



Decay of Correlations in Finite Abelian Lattice Gauge Theories

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Abstract: In this paper, we study lattice gauge theory on \mathbb{Z}^4 with finite Abelian structure group. When the inverse coupling strength is sufficiently large, we use ideas from disagreement percolation to give an upper bound on the decay of correlations of local functions. We then use this upper bound to compute the leading-order term for both the expected value of the spin at a given plaquette as well as for the two-point correlation function. Moreover, we give an upper bound on the dependency of the size of the box on which the model is defined. The results in this paper extend and refine results by Chatterjee and Borgs.

1. Introduction

1.1. Background. Gauge theories are crucial tools in modern physics. For instance, they are used to formulate the Standard Model. These quantum field theories describe how different types of elementary particles interact. Even though such models have proved to be very successful in physics, they are not mathematically well-defined. This problem was considered important enough to be chosen as one of the Millennium Problems by the Clay Mathematics Institute [11].

Euclidean lattice gauge theories, with underlying structure group given by e.g. $U(1)$, $SU(2)$ or $SU(3)$, appear as natural and well-defined discretisations of gauge theories on hyper-cubic lattices [17]. These discretizations have been proven to be very useful as tools to study the corresponding quantum field theories using e.g. simulations, high temperature expansions and low temperature expansions [12]. However, there is also hope that one would be able to take a scaling limit and in this way obtain a rigorously defined continuum gauge theory. As a first step in this direction, it is often instructive to try to understand relevant properties of slightly simpler models of the same type. The decay of correlations is an important property to try to understand in any model in statistical physics. For lattice gauge theories, this type of property is given further relevance due its connection with the mass gap problem in Yang–Mills theories. This is

the main motivation for the current paper, where we study the decay of correlations in Abelian lattice gauge theories.

While lattice gauge theories with a finite Abelian structure group are not of known direct physical significance in the context of the Standard Model, they provide toy models for development of tools and ideas which can later be generalized to more physically relevant models. For this reason, they have been studied in the physics literature, see, e.g., [2, 8, 13] and the references therein, as well as in the mathematical literature, see e.g. [2, 6, 7, 16].

1.2. Lattice gauge theories with Wilson action. The lattice \mathbb{Z}^4 has a vertex at each point in \mathbb{R}^4 with integer coordinates, and an edge between nearest neighbors, oriented in the positive direction, so that there are exactly four positively oriented edges emerging from each vertex x , denoted by dx_i , $i = 1, \dots, 4$. We will let $-dx_i$ denote the edge with the same end points as dx_i but with opposite orientation. Each pair dx_i and dx_j of directed edges defines an *oriented plaquette* $dx_i \wedge dx_j$. If $i < j$, we say that the plaquette $dx_i \wedge dx_j$ is positively oriented, and if $i > j$, we say that the plaquette $dx_i \wedge dx_j = -dx_j \wedge dx_i$ is negatively oriented.

Given real numbers $a_1 < b_1, a_2 < b_2, a_3 < b_3$ and $a_4 < b_4$, we say that $B = ([a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \times [a_4, b_4]) \cap \mathbb{Z}^4$ is a box. When B is a box, we write E_B for the set of (positively and negatively) oriented edges both of whose endpoints are contained in B , and P_B for set of the oriented plaquettes whose edges are all contained in E_B . We will often write e and p for elements of E_B and P_B , respectively.

In this paper we will always assume that a finite and Abelian group G has been given. This group will be referred to as the *structure group*. We let Σ_{E_B} be the set of G -valued 1-forms on E_B , i.e., the set of functions $\sigma : E_B \rightarrow G$ with the property that $\sigma(e) = -\sigma(-e)$. Whenever $\sigma \in \Sigma_{E_B}$ and $e \in E_B$, we write $\sigma_e := \sigma(e)$. Each element $\sigma \in \Sigma_{E_B}$ induces a configuration $d\sigma$ on P_B by assigning

$$(d\sigma)_p := \sigma_{e_1} + \sigma_{e_2} + \sigma_{e_3} + \sigma_{e_4}, \quad p \in P_B, \tag{1}$$

where e_1, e_2, e_3, e_4 are the edges in the boundary ∂p of p , directed according to the orientation of the plaquette p , see Sect. 2.1.3. The set of all configurations on P_B which arise in this way will be denoted by Σ_{P_B} .

Next, we let ρ be a faithful, irreducible and unitary representation of G . With G and ρ fixed, we define the *Wilson action* by

$$S(\sigma) := - \sum_{p \in P_B} \Re \operatorname{tr} \rho((d\sigma)_p), \quad \sigma \in \Sigma_{E_B}. \tag{2}$$

Letting μ_H denote the uniform measure on Σ_{E_B} and fixing some $\beta \geq 0$, we obtain an associated probability measure $\mu_{B,\beta}$ on Σ_{E_B} by weighting μ_H by the Wilson action:

$$\mu_{B,\beta}(\sigma) := Z_{B,\beta}^{-1} e^{-\beta S(\sigma)} \mu_H(\sigma), \quad \sigma \in \Sigma_{E_B}, \tag{3}$$

where $Z_{B,\beta}$ is a constant that ensures that $\mu_{B,\beta}$ is a probability measure. The probability measure $\mu_{B,\beta}$ describes lattice gauge theory on B with structure group G , representation ρ , coupling parameter β and *free boundary conditions*. We let $\mathbb{E}_{B,\beta}$ denote expectation with respect to $\mu_{B,\beta}$.

1.3. *A distance between sets.* Let B and B' be two boxes in \mathbb{Z}^4 with $B' \subseteq B$. In all of our results, we need a measure of the distance between sets of plaquettes $P_1, P_2 \subseteq P_B$. To be able to define such a measure, we now introduce the following graph. Given $\omega \in \Sigma_{P_B}$ and $\omega' \in \Sigma_{P_{B'}}$, let $\mathcal{G}(\omega, \omega')$ be the graph with vertex set $\text{supp } \omega \cup \text{supp } \omega'$, and an edge between two distinct plaquettes $p_1, p_2 \in \text{supp } \omega \cup \text{supp } \omega'$ if p_1 and $\pm p_2$ are both in the boundary of some common 3-cell (see also Definition 1). For distinct $p_1, p_2 \in P_B$, let

$$\text{dist}_{B,B'}(p_1, p_2) := \frac{1}{2} \min \left\{ |\text{supp } \omega| + |\text{supp } \omega'| : \omega \in \Sigma_{P_B}, \omega' \in \Sigma_{P_{B'}} \text{ s.t.} \right. \\ \left. p_1 \text{ and } p_2 \text{ are in the same connected component of } \mathcal{G}(\omega, \omega') \right\}, \tag{4}$$

for $p \in P_B$, let $\text{dist}_{B,B'}(p, p) := 0$, and for sets $P_1, P_2 \subseteq P_B$, let

$$\text{dist}_{B,B'}(P_1, P_2) := \min_{p_1 \in P_1, p_2 \in P_2} \text{dist}_{B,B'}(p_1, p_2). \tag{5}$$

When $B' = B$, we write dist_B instead of $\text{dist}_{B,B}$. We mention that for any two distinct plaquettes p_1 and p_2 , one can show that $\text{dist}_{B,B'}(p_1, p_2)$ is bounded from above and below by some constant times the graph distance (in the lattice \mathbb{Z}^4) between the corners of p_1 and the corners of p_2 .

1.4. *Preliminary notation.* To simplify notation, for $\beta \geq 0$ and $g \in G$, we let

$$\phi_\beta(g) := \frac{e^{\beta \Re \text{tr } \rho(g)}}{e^{\beta \Re \text{tr } \rho(0)}}, \tag{6}$$

and

$$\alpha(\beta) := \sum_{g \in G \setminus \{0\}} \phi_\beta(g)^2.$$

The function $\alpha(\beta)$ will be used to express upper bounds on error terms in our main results. We mention that for any finite Abelian group G with a faithful representation ρ , there are constants $C > 0$ and $\xi > 0$ such that $\alpha(\beta) \leq C e^{-\beta \xi}$. In other words, $\alpha(\beta)$ decays exponentially in β .

Next, for $\beta \geq 0$ such that $30\alpha(\beta) < 1$, we define

$$C_1(\beta) := \frac{20}{15^2(1 - 5\alpha(\beta))^2} \left[1 + \frac{2}{1 - 30\alpha(\beta)} \right], \tag{7}$$

and

$$C_2 := 30. \tag{8}$$

We note that as $\beta \rightarrow \infty$, $C_1(\beta) \searrow 4/9$.

When P is a set of plaquettes, we let δP denote the set of all plaquettes in P which shares a 3-cell with some plaquette which does not belong to P .

1.5. Main results. In several recent papers, the expected value of Wilson loop observables have been rigorously analyzed with probabilistic techniques, for different structure groups [4, 6, 7, 9]. The Wilson loop is an important observable in lattice gauge theories because it is believed to be related to the energy required to separate a pair of quarks [5, 17]. Another important observable is the spin-spin-correlation function, which is thought to be related to the so-called mass gap of the model [5]. In the first three main results of this paper, we study variants of this function in the low-temperature regime, by giving results which describe the decay of correlations of local functions. The first of these results is the following theorem. To give the statement, when $\omega \in \Sigma_{P_B}$ and $P \subseteq P_B$ and $P = -P$, we let $\omega|_P := (\omega_p \cdot \mathbb{1}_{p \in P})$ denote the restriction of ω to P in the natural way (see also Sect. 2.1.10).

Theorem 1. *Let B be a box in \mathbb{Z}^4 , and let $\beta \geq 0$ be such that $30\alpha(\beta) < 1$. Further, let $f_1, f_2: \Sigma_{P_B} \rightarrow \mathbb{C}$ and assume that there are disjoint sets $P_1, P_2 \subseteq P_B$ such that for all $\omega \in \Sigma_{P_B}$ we have $f_1(\omega) = f_1(\omega|_{P_1})$ and $f_2(\omega) = f_2(\omega|_{P_2})$. Then, if $\sigma \sim \mu_{B,\beta}$, we have*

$$|\text{Cov}(f_1(d\sigma), f_2(d\sigma))| \leq C_1 \|f_1\|_\infty \|f_2\|_\infty (C_2\alpha(\beta))^{\text{dist}_B(P_1, P_2)}, \tag{9}$$

where $C_1 = C_1(\beta)$ and C_2 are given by in (7) and (8) respectively.

In particular, if $p_1 \in P_B$ and $p_2 \in P_B$ are distinct, then

$$|\text{Cov}(\text{tr } \rho((d\sigma)_{p_1}), \text{tr } \rho((d\sigma)_{p_2}))| \leq C_1 (\dim \rho)^2 (C_2\alpha(\beta))^{\text{dist}_B(p_1, p_2)}. \tag{10}$$

Remark 1. Since $\alpha(\beta)$ decays exponentially in β , Theorem 1 shows that the covariance of two local functions decays exponentially both in β and in the distance between the corresponding sets P_1 and P_2 .

Remark 2. We mention that if one for disjoint plaquettes $p_1, p_2 \in P_B$ knew which configurations attained the minimum in (4), then the methods used in this paper could be adapted slightly to describe the first order behaviour of $\text{Cov}(\text{tr } \rho((d\sigma)_{p_1}), \text{tr } \rho((d\sigma)_{p_2}))$ for sufficiently large β .

With Theorem 1 at hand, it is natural to ask how the decay of the covariance in (10) relates to the so-called spin-spin correlation $\mathbb{E}_{B,\beta}[\text{tr } \rho((d\sigma)_p) \text{tr } \rho((d\sigma)_{p'})]$. The next theorem, which improves upon a special case of Lemma 4.2 in [2], is a first step towards answering this question.

Theorem 2. *Let B be a box in \mathbb{Z}^4 , and let $\beta \geq 0$ be such that $5\alpha(\beta) < 1$.*

Further, let $p \in P_B$ and $f: G \rightarrow \mathbb{C}$. Then, if $\text{dist}_B(\{p\}, \delta P_B) > 11$, we have

$$\begin{aligned} & \left| \mathbb{E}_{B,\beta}[f((d\sigma)_p)] - \left(f(0) + \sum_{e \in \partial p} \sum_{g \in G} (f(g) - f(0)) \phi_\beta(g)^{12} \right) \right| \\ & \leq \frac{(5\alpha(\beta))^{11}}{1 - 5\alpha(\beta)} \max_{g \in G} |f(g) - f(0)|. \end{aligned} \tag{11}$$

Remark 3. When β tends to infinity, we have $\sum_{e \in \partial p} \sum_{g \in G} \phi_\beta(g)^{12} \asymp \alpha(\beta)^6$, and hence the right hand side of (11) will in general tend to zero much faster than the term $\sum_{e \in \partial p} \sum_{g \in G} (f(g) - f(0)) \phi_\beta(g)^{12}$ on the left hand side of the same equation. Consequently, Theorem 2 captures the first- and second-order behaviour of $\mathbb{E}_{B,\beta}[f((d\sigma)_p)]$.

Combining Theorem 1 and Theorem 2, we obtain the following result as a corollary.

Theorem 3. *Let B be a box in \mathbb{Z}^4 , and let $\beta \geq 0$ be such that $30\alpha(\beta) < 1$. Further, let $p_1, p_2 \in P_B$ be distinct, and let $f_1, f_2: G \rightarrow \mathbb{C}$. Then, if $\text{dist}_B(\{p_1, p_2\}, \delta P_B) > 11$, we have*

$$\left| \mathbb{E}_{B,\beta} [f_1((d\sigma)_{p_1}) f_2((d\sigma)_{p_2})] - \prod_{j \in \{1,2\}} \left(f_j(0) + \sum_{e \in \partial p} \sum_{g \in G} (f_j(g) - f_j(0)) \phi_\beta(g)^{12} \right) \right| \tag{12}$$

$$\leq C_1 \|f_1\|_\infty \|f_2\|_\infty (C_2 \alpha(\beta))^{\text{dist}_B(\{p_1\}, \{p_2\})} + 8 \|f_1\|_\infty \|f_2\|_\infty \frac{(5\alpha(\beta))^{11}}{1 - 5\alpha(\beta)}.$$

where $C_1 = C_1(\beta)$ and C_2 , are given by (7), and (8) respectively.

Remark 4. For $N \geq 1$, let B_N be the box $[-N, N]^4 \cap \mathbb{Z}^4$. By applying Ginibre’s inequality [10], one can show that whenever f is a real-valued function which depends only on a finite number of plaquettes, then the limit

$$\lim_{N \rightarrow \infty} \mathbb{E}_{B_N, \beta} [f(d\sigma)] \tag{13}$$

exists and is translation invariant (see e.g. Section 2.3 in [7]). From this result, it follows that Theorems 1, 2 and 3 holds also in this limit.

By using the same strategy as for the proof of Theorem 1, we obtain the following result, which extends Theorem 5.3 in [6]. In this result, $\text{dist}_{TV}(X, Y)$ denotes the total variation distance between two random variables X and Y .

Theorem 4 (Compare with Theorem 5.3 in [6] and Theorem 2.4 in [2]). *Let B and B' be two boxes in \mathbb{Z}^4 with $B' \subsetneq B$, and let $\beta \geq 0$ be such that $30\alpha(\beta) < 1$. Further, let $P \subseteq P_{B'}$, and let $\sigma \sim \mu_{B,\beta}$ and $\sigma' \sim \mu_{B',\beta}$. Then*

$$\text{dist}_{TV}((d\sigma)|_P, (d\sigma')|_P) \leq C_1 |P| (C_2 \alpha(\beta))^{\text{dist}_{B,B'}(P, P_B \setminus P_{B'})},$$

where $C_1 = C_1(\beta)$ and C_2 are given by (7) and (8) respectively.

Remark 5. In contrast to Theorem 5.3 in [6], which hold only for $G = \mathbb{Z}_2$, Theorem 4 is valid for any finite Abelian group. Moreover, our proof can easily be adapted to work for other lattices such as \mathbb{Z}^n for $n \geq 3$, as well as for other actions such as the Villain action. Moreover, we mention that even in the case of $G = \mathbb{Z}_2$, we use a completely different proof strategy than the strategy used in the corresponding proof in [6].

Remark 6. By [1], a critical value for β in the case $G = \mathbb{Z}_2$ is given by 0.22. In comparison, when $G = \mathbb{Z}_2$, the assumption on β in the above results is either that $5e^{-4\beta} < 1$ (equivalently, $\beta \geq 0.40$) or that $30e^{-4\beta} < 1$ (equivalently, $\beta \geq 0.8$).

Remark 7. In this paper, we always use the measure given by (3), corresponding to free boundary conditions. However, with minor changes to the proofs in the paper, one can obtain results analogous to our main results for zero or periodic boundary conditions.

Remark 8. Using the theory as outlined in [4], some of the ideas in this paper might extend to finite non-Abelian structure groups as well. However, in some of the proofs, we use tools from discrete exterior calculus which are not valid in a non-Abelian setting. Consequently, such a generalization would be non-trivial.

Remark 9. The main novelty of this paper is the use of a coupling argument, similar to arguments in e.g. [3, 15], in to obtain upper bounds on both the covariance of local functions and on the total variation distance. Also, we give natural extensions and generalizations of several technical but useful lemmas from [7].

1.6. Structure of the paper. In Sect. 2, we give a brief summary of the foundations of discrete exterior calculus on hypercubic lattices, which we will use throughout the rest of this paper. Next, in Sect. 3, we give upper and lower bounds for the probability that certain plaquette configurations arise in spin configurations. In Sect. 4, we discuss the structure and properties of paths in the graph $\mathcal{G}(\omega^{(0)}, \omega^{(1)})$ for $\omega^{(0)}, \omega^{(1)} \in \Sigma_{P_B}$. In Sect. 5, we define a notion of two sets being connected, and prove a lemma suggesting the usefulness of this concept. In Sect. 6, we define another measure $\text{dist}_B^*(p_1, p_2)$ of the distance between two plaquettes, and state and prove a lemma which gives a relationship between this function and the function $\text{dist}_{B, B'}(p_1, p_2)$ defined in the introduction. These results are then used in Sect. 7 to give an upper bound on the probability that two sets are connected. In Sect. 8, we give a connection between the event that two sets are connected and the covariance of local functions supported on these sets, and then use this connection to give a proof of Theorem 1. In Sect. 9, we give a similar connection between the event that two sets are connected and the total variation distance, and then use this observation to give a proof of Theorem 4. Finally, in Sect. 10, we prove Theorems 2 and 3.

2. Preliminaries

2.1. Discrete exterior calculus. In this section, we give a very brief overview of discrete exterior calculus on the cell complexes of \mathbb{Z}^n for $n \in \mathbb{N}$. For a more thorough background on discrete exterior calculus, we refer the reader to [6].

All of the results in this section are obtained under the assumption that an Abelian group G , which is not necessarily finite, has been given. In particular, they all hold for $G = \mathbb{Z}$.

2.1.1. Oriented edges (1-cells) The graph \mathbb{Z}^n has a vertex at each point $x \in \mathbb{Z}^n$ with integer coordinates and an (undirected) edge between nearest neighbors. We associate to each undirected edge \bar{e} in \mathbb{Z}^n exactly two directed or *oriented* edges e and $-e$ with the same endpoints as \bar{e} ; e is directed so that the coordinate increases when traversing e , and $-e$ is directed in the opposite way.

Let $\mathbf{e}_1 := (1, 0, 0, \dots, 0)$, $\mathbf{e}_2 := (0, 1, 0, \dots, 0)$, ..., $\mathbf{e}_n := (0, \dots, 0, 1)$ and let $d\mathbf{e}_1, \dots, d\mathbf{e}_n$ denote the n oriented edges with one endpoint at the origin which naturally correspond to these unit vectors (oriented away from the origin). We say that an oriented edge e is *positively oriented* if it is equal to a translation of one of these unit vectors, i.e., if there exists a point $x \in \mathbb{Z}^n$ and an index $j \in \{1, 2, \dots, n\}$ such that $e = x + d\mathbf{e}_j$. If $x \in \mathbb{Z}^n$ and $j \in \{1, 2, \dots, n\}$, then we let $dx_j := x + d\mathbf{e}_j$.

Given a box B , we let E_B denote the set of oriented edges whose end-points are both in B , and let E_B^+ denote the set of positively oriented edges in E_B .

2.1.2. Oriented k-cells For any two oriented edges $e_1 \in E_B$ and $e_2 \in E_B$, we consider the wedge product $e_1 \wedge e_2$ satisfying $e_1 \wedge e_1 = 0$ and

$$e_1 \wedge e_2 = -(e_2 \wedge e_1) = (-e_2) \wedge e_1 = e_2 \wedge (-e_1). \tag{14}$$

If e_1, e_2, \dots, e_k are oriented edges which do not share a common endpoint, we set $e_1 \wedge e_2 \wedge \dots \wedge e_k = 0$.

If e_1, e_2, \dots, e_k are oriented edges and $e_1 \wedge \dots \wedge e_k \neq 0$, we say that $e_1 \wedge \dots \wedge e_k$ is an oriented k -cell. If there exists an $x \in \mathbb{Z}^n$ and $j_1 < j_2 < \dots < j_k$ such that $e_i = dx_{j_i}$, then we say that $e_1 \wedge \dots \wedge e_k$ is *positively oriented* and that $-(e_1 \wedge \dots \wedge e_k)$ is *negatively oriented*.

oriented. Using (14), this defines an orientation for all k -cells. If not stated otherwise, we will always consider k -cells as being oriented.

If A is a set of oriented k -cells, we let A^+ denote the set of positively oriented k -cells in A . If $A = -A$, then we say that A is *symmetric*.

2.1.3. Oriented plaquettes We will usually say *oriented plaquette* instead of oriented 2-cell. If $x \in \mathbb{Z}^n$ and $1 \leq j_1 < j_2 \leq n$, then $p := dx_{j_1} \wedge dx_{j_2}$ is a positively oriented plaquette, and we define

$$\partial p := \{dx_{j_1}, (d(x + \mathbf{e}_{j_1}))_{j_2}, -(d(x + \mathbf{e}_{j_2}))_{j_1}, -dx_{j_2}\}.$$

If e is an oriented edge, we let $\hat{\partial}e$ denote the set of oriented plaquettes p such that $e \in \partial p$.

We let P_B denote the set of oriented plaquettes whose edges are all in E_B .

2.1.4. Discrete differential forms A G -valued function f defined on a subset of the set of k -cells in \mathbb{Z}^n with the property that $f(c) = -f(-c)$ is called a k -form. If f is a k -form which takes the value $f_{j_1, \dots, j_k}(x)$ on $dx_{j_1} \wedge \dots \wedge dx_{j_k}$, it is useful to represent its values on the k -cells at $x \in \mathbb{Z}^n$ by the formal expression

$$f(x) = \sum_{1 \leq j_1 < \dots < j_k \leq n} f_{j_1, \dots, j_k}(x) dx_{j_1} \wedge \dots \wedge dx_{j_k}.$$

To simplify notation, if $c := dx_{j_1} \wedge \dots \wedge dx_{j_k}$ is a k -cell and f is a k -form we often write f_c instead of $f_{j_1, \dots, j_k}(x)$.

Given a k -form f , we let $\text{supp } f$ denote the support of f , i.e. the set of all oriented k -cells c such that $f(c) \neq 0$.

Now let B be a box, and recall that for $k \in \{1, 2, \dots, n\}$, a k -cell c is said to be in B if all its corners are in B . The set of G -valued k -forms with support in the set of k -cells that are contained in a B will be denoted by $\Sigma_{B,k}$. The set of G -valued 1-forms with support in E_B will also be denoted by $\Sigma_{E_B} = \Sigma_{B,k}$, and will be referred to as *spin configurations*, and the set of G -valued 2-forms with support in Σ_{P_B} will be referred to as *plaquette configurations*.

2.1.5. The exterior derivative Given $h: \mathbb{Z}^n \rightarrow G$, $x \in \mathbb{Z}^n$, and $i \in \{1, 2, \dots, n\}$, we let

$$\partial_i h(x) := h(x + \mathbf{e}_i) - h(x).$$

If $k \in \{0, 1, 2, \dots, n-1\}$ and f is a G -valued k -form, we define the $(k+1)$ -form df via the formal expression

$$df(x) = \sum_{1 \leq j_1 < \dots < j_k \leq n} \sum_{i=1}^n \partial_i f_{j_1, \dots, j_k}(x) dx_i \wedge (dx_{j_1} \wedge \dots \wedge dx_{j_k}), \quad x \in \mathbb{Z}^n.$$

The operator d is called the *exterior derivative*.

2.1.6. *Boundary operators* If $\hat{x} \in \mathbb{Z}^n$ and $j \in \{1, 2, \dots, n\}$, then

$$\begin{aligned} d(\mathbb{K}_{x=\hat{x}} dx_j) &= \sum_{i=1}^n (\partial_i \mathbb{K}_{x=\hat{x}}) dx_i \wedge dx_j = \sum_{i=1}^n (\mathbb{K}_{x+\mathbf{e}_i=\hat{x}} - \mathbb{K}_{x=\hat{x}}) dx_i \wedge dx_j \\ &= \sum_{i=1}^{j-1} (\mathbb{K}_{x+\mathbf{e}_i=\hat{x}} - \mathbb{K}_{x=\hat{x}}) dx_i \wedge dx_j - \sum_{i=j+1}^n (\mathbb{K}_{x+\mathbf{e}_i=\hat{x}} - \mathbb{K}_{x=\hat{x}}) dx_j \wedge dx_i. \end{aligned}$$

Here we are writing $\mathbb{K}_{x=\hat{x}}$ for the Dirac delta function of x with mass at \hat{x} . From this equation, it follows that whenever $e = \hat{x} + d\mathbf{e}_j = d\hat{x}_j$ is an oriented edge and p is an oriented plaquette, we have

$$(d(\mathbb{K}_{x=\hat{x}} dx_j))_p = \begin{cases} 1 & \text{if } e \in \partial p, \\ -1 & \text{if } -e \in \partial p, \\ 0 & \text{else.} \end{cases} \tag{15}$$

Note that this implies in particular that if $1 \leq j_1 < j_2 \leq n$, $p = dx_{j_1} \wedge dx_{j_2}$ is a plaquette, and f is a 1-form, then

$$(df)_p = (df)_{j_1, j_2}(x) = \sum_{e \in \partial p} f_e.$$

Analogously, if $k \in \{1, 2, \dots, n\}$ and c is a k -cell, we define ∂c as the set of all $(k - 1)$ -cells $\hat{c} = d\hat{x}_{j_1} \wedge \dots \wedge d\hat{x}_{j_{k-1}}$ such that

$$(d(\mathbb{K}_{x=\hat{x}} d\hat{x}_{j_1} \wedge \dots \wedge d\hat{x}_{j_{k-1}}))_c = 1.$$

Using this notation, one can show that if f is a k -form and c_0 is a $(k + 1)$ -cell, then

$$(df)_{c_0} = \sum_{c \in \partial c_0} f_c.$$

If $k \in \{1, 2, \dots, n\}$ and \hat{c} is a k -cell, the set $\partial \hat{c}$ will be referred to as the *boundary* of \hat{c} . When $k \in \{0, 1, 2, 3, \dots, n - 1\}$ and c is a k -cell, we also define the *co-boundary* $\hat{\partial} c$ of c as the set of all $(k + 1)$ -cells \hat{c} such that $c \in \partial \hat{c}$.

Finally, when $k \in \{1, 2, \dots, n\}$ and c is a k -cell, we will abuse notation and let

$$\hat{\partial}(\partial c) := \bigcup_{c' \in \partial c} \hat{\partial} c'.$$

Similarly, when $k \in \{0, 1, 2, \dots, n - 1\}$ and c is a k -cell, we let

$$\partial(\hat{\partial} c) := \bigcup_{c' \in \hat{\partial} c} \partial c'.$$

The following lemma will be useful to us.

Lemma 5 (The Bianchi lemma, see e.g. Lemma 2.5 in [7]). *Let B be a box in \mathbb{Z}^n for some $n \geq 3$, and let $\omega \in \Sigma_{P_B}$. Then, for any oriented 3-cell c in B , we have*

$$\sum_{p \in \partial c} \omega_p = 0. \tag{16}$$

2.1.7. *Boundary cells* Let $k \in \{1, 2, \dots, n\}$. When A is a symmetric set of k -cells, we define the boundary of A by

$$\delta A := \{c \in A : \partial \hat{\partial} c \not\subseteq A\}.$$

If B is a box and $e \in \delta E_B$, we say that e is a *boundary edge* of B . Analogously, a plaquette $p \in \delta P_B$ is said to be a *boundary plaquette* of B . More generally, for $k \in \{0, 1, \dots, n - 1\}$, a k -cell c in B is said to be a *boundary cell* of B , or equivalently to be in *the boundary* of B , if there is a $(k + 1)$ -cell $\hat{c} \in \hat{\partial} c$ which contains a k -cell that is not in B .

2.1.8. *Closed forms* If $k \in \{1, \dots, n - 1\}$ and f is a k -form such that $df = 0$, then we say that f is *closed*.

2.1.9. *The Poincaré lemma*

Lemma 6 (The Poincaré lemma, Lemma 2.2 in [6]). *Let $k \in \{0, 1, \dots, n - 1\}$ and let B be a box in \mathbb{Z}^n . Then the exterior derivative d is a surjective map from the set of G -valued k -forms with support contained in B onto the set of G -valued closed $(k + 1)$ -forms with support contained in B . Moreover, if G is finite and m is the number of closed G -valued k -forms with support contained in B , then this map is an m -to-1 correspondence. Lastly, if $k \in \{0, 1, 2, \dots, n - 1\}$ and f is a closed $(k + 1)$ -form that vanishes on the boundary of B , then there is a k -form h that also vanishes on the boundary of B and satisfies $dh = f$.*

Recall from the introduction that when B is a box in \mathbb{Z}^n , we defined

$$\Sigma_{P_B} = \{\omega \in \Sigma_{B,2} : \exists \sigma \in \Sigma_{E_B} \text{ such that } \omega = d\sigma\}.$$

From Lemma 6, it follows that $\omega \in \Sigma_{P_B}$ if and only if $d\omega = 0$.

2.1.10. *Restrictions of forms* If $\sigma \in \Sigma_{E_B}$, $E \subseteq E_B$ is symmetric, we define $\sigma|_E \in \Sigma_{E_B}$ for $e \in E_B$ by

$$(\sigma|_E)_e := \begin{cases} \sigma_e & \text{if } e \in E \\ 0 & \text{else.} \end{cases}$$

Similarly, if $\nu \in \Sigma_{P_B}$, $P \subseteq P_B$ is symmetric, we define $\omega|_P \in \Sigma_{B,2}$ for $p \in P_B$ by

$$(\nu|_P)_p := \begin{cases} \nu_p & \text{if } p \in P \\ 0 & \text{else.} \end{cases}$$

2.1.11. *Non-trivial forms* Let $k \in \{1, 2, \dots, n\}$. A k -form f is said to be *non-trivial* if it is not identically equal to zero.

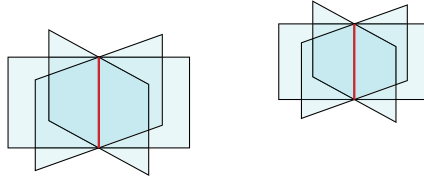


Fig. 1. Above, we illustrate the setting of Example 1; the red edges represent the edges e and e' , and the blue plaquettes represent the plaquettes in the supports of ω and ω' .

2.1.12. Irreducible forms Let B be a box in \mathbb{Z}^n , and let $\omega \in \Sigma_{P_B}$. If $P \subseteq \text{supp } \omega$ is symmetric, and there is no symmetric set $P_0 \subseteq P_B$ such that

- (i) $P \subseteq P_0 \subsetneq \text{supp } \omega$, and
- (ii) $\omega|_{P_0} \in \Sigma_{P_B}$ (equivalently $d(\omega|_{P_0}) = 0$),

then ω is said to be P -irreducible (see Fig. 1). If $\omega \in \Sigma_{P_B}$ is P -irreducible for all non-empty, symmetric sets $P \subseteq \text{supp } \omega$, then we say that ω is *irreducible*. Finally, note that if $\omega \in \Sigma_{P_B}$ is \emptyset -irreducible, then $\omega \equiv 0$.

Example 1. Assume that we are given two edges $e, e' \in \Sigma_{E_N}$ with $\hat{\partial}e \cap \pm\hat{\partial}e' = \emptyset$, that are not in the boundary of B_N . For $g \in G \setminus \{0\}$, let $\omega := d(g\mathbb{1}_{e'} - g\mathbb{1}_{-e})$ and $\omega' := d(g\mathbb{1}_{-e'} - g\mathbb{1}_{-e})$ (see Fig. 1). Then, by definition, we have $\omega, \omega' \in \Sigma_{P_N}$, and one easily verifies that ω and ω' are both irreducible.

On the other hand, if $p \in \text{supp } \omega$, and we define $P := \{p, -p\}$ and $P_0 := \text{supp } \omega' \sqcup P$, then $\omega'|_{P_0} = \omega' \in \Sigma_{P_{B_N}}$, and hence $\omega + \omega'$ is not P -irreducible, implying in particular that it is not irreducible.

More generally, one can show that if $P \subseteq \text{supp } \omega + \omega'$ is symmetric, then $\omega + \omega'$ is P -irreducible if and only if P has non-empty intersection with both $\text{supp } \omega$ and $\text{supp } \omega'$.

Lemma 7. *Let B be a box in \mathbb{Z}^n , let $\omega \in \Sigma_{P_B}$, and let $P \subseteq \text{supp } \omega$ be non-empty and symmetric. Then there is a symmetric set $P' \subseteq \text{supp } \omega$ such that $\omega|_{P'} \in \Sigma_{P_B}$ and $\omega|_{P'}$ is P -irreducible.*

Proof. Consider the set \mathcal{S} of all symmetric sets $P' \subseteq P_B$ which are such that

- (1) $P \subseteq P' \subseteq \text{supp } \omega$, and
- (2) $\omega|_{P'} \in \Sigma_{P_B}$.

Since $\text{supp } \omega$ is symmetric and $\omega \in \Sigma_{P_B}$, we have $\omega \in \mathcal{S}$, and hence \mathcal{S} is non-empty. Moreover, if we order the elements in \mathcal{S} using set inclusion, \mathcal{S} is a partially ordered set. Since P_B is finite, \mathcal{S} is finite, and hence there is a minimal element $P' \in \mathcal{S}$. By definition, any such minimal element is P -irreducible, and hence the desired conclusion follows. □

2.1.13. The dual lattice The lattice \mathbb{Z}^n has a natural dual, called the *dual lattice* and denoted by $*\mathbb{Z}^n$. In this context, the lattice \mathbb{Z}^n is called the *primal lattice*.

The vertices of the dual lattice $*\mathbb{Z}^n$ are placed at the centers of the n -cells of the primal lattice.

For $k \in \{0, 1, \dots, n\}$, there is a bijection between the set of k -cells of \mathbb{Z}^n and the set of $(n - k)$ -cells of $*\mathbb{Z}^n$ defined as follows. For each $x \in \mathbb{Z}^n$, let $y := *(dx_1 \wedge \dots \wedge dx_n) \in *\mathbb{Z}^n$ be the point at the centre of the primal lattice n -cell $dx_1 \wedge \dots \wedge dx_n$. Let $dy_1 = y - \mathbf{de}_1, \dots, dy_n = y - \mathbf{de}_n$ be the edges coming out of y in the *negative*

direction. Next, let $k \in \{0, 1, \dots, n\}$ and assume that $1 \leq i_1 < \dots < i_k \leq n$ are given. If $x \in \mathbb{Z}^n$, then $c = dx_{i_1} \wedge \dots \wedge dx_{i_k}$ is a k -cell in \mathbb{Z}^n . Let j_1, \dots, j_{n-k} be any enumeration of $\{1, 2, \dots, n\} \setminus \{i_1, \dots, i_k\}$, and let $\text{sgn}(i_1, \dots, i_k, j_1, \dots, j_{n-k})$ denote the sign of the permutation that maps $(1, 2, \dots, n)$ to $(i_1, \dots, i_k, j_1, \dots, j_{n-k})$. Define

$$*(dx_{i_1} \wedge \dots \wedge dx_{i_k}) = \text{sgn}(i_1, \dots, i_k, j_1, \dots, j_{n-k}) dy_{j_1} \wedge \dots \wedge dy_{j_{n-k}}$$

and, analogously, define

$$\begin{aligned} *(dy_{j_1} \wedge \dots \wedge dy_{j_{n-k}}) &= \text{sgn}(j_1, \dots, j_{n-k}, i_1, \dots, i_k) dx_{i_1} \wedge \dots \wedge dx_{i_k} \\ &= (-1)^{k(n-k)} \text{sgn}(i_1, \dots, i_k, j_1, \dots, j_{n-k}) dx_{i_1} \wedge \dots \wedge dx_{i_k}. \end{aligned}$$

2.1.14. Minimal non-trivial configurations The purpose of the next two lemmas is to describe the non-trivial plaquette configurations in Σ_{P_B} with smallest support.

Lemma 8. *Let B be a box in \mathbb{Z}^4 , and let $\omega \in \Sigma_{P_B}$. If $\omega \neq 0$ and the support of ω does not contain any boundary plaquettes of P_B , then either $|\text{supp } \omega| = 12$, or $|\text{supp } \omega| \geq 22$.*

Proof. Let $P := \text{supp } \omega$.

By Lemma 5, if c is an oriented 3-cell in the primary lattice, then

$$|\partial c \cap P| \in \{0, 2, 3, 4, 5, 6\}.$$

Consequently, if $e = *c$ in an oriented edge in the dual lattice, then

$$|\hat{\partial} e \cap *P| \in \{0, 2, 3, 4, 5, 6\}. \tag{17}$$

Let $\overline{*P}$ be the set of unoriented plaquettes obtained from $*P$ by identifying p and $-p$ for each $p \in *P$. It then follows from (17) that each unoriented edge \bar{e} in the dual lattice which is in the boundary of a plaquette in $\overline{*P}$ must be in the boundary of at least two plaquettes in $\overline{*P}$.

In other words, the set $\overline{*P}$ is a closed surface in the dual lattice. One easily verifies that the closed (non-empty) surfaces in the dual lattice which contains the fewest number of plaquettes are 3-dimensional cubes (see Fig. 2), and hence we must have $|\overline{*P}| \geq 6$. If $|\overline{*P}| > 6$, then by the same argument we must have $|\overline{*P}| \geq 11$ (see Fig. 2).

Since $|\text{supp } \omega| = |P| = 2|\overline{*P}|$, the desired conclusion follows. □

Lemma 9 (Lemma 4.6 in [7]). *Let B be a box in \mathbb{Z}^4 , and let $\omega \in \Sigma_{P_B}$. If the support of ω does not contain any boundary plaquettes of P_B and $|\text{supp } \omega| = 12$, then there is an edge $dx_j \in E_B$ and $g \in G \setminus \{0\}$ such that*

$$\omega = d(g dx_j).$$

When B is a box in \mathbb{Z}^4 and $\omega \in \Sigma_{P_B}$, we say that a plaquette $p \in P_B$ is *frustrated* (in ω) if $\omega_p \neq 0$.


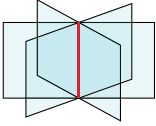
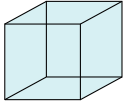

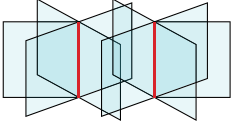
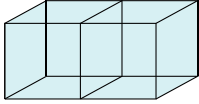

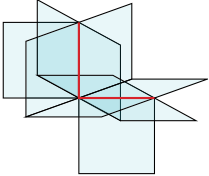
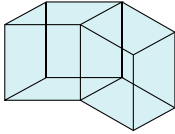
$\overline{\text{supp } \sigma}$	$\bar{P} = \overline{\text{supp } d\sigma}$	$*P$	$ \bar{P} $
			6
			11
			11

Fig. 2. The above table shows projections of the supports of the non-trivial and irreducible plaquette configurations in \mathbb{Z}^4 which has the smallest support (up to translations and rotations), using the notation of the proof of Lemma 8

2.2. $\mu_{B,\beta}$ as a measure on plaquette configurations. Let B be a box in \mathbb{Z}^4 , and let $\beta \geq 0$. In (3), we introduced $\mu_{B,\beta}$ as a measure on Σ_{E_B} . Using Lemma 6, this induces a measure on Σ_{P_B} as follows. For $\omega \in \Sigma_{P_B}$, by definition, we have

$$\mu_{B,\beta}(\{\sigma \in \Sigma_{E_B} : d\sigma = \omega\}) = \frac{\sum_{\sigma \in \Sigma_{E_B} : d\sigma = \omega} \prod_{p \in P_B} \phi_\beta((d\sigma)_p)}{\sum_{\sigma \in \Sigma_{E_B}} \prod_{p \in P_B} \phi_\beta((d\sigma)_p)}.$$

If $\sigma \in \Sigma_{E_B}$, then $d\sigma \in \Sigma_{P_B}$. Consequently, the previous equation is equal to

$$\frac{\sum_{\sigma \in \Sigma_{E_B} : d\sigma = \omega} \prod_{p \in P_B} \phi_\beta(\omega_p)}{\sum_{\omega' \in \Sigma_{P_B}} \sum_{\sigma \in \Sigma_{E_B} : d\sigma = \omega'} \prod_{p \in P_B} \phi_\beta(\omega'_p)}$$

Changing the order of summation, we get

$$\frac{(\prod_{p \in P_B} \phi_\beta(\omega_p)) |\{\sigma \in \Sigma_{E_B} : d\sigma = \omega\}|}{(\sum_{\omega' \in \Sigma_{P_B}} \prod_{p \in P_B} \phi_\beta(\omega'_p)) |\{\sigma \in \Sigma_{E_B} : d\sigma = \omega'\}|}$$

By Lemma 6, the term $|\{\sigma \in \Sigma_{E_B} : d\sigma = \omega'\}|$ is equal for all $\omega' \in \Sigma_{P_B}$, and hence in particular, for all $\omega' \in \Sigma_{P_B}$ we have $|\{\sigma \in \Sigma_{E_B} : d\sigma = \omega'\}| = |\{\sigma \in \Sigma_{E_B} : d\sigma = \omega\}|$. Combining the above equations, we thus obtain

$$\mu_{B,\beta}(\{\sigma \in \Sigma_{E_B} : d\sigma = \omega\}) = \frac{\prod_{p \in P_B} \phi_\beta(\omega_p)}{\sum_{\omega' \in \Sigma_{P_B}} \prod_{p \in P_B} \phi_\beta(\omega'_p)}. \tag{18}$$

Consequently, $\mu_{B,\beta}$ induces a measure on plaquette configurations. In order to simplify notation, we will abuse notation and use $\mu_{B,\beta}$ and $\mathbb{E}_{B,\beta}$ for both the measure on Σ_{E_B} and for the induced measure on Σ_{P_B} .

2.3. *The activity of plaquette configurations.* When B is a box in \mathbb{Z}^4 , $\omega \in P_B$ and $\beta \geq 0$, then, recalling the definition of ϕ_β from (6), we abuse notation and write

$$\phi_\beta(\omega) := \prod_{p \in \text{supp } \omega} \phi_\beta(\omega_p).$$

The quantity $\phi_\beta(\omega)$ is called the *activity* of ω (see e.g. [2]). Using this notation, since $\phi_\beta(0) = 1$, for any $\omega \in \Sigma_{P_B}$, we also have

$$\prod_{p \in P_B} \phi_\beta(\omega_p) = \prod_{p \in \text{supp } \omega} \phi_\beta(\omega_p) = \phi_\beta(\omega)$$

and hence, using (18), we can write

$$\mu_{B,\beta}(\{\omega\}) = \frac{\phi_\beta(\omega)}{\sum_{\omega' \in \Sigma_{P_B}} \phi_\beta(\omega')}. \tag{19}$$

The following lemma is referred to as the factorization property of ϕ_β in e.g. [4].

Lemma 10. *Let B be a box in \mathbb{Z}^4 , and let $\beta \geq 0$. Further, let $\omega, \omega' \in \Sigma_{P_B}$ be such that $\text{supp } \omega \cap \text{supp } \omega' = \emptyset$. Then*

$$\phi_\beta(\omega + \omega') = \phi_\beta(\omega)\phi_\beta(\omega').$$

Proof. By definition, we have

$$\phi_\beta(\omega + \omega') = \prod_{p \in \text{supp}(\omega + \omega')} \phi_\beta((\omega + \omega')_p) = \prod_{p \in \text{supp}(\omega + \omega')} \phi_\beta(\omega_p + \omega'_p).$$

Since ω and ω' have disjoint supports, we have

$$\begin{aligned} \prod_{p \in \text{supp}(\omega + \omega')} \phi_\beta(\omega_p + \omega'_p) &= \prod_{p \in \text{supp } \omega} \phi_\beta(\omega_p + 0) \prod_{p \in \text{supp } \omega'} \phi_\beta(0 + \omega'_p) \\ &= \prod_{p \in \text{supp } \omega} \phi_\beta(\omega_p) \prod_{p \in \text{supp } \omega'} \phi_\beta(\omega'_p). \end{aligned}$$

Since by definition, we have

$$\prod_{p \in \text{supp } \omega} \phi_\beta(\omega_p) \prod_{p \in \text{supp } \omega'} \phi_\beta(\omega'_p) = \phi_\beta(\omega)\phi_\beta(\omega'),$$

the desired conclusion immediately follows. □

Combining (19) and Lemma 10, we obtain the following lemma. This lemma can be extracted from the proof of Lemma 4.7 in [7], but we state and prove it here as an independent lemma for easier reference.

Lemma 11. *Let B be a box in \mathbb{Z}^4 , let $\beta \geq 0$, and let $v \in \Sigma_{P_B}$. Then*

$$\frac{\mu_{B,\beta}(\{\omega \in \Sigma_{P_B} : \omega|_{\text{supp } v} = v\})}{\mu_{B,\beta}(\{\omega \in \Sigma_{P_B} : \omega|_{\text{supp } v} = 0\})} = \phi_\beta(v). \tag{20}$$

Proof. Let $P = \text{supp } v$. Further, let

$$\mathcal{E}_{P,v} := \{\omega \in \Sigma_{P_B} : \omega|_P = v\},$$

and, similarly, let

$$\mathcal{E}_{P,0} := \{\omega \in \Sigma_{P_B} : \omega|_P = 0\}.$$

By (19), we then have

$$\frac{\mu_{B,\beta}(\{\omega \in \Sigma_{P_B} : \omega|_{\text{supp } v} = v\})}{\mu_{B,\beta}(\{\omega \in \Sigma_{P_B} : \omega|_{\text{supp } v} = 0\})} = \frac{\sum_{\omega \in \mathcal{E}_{P,v}} \phi_\beta(\omega)}{\sum_{\omega \in \mathcal{E}_{P,0}} \phi_\beta(\omega)}. \tag{21}$$

Since $v \in \Sigma_{P_B}$ by assumption, we have $dv = 0$. Consequently, for any $\omega \in \Sigma_{P_B}$, we have

$$d(\omega - v) = d\omega - dv = 0 - 0 = 0,$$

and hence $(\omega - v) \in \Sigma_{P_B}$. This implies in particular that the mapping $\omega \mapsto \omega - v$ is a bijection from $\mathcal{E}_{P,v}$ to $\mathcal{E}_{P,0}$, and hence the right-hand side of (21) is equal to

$$\frac{\sum_{\omega \in \mathcal{E}_{P,v}} \phi_\beta(\omega)}{\sum_{\omega \in \mathcal{E}_{P,v}} \phi_\beta(\omega - v)} = \frac{\sum_{\omega \in \mathcal{E}_{P,v}} \phi_\beta((\omega - v) + v)}{\sum_{\omega \in \mathcal{E}_{P,v}} \phi_\beta(\omega - v)}. \tag{22}$$

Next, note that if $\omega \in \mathcal{E}_{P,v}$, then $(\omega - v)$ and v have disjoint supports. Consequently, for such ω we can apply Lemma 10 to obtain

$$\phi_\beta((\omega - v) + v) = \phi_\beta(\omega - v)\phi_\beta(v).$$

Plugging this into the left hand side of (22), we obtain

$$\frac{\sum_{\omega \in \mathcal{E}_{P,v}} \phi_\beta((\omega - v) + v)}{\sum_{\omega \in \mathcal{E}_{P,v}} \phi_\beta(\omega - v)} = \frac{\sum_{\omega \in \mathcal{E}_{P,v}} \phi_\beta(\omega - v)\phi_\beta(v)}{\sum_{\omega \in \mathcal{E}_{P,v}} \phi_\beta(\omega - v)} = \phi_\beta(v).$$

Combining the previous equations, we obtain (20) as desired. □

2.4. Notation and standing assumptions. Throughout the remainder of this paper, we assume that a finite Abelian structure group G , and a faithful, irreducible and unitary representation of ρ has been fixed.

3. The Probability of Sets of Plaquettes Being Frustrated

In this section we will state and prove three propositions which will be useful in later sections. We mention that although these are technical, they might be useful outside the scope of this paper. The first of these results is the following proposition, which extends Proposition 4.9 in [7].

Proposition 12. *Let B be a box in \mathbb{Z}^4 , let $\beta \geq 0$ be such that $5\alpha(\beta) < 1$, and let $P \subseteq P_B$ be non-empty and symmetric. For $M \geq |P^+|$, let $\Pi_{P,M}^{\geq} := \Pi_{\tilde{B},P,M}^{\geq}$ be the set of all $\omega \in \Sigma_{P_B}$ such that there is $v \in \Sigma_{P_B}$ with*

- (1) $P \subseteq \text{supp } v$,
- (2) v is P -irreducible,
- (3) $|\text{supp } v| \geq 2M$, and
- (4) $\omega|_{\text{supp } v} = v$.

Then

$$\mu_{\beta,N}(\Pi_{P,M}^{\geq}) \leq \frac{5^{M-|P^+|}\alpha(\beta)^M}{1 - 5\alpha(\beta)}.$$

Before we give a proof of Proposition 12, we state and prove a few lemmas which will be used in the proof. The first of these lemmas is Lemma 13 below, which essentially is identical to Lemma 4.7 in [7].

Lemma 13. *Let B be a box in \mathbb{Z}^4 , let $\beta \geq 0$, and let $v \in \Sigma_{P_B}$. Then*

$$\mu_{B,\beta}(\{\omega \in \Sigma_{P_B} : \omega|_{\text{supp } v} = v\}) \leq \phi_{\beta}(v). \tag{23}$$

Proof. Since

$$\begin{aligned} &\mu_{B,\beta}(\{\omega \in \Sigma_{P_B} : \omega|_{\text{supp } v} = v\}) \\ &= \frac{\mu_{B,\beta}(\{\omega \in \Sigma_{P_B} : \omega|_{\text{supp } v} = v\})}{\mu_{B,\beta}(\{\omega \in \Sigma_{P_B} : \omega|_{\text{supp } v} = 0\})} \cdot \mu_{B,\beta}(\{\omega \in \Sigma_{P_B} : \omega|_{\text{supp } v} = 0\}), \end{aligned}$$

and $\mu_{B,\beta}$ is a probability measure, we have

$$\mu_{B,\beta}(\{\omega \in \Sigma_{P_B} : \omega|_{\text{supp } v} = v\}) \leq \frac{\mu_{B,\beta}(\{\omega \in \Sigma_{P_B} : \omega|_{\text{supp } v} = v\})}{\mu_{B,\beta}(\{\omega \in \Sigma_{P_B} : \omega|_{\text{supp } v} = 0\})}.$$

Using Lemma 11, we obtain (13) as desired. □

The next result we will need in the proof of Proposition 12 is the following lemma, which gives an upper bound on the sum of the activity of the plaquette configurations v which satisfies (1), (2), and (3) of Proposition 12.

Lemma 14. *Let B be a box in \mathbb{Z}^4 , and let $\beta \geq 0$. Further, let $P \subseteq P_B$ be non-empty and symmetric, let $m \geq |P^+|$, and let $\Pi_{P,m}$ be the set of all plaquette configurations $v \in \Sigma_{P_B}$ such that*

- (1) $P \subseteq \text{supp } v$,
- (2) v is P -irreducible
- (3) $|\text{supp } v| = 2m$.

Then

$$\sum_{\nu \in \Pi_{P,m}} \phi_\beta(\nu) \leq 5^{m-|P^+|} \alpha(\beta)^m.$$

Remark 10. Lemma 14 is very similar to Lemma 4.9 in [7], and is also similar to Lemma 3.11 in [14] and Lemma 1.5 in [2]. We remark however that our proof strategy yields a strictly better upper bound than the corresponding proofs in [2] and [14]. This would yield a strictly larger lower bound on β in the main results of this paper, and hence we give an alternative proof here.

Proof of Lemma 14. Let $\ell := |P_+|$. We will prove that Lemma 14 holds by giving an injective map from the set $\Pi_{P,m}$ to a set of sequences $\nu^{(\ell)}, \nu^{(\ell+1)}, \dots, \nu^{(m)}$ of G -valued 2-forms on P_B , and then use this map to obtain the desired upper bound. To this end, assume that a total ordering of the plaquettes in P_B and a total ordering of the 3-cells in B are given. If $\Pi_{P,m}$ is empty, then the desired conclusion trivially holds, and hence we can assume that this is not the case.

Fix some $\nu \in \Pi_{P,m}$.

Let $\{p_1, p_2, \dots, p_\ell\} := P^+$, and define

$$\nu^{(\ell)} := \nu|_P.$$

Now assume that for some $k \in \{\ell, \ell + 1, \dots, m\}$, we are given 2-forms $\nu^{(\ell)}, \nu^{(\ell+1)}, \dots, \nu^{(k)}$ such that

- (i) for each $j \in \{\ell + 1, \ell + 2, \dots, k\}$, we have $\text{supp } \nu^{(j)} = \text{supp } \nu^{(j-1)} \sqcup \{p_j, -p_j\}$ for some $p_j \in P_B$, and
- (ii) for each $j \in \{\ell, \ell + 1, \dots, k\}$ we have $\nu|_{\text{supp } \nu^{(j)}} = \nu^{(j)}$.

Consider first the case that $d\nu^{(k)} = 0$. Since, by (ii), we have $\nu|_{\text{supp } \nu^{(k)}} = \nu^{(k)}$, it follows from (2) that we must have $\nu^{(k)} = \nu$. On the other hand, by (i), we have $|\text{supp } \nu^{(k)}| = 2k$, and hence using (3) we obtain $k = m$. Consequently, if $k < m$, then $d\nu^{(k)} \neq 0$. Equivalently, in this case there is at least one oriented 3-cell c in B for which $(d\nu^{(k)})_c \neq 0$. Let c_{k+1} be the first oriented 3-cell (with respect to the ordering of the 3-cells) for which $(d\nu^{(k)})_{c_{k+1}} \neq 0$. Since $\nu \in \Sigma_{P_B}$, we have $(d\nu)_{c_{k+1}} = 0$, and consequently there must be at least one plaquette $p \in \text{supp } \nu \cap (\partial c_{k+1} \setminus \text{supp } \nu^{(k)})$. Let p_{k+1} be the first such plaquette (with respect to the ordering of the plaquettes).

Define

$$\nu_p^{(k+1)} := \begin{cases} \nu_p & \text{if } p = \pm p_{k+1} \\ \nu_p^{(k)} & \text{otherwise.} \end{cases}$$

Note that if $\nu^{(\ell)}, \nu^{(\ell+1)}, \dots, \nu^{(k)}$ satisfies (i) and (ii), then so does $\nu^{(k+1)}$. Using induction, we obtain a sequence $\nu^{(\ell)}, \nu^{(\ell+1)}, \dots, \nu^{(m)}$ of 2-forms with $\text{supp } \nu^{(\ell)} = P$ which satisfies (i) and (ii). We now show that such a sequence must satisfy $\nu^{(m)} = \nu$. To this end, note that by (i), $|\text{supp } \nu^{(m)}| = 2m$ and by (ii), $\nu|_{\text{supp } \nu^{(m)}} = \nu^{(m)}$. Since $|\text{supp } \nu| = 2m$, it follows that $\nu^{(m)} = \nu$.

We now use the above construction of sequences $(\nu^{(\ell)}, \nu^{(\ell+1)}, \dots, \nu^{(m)})$, each corresponding to some $\nu \in \Pi_{P,m}$, to get an upper bound on $\sum_{\nu \in \Pi_{P,m}} \phi_\beta(\nu)$. To obtain such an upper bound, note first that for each $k \in \{\ell, \ell + 1, \dots, m - 1\}$, given $\nu^{(k)}$, the 3-cell c_{k+1} is uniquely determined. Next, given $\nu^{(k)}$ and c_{k+1} , there are at most five

possible choices for p_{k+1} . Finally, note that for each $k \in \{1, 2, \dots, m\}$, we must choose $v_{p_k} \in G \setminus \{0\}$. Since for $v \in \Pi_{P,m}$ the mapping $v \mapsto (v^{(\ell)}, v^{(\ell+1)}, \dots, v^{(m)})$ is injective, we obtain

$$\sum_{v \in \Pi_{P,m}} \phi_\beta(v) = \sum_{v \in \Pi_{P,m}} \prod_{p \in \text{supp } v} \phi_\beta(v_p) \leq 5^{m-|P^+|} \left[\sum_{g \in G \setminus \{0\}} \phi(g)^2 \right]^m.$$

Recalling the definition of $\alpha(\beta)$, the desired conclusion now follows. □

Proof of Proposition 12. For each $m \geq |P^+|$ let $\Pi_{P,m}$ be defined as in Lemma 14. Then, by definition,

$$\Pi_{P,M}^{\geq} = \bigcup_{m \geq M} \Pi_{P,m}.$$

Consequently, by a union bound, we have

$$\mu_{B,\beta}(\Pi_{P,M}^{\geq}) \leq \sum_{m=M}^{\infty} \mu_{B,\beta}(\Pi_{P,m}) \leq \sum_{m=M}^{\infty} \sum_{v \in \Pi_{P,m}} \mu_{B,\beta}(\{\omega \in \Sigma_{P_B} : \omega|_{\text{supp } v} = v\}).$$

By Lemma 13, for any $m \geq M$ and any $v \in \Pi_{P,m} \subseteq \Sigma_{P_B}$, we have

$$\mu_{B,\beta}(\{\omega \in \Sigma_{P_B} : \omega|_{\text{supp } v} = v\}) \leq \phi_\beta(v).$$

Applying Lemma 14, we thus obtain

$$\sum_{m=M}^{\infty} \sum_{v \in \Pi_{P,m}} \mu_{B,\beta}(\{\omega \in \Sigma_{P_B} : \omega|_{\text{supp } v} = v\}) \leq \sum_{m=M}^{\infty} 5^{m-|P^+|} \alpha(\beta)^m.$$

The right-hand side in the previous equation is a geometric sum, which converges exactly if $5\alpha(\beta) < 1$.

In this case, by combining the previous equations, we obtain

$$\mu_{B,\beta}(\Pi_{P,M}^{\geq}) \leq \frac{5^{M-|P^+|} \alpha(\beta)^M}{1 - 5\alpha(\beta)}$$

as desired. □

The next result is a small variation of Proposition 12 which turns out to be useful when we work with pairs of plaquette configurations.

Proposition 15. *Let B and B' be two boxes in \mathbb{Z}^4 with $B' \subseteq B$, let $\beta \geq 0$ be such that $5\alpha(\beta) < 1$, and let $P_0 \subseteq P_{B'}$ be non-empty and symmetric. For $M \geq |P_0^+|$, let $\bar{\Pi}_{P_0,M}^{\geq}$ be the set of all pairs $(\omega, \omega') \in \Sigma_{P_B} \times \Sigma_{P_{B'}}$ such that there is $v \in \Sigma_{P_B}$ and $v' \in \Sigma_{P_{B'}}$ with*

- (1) $\omega|_{\text{supp } v} = v$ and $\omega'|_{\text{supp } v'} = v'$,
- (2) $P_0 \subseteq \text{supp } v \cup \text{supp } v'$,
- (3) for all symmetric sets $P \subseteq \text{supp } v$ and $P' \subseteq \text{supp } v'$ which satisfies satisfies $P \sqcup P' = P_0$, v is P -irreducible and v' is P' -irreducible.
- (4) $|\text{supp } v| + |\text{supp } v'| \geq 2M$.

Then

$$\mu_{B,\beta} \times \mu_{B',\beta}(\bar{\Pi}_{P_0,M}^{\geq}) \leq \frac{2^{|P_0^+|}(M - |P_0^+|)5^{M-|P_0^+|}\alpha(\beta)^M}{(1 - 5\alpha(\beta))^2}. \tag{24}$$

Proof. For non-empty and symmetric sets $P \subseteq P_B$, and $m \geq |P^+|$, recall the definition of $\Pi_{B,P,m}^{\geq}$ from Proposition 12.

For $m_0 \geq 0$, define

$$\Pi_{B,\emptyset,m_0}^{\geq} := \begin{cases} \Sigma_{P_B} & \text{if } m_0 = 0, \\ \emptyset & \text{else.} \end{cases}$$

Analogously, for non-empty and symmetric sets $P \subseteq P_{B'}$, and $m \geq |P^+|$, recall the definition of $\Pi_{B',P,m}^{\geq}$ from Proposition 12, and for $m_0 \geq 0$, define

$$\Pi_{B',\emptyset,m_0}^{\geq} := \begin{cases} \Sigma_{P_{B'}} & \text{if } m_0 = 0, \\ \emptyset & \text{else.} \end{cases}$$

Then, by definition, we have

$$\bar{\Pi}_{P_0,M}^{\geq} \subseteq \bigcup_{\substack{P \subseteq P_0: \\ P = -P}} \bigcup_{m_0 = |P^+|}^{M - |P^+ \setminus P^+|} \Pi_{B,P,m_0}^{\geq} \times \Pi_{B',P_0 \setminus P, M - m_0}^{\geq}.$$

Consequently, by a union bound, we have

$$\mu_{B,\beta} \times \mu_{B',\beta}(\bar{\Pi}_{P_0,M}^{\geq}) \leq \sum_{\substack{P \subseteq P_0: \\ P = -P}} \sum_{m_0 = |P^+|}^{M - |P_0^+ \setminus P^+|} \mu_{B,\beta}(\Pi_{B,P,m_0}^{\geq}) \mu_{B',\beta}(\Pi_{B',P_0 \setminus P, M - m_0}^{\geq}). \tag{25}$$

If $P = \emptyset$, then

$$\begin{aligned} & \sum_{m_0 = |P^+|}^{M - |P_0^+ \setminus P^+|} \mu_{B,\beta}(\Pi_{B,P,m_0}^{\geq}) \mu_{B',\beta}(\Pi_{B',P_0 \setminus P, M - m_0}^{\geq}) = \mu_{B,\beta}(\Pi_{B,\emptyset,0}^{\geq}) \mu_{B',\beta}(\Pi_{B',P_0,M}^{\geq}) \\ & = \mu_{B',\beta}(\Pi_{B',P_0,M}^{\geq}). \end{aligned}$$

Similarly, if $P = P_0$, then

$$\begin{aligned} & \sum_{m_0 = |P^+|}^{M - |P_0^+ \setminus P^+|} \mu_{B,\beta}(\Pi_{B,P,m_0}^{\geq}) \mu_{B',\beta}(\Pi_{B',P_0 \setminus P, M - m_0}^{\geq}) = \mu_{B,\beta}(\Pi_{B,P_0,M}^{\geq}) \mu_{B',\beta}(\Pi_{B',\emptyset,0}^{\geq}) \\ & = \mu_{B,\beta}(\Pi_{B,P_0,M}^{\geq}). \end{aligned}$$

Consequently, we have

$$\begin{aligned}
 & \sum_{\substack{P \subseteq P_0 \\ P = -P}} \sum_{m_0 = |P^+|}^{M - |P_0^+ \setminus P^+|} \mu_{B,\beta}(\Pi_{B,P,m_0}^{\geq}) \mu_{B',\beta}(\Pi_{B',P_0 \setminus P, M - m_0}^{\geq}) \\
 &= \mu_{B,\beta}(\Pi_{B,P_0,M}^{\geq}) + \mu_{B',\beta}(\Pi_{B',P_0,M}^{\geq}) \\
 &+ \sum_{\substack{\emptyset \subsetneq P \subsetneq P_0 \\ P = -P}} \sum_{m_0 = |P^+|}^{M - |P_0^+ \setminus P^+|} \mu_{B,\beta}(\Pi_{B,P,m_0}^{\geq}) \mu_{B',\beta}(\Pi_{B',P_0 \setminus P, M - m_0}^{\geq}).
 \end{aligned} \tag{26}$$

Since $5\alpha(\beta) < 1$, we can now apply Proposition 12 to obtain

$$\begin{aligned}
 & \mu_{B,\beta}(\Pi_{B,P_0,M}^{\geq}) + \mu_{B',\beta}(\Pi_{B',P_0,M}^{\geq}) \\
 &+ \sum_{\substack{\emptyset \subsetneq P \subsetneq P_0 \\ P = -P}} \sum_{m_0 = |P^+|}^{M - |P_0^+ \setminus P^+|} \mu_{B,\beta}(\Pi_{B,P,m_0}^{\geq}) \mu_{B',\beta}(\Pi_{B',P_0 \setminus P, M - m_0}^{\geq}) \\
 &\leq 2 \frac{5^{M - |P_0^+|} \alpha(\beta)^M}{1 - 5\alpha(\beta)} \\
 &+ \sum_{\substack{\emptyset \subsetneq P \subsetneq P_0 \\ P = -P}} \sum_{m_0 = |P^+|}^{M - |P_0^+ \setminus P^+|} \frac{5^{m_0 - |P^+|} \alpha(\beta)^{m_0}}{1 - 5\alpha(\beta)} \frac{5^{(M - m_0) - |P_0^+ \setminus P^+|} \alpha(\beta)^{M - m_0}}{1 - 5\alpha(\beta)} \\
 &= 2 \frac{5^{M - |P^+|} \alpha(\beta)^M}{1 - 5\alpha(\beta)} + \sum_{\substack{\emptyset \subsetneq P \subsetneq P_0 \\ P = -P}} \sum_{m_0 = |P^+|}^{M - |P_0^+ \setminus P^+|} \frac{5^{M - |P_0^+|} \alpha(\beta)^M}{(1 - 5\alpha(\beta))^2} \\
 &= 2 \frac{5^{M - |P_0^+|} \alpha(\beta)^M}{1 - 5\alpha(\beta)} + (2^{|P_0^+|} - 2)(M - |P_0^+| + 1) \frac{5^{M - |P_0^+|} \alpha(\beta)^M}{(1 - 5\alpha(\beta))^2} \\
 &\leq 2^{|P_0^+|} (M - |P_0^+| + 1) \frac{5^{M - |P_0^+|} \alpha(\beta)^M}{(1 - 5\alpha(\beta))^2}.
 \end{aligned} \tag{27}$$

Combining (25), (26) and (27), we finally obtain (24) as desired. □

The last result of this section is the following proposition, which provides a matching lower bound for the inequality in Lemma 13.

Proposition 16. *Let B be a box in \mathbb{Z}^4 , let $\beta > 0$ be such that $5\alpha(\beta) < 1$, and let $v \in \Sigma_{P_B}$ be such that $\text{dist}_B(\text{supp } v, \delta B) \geq 7$.*

Then

$$\left(1 - \frac{5^5 \alpha(\beta)^6 |\text{supp } v|}{2(1 - 5\alpha(\beta))}\right) \phi_\beta(v) \leq \mu_{B,\beta}(\{\omega \in \Sigma_{P_B} : \omega|_{\text{supp } v} = v\}) \leq \phi_\beta(v). \tag{28}$$

Remark 11. For any given $v \in \Sigma_{P_B}$, the ratio between the upper and lower bound in (28) can be made arbitrarily close to one by taking β sufficiently large.

Proof. Since the upper bound in (28) is a direct consequence of Lemma 13, we only need to show that the lower bound in (28) holds. To this end, note first that

$$\begin{aligned} &\mu_{B,\beta}(\{\omega \in \Sigma_{P_B} : \omega|_{\text{supp } v} = v\}) \\ &= \frac{\mu_{B,\beta}(\{\omega \in \Sigma_{P_B} : \omega|_{\text{supp } v} = v\})}{\mu_{B,\beta}(\{\omega \in \Sigma_{P_B} : \omega|_{\text{supp } v} = 0\})} \cdot \mu_{B,\beta}(\{\omega \in \Sigma_{P_B} : \omega|_{\text{supp } v} = 0\}), \end{aligned} \tag{29}$$

Assume that $p \in P_B$ and that $\omega' \in \Sigma_{P_B}$ is such that $\omega'|_p \neq 0$. By Lemma 7 there exists an irreducible and non-trivial plaquette configuration $v' \in \Sigma_{P_B}$ with $p \in \text{supp } v'$ and $\omega'|_{\text{supp } v'} = v'$. By Lemma 8, any non-trivial plaquette configuration $v' \in \Sigma_{P_B}$ must satisfy $|\text{supp } v'|^+ \geq 6$. Consequently, by Proposition 12, applied with $P := \{p, -p\}$ and $M = 6$, we have

$$\mu_{B,\beta}(\{\omega \in \Sigma_{P_B} : \omega|_p \neq 0\}) \leq \frac{5^{6-1}\alpha(\beta)^6}{1 - 5\alpha(\beta)}.$$

Next note that for $\omega \in \Sigma_{P_B}$, the event $\omega|_{\text{supp } v} \neq 0$ is equivalent to that $\text{supp } \omega \cap (\text{supp } v)^+ \neq \emptyset$.

Consequently, by a union bound, we obtain

$$\mu_{B,\beta}(\{\omega \in \Sigma_{P_N} : \omega|_{\text{supp } v} = 0\}) \geq 1 - \sum_{p \in (\text{supp } v)^+} \frac{5^{6-1}\alpha(\beta)^6}{1 - 5\alpha(\beta)}. \tag{30}$$

On the other hand, by Lemma 11, we have

$$\frac{\mu_{B,\beta}(\{\omega \in \Sigma_{P_B} : \omega|_{\text{supp } v} = v\})}{\mu_{B,\beta}(\{\omega \in \Sigma_{P_B} : \omega|_{\text{supp } v} = 0\})} = \phi_\beta(v). \tag{31}$$

Combining (29), (30) and (31), we obtain the desired lower bound. □

4. Paths, Optimal Paths and Geodesics

The following definition, which we recall from the introduction, will be central in the rest of this paper.

Definition 1. Given two boxes B and B' in \mathbb{Z}^4 , $\omega \in \Sigma_{P_B}$ and $\omega' \in \Sigma_{P_{B'}}$, let $\mathcal{G}(\omega, \omega')$ be the graph with vertex set $\text{supp } \omega \cup \text{supp } \omega'$ and an edge between two distinct plaquettes $p_1, p_2 \in \text{supp } \omega \cup \text{supp } \omega'$ if either $\hat{\partial} p_1 \cap \hat{\partial} p_2 \neq \emptyset$ or $\hat{\partial} p_1 \cap \hat{\partial}(-p_2) \neq \emptyset$.

If $p_1, p_2 \in \text{supp } \omega \cup \text{supp } \omega'$ are neighbors in $\mathcal{G}(\omega, \omega')$, we write $p_1 \sim p_2$.

The main reason for introducing the graph $\mathcal{G}(\omega, \omega')$ is to be able to talk about paths and geodesics in this graph. To be able to talk about general properties of such paths, we introduce the following notation.

Definition 2. Let B be a box in \mathbb{Z}^4 , and let the plaquettes $p_1, p_2, \dots, p_m \in P_B$ be distinct. If for each $k \in \{1, 2, \dots, m - 1\}$ we have $\hat{\partial} p_k \cap \pm \hat{\partial} p_{k+1} \neq \emptyset$, then $\mathcal{P} := (p_1, p_2, \dots, p_m)$ is said to be a *path* (from p_1 to p_m) in P_B .

When $\mathcal{P} = (p_1, p_2, \dots, p_m)$ is a path, we let $\text{Image}(\mathcal{P})$ denote the set $\{p_1, p_2, \dots, p_m\}$, and let $|\mathcal{P}| := |\text{Image}(\mathcal{P})| = m$.

Note that if B and B' are two boxes in \mathbb{Z}^4 with $B' \subseteq B$ and \mathcal{P} is a path in $P_{B'}$, then \mathcal{P} is a path in P_B . Also, if $\omega \in \Sigma_{P_B}$, $\omega' \in \Sigma_{P_{B'}}$, and $\overline{\mathcal{P}}$ is a path in $\mathcal{G}(\omega, \omega')$, then \mathcal{P} is a path in P_B . Conversely, if \mathcal{P} is a path in P_B , $g \in G \setminus \{0\}$, and we define $\omega \in \Sigma_{P_B}$ and $\omega' \in \Sigma_{P_{B'}}$ by letting $\omega' \equiv 0$ and $\omega_p = g$ for all $p \in P_B^+$, then \mathcal{P} is a path in $\mathcal{G}(\omega, \omega')$. Note that in this example, ω has full support. The next lemma shows that in some cases, there is $\omega \in \Sigma_{P_B}$ and $\omega' \in \Sigma_{P_{B'}}$ with relatively small supports such that a given path \mathcal{P} is a path in $\mathcal{G}(\omega, \omega')$.

Lemma 17. *Let B be a box in \mathbb{Z}^4 , and let $\mathcal{P} = (p_1, p_2, \dots, p_m)$ be a shortest path from p_1 to p_m in P_B . Then there is $\omega^{(0)}, \omega^{(1)} \in \Sigma_{P_B}$ such that*

- (1) \mathcal{P} is a path in $\mathcal{G}(\omega^{(0)}, \omega^{(1)})$, and
- (2) $|\text{supp } \omega^{(0)}| + |\text{supp } \omega^{(1)}| \leq 12|\mathcal{P}|$.

Proof. We first choose a sequence of oriented edges as follows.

- (i) Let $k_1 := 1$. If $m \geq 2$ and $\partial p_1 \cap \pm \partial p_2 \neq \emptyset$, pick $e_1 \in \partial p_1 \cap \pm \partial p_2$. Otherwise, pick any $e_1 \in \partial p_1$.

Now assume that e_1, e_2, \dots, e_{j-1} are given for some $j \geq 2$. If there is $k \in \{1, \dots, m\}$ such that $\partial p_k \cap \pm \{e_1, e_2, \dots, e_{j-1}\} = \emptyset$, we choose e_j as follows.

- (ii) Let $k_j \geq 1$ be the smallest integer such that $\partial p_{k_j} \cap \pm \{e_1, e_2, \dots, e_{j-1}\} = \emptyset$. If $k_j + 1 \leq m$ and $\partial p_{k_j} \cap \pm \partial p_{k_j+1} \neq \emptyset$, pick $e_j \in \partial p_{k_j} \cap \pm \partial p_{k_j+1}$. Otherwise, pick any $e_j \in \partial p_{k_j}$.

Let ℓ be the smallest positive integer which is such that $\partial p_k \cap \pm \{e_1, e_2, \dots, e_\ell\} \neq \emptyset$ for all $k \in \{1, 2, \dots, m\}$.

Fix any $g \in G \setminus \{0\}$ and define $\sigma^{(0)}, \sigma^{(1)} \in \Sigma_{E_N}$ by

$$\sigma_e^{(0)} = \sum_{\substack{j \in \{1, 2, \dots, \ell\}: \\ j \text{ is even}}} (g \mathbb{1}_{e=e_j} - g \mathbb{1}_{e=-e_j}), \quad e \in E_B$$

and

$$\sigma_e^{(1)} = \sum_{\substack{j \in \{1, 2, \dots, \ell\}: \\ j \text{ is odd}}} (g \mathbb{1}_{e=e_j} - g \mathbb{1}_{e=-e_j}), \quad e \in E_B.$$

We will show that $\omega^{(0)} := d\sigma^{(0)}$ and $\omega^{(1)} := d\sigma^{(1)}$ have the desired properties.

Since \mathcal{P} is a path, (1) is equivalent to that

$$\{p_1, p_2, \dots, p_m\} \subseteq \text{supp } \omega^{(0)} \cup \text{supp } \omega^{(1)}. \tag{32}$$

To see that this holds, fix some $k \in \{1, 2, \dots, m\}$, and define

$$J_k := \{j \in \{1, 2, \dots, \ell\} : e_j \in \pm \partial p_k\}.$$

By the choice of the edges e_1, e_2, \dots, e_ℓ , the set J_k is non-empty.

Now let $j, j' \in J_k$ be such that $j \leq j'$.

By the choice of e_j , we have $e_j \in \partial p_{k_j}$, and since $j \in J_k$, we also have $e_j \in \pm \partial p_k$. Consequently, $e_j \in \partial p_{k_j} \cap \pm \partial p_k$, and thus $p_{k_j} \sim p_k$. By the same argument, we also

have $p_{k_{j'}} \sim p_k$. Consequently, $(p_{k_j}, p_k, p_{k_{j'}})$ is a path in $\mathcal{G}(\omega^{(0)}, \omega^{(1)})$. Since \mathcal{P} is a shortest path, \mathcal{P} contains $p_{k_j} p_k$, and $p_{k_{j'}}$ and $k_j \leq k_{j'}$ (since $j \leq j'$), we must have $k_j \in \{k - 1, k\}$ and $k_{j'} \in \{k, k + 1\}$.

Since $e_j \in \pm \partial p_k$, we cannot have $k_{j+1} = k$, and since $k_j \in \{k - 1, k\}$, we thus have $k_{j+1} \geq k + 1$. Consequently, if $j' \neq j$, we must have $j' = j + 1$.

In other words, we have either $J_k = \{j\}$ for some $j \in \{1, 2, \dots, \ell\}$, or $J_k = \{j, j + 1\}$ for some $j \in \{1, 2, \dots, \ell - 1\}$.

This implies in particular that

$$p_k \in \begin{cases} \text{supp } \omega^{(0)} & \text{if } j \text{ is even} \\ \text{supp } \omega^{(1)} & \text{if } j \text{ is odd,} \end{cases}$$

and thus (1) holds.

To see that (2) holds, note simply that

$$|\text{supp } \omega^{(0)}| + |\text{supp } \omega^{(1)}| \leq 6(|\text{supp } \sigma^{(0)}| + |\text{supp } \sigma^{(1)}|) = 12\ell \leq 12|\mathcal{P}|.$$

This concludes the proof. □

Lemma 18. *Let B and B' be two boxes in \mathbb{Z}^4 , let $P_1, P_2 \subseteq P_B \cup P_{B'}$ be disjoint, and assume that $\omega \in \Sigma_{P_B}$ and $\omega' \in \Sigma_{P_{B'}}$ are such that that $P_1 \leftrightarrow P_2$ in $\mathcal{G}(\omega, \omega')$. Further, let $\mathcal{P} := (p_1, p_2, \dots, p_\ell)$ be a geodesic in $\mathcal{G}(\omega, \omega')$ between $p_1 \in P_1$ and $p_\ell \in P_2$. Then*

$$|\mathcal{P}| \leq (|\text{supp } \omega| + |\text{supp } \omega'|)/2.$$

Proof. Since \mathcal{P} is a geodesic in \mathcal{G} , we must have

$$\text{Image}(\mathcal{P}) \cap (-\text{Image}(\mathcal{P})) = \emptyset,$$

and

$$\pm \text{Image}(\mathcal{P}) \subseteq \text{supp } \omega \cup \text{supp } \omega'.$$

Consequently, we have

$$2|\mathcal{P}| \leq |\text{supp } \omega \cup \text{supp } \omega'| \leq |\text{supp } \omega| + |\text{supp } \omega'|. \tag{33}$$

From this the desired conclusion immediately follows. □

Remark 12. If we either assume that $\partial \hat{\partial} p \cap \delta P_B = \emptyset$ for all $p \in \text{Image } P$ or consider lattice gauge theory with zero boundary conditions, then one can quite easily show that (33) can be replaced with

$$10|\mathcal{P}|/3 \leq |\text{supp } \omega| + |\text{supp } \omega'|.$$

If we have zero boundary conditions, then this stronger inequality can be used to replace the constant C_2 defined in (8) with the smaller constant $3^{3/5} \cdot 10$.

Definition 3. A path $\mathcal{P} = (p_1, p_2, \dots, p_m)$ is said to be *optimal* if

- (i) p_1, p_2, \dots, p_{m-1} all have the same orientation, and
- (ii) for any $i, j \in \{1, 2, \dots, m\}$, if $\hat{\partial} p_i \cap (\hat{\partial} p_j \cup \hat{\partial}(-p_j)) \neq \emptyset$, then $|i - j| \leq 1$.

The next lemma gives an upper bound on the number of optimal paths of a given length.

Lemma 19. *Let B be a box in \mathbb{Z}^4 , let $p_1 \in P_B$, and let $m \geq 2$. Then there are at most $40 \cdot 15^{m-2}$ optimal paths $\mathcal{P} = (p_1, p_2, \dots, p_m)$ with $p_m \neq \pm p_1$.*

Proof. Note first that for any $p \in P_B$, we have $|\hat{\partial}p| = 4$, and if $c \in \hat{\partial}p$, then $|(\partial c) \setminus \{p\}| = 5$.

Since \mathcal{P} is a path, we have

$$p_2 \in \pm(\partial\hat{\partial}p_1 \setminus \{p_1\}).$$

Since

$$\left| \pm(\partial\hat{\partial}p_1 \setminus \{p_1\}) \right| = 2 \cdot 4 \cdot 5 = 40,$$

the desired conclusion holds in the case $m = 2$.

On the other hand, for each plaquette $p \in \pm(\partial\hat{\partial}p_1 \setminus \{p_1\})$, we also have $-p \in \pm(\partial\hat{\partial}p_1 \setminus \{p_1\})$, and exactly one of p and $-p$ has the same orientation as p_1 . Since \mathcal{P} is optimal, if $m \geq 3$ then the plaquettes p_1 and p_2 must have the same orientation, and hence in this case, given p_1 there are at most $40/2 = 20$ possible choices of p_2 .

Next, assume that $m \geq 4$ and that $j \in \{3, \dots, m - 1\}$. Since \mathcal{P} is a path, we have

$$p_j \in \pm(\partial\hat{\partial}p_{j-1} \setminus \{p_{j-1}\}),$$

and since \mathcal{P} is optimal, we have

$$p_j \notin \pm\partial\hat{\partial}p_{j-2}.$$

Since \mathcal{P} is a path, we have $|\hat{\partial}p_{j-1} \cap \pm\hat{\partial}p_{j-2}| \geq 1$. Since $j < m$, the plaquettes p_j and p_{j-1} must have the same orientation, and hence, given p_1, p_2, \dots, p_{j-1} , there are at most $2 \cdot (4 - 1) \cdot 5 \cdot \frac{1}{2} = 15$ possible choices of p_j . By the same argument, since the last plaquette can have any orientation, it follows that given p_1, p_2, \dots, p_{m-1} , there can be at most $2 \cdot (4 - 1) \cdot 5 = 30$ possible choices of p_m .

To sum up, we have showed that there are exactly 40 optimal paths $\mathcal{P} = (p_1, p_2)$ with $p_2 \neq \pm p_1$, and if $m \geq 3$, there are at most $20 \cdot 15^{m-3} \cdot 30$ optimal paths $\mathcal{P} = (p_1, p_2, \dots, p_m)$ with $p_m \neq \pm p_1$. This concludes the proof. \square

Lemma 20. *Let B and B' be two boxes in \mathbb{Z}^4 , and let $p_1, p_2 \in P_B$ be disjoint. Further, let $\omega \in \Sigma_{P_B}$ and $\omega' \in \Sigma_{P_{B'}}$, and assume that $\{p_1\} \leftrightarrow \{p_2\}$ in $\mathcal{G}(\omega, \omega')$. Then there is an optimal geodesic \mathcal{P} from p_1 to p_2 in $\mathcal{G}(\omega, \omega')$.*

Proof. Since $\{p_1\} \leftrightarrow \{p_2\}$ in $\mathcal{G}(\omega, \omega')$, there exist at least one geodesic from p_1 to p_2 in $\mathcal{G}(\omega, \omega')$. Let $(p^{(1)}, p^{(2)}, \dots, p^{(m)})$ be such a geodesic.

Then, for any $\tau_2, \dots, \tau_{m-1} \in \{-1, 1\}$, the path

$$\mathcal{P} := (p^{(1)}, \tau_2 p^{(2)}, \dots, \tau_{m-1} p^{(m-1)}, p^{(m)})$$

is also a geodesic from p_1 to p_2 in $\mathcal{G}(\omega, \omega')$. In particular, we can choose $\tau_2, \dots, \tau_{m-1} \in \{-1, 1\}$ such that \mathcal{P} is an optimal geodesic from p_1 to p_2 in $\mathcal{G}(\omega, \omega')$. This concludes the proof. \square

5. Connected Sets of Plaquettes

Definition 4. Let B and B' be two boxes in \mathbb{Z}^4 , and let $\omega \in \Sigma_{P_B}$ and $\omega' \in \Sigma_{P_{B'}}$.

Let $P_1, P_2 \subseteq P_B \cup P_{B'}$ be two disjoint sets. If there is $p_1 \in P_1$ and $p_2 \in P_2$ and a path from p_1 to p_2 in $\mathcal{G}(\omega, \omega')$, then we say that P_1 and P_2 are *connected* in $\mathcal{G}(\omega, \omega')$ and write $P_1 \leftrightarrow P_2$ (in $\mathcal{G}(\omega, \omega')$). Otherwise, we say that P_1 and P_2 are *disconnected*, and write $P_1 \nleftrightarrow P_2$ (in $\mathcal{G}(\omega, \omega')$).

Note that with this definition, if $p_1, p_2 \in P_B \cup P_{B'}$, $\omega \in \Sigma_{P_B}$ and $\omega' \in \Sigma_{P_{B'}}$, then there is a path from p_1 to p_2 in $\mathcal{G}(\omega, \omega')$ if and only if $\{p_1\} \leftrightarrow \{p_2\}$ in $\mathcal{G}(\omega, \omega')$. Note also that if $P_1, P_2 \subseteq P_B \cup P_{B'}$, $\omega \in \Sigma_{P_B}$ and $\omega' \in \Sigma_{P_{B'}}$ then, by definition, $P_1 \leftrightarrow P_2$ in $\mathcal{G}(\omega, \omega')$ if and only if $P_2 \leftrightarrow P_1$ in $\mathcal{G}(\omega, \omega')$. Moreover, $P_1 \leftrightarrow P_2$ in $\mathcal{G}(\omega, \omega')$ if and only if $\delta P_1 \leftrightarrow \delta P_2$ in $\mathcal{G}(\omega, \omega')$.

Re remark that if $\omega \in \Sigma_{P_B}$, then the connected components of $\mathcal{G}(\omega, 0)$ are called *vertices* in [4,6], and correspond to the support of *polymers* in [2].

Our main reason for introducing a notion of sets of plaquettes being connected is the following lemma.

Lemma 21. Let B be a box in \mathbb{Z}^4 , and let $\beta \geq 0$. Further, let $P_1, P_2 \subseteq P_B$ be symmetric and disjoint, and let $f : \Sigma_{P_B} \rightarrow \mathbb{C}$ and $g : \Sigma_{P_B} \rightarrow \mathbb{C}$ be such that $f(\omega) = f(\omega|_{P_1})$ and $g(\omega) = g(\omega|_{P_2})$ for all $\omega \in \Sigma_{P_B}$. Finally, let $\omega^{(0)}, \omega^{(1)} \sim \mu_{B,\beta}$ be independent. Then

$$\begin{aligned} & \mathbb{E}_{B,\beta} \times \mathbb{E}_{B,\beta} \left[f(\omega^{(0)}|_{P_1})g(\omega^{(0)}|_{P_2}) \cdot \mathbb{1}[P_1 \leftrightarrow P_2 \text{ in } \mathcal{G}(\omega^{(0)}, \omega^{(1)})] \right] \\ &= \mathbb{E}_{B,\beta} \times \mathbb{E}_{B,\beta} \left[f(\omega^{(0)}|_{P_1})g(\omega^{(1)}|_{P_2}) \cdot \mathbb{1}[P_1 \leftrightarrow P_2 \text{ in } \mathcal{G}(\omega^{(0)}, \omega^{(1)})] \right]. \end{aligned} \tag{34}$$

Proof. Given $\omega^{(0)}, \omega^{(1)} \in \Sigma_{P_B}$, let

$$\hat{P}_1 := \hat{P}_1(\omega, \omega') := \{p \in P_B : \partial(\hat{\partial} p) \leftrightarrow P_1 \text{ in } \mathcal{G}(\omega, \omega')\}.$$

Then \hat{P}_1 is the set of all plaquettes in P_B which shares a 3-cell with some plaquette in P_B which is connected to P_1 .

Note that the set \hat{P}_1 can be determined without looking outside \hat{P}_1 in $\omega^{(0)}$ and $\omega^{(1)}$, and that

$$d(\omega^{(0)}|_{\hat{P}_1}) = d(\omega^{(1)}|_{\hat{P}_1}) = 0.$$

This implies that even if we know $\omega^{(0)}|_{\hat{P}_1}$ and $\omega^{(1)}|_{\hat{P}_1}$, we cannot distinguish between $\omega^{(0)}|_{P_B \setminus \hat{P}_1}$ and $\omega^{(0)}|_{P_B \setminus \hat{P}_1}$. Consequently, we have

$$\begin{aligned} & \mathbb{E}_{B,\beta} \times \mathbb{E}_{B,\beta} \left[g(\omega^{(0)}|_{P_2}) \mid P_2 \cap \hat{P}_1 = \emptyset, \omega^{(0)}|_{\hat{P}_1}, \omega^{(1)}|_{\hat{P}_1} \right] \\ &= \mathbb{E}_{B,\beta} \times \mathbb{E}_{B,\beta} \left[g(\omega^{(1)}|_{P_2}) \mid P_2 \cap \hat{P}_1 = \emptyset, \omega^{(0)}|_{\hat{P}_1}, \omega^{(1)}|_{\hat{P}_1} \right]. \end{aligned}$$

This implies in particular that

$$\begin{aligned} & \mathbb{E}_{B,\beta} \times \mathbb{E}_{B,\beta} \left[f(\omega^{(0)}|_{P_1})g(\omega^{(0)}|_{P_2}) \cdot \mathbb{1}[P_2 \cap \hat{P}_1 = \emptyset] \right] \\ &= \mathbb{E}_{B,\beta} \times \mathbb{E}_{B,\beta} \left[f(\omega^{(0)}|_{P_1})g(\omega^{(1)}|_{P_2}) \cdot \mathbb{1}[P_2 \cap \hat{P}_1 = \emptyset] \right]. \end{aligned}$$

Noting that

$$P_1 \leftrightarrow P_2 \text{ in } \mathcal{G}(\omega^{(0)}, \omega^{(1)}) \Leftrightarrow P_2 \cap \hat{P}_1 = \emptyset,$$

the desired conclusion follows. □

6. Distances Between Plaquettes

Assume that two boxes B and B' in \mathbb{Z}^4 with $B' \subseteq B$, and sets $P_1, P_2 \subseteq P_B \cup P_{B'}$ are given. Recall the definition of $\text{dist}_{B,B'}(P_1, P_2)$ from (5). Using the notation of the previous section, we have

$$\text{dist}_{B,B'}(P_1, P_2) = \frac{1}{2} \min \left\{ |\text{supp } \omega| + |\text{supp } \omega'| : \omega \in \Sigma_{P_B}, \omega' \in \Sigma_{P_{B'}} \text{ s.t. } P_1 \leftrightarrow P_2 \text{ in } \mathcal{G}(\omega, \omega') \right\}.$$

We now define another measure of the distance between sets of plaquettes which will be useful to us.

$$\text{dist}_B^*(P_1, P_2) := \min_{p_1 \in P_1, p_2 \in P_2} \min \{ |\mathcal{P}| : \mathcal{P} \text{ is a path from } p_1 \text{ to } p_2 \text{ in } P_B \}.$$

We mention that we do neither claim nor prove that the functions $\text{dist}_{B,B'}$ and dist_B^* defined above are distance functions, and this will not be needed in the rest of this paper.

The following lemma gives a relationship between the distances $\text{dist}_{B,B'}$ and dist_B^* .

Lemma 22. *Let B and B' be two boxes in \mathbb{Z}^4 with $B' \subseteq B$, and let $P_1, P_2 \subseteq P_B$ be disjoint. Then*

$$\text{dist}_B^*(P_1, P_2) \leq \text{dist}_{B,B'}(P_1, P_2). \tag{35}$$

and

$$\text{dist}_B(P_1, P_2) \leq 6 \text{dist}_B^*(P_1, P_2). \tag{36}$$

Proof. We first prove that (36) holds.

To this end, let \mathcal{P} be a path from $p_1 \in P_1$ to $p_2 \in P_2$ in P_B with $|\mathcal{P}| = \text{dist}_B^*(P_1, P_2)$. Then \mathcal{P} is a shortest path from p_1 to p_2 , and hence by Lemma 17 there is $\omega^{(0)}, \omega^{(1)} \in \Sigma_{P_B}$ such that \mathcal{P} is a path in $\mathcal{G}(\omega^{(0)}, \omega^{(1)})$ and $|\text{supp } \omega^{(0)}| + |\text{supp } \omega^{(1)}| \leq 12|\mathcal{P}|$. Since \mathcal{P} is a path in $\mathcal{G}(\omega^{(0)}, \omega^{(1)})$ from $p_1 \in P_1$ to $p_2 \in P_2$, we have $P_1 \leftrightarrow P_2$ in $\mathcal{G}(\omega^{(0)}, \omega^{(1)})$, and hence

$$\text{dist}_B(P_1, P_2) \leq \frac{1}{2} (|\text{supp } \omega^{(0)}| + |\text{supp } \omega^{(1)}|) \leq \frac{12|\mathcal{P}|}{2} = 6|\mathcal{P}| = 6 \text{dist}_B^*(P_1, P_2).$$

This concludes the proof of (36).

We now show that (35) holds. To this end, assume that $\omega \in \Sigma_{P_B}$ and $\omega' \in \Sigma_{P_{B'}}$ are such that that $P_1 \leftrightarrow P_2$ in $\mathcal{G}(\omega, \omega')$ and $|\text{supp } \omega| + |\text{supp } \omega'| = 2 \text{dist}_{B,B'}(P_1, P_2)$. By the definition of $\text{dist}_{B,B'}(P_1, P_2)$, such plaquette configurations ω and ω' exists.

Since $P_1 \leftrightarrow P_2$ in $\mathcal{G}(\omega, \omega')$, there exists at least one path in $\mathcal{G}(\omega, \omega')$ between some $p_1 \in P_1$ and some $p_2 \in P_2$. Let \mathcal{P} be such a path. Then by definition, we have

$$\text{dist}_{B,B'}^*(P_1, P_2) \leq |\mathcal{P}|.$$

Moreover, by Lemma 18, we have

$$|\mathcal{P}| \leq (|\text{supp } \omega| + |\text{supp } \omega'|)/2 = \text{dist}_{B,B'}(P_1, P_2).$$

Combining these equations, we obtain

$$\text{dist}_B^*(P_1, P_2) \leq \text{dist}_{B,B'}(P_1, P_2)$$

as desired. □

7. The Probability of Two Sets Being Connected

The purpose of this section is to give a proof of the following result, which gives an upper bound on the probability that two given sets are connected.

Proposition 23. *Let B and B' be two boxes in \mathbb{Z}^4 with $B' \subseteq B$, let $\beta \geq 0$ be such that $30\alpha(\beta) < 1$, and let $P_1, P_2 \subseteq P_B$ be disjoint. Then*

$$\begin{aligned} &\mu_{B,\beta} \times \mu_{B',\beta} \left(\left\{ \omega \in \Sigma_{P_B}, \omega' \in \Sigma_{P_B} : P_1 \leftrightarrow P_2 \text{ in } \mathcal{G}(\omega, \omega') \right\} \right) \\ &\leq \frac{C_1 |P_1| (C_2 \alpha(\beta))^{\text{dist}_{B,B'}(P_1, P_2)}}{2}. \end{aligned}$$

Before giving a proof of this result at the end of this section, we will state and prove the following lemma, which gives an upper bound on the probability that a given path \mathcal{P} is a path in a random graph $\mathcal{G}(\omega, \omega')$.

Lemma 24. *Let B and B' be two a boxes in \mathbb{Z}^4 with $B' \subseteq B$, and let $\beta \geq 0$ be such that $5\alpha(\beta) < 1$. Further, let $p_1, p_2 \in P_B$ be distinct, and let \mathcal{P} be a path from p_1 to p_2 in P_B .*

Then

$$\begin{aligned} &\mu_{B,\beta} \times \mu_{B',\beta} \left(\left\{ \omega \in \Sigma_{P_B}, \omega' \in \Sigma_{P_{B'}} : \mathcal{P} \text{ is a geodesic in } \mathcal{G}(\omega, \omega') \right\} \right) \\ &\leq \frac{5^{-|\mathcal{P}|} (10\alpha(\beta))^M}{2(1 - 5\alpha(\beta))^2}, \end{aligned}$$

where $M := \max(|\mathcal{P}|, \text{dist}_{B,B'}(p_1, p_2))$.

Proof. Let $\omega \in \Sigma_{P_B}$ and $\omega' \in \Sigma_{P_{B'}}$ be such that \mathcal{P} is a geodesic from p_1 to p_2 in $\mathcal{G}(\omega, \omega')$, and let $P_0 := \pm \text{Image } \mathcal{P}$. □

We now claim that the following holds.

Claim 25. *There are symmetric sets $\hat{P} \subseteq \text{supp } \omega$ and $\hat{P}' \subseteq \text{supp } \omega'$ such that*

- (a) $P_0 \subseteq \hat{P} \cup \hat{P}'$,
- (b) *for all symmetric sets $P \subseteq \hat{P}$ and $P' \subseteq \hat{P}'$ which satisfies satisfies $P \sqcup P' = P_0$, $\omega|_P$ is P -irreducible and $\omega'|_{P'}$ is P' -irreducible.*

Proof of Claim. We will give a proof by contradiction. To this end, and assume that no symmetric sets $\hat{P} \subseteq \text{supp } \omega$ and $\hat{P}' \subseteq \text{supp } \omega'$ which satisfy both (a) and (b) exist.

1. Let $\hat{P}_0 := \text{supp } \omega$ and $\hat{P}'_0 := \text{supp } \omega'$. Note that, by assumption, since \mathcal{P} is a geodesic in $\mathcal{G}(\omega, \omega')$ and $P_0 = \pm \text{Image } \mathcal{P}$, (a) holds with $\hat{P} = \hat{P}_0$ and $\hat{P}' = \hat{P}'_0$.
2. For $k = 1, 2, 3, \dots$, assume that sets \hat{P}_{k-1} and \hat{P}'_{k-1} such that (a) holds with $\hat{P} = \hat{P}_{k-1}$ and $\hat{P}' = \hat{P}'_{k-1}$ are given.

Since (a) holds with $\hat{P} = \hat{P}_{k-1}$ and $\hat{P}' = \hat{P}'_{k-1}$, by assumption, (b) must fail.

Consequently, there must exist symmetric sets $P \subseteq \hat{P}_{k-1}$ and $P' \subseteq \hat{P}'_{k-1}$ such that $P \sqcup P' = P_0$, and such that either $\omega|_P$ is not P -irreducible or $\omega'|_{P'}$ is not P' -irreducible.

At the same time, by Lemma 7, there is $\hat{P}_k \subseteq \hat{P}_{k-1}$ and $\hat{P}'_k \subseteq \hat{P}'_{k-1}$ such that $\omega|_{\hat{P}_k}$ is P -irreducible and $\omega|_{\hat{P}'_k}$ is P' -irreducible.

Since $P_0 = P \sqcup P'$, we have $P_0 \subseteq \text{supp } \omega|_{\hat{P}_k} \cup \text{supp } \omega'|_{\hat{P}'_k} = \hat{P}_k \cap \hat{P}'_k$ and hence (a) holds with $\hat{P} = \hat{P}_k$ and $\hat{P}' = \hat{P}'_k$.

Also, since either $\omega|_{\hat{P}_{k-1}}$ is not P -irreducible or $\omega'|_{\hat{P}'_{k-1}}$ is not P' irreducible, we must have $|\hat{P}_k| + |\hat{P}'_k| < |\hat{P}_{k-1}| + |\hat{P}'_{k-1}|$.

Repeating the above argument, we obtain an infinite sequence of sets $(\hat{P}_0, \hat{P}'_0), (\hat{P}_1, \hat{P}'_1), \dots$ with

$$|\hat{P}_0| + |\hat{P}'_0| > |\hat{P}_1| + |\hat{P}'_1| > |\hat{P}_2| + |\hat{P}'_2| > \dots$$

However, since P_N is finite and $P_0 \subseteq \hat{P}_k \cup \hat{P}'_k$, this is impossible, and hence our assumption must be false. This concludes the proof. \square

Using Claim 25, we let $\hat{P} \subseteq \text{supp } \omega$ and $\hat{P}' \subseteq \text{supp } \omega'$ be symmetric sets which satisfy (a) and (b) of this claim, and define $\nu := \omega|_{\hat{P}}$ and $\nu' := \omega'|_{\hat{P}'}$.

Note that by definition, ν and ν' satisfy (1), (2), and (3) of Proposition 15.

Next, since \mathcal{P} is a geodesic in $\mathcal{G}(\omega, \omega')$, it follows that \mathcal{P} is a geodesic in $\mathcal{G}(\nu, \nu')$. Using Lemma 18, it follows from (a) and the definition of ν and ν' that

$$\frac{|\text{supp } \nu| + |\text{supp } \nu'|}{2} \geq |\mathcal{P}|.$$

On the other hand, by the definition of $\text{dist}_{B, B'}$, we must also have

$$\frac{|\text{supp } \nu| + |\text{supp } \nu'|}{2} \geq \text{dist}_{B, B'}(p_1, p_2).$$

To sum up, we have showed that if $\omega \in \Sigma_{P_B}$ and $\omega' \in \Sigma_{P_{B'}}$ are such that \mathcal{P} is a geodesic from p_1 to p_2 in $\mathcal{G}(\omega, \omega')$, and we let $\bar{\Pi}_{P_0, M}^{\geq}$ be as in Proposition 15, then $(\omega, \omega') \in \bar{\Pi}_{P_0, M}^{\geq}$. Consequently,

$$\begin{aligned} &\mu_{B, \beta} \times \mu_{B', \beta}(\{\omega \in \Sigma_{P_B}, \omega' \in \Sigma_{P_{B'}} : \mathcal{P} \text{ is a geodesic in } \mathcal{G}(\omega, \omega')\}) \\ &\leq \mu_{B, \beta} \times \mu_{B', \beta}(\bar{\Pi}_{P_0, M}^{\geq}) \end{aligned}$$

Applying Proposition 15, we thus obtain

$$\begin{aligned} &\mu_{B, \beta} \times \mu_{B', \beta}(\{\omega \in \Sigma_{P_B}, \omega' \in \Sigma_{P_{B'}} : \mathcal{P} \text{ is a geodesic in } \mathcal{G}(\omega, \omega')\}) \\ &\leq \frac{2^{|\mathcal{P}|} (M - |\mathcal{P}|) 5^{M - |\mathcal{P}|} \alpha(\beta)^M}{(1 - 5\alpha(\beta))^2}. \end{aligned}$$

Since p_1 and p_2 are distinct, we have $|\mathcal{P}| \geq 2$, and hence $M - |\mathcal{P}| \geq 1$. Using the inequality $x \leq 2^{x-1}$, valid for all $x \geq 1$, the desired conclusion follows. \square

We now give a proof of Proposition 23.

Proof of Proposition 23. Let $\omega \in \Sigma_{P_B}$ and $\omega' \in \Sigma_{P_{B'}}$, and assume that $P_1 \leftrightarrow P_2$ in $\mathcal{G}(\omega, \omega')$. Then, by Lemma 20, there is $p_1 \in P_1, p_2 \in P_2$, and an optimal geodesic from p_1 to p_2 in $\mathcal{G}(\omega, \omega')$.

Consequently, we have the upper bound

$$\begin{aligned} &\mu_{B,\beta} \times \mu_{B',\beta} \left(\left\{ \omega \in \Sigma_{P_B}, \omega' \in \Sigma_{P_{B'}} : P_1 \leftrightarrow P_2 \text{ in } \mathcal{G}(\omega, \omega') \right\} \right) \\ &\leq \mu_{B,\beta} \times \mu_{B',\beta} \left(\left\{ \omega \in \Sigma_{P_B}, \omega' \in \Sigma_{P_{B'}} : \exists p_1 \in P_1, p_2 \in P_2 \text{ and} \right. \right. \quad (37) \\ &\quad \left. \left. \text{an optimal geodesic } \mathcal{P} \text{ from } p_1 \text{ to } p_2 \text{ in } \mathcal{G}(\omega, \omega') \right\} \right) \end{aligned}$$

By Lemma 19, given $m \geq 2$ and $p_1 \in P_1$, there are at most $40 \cdot 15^{m-2}$ optimal paths \mathcal{P} of length m which starts at p_1 and ends at some $p \in P_B$.

Since P_1 and P_2 are disjoint, we have $\text{dist}_B^*(P_1, P_2) \geq 2$. Consequently, for any $m \geq \text{dist}_B^*(P_1, P_2)$, there can be at most $|P_1| \cdot 40 \cdot 15^{m-2}$ optimal paths which starts at some $p_1 \in P_1$ and ends at some $p_2 \in P_2$.

In particular, this implies that for any $m \geq \text{dist}_B^*(P_1, P_2)$, there are at most $|P_1| \cdot 40 \cdot 15^{m-2}$ paths which starts at some $p_1 \in P_1$ and ends at some $p_2 \in P_2$ which can be an optimal geodesic in $\mathcal{G}(\omega, \omega')$ for some $\omega \in \Sigma_{P_B}$ and $\omega' \in \Sigma_{P_{B'}}$.

On the other hand, given a path \mathcal{P} from p_1 to p_2 , by Lemma 24 we have

$$\begin{aligned} &\mu_{B,\beta} \times \mu_{B',\beta} \left(\left\{ \omega \in \Sigma_{P_B}, \omega' \in \Sigma_{P_{B'}} : \mathcal{P} \text{ is a geodesic in } \mathcal{G}(\omega, \omega') \right\} \right) \\ &\leq \frac{5^{-|\mathcal{P}|} (10\alpha(\beta))^{\max(|\mathcal{P}|, \text{dist}_{B,B'}(p_1, p_2))}}{2(1 - 5\alpha(\beta))^2}. \end{aligned}$$

Combining these observations, and summing over all $m \geq \text{dist}_B^*(P_1, P_2)$, we thus obtain

$$\begin{aligned} &\mu_{B,\beta} \times \mu_{B',\beta} \left(\left\{ \omega \in \Sigma_{P_B}, \omega' \in \Sigma_{P_{B'}} : \exists p_1 \in P_1, p_2 \in P_2 \text{ and} \right. \right. \quad (38) \\ &\quad \left. \left. \text{an optimal geodesic } \mathcal{P} \text{ from } p_1 \text{ to } p_2 \text{ in } \mathcal{G}(\omega, \omega') \right\} \right) \\ &\leq \sum_{m=\text{dist}_B^*(P_1, P_2)}^{\infty} 40|P_1| \cdot 15^{m-2} \cdot \frac{5^{-m} (10\alpha(\beta))^{\max(m, \text{dist}_{B,B'}(P_1, P_2))}}{2(1 - 5\alpha(\beta))^2} \\ &= \frac{20|P_1|}{15^2(1 - 5\alpha(\beta))^2} \sum_{m=\text{dist}_B^*(P_1, P_2)}^{\infty} 3^m (10\alpha(\beta))^{\max(m, \text{dist}_{B,B'}(P_1, P_2))}. \quad (39) \end{aligned}$$

We now rewrite the sum in (39) as follows. First, note that by Lemma 22, we have

$$\text{dist}_B^*(P_1, P_2) \leq \text{dist}_{B,B'}(P_1, P_2).$$

Consequently,

$$\begin{aligned} &\sum_{m=\text{dist}_B^*(P_1, P_2)}^{\infty} 3^m (10\alpha(\beta))^{\max(m, \text{dist}_{B,B'}(p_1, p_2))} \\ &= \sum_{m=\text{dist}_B^*(P_1, P_2)}^{\text{dist}_{B,B'}(P_1, P_2)-1} 3^m (10\alpha(\beta))^{\text{dist}_{B,B'}(P_1, P_2)} + \sum_{m=\text{dist}_{B,B'}(P_1, P_2)}^{\infty} (30\alpha(\beta))^m. \quad (40) \end{aligned}$$

The first term in (40) is a finite geometric sum, and the second term is an infinite geometric sum which converge since $30\alpha(\beta) < 1$. Hence the previous equation is equal to

$$\frac{3^{\text{dist}_{B,B'}(P_1, P_2)} - 3^{\text{dist}_B^*(P_1, P_2)}}{3 - 1} (10\alpha(\beta))^{\text{dist}_{B,B'}(P_1, P_2)} + \frac{(30\alpha(\beta))^{\text{dist}_{B,B'}(P_1, P_2)}}{1 - 30\alpha(\beta)}.$$

Since $30\alpha(\beta) < 1$, we can bound the previous equation from above by

$$\left[\frac{1}{2} + \frac{1}{1 - 30\alpha(\beta)} \right] (30\alpha(\beta))^{\text{dist}_{B,B'}(P_1, P_2)}.$$

Combining the previous equations, we obtain

$$\begin{aligned} & \mu_{B,\beta} \times \mu_{B',\beta} \left(\{ \omega \in \Sigma_{P_B}, \omega' \in \Sigma_{P_{B'}} : P_1 \leftrightarrow P_2 \text{ in } (\omega, \omega') \} \right) \\ & \leq \frac{20|P_1|}{15^2(1 - 5\alpha(\beta))^2} \left[\frac{1}{2} + \frac{1}{1 - 30\alpha(\beta)} \right] (30\alpha(\beta))^{\text{dist}_{B,B'}(P_1, P_2)}. \end{aligned}$$

Recalling the definition of C_1 from (7) and the definition of C_2 from (8), we obtain the desired conclusion. □

8. Covariance and Connected Sets

We now use the notion of connected sets to obtain an upper bound on the covariance of local functions.

Lemma 26. *Let B be a box in \mathbb{Z}^4 , and let $\beta \geq 0$. Further, let $P_1 \subseteq P_B$ and $P_2 \subseteq P_B$ be symmetric and disjoint, and let $f_1, f_2: \Sigma_{B,2} \rightarrow \mathbb{C}$ be such that $f_1(\omega) = f_1(\omega|_{P_1})$ and $f_2(\omega) = f_2(\omega|_{P_2})$ for all $\omega \in \Sigma_{P_B}$. Finally, let $\omega, \omega' \sim \mu_{B,\beta}$ be independent. Then*

$$\begin{aligned} & \left| \text{Cov}(f_1(\omega), f_2(\omega)) \right| \\ & \leq 2 \|f_1\|_\infty \|f_2\|_\infty \mu_{B,\beta} \times \mu_{B,\beta} \left(\{ \omega, \omega' \in \Sigma_{P_B} : P_1 \leftrightarrow P_2 \text{ in } \mathcal{G}(\omega, \omega') \} \right). \end{aligned}$$

Proof. Note first that

$$\begin{aligned} \mathbb{E}_{B,\beta} [f_1(\omega) f_2(\omega)] &= \mathbb{E}_{B,\beta} \times \mathbb{E}_{B,\beta} \left[f_1(\omega) f_2(\omega) \cdot \mathbb{1}[P_1 \leftrightarrow P_2 \text{ in } \mathcal{G}(\omega, \omega')] \right] \\ & \quad + \mathbb{E}_{B,\beta} \times \mathbb{E}_{B,\beta} \left[f_1(\omega) f_2(\omega) \cdot \mathbb{1}[P_1 \nleftrightarrow P_2 \text{ in } \mathcal{G}(\omega, \omega')] \right]. \end{aligned}$$

Since $f_1(\omega'') = f_1(\omega''|_{P_1})$ and $f_2(\omega'') = f_2(\omega''|_{P_2})$ for all $\omega'' \in \Sigma_{P_B}$, we can apply Lemma 21 to obtain

$$\begin{aligned} & \mathbb{E}_{B,\beta} \times \mathbb{E}_{B,\beta} \left[f_1(\omega) f_2(\omega) \cdot \mathbb{1}[P_1 \leftrightarrow P_2 \text{ in } \mathcal{G}(\omega, \omega')] \right] \\ & = \mathbb{E}_{B,\beta} \times \mathbb{E}_{B,\beta} \left[f_1(\omega) f_2(\omega') \cdot \mathbb{1}[P_1 \leftrightarrow P_2 \text{ in } \mathcal{G}(\omega, \omega')] \right] \\ & = \mathbb{E}_{B,\beta} \times \mathbb{E}_{B,\beta} [f_1(\omega) f_2(\omega')] \\ & \quad - \mathbb{E}_{B,\beta} \times \mathbb{E}_{B,\beta} \left[f_1(\omega) f_2(\omega') \cdot \mathbb{1}[P_1 \leftrightarrow P_2 \text{ in } \mathcal{G}(\omega, \omega')] \right] \\ & = \mathbb{E}_{B,\beta} [f_1(\omega)] \mathbb{E}_{B,\beta} [f_2(\omega')] \\ & \quad - \mathbb{E}_{B,\beta} \times \mathbb{E}_{B,\beta} \left[f_1(\omega) f_2(\omega') \cdot \mathbb{1}[P_1 \leftrightarrow P_2 \text{ in } \mathcal{G}(\omega, \omega')] \right]. \end{aligned}$$

Combining the previous equations, we obtain

$$\begin{aligned} & \mathbb{E}_{B,\beta}[f_1(\omega)f_2(\omega)] \\ &= \mathbb{E}_{B,\beta}[f_1(\omega)]\mathbb{E}_{B,\beta}[f_2(\omega')] \\ &\quad - \mathbb{E}_{B,\beta} \times \mathbb{E}_{B,\beta} \left[f_1(\omega)f_2(\omega') \cdot \mathbb{K}[P_1 \leftrightarrow P_2 \text{ in } \mathcal{G}(\omega, \omega')] \right] \\ &\quad + \mathbb{E}_{B,\beta} \times \mathbb{E}_{B,\beta} \left[f_1(\omega)f_2(\omega) \cdot \mathbb{K}[P_1 \leftrightarrow P_2 \text{ in } \mathcal{G}(\omega, \omega')] \right], \end{aligned}$$

and hence

$$\begin{aligned} \text{Cov}(f_1(\omega), f_2(\omega)) &= \mathbb{E}_{B,\beta} \times \mathbb{E}_{B,\beta} \left[f_1(\omega)f_2(\omega) \cdot \mathbb{K}[P_1 \leftrightarrow P_2 \text{ in } \mathcal{G}(\omega, \omega')] \right] \\ &\quad - \mathbb{E}_{B,\beta} \times \mathbb{E}_{B,\beta} \left[f_1(\omega)f_2(\omega') \cdot \mathbb{K}[P_1 \leftrightarrow P_2 \text{ in } \mathcal{G}(\omega, \omega')] \right]. \end{aligned}$$

From this the desired conclusion immediately follows. □

We now combine the previous lemma with Proposition Proposition 23 to obtain our first main result.

Proof of Theorem 1. By combining Lemma 26 and Proposition 23, we immediately obtain

$$\left| \text{Cov}(f_1(d\sigma), f_2(d\sigma)) \right| \leq C_1(C_2\alpha(\beta))^{\text{dist}_B(P_1, P_2)} \|f_1\|_\infty \|f_2\|_\infty.$$

This concludes the proof of the first part of Theorem 1.

To see that the second claim of the theorem holds, we apply the first part of the theorem with $f_1(\omega) = \text{tr } \rho(\omega_{p_1})$, $f_2(\omega) = \text{tr } \rho(\omega_{p_2})$, $P_1 = \{p_1\}$, and $P_2 = \{p_2\}$, and note that since ρ is unitary, we have $\|f_1\|_\infty \leq \dim \rho$ and $\|f_2\|_\infty \leq \dim \rho$. □

9. Total Variation and Connected Sets

The next lemma gives a connection between the total variation distance of two restricted configurations and the property of two sets being connected in the same configurations. The proof of this lemma uses a coupling of the two configurations. We mention that this coupling is the same type of coupling that is used in e.g. [15] and [3].

Lemma 27. *Let $B' \subsetneq B$ be two boxes in \mathbb{Z}^4 , and let $\beta \geq 0$. Let $P_0 \subseteq P_{B'}$. Further, let $\omega \sim \mu_{B,\beta}$ and $\omega' \sim \mu_{B',\beta}$. Then*

$$\begin{aligned} & \text{dist}_{TV}(\omega|_{P_0}, \omega'|_{P_0}) \\ & \leq \mu_{B,\beta} \times \mu_{B',\beta} \left(\left\{ \omega \in \Sigma_{P_B}, \omega' \in \Sigma_{P_{B'}} : P_0 \leftrightarrow (P_B \setminus P_{B'}) \text{ in } \mathcal{G}(\omega, \omega') \right\} \right). \end{aligned}$$

Proof. Given $\omega \in \Sigma_{P_B}$ and $\omega' \in \Sigma_{P_{B'}}$, let

$$\hat{P} := \{p \in P_{B'} : \{p\} \leftrightarrow P_B \setminus P_{B'} \text{ in } \mathcal{G}(\omega, \omega')\}$$

(see Fig. 3B).

Next, define

$$\hat{P} := (P_B \setminus P_{B'}) \cup \{p \in P_{B'} : \partial(\hat{p}) \leftrightarrow P_B \setminus P_{B'} \text{ in } \mathcal{G}(\omega, \omega')\}.$$

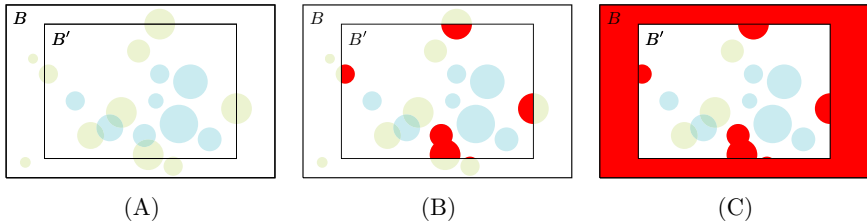


Fig. 3. In the three figures above we let the green and blue disks represent the connected components of $\mathcal{G}(\omega, 0)$ and $\mathcal{G}(0, \omega')$ respectively for some $\omega \in \Sigma_{P_B}$ and $\omega' \in \Sigma_{P_{B'}}$ (see A). The clusters of partially overlapping disks correspond to connected components in $\mathcal{G}(\omega, \omega')$. In B the red area corresponds to the set \hat{P} , and in C the red area corresponds to the set \hat{P}

(see Fig. 3C) and note that $P_B \setminus P_{B'} \subseteq \hat{P}$ and that

$$\text{supp } \omega|_{\hat{P}} \cup \text{supp } \omega'|_{\hat{P}} = \hat{P}.$$

Note also that the set \hat{P} can be determined without looking outside \hat{P} in ω and ω' . Together, these observations imply that if we let $\omega \sim \mu_{B,\beta}$ and $\omega' \sim \mu_{B',\beta}$ be independent, then, conditioned on $\hat{P} = \hat{P}(\omega, \omega')$, we have

$$\omega|_{P_{B'} \setminus \hat{P}} \stackrel{d}{=} \omega'|_{P_{B'} \setminus \hat{P}}. \tag{41}$$

We now use this observation to construct a coupling $(\hat{\omega}, \hat{\omega}')$ of $\hat{\omega} \sim \mu_{B,\beta}$ and $\hat{\omega}' \sim \mu_{B',\beta}$. To this end, let $\omega \sim \mu_{B,\beta}$ and $\omega' \sim \mu_{B',\beta}$ be independent.

Given $\hat{P} = \hat{P}(\omega, \omega')$, define

$$\begin{cases} \hat{\omega} := \omega \\ \hat{\omega}' := \omega'|_{\hat{P}} + \omega|_{P_{B'} \setminus \hat{P}}. \end{cases}$$

By (41), we have

$$\hat{\omega}|_{P_{B'} \setminus \hat{P}} \stackrel{d}{=} \hat{\omega}'|_{P_{B'} \setminus \hat{P}}, \tag{42}$$

and hence $\hat{\omega}' \sim \mu_{B',\beta}$. Since $\hat{\omega} \sim \mu_{B,\beta}$ by definition, it follows that $(\hat{\omega}, \hat{\omega}')$ is a coupling of $\hat{\omega} \sim \mu_{B,\beta}$ and $\hat{\omega}' \sim \mu_{B',\beta}$.

By the coupling characterization of total variation distance, we therefore have

$$\text{dist}_{TV}(\omega|_{P_0}, \omega'|_{P_0}) \leq \mu_{B,\beta} \times \mu_{B',\beta} \left(\{ \omega \in \Sigma_{P_B}, \omega' \in \Sigma_{B',\beta} : \hat{\omega}|_{P_0} \neq \hat{\omega}'|_{P_0} \} \right).$$

If $\hat{\omega}|_{P_0} \neq \hat{\omega}'|_{P_0}$, then

$$P_0 \leftrightarrow \hat{P}_B \setminus P_{B'},$$

and hence it follows that

$$\text{dist}_{TV}(\omega|_{P_0}, \omega'|_{P_0}) \leq \mu_{B,\beta} \times \mu_{B',\beta} \left(\{ \omega, \omega' \in \Sigma_{P_B} : P_0 \leftrightarrow (P_B \setminus P_{B'}) \text{ in } \mathcal{G}(\omega, \omega') \} \right).$$

This completes the proof. \square

We now use the previous lemma to give a proof of Theorem 4.

Proof of Theorem 4. By Lemma 27, we have

$$\begin{aligned} \text{dist}_{TV}(\omega|_{P_0}, \omega'|_{P_0}) &\leq \mu_{B,\beta} \\ &\times \mu_{B',\beta} \left(\left\{ \omega \in \Sigma_{P_B}, \omega' \in \Sigma_{P_{B'}} : P_0 \leftrightarrow (P_B \setminus P_{B'}) \text{ in } (\omega, \omega') \right\} \right). \end{aligned}$$

On the other hand, by Proposition 23, we have

$$\begin{aligned} \mu_{B,\beta} \times \mu_{B',\beta} \left(\left\{ \omega \in \Sigma_{P_B}, \omega' \in \Sigma_{P_{B'}} : P_0 \leftrightarrow (P_B \setminus P_{B'}) \text{ in } (\omega, \omega') \right\} \right) \\ \leq C_1 |P_0| (C_2 \alpha(\beta))^{\text{dist}_{B,B'}(P_0, P_B \setminus P_{B'})}. \end{aligned}$$

Combining the previous equations, we thus obtain

$$\text{dist}_{TV}(\omega|_{P_0}, \omega'|_{P_0}) \leq C_1 |P_0| (C_2 \alpha(\beta))^{\text{dist}_{B,B'}(P_0, P_B \setminus P_{B'})}$$

as desired. □

10. The Expected Spin at a Plaquette

In this section, we give proofs of Theorem 2 and Theorem 3.

Proof of Theorem 2. We first define two useful events.

Let

$$\begin{aligned} A := \left\{ \sigma \in \Sigma_{E_B} : \exists \text{ irreducible } \nu \in \Sigma_{P_B} \text{ with } (d\sigma)|_{\text{supp } \nu} = \nu, \right. \\ \left. p \in \text{supp } \nu \text{ and } |(\text{supp } \nu)^+| \geq 11 \right\}, \end{aligned}$$

and, for $e \in \partial p$ and $g \in G$, let

$$A_{e,g} := \left\{ \sigma \in \Sigma_{E_B} : \forall p' \in \partial e \text{ we have } (d\sigma)_{p'} = g \right\}.$$

Note that since $\text{dist}_B(\{p\}, \delta P_B) > 6$, these events are all well-defined. Moreover, by Lemma 8 and Lemma 9, since $\text{dist}_B(\{p\}, \delta P_B) > 11$, we have

$$\left\{ \sigma \in \Sigma_{E_N} : (d\sigma)_p \neq 0 \right\} = A \cup \bigcup_{e \in \partial p} \bigsqcup_{g \in G \setminus \{0\}} A_{e,g} \tag{43}$$

We now give upper bounds of the $\mu_{B,\beta}$ -measure of the events defined above.

To this end, note first that by Proposition 12, applied with $P = \{p, -p\}$ and $M = 11$, we have

$$\mu_{B,\beta}(A) \leq \frac{5^{10} \alpha(\beta)^{11}}{1 - 5\alpha(\beta)}. \tag{44}$$

If for some $e \in \partial p$ the event $A_{e,0}^c$ happen, then there must exist an irreducible plaquette configuration $\nu \in \Sigma_{P_B}$ with $(d\sigma)|_{\text{supp } \nu} = \nu$ and $\text{supp } \nu \cap \hat{\partial} e \neq \emptyset$. From Proposition 12, applied with $p' \in \hat{\partial} e$ and $M = 6$, and a union bound, we obtain

$$0 \leq 1 - \mu_{B,\beta}(A_{e,0}) \leq \sum_{p' \in \hat{\partial} e} \frac{5^5 \alpha(\beta)^6}{1 - 5\alpha(\beta)}. \tag{45}$$

Next, note that if $\{e, e'\} \subseteq \partial p$ and $g \in G \setminus \{0\}$ then it follows from Proposition 12, applied with $P = (\hat{\partial}e \cup \hat{\partial}(-e)) \cup (\hat{\partial}e' \cup \hat{\partial}(-e'))$ and $M = |P^+| = 6 + 6 - 1 = 11$, that

$$\sum_{g \in G \setminus \{0\}} \mu_{B,\beta}(A_{e,g} \cap A_{e',g}) \leq \frac{\alpha(\beta)^{11}}{1 - 5\alpha(\beta)}. \tag{46}$$

Finally, if we for $e = dx_j \in \partial p$ and $g \in G \setminus \{0\}$ define $v^{(e,g)}$ by

$$v_{e'}^{(e,g)} := d(g dx_j)$$

then $d(v^{(e,g)}) = 0$, $\text{supp } v^{(e,g)} = \pm \hat{\partial}e$, and by Lemma 9, we have

$$\sigma \in A_{e,g} \Leftrightarrow (d\sigma)|_{\pm \hat{\partial}e} = v^{(e,g)}.$$

Moreover, we have

$$\sigma \in A_{e,0} \Leftrightarrow (d\sigma)|_{\pm \hat{\partial}e} = 0.$$

Consequently, we can apply Lemma 11 with $\nu = v^{(e,g)}$ to obtain

$$\frac{\mu_{B,\beta}(A_{e,g})}{\mu_{B,\beta}(A_{e,0})} = \frac{\mu_{B,\beta}(\{\sigma \in \Sigma_{E_B} : (d\sigma)|_{\text{supp } v^{(e,g)}} = v^{(e,g)}\})}{\mu_{B,\beta}(\{\sigma \in \Sigma_{E_B} : (d\sigma)|_{\text{supp } v^{(e,g)}} = 0\})} = \phi_\beta(g)^{12} \tag{47}$$

We now combine the above equations to obtain (11). To this end, note first that by the triangle inequality we have

$$\begin{aligned} & \left| \mathbb{E}_{B,\beta} \left[f((d\sigma)_p) \right] - \left(f(0) + \sum_{e \in \partial p} \sum_{g \in G} (f(g) - f(0)) \phi(g)^{12} \right) \right| \\ &= \left| \mathbb{E}_{B,\beta} [f((d\sigma)_p) - f(0)] - \sum_{e \in \partial p} \mu_{B,\beta}(A_{e,0}) \sum_{g \in G} (f(g) - f(0)) \phi(g)^{12} \right. \\ & \quad \left. + \sum_{e \in \partial p} (\mu_{B,\beta}(A_{e,0}) - 1) \sum_{g \in G} (f(g) - f(0)) \phi(g)^{12} \right| \\ &\leq \left| \mathbb{E}_{B,\beta} [f((d\sigma)_p) - f(0)] - \sum_{e \in \partial p} \mu_{B,\beta}(A_{e,0}) \sum_{g \in G} (f(g) - f(0)) \phi(g)^{12} \right| \tag{48} \end{aligned}$$

$$+ \left| \sum_{e \in \partial p} (1 - \mu_{B,\beta}(A_{e,0})) \sum_{g \in G} (f(g) - f(0)) \phi(g)^{12} \right|. \tag{49}$$

To get an upper bound on (48), we first rewrite (43) as a union of disjoint events;

$$\{\sigma \in \Sigma_{E_N} : (d\sigma)_p \neq 0\} = A \sqcup \bigsqcup_{g \in G \setminus \{0\}} \left(\bigsqcup_{\substack{E \subseteq \partial p: \\ E \neq \emptyset}} (A^c \cap A_{e',g} \cap A_{e'',g}^c) \right). \tag{50}$$

Using first (47) and then (50), we immediately obtain

$$\begin{aligned} & \left| \mathbb{E}_{B,\beta} [f((d\sigma)_p) - f(0)] - \sum_{e \in \partial p} \mu_{B,\beta}(A_{e,0}) \sum_{g \in G} (f(g) - f(0)) \phi_{B,\beta}(g) \right| \\ & \stackrel{(47)}{=} \left| \mathbb{E}_{B,\beta} [f((d\sigma)_p) - f(0)] - \sum_{e \in \partial p} \sum_{g \in G} (f(g) - f(0)) \mu_{B,\beta}(A_{e,g}) \right| \\ & \stackrel{(50)}{\leq} \left(\mu_{B,\beta}(A) + \sum_{\{e_1, e_2\} \subseteq \partial p} \sum_{g \in G \setminus \{0\}} \mu_{B,\beta}(A_{e_1,g} \cap A_{e_2,g}) \right) \cdot \max_{g \in G} |f(g) - f(0)| \\ & \stackrel{(44),(46)}{\leq} \left(\frac{5^{10} \alpha(\beta)^{11}}{1 - 5\alpha(\beta)} + \binom{|\partial p|}{2} \frac{\alpha(\beta)^{11}}{1 - 5\alpha(\beta)} \right) \cdot \max_{g \in G} |f(g) - f(0)|. \end{aligned}$$

To obtain an upper bound on (49), we use (45) to get

$$\begin{aligned} & \left| \sum_{e \in \partial p} (1 - \mu_{B,\beta}(A_{e,0})) \sum_{g \in G} (f(g) - f(0)) \phi(g)^{12} \right| \\ & \leq \sum_{e \in \partial p} \sum_{p' \in \hat{\partial} e} \frac{5^5 \alpha(\beta)^6}{1 - 5\alpha(\beta)} \cdot \max_{g \in G} |f(g) - f(0)| \cdot \sum_{g \in G \setminus \{0\}} \phi_\beta(g)^{12}. \end{aligned}$$

Finally, note that $\binom{|\partial p|}{2} = 6$, that $\sum_{e \in \partial p} |\hat{\partial} e| = 4 \cdot 6$, and that

$$\sum_{g \in G \setminus \{0\}} \phi_\beta(g)^{12} \leq \left(\sum_{g \in G \setminus \{0\}} \phi_\beta(g)^2 \right)^6 = \alpha(\beta)^6$$

Combining the above equations, we thus get

$$\begin{aligned} & \left| \mathbb{E}_{B,\beta} [f((d\sigma)_p)] - \left(f(0) + \sum_{e \in \partial p} \sum_{g \in G} (f(g) - f(0)) \phi(g)^{12} \right) \right| \\ & \leq \left(\frac{5^{10} \alpha(\beta)^{11}}{1 - 5\alpha(\beta)} + \frac{6\alpha(\beta)^{11}}{1 - 5\alpha(\beta)} + \frac{24 \cdot 5^5 \alpha(\beta)^{12}}{1 - 5\alpha(\beta)} \right) \cdot \max_{g \in G} |f(g) - f(0)|. \end{aligned}$$

Since $5\alpha(\beta) < 1$, we have $5^{10} + 6 + 24 \cdot 5^5 \alpha(\beta) < 5^{11}$, and hence the desired conclusion follows. □

Proof of Theorem 3. Note first that, by definition, we have

$$\begin{aligned} & \mathbb{E}_{B,\beta} [f_1((d\sigma)_{p_1}) f_2((d\sigma)_{p_2})] \\ & = \text{Cov}(f_1((d\sigma)_{p_1}), f_2((d\sigma)_{p_2})) + \mathbb{E}_{B,\beta} [\rho((d\sigma)_{p_1})] \mathbb{E}_{B,\beta} [\rho((d\sigma)_{p_2})]. \end{aligned}$$

Consequently, by the triangle inequality we have

$$\begin{aligned} & \left| \mathbb{E}_{B,\beta} [f_1((d\sigma)_{p_1}) f_2((d\sigma)_{p_2})] - \prod_{j \in \{1,2\}} \left(f_j(0) + \sum_{e \in \partial p_j} \sum_{g \in G} (f_j(g) - f_j(0)) \phi_\beta(g)^{12} \right) \right| \\ & \leq \left| \text{Cov} \left(f_1((d\sigma)_{p_1}), f_2((d\sigma)_{p_2}) \right) \right| \\ & \quad + \left| \mathbb{E}_{B,\beta} [f_1((d\sigma)_p)] \mathbb{E}_{B,\beta} [f_2((d\sigma)_{p'})] \right. \\ & \quad \left. - \prod_{j \in \{1,2\}} \left(f_j(0) + \sum_{e \in \partial p} \sum_{g \in G} (f_j(g) - f_j(0)) \phi_\beta(g)^{12} \right) \right|. \end{aligned}$$

For the second term, note that for any $x, y, a, b \in \mathbb{C}$, we have $|xy - ab| \leq |(x - a)(y - b)| + |a(y - b)| + |b(x - a)|$. Using this inequality and Theorem 2, we obtain

$$\begin{aligned} & \left| \mathbb{E}_{B,\beta} [f_1((d\sigma)_{p_1})] \mathbb{E}_{B,\beta} [f_2((d\sigma)_{p_2})] \right. \\ & \quad \left. - \prod_{j \in \{1,2\}} \left(f_j(0) + \sum_{e \in \partial p_j} \sum_{g \in G} (f_j(g) - f_j(0)) \phi_\beta(g)^{12} \right) \right| \\ & \leq \prod_{j \in \{0,1\}} \frac{(5\alpha(\beta))^{11}}{1 - 5\alpha(\beta)} \cdot \max_{g \in G} |f_j(g) - f_j(0)| \\ & \quad + \sum_{j \in \{0,1\}} \frac{(5\alpha(\beta))^{11}}{1 - 5\alpha(\beta)} \cdot \max_{g \in G} |f_j(g) - f_j(0)| \cdot \max_{g \in G} |f_{1-j}(g)|. \end{aligned}$$

Combining the previous equations and using Theorem 1, we get

$$\begin{aligned} & \left| \mathbb{E}_{B,\beta} [f_1((d\sigma)_{p_1}) f_2((d\sigma)_{p_2})] - \prod_{j \in \{1,2\}} \left(f_j(0) + \sum_{e \in \partial p_j} \sum_{g \in G} (f_j(g) - f_j(0)) \phi_\beta(g)^{12} \right) \right| \\ & \leq C_1 \|f_1\|_\infty \|f_2\|_\infty (C_2 \alpha(\beta))^{\text{dist}_B(\{p_1\}, \{p_2\})} \\ & \quad + \left(\frac{(5\alpha(\beta))^{11}}{1 - 5\alpha(\beta)} \right)^2 \prod_{j \in \{0,1\}} \max_{g \in G} |f_j(g) - f_j(0)| \\ & \quad + \frac{(5\alpha(\beta))^{11}}{1 - 5\alpha(\beta)} \sum_{j \in \{0,1\}} \max_{g \in G} |f_j(g) - f_j(0)| \cdot \max_{g \in G} |f_{1-j}(g)|. \end{aligned}$$

Noting that, for $j \in \{0, 1\}$, we have

$$\max_{g \in G} |f_j(g) - f_j(0)| \leq 2 \max_{g \in G} |f_j(g)|,$$

we obtain (12) as desired. □

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