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Asymptotic Analysis of von Neumann Entropy in Conformal Field Theory

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Abstract: Given a QFT net $\mathcal{A}$ of local von Neumann algebras $\mathcal{A}(O)$, we consider the von Neumann entropy $S_{\mathcal{A}}(O, \widetilde{O})$ of the restriction of the vacuum state to the canonical intermediate type I factor for the inclusion of von Neumann algebras $\mathcal{A}(O) \subset \mathcal{A}(\widetilde{O})$ (split property). This canonical entanglement entropy $S_{\mathcal{A}}(O, \widetilde{O})$ is finite for the chiral conformal net on the circle generated by finitely many free Fermions (here double cones are intervals). The finiteness property is derived by an explicit formula of entropy and an observation that the operators in the definition are closely related to Hankel operators. In this paper we give further analysis of this entropy using a variety of techniques that have been developed in different context, and in particular we show that there is an upper bound given by a positive constant multiply by $|\ln \eta|$, where η is the cross ratio of the underlying system, when $\eta \to 0$.

1. Introduction

von Neumann entropy is the basic concept in quantum information and extends the classical Shannon's information entropy notion to the non commutative setting. The role of entanglement in Quantum Field Theory is more recent and increasingly important; it represents a piece of the quantum information framework in this subject. It appears in relation with several primary research topics in theoretical physics as area theorems, c-theorems, quantum null energy inequality, etc. (see for instance [5,6,40] and refs. therein).

Despite the rich physical literature on the subject, the rigorous definition of entanglement entropy in QFT is however not obvious. The point is that the von Neumann algebra $\mathcal{A}(O)$ associated with a double cone spacetime region O is typically a factor of type III, so no trace exists on $\mathcal{A}(O)$ and one cannot naively extends the definition of entropy as one would do with $A = \mathcal{A}(O)$, $B = \mathcal{A}(O')$, where O is a double cone and O' is its causal complement and ω the vacuum state. Due to ultraviolet divergence, such a measure of the vacuum entanglement would always result to be infinite. By Haag duality, that holds in much generality, $\mathcal{A}(O')$ is the commutant $\mathcal{A}(O)'$ of $\mathcal{A}(O)$ on the vacuum Hilbert space $\mathcal{H}$, so the von Neumann algebra $\mathcal{A}(O) \vee \mathcal{A}(O')$ generated by $\mathcal{A}(O)$ and $\mathcal{A}(O')$ is equal to $B(\mathcal{H})$, a type I factor, and cannot be naturally isomorphic to the von Neumann tensor product $\mathcal{A}(O) \otimes \mathcal{A}(O')$ which is type III.

To get rid of short distance divergences, one may however consider a slightly larger double cone $O \subset \widetilde{O}$, namely the closure of O is contained in the interior of $\widetilde{O}$. The split property states that there is a natural isomorphism of von Neumann algebras

$$\mathcal{A}(O) \vee \mathcal{A}(\widetilde{O}') \simeq \mathcal{A}(O) \otimes \mathcal{A}(\widetilde{O}'),$$

that identifies $\mathcal{A}(O)$ with $\mathcal{A}(O) \otimes 1$ and $\mathcal{A}(O')$ with $1 \otimes \mathcal{A}(O')$.

The split property expresses the statistical independence of $\mathcal{A}(O)$ and $\mathcal{A}(\widetilde{O}')$; it was verified for the free, neutral Boson QFT case in [2]. It was studied in [11] and led to important structural features both in Mathematics and in Physics. It follows under natural, general physical requirements [4]. It holds automatically in chiral conformal QFT [28]. (See [16] for a discussion of its validity in topologically non trivial spacetimes).

Approaches to the entanglement entropy by means of the split property are studied in [8,12,17,27,29,40].

The split property is a local property, in fact it is equivalent to the existence of an intermediate type I factor $\mathcal{F}$ between $\mathcal{A}(O)$ and $\mathcal{A}(\widetilde{O})$

$$\mathcal{A}(O) \subset \mathcal{F} \subset \mathcal{A}(\widetilde{O}) . \tag{1}$$

A type I factor $\mathcal{F}$ is a von Neumann algebra isomorphic to $B(\mathcal{K})$, the algebra of all bounded linear operators on some Hilbert space $\mathcal{K}$.

We may then define the entanglement entropy of the net $\mathcal{A}$ associated with the double cones $O \subset \widetilde{O}$ as the vacuum von Neumann entropy associated with the $\mathcal{F}$ where the global systems is $B(\mathcal{H})$, the factorization is given by $\mathcal{F}$, namely $A = \mathcal{F}$, $B = \mathcal{F}'$ with a tensor product decomposition

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B , \quad A \simeq B(\mathcal{H}_A) \otimes 1, \quad B \simeq 1 \otimes B(\mathcal{H}_B),$$

and the pure state is the vacuum state.

This definition however depends on the choice of $\mathcal{F}$. Actually, if the split property holds, there are infinitely many intermediate type I factors $\mathcal{F}$ in (1). Yet, as shown in [11], there is a canonical intermediate type I factor $\mathcal{F}$, associated with the O, $\widetilde{O}$ and the vacuum vector Ω, given by the formula

$$\mathcal{F} = \mathcal{A}(O) \vee J\mathcal{A}(O)J = \mathcal{B}(\widetilde{O}) \cap J\mathcal{B}(\widetilde{O})J \tag{2}$$

(if the local von Neumann algebras are factors), with J is the modular conjugation of the relative commutant von Neumann algebra $\mathcal{A}(O)' \cap \mathcal{A}(\widetilde{O})$ associated with Ω.

We then define the (canonical) entanglement entropy of $\mathcal{A}$ with respect to O, $\widetilde{O}$ as

$$S_{\mathcal{A}}(O, \widetilde{O}) = -\mathrm{tr}(\rho_{\mathcal{F}} \log \rho_{\mathcal{F}}), \tag{3}$$

where $\mathcal{F}$ is the canonical intermediate type I factor (2). Here Tr is the trace of $\mathcal{F}$ (namely $\mathcal{F} = B(\mathcal{H}_A) \otimes 1_{\mathcal{H}_B}$ and tr corresponds to the usual trace on $B(\mathcal{H}_A)$) and $\rho_{\mathcal{F}}$ is the vacuum density matrix relative to $\mathcal{F}$.

The above definition concerns a local net $\mathcal{A}$. If $\mathcal{A}$ if a Fermi net, graded locality rather than locality holds. In this case, the split property is still defined by (1) and the entanglement entropy by (3). However, the canonical intermediate type I factor is to be defined by a twisted version of formula (2), cf. equation 50 of [25].

A main result in [25] is that above defined canonical entanglement entropy is finite for the chiral conformal net $\mathcal{M}$ generated by a complex free fermion on S^1. Here, double cones are intervals $I \subset \widetilde{I}$ of S^1.

In fact in [25] an explicit formula for the von Neumann entropy is given, and its finiteness follows from observing the connection to the theory of Hankel operators . It is in fact the first known case where such canonical entropy is proved to be finite. It is therefore a natural question to estimate this finite entropy, in particular its asymptotic property as the cross ratio $(\sin(\eta/2))^2$ goes to zero or equivalently when the end points of the interval get close to each other (cf. Remark 3.9 in [25]) . Note that due to the monoticity of relative entropy the entropy is bounded below (cf. Lemma 2.4) by $\frac{1}{6}|\ln \sin(\eta/2)|$, so the real interest is about its upper bound. Our result (cf. Cor. 3.17) is that the upper bound is again a constant multiplied by $|\ln \eta|$ as $\eta \to 0$. The proof of this result is surprisingly delicate and rely on deep results in [37], [21] and [38]. The results of [37], [21] and [38] are motivated by questions of semi-classical analysis of entropy in QFT, and the context of these questions are very different from ours. In fact since our functions are not smooth on the circle, we have to modify the proof of some of the results in these papers for our analysis. These modifications include Lemma 3.11 which is based on a result in [38], but now applied in three different scales in the proof of Th. 3.15. By using properties of Hankel operators, it turns out that we can do our estimates by removing the poles of our functions inside the unit disk. We then use a change of part of the path to evaluate the fourier coefficients of our functions, first in the relatively easy case when our functions have no branch cuts. When our functions have branch cuts inside the unit disk, we reduce our analysis to the estimation of Besov quasinorm of these functions (cf. Section 3.3). We expect that our techniques will have applications in more general cases.

There are some similarities between our entropy and reflective entropy discussed in the physics literature (cf. [9] and references therein). In [9] there are also numerical computations of such reflective entropy and their numerical data agrees with our asymptotic analysis, but it is not clear at all that those numerical computations on finite lattices in [9] actually converge to our entropy. It is an interesting question to further understand this similarity. It is also an interesting question to improve our estimates in this paper, in particular to determine the constant in Cor. 3.17. See Remark (3.18) and Remark (3.8).

The rest of this paper is organized as follows. In Section 2 we recall the entropy defined in [25] in the context of chiral net of free fermion, and recall some basic facts related to Hankel operators in [30]. We begin our asymptotic analysis in Section 3. Our basic idea is explained at the beginning of Section 3.1. Roughly speaking we deform the path of integration, removing the poles of our functions inside the unit disk, and then estimate Besov quasi-norms of these functions (cf. the proof of Th. 3.7). We then use these results in Section 3.4 to give the Schatten norm of our functions. This is based on Lemma 3.11, and a key result in Th. 3.15. In Section 3.5 we prove Cor. 3.17 as a consequence of our results in the previous two sections and results of [21]. In the last Section we show that our entropy function is continuous in η and goes to 0 as η goes to π.

2. Preliminaries

2.1. Schatten-von Neumann Ideals. This paper relies on the results for general quasi-normed ideals of compact operators. Here we limit our attention to the case of Schatten-von Neumann operator ideals $S_q, q > 0$. Detailed information on these ideals can be found e.g. in [30] and [36]. We shall point out only some basic facts. For a compact operator A on a separable Hilbert space H, denote by $s_n(A)$, $n = 1, 2, \ldots$ its n-th singular values, that is, the eigenvalues of the operator $|A| := \sqrt{A^*A}$. Note that if R_1, R_2 are bounded operators, Then (cf. [36])

$$s_n(R_1 A R_2) \leq ||R_1|| s_n(A) ||R_2|| \tag{4}$$

where $||A||$ to denote the norm of an operator A. We denote the identity operator on H by 1. The Schatten-von Neumann ideal $S_q, q > 0$ consists of all compact operators A, for which $|A|_{S_q} := (\sum_{k=1}^{\infty} s_k(A)^q)^{\frac{1}{q}} < \infty$. Note that $|A|_{S_q} = |A^*|_{S_q}$.

If $q \geq 1$, then the above functional defines a norm; if $0 < q < 1$, then it is a so-called quasi-norm. There is nevertheless a convenient analogue of the triangle inequality, which is called the q-triangle inequality:

$$|A_1 + A_2|_{S_q}^q \leq |A_1|_{S_q}^q + |A_2|_{S_q}^q, 0 < q \leq 1 \tag{5}$$

We also have the Holder inequality:

$$|A_1 A_2|_{S_q} \leq |A_1|_{S_{q_1}} |A_2|_{S_{q_2}}, 1/q = 1/q_1 + 1/q_2, 0 < q_1, q_2 \leq \infty \tag{6}$$

See [19] and also [3]. In what follows we focus on the case $q \in (0, 1]$. We will use $||A||$ to denote the norm of an operator, and $||A||_1$ the trace of $|A|$. By definition

$$|||A|^q||_1 = |A|_{S_q}^q$$

Note that for a nonzero operator A, the singular values of $A/||A||$ is bounded above by 1, therefore if $0 < p < q \leq 1$ we have $|A/||A|||_{S_q}^q \leq |A/||A|||_{S_p}^p$, from which we have

$$|A|_{S_q}^q \leq |A|_{S_p}^p ||A||^{q-p} \tag{7}$$

2.2. Basic Representation of LU_1 and Free Fermion net. Let H denote the Hilbert space $L^2(S^1; \mathbb{C})$ of square-summable $\mathbb{C}$-valued functions on the circle. The group LU_1 of smooth maps $S^1 \to U_1$, with U_1 the set of the unit circle in $\mathbb{C}$, acts on H as multiplication operators.

Let us decompose $H = H_+ \oplus H_-$, where

$$H_+ = \{\text{functions whose negative Fourier coeffients vanish}\}.$$

We denote by P the Hardy projection from H onto H_+.

Denote by $U_{\text{res}}(H)$ the group consisting of unitary operator A on H such that the commutator $[P, A]$ is a Hilbert-Schmidt operator. Denote by $\text{Diff}^+(S^1)$ the group of orientation preserving diffeomorphism of the circle. It follows from Proposition 6.3.1 and Proposition 6.8.2 in [31] that LU_1 and $\text{Diff}^+(S^1)$ are subgroups of $U_{\text{res}}(H)$. The basic representation of LU_1 is the representation on Fermionic Fock space $F_p = \Lambda(PH) \otimes \Lambda((1-P)H)^*$ as defined in Section 10.6 of [31]. For more details, see [31] or [39]. Such a representation gives rise to a graded net as follows. Denote by $\mathcal{A}(I)$ the von Neumann algebra generated by $c(\xi)'s$, with $\xi \in L^2(I, \mathbb{C})$. Here $c(\xi) = a(\xi) + a(\xi)^*$ and $a(\xi)$ is the creation operator defined as in Chapter 1 of [39]. Let $Z : F_p \to F_p$ be the Klein transformation given by multiplication by 1 on even forms and by i on odd forms. It

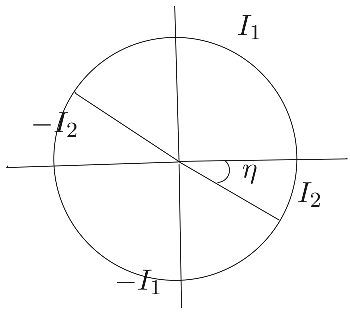

Fig. 1. The symmetric intervals

follows from Section 15 of chapter 2 of [39] that $\mathcal{A}$ is a graded Möbius covariant net, and $\mathcal{A}$ will be called the *net of free fermion*. $\mathcal{A}$ is strongly additive, and the commutant of $\mathcal{A}(I)$ is $Z\mathcal{A}(I')Z^{-1}$ where I' is the complement of I on the circle.

Let I_1, I_2 be two open intervals on the circle, and I_1, I_2 are disjoint, that is $\bar{I}_1 \cap \bar{I}_2 = \varnothing$, and $I = I_1 \cup I_2$.

For bounded operators $A, B : F_p \to F_p$, we define $A^+ = \Gamma A \Gamma,\ A^- = A - A^+$, where Γ is an operator on F_p given by multiplication by 1 on even forms and -1 on odd forms. Note that $Z = \frac{1-i\Gamma}{1-i}$.

An operator A is called even (resp. odd) if $A = A^+$ (resp. $A = A^-$). Note that $\omega(a) = 0$ if a is odd, where ω is the vacuum state corresponding to the vacuum vector Ω.

We set

$$\omega_1 \otimes_2 \omega_2(AB) = \langle A\Omega, \Omega\rangle\langle B\Omega, \Omega\rangle, \quad \forall A \in \mathcal{A}(I_1),\ B \in \mathcal{A}(I_2)\,.$$

By (1) Lemma 3.1 in [24] this defines a normal state on the von Neumann algebra generated by $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$.

The mutual information $S(\omega, \omega_1 \otimes_2 \omega_2)$ (cf. Definition 2.1 of [24]) is computed in Section 3 of [24].

2.3. von Neumann Entropy from Split Property. In this section we recall the von Neumann entropy defined in [25] from split property that we aim to compute in this paper.

2.3.1. General Symmetric Interval We will focus on the one particle structure on $L^2(S^1; \mathbb{C})$ in this section. On S^1, we consider the following general four "symmetric intervals"

$$I_1 = \{e^{i\theta} : 0 < \theta < \phi\}, \quad I_2 = \{e^{i\theta} : \phi - \pi < \theta < 0\}, \tag{8}$$

$$-I_1 = \{e^{i(\pi+\theta)} : 0 < \theta < \phi\}, \quad -I_2 = \{e^{i(\pi+\theta)} : \phi - \pi < \theta < 0\}, \quad 0 < \phi < \pi\,. \tag{9}$$

We will denote by $\eta = \pi - \phi$. See Figure 1.

Denote by $I_0 := \{e^{i\theta} : 0 < \theta < 2\phi\}$. For any interval I if we denote by I^2 the set of z such that $z = w^2$ for some $w \in I$, then it is clear that $I_0 = I_1^2$. We shall consider the action of $SU(1, 1)$ on S^1 which is given by $z \to \frac{az+b}{\bar{b}z+\bar{a}}$ with $|a|^2 - |b|^2 = \pm 1$. The Möbius group Mob is the subgroup of $SU(1,1)$ of elements with determinant $|a|^2 - |b|^2 = 1$. The action $z \to \frac{1}{z}$ is orientation reversing. This element has $a = d = 0, b = c = -1$.

If $m(z) = \frac{az+b}{\bar{b}z+\bar{a}}$, the unitary action of m on S^1 is given by (See Section 4 of [39])

$$(U_m f)(z) = (a - \bar{b}z)^{-1} f(m^{-1}z)\,. \tag{10}$$

Since $(a - \bar{b}z)^{-1}$ is holomorphic for $|z| < 1$ and $|a| > |b|$, U_m and its inverse preserves PH, and so U_m commutes with the Hardy space projection P. The flip map $(F_1 f)(z) = \frac{1}{z} f(\frac{1}{z})$ clearly satisfies $PF_1P = 1 - P$. By sending the orientation reversing element $z \to \frac{1}{z}$ in $SU(1,1)$ to F_1, we get an action of $SU(1,1)$ on $\mathcal{H}$ which is of the form

$$(U_m f)(z) = \alpha_m(z) f(m^{-1}z)\,, \tag{11}$$

where $m(z) = \frac{az+b}{\bar{b}z+\bar{a}}$, $\alpha_m(z) = (a - \bar{b}z)^{-1}$.

Let $m \in$ Mob be such that mI_0 is the upper half circle. Let $m_1 = m^{-1}F_1m$. It is straightforward to see that

$$m_1(z) = \frac{z(1+e^{2i\phi})/2 - e^{2i\phi}}{z - (1+e^{2i\phi})/2}\,. \tag{12}$$

Define

$$(F_0 f)(z) = \alpha_{m_1}(z) f(m_1^{-1}z)\,. \tag{13}$$

We have the following definition

$$(jf)(z) := \alpha_{m_1}(z^2)\left(\frac{1}{2} + \frac{z}{2u}\right) f(u) + \alpha_{m_1}(z^2)\left(\frac{1}{2} - \frac{z}{2u}\right) f(-u)\,, \tag{14}$$

where

$$u^2 = m_1(z^2) \tag{15}$$

Note that $m_1(z^2)$ depends on η.

Note that j maps $L^2(I_1)$ to $L^2(I_2 \cup -I_2)$. We will denote by M_I the multiplication operator by χ_I, the characteristic function of interval I.

By definition, M_{I_1} and $jM_{I_1}j$ are orthogonal projections whose ranges $L^2(I_1)$ and $jL^2(I_1)$ are also orthogonal. Hence $P_{12} := M_{I_1} + jM_{I_1}j$ is a projection. If we wish to emphasize the dependence on η we will write P_{12} as $P_{12}(\eta)$.

Note that $L^2(I_1) \oplus jL^2(I_1)$ is the canonical type I standard subspace which is intermediate between $L^2(I_1)$ and $L^2(I_1 \cup I_2 \cup -I_2)$. The following is theorem 3.3 from [25]:

Theorem 2.1.

$$P_{12} = M_g + M_h R \quad \text{on } L^2(S^1)\,,$$

is a projection onto $L^2(I_1) \oplus jL^2(I_1)$ *where*

$$g(z) = \frac{(u+z)^2}{4uz}\chi_{I_1}(u) - \frac{(u-z)^2}{4uz}\chi_{I_1}(-u) + \chi_{I_1}(z)\,, \tag{16}$$

$$h(z) = \frac{z^2 - u^2}{4uz}(\chi_{I_1}(u) - \chi_{I_1}(-u)), \quad (17)$$

$$(R(f)(z) = f(-z), u^2 = m_1(z^2), \quad (18)$$

$$m_1(z) = \frac{z(1 + e^{2i\phi})/2 - e^{2i\phi}}{z - (1 + e^{2i\phi})/2}. \quad (19)$$

We note that since g, h are invariant under $u \to -u$, these functions are independent of the choice of the square root u of $m_1(z^2)$ in Th. 2.1.

For simplicity we will write multiplication operator such as M_g simply as g when no confusion arises. For an example we can write $P_{12} = g + hR$. Similarly we write g^* the complex conjugate of a function such as g.

The von Neumann entropy $S(F, \eta)$ that comes from the canonical type I standard subspace $L^2(I_1) \oplus jL^2(I_1)$ is defined as follows. Let $f_0 := -x \ln x - (1 - x) \ln(1 - x)$, $0 < x < 1$, and we define $f_0(0) = f_0(1) = 0$. We make the following definition:

Definition 2.2.

$$S(F, \eta) = \mathrm{tr}(f_0(PP_{12}(\eta)P))$$

where P_{12} is as in Th. 2.1. For any operator self adjoint T and continuous f defined on the spectrum of T and PTP, if $f(PTP) - Pf(T)P$ is trace class, we use $\tau(T, f)$ to denote the trace of $f(PTP) - Pf(T)P$.

Since $P_{12}(\eta)$ is a projection and $f_0(P_{12}(\eta)) = 0$, $S(F, \eta) = \tau(P_{12}(\eta), f_0)$. We choose the notation $S(F, \eta)$ since $S(F)$ is what we used in [25], and we put in extra η to emphasize the dependence on η. Let us first explain why $S(F, \eta) = S(F)$ where $S(F)$ is as in Th. 3.8 of [25]. Let us first recall how $S(F)$ is defined. We shall denote by $F := L^2(I_1) \oplus jL^2(I_1)$ the canonical type I standard subspace. Recall that on $\mathcal{H} = L^2(S^1; \mathbb{C})$, the complex structure on $\mathcal{H}$ is given by $i(2P - 1)$, with P the projection onto the Hardy space. Let F' be the orthogonal complement of $i(2P - 1)F$. Let $P_F := P_{12}(\eta)$, $P_{F'}$ be the projections onto F, F' respectively. Then $P_{F'} = (2P - 1)(1 - P_F)(2P - 1)$, and

$$P_F P_{F'} P_F = P_F(2P - 1)(1 - P_F)(2P - 1)P_F = 4(P_F P P_F - (P_F P P_F)^2)$$

By Lemma 2.4 of [25] $P_F P_{F'} P_F = 4\frac{\Delta_F}{(\Delta_F+1)^2} P_F$ where Δ_F is the modular operator associated with F, it follows that $P_F P P_F - (P_F P P_F)^2 = \frac{\Delta_F}{(\Delta_F+1)^2} P_F$. Let $0 < \lambda_i < 1$, $1 \leq i < \infty$ be the list (counting multiplicities) of eigenvalues of Δ_F that is in the interval $(0, 1)$. Define $\mu_i = \frac{1}{1+\lambda_i}$. By definition $S(F)$ is the von Neumann entropy of state $\rho'_F = \frac{\widehat{\rho'_F}}{\mathrm{tr}\widehat{\rho'_F}}$, where $\widehat{\rho'_F}$ is $\Lambda(\Delta_F|_{\mathcal{H}_F(0,1)})$ by Cor. 2.10 of [25]. By using Prop. 2.11 of [25] and a straightforward computation we have

$$S(F) = -\sum_{1 \leq i < \infty} (\mu_i \ln \mu_i + (1 - \mu_i) \ln(1 - \mu_i))$$

By examing the spectrum of $\frac{\Delta_F}{(\Delta_F+1)^2} P_F$ as in Lemma 2.7 in [25] we see that the list (counting multiplicities) of eigenvalues of $\frac{\Delta_F}{(\Delta_F+1)^2} P_F$ is $\frac{\lambda_i}{(\lambda_i+1)^2} = \mu_i(1 - \mu_i)$, $1 \leq i <$

∞. Since $(P_F P P_F - (P_F P P_F)^2) = \frac{\Delta_F}{(\Delta_F+1)^2} P_F$, the list (counting multiplicities) of eigenvalues of $P_F P P_F$ is μ_i', where $\mu_i' = \mu_i$ or $\mu_i' = 1 - \mu_i$, $1 \le i < \infty$. Therefore

$$S(F) = -\sum_{1\le i<\infty} (\mu_i' \ln \mu_i' + (1-\mu_i')\ln(1-\mu_i'))$$

Since (cf. Chapter 1 of [36]) the set of nonzero elements in the spectrum of T^*T is the same as the set of nonzero elements in the spectrum of TT^* for a bounded operator T, it follows that the list of nonzero eigenvalues (counting multiplicities) of $PP_FP = PP_{12}(\eta)P = PP_FP_FP$ is the same as that of $P_FPPP_F = P_FPP_F$. So we have $S(F,\eta) = S(F)$.

In [25] we proved that $S(F,\eta)$ is finite by observing its connection with Hankel operators. This relies on the growth of fourier coefficients of g, h. We recall the following result which is proved in [25] and follows essentially from an observation of [18], see Th. 3.7 in [25].

Lemma 2.3. *Suppose* $f = \sum_n f_n z^n$.

If $|f_n| \le Cn^{-\alpha}$ *with* $\alpha > \frac{3}{2}, n \ge 0$, *then* $|Pf(1-P)|_{S_q}$ *is bounded by a constant which only depends on C and* $1 > q > \frac{1}{\alpha - \frac{1}{2}}$ *; If* $|f_n| \le C|n|^{-\alpha}$ *with* $\alpha > \frac{3}{2}, n < 0$, *then* $|(1-P)fP|_{S_q}$ *is bounded by a constant which only depends on C and* $1 > q > \frac{1}{\alpha-\frac{1}{2}}$.

Note that g, h are invariant under $u \to -u$. Note that $u(z) \in I_1 \cup -I_1$ if an only if $u(z)^2 \in I_1^2 = I_0$. Since $u(z)^2 = m_1(z^2)$, it follows that $u(z)^2 \in I_1^2 = I_0$ if and only if $m_1(z^2) \in I_0$. By definition of $m_1(z)$, we have $m_1^{-1}(I_0)$ is the complement of I_0, which is I_2^2. It follows that $u(z) \in I_1 \cup -I_1$ if an only if $z \in I_2 \cup -I_2$. So $h = \frac{z^2-u^2}{4uz}\chi_{I_2 \cup -I_2}(z)$. Note that this matches with equation (34) of [25]. For the reader who may be confused with the equation (34) of [25], we note that in the definition of h in 2.1 it is important that we have $\chi_{I_1}(u)$, not $\chi_{I_1}(z)$.

If $u(z) \in I_1$, then $g(z) = \frac{(u+z)^2}{4uz} = \frac{1}{2} + \frac{1}{4}(\frac{u}{z} + \frac{z}{u})$. Note that $|u| = |z| = 1$, it follows that $g \ge 0$. Similarly $g \ge 0$ if $u(z) \in -I_1$. When $z \in I_1$, $g = 1$, and $g = 0$ when $z \in -I_1$. So $g \ge 0$. Similarly we can check that ih is real. $g - 1/2$ and h are both odd functions of z, and $g^2 - g = h^2$. To do computations for η close to 0, it is convenient to choose an analytic continuation of u inside the unit disk. Recall that $u^2 = m_1(z^2) = \frac{z^2\cos\eta - e^{-i\eta}}{e^{i\eta}z^2 - \cos\eta}$. Note that the roots of $z^2\cos\eta - e^{-i\eta}$ are outside the unit disk. For the square root of $z^2\cos\eta - e^{-i\eta}$, we can choose any branch cut outside the closed unit disk, for an example, two half lines coming out of the two roots of the equation $z^2\cos\eta - e^{-i\eta} = 0$ which do not intersect the closed unit disk. The two roots of $e^{i\eta}z^2 - \cos\eta$ are inside the disk, and for the square root of $e^{i\eta}z^2 - \cos\eta$, we can choose the branch cut to be the closed line segment connecting $\pm e^{-i\eta/2}\sqrt{\cos\eta}$ which are the poles of u (cf. Page 128 of [1] for such a choice of branch cut for square root of a quadratic function). u is then the quotient of these two functions. u is an analytic function in the unit disk minus the branch cut. We will see that this branch cut is important for our analysis when $\eta \to 0$.

On I_2

$$g'(z) = \frac{(u^2 - z^2)(zu' - u)}{4z^2u^2}, h'(z) = \frac{(u^2+z^2)(zu'-u)}{4z^2u^2} \tag{20}$$

$$m_1(z^2) = \frac{z^2\cos\eta - e^{-i\eta}}{e^{i\eta}z^2 - \cos\eta}, m_1(z)' = \frac{-\sin^2(\eta)e^{i\eta}}{(e^{i\eta}z - \cos\eta)^2}, m_1(z)'' = \frac{-\sin^2(\eta)e^{2i\eta}}{(e^{i\eta}z - \cos\eta)^3} \tag{21}$$

Let $L(z) := |z^2 - e^{-i\eta}\cos\eta|$. Let us explain how to estimate the derivatives of g, h when η is sufficiently small. We will use g as an example since h is similar. If $u(z) \in I_1$, then $g(z) = \frac{(u+z)^2}{4uz} = \frac{1}{2} + \frac{1}{4}(\frac{u}{z} + \frac{z}{u})$. Hence if $u(z) \in I_1$ the derivatives of g are linear combinations of derivatives of $\frac{u}{z}$ and $\frac{z}{u}$. To compute the derivatives of $u(z)$, by definition $u(z)^2 = m_1(z^2)$. So by Chain Rule $u(z)'u(z) = m_1'(z^2)z$. Keep in mind $|u| = |z| = 1$. It follows that g' is up to addition by a bounded function $m_1'(z^2)$ multiplied by a bounded function. g'' is the sum of a bounded function, $((m_1'(z^2)))^2$ multiplied by a bounded function and $m_1''(z^2)$ multiplied by a bounded function. The same idea applies to all other cases. Note that when η is sufficiently small $L(z) \geq 1/2\eta^2$, and therefore from (21) we have

$$|((m_1'(z^2)))^2| = \frac{(\sin\eta)^4}{L(z)^4} \leq 2\frac{(\sin\eta)^2}{L(z)^3}$$

From this we have

$$|h'| \leq O(1) + C\frac{\sin^2(\eta)}{L(z)^2}, |h''| \leq O(1) + C\frac{\sin^2(\eta)}{L(z)^3}, |g'| \leq O(1) + C\frac{\sin^2(\eta)}{L(z)^2}, |g''| \leq O(1) + C\frac{\sin^2(\eta)}{L(z)^3} \tag{22}$$

where $O(1)$ and C are constants independent of η.

Note that the minimal value of $L(z^2)$ is when $z = e^{-i\eta/2}$, i.e. when z is at the middle point of I_2, and $L(e^{-i\eta/2}) = |1 - \cos(\eta)| \sim \frac{1}{2}\eta^2$ when η is close to 0, it follows that on $I_2 \cup -I_2$

$$|h'| \leq C\frac{1}{L(z)}, |h''| \leq C\frac{1}{L(z)^2}, |g'| \leq C\frac{1}{L(z)}, |g''| \leq C\frac{1}{L(z)^2} \tag{23}$$

where C is independent of η and η is close to 0. Note that since $g = 1$ on I_1, the above formula for g also holds on I_1.

Note that $m_1(z)$ is conjugate to the flip, and fix the end points of I_0. When z is at the end points of I_2 or $-I_2$, z^2 takes values at the end points of I_0. It follows that at the end points of I_2 or $-I_2$, $u(z)^2 = m_1(z^2) = z^2$. From formula (20) we can see that g' is continuous, and $g' = 0$ on the boundary of $I_2, -I_2$. g'' exists at all points on the circle except the four boundary points of $I_2, -I_2$ and is bounded. Since $g' = 0$ on the boundary of $I_2, -I_2$, it follows that the second derivative of g in the distribution sense agrees with g'', and in particular it is essentially bounded. Hence $g \in W^{2,\infty}$. Similarly from formula (20) we see that h' is not continuous, but $h'(z)$ on $I_2 \cup -I_2$ is bounded when z is close to the boundary of $I_2, -I_2$.

The image of I_2 (resp. $-I_2$) under V is interval $(1, \tan\psi)$ (resp. $(-1, -\cot\psi)$) with $\psi = (\pi/4 + \eta/2)$. The cross ratio of the interval $-I_1, I_1$ in the clockwise order is $\frac{1}{\sin^2(\eta/2)}$.

First we recall the lower bound of $S(F, \eta)$:

Lemma 2.4. $S(F, \eta) \geq \frac{-1}{6}\ln(\sin(\eta/2))$.

Proof. Since $\mathcal{A}(I_1) \subset F \subset \mathcal{A}(I_1 \cup I_2 \cup -I_2)$, and the commutant of $\mathcal{A}(I_1 \cup I_2 \cup -I_2)$ is $Z\mathcal{A}(-I_1)Z^{-1}$, where $Z = \frac{1-i\Gamma}{1-i}$ is the Klein operator (cf. Section 2.2), we have $\mathcal{A}(-I_1) \subset ZF'Z^{-1}$.

Denote by $\omega_F, \omega_{F'}$ the restriction of the vacuum state to F and its commutant F'. Let $\hat{\omega}$ be the tensor state $\omega_F \otimes \omega_{F'}$, namely $\hat{\omega}(m_1 m_2) = \omega(m_1)\omega(m_2), \forall m_1 \in F, m_2 \in F'$. Let us show that when restricting $\hat{\omega}$ to $\mathcal{A}(I_1) \vee \mathcal{A}(-I_1)$, this is the same as $\omega_1 \otimes_2 \omega_2$ as in Sect. 2.2. Since F, F' are type I factors and $\mathrm{Ad}\Gamma$ are automorphisms of F (resp. F') of order two, it follows that $\Gamma = u_1 u_2$ where u_1 (resp. u_2) is unitary element in F (resp. F'). Multiplying by a phase factor if necessary we can choose $u_1^2 = 1, u_2^2 = 1$. Since $\Gamma u_1 \Gamma = u_1^3 = u_1$, it follows that u_1 is an even element, and similarly u_2 is an even element.

By definition $\hat{\omega}$ is the same as $\omega_1 \otimes_2 \omega_2$ on elements of the form ab^+, where $a \in \mathcal{A}(I_1)$, and b^+ is an even element of $\mathcal{A}(-I_1)$, i.e., $\Gamma b^+ \Gamma = b^+$. It is sufficient to check $\hat{\omega}(ab^-) = 0$ if $\Gamma b^- \Gamma = -b^-, b^- \in \mathcal{A}(-I_1)$, i.e., b^- is an odd element.

Note that $Zb^- Z^{-1} = -i\Gamma b^- \in F'$, and so $b^- = i\Gamma Z b^- Z^{-1} = iu_1 u_2 Z b^- Z^{-1}$. By definition of $\hat{\omega}$ we have

$$\hat{\omega}(ab^-) = \hat{\omega}(iau_1u_2Zb^-Z^{-1}) = \omega(iau_1)\omega(u_2Zb^-Z^{-1})$$

Note that $u_2 Z b^- Z^{-1}$ is an odd element in F' and so $\omega(u_2 Z b^- Z^{-1}) = 0$.

By monotonicity of relative entropy (cf. Chapter 5 of [26])

$$S(\omega, \omega_1 \otimes_2 \omega_2) \leq S(\omega, \omega_F \otimes \omega_{F'}) = 2S(F, \eta)$$

By Th. 3.16 in [24] we have proved the Lemma. □

2.4. An Inequality from Besov Quasinorm. We proceed now to the Besov classes $B_p^{\frac{1}{p}}$ for $0 < p < 1$. Let F be an infinitely differentiable function on the real line such that $F \geq 0$, with support in $[1/2, 2]$, and

$\sum_{n\geq 0} F(\frac{x}{2^n}) = 1, \forall x \geq 1$. It is very easy to construct such a function. We can take a nonnegative smooth function F on the interval $[1/2, 1]$ such that $F(1/2) = 0, F(1) = 1, F^{(k)}(1/2) = F^{(k)}(1) = 0, \forall k \geq 1$. Then we can put $F(x) = 1 - F(x/2), x \in [1, 2]$ and $F(x) = 0$, when x is outside $[1/2, 2]$. Given an analytic function G in the unit disk with continuous extension to the boundary, assume that $G(z) - G(0) = \sum_{n>0} G_n z^n$, define $F_n * G(z) = \sum_{j\geq 1} F(j/n) G_j z^j$ where $n \geq 1$ is an integer. Note that $F_n * G(z)$ is a trignometric polynomial of degree less than $2n$. By definition we have $G(z) - G(0) = \sum_{m\geq 0, n=2^m} F_n * G$. Let $z = e^{2\pi i t}$.

It follows from Page 250 of [30] that if $0 < q < 1$

$$|PG(1-P)|_{S_q}^q \leq \sum_{m=0,n=2^m}^{\infty} 2^{1-q} 2^m \int_{-1/2\leq t\leq 1/2} |F_n * G(z)|^q dt \tag{24}$$

This inequality will play a crucial role in our paper. We shall refer to $\sum_{m=0,n=2^m}^{\infty} 2^m \int_{-1/2\leq t\leq 1/2} |F_n * G(z)|^q dt$ as the *Besov quasinorm* of G. The definition depends on the choice of F, and all choices give equivalent seminorms, but we shall not make use of this fact. The other direction that

$$\sum_{m=0,n=2^m}^{\infty} 2^m \int_{-1/2\leq t\leq 1/2} |F_n * G(z)|^q dt < \infty$$

implies $|PG(1-P)|_{S_q}^q < \infty$ is also true, see Th. 3.1 of [30].

2.5. Poisson Summation. Let F be an infinitely differentiable function on the real line with support in $[1/2, 2]$, and let m be a positive integer. Consider $F_m(z) = \sum_{k\in\mathbb{Z}} F(k/m)$ z^k where $z = e^{2\pi it}$ with t a point on the upper half plane (including the boundary real line).

Recall that $\mathcal{F}F$ is the Fourier transform of F, $\mathcal{F}F(s) = \int F(x)e^{-2\pi ixs}dx$. The proof of the following Lemma is essentially the same as the proof of Lemma 3.3 in [30]:

Lemma 2.5. *Let $G_m(t) = F_m(e^{2\pi it}) - m(\mathcal{F}F)(-mt)$ where t is on the upper half plane with real part in* $[-1/2, 1/2]$. *Denote by G_m the maximum of $|G_m(t)|$ on* $[-1/2, 1/2]$. *Then $G_m m^N \to 0$ for all $N > 0$.*

Proof. Let $\psi(x) = F(x/m)e^{2\pi xit}$. Note that $\psi(x)$ is a Schwartz function since t is on the upper half plane, and the support of F is in $[1/2, 2]$. The rest of the proof using Poisson summation formula is exactly the same as the proof of Lemma 3.3 in [30]. □

It is interesting to note that if t is real, $F_m(e^{2it})$ is a periodic function in t and therefore can be thought as a function on the unit circle, but $m(\mathcal{F}F)(-mt)$ is not. Nevertheless $m(\mathcal{F}F)(-mt)$ captures the dominating part of $F_m(z)$ when $m \to \infty$ as the above Lemma shows. By repeatedly using integration by parts we have

$$(\mathcal{F}F)(-mt) \leq C_N \frac{e^{-ms}}{1 + m^N|t|^N}, \forall N > 0 \tag{25}$$

where s is the imaginary part of t, and C_N is a constant which only depends on F.

3. Asymptotic Analysis

We will determine the upper bound of $S(F, \eta)$ in the next few sections. Let us first describe the basic ideas.

We note that

$$PP_{12}P = PgP + PhPR$$

, and both PgP and PhP (cf. Th. 2.1 for definitions) are Toeplitz operators. When η is small, the support of h shrinks to zero size, so we expect the main contribution to $S(F, \eta)$ should come from PgP. To do this we first need to have a good control on the Schattern-von Neumann norm of $Ph(1-P)$, this is done in Sect. 3.3. There is also a further complication concerning PgP. It turns out that $f_0(PgP)$ is not trace class, but $f_0(PgP) - Pf_0(g)P$ is. This problem is addressed in Sect. 3.4.

Suppose a function $F(z, \eta)$, $z \in S^1$ is defined on the circle which depends also on a parameter $0 < \eta < \pi$. We always assume that F is bounded, i.e., $|F(z, \eta)| \leq M$ for some constant M which is independent of z, η. We are interested in the property of $F(z, \eta)$ when $\eta \to 0$.

Definition 3.1. A bounded function $F(z, \eta)$ is said to be **very good** if both $|PF(1-P)|_{S_q}$ $|(1-P)FP|_{S_q}$ are $O(1)$ when $\eta \to 0$ for a $0 < q < 1$. A function $F(z, \eta)$ is said to be **good** if both $|PF(1-P)|_{S_q}^q$ and $|(1-P)FP|_{S_q}^q$ are $o(-\ln \eta)$ when $\eta \to 0$. We write $F(z, \eta) \sim G(z, \eta)$ if there exist two positive constants C_1, C_2 such that

$$C_1|G(z, \eta)| \leq |F(z, \eta)| \leq C_2|G(z, \eta)|$$

Proposition 3.2. *(1) If F is good (resp. very good) , then $|FP - PF|^q_{S_q} = o(-\ln \eta)$ (resp. $O(1)$) ;*
(2) If F and G are good (resp. very good), then both $F+G$ and FG are good (resp. very good).

Proof. By equation (7) we may assume that F, G are good for the same q.
Ad (1): $FP - PF = -PF(1-P) + (1-P)FP$
So by q-triangle inequality

$$|FP - PF|^q_{S_q} \le |(1-P)FP|^q_{S_q} + |PF(1-P)|^q_{S_q}$$

and (1) follows.
Ad (2): The statement for $F+G$ follows from the q-triangle inequality as in (5).
Note that $PFG(1-P) = (PF - FP)G(1-P) + FPG(1-P)$. So we have

$$\begin{aligned}|PFG(1-P)|^q_{S_q} &\le |(PF - FP)G(1-P)|^q_{S_q} + |FPG(1-P)|^q_{S_q}\\ &\le |(PF-FP)|^q_{S_q}||G||^q + |PG(1-P)|^q_{S_q}||F||^q\end{aligned}$$

and (2) is proved. □

3.1. Deformation of path. We'd like to use Lemma 2.3 to do estimation. For this purpose it is important to estimate the growth of the Fourier coefficients of our functions such as h, g. Unfortunately h'' grows like (cf. equation (22)) $\frac{\eta^2}{L(z)^3}$ on I_2, and $L(z) \sim (\theta + \eta^2)$ where θ is the distance between z and the middle point of I_2. This makes it difficult or even impossible to obtain $O(n^{-2})$ type estimate. One simple idea is to see if we can use Cauchy's theorem to deform the path I_2 to a path where h'' is better controlled. A natural such path is the path N which join the ends of I_2 inside the unit disk with property $|L(z)| = \sin \eta$. On this path N, $h'' \sim \frac{1}{\eta}$, and when integrated over N which has length $\sim \pi\eta$ will give us $O(1)$. But we have to pay close attention to possible poles and branch cuts enclosed by I_2 and N. We will see that ultimately it is the branch cut that is responsible for the asymptotic growth of our entropy.

3.2. Deformation of path: The case with no Branch cut.

Lemma 3.3. *Assume that $F(z, \eta)$ is analytic in the interior bounded by I_2 and N in the unit disk, and has continuous first derivative on I_2. In addition assume on the circle F is 0 at the boundary of I_2, and F' is $O(1)$ on the boundary of I_2. If $\int_N |F''| = O(1)$ where N is the path which join the ends of I_2 inside the unit disk with property $|L(z)| = \sin \eta$. Then $a_n(F) := \int_{I_2} Fz^n dz = O(n^{-2})$, $\forall n \ge 0$.*

Proof. By our assumptions on F and integration by parts

$$a_n = \int_{I_2} Fz^n dz = \frac{1}{(n+1)(n+2)} \int_{I_2} F'' z^n dz + O(n^{-2})$$

It is sufficient to check that

$$\int_{I_2} F'' z^n dz = O(1)$$

Since F is analytic in the unit disk, deforming the path I_2 to N, and keep in mind $|z^n| \leq 1, n \geq 0$ when $|z| \leq 1$ we have

$$|\int_{I_2} F'' z^n dz| = |\int_N F'' z^n dz| \leq \int_N |F''| = O(1)$$

□

Proposition 3.4. *Both u^2 and u^{-2} are very good.*

Proof. By definition

$$u^2 = m_1(z^2) = \frac{z^2 \cos\eta - e^{-i\eta}}{e^{i\eta} z^2 - \cos\eta}$$

$$m_1(z^2) = \frac{z^2 \cos\eta - e^{-i\eta}}{e^{i\eta} z^2 - \cos\eta} = e^{-i\eta}\cos\eta - \frac{e^{-i2\eta}\sin^2(\eta)}{z^2 - e^{-i\eta}\cos\eta}$$

It follows that

$$Pu^2(1-P) = 0$$

since u^2 is analytic outside the unit disk, and by using Laurent series for u^2

$$(1-P)u^2 P = (1-P)T_1 P - (1-P)T_2 P$$

where

$$T_1 = \frac{1/2 e^{-3/2 i\eta}(\sin(\eta))^2 (\cos\eta)^{-1/2}}{z - e^{-1/2 i\eta}\sqrt{\cos\eta}}$$

$$T_2 = \frac{1/2 e^{-3/2 i\eta}(\sin(\eta))^2 (\cos\eta)^{-1/2}}{z + e^{-1/2 i\eta}\sqrt{\cos\eta}}$$

Note that $(1-P)T_1 P$ is a rank one operator by using Laurent series for T_1, and the norm of $PT_1(1-P)$ is given by the maximum of $|T_1|$ on the circle. These are very special cases of finite rank Hankel operators, cf. 1.3 of [30] for more details. The maximum of $|T_1|$ on the circle is

$$(\cos\eta)^{-1/2}\frac{(\sin(\eta))^2}{1-\sqrt{\cos\eta}} = O(1)$$

as $\eta \to 0$. It follows that

$$|(1-P)T_1 P)|_{S_q}^q = O(1)$$

for any $q > 0$. Similarly

$$|(1-P)T_2 P|_{S_q}^q = O(1)$$

for any $q > 0$ and we have proved u^2 is very good. Note that $[Pu^{-2}(1-P)]^* = (1-P)u^2 P$ It follows that

$$|Pu^{-2}(1-P)|_{S_q}^q = |(1-P)u^2 P|_{S_q}^q$$

and the Proposition is proved. □

Proposition 3.5. *$(z^{-2} - u^{-2})(\chi_{I_2} + \chi_{-I_2})$ and its complex conjugate are very good.*

Proof. It is sufficient to prove that $(z^{-2}-u^{-2})\chi_{I_2}$ and its complex conjugate $(z^2-u^2)\chi_{I_2}$ are very good. The proof for $(z^{-2} - u^{-2})\chi_{-I_2}$ and its complex conjugate is similar. To simplify writing we will denote the function $(z^{-2} - u^{-2})\chi_{I_2}$ by h_1 only in the proof of this proposition. We first show that $|(1 - P)h_1 P|_{S_q} = O(1)$ for some $0 \le q < 1$. By Lemma 2.3 it is enough to show that $\int_{I_2} h_1 z^n dz = O(n^{-2}), \forall n \ge 0$.

On N we shall need an estimate of derivatives of h_1 similar to that of formula (22) for h on the circle. Note that when η is small enough, $|z|$ is close to 1 on N. Since $|z^2 - e^{-i\eta}\cos\eta| = \sin\eta$ on N, $\cos\eta \sim 1 - \frac{1}{2}\eta^2$, $\cos\eta - (\cos\eta)^{-1} \sim \eta^2$, we have $|z^2 e^{i\eta} - (\cos\eta)^{-1}| \sim \eta$ when η is sufficiently small, and hence

$|m_1(z^2)| = |\frac{z^2 e^{i\eta}-(\cos\eta)^{-1}}{\sin\eta}|$ is close to 1 when η is sufficiently small. Now the comments before formula (22) applies verbatim, We have both h_1 and h_1' are $O(1)$ and $|h_1''| \sim 1/\eta$, and it follows that

$$\int_N |h_1''| ds \le C\pi\eta/\eta = O(1)$$

By Lemma 3.3 it follows that $|(1 - P)h_1 P|_{S_q} = O(1), 2/3 < q < 1$. Note that h_1^* has poles inside the unit disk, so our deformation of path argument above does not work for h^*. But $h_1^* = -z^2u^2h_1$, hence if we multiply h_1^* by $\frac{1}{-z^2u^2}$ then we get h_1, and we have removed the poles of h_1^*. The function $\frac{1}{-z^2u^2}$ has modulus 1 and both $\frac{1}{-z^2u^2}$ and its complex conjugate are very good by Prop. 3.4, it will follow that h_1^* is very good. This will be called as “the trick of removing poles ". In more explicit terms,

$(Ph_1(1 - P))^* = (1 - P)(-z^2u^2)h_1 P,$

$$|Ph_1(1 - P)|_{S_q} = |[(Ph_1(1 - P))]^*|_{S_q} = |(1 - P)(-z^2u^2)h_1 P|_{S_q} = O(1)$$

by Prop. 3.2, Prop. 3.4 and $|(1 - P)h_1 P|_{S_q} = O(1), 2/3 < q < 1$ that has already been proved.

Note that $|Ph_1^*(1-P)|_{S_q} = |(1-P)h_1 P|_{S_q}$ and similarly with P replaced by $1-P$, and the proposition is proved. □

3.3. Deformation of path: The case with Branch cut. Our goal in this section is to show that h is good. Note that h is independent of the choice of branch cut of u inside unit disk , and in this section we choose the branch cut to be the closed line segment with end points $e^{-i\eta/2}\sqrt{\cos\eta}$ and $-e^{-i\eta/2}\sqrt{\cos\eta}$. Here η is very close to 0 so that $\cos\eta \sim 1$.

Note that h has poles inside the unit disk, but since by Prop. 3.4 u^{-2} and u^2 are very good, by Prop. 3.2 it is sufficient to show that $h_0 := u^{-2}h$ is good, and $u^{-2}h$ has no poles in the unit disk, but has branch cut. This is another example of the trick of removing poles. The branch cut is important, because without the branch cut we c could conclude as in the previous section that by using the trick of removing poles, both g, h are very good, but this would contradict the lower bound in Lemma 2.4. First we have :

$$z^2 - u^2 = -\frac{(z^2 - e^{-2i\eta})(z^2 - 1)e^{i\eta}}{e^{i\eta}z^2 - \cos\eta}$$

Let $z_1 = e^{i\theta_1}$ with $\theta_1 = \eta + 2\theta$, then

$$h_0 = Cz^{-3}(\cos\eta)^{-1/2}\frac{\frac{1}{2}(z_1 + z_1^{-1}) - \cos\eta}{1 - z_1\cos\eta}\sqrt{\frac{z_1 - \cos\eta}{z_1 - (\cos\eta)^{-1}}}$$

where C is a constant with $|C| = 1$. Hence it is enough to check that

$$h_1 = \frac{\frac{1}{2}(z_1 + z_1^{-1}) - \cos\eta}{1 - z_1\cos\eta}\sqrt{\frac{z_1 - \cos\eta}{z_1 - (\cos\eta)^{-1}}}$$

is good. Note that $z_1 = e^{i\eta}z^2$ and we think of h_1 as a function of z. Note that $h_0^* = u^2 h = -u^4 h_0$ since $h^* = -h$, it follows that $h_1^* = C_1 z^6 u^4 h_1$ where C_1 is a nonzero constant. Hence if we can show that $|(1-P)h_1 P|_{S_q}^q = o(-\ln\eta)$, then it follows

$$|Ph_1(1-P)|_{S_q}^q = |(1-P)h_1^* P|_{S_q}^q = |(1-P)z^6u^4h_1P|_{S_q}^q = o(-\ln\eta)$$

since z^6u^4 is very good by Prop. 3.4 and Prop. 3.2.

In the following we will only consider h_1 restricted to I_2 and show $|(1-P)h_1P|_{S_q}^q = o(-\ln\eta)$. Exactly the same argument also shows that h_1 restricted to $-I_2$ verifies similar inequality.

Denote by $a_n = \int_{I_2} h_1 z^n dz, n \geq 0$. Consider the function $h_2(z) = \sum_{n\geq 0} a_n z^{-n-1}$ on the circle. We will write h_2 as a sum of three functions. Note that $h_1 z^n$ is analytic in the unit disk except along the branch cut. We will write the $\int_{I_2} h_1 z^n dz$ as the integral of $h_1 z^n$ on three paths on the z plane. To describe these paths, note that we will be doing integrals in a small neighborhood of 1 when η is close to 0. In this small neighborhood the map $z \to z_1 = e^{i\eta}z^2$ is certainly one to one. Hence it is enough to describe these paths under the map $z \to z_1 = e^{i\eta}z^2$.

The imagine of these three paths are easier to describe in terms of $z_1 = e^{i\eta}z^2$ on the z_1 plane: first the path on the upper half of z_1 plane with $|z_1 - \cos\eta| = \sin\eta$ from $e^{i\eta}$ to $\cos\eta - \sin\eta$; We denote this quarter of the circle by $\hat{J}_1$.

The second path is along part of the branch cut $[\cos\eta - \sin\eta, \cos\eta]$, and then turning in the opposite direction along the same closed interval. We denote this interval by $\hat{J}_2$. The last part is in the lower half of z_1 plane from $\cos\eta - \sin\eta$ to $e^{-i\eta}$, and we denote this quarter of the circle by $\hat{J}_3$. See Figure 2 for the image of the three paths on the z_1 plane. In Figure 2 points 2, 3 correspond to $\cos\eta - \sin\eta, \cos\eta$ respectively on the z_1 plane. The small arc part of the unit circle from $e^{-i\eta}$ to $e^{i\eta}$ is the image of I_2 on the z_1 plane. We will denote by J_1, J_2, J_3 the pre-images of $\hat{J}_1, \hat{J}_2, \hat{J}_3$ on the z plane.

3.3.1. The part from Integral Along a Quarter of a Circle Let us first show that $b_n = \int_{J_1} h_1 z^n dz = O(n^{-2})$. Note that h_1 is equal to 0 at $z_1 = e^{i\eta}$.

$$\cos\eta - \sin\eta + \frac{1}{\cos\eta - \sin\eta} = 2 + \eta^2 + o(\eta^2)$$

Recall that

$$h_1 = \frac{\frac{1}{2}(z_1 + z_1^{-1}) - \cos\eta}{1 - z_1\cos\eta}\sqrt{\frac{z_1 - \cos\eta}{z_1 - (\cos\eta)^{-1}}}$$

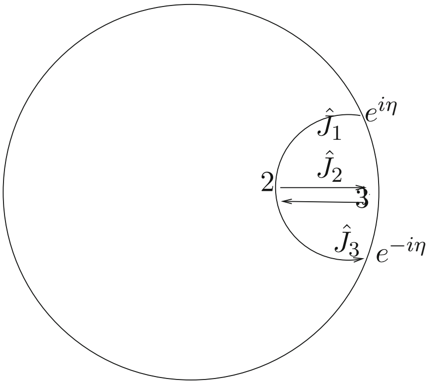

Fig. 2. Image of a contour

When $z_1 = \cos\eta - \sin\eta$ and η is sufficiently close to 0, we have $z_1 \sim 1 - \eta - \frac{1}{2}\eta^2$, $\cos\eta \sim 1 - \frac{1}{2}\eta^2$. It follows that $\frac{1}{2}(z_1 + z_1^{-1}) - \cos\eta \sim \eta^2$, $1 - z_1\cos\eta \sim \eta$, $z_1 - \cos\eta \sim \eta$, $z_1 - (\cos\eta)^{-1} \sim \eta$, $|\cos\eta - \sin\eta| \leq 1 - \frac{1}{2}\eta$,

we conclude that the value of $h_1 z^n$ at $z_1 = z^2 e^{i\eta} = \cos\eta - \sin\eta$ is bounded by an absolute constant multiplied by

$$\eta(1 - \frac{\eta}{2})^{\frac{n}{2}}$$

We need the following:

Lemma 3.6. $(1-\eta)^n\eta = O(n^{-1})$ *uniformly in* $\forall 0 < \eta < 1$.

Proof. Let $f(\eta) = (1-\eta)^n\eta$. Note that $f(0) = f(1) = 0$, and so the maximum of f is attained at the critical point of f. Let $f' = (1-\eta)^n - n\eta(1-\eta)^{n-1} = 0$ we get $\eta = \frac{1}{n+1}$, and so the maximum of f is

$$(1 - \frac{1}{n+1})^n \frac{1}{n+1}$$

Note that

$$\lim_{n\to\infty}(1 - \frac{1}{n+1})^n = e^{-1}$$

and the Lemma is proved. □

By using integration by parts, h_1 is equal to 0 at $z_1 = e^{i\eta}$, and Lemma 3.6 we have

$$\int_{J_1} h_1 z^n dz = O(n^{-2}) - \frac{1}{n+1}\int_{J_1} h_1' z^{n+1} dz$$

On J_1 we shall need an estimate of derivatives of h_1 similar to that of formula (22) for h on the circle. Note that when η is small enough, $|z|$ is close to 1 on J_1. Since $|z_1 - \cos\eta| = \sin\eta$ on J_1, $|z_1 - (\cos\eta)^{-1}| \sim \eta$ when η is sufficiently small, and hence

$|m_1(z^2)| = |\frac{z_1 \cos\eta - 1}{z_1 - \cos\eta}|$ is close to 1 when η is sufficiently small. Now the comments before formula (22) applies verbatim, and we find that $h_1' z^n$ is O(1) on the boundary of J_2, and

$$\int_{J_1} h_1 z^n dz = O(n^{-2}) - \frac{1}{(n+1)(n+2)} \int_{J_1} h_1'' z^{n+2} dz$$

Note that

$$\int_{J_1} |h_1''| \sim \int_0^{\eta} \frac{\eta^2}{\eta^3} d\theta = O(1)$$

we have shown that

$$\int_{J_1} h_1 z^n dz = O(n^{-2})$$

Similarly we have

$$\int_{J_3} h_1 z^n = O(n^{-2})$$

3.3.2. The part from Integrals Along the Branch cut Set $c_n := \int_{J_2} h_1 z^n dz$. Since u changes signs from the upper part of the branch cut to the lower part, we should actually consider $2c_n$. But since our estimate is up to multiplication by a positive constant, we can ignore this constant 2 in the following. We need to show $h_3(z) := \sum_{n\geq 0} c_n z^{-n-1}$ is good.

Since $|(1-P)h_3 P|_{S_q}^q = |P h_3^*(1-P)|_{S_q}^q$, We will use inequality (24) for h_3^*.

Let $\hat{f}_n(t) = F_n * h_3^*$, and $f_n(t) = \hat{f}_n^{\,*}$. Then

$$f_n(t) = \int_{J_2} \sum_{j\geq 1} h_1 z^{j-1} F(j/n) e^{-2\pi i j t} dz$$

Note that $z_1 = z^2 e^{i\eta}$, $z = e^{-i\eta/2}\sqrt{z_1}$, $dz = \frac{e^{-i\eta/2}}{2\sqrt{z_1}} dz_1$, and

$$f_n(t) = \int_{\cos\eta - \sin\eta}^{\cos\eta} \frac{\frac{1}{2}(z_1 + z_1^{-1}) - \cos\eta}{1 - z_1\cos\eta} \sqrt{\frac{z_1 - \cos\eta}{z_1 - (\cos\eta)^{-1}}} \sum_{j\geq 1} [\sqrt{z_1} e^{-i\eta/2}]^j F(j/n) e^{-2\pi i j t} \frac{1}{2z_1} dz_1$$

Set $t_1 = -t - \frac{\eta}{4\pi}$. Note we have

$$\int_{-1/2\leq t\leq 1/2} |\hat{f}_n(t)|^p dt = \int_{-1/2\leq t\leq 1/2} |f_n(t)|^p dt = \int_{-1/2\leq t_1\leq 1/2} |f_n(t)|^p dt$$

where the second equality follows since $f_n(t)$ is a function of t with period 1.

We need to estimate $\int_{-1/2\leq t_1\leq 1/2} |f_n(t)|^p dt$. On J_2,

$$\sqrt{\frac{z_1 - \cos\eta}{z_1 - (\cos\eta)^{-1}}} = O(1), \; \frac{1}{2}(z_1 + z_1^{-1}) - \cos\eta = O(\eta^2)$$

and

$$\int_{\cos\eta-\sin\eta}^{\cos\eta} \frac{1}{1-z_1\cos\eta} dz_1 = O(-\ln\eta)$$

When η is sufficiently small we have $\frac{1}{2z_1} \le 1$.

Note that $\sum_{j\ge1}[\sqrt{z_1}e^{-i\eta/2}]^j F(j/n)e^{-2\pi ijt} = \sum_{j\ge1} F(j/n)e^{2\pi ij\hat{t}}$ with $\hat{t} = t_1 + \frac{s}{2\pi}$, $s = -i\frac{1}{2}\ln z_1$, and on J_2, $\frac{1}{2}\ln z_1 < 0$. When $-1/2 \le t_1 \le 1/2$, apply Lemma 2.5 to $\sum_{j\ge1} F(j/n)e^{2\pi ij\hat{t}}$, we have that up to $O(n^{-N})$ term for any $N > 0$, we can replace $f_n(t)$ by

$$g_n(t_1) = \int_{\cos\eta-\sin\eta}^{\cos\eta} \frac{\frac{1}{2}(z_1+z_1^{-1})-\cos\eta}{1-z_1\cos\eta}\sqrt{\frac{z_1-\cos\eta}{z_1-(\cos\eta)^{-1}}}\, n\mathcal{F}F(-n(t_1+\frac{s}{2\pi}))\frac{1}{2z_1}dz_1$$

where $s = -i\frac{1}{2}\ln z_1$. We will choose N large enough such that

$$\sum_{m=0,n=2^m}^{\infty} n\int_{|t_1|\le\frac{1}{2}} |f_n(t)|^p dt = O(1) + \sum_{m=0,n=2^m}^{\infty} n\int_{|t_1|\le\frac{1}{2}} |g_n(t_1)|^p dt_1$$

Now it is sufficient to evaluate $\sum_{m=0,n=2^m}^{\infty} n\int_{|t_1|\le\frac{1}{2}} |g_n(t_1)|^p dt_1$.

Note that by inequality (25)

$$|\mathcal{F}F(-n(t_1+\frac{s}{2\pi}))| \le C_N \frac{e^{\frac{n}{2}\ln(z_1)}}{1+n^N(|t_1+\frac{s}{2\pi}|))^N}, \forall N \ge 0$$

where the constant C_N depends on N and F.

It follows that when η is sufficiently small

$$|g_n(t_1)| \le n\int_{\cos\eta-\sin\eta}^{\cos\eta} \frac{|\frac{1}{2}(z_1+z_1^{-1})-\cos\eta|}{1-z_1\cos\eta}\sqrt{\frac{z_1-\cos\eta}{z_1-(\cos\eta)^{-1}}}$$
$$C_N\frac{e^{\frac{n}{2}\ln(\cos\eta)}}{(1+n^N(|t_1|+\frac{-\ln(\cos\eta)}{2\pi}))^N}dz_1$$

We note that the exponential decay factor $e^{\frac{n}{2}\ln(\cos\eta)}$ is due to the fact that the branch cut is inside the unit disk.

Recall that on J_2,

$$\sqrt{\frac{z_1-\cos\eta}{z_1-(\cos\eta)^{-1}}} = O(1), \frac{1}{2}(z_1+z_1^{-1})-\cos\eta = O(\eta^2)$$

and

$$\int_{\cos\eta-\sin\eta}^{\cos\eta} \frac{1}{1-z_1\cos\eta} = O(-\ln\eta)$$

It follows that

$$|g_n(t_1)| \le C_N n(-\ln\eta)\eta^2 \frac{e^{\frac{n}{2}\ln(\cos\eta)}}{1+n^N(|t_1|+\frac{-\ln(\cos\eta)}{2\pi})^N} \tag{26}$$

To evaluate $\int_{-1/2\le t_1\le 1/2}|g_n(t_1)|^p dt_1$, $t_1 = t - \frac{\eta}{4\pi}$, we break this integral into two parts. Set $\delta := -\ln(\cos\eta)$. First we evaluate

$$\int_{|t_1|\le\delta}|g_n(t_1)|^p dt_1$$

Choose $N = 1$ in (26). Since $p < 1$ we have $\int_{|t_1|\le\delta}|g_n(t_1)|^p dt_1 \le C_1\eta^{2p}(-\ln\eta)^p\delta^{1-p}e^{-\delta\frac{n}{2}p}$

Hence

$$\sum_{m=0,n=2^m}^{\infty} n\int_{|t_1|\le\delta}|g_n(t_1)|^p dt_1 \le C_1\sum_{m=0,n=2^m}^{\infty} n\eta^{2p}(-\ln\eta)^p\delta^{1-p}e^{-\delta\frac{n}{2}p}$$

Note that $\eta^2 = O(\delta)$ and by Lemma 3.10 we have proved that

$$\sum_{m=0,n=2^m}^{\infty} n\int_{|t_1|\le\delta}|g_n(t_1)|^p dt_1 = O((-\ln\eta)^p)$$

Next we evaluate $\int_{\frac{1}{2}\ge|t_1|\ge\delta}|g_n(t_1)|^p dt_1$. This time we choose N in (26) such that $1 + p > Np > 1$. Note that when η is small enough we have

$$\int_{\frac{1}{2}\ge|t_1|\ge\delta}|t_1|^{-Np}dt_1 \le 2\delta^{1-Np}$$

We get

$$\sum_{m=0,n=2^m}^{\infty} n\int_{\frac{1}{2}\ge|t_1|\ge\delta}|g_n(t_1)|^p dt_1 \le 2C_N\sum_{m=1,n=2^m}^{\infty} n^{1+p-Np}\eta^{2p}(-\ln\eta)^p\delta^{1-Np}e^{-\delta\frac{n}{2}p}$$

Note that $\eta^2 = O(\delta)$ and by Lemma 3.10 we prove that

$$\sum_{m=0,n=2^m}^{\infty} n\int_{\frac{1}{2}\ge|t_1|\ge\delta}|g_n(t_1)|^p dt_1 = O((-\ln\eta)^p)$$

By inequality (24) we have proved

$$|(1-P)h_3P|_{S_p}^p = O((-\ln\eta)^p), \forall 0 < p < 1.$$

Putting together these three parts from Sects. 3.3.1 and 3.3.2, and use Lemma 2.3, we prove the following Theorem:

Theorem 3.7. *If* $2/3 < p < 1$, *then*

$$|(1-P)hP|_{S_p}^p = O((-\ln\eta)^p), |Ph(1-P)|_{S_p}^p = O((-\ln\eta)^p)$$

Remark 3.8. Though the above theorem is sufficient for our purpose, we can actually show that $||(1-P)hP||_1 = O(1)$. It is an interesting question to see if one can improve the above theorem to show that h is very good.

Corollary 3.9. *If* $2/3 < p < 1$, *then*

$$|(1-P)ghP|_{S_p}^p = O((-\ln\eta)^p), |Phg(1-P)|_{S_p}^p = O((-\ln\eta)^p)$$

Proof. By definition

$$gh = \frac{z^2+u^2}{16u^2z^2}(z^2-u^2)\chi_{I_2\cup -I_2} + \frac{1}{2}h$$

The corollary follows from Th. 3.7 and Proposition 3.4, Prop. 3.5 and Prop. 3.2. □

Lemma 3.10. *Assume that* $x > 0$, $p > 0$, $\delta > 0$. *Then* $\sum_{m=0}^{\infty} 2^{mx} e^{-2^m \delta p} \delta^x = O(1)$ *when* $\delta \to 0$.

Proof. Let $F(y) := y^x e^{-yp}$. Then $\lim_{y\to 0} F(y) = 0 = \lim_{y\to\infty} F(y)$. The only critical point of $F(y)$ is at $y = x/p$ and the maximum of $F(y)$ is $F(x/p) < \infty$. F is increasing when $y < x/p$ and decreasing when $y > x/p$.

It follows that

$$\sum_{m=0}^{\infty} 2^{mx} e^{-2^m\delta p}\delta^x \leq 2F(x/p)\delta^x + \int_0^{\infty} 2^{wx} e^{-2^w\delta p}\delta^x dw$$

Set $2^w\delta = w_1$, and since $x > 0$ we have

$$\int_0^{\infty} 2^{wx} e^{-2^w\delta p}\delta^x dw = \int_{\delta}^{\infty} w_1^{x-1} e^{-w_1 p}\frac{1}{\ln 2} dw_1 = O(1)$$

and the Lemma follows.

3.4. Estimation of Entropy. Since $f_0(PgP)$, $P_0 f(g)P$ are not trace class operators, but $f_0(PgP) - Pf_0(g)P$ is, this makes it very delicate to show that $S(F,\eta) - \mathrm{tr}(f_0(PgP) - Pf_0(g)P)$ is small. This is proved in several steps in this section.

We begin this section with a generalization of Lemma 2.2 in [38].

Lemma 3.11. *Suppose that* $R_1(z)$, $R_2(z)$, $z > 0$, V *are bounded operators, and* $W = R_1(z)VR_2(z)$. *Assume that as* $z \to 0$, $||R_1(z)|| \sim \frac{1}{z^{t_1}}$, $||R_2(z)|| \sim \frac{1}{z^{t_2}}$, $||W|| \sim \frac{1}{z^{t_0}}$, *where* t_0, t_1, t_2 *are positive. Let* $0 < \sigma < 1$. *Then*

$$||W||_1 \leq ||W||^{1-\sigma}||R_1||^{\sigma}||R_2||^{\sigma}|V|_{S_\sigma}^{\sigma} \sim \frac{1}{z^{(t_0(1-\sigma)+(t_1+t_2)\sigma)}}|V|_{S_\sigma}^{\sigma}$$

Proof. First by (6)

$||W||_1 = |||W|^{1-\sigma}|W|^{\sigma}||_1 \leq ||W||^{1-\sigma}|||W|^{\sigma}||_1$

Note that by (4) $s_n(R_1VR_2) \leq ||R_1||s_n(V)||R_2||$, and so $s_n(R_1VR_2)^{\sigma} \leq ||R_1||^{\sigma} s_n(V)^{\sigma}||R_2||^{\sigma}$ and the Lemma follows. □

In our applications in this section $0 \leq t_0 \leq 2$, $0 \leq t_1 \leq 1$, $0 \leq t_2 \leq 1$. So the maximum of $t_0(1-\sigma) + (t_1+t_2)\sigma)$ is 2. There are two different cases that are important in the following: The first case is when we need $t_0(1-\sigma) + (t_1+t_2)\sigma < 2$ to make sure our integral is convergent: in this case if we can manage to find one of the t_i, $i = 0, 1, 2$ which do not take their maximal value then we will achieve our goal. The second case is when $0 \leq t_0 \leq 1$, $0 \leq t_1 + t_2 \leq 1$. In this case the maximum of $t_0(1-\sigma) + (t_1+t_2)\sigma \leq 1$, and we need to get $t_0(1-\sigma)+(t_1+t_2)\sigma < 1$. Again this can be done if we can manage to get t_0 or t_1+t_2 to take values less than their allowed maximum , then we can make sure that our integral is convergent. We will see three different such "savings" of the exponents in the following.

First we will use an integral representation for $f_0(T)$ (cf. [7]).

Lemma 3.12. *Suppose* $0 \le T \le 1$ *is an operator. Then*

$$f_0(T) = \int_{\frac{1}{2}}^{\infty} \frac{2\beta}{\beta+\frac{1}{2}} \frac{\widetilde{T}}{z+\widetilde{T}} d\beta$$

where $\widetilde{T} = T(1-T)$, $z = \beta^2 - \frac{1}{4}$.

Proof. Using Fundamental Theorem of Calculus one checks that if $0 \le x \le 1$,

$$f_0(x) = \int_{\frac{1}{2}}^{\infty} \frac{2\beta}{\beta+\frac{1}{2}} \frac{x(1-x)}{\beta^2-\frac{1}{4}+x(1-x)} d\beta$$

and the Lemma follows from functional calculus for self-adjoint operators. □

Let $A = PP_{12}P$, $\widetilde{A} = PP_{12}P(1-PP_{12}P)$. See Th. 2.1 for definition of P_{12}. $\widetilde{B_1} = Pg^2P - (PgP)^2$, $\widetilde{B} = PgP - (PgP)^2$.

Note that since P_{12} is a projection, $\widetilde{A} = PP_{12}P(1-PP_{12}P) = PP_{12}(1-P)P_{12}P$. By Prop. 3.5 and Th. 3.7 of [25] $\widetilde{A}$ is of trace class. Similarly $\widetilde{B_1}$ if of trace class but $\widetilde{B}$ is not.

By definition 2.2 we have

$$S(F,\eta) = \tau(P_{12}(\eta), f_0) = \mathrm{tr}(f_0(PAP))$$

Lemma 3.13.

$$|\widetilde{B_1} - \widetilde{A}|_{S_q}^q = O((-\ln\eta)^q),\ 2/3 < q < 1$$

Proof. Note that since P_{12} is a projection, $\widetilde{A} = PP_{12}^2P - (PP_{12}P)^2 = PP_{12}(1-P)P_{12}P$. Also $\widetilde{B_1} = Pg(1-P)gP$.

So we have

$$\widetilde{B_1} - \widetilde{A} = -PhR(1-P)P_{12}P - Pg(1-P)hRP$$

Note $[R,P]=0$, and $Pg(1-P)hRP = Pgh(1-P)RP + Pg[1-P,h]RP$. Note that $[1-P,h] = -[P,h] = (1-P)hP - Ph(1-P)$ so Th. 3.7 applies.

By Th. 3.7, Cor. 3.9 and Prop. 3.2 the lemma is proved. □

Note that

$$\frac{\widetilde{A}}{z+\widetilde{A}} - \frac{\widetilde{B_1}}{z+\widetilde{B_1}} = z \times \frac{1}{z+\widetilde{A}}(\widetilde{B_1}-\widetilde{A})\frac{1}{z+\widetilde{B_1}}$$

Apply Lemma 3.11 for $W = \frac{1}{z+\widetilde{A}} - \frac{1}{z+\widetilde{B_1}} = \frac{1}{z+\widetilde{A}}(\widetilde{B_1}-\widetilde{A})\frac{1}{z+\widetilde{B_1}}$, with $t_0 = 1, t_1 = t_2 = 1$, we have

$$\left|\left|\frac{\widetilde{A}}{z+\widetilde{A}} - \frac{\widetilde{B_1}}{z+\widetilde{B_1}}\right|\right|_1 \le C|\widetilde{B_1}-\widetilde{A}|_{S_\sigma}^{\sigma} \frac{1}{z^\sigma}$$

It follows that by Lemma 3.13

$$\int_{1/2}^{\infty} |\mathrm{tr}(\frac{2\beta}{\beta+1/2}(\frac{\widetilde{A}}{z+\widetilde{A}} - \frac{\widetilde{B_1}}{z+\widetilde{B_1}}))|d\beta \le C\int_{1/2}^{\infty} \frac{1}{z^\sigma} \times O((-\ln\eta)^\sigma)$$

Recall that $z = \beta^2 - 1/4$, and so $\int_{1/2}^{\infty} \frac{1}{z^\sigma} = O(1)$ if $1 > \sigma > 1/2$. It follows that

$$S(F,\eta) - \int_{1/2}^{\infty} \mathrm{tr}(\frac{2\beta}{\beta+1/2}\frac{\widetilde{B_1}}{z+\widetilde{B_1}})d\beta = O((-\ln\eta)^\sigma),\ 2/3 < \sigma < 1 \tag{27}$$

where $S(F,\eta)$ is the first term in above integral by Lemma 3.12.

Next we estimate

$$\int_{1/2}^{\infty} \mathrm{tr}(\frac{2\beta}{\beta+1/2}\frac{\widetilde{B_1}}{z+\widetilde{B_1}})d\beta - \tau(g, f_0)$$

First we introduce some notations that will simplify writing. These notations will only be used in this section.

Let $X = PgP - (PgP)^2$, $Y := \widetilde{B_1} = Pg(1-P)gP$. $X - Y = P(g-g^2)P = Ph_1^2P$, $h_1 = ih$. $h_1^* = h_1$. Here h is as in Th. 2.1. Note that $X \ge 0$, $Y \ge 0$.

$$h_1Y = h_1Pg(1-P)gP = [h_1, P]g(1-P)gP + Ph_1g(1-P)gP$$

. It follows from Th. 3.7, Cor. 3.9 and Prop. 3.2 that

$$|h_1Y|_{S_q}^q = O((-\ln\eta)^q),\ 2/3 < q < 1$$

Recall by definition

$$\tau(g, f_0) = \int_{1/2}^{\infty} \frac{2\beta}{\beta+1/2}\mathrm{tr}\left(\frac{X}{X+z} - P\frac{W}{W+z}P\right)$$

The estimate of

$$\int_{1/2}^{\infty} \frac{2\beta}{\beta+1/2}\mathrm{tr}(\frac{\widetilde{B_1}}{z+\widetilde{B_1}})d\beta - \tau(g, f_0)$$

reduces to estimate

$$\frac{Y}{Y+z} - (\frac{X}{X+z} - P\frac{W}{W+z}P)$$

where $W = h_1^2$.

As a first step we estimate

$$\frac{X}{X+z} - P\frac{W}{W+z}P = zP\frac{1}{X+z}(X - PW)\frac{1}{W+z}P$$

We have $X - PW = Y + PWP - PW$.

First we need a simple Lemma:

Lemma 3.14. *If S is a positive operator, and $TT^* \le S$, $z > 0$, then*

$$||\frac{1}{z+S}T|| = ||T^*\frac{1}{z+S}|| \le \frac{1}{\sqrt{2}z^{\frac{1}{2}}}$$

Proof. Since $||Q|| = ||Q^*||$, it is sufficient to prove

$$||\frac{1}{z+S}T|| \leq \frac{1}{\sqrt{2}z^{\frac{1}{2}}}$$

We have

$$\frac{1}{z+S}TT^*\frac{1}{z+S} \leq \frac{1}{z+S}S\frac{1}{z+S} \leq \frac{1}{2z}$$

Hence

$$||\frac{1}{z+S}T||^2 = ||\frac{1}{z+S}TT^*\frac{1}{z+S}|| \leq \frac{1}{2z}$$

and the Lemma is proved. □

Let us show that

$$\int_{1/2}^{\infty} z\frac{2\beta}{\beta+1/2}\mathrm{tr}(P\frac{1}{X+z}(PWP-PW)\frac{1}{W+z}P)dz = O((-\ln\eta)^{\sigma})$$

We will apply Lemma 3.11 with $R_1 = \frac{1}{X+z}$, $V = PWP - PW$, $R_2 = \frac{1}{W+z}P$. It is clear that $t_1 = t_2 = 1$, and we need choose t_0 small enough. The key observation is that

$$R_1VR_2 = \frac{1}{X+z}(Ph_1h_1P)\frac{1}{W+z}P - \frac{1}{X+z}PW\frac{1}{W+z}P$$

and since $X \geq Ph_1^2P$, from Lemma 3.14 we have

$$||\frac{1}{X+z}(Ph_1h_1P)\frac{1}{W+z}P|| \leq \frac{1}{\sqrt{2}z^{1/2+1}}$$

It is also clear that

$$||\frac{1}{X+z}PW\frac{1}{W+z}P|| \leq \frac{1}{z}$$

and we achieve our goal with

$$||R_1VR_2|| \leq C\frac{1}{z^{3/2}}$$

Now we can apply Lemma 3.11 with $t_0 = 3/2$, $t_1 = t_2 = 1$ to obtain

$$|z\mathrm{tr}(R_1VR_2)| \leq \frac{1}{z^{\frac{1+\sigma}{2}}}|||V|^{\sigma}||_1$$

Since $h_1^2 = -h^2$, By Th. 3.7 we get

$$\int_{1/2}^{\infty}\frac{2\beta}{\beta+1/2}\mathrm{tr}(\frac{1}{X+z}(Ph_1h_1P)\frac{1}{W+z}P - \frac{1}{X+z}PW\frac{1}{W+z}P)d\beta$$
$$= O((-\ln\eta)^{\sigma})), 2/3 < \sigma < 1 \quad (28)$$

Let us consider

$$P\frac{1}{X+z}Y(1-\frac{W}{z+W})P$$

We write

$$P\frac{1}{X+z}Y\frac{W}{z+W}P = P\frac{1}{X+z}Pg(1-P)(1-P)gPh_1\frac{h_1}{z+h_1^2}P$$

and apply Lemma 3.11 with $R_1 = P\frac{1}{X+z}Pg(1-P)$, $V = (1-P)gPh_1$, $R_2 = \frac{h_1}{z+h_1^2}P$

Note that by Lemma 3.14 we have

$$||R_1|| \sim z^{-1/2}, ||R_2|| \sim z^{-1/2}, ||R_1VR_2|| \sim z^{-1/2}$$

We have $t_0 = 1/2, t_1 = t_2 = 1/2$, again with savings on exponents. We have

$$||P\frac{1}{X+z}Y\frac{W}{z+W}P||_1 \leq C|(1-P)gPh_1|^{\sigma}_{S_\sigma}\frac{1}{z^{\frac{1}{2}+\frac{\sigma}{2}}}$$

Note that $|(1-P)gPh_1|^{\sigma}_{S_\sigma} \leq |(1-P)gh_1P|^{\sigma}_{S_\sigma} + ||(1-P)g|||[P,h_1]|^{\sigma}_{S_\sigma}$

By Th. 3.7 and Cor. 3.9, the same argument as above shows that

$$\int_{1/2}^{\infty}\frac{2\beta}{\beta+1/2}\mathrm{tr}\left(P\frac{1}{X+z}Y\frac{W}{z+W}P\right)d\beta = O((-\ln\eta)^{\sigma}), 2/3 < \sigma < 1 \qquad (29)$$

Finally we are left with

$$\left(\frac{1}{z+Y}-\frac{1}{z+X}\right)Y = \frac{1}{z+X}(z+X-z-Y)\frac{1}{z+Y}Y = \frac{1}{z+X}PWP\frac{Y}{z+Y}$$

By choosing $R_1 = \frac{1}{z+X}Ph_1$, $V = h_1Pg(1-P)$, $R_2 = (1-P)gP\frac{1}{z+Y}$ and use Lemma 3.14 we find that $t_0 = 1/2, t_1 = t_2 = 1/2$, again with savings as the preceding case to complete the proof that

$$\int_{1/2}^{\infty}\frac{2\beta}{\beta+1/2}\mathrm{tr}\left(\left(\frac{1}{Y+z}-\frac{1}{z+X}\right)Y\right)d\beta = O((-\ln\eta)^{\sigma}), 2/3 < \sigma < 1 \qquad (30)$$

To summarize, we first prove that

$$S(F,\eta) - \int_{1/2}^{\infty}\mathrm{tr}(\frac{2\beta}{\beta+1/2}\frac{\widetilde{B_1}}{z+\widetilde{B_1}})d\beta = O((-\ln\eta)^{\sigma}), 2/3 < \sigma < 1$$

which is equation (27).

Next we estimate

$$\int_{1/2}^{\infty}\frac{2\beta}{\beta+1/2}\mathrm{tr}(\frac{\widetilde{B_1}}{z+\widetilde{B_1}})d\beta - \tau(g, f_0)$$

In addition to multiplication by $\frac{2\beta}{\beta+1/2}$, the integrand in the above integral is

$$\frac{Y}{Y+z} - \left(\frac{X}{X+z} - P\frac{W}{W+z}P\right)$$

where $W = h_1^2$. We show that

$$\frac{X}{X+z} - P\frac{W}{W+z}P = zP\frac{1}{X+z}(Y + PWP - PW)\frac{1}{W+z}P$$

and the integral corresponds to the integrand

$$zP\frac{1}{X+z}(PWP - PW)\frac{1}{W+z}P$$

is $O((-\ln\eta)^{\sigma})$, $2/3 < \sigma < 1$ as in equation (28).

Then we show the integral corresponds to the integrand

$$P\frac{1}{X+z}YW\frac{1}{W+z}P$$

is $O((-\ln\eta)^{\sigma})$, $2/3 < \sigma < 1$ as in equation (29). This shows that up to $O((-\ln\eta)^{\sigma})$, $2/3 < \sigma < 1$, the integrand $(\frac{X}{X+z} - P\frac{W}{W+z}P)$ can be replaced by $-\frac{1}{z+X}Y$. And finally we show that the integral corresponds to the integrand $(\frac{1}{z+Y} - \frac{1}{z+X})Y$ is also $O((-\ln\eta)^{\sigma})$, $2/3 < \sigma < 1$ as in equation (30).

So we have proved the following theorem:

Theorem 3.15.

$$S(F,\eta) - \tau(g, f_0) = O((-\ln\eta)^{\sigma}),\ 2/3 < \sigma < 1$$

where $S(F,\eta)$ and $\tau(g, f_0)$ are defined as in definition 2.2.

3.5. Upper Bound for Entropy. Th. 3.15 reduce the estimation of $S(F,\eta)$ to $\tau(g; f_0)$. $f_0(PgP) - Pf_0(g)P$ is called truncated Wiener-Hopf operators in [21]. There is a remarkable formula for $\tau(g; f_0)$ going back to H. Widom (cf. [21] and references therein). The more general version that we will use can be found in [37] and [21]. To describe this formula, we recall some basic definitions from [37].

For any complex valued function $f : \mathbb{C} \to \mathbb{C}$ and s_1, s_2 define

$$U(s_1, s_2; f) = \int_0^1 \frac{f((1-t)s_1 + ts_2) - ((1-t)f(s_1) + tf(s_2))}{t(1-t)}dt$$

and introduce

$$B(a; f) = \frac{1}{8\pi^2}\int\int \frac{U(a(\psi_1), \alpha(\psi_2); f)}{|\psi_1 - \psi_2|^2}d\psi_1 d\psi_2$$

where a is another function $a : \mathbb{C} \to \mathbb{C}$.

The quantity $B(a; f)$ is an object that appears very often in the theory of Wiener-Hopf operators.

We denote by

$$B_\epsilon(a; f) := \frac{1}{8\pi^2}\int\int_{|\psi_1-\psi_2|<\epsilon} \frac{U(a(\psi_1, \psi_2; f)}{|\psi_1 - \psi_2|^2}d\psi_1 d\psi_2$$

and

$$B_{\epsilon_1,\epsilon_2}(a; f) := \frac{1}{8\pi^2}\int\int_{\epsilon_1<|\psi_1-\psi_2|<\epsilon_2} \frac{U(a(\psi_1,\psi_2; f)}{|\psi_1-\psi_2|^2} d\psi_1 d\psi_2$$
$$B_{\geq\epsilon}(a; f) := \frac{1}{8\pi^2}\int\int_{\epsilon\leq|\psi_1-\psi_2|} \frac{U(a(\psi_1,\psi_2; f)}{|\psi_1-\psi_2|^2} d\psi_1 d\psi_2$$

Now we will use Cayley transformation to identify the unit circle with the extended real line, and to think our function g as a function $\hat{g}$ on the real line, that is $\hat{g}(x) = g(C(x))$. Recall Cayley transform $V(x) = i(x+i)/(x-i)$, which carries the (one point compactification of the) real line onto the circle and the upper half plane onto the unit disk. It induces a unitary map

$$Uf(x) = \pi^{-\frac{1}{2}}(x-i)^{-1} f(V(x))$$

of $L^2(S^1, \mathbb{C})$ onto $L^2(\mathbb{R}, \mathbb{C})$. The operator U carries the Hardy space on the circle onto the Hardy space on the real line (cf. Chapter one of [30]). We will use the Cayley transform to identify intervals on the circle with one point removed to intervals on the real line. Note that $\hat{g} \in W^{2,\infty}$, and has compact support.

The length function is $L(z) = |z^2 - e^{-i\eta}\cos\eta|$. Notice that $|L(z_1) - L(z_2)| \leq |z_1^2 - z_2^2| \leq 2|z_1 - z_2|$.

Define scale function τ and amplitude function v (cf. Sect. 3 of [21]) as follows:

$$\tau(x) = \frac{1}{5}L(C(x)), v(x) = (1+|x|)^{-2}$$

Note that

$$C(x) = i\frac{x+i}{x-i}$$
$$|C'(x)| = |\frac{2}{(x-i)^2}| \leq 2, |C''(x)| = |\frac{4}{(x-i)^3}| \leq 4$$
$$|\tau(x_1) - \tau(x_1)| \leq \frac{2}{5}|C(x_1) - C(x_2)| \leq \frac{4}{5}|x_1 - x_2|$$

One can check directly from (23) that

$$|\hat{g}(x)| \leq Cv(x), |\hat{g}^{(k)}(x)| \leq C_k\frac{v(x)}{\tau(x)^k}, k = 1, 2$$

where C, C_k are constants. Note that the minimum of $\tau(x)$ is $\tau_{\min} = 1 - \cos\eta \sim \eta^2$.

Our $\hat{g} \in W^{2,\infty}$, has support in $[-2, 2]$ and verifies conditions 4.1 in [21]. Hence Th. 3.2 in [37] applies to $\hat{g}$.

Lemma 3.16.

$$\tau(g; f_0) = B(\hat{g}; f_0)$$

Proof. By Prop. 4.1 of [21] $\tau(g; f) = B(\hat{g}; f)$ if $f \in C_0^4(\mathbb{R})$. Now choose a sequence of smooth functions f_n to approximate f_0 in the norm defined in Th. 3.2 of [37], and our lemma follows from Prop. 2.2 of [21] and Th. 3.2 of [37]. □

By Th. 6.1 of [37] we have

$$|B(\hat{g};\, f_0)| \leq C \int \frac{v(x)}{\tau(x)} dx$$

for some constant $C > 0$.

Let us evaluate

$$\int \frac{v(x)}{\tau(x)} dx$$

Change coordinate to $z = C(x) = e^{i\theta}$, we have

$$\int \frac{v(x)}{\tau(x)} dx \leq C \int_0^{2\pi} \frac{1}{|z^2 - e^{-i\eta}\cos\eta|} d\theta = 2C \int_0^{2\pi} \frac{1}{|z - e^{-i\eta}\cos\eta|} d\theta$$

By Prop. 1.4.10 of [35] we have

$$\int \frac{1}{|z - e^{-i\eta}\cos\eta|} d\theta \sim -\ln(1 - \cos^2(\eta)) \sim -2\ln\eta$$

Hence

$$|\tau(C_1)| \leq C(-\ln\eta)$$

for some constant $C > 0$.

By Th. 3.15 and Lemma 3.16 we have therefore proved

Corollary 3.17. *$S(F, \eta) \leq C(-\ln\eta)$ for some constant $C > 0$ when $\eta \to 0$, where $S(F, \eta)$ is as in definition 2.2.*

Remark 3.18. In fact we can write

$$B(\hat{g};\, f_0) = B_{\eta^2}(\hat{g},\, f_0) + B_{\eta^2,\eta}(\hat{g},\, f_0) + B_{\geq\eta}(\hat{g},\, f_0)$$

By Section 9 of [21] we have $B_{\eta^2}(\hat{g};\, f_0) = O(1)$, and $B_{\geq\eta}(\hat{g};\, f_0) = \frac{-1}{6}\ln\eta + O(1)$. Unfortunately it is not clear if one can show $B_{\eta^2,\eta}(\hat{g};\, f_0) = o(-\ln\eta)$ since our function $\hat{g}$ is not smooth as the functions considered in Section 9 of [21]. If $B_{\eta^2,\eta}(\hat{g},\, f_0) = o(-\ln\eta)$, then it follows that

$$S(F, \eta) = \frac{1}{6}(-\ln\eta) + o(-\ln\eta)$$

as $\eta \to 0$.

3.6. Continuity. Note that in general von Neumann entropy does not behave as well as relative entropy. It is therefore interesting to examine the properties of $S(F, \eta)$ as functions of η using our explicit formula. In this section we prove that $S(F, \eta)$ is continuous and $\lim_{\eta \to \pi} S(F, \eta) = 0$. First we have the following Lemma:

Lemma 3.19. *Suppose* $f(z, v) = \sum_n f_n(v) z^n$, $|f_n(v)| \leq C|n|^{-\alpha}$ *with* $\alpha > \frac{3}{2}$ *where* C *only depends on the neighborhood* V *of* v_0*. In addition assume that for each* $n \neq 0$, $\lim_{v \to v_0} f_n = 0$*. Then* $\lim_{v \to v_0} |Pf(1-P)|_{S_p}^p = 0$, $\lim_{v \to v_0} |(1-P) f P|_{S_p}^p = 0$, $1 > p > \frac{1}{\alpha - \frac{1}{2}}$.

Proof. Note that $P(f - C)(1-P) = Pf(1-P)$ for any constant C. That is why $n \neq 0$ in the Lemma. We prove $\lim_{v \to v_0} |Pf(1-P)|_{S_p}^p = 0$. The proof of $\lim_{v \to v_0} |(1 - P) f P|_{S_p}^p = 0$ is similar. Given any $\epsilon > 0$. We first write $f = f_N + f_{\geq N}$ where $f_N = \sum_{|n| \leq N} f_n(v) z^n$.

As in [18],

$$Pf(1-P)(z^n) = \sum_{k \geq 0} f_{k-n}(v) z^k = \xi_{-n}, \quad n < 0\,.$$

It follows that

$$Pf(1-P) = \sum_{n<0} (\cdot\,, z^n) \xi_{-n}$$
$$||Pf(1-P)||_{S_p}^p \leq \sum_{n<0} ||\xi_{-n}||^p$$

where $(\cdot\,, z^n)$ is the inner product with z^n.

Note that $||\xi_{-n}|| = \left(\sum_{k \geq 0} |f_{k-n}(v)|^2\right)^{\frac{1}{2}} = O(|n|^{-\alpha + \frac{1}{2}})$, by choosing N sufficiently large we have $|Pf_{\geq N}(1-P)|_{S_p}^p < \epsilon/2$. Since

$$|Pf_N(1-P)|_{S_p}^p \leq 2N\Big(\sum_{1 \leq |n| \leq N} |f_n(v)|\Big)^p$$

By assumption we can choose v close enough to v_0 such that

$$|Pf_N(1-P)|_{S_p}^p \leq \epsilon/2$$

and the Lemma is proved. □

Proposition 3.20. *Suppose* T_1, T_2 *are projections, and let* $\gamma := |P(T_1 - T_2)(1-P)|_p^p + |(1-P)(T_1 - T_2)P|_p^p < \infty$*, for some* $0 < p < 1$*. Then* $f_0(PT_1P) - f_0(PT_2P)$ *is trace class and*

$$|tr(f_0(PT_1P) - f_0(PT_2P))| \leq C_p \gamma$$

where C_p *is a constant which only depends on* p.

Proof. By inequality (7) we can assume that $p > 1/2$.

The idea of the proof is already present after Lemma 3.13.

By using Lemma 3.12

$$f_0(PT_1P) - f_0(PT_2P) = \int_{1/2}^{\infty} (\frac{2\beta}{\beta + 1/2} (\frac{\widetilde{PT_1P}}{z + \widetilde{PT_1P}} - \frac{\widetilde{PT_2P}}{z + \widetilde{PT_2P}}))d\beta$$

Where $\widetilde{T} := T(1-T)$. Apply Lemma 3.11 for $W = \frac{1}{z+\widetilde{PT_1P}} - \frac{1}{z+\widetilde{PT_2P}} = \frac{1}{z+\widetilde{PT_1P}} (\widetilde{PT_2P} - \widetilde{PT_1P})\frac{1}{z+\widetilde{PT_2P}}$, with $t_0 = 1, t_1 = t_2 = 1$ exactly as after Lemma 3.13, we have

$$\left\| \frac{\widetilde{PT_1P}}{z + \widetilde{PT_1P}} - \frac{\widetilde{PT_2P}}{z + \widetilde{PT_2P}} \right\|_1 \leq C|\widetilde{PT_1P} - \widetilde{PT_2P}|_{S_p}^p \frac{1}{z^p}$$

Recall that $z = \beta^2 - 1/4$, and so $\int_{1/2}^{\infty} \frac{1}{z^p} = O(1)$ if $p > 1/2$. Finally notice that since T_1 is a projection,

$$PT_1P(1 - PT_1P) = PT_1^2P - (PT_1P)^2 = PT_1(1-P)T_1P$$

Denote by $T := T_1 - T_2$. Then we have $T_1 = T + T_2$, and $PT_1(1-P)T_1P - PT_2(1-P)T_2P = PT_2(1-P)TP + PT(1-P)T_1P$.

It follows that

$$|\widetilde{PT_1P} - \widetilde{PT_2P}|_{S_p}^p \leq \gamma$$

and the Proposition is proved. □

Theorem 3.21. *$S(F, \eta)$ is a continuous function of $\eta \in (0, \pi)$ and* $\lim_{\eta \to \pi^-} S(F, \eta) = 0$.

Proof. Let $\eta = \pi - \phi$, and assume that $\phi \to 0$.

First from the formula (22) we see that both g'', h'' are bounded, up to addition of constants, by constants multiplied by $\frac{\phi^2}{L^3}$ where L is the distance between z^2 and $e^{-i\eta}\cos\eta$. Note that as $\phi \to 0$, the smallest L is reached at end points of I_2 and this value is $\sim \phi$. If we use angle θ between points in I_2 and the end points of I_2 where L attains its minimum as an integration parameter, then we have as $\phi \to 0$

$$\int_{I_2} |h''|d\theta \sim \int_0^{\pi} \frac{\phi^2}{(\theta + \phi)^3} d\theta = O(1)$$

Similarly

$$\int_{I_2} |g''|d\theta \sim \int_0^{\pi} \frac{\phi^2}{(\theta + \phi)^3} d\theta = O(1)$$

The same is also true for the integrals of h'', g'' over $-I_2$.

It follows by integration by parts that the Fourier coefficients of h, g are of $O(n^{-2})$ as $n \to \infty$. Moreover for $n \geq 0$ and remember h is an odd function we have

$$2\int_{I_2} h(z)z^{2n}dz = \int_{I_2 \cup -I_2} h(z)z^{2n}dz = -\int_{J \cup -J} h(z)z^{2n}dz$$

where J is a path connecting end points of I_2 and $-I_2$ with $|e^{i\eta}z^2 - \cos\eta| = \sin(\phi)$, since $h(z)z^{2n}$ is analytic in the region bounded by $I_2 \cup -I_2 \cup J \cup -J$. Here we have used the fact that h is independent of the choice of analytical continuation of u and we can choose branch cut of u which is outside the region bounded by $I_2 \cup -I_2 \cup J \cup -J$.

It follows that for

$$\left|\int_{I_2} h(z)z^{2n}dz\right| = \left|\int_J h(z)z^{2n}dz\right| \leq C\phi$$

as $\phi \to 0$, where C is a constant. It is also clear that

$$\int_{I_2} h(z)z^{2n}dz$$

is continuous in η. Since ih is real it follows all fourier coefficients of h goes to 0 when $\phi \to 0$. Similarly since $g - \frac{1}{2}$ is odd we have

$$\left|\int_{I_2} (g - \frac{1}{2})z^{2n}dz\right| = \left|\int_J (g - \frac{1}{2})z^{2n}dz\right| \to 0$$

as $\phi \to 0$. Since on I_1, $g = 1$, $|\int_{I_1} z^{2n}dz| = O(\phi)$. Moreover since g is real, it follows that all fourier coefficients of $(g - \frac{1}{2})$ goes to 0 when $\phi \to 0$. By applying Lemma 3.19 (note that $n \neq 0$ in Lemma 3.19) and Prop. 3.20 with $T_1 = P_{12}(\phi)$, $T_2 = 0$ we conclude that $\lim_{\eta\to\pi^-} S(F, \eta) = 0$. To prove continuity, we observe if we fix a small neighborhood V of η_0 in $(0, \pi)$, then on V we have $|h_n| \leq Cn^{-2}$, $|g_n| \leq Cn^{-2}$ where C only depends on the neighborhood V. Since h_n, g_n are obviously continuous in η, the continuity of $S(F)(\eta)$ follows again by applying Lemma 3.19 and Prop. 3.20, with $T_1 = P_{12}(\phi)$, $T_2 = P_{12}(\eta_0)$. □

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