{Ric}_f := \mathrm{Ric} + \mathrm{Hess}_f$
Scalar curvature	R	$\mathrm{R}_f := \mathrm{R} + 2\Delta f - \lvert\nabla f\rvert^2$
Hilbert-Einstein fct.	$\mathrm{HE} := \int_M \mathrm{R}\,dV$	$\mathcal{F}(f) := \int_M \mathrm{R}_f\, e^{-f}\,dV$
Einstein's tensor	$\mathrm{E} := \mathrm{Ric} - \frac{\mathrm{R}}{2} g$	$\mathrm{E}_f := \mathrm{Ric}_f - \frac{\mathrm{R}_f}{2} g$
Divergence	div	$\mathrm{div}_f(h) := \mathrm{div}(h) - h(\nabla f, \cdot)$
Bianchi identity	$\mathrm{div}(\mathrm{E}) = 0$	$\mathrm{div}_f(\mathrm{E}_f) = 0$
Einstein metric	$\mathrm{Ric} = \Lambda g$	$\mathrm{Ric}_f = \Lambda g$
Mean curvature	H	$H_f := H - \nabla_\nu f$
Dirac operator*	D	$D_f := D - \frac{1}{2}(\nabla f)\cdot$
Lichnerowicz formula*	$D^2 = -\Delta + \frac{1}{4}\mathrm{R}$	$D_f^2 = -\Delta_f + \frac{1}{4}\mathrm{R}_f$
Ricci identity*	$[D, \nabla_X] = \frac{1}{2}\mathrm{Ric}(X)\cdot$	$[D_f, \nabla_X] = \frac{1}{2}\mathrm{Ric}_f(X)\cdot$
Dirac spinor*	ψ s.t. $D\psi = 0$	$\psi_f := e^{-\frac{f}{2}}\psi$ s.t. $D_f\psi_f = 0$
Eigenvalue bound*	$\lambda(D)^2 \geqslant \frac{n}{4(n-1)} \min \mathrm{R}$	$\lambda(D)^2 = \lambda(D_f)^2 \geqslant \frac{n}{4(n-1)} \min \mathrm{R}_f$
ADM mass*	$\mathfrak{m} = \lim_{\rho\to\infty} \int_{S_\rho} (\partial_i g_{ij} - \partial_j g_{ii})\, dA_j$	$\mathfrak{m}_f := \mathfrak{m} + 2 \lim_{\rho\to\infty} \int_{S_\rho} \langle \nabla f, \nu\rangle\, e^{-f}\, dA$
Witten formula*	$\mathfrak{m} = 4\int_M \left(\lvert\nabla\psi\rvert^2 + \frac{1}{4}\mathrm{R}\,\lvert\psi\rvert^2\right) dV$	$\mathfrak{m}_f = 4\int_M \left(\lvert\nabla\psi\rvert^2 + \frac{1}{4}\mathrm{R}_f\,\lvert\psi\rvert^2\right) e^{-f}\, dV$

Contributions from this paper are labeled with an asterisk (*)

of the positive mass theorem [W1], Gromov-Lawson's obstructions to positive scalar curvature [GL], and the Seiberg-Witten theory [W3]. Here, the first two of said theorems are generalized and then applied to the Ricci flow.

Aside from their applications in Ricci flow, weighted manifolds have proven extremely useful in the context of diffusion operators in analysis and probability theory, starting with Bakry and Émery's celebrated article [BE]. In a more classical Riemannian geometry context, Cheeger-Colding showed that limits of collapsing manifolds are naturally endowed with densities. Such densities differ from those defined by the Riemannian volume form, and the natural object of study is a *metric measure space*. See also the many extensions to the theory of (R)CD spaces started in [LV, S].

In physics, manifolds with density appear in a number of theories arising from Kaluza-Klein compactifications, via the mechanism of dimensional reduction. The closest to the purpose of this paper is probably Brans-Dicke theory, which motivates the study of manifolds with density and Bakry-Émery's notion of (weighted) Ricci curvature in [GW, WW, LMO]. Also, the weighted version of the Hilbert-Einstein action, introduced by Perelman, appears as the Lagrangian in several gravitational theories; this fact was noted in [CCD+], for instance.

Table 1 gives a summary comparison between classical and weighted quantities. The weighted quantities are typically better behaved than their Riemannian counterparts as one can choose a geometrically meaningful density. This idea can be seen as the core of Perelman's proofs [P]. In the context of scalar curvature and mass questions, proofs often employ a conformal change of the metric to reach *constant* scalar curvature, significantly changing the geometry; see [CP] for a survey of this technique. In contrast, on weighted manifolds, the idea is rather to fix the background geometry while varying the weight in order to obtain a metric with *constant* weighted scalar curvature.

0.1. Weighted Dirac Operator. Section 1 extends classical spin geometry theory to weighted manifolds. The new mathematical object introduced in this section is the weighted Dirac operator,

$$D_f = D - \frac{1}{2}(\nabla f)\cdot \tag{0.1}$$

The ∇f term acts by Clifford multiplication, and D denotes the standard (unweighted) Dirac operator. The weighted Dirac operator is self-adjoint with respect to the weighted L^2-inner product and is unitarily equivalent to the standard Dirac operator; see Proposition 1.20.

Differential operators naturally associated with weighted measures have proven invaluable in analysis and geometry. Of particular note is the weighted Laplacian, $\Delta_f = \Delta - \nabla_{\nabla f}$, also called the drift Laplacian, f-Laplacian, or Witten Laplacian. When $f = \frac{|x|^2}{4}$ on $\mathbb{R}^n$, then Δ_f is the Ornstein-Uhlenbeck operator. Weighted Laplace operators have been used in Ricci and mean curvature flow to analyze solitons [CM,CZ,MW], and by Witten in his study of Morse theory [W2], for example.

Proposition 1.8 proves a weighted Lichnerowicz formula involving the weighted scalar curvature,

$$D_f^2 = -\Delta_f + \frac{1}{4}\mathrm{R}_f\,. \tag{0.2}$$

Proposition 1.15 proves a weighted Ricci identity involving the Bakry-Émery Ricci curvature,

$$[D_f, \nabla_X] = \frac{1}{2}\mathrm{Ric}_f(X)\cdot \tag{0.3}$$

Theorem 1.23 generalizes the classical lower bound for Dirac eigenvalues to the weighted setting: on a closed, weighted spin manifold, any eigenvalue λ of D_f satisfies

$$\lambda^2 \geqslant \frac{n}{4(n-1)}\min \mathrm{R}_f\,. \tag{0.4}$$

Furthermore, the same lower bound also holds for eigenvalues of the standard Dirac operator.

Forthcoming work will study weighted spin manifolds with boundary [BO2].

0.2. Weighted Asymptotically Euclidean Manifolds. A fundamental quantity associated with an asymptotically Euclidean (AE) manifold (M^n, g) is the ADM mass [ADM], denoted $\mathfrak{m}(g)$. Section 2 introduces a quantity extending the ADM mass to the weighted setting: the *weighted mass* of an AE manifold with weight function f is defined as

$$\mathfrak{m}_f(g) := \mathfrak{m}(g) + 2\lim_{\rho\to\infty}\int_{S_\rho}\langle\nabla f, \nu\rangle\, e^{-f}\, dA, \tag{0.5}$$

where S_ρ is a coordinate sphere of radius ρ with outward normal ν and area form dA. The normalization for $\mathfrak{m}$ used in this paper is related to Bartnik's [B] by $\mathfrak{m} = c_n m_{\mathrm{ADM}}$, where $c_n = 2(n-1)\omega_{n-1}$ and ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$; this simplifies the formulas to follow.

Theorem 2.5 shows that the weighted mass of a spin manifold satisfies a weighted Witten formula: if the weighted scalar curvature is nonnegative and f decays suitably rapidly at infinity, there exists an asymptotically constant weighted-harmonic spinor ψ of norm 1 at infinity and satisfying

$$\mathfrak{m}_f(g) = 4\int_M \left(|\nabla\psi|^2 + \frac{1}{4}\,\mathrm{R}_f\,|\psi|^2\right) e^{-f}\,dV_g. \tag{0.6}$$

Moreover, Theorem 2.13 proves a positive weighted mass theorem on spin manifolds: if the weighted scalar curvature is nonnegative and f decays suitably rapidly at infinity, then

$$\mathfrak{m}_f(g) \geq 0, \quad \text{with equality iff } (M^n, g) \cong (\mathbb{R}^n, g_{\mathrm{euc}}) \text{ and } \int_{\mathbb{R}^n} (\Delta_f f)\, e^{-f}\, dV_{g_{\mathrm{euc}}} = 0. \tag{0.7}$$

By way of a parenthetical remark: using work of Nakajima [N] (see [DO1]), the results of this section have straightforward extensions to asymptotically locally Euclidean spaces of dimension 4 with subgroup SU(2) at infinity, though they are not pursued in this paper.

0.2.1. Weighted Mass and Ricci Flow ADM mass does not measure how far a manifold is from the Euclidean metric, except in an asymptotic way at infinity. Indeed, one striking way to see this is that 3-dimensional Ricci flow (with surgery) starting at an AE metric with nonnegative scalar curvature converges to Euclidean space [Li]; however, *mass is constant* along the flow and thus does not detect the improvement of the geometry [DM,OW,Ha2,Li].

On the other hand, with a suitable choice of weight function f, the weighted mass indeed measures how far an AE manifold is from Euclidean space: the most natural choice for f is the unique f_g decaying at infinity and solving $\mathrm{R}_{f_g} \equiv 0$. Theorem 2.17 shows that such an f_g exists on any AE manifold with nonnegative scalar curvature. This surprisingly yields the formula

$$\mathfrak{m}_{f_g}(g) = -\lambda_{\mathrm{ALE}}(g), \tag{0.8}$$

where $\lambda_{\mathrm{ALE}}(g)$ is the renormalized Perelman functional introduced by Deruelle and the second author [DO1]. Equality (0.8) is the content of Theorem 2.17, and is unexpected at first sight since λ_{ALE} stems from a variational principle on the whole manifold, and a priori is not a boundary term. (The notation for λ_{ALE} is adopted from [DO1], since the results here also apply to ALE spaces.)

The renormalized Perelman functional is the correct modification of Perelman's λ-functional (for closed manifolds) to AE manifolds: it has the crucial property that Ricci flow, $\partial_t g = -2\mathrm{Ric}$, is its gradient flow [DO1,Ha1]. Thus equality (0.8) implies that a Ricci flow on an AE manifold with nonnegative scalar curvature is the gradient flow of the weighted mass (see Corollary 2.20):

$$\frac{d}{dt}\mathfrak{m}_{f_g}(g) = -2\int_M |\mathrm{Ric} + \mathrm{Hess}_{f_g}|^2 e^{-f_g}\, dV \leq 0, \tag{0.9}$$

and equality implies Ricci-flatness. Together, (0.6), (0.8), and (0.9) imply the following monotonicity formula along Ricci flow for the weighted *spinorial* Dirichlet energy of a weighted Witten spinor:

$$\frac{d}{dt}\int_M |\nabla\psi|^2 e^{-f_g}\, dV = -\frac{1}{2}\int_M |\mathrm{Ric} + \mathrm{Hess}_{f_g}|^2 e^{-f_g}\, dV. \tag{0.10}$$

This monotonicity formula stands in contrast to the constancy of ADM mass along Ricci flow, which implies that for an (unweighted) Witten spinor φ, the integral $\int_M (|\nabla\varphi|^2 + \frac{1}{4}\mathrm{R}\,|\varphi|^2)dV$ is constant along Ricci flow. Further applications of spin geometry to the Ricci flow, including a direct proof of (0.10) via the first variation, will be presented in forthcoming work [BO1].

Equality (0.8) additionally implies that all of the advantages of λ_{ALE} over the ADM mass also hold for the weighted mass. In addition to those already stated, the key advantages of the weighted mass over the ADM mass are as follows: like ADM mass, $\mathfrak{m}_{f_g}(g)$ is nonnegative on any *spin* AE manifold, and vanishes only on Euclidean space; $\mathfrak{m}_{f_g}(g)$ satisfies a Łojasiewicz inequality measuring the distance to Euclidean space; $\mathfrak{m}_{f_g}(g)$ is real-analytic on weighted Hölder spaces, where neither mass, nor the L^1-norm of scalar curvature are defined; even when an AE manifold has some negative scalar curvature, $\mathfrak{m}_{f_g}(g)$ is nonnegative and detects how far from Euclidean space the geometry is, allowing for stability analysis of gravitational instantons under general perturbations [DO2].

1. Weighted Dirac Operator

Let (M^n, g) be a complete Riemannian spin n-manifold without boundary. The spin bundle $\Sigma M \to M$ is a complex vector bundle of rank $2^{\lfloor \frac{n}{2} \rfloor}$, equipped with a Hermitian metric, Clifford multiplication, and connection. These objects satisfy compatibility conditions which are stated below. A spinor field, or simply spinor, is a section of the bundle ΣM. For background on spin geometry, see the book [P], whose notation and conventions are adopted here.

Let $f \in C^\infty(M)$. The weighted Dirac operator $D_f : \Gamma(\Sigma M) \to \Gamma(\Sigma M)$ is defined as

$$D_f = D - \frac{1}{2}(\nabla f), \tag{1.1}$$

where $D = e_i \cdot \nabla_i$ is the standard (Atiyah-Singer) Dirac operator and $\cdot$ denotes Clifford multiplication. (Throughout this paper, 1-forms and vector fields will often be identified without explicit mention.) The weighted Dirac operator is the Dirac operator associated with the modified spin connection $\nabla^f : \Gamma(\Sigma M) \to \Gamma(T^*M \otimes \Sigma M)$, defined by

$$\nabla^f_X \psi = \nabla_X \psi - \frac{1}{2}(\nabla_X f)\psi, \tag{1.2}$$

where ∇ is the standard spin connection induced by the Levi-Civita connection. The modified spin connection ∇^f is *not* metric compatible with the standard metric [BHM+, Proposition 2.5] on the spin bundle, $\langle\cdot,\cdot\rangle$, however, it is compatible with the modified metric $\langle\cdot,\cdot\rangle_f := \langle\cdot,\cdot\rangle e^{-f}$, that is

$$X(\langle\psi,\varphi\rangle e^{-f}) = \langle\nabla^f_X\psi,\varphi\rangle e^{-f} + \langle\psi,\nabla^f_X\varphi\rangle e^{-f}, \tag{1.3}$$

for any vector field X and spinors ψ, φ. Moreover, since Clifford multiplication is parallel with respect to the standard spin connection, it is also parallel with respect to ∇^f. This means that

$$\nabla^f_X(Y \cdot \psi) = Y \cdot \nabla^f_X \psi + (\nabla_X Y) \cdot \psi, \tag{1.4}$$

for any vector fields X, Y and spinor ψ.

The weighted Dirac operator satisfies the following weighted integration by parts formula on $W^{1,2}(e^{-f}\,dV)$,

$$\int_M \langle \psi, D_f \varphi \rangle e^{-f}\,dV = \int_M \langle D_f \psi, \varphi \rangle e^{-f}\,dV \tag{1.5}$$

and hence is self-adjoint on $W^{1,2}(e^{-f}\,dV)$. Furthermore, a weighted Lichnerowicz formula holds, which was observed by Perelman [P, Rem. 1.3]. To state it, let

$$\Delta_f = \Delta - \nabla_{\nabla f} \tag{1.6}$$

be the weighted Laplacian acting on spinors and let

$$\mathrm{R}_f = \mathrm{R} + 2\Delta f - |\nabla f|^2 \tag{1.7}$$

be Perelman's weighted scalar curvature (or P-scalar curvature).

Proposition 1.8 (Weighted Lichnerowicz). *The square of the weighted Dirac operator D_f satisfies*

$$D_f^2 = -\Delta_f + \frac{1}{4}\mathrm{R}_f\,. \tag{1.9}$$

Proof. The proof is a consequence of the standard Lichnerowicz formula and the properties of Clifford multiplication. Recall that if $e_1, \ldots, e_n$ is a local orthonormal basis of TM, then for any symmetric 2-tensor A,

$$\sum_{i,j=1}^{n} A(e_i, e_j) e_i \cdot e_j \cdot = -\mathrm{tr}(A)\mathbb{1}. \tag{1.10}$$

(The proof is immediate from the Clifford algebra relation $e_i \cdot e_j + e_j \cdot e_i = -2\delta_{ij}\mathbb{1}$). Combined with the standard Lichnerowicz formula and the Clifford algebra relation, it follows that for any smooth spinor ψ,

$$\begin{aligned}
D_f^2\psi &= \left(D - \frac{1}{2}(\nabla f)\cdot\right)\left(D - \frac{1}{2}(\nabla f)\cdot\right)\psi \\
&= D^2\psi - \frac{1}{2}D((\nabla f)\cdot\psi) - \frac{1}{2}(\nabla f)\cdot D\psi - \frac{1}{4}|\nabla f|^2\psi \\
&= D^2\psi - \frac{1}{2}e_i \cdot \nabla_i((\nabla_j f)e_j \cdot \psi) - \frac{1}{2}(\nabla_j f)e_j \cdot e_i \cdot \nabla_i\psi - \frac{1}{4}|\nabla f|^2\psi \\
&= D^2\psi - \frac{1}{2}(\nabla_i\nabla_j f)e_i \cdot e_j \cdot \psi - \frac{1}{2}(\nabla_j f)(e_i \cdot e_j + e_j \cdot e_i)\cdot \nabla_i\psi - \frac{1}{4}|\nabla f|^2\psi \\
&= -\Delta\psi + \frac{1}{4}\mathrm{R}\,\psi + \frac{1}{2}(\Delta f)\psi + \langle \nabla f, \nabla\psi \rangle - \frac{1}{4}|\nabla f|^2\psi \\
&= -\Delta_f\psi + \frac{1}{4}(\mathrm{R} + 2\Delta f - |\nabla f|^2)\psi \\
&= -\Delta_f\psi + \frac{1}{4}\mathrm{R}_f\,\psi.
\end{aligned} \tag{1.11}$$

□

Remark 1.12. The weighted Lichnerowicz formula also follows from the Lichnerowicz formula for spin-c Dirac operators [Fr2, §3.3],

$$D_A^2 = -\Delta_A + \frac{1}{4}\mathrm{R} + \frac{1}{2}dA, \tag{1.13}$$

by choosing the spin-c connection ∇^A for which $A = -\frac{1}{2}df$. Indeed, with this connection,

$$\Delta_A = (\nabla^A)^*\nabla^A = \Delta_f - \frac{1}{4}(2\Delta f - |\nabla f|^2) \tag{1.14}$$

and $dA = -\frac{1}{2}d^2 f = 0$, from which the weighted Lichnerowicz formula (1.9) follows immediately. In this sense, the weighted Dirac operator can also be thought of as the twisted Dirac operator D_A.

Proposition 1.15 (Weighted Ricci identity). *The weighted Ricci curvature* $\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess}_f$ *is proportional to the commutator of* D_f *and* ∇*: for any vector field* X *and spinor* ψ*,*

$$[D_f, \nabla_X]\psi = \frac{1}{2}\mathrm{Ric}_f(X)\cdot\psi. \tag{1.16}$$

Proof. Recall the unweighted Ricci identity, $[D, \nabla_X] = \frac{1}{2}\mathrm{Ric}(X)\cdot$. (For a proof, see for example [BHM+, Rem. 2.50]). Using this identity and the fact that Clifford multiplication is parallel with respect to the weighted spin connection (1.4), it follows that, for any spinor ψ,

$$\begin{aligned} D_f\nabla_X\psi - \nabla_X D_f\psi &= D\nabla_X\psi - \frac{1}{2}(\nabla f)\cdot\nabla_X\psi - \nabla_X D\psi + \frac{1}{2}\nabla_X((\nabla f)\cdot\psi) \\ &= [D, \nabla_X]\psi + \frac{1}{2}(\nabla_X\nabla f)\cdot\psi \\ &= \frac{1}{2}\mathrm{Ric}(X)\cdot\psi + \frac{1}{2}\mathrm{Hess}_f(X)\cdot\psi. \end{aligned} \tag{1.17}$$

□

In what follows, denote the space of weighted L^2-spinors by $L^2_f = L^2(\Sigma M, e^{-f}dV)$ and let L^2 be the space of unweighted L^2-spinors. Define the linear operator

$$U_f : L^2 \to L^2_f, \qquad \psi \mapsto e^{f/2}\psi. \tag{1.18}$$

This operator is an isomorphism of Hilbert spaces with inverse given by $U_f^{-1} = U_{-f}$; it preserves norms since

$$\|U_f\psi\|_{L^2_f} = \int_M |e^{f/2}\psi|^2\, e^{-f}dV = \|\psi\|_{L^2}. \tag{1.19}$$

In particular, U_f is a unitary operator. Recall that two operators A, B acting on Hilbert spaces with domains of definition $\mathcal{D}_A$ and $\mathcal{D}_B$ are unitarily equivalent if there exists a unitary operator U such that $U\mathcal{D}_A = \mathcal{D}_B$ and $UAU^{-1}x = Bx$ for all $x \in \mathcal{D}_B$.

Proposition 1.20 (Unitary equivalence). *The Dirac operator D and the weighted Dirac operator D_f are unitarily equivalent and hence isospectral; on C^1-spinors, these operators are related by*

$$U_f D U_f^{-1} = D_f. \tag{1.21}$$

In particular, $D\psi = 0$ if and only if $D_f(e^{f/2}\psi) = 0$.

Proof. For any C^1-spinor ψ,

$$\begin{aligned} U_f D U_f^{-1}\psi &= e^{f/2} D(e^{-f/2}\psi) = e^{f/2}\left(e^{-f/2}D\psi + (\nabla e^{-f/2})\cdot\psi\right) \\ &= D\psi - \frac{1}{2}(\nabla f)\cdot\psi = D_f\psi. \end{aligned} \tag{1.22}$$

This proves (1.21), and it follows immediately from this equation and the fact that U_f is an isomorphism, that $D\psi = \lambda\psi$ if and only if $D_f(U_f\psi) = \lambda U_f\psi$. In particular, U_f is an isomorphism between the eigenspaces $E_\lambda(D)$ and $E_\lambda(D_f)$, for any $\lambda \in \mathbb{R}$. Hence, (when defined) the multiplicities of the eigenvalues coincide. □

The following eigenvalue inequality is a generalization of Friedrich's inequality [Fr1] and the proof below generalizes his proof. See [Fr2, §5.1] for an insightful exposition of the classical proof, whose outline will be followed below. The weighted Friedrich inequality proved below is sharp. Indeed, on the round sphere with constant scalar curvature R and with f a constant function, equality is obtained.

Theorem 1.23. *Suppose that (M^n, g) is closed, let $f \in C^\infty(M)$, and let λ be an eigenvalue of the Dirac operator D. Then*

$$\lambda^2 \geq \frac{n}{4(n-1)} \min \mathrm{R}_f, \tag{1.24}$$

with equality if and only if f is constant and (M^n, g) admits a Killing spinor, in which case (M^n, g) is Einstein.

Proof. Let ψ be an eigenspinor of the Dirac operator with $D\psi = \lambda\psi$.

Define the connection

$$\nabla_X^{f,\lambda} = \nabla_X + \frac{1}{2}(\nabla_X f) + \frac{1}{2n} X\cdot(\nabla f)\cdot + \frac{\lambda}{n} X\cdot \tag{1.25}$$

A calculation employing a local orthonormal frame shows that the assumption $D\psi=\lambda\psi$ implies

$$|\nabla^{f,\lambda}\psi|^2 = |\nabla\psi|^2 - \frac{\lambda^2}{n}|\psi|^2 + \frac{1}{4}\left(1-\frac{1}{n}\right)|\nabla f|^2|\psi|^2 + \frac{1}{2}\langle\nabla f, \nabla|\psi|^2\rangle. \tag{1.26}$$

Integrating the above equation over M and integrating the last term by parts implies

$$\int_M |\nabla^{f,\lambda}\psi|^2\, dV = \int_M \left(|\nabla\psi|^2 - \frac{\lambda^2}{n}|\psi|^2 + \frac{1}{4}\left(1-\frac{1}{n}\right)|\nabla f|^2|\psi|^2 - \frac{1}{2}(\Delta f)|\psi|^2\right) dV. \tag{1.27}$$

The standard (unweighted) Lichnerowicz formula, the self-adjointness of D on L^2, and the definition of the weighted scalar curvature then imply

$$\begin{aligned}\int_M |\nabla^{f,\lambda}\psi|^2\, dV &= \int_M \left(|D\psi|^2 - \frac{1}{4}\mathrm{R}|\psi|^2 - \frac{\lambda^2}{n}|\psi|^2\right.\\ &\quad \left.+\frac{1}{4}\left(1-\frac{1}{n}\right)|\nabla f|^2|\psi|^2 - \frac{1}{2}(\Delta f)|\psi|^2\rangle\right) dV\\ &= \int_M \left(\left(\frac{n-1}{n}\right)\lambda^2|\psi|^2 - \frac{1}{4}\mathrm{R}_f|\psi|^2 - \frac{1}{4n}|\nabla f|^2|\psi|^2\right) dV,\end{aligned} \tag{1.28}$$

which, after rearranging, implies

$$\begin{aligned}\lambda^2\left(\frac{n-1}{n}\right)\|\psi\|^2_{L^2} &= \|\nabla^{f,\lambda}\psi\|^2_{L^2} + \frac{1}{4}\int_M \left(\mathrm{R}_f + \frac{1}{n}|\nabla f|^2\right)|\psi|^2\, dV \\ &\geq \frac{1}{4}\min_M \mathrm{R}_f\|\psi\|^2_{L^2}.\end{aligned} \tag{1.29}$$

This was to be shown.

If equality occurs in the previous inequality, then R_f is constant, $\nabla^{f,\lambda}\psi = 0$ and $\nabla f = 0$. In particular, f is constant, so $0 = \nabla^{f,\lambda}\psi = \nabla^{0,\lambda}\psi$. This is equivalent to the condition that, for all vector fields X

$$\nabla_X\psi = -\frac{\lambda}{n}X\cdot\psi. \tag{1.30}$$

Hence ψ s a Killing spinor.

Finally, a manifold admitting a Killing spinor must be Einstein; see for example [Fr2, §5.2]. The converse is immediate. □

Whenever the scalar curvature is not constant, Theorem 1.23 implies a strict improvement of Friedrich's inequality. This is because the weight f can always be chosen to make R_f constant, while if R is not constant, then it follows that $\mathrm{R}_f > \mathrm{R}_{\min}$. To show this, recall that Perelman's entropy λ_{P} is defined as the first eigenvalue of the operator $-4\Delta + \mathrm{R}$, or equivalently, as the minimum of the weighted Hilbert-Einstein functional [P]:

$$\lambda_{\mathrm{P}} = \inf_u \frac{\int_M \left(4|\nabla u|^2 + \mathrm{R}\,u^2\right) dV}{\int_M u^2\, dV} = \inf_f \frac{\int_M \mathrm{R}_f\, e^{-f}\, dV}{\int_M e^{-f}\, dV}. \tag{1.31}$$

If f is the minimizer of λ_{P}, the *weighted* scalar curvature is constant, with $\mathrm{R}_f = \lambda_{\mathrm{P}}$. On the other hand, if the scalar curvature is not constant, then $\mathrm{R}_f = \lambda_{\mathrm{P}} > \mathrm{R}_{\min}$, and thus the weighted Friedrich inequality (1.24) implies a strict improvement of Friedrich's inequality.

Corollary 1.32. *Any eigenvalue λ of the Dirac operator D on a closed manifold (M^n, g) satisfies*

$$\lambda^2 \geq \frac{n}{4(n-1)}\lambda_{\mathrm{P}}(g), \tag{1.33}$$

with equality if and only if (M^n, g) admits a Killing spinor, in which case (M^n, g) is Einstein.

The bound (1.32) gives another proof of the *stability* of hyperkähler metrics on the $K3$ surface along Ricci flow. Indeed, all metrics on $K3$ satisfy the above inequality with $\lambda = 0$ since $\hat{A}(K3) \neq 0$. Consequently, Corollary 1.32 implies that $\lambda_P(g) \leqslant 0$ for *all* metrics g on $K3$, with equality exactly on hyperkähler metrics. These metrics are consequently stable by [Ha1].

Remark 1.34. Hijazi [Hi, Eqn. (5.1)] proved an inequality closely related to that of Theorem 1.23. Hijazi's proof employs the Dirac operator of a conformally related metric, whereas the proof of Theorem 1.23 keeps the metric fixed and uses the weighted Lichnerowicz formula (1.9). Hijazi's inequality implies that any eigenvalue λ of the Dirac operator satisfies $\lambda^2 \geq \frac{n}{4(n-1)}\mu_1(g)$, where $\mu_1(g)$ is the smallest eigenvalue of the conformal Laplace operator $-4\frac{n-1}{n-2}\Delta + \mathrm{R}$. Since $\lambda_\mathrm{P}(g)$ is the first eigenvalue of the operator $-4\Delta + \mathrm{R}$, it follows that

$$\mu_1(g) \geq \lambda_\mathrm{P}(g). \tag{1.35}$$

In this sense, Hijazi's inequality [Hi, Eqn. (5.1)] is sharper than the inequality of Theorem 1.23. On the other hand, the inequality in Corollary 1.32 improves along Ricci flow.

2. Weighted Asymptotically Euclidean Manifolds

A smooth orientable Riemannian manifold (M^n, g) is called asymptotically Euclidean (AE) of order τ if there exists a compact subset $K \subset M$ and a diffeomorphism $\Phi : M \setminus K \to \mathbb{R}^n \setminus B_\rho(0)$, for some $\rho > 0$, with respect to which

$$g_{ij} = \delta_{ij} + O(r^{-\tau}), \qquad \partial^k g_{ij} = O(r^{-\tau-k}), \tag{2.1}$$

for any partial derivative of order k as $r \to \infty$, where $r = |\Phi|$ is the Euclidean distance function. The set $M \setminus K$ is called the end of M^n. (The results of this section extend in a straightforward manner to AE manifolds with multiple ends, though they are not pursued here.)

The ADM mass [ADM] of (M^n, g) is defined by

$$\mathfrak{m}(g) = \lim_{\rho\to\infty} \int_{S_\rho} (\partial_i g_{ij} - \partial_j g_{ii})\, \partial_j \lrcorner\, dV_g, \tag{2.2}$$

where $S_\rho = r^{-1}(\rho)$ is a coordinate sphere of radius ρ.[1] Although the definition of mass involves a choice of AE coordinates, if $\tau > (n-2)/2$ and the scalar curvature is integrable, then the mass is finite and independent of the choice of AE coordinates [B,C]. If $n \leq 7$ or (M^n, g) admits a spin structure, then the assumptions $\mathrm{R} \geq 0, \mathrm{R} \in L^1(M, g)$, and $\tau > \frac{n-2}{2}$, imply that $\mathfrak{m}(g)$ is nonnegative and is zero if and only if (M^n, g) is isometric to $(\mathbb{R}^n, g_{\mathrm{euc}})$, by the positive mass theorem [SY,W1].

The AE structure defines a trivialization of the spin bundle at infinity. Indeed, choose an asymptotic coordinate system $\Phi^{-1} : \mathbb{R}^n \setminus B_R(0) \to M \setminus K$. The pullback bundle $(\Phi^{-1})^*\Sigma M$ differs from the trivial spin bundle $\mathbb{R}^n \times \Sigma$ by an element of $H^1(\mathbb{R}^n \setminus B_R(0); \mathbb{Z}) = 0$. Hence the spin structure is trivial over the end of M and the bundle $(\Phi^{-1})^*\Sigma M$ extends trivially over all of $\mathbb{R}^n$. This trivialization of the spin bundle allows for the definition of "constant spinors" on the end of M: a spinor ψ defined on the end

[1] The ADM mass as defined in [B] equals $(2(n-1)\omega_{n-1})^{-1}\mathfrak{m}(g)$, where ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$.

M is called *constant* (with respect to the asymptotic coordinates Φ) if $\psi = (\Phi^{-1})^*\psi_0$, for some constant spinor ψ_0 on $\mathbb{R}^n$.

Witten argued that for any such constant spinor ψ_0 on $M \setminus K$ with $|\psi_0| \to 1$ at infinity, there exists a harmonic spinor ψ on M which is asymptotic to ψ_0, in the sense that $|\psi - \psi_0| = O(r^{-\tau})$ and $|\nabla\psi| = O(r^{-\tau-1})$. Such a spinor ψ is called a *Witten spinor*. Moreover, the ADM mass of (M^n, g) is given by

$$\mathfrak{m}(g) = 4\int_M \left(|\nabla\psi|^2 + \frac{1}{4}\mathrm{R}\,|\psi|^2\right) dV_g, \tag{2.3}$$

which is called *Witten's formula* for the mass. A rigorous proof of the existence of Witten spinors is given by Parker-Taubes [PT] and Lee-Parker [LP]; their proofs are generalized below and in Appendix A.

2.1. Weighted Mass. The *weighted ADM mass* of a weighted AE manifold (M^n, g, f) is defined by

$$\mathfrak{m}_f(g) := \mathfrak{m}(g) + 2\lim_{\rho\to\infty}\int_{S_\rho} \langle\nabla f, \nu\rangle\, e^{-f}\, dA. \tag{2.4}$$

This definition is motivated by the weighted Witten formula (2.7) below, and manifestly extends to non-spin manifolds. Like ADM mass, the weighted mass is independent of the choice of asymptotic coordinates if $\tau > \frac{n-2}{2}$ and $\mathrm{R} \in L^1(M)$: indeed, the ADM mass is coordinate independent under said assumptions [B,C], and by the divergence theorem, the second term in (2.4) equals $2\int_M (\Delta_f f)\, e^{-f}\, dV$, which is manifestly coordinate independent.

The appropriate analytic tools for studying AE manifolds are the *weighted Hölder spaces* $C^{k,\alpha}_\beta(M)$, whose precise definitions are stated in Appendix A. These spaces share many of the global elliptic regularity results which hold for the usual Hölder spaces on compact manifolds. The index β is important because it denotes the *order of growth*: functions in $C^{k,\alpha}_\beta(M)$ grow at most like r^β. In particular, if the metric g is AE of order τ on $M = \mathbb{R}^n$, then in the AE coordinate system, $g - \delta$ lies in $C^{k,\alpha}_{-\tau}(M)$ for all $k \in \mathbb{N}$ and the scalar curvature of g lies in $C^{k,\alpha}_{-\tau-2}(M)$ for all $k \in \mathbb{N}$.

In what follows, let D_f be the weighted Dirac operator associated with the weighted spin connection (1.2) defined by f, which satisfies the weighted Lichnerowicz formula (1.9).

Theorem 2.5 (Weighted Witten formula). *Let (M^n, g, f) be a weighted, spin, AE manifold of order τ. Suppose that $f \in C^{2,\alpha}_{-\tau}(M)$, that*

$$\mathrm{R}_f \geqslant 0, \qquad \mathrm{R}_f \in L^1(M, g), \qquad \frac{n-2}{2} < \tau < n-2, \tag{2.6}$$

and that ψ_0 is a spinor on (M^n, g) which is constant at infinity, with $|\psi_0| \to 1$. Then there exists a D_f-harmonic spinor ψ which is asymptotic to ψ_0 in the sense that $\psi - \psi_0 \in C^{2,\alpha}_{-\tau}(M)$ and

$$\mathfrak{m}_f(g) = 4\int_M \left(|\nabla\psi|^2 + \frac{1}{4}\mathrm{R}_f\,|\psi|^2\right) e^{-f}\, dV_g. \tag{2.7}$$

Proof. Here the proof is given under the additional natural assumptions that $\mathrm{R} \geq 0$, $\mathrm{R} \in L^1(M,g)$ and $|\nabla f| = O(r^{-(n-1)})$. The additional assumptions $\mathrm{R} \geq 0$, $\mathrm{R} \in L^1(M,g)$ ensure the existence of an (unweighted) Witten spinor ψ [LP]. Further, the assumption $|\nabla f| = O(r^{-(n-1)})$ is satisfied if $\mathrm{R}_f = 0$; see [DO1, Prop. (2.2)]. In Appendix A.1, a proof of the general case is given.

By (1.21), if $D\psi = 0$, then the spinor $\psi_f = e^{f/2}\psi$ is D_f-harmonic. Since

$$\nabla\psi = \nabla(e^{-f/2}\psi_f) = e^{-f/2}\left(\nabla\psi_f - \frac{1}{2}df \otimes \psi_f\right), \tag{2.8}$$

$$\nabla\psi_f = \nabla(e^{f/2}\psi) = e^{f/2}\nabla\psi + \frac{1}{2}df \otimes \psi_f, \tag{2.9}$$

it follows that

$$\begin{aligned}
|\nabla\psi|^2 &= e^{-f}\left(|\nabla\psi_f|^2 + \frac{1}{4}|\nabla f|^2|\psi_f|^2 - \operatorname{Re}\langle\nabla\psi_f, df\otimes\psi_f\rangle\right)\\
&= e^{-f}\left(|\nabla\psi_f|^2 + \frac{1}{4}|\nabla f|^2|\psi_f|^2 - \frac{1}{2}|\nabla f|^2|\psi_f|^2 - e^f \operatorname{Re}\langle\nabla\psi, df\otimes\psi\rangle\right)\\
&= e^{-f}\left(|\nabla\psi_f|^2 - \frac{1}{4}|\nabla f|^2|\psi_f|^2\right) - \operatorname{Re}\langle\nabla_{\nabla f}\psi, \psi\rangle.
\end{aligned} \tag{2.10}$$

By the definition of the weighted scalar curvature (1.7) and Witten's formula for the mass,

$$\begin{aligned}
\frac{1}{4}\mathfrak{m}(g) &= \int_M \left(|\nabla\psi|^2 + \frac{1}{4}\mathrm{R}\,|\psi|^2\right) dV_g\\
&= \int_M \left(|\nabla\psi_f|^2 + \frac{1}{4}(\mathrm{R} - |\nabla f|^2)|\psi_f|^2\right) e^{-f}\,dV_g - \operatorname{Re}\int_M \langle\nabla_{\nabla f}\psi, \psi\rangle\, dV_g\\
&= \int_M \left(|\nabla\psi_f|^2 + \frac{1}{4}\mathrm{R}_f\,|\psi_f|^2 - \frac{1}{2}(\Delta f)|\psi_f|^2\right) e^{-f}\,dV_g - \operatorname{Re}\int_M \langle\nabla_{\nabla f}\psi, \psi\rangle\, dV_g\\
&= \int_M \left(|\nabla\psi_f|^2 + \frac{1}{4}\mathrm{R}_f\,|\psi_f|^2\right) e^{-f}\,dV_g - \int_M \left(\frac{1}{2}(\Delta f)|\psi|^2 + \operatorname{Re}\langle\nabla_{\nabla f}\psi, \psi\rangle\right) dV_g.
\end{aligned} \tag{2.11}$$

Integrating the last term by parts and using the fact that $|\psi_f| \to 1$ at infinity gives

$$\begin{aligned}
\frac{1}{4}\mathfrak{m}(g) &= \int_M \left(|\nabla\psi_f|^2 + \frac{1}{4}\mathrm{R}_f\,|\psi_f|^2\right) e^{-f}\,dV_g - \lim_{\rho\to\infty}\frac{1}{2}\int_{S_\rho}\langle\nabla f, \nu\rangle|\psi_f|^2\,e^{-f}\,dA\\
&= \int_M \left(|\nabla\psi_f|^2 + \frac{1}{4}\mathrm{R}_f\,|\psi_f|^2\right) e^{-f}\,dV_g - \lim_{\rho\to\infty}\frac{1}{2}\int_{S_\rho}\langle\nabla f, \nu\rangle\,e^{-f}\,dA.
\end{aligned} \tag{2.12}$$

By the assumption $f \to 0$ at infinity and $|\nabla f| = O(r^{-(n-1)})$, the latter limit exists and is finite, since the area of S_ρ is of order ρ^{n-1}. □

The following theorem generalizes Schoen-Yau [SY] and Witten's [W1] positive mass theorem to the weighted (spin) setting.

Theorem 2.13 (Positive weighted mass theorem). *Let (M^n, g, f) be a weighted, spin, AE manifold satisfying the assumptions of Theorem 2.5. Then $\mathfrak{m}_f(g) \geqslant 0$, with equality if and only if (M^n, g) is isometric to $(\mathbb{R}^n, g_{\text{euc}})$ and $\int_{\mathbb{R}^n} (\Delta_f f)\, e^{-f}\, dV = 0$.*

Proof. Theorem 2.5 provides the existence of a weighted Witten spinor ψ satisfying the weighted Witten formula (2.7). This shows that $\mathfrak{m}_f(g) \geq 0$ if $\mathrm{R}_f \geq 0$. The proof of the equality statement resembles Witten's proof of the equality statement for the positive mass theorem: equality implies that $\nabla\psi = 0$, and since there exist $\operatorname{rank}(\Sigma M)$ possible linearly independent constant spinors at infinity ψ_0 to which ψ is asymptotic, ΣM admits a basis of parallel spinors. Since the map $\Sigma M \to TM$ sending a spinor φ to the vector field V_φ defined by

$$\langle V_\varphi, X\rangle = \operatorname{Im}\langle \varphi, X\cdot\varphi\rangle \qquad \text{for all } X \in \Gamma(TM), \tag{2.14}$$

is surjective, and since V_φ is a parallel vector field if φ is a parallel spinor, TM admits a basis of parallel vector fields. Thus (M^n, g) is flat. Finally, since $\mathfrak{m}(g_{\text{euc}}) = 0$, integration by parts and $\mathfrak{m}_f(g_{\text{euc}}) = 0$ imply that

$$0 = \mathfrak{m}_f(g_{\text{euc}}) = \lim_{\rho\to\infty} 2\int_{S_\rho} \langle \nabla f, \nu\rangle\, e^{-f}\, dA = -2\int_{\mathbb{R}^n} (\Delta_f f)\, e^{-f}\, dV. \tag{2.15}$$

□

2.2. Weighted Mass and Ricci Flow. Given an asymptotically Euclidean manifold (M^n, g), define the *renormalized Perelman entropy* as

$$\lambda_{\text{ALE}}(g) = \inf_{u-1\in C_c^\infty(M)} \int_M \left(4|\nabla u|^2 + \mathrm{R}\, u^2\right) dV - \mathfrak{m}(g). \tag{2.16}$$

Note that $\lambda_{\text{ALE}}(g)$ can equivalently be defined as the infimum of $\int_M \mathrm{R}_f\; e^{-f}\, dV - \mathfrak{m}_f(g)$, over all $f \in C_c^\infty(M)$. If (M^n, g) admits a Witten spinor ψ, then testing the right-hand-side of the above equation with $u = |\psi|$ gives that $\lambda_{\text{ALE}}(g) \leq 0$, by Kato's inequality, $|\nabla|\psi|| \leq |\nabla\psi|$. As mentioned in the Introduction, Ricci flow is the gradient flow of λ_{ALE} on AE manifolds and λ_{ALE} has various advantages over the ADM mass in the context of Ricci flow; see the Introduction and also [DO1].

Theorem 2.17. *Let (M^n, g) be an asymptotically Euclidean manifold of order $\tau > \frac{n-2}{2}$ and with nonnegative scalar curvature. Then there exists a solution $f \in C^{2,\alpha}_{-\tau}(M)$ of the elliptic equation $\mathrm{R}_f = 0$, and the f-weighted mass satisfies*

$$\mathfrak{m}_f(g) = -\lambda_{\text{ALE}}(g). \tag{2.18}$$

Proof. By [DO1, (2.3)], there exists a strictly positive minimizer $w = e^{-f/2}$ of (2.16) with $w - 1 \in C^{2,\alpha}_{-\tau}(M)$ satisfying $-4\Delta w + \mathrm{R}\, w = 0$. Since $w \to 1$ at infinity, integration by parts implies

$$\begin{aligned}\inf_{u-1\in C_c^\infty(M)} \int_M \left(4|\nabla u|^2 + \mathrm{R}\, u^2\right) dV &= \int_M \left(4|\nabla w|^2 + \mathrm{R}\, w^2\right) dV \\ &= \lim_{\rho\to\infty} \int_{S_\rho} 4\langle \nabla w, \nu\rangle w\, dA\end{aligned}$$

$$= -2 \lim_{\rho\to\infty} \int_{S_\rho} \langle \nabla f, \nu\rangle \, e^{-f} dA. \tag{2.19}$$

The result now follows immediately from the definition (2.4) of $\mathfrak{m}_f(g)$ and that of λ_{ALE}, (2.16).

Note that [DO1, Eqn. (2.3)] is stated for ALE manifolds in the neighborhood of a Ricci-flat ALE manifold, to ensure the existence and uniqueness of f by the positivity of $-4\Delta + \mathrm{R}$ thanks to a Hardy inequality; see [DO1, Prop. 1.12]. However, the same proof holds under the above assumptions on (M^n, g) since the scalar curvature is nonnegative and the operator $-4\Delta + \mathrm{R}$ is therefore positive; see the proof of [Ha1, Thm. 2.6] for a similar argument. □

It has been proven in [Li, Thm. 2.2] that the AE conditions are preserved along Ricci flow (with the same coordinate system) as long as the flow is nonsingular. An *asymptotically Euclidean Ricci flow* is defined to be any Ricci flow starting at an AE manifold.

Corollary 2.20 (Monotonicity of weighted mass). *Let $(M^n, g(t))_{t\in I}$ be an asymptotically Euclidean Ricci flow with nonnegative scalar curvature. Let $f : M \times I \to \mathbb{R}$ be the time-dependent family of functions solving $\mathrm{R}_f = 0$ and $f \to 0$ at infinity, at each time $t \in I$. Then*

$$\frac{d}{dt}\mathfrak{m}_f(g) = -2\int_M |\mathrm{Ric} + \mathrm{Hess}_f|^2 e^{-f} dV \leq 0. \tag{2.21}$$

In particular, $\mathfrak{m}_f(g)$ is monotone decreasing along the Ricci flow, and is constant only if $(M^n, g(t))$ is Ricci-flat.

Proof. Since $\mathfrak{m}_f(g) = -\lambda_{ALE}(g)$, equation (2.21) follows from the formula for the first variation of λ_{ALE}, which can be found in [DO1, Prop. 2.3 and 3.13]. Once again, the assumptions of closeness to a Ricci-flat ALE metric of Deruelle-Ozuch can be replaced by the nonnegativity of scalar curvature. Their closeness assumption is again only used to ensure the existence of f. Note that in contrast with Perelman's monotonicity for closed manifolds, which is proved by letting f evolve *parabolically* backwards in time, the monotonicity formula (2.21) uses the fact that f solves the *elliptic* equation $\mathrm{R}_f = 0$ at each time.

To prove the equality statement, note that formula (2.21) implies that $\mathfrak{m}_f(g)$ is constant if and only if (M^n, g, f) is a steady Ricci soliton. The proof is completed by using [DK, Prop. 2.6]: any ALE steady soliton with $\nabla f \to 0$ at infinity is Ricci flat. □

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A. Appendix

This appendix provides a proof of the general case of Theorem 2.5 on the existence of a weighted Witten spinor satisfying the weighted Witten formula. In Section 2.1, a simple and illustrative proof was given under natural, albeit more restrictive assumptions.

Let (M^n, g) be an asymptotically Euclidean (AE), Riemannian spin manifold of order τ. The asymptotic coordinates define a positive function r on M, which equals the Euclidean distance to the origin on $M \setminus K$, and which can be extended to a smooth function which is bounded below by 1 on all of M.

Using r, the weighted C^k space $C^k_\beta(M)$ is defined for $\beta \in \mathbb{R}$ as the set of C^k functions u on M for which the norm

$$\|u\|_{C^k_\beta} = \sum_{i=0}^{k} \sup_M r^{-\beta+i} |\nabla^i u| \tag{A.1}$$

is finite. The weighted Hölder space $C^{k,\alpha}_\beta(M)$ is defined for $\alpha \in (0, 1)$ as the set of $u \in C^k_\beta(M)$ for which the norm

$$\|u\|_{C^{k,\alpha}_\beta} = \|u\|_{C^k_\beta} + \sup_{x,y} (\min\{r(x), r(y)\})^{-\beta+k+\alpha} \frac{|\nabla^k u(x) - \nabla^k u(y)|}{d(x, y)^\alpha} \tag{A.2}$$

is finite.[2] These definitions of weighted Hölder spaces coincide with those of [LP, §9]. In particular, the index β denotes the *order of growth*: functions in $C^{k,\alpha}_\beta(M)$ grow at most like r^β. Note that the definitions of the weighted function spaces depend on the "distance function" r, and thereby on the choice of asymptotic coordinates. However, it is easy to see that r is uniformly equivalent to the geodesic distance from an arbitrary fixed point in M as $r \to \infty$, hence all choices of r define equivalent norms. For the remainder of this appendix, fix $\alpha \in (0, 1)$.

A.1. Existence of weighted Witten spinors. Let $f \in C^\infty(M)$ and let D_f be the weighted Dirac operator associated with the weighted spin connection (1.2) defined by f, which satisfies the weighted Lichnerowicz formula (1.9).

Lemma A.3. *On a weighted, AE, spin manifold (M^n, g, f) satisfying the hypotheses of Theorem 2.5, the operator*

$$D_f^2 : C^{2,\alpha}_{-\tau}(M) \to C^{0,\alpha}_{-\tau-2}(M) \tag{A.4}$$

is an isomorphism.

[2] The meaning of "weighted" in "weighted Hölder spaces" is distinct from its meaning in "weighted manifolds."

Proof. To show injectivity, suppose $D_f^2\xi = 0$ for some $\xi \in C^{2,\alpha}_{-\tau}(M)$. Then $\xi = O(r^{-\tau})$ and $\nabla\xi = O(r^{-\tau-1})$. Applying the weighted Lichnerowicz formula and integration by parts (the boundary term vanishes because $\tau > (n-2)/2$), it follows that

$$\begin{aligned} 0 &= \int_M \langle D_f^2\xi, \xi\rangle\, e^{-f} dV_g = \int_M \left(-\langle \Delta_f \xi, \xi\rangle + \frac{1}{4}\mathrm{R}_f\, |\xi|^2\right) e^{-f} dV_g \\ &= \int_M \left(|\nabla\xi|^2 + \frac{1}{4}\mathrm{R}_f\, |\xi|^2\right) e^{-f} dV_g. \end{aligned} \tag{A.5}$$

Since $\mathrm{R}_f \geq 0$, this shows that $\nabla\xi = 0$, so $\nabla|\xi|^2 = 0$. Thus $|\xi|$ is a constant, which is zero since ξ vanishes at infinity. Thus D_f^2 is injective.

The weighted Lichnerowicz formula implies that $D_f^2 = -\Delta + \nabla_{\nabla f} + \frac{1}{4}\mathrm{R}_f$. Since $f \in C^{2,\alpha}_{-\tau}$ and g is smooth and AE of order τ, it follows that $\nabla f \in C^{1,\alpha}_{-\tau-1}$ and $\mathrm{R}_f \in C^{0,\alpha}_{-\tau-2}$. Since $\frac{n-2}{2} < \tau < n-2$, it follows from [CSCB] that D_f^2 is an isomorphism if it is injective. With injectivity proven above, the proof is complete; see [LP, Thm. 9.2d] for the proof for the unweighted Dirac operator. □

As explained in Section 2, the asymptotically Euclidean structure defines a trivialization of the spin bundle at infinity, allowing for the notion of a spinor which is "constant" in the asymptotic coordinate system. In what follows, for $\rho > 0$, let $S_\rho = r^{-1}(\rho)$ be the ρ-level set of r, that is, a coordinate sphere of radius ρ.

Proof of Theorem 2.5. With respect to the trivialization of the spin bundle at infinity, the weighted Dirac operator may be written as

$$D_f = e^i \cdot \partial_i - \frac{1}{2}(\nabla f)\cdot - \frac{1}{8}(\partial_k g_{ij}) e^i \cdot [e^j \cdot, e^k \cdot] + O(r^{-2\tau-1}). \tag{A.6}$$

Choose a spinor ψ_0 which is constant at infinity and with $|\psi_0| \to 1$ at infinity, and extend it to a smooth spinor on M. It follows from the above equation and the assumption $f \in C^{2,\alpha}_{-\tau}(M)$ that $D_f^2\psi_0 \in C^{0,\alpha}_{-\tau-2}(M)$. By Lemma (A.3), there exists $\xi \in C^{2,\alpha}_{-\tau}(M)$ with $D_f^2\xi = D_f^2\psi_0$. The spinor $\psi = \psi_0 - \xi$ then satisfies $D_f^2\psi = 0$ and $\varphi := D_f\psi = D_f\psi_0 - D_f\xi$ satisfies $D_f\varphi = 0$ and lies in $C^{1,\alpha}_{-\tau-1}(M)$, so integrating by parts as in the proof of the Lemma above shows that $\varphi = 0$. Thus ψ is a weighted harmonic spinor which is asymptotic to ψ_0.

Let X be the vector field on $M \setminus K$ defined by

$$X = \mathrm{Re}\,\langle \nabla_i\psi, \psi\rangle e^{-f} e_i. \tag{A.7}$$

Let $\lambda_i = \mathrm{Re}\,\langle\nabla_i\psi, \psi\rangle e^{-f}$ so that $X = \lambda_i e_i$. Define the $(n-1)$-form

$$\alpha = \iota_X(dV_g). \tag{A.8}$$

Then $d\alpha = \mathrm{div}_g(X) dV_g$ and

$$\begin{aligned} \mathrm{div}_g(X) &= \lambda_i \mathrm{div}_g(e_i) + \langle \nabla\lambda_i, e_i\rangle \\ &= \nabla_i\lambda_i \\ &= \mathrm{Re}\,\nabla_i(\langle\nabla_i\psi, \psi\rangle e^{-f}) \\ &= \left(\mathrm{Re}\,\langle\nabla_i\nabla_i\psi, \psi\rangle - \mathrm{Re}\,\langle\nabla_{\nabla f}\psi, \psi\rangle + |\nabla\psi|^2\right) e^{-f} \end{aligned}$$

$$= \left(\mathrm{Re}\, \langle \Delta_f \psi, \psi \rangle + |\nabla \psi|^2 \right) e^{-f}, \tag{A.9}$$

hence

$$d\alpha = \left(\mathrm{Re}\, \langle \Delta_f \psi, \psi \rangle + |\nabla \psi|^2 \right) e^{-f} dV_g. \tag{A.10}$$

Stokes' theorem then gives, with $M_\rho = \{r \leq \rho\} \subset M$ and $S_\rho = \partial M_\rho$, that

$$\begin{aligned} &\int_{M_\rho} \left(\mathrm{Re}\, \langle \Delta_f \psi, \psi \rangle + |\nabla \psi|^2 \right) e^{-f} dV_g \\ &= \int_{M_\rho} d\alpha = \int_{S_\rho} \alpha = \mathrm{Re} \int_{S_\rho} \langle \nabla_i \psi, \psi \rangle e^{-f} \iota_{e_i}(dV_g). \end{aligned} \tag{A.11}$$

Since $\psi = \psi_0 - \xi$, the latter boundary term equals

$$\mathrm{Re} \int_{S_\rho} \left(\langle \nabla_i \psi_0, \psi_0 \rangle - \langle \nabla_i \xi, \psi_0 \rangle - \langle \xi, \nabla_i \psi_0 \rangle + \langle \nabla_i \xi, \xi \rangle \right) e^{-f} \iota_{e_i}(dV_g). \tag{A.12}$$

Since $[e_j \cdot, e_k \cdot]$ is skew-Hermitian, as in (A.6), it follows that

$$\mathrm{Re}\, \langle \nabla_i \psi_0, \psi_0 \rangle = -\frac{1}{8} \mathrm{Re}\, (\partial_k g_{ij}) \langle [e_j \cdot, e_k \cdot] \psi_0, \psi_0 \rangle + O(r^{-2\tau-1}) = O(r^{-2\tau-1}), \tag{A.13}$$

and so the first term in (A.12) vanishes as $\rho \to \infty$. Also, since $\xi = O(r^{-\tau})$, $\nabla \xi = O(r^{-\tau-1})$, and $\nabla \psi_0 = O(r^{-\tau-1})$, the third and fourth terms in (A.12) also vanish as $\rho \to \infty$. Thus only the second term in (A.12) contributes to the limit $\rho \to \infty$; the remainder of the proof consists in showing that said term equals the weighted mass.

To analyze the remaining term, let L_i^f denote the operator

$$\begin{aligned} L_i^f &= \frac{1}{2} [e_i \cdot, e_j \cdot] (\nabla_j - \frac{1}{2} (\nabla_j f)) \\ &= (\delta_{ij} + e_i \cdot e_j \cdot) (\nabla_j - \frac{1}{2} (\nabla_j f)) \\ &= \nabla_i - \frac{1}{2} (\nabla_i f) + e_i \cdot D - e_i \cdot \frac{1}{2} (\nabla f) \cdot \\ &= \nabla_i^f + e_i \cdot D_f. \end{aligned} \tag{A.14}$$

If β is the $(n-2)$-form

$$\beta = e^{-f} \langle [e_i \cdot, e_j \cdot] \psi_0, \xi \rangle \iota_{e_i} \iota_{e_j} dV_g, \tag{A.15}$$

then since $e^k \wedge \iota_{e_i} \iota_{e_j} dV_g = \delta_{ik} \iota_{e_j} dV_g - \delta_{jk} \iota_{e_i} dV_g$,

$$\begin{aligned} d\beta &= 2e^{-f} ((\nabla_j f) \langle [e_i \cdot, e_j \cdot] \psi_0, \xi \rangle \\ &\qquad - \langle [e_i \cdot, e_j \cdot] \nabla_j \psi_0, \xi \rangle + \langle [e_i \cdot, e_j \cdot] \psi_0, \nabla_j \xi \rangle)) \iota_{e_i} dV_g \\ &= -2e^{-f} (\langle [e_i \cdot, e_j \cdot] (\nabla_j \psi_0 - \frac{1}{2} (\nabla_j f) \psi_0), \xi \rangle \\ &\qquad - \langle \psi_0, [e_i \cdot, e_j \cdot] (\nabla_j \xi - \frac{1}{2} (\nabla_j f) \psi_0) \rangle) \iota_{e_i} dV_g \end{aligned}$$

$$= -4e^{-f}(\langle L_i^f \psi_0, \xi\rangle - \langle \psi_0, L_i^f \xi\rangle)\iota_{e_i} dV_g. \tag{A.16}$$

Therefore, by Stokes' theorem and the fact that $D_f \xi = D_f \psi_0$, the second term in (A.12) is

$$\begin{aligned} -\operatorname{Re}\int_{S_\rho} \langle \nabla_i \xi, \psi_0\rangle e^{-f}\iota_{e_i}(dV_g) &= \operatorname{Re}\int_{S_\rho} \langle e_i \cdot D_f \xi - L_i^f \xi - \frac{1}{2}(\nabla_i f)\xi, \psi_0\rangle e^{-f}\iota_{e_i}(dV_g) \\ &= \operatorname{Re}\int_{S_\rho}\left(\langle e_i \cdot D_f \psi_0, \psi_0\rangle - \langle \xi, L_i^f \psi_0\rangle - \frac{1}{2}\langle(\nabla_i f)\xi, \psi_0\rangle\right) e^{-f}\iota_{e_i}(dV_g). \end{aligned} \tag{A.17}$$

Since $f \to 0$ at infinity, $\nabla f = O(r^{\delta-1})$, where $\delta - 1 < \tau - (n-1)$ by (2.6), and $\xi = O(r^{-\tau})$, the last term above vanishes as $\rho \to \infty$. Similarly, the second term above vanishes in the limit. On the other hand, (A.6) gives

$$\begin{aligned} e_i \cdot D_f \psi_0 &= -\frac{1}{8}(\partial_k g_{lj}) e_i \cdot e_l \cdot [e_j \cdot, e_k \cdot]\psi_0 - \frac{1}{2} e_i \cdot (\nabla f) \cdot \psi_0 + O(r^{-2\tau-1})\psi_0 \\ &= -\frac{1}{4}(\partial_k g_{lj}) e_i \cdot e_l \cdot (\delta_{jk} + e_j \cdot e_k \cdot)\psi_0 - \frac{1}{2} e_i \cdot (\nabla f) \cdot \psi_0 + O(r^{-2\tau-1})\psi_0 \\ &= -\frac{1}{4}(\partial_j g_{kj} - \partial_k g_{jj}) e_i \cdot e_k \cdot \psi_0 - \frac{1}{2} e_i \cdot (\nabla f) \cdot \psi_0 + O(r^{-2\tau-1})\psi_0. \end{aligned} \tag{A.18}$$

Writing $e_i \cdot e_k \cdot = \frac{1}{2}[e_i \cdot, e_k \cdot] - \delta_{ik}$ and noting that $[e_i \cdot, e_k \cdot]$ is skew, it follows that

$$\operatorname{Re}\langle e_i \cdot D_f \psi_0, \psi_0\rangle = \frac{1}{4}(\partial_j g_{ij} - \partial_i g_{jj} + 2(\nabla_i f) + O(r^{-2\tau-1}))|\psi_0|^2. \tag{A.19}$$

and hence (A.17) becomes

$$\frac{1}{4}\int_{S_\rho}\left(\partial_j g_{ij} - \partial_i g_{jj} + 2(\nabla_i f) + O(r^{-2\tau-1})\right)|\psi_0|^2 e^{-f}\iota_{e_i} dV_g. \tag{A.20}$$

Putting this into (A.11), letting $\rho \to \infty$ and using the definition of mass (2.2) gives the formula

$$\int_M \left(|\nabla\psi|^2 + \frac{1}{4}\mathrm{R}_f\,|\psi|^2\right) e^{-f} dV_g = \frac{1}{4}\mathfrak{m}(g) + \frac{1}{2}\lim_{\rho\to\infty}\int_{S_\rho} |\psi_0|^2 e^{-f}\iota_{\nabla f} dV_g. \tag{A.21}$$

Finally, a coordinate calculation shows that

$$\int_{S_\rho} \langle \nabla f, \nu\rangle\, e^{-f} dA = \int_{S_\rho} e^{-f}\iota_{\nabla f} dV_g, \tag{A.22}$$

and since $|\psi_0| \to 1$ at infinity, the second-to-last equation gives the weighted Witten formula

$$\int_M \left(|\nabla\psi|^2 + \frac{1}{4}\mathrm{R}_f\,|\psi|^2\right) e^{-f} dV_g = \frac{1}{4}\mathfrak{m}(g) + \frac{1}{2}\lim_{\rho\to\infty}\int_{S_\rho} \langle \nabla f, \nu\rangle\, e^{-f} dA. \tag{A.23}$$

□

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