



The Hawking–Penrose Singularity Theorem for C^1 -Lorentzian Metrics

Michael Kunzinger , Argam Ohanyan, Benedict Schinnerl, Roland Steinbauer

Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria.
E-mail: michael.kunzinger@univie.ac.at; argam.ohanyan@univie.ac.at; benedict.schinnerl@univie.ac.at; roland.steinbauer@univie.ac.at

Received: 18 October 2021 / Accepted: 18 January 2022
Published online: 26 February 2022 – © The Author(s) 2022

Abstract: We extend both the Hawking–Penrose theorem and its generalisation due to Galloway and Senovilla to Lorentzian metrics of regularity C^1 . For metrics of such low regularity, two main obstacles have to be addressed. On the one hand, the Ricci tensor now is distributional, and on the other hand, unique solvability of the geodesic equation is lost. To deal with the first issue in a consistent way, we develop a theory of tensor distributions of finite order, which also provides a framework for the recent proofs of the theorems of Hawking and of Penrose for C^1 -metrics (Graf in *Commun Math Phys* 378(2):1417–1450, 2020). For the second issue, we study geodesic branching and add a further alternative to causal geodesic incompleteness to the theorem, namely a condition of maximal causal non-branching. The genericity condition is re-cast in a distributional form that applies to the current reduced regularity while still being fully compatible with the smooth and $C^{1,1}$ -settings. In addition, we develop refinements of the comparison techniques used in the proof of the $C^{1,1}$ -version of the theorem (Graf in *Commun Math Phys* 360:1009–1042, 2018). The necessary results from low regularity causality theory are collected in an appendix.

1. Introduction

The classical singularity theorems¹ of General Relativity (GR) collect sufficient and physically reasonable conditions that lead to causal geodesic incompleteness of space-time, hence to the occurrence of a singularity in the spirit of Roger Penrose’s Nobel Prize-winning approach [40]. Being rigorous statements in pure Lorentzian geometry, they were first formulated within the smooth category. As they form a body of essential results in GR, the quest for low regularity versions of the theorems is eminent and was already explicitly discussed in [14, Sec. 8.4]. In fact, the (smooth) theorems can be read as predicting a mere drop of the differentiability of the metric below C^2 , rather than incompleteness.

¹ See [14, Ch. 8], [21, Ch. 9], [46–48] for extensive treatments.

However, only with the rather recent advent of systematic studies in low regularity GR [6] and, in particular, causality theory [5,24,31,41], a rigorous extension of the singularity theorems for metrics below the C^2 -class became feasible, cf. [46, Sec. 6.2]. This applies first of all to metrics of regularity $C^{1,1}$, which model situations where the matter variables possess finite jumps. Indeed the Hawking, the Penrose, and, finally, the Hawking-Penrose theorem were proven in this regularity [8,25,26]. These results rule out that a smooth spacetime that is incomplete by the classical theorems, can be extended to a complete $C^{1,1}$ -spacetime. Also, they imply that the spacetime either is incomplete or the differentiability of the metric is below $C^{1,1}$, and consequently the curvature becomes unbounded.

Technically, in $C^{1,1}$ the exponential map is still available [23,31] and the curvature is still locally bounded, which allows for a natural extension of the energy and the genericity conditions. However, the curvature tensor is only defined almost everywhere, which forbids the use of Jacobi fields and conjugate points, both essential tools in the classical proofs. Instead one uses regularisation techniques which allow one to derive weakened versions of the energy conditions for approximating smooth metrics with controlled causality. Using careful comparison techniques for the (matrix) Riccati equation, it is then possible to show that these (still) lead to the occurrence of conjugate or focal points along causal geodesics for the approximating metrics. This in turn forces the geodesics of the $C^{1,1}$ -metric to stop maximising. Similarly, a natural extension of the initial and the energy conditions leads to the formation of a trapped set. This, together with an extension of the causal parts of the classical proofs, allows one to finally derive the results.

The next step in lowering the regularity assumptions was undertaken by Graf [7], who extended the Hawking and the Penrose theorems to C^1 -metrics, with the Gannon-Lee theorem following in [43]. C^1 -regularity is at the moment the lowest possible class where classical singularity theorems have been established, and in this paper we complete this effort by proving the most refined of these statements, namely the Hawking-Penrose theorem.

In this regularity class one faces the following added severe complications arising from the fact that the Levi-Civita connection is only continuous:

- (a) The curvature is no longer locally bounded, but is merely a distribution *of order one*.
- (b) The initial value problem for the geodesic equation is solvable, but *not uniquely* so.
- (c) The exponential map is no longer defined.

The first item is especially relevant when formulating the energy conditions. While the strong energy condition can be extended in a straightforward manner, a formulation of the null energy condition is more subtle, cf. [7, Sec. 5], see also Sect. 2.3, below. The genericity condition needed for the Hawking-Penrose theorem is still more delicate. Actually it turns out to be necessary to apply the Ricci tensor to vector fields constructed via parallel transport, which means that they are merely of C^1 -regularity, see Definition 2.15. This can only be done since the curvature is a distribution of order one, and we present the required distributional setting in Sect. 2.1, below. Whereas in the $C^{1,1}$ -context the use of distributional methods could be avoided, it becomes essential in the present work, and at the same time has the benefit of providing a global framework also for the earlier results by Graf [7] on which we build.

Then, while we can use the weakened versions of the strong and null energy conditions for approximating smooth metrics derived in [7] (employing a refined version of the Friedrichs Lemma), we have to derive from the distributional genericity condition an appropriate weakened version for smooth approximations that allows us to employ an

extension of the Riccati comparison techniques developed in [8], see Lemmas 2.17, and 2.18. The key technique of regularisation by smooth metrics with adapted causality as put forward in [5] is recalled in Sect. 2.2.

The first issue connected to item (b) is that it necessitates a decision on how to extend the notion of geodesic completeness, and following [7] we say that a spacetime is complete if *every* inextendible solution to the geodesic equation is complete (rather than merely demanding only one complete solution for any choice of initial data).

Next, item (b) together with item (c) forbids the use of an essential argument in the context of approximating maximising causal geodesics by maximising causal geodesics of the approximating metrics. This issue has been solved in [7, Sec. 2] in the *globally hyperbolic* case. In the context of the present work, however, we have to establish more general results, see Sect. 3. In fact, we have to introduce an additional assumption, namely a non-branching condition for maximising causal geodesics (Definition 3.1), to derive the corresponding approximation results in Proposition 3.4. Here it is also essential to apply a classical ODE result (Proposition 2.5), which we reformulate for our purpose in Corollary 2.6. Moreover, since the interrelation between causal geodesics (i.e., solutions of the geodesic equation) and maximising curves becomes more subtle below $C^{1,1}$ [16, 42], we rely in particular on the fact that maximising causal curves are indeed (unbroken) geodesics and hence C^2 -curves [27, 43].

The new non-branching assumption is well motivated by similar conditions used in metric geometry, where due to the lack of a differentiable structure the geodesic equation is not available. Also a null version of the condition was essential in the proof of the C^1 -Gannon-Lee theorem, see [43, Sec. 3]. Moreover it adds a novel facet to the interpretation of the C^1 -version of the theorem: It predicts either geodesic incompleteness or branching of maximising causal geodesics, with both alternatives physically signifying a catastrophic event for the observer corresponding to the geodesic. Of course, there is the alternative that the regularity of the metric drops below C^1 , which renders the curvature a distribution of higher order. Again, the result also forbids the extension of the spacetime to a complete one of regularity C^1 without (maximal causal) geodesic branching. This once more complements the recent C^0 -inextendibility results of [9, 45].

Next we briefly discuss the further prospects of low-regularity singularity theorems. Concerning the causality part of the results it is expected that they extend to $C^{0,1}$ -metrics², while some features of causality theory fail below this class [5, 12]. Nevertheless, the causal core of some of the singularity theorems has been established in the very general setting of closed cone structures, see [35, Sec. 2.15].

On the analytic side, already in $C^{0,1}$ one faces the problem that the right hand side of the geodesic equation is merely locally bounded and one would have to resort to non-classical solution concepts. Also another analytical technique at the core of the arguments, i.e. the Friedrichs Lemma which is used to go from the distributional energy conditions of the singular metric to useful surrogate conditions for the smooth approximations, appears to be sharp, see [7, Lem. 4.8] for the currently most advanced version. This is actually a long way from the largest possible class where the curvature can be (stably) defined in an analytical (distributional) way, i.e. $H^1 \cap L^\infty$ locally [11, 28, 50]. However, the quest is there to at least go into the direction of regularity classes more closely linked to the classical existence results for the field equations, H^s locally for $s > 5/2$, or to current formulations of cosmic censorship, i.e. the connection lying locally in L^2 .

² A class for which Hawking and Ellis [14, p. 268, 287] still speculate the singularity theorems to hold.

Very recently, also synthetic formulations of the singularity theorems have appeared: In [1], Lorentzian length spaces that are warped products are studied and a Hawking singularity theorem is derived from suitable sectional curvature bounds (implying Ricci curvature bounds in such geometries), based on triangle comparison. Moreover, Cavalletti and Mondino prove a version of Hawking's singularity theorem in Lorentzian metric measure spaces, where Ricci curvature bounds are implemented using methods from optimal transport [4]. However, it remains unclear to date how these results precisely relate to the analytical approach to low regularity pursued in this work.

The Hawking-Penrose theorem classically comes in a causal version [17, p. 538, Thm.], which asserts the incompatibility of the following three conditions for a C^2 -spacetime (M, g) : chronology, the fact that every inextendible causal geodesic stops maximising, and the existence of a (future or past) trapped set. The analytic main result is then a corollary [17, Sec. 3, Cor.] which collects sufficient conditions (energy and genericity conditions, and initial conditions) to derive the above incompatible items under the assumption of causal geodesic completeness. We will generally follow this path, but prove a more general version of the theorem due to Galloway and Senovilla [10], which adds the existence of a trapped submanifold of arbitrary codimension to the list of initial conditions.

Our version of the causal result is Theorem 6.2 and, for the sake of completeness and for the convenience of the reader we collect all results from C^1 -causality theory³ needed here (and elsewhere) in Appendix A. The main result is then Theorem 6.3. The results needed to infer from its analytic conditions (the distributional energy and genericity conditions) and the non-branching assumptions the inexistence of lines are proven in Sect. 4. The existence of a trapped set is derived from the various sets of initial conditions (and the energy conditions) in Sect. 5. Finally, some technical results on extending vector fields in C^1 -spacetimes are collected in Appendix B. They allow us to put to use the distributional genericity condition in the context of regularisations.

To conclude this introduction, we fix the notations and conventions used throughout the paper. By a manifold M we will mean a smooth, connected, second countable Hausdorff manifold of dimension n (with $n \geq 3$). When such an M is endowed with a Lorentzian metric g of regularity C^1 of signature $(-, +, \dots, +)$ and is time-oriented via a smooth timelike vector field we call it a C^1 -spacetime (hence lowering the regularity will always apply to the spacetime metric only, not to the underlying manifold). The Levi-Civita connection of a smooth or C^1 -spacetime will be denoted by ∇ (for its distributional definition in the C^1 -case, see the next section). We will always assume that M is also endowed with a smooth (complete, background) Riemannian metric h with associated Riemannian distance function d_h and norm $\|\cdot\|_h$ which we use to estimate tensor fields. Since all such estimates will be on compact sets only, they are independent of the choice of h .

A curve $\gamma : I \rightarrow M$ is called timelike (causal, null, future or past directed) if it is locally Lipschitz and $\dot{\gamma}(t)$ is timelike (causal, null, future or past directed) almost everywhere. Concerning causality theory we use standard notation, cf. e.g., [34, 39]. Thus $p \ll q$ (resp. $p \leq q$) means that there exists a future directed timelike (resp. causal) curve from p to q , $I^+(A) := \{q \in M : p \ll q \text{ for some } p \in A\}$ and $J^+(A) := \{q \in M : p \leq q \text{ for some } p \in A\}$. These definitions do not change if one defines the causality relations via piecewise smooth curves [24, 31]. The Lorentzian distance (or time-separation function) associated to a Lorentzian metric g will be denoted by d_g . A C^1 -spacetime (M, g) is globally hyperbolic if it is causal (i.e., contains no closed

³ It should be mentioned that these results all follow from the more general approaches of [5, 35, 41].

causal curves) and $J(p, q) := J^+(p) \cap J^-(q)$ is compact for all $p, q \in M$. A Cauchy hypersurface in (M, g) is a closed acausal set that is met by each inextendible causal curve.

For the Riemann curvature tensor we use the convention $R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z$ and the Ricci tensor is given by $\text{Ric}(X, Y) = \sum_{i=1}^n \langle E_i, E_i \rangle \langle R(E_i, X)Y, E_i \rangle$ (again we refer to the next section for a distributional interpretation in the case of C^1 -metrics). Here $(E_i)_{i=1}^n$ is a (local) orthonormal frame field, while by $(e_i)_{i=1}^n$ we will denote frames in individual tangent spaces $T_p M$. For an embedded submanifold N of M of codimension m we define the second fundamental form by $\text{II}(V, W) = \text{nor}(\nabla_V W)$ and the shape operator derived from a normal unit field ν by $S(X) = \nabla_X \nu$. Using (2.13) below it follows that for $g \in C^1$ and $V, W \in C^1$, $\text{II}(V, W)$ is continuous.

For g smooth (resp. C^1), $1 < m < n$ and S a smooth (resp. C^2) spacelike $(n - m)$ -dimensional submanifold let $e_1(q), \dots, e_{n-m}(q)$ be an orthonormal basis for $T_q S$, varying smoothly (resp. C^1) with q in a neighborhood (in S) of $p \in S$. Then $H_S := \frac{1}{n-m} \sum_{i=1}^{n-m} \text{II}(e_i, e_i)$ denotes the mean curvature vector field of S , and $\mathbf{k}_S(\nu) := g(H_S, \nu)$ the convergence of $\nu \in TM|_S$. For g in C^1 , both H_S and \mathbf{k}_S are continuous. A closed spacelike submanifold S is called (*future*) *trapped* if for any future-directed null vector $\nu \in TS^\perp$ the convergence $\mathbf{k}_S(\nu)$ is positive, or equivalently that the mean curvature vector field H_S is past pointing timelike on S .

One final convention we shall require is that of tensor classes (cf. [21, Sec. 4.6.3]): Given a causal geodesic γ in a C^1 -spacetime (M, g) , we set $[\dot{\gamma}(t)]^\perp := (\dot{\gamma}(t))^\perp / \mathbb{R}\dot{\gamma}(t)$. For γ null, $[\dot{\gamma}(t)]^\perp$ is an $(n - 2)$ -dimensional subspace of the hypersurface $(\dot{\gamma}(t))^\perp$. On the other hand, for γ timelike, $[\dot{\gamma}(t)]^\perp$ equals $(\dot{\gamma}(t))^\perp$. To obtain a unified notation, throughout the paper we will denote the dimension of $[\dot{\gamma}(t)]^\perp$ by d , which amounts to setting $d = n - 2$ in the null case and $d = n - 1$ in the timelike case. Furthermore, we set $[\dot{\gamma}]^\perp = \bigcup_t [\dot{\gamma}(t)]^\perp$. Then for every normal tensor field A along γ we obtain a well-defined tensor class $[A]$ along γ , as well as a well-defined tensor class representing the induced covariant derivative $\nabla_{\dot{\gamma}}$. We then set $[\dot{A}] = [\nabla_{\dot{\gamma}} A]$. The metric $g|_{[\dot{\gamma}]^\perp}$ is positive definite in both the null and the timelike case. For smooth metrics, the curvature (or tidal force) operator is given by $[R](t) : [\dot{\gamma}(t)]^\perp \rightarrow [\dot{\gamma}(t)]^\perp, [\nu] \mapsto [R(\nu, \dot{\gamma}(t))\dot{\gamma}(t)]$.

2. Distributional Curvature of C^1 -Spacetimes

2.1. *Curvature tensors of C^1 -metrics.* In what follows we discuss the general distributional framework in which to understand the curvature quantities associated to metric tensors (of arbitrary signature) of regularity below C^2 , as is required for our further analysis. The general mathematical framework goes back to [29], while [11] provided the first study specific to GR. We mainly follow [28], cf. also [13, 49], but put a special emphasis on metrics of regularity C^1 and distributions of finite order.

As in [13, Sec. 3.1], for $k \in \mathbb{N}_0 \cup \{\infty\}$ we denote by $\text{Vol}(M)$ the volume bundle over M , and by $\Gamma_c^k(M, \text{Vol}(M))$ the space of compactly supported one-densities on M (i.e., sections of $\text{Vol}(M)$) that are k -times continuously differentiable. Then the space of distributions of order k on M is the topological dual of $\Gamma_c^k(M, \text{Vol}(M))$,

$$\mathcal{D}'^{(k)}(M) := \Gamma_c^k(M, \text{Vol}(M))'.$$

For $k = \infty$ we omit the superscript (k) . Clearly there are topological embeddings $\mathcal{D}'^{(k)}(M) \hookrightarrow \mathcal{D}'^{(k+1)}(M) \hookrightarrow \mathcal{D}'(M)$ for all k . As we shall see, all curvature quantities

of C^1 -metrics are distributional tensor fields of order 1. The space of distributional (r, s) -tensor fields of order k is defined as

$$\mathcal{D}'^{(k)}\mathcal{T}_s^r(M) \equiv \mathcal{D}'^{(k)}(M, T_s^r M) := \Gamma_c^k(M, T_s^r(M) \otimes \text{Vol}(M))'. \tag{2.1}$$

By [13, 3.1.15],

$$\mathcal{D}'\mathcal{T}_s^r(M) \cong \mathcal{D}'(M) \otimes_{C^\infty(M)} T_s^r(M) \cong L_{C^\infty(M)}(\Omega^1(M)^r \times \mathfrak{X}(M)^s; \mathcal{D}'(M)). \tag{2.2}$$

This algebraic isomorphism ultimately rests on the fact that the $C^\infty(M)$ -module of C^∞ -sections $\Gamma(M, F)$ in any smooth vector bundle $F \rightarrow M$ is finitely generated and projective, which allows one to apply [3, Ch. II §4.2, Prop. 2]. These properties persist if we consider $\Gamma_{C^k}(M, F)$ as a $C^k(M)$ -module for any finite k , so we also have

$$\begin{aligned} \mathcal{D}'^{(k)}\mathcal{T}_s^r(M) &\cong \mathcal{D}'^{(k)}(M) \otimes_{C^k(M)} (T_s^r)_{C^k}(M) \\ &\cong L_{C^k(M)}(\Omega_{C^k}^1(M)^r \times \mathfrak{X}_{C^k}(M)^s; \mathcal{D}'^{(k)}(M)). \end{aligned} \tag{2.3}$$

In other words, distributional tensor fields of order k act as C^k -balanced multilinear maps on one forms and vector fields of regularity C^k to give a scalar distribution of order k . For finite k , the multiplication $C^k(M) \times \mathcal{D}'^{(k)}(M) \rightarrow \mathcal{D}'^{(k)}(M)$ is jointly continuous (w.r.t. the strong topology), whereas for $k = \infty$ it is only hypocontinuous [20, p. 362], but still jointly sequentially continuous [37, p. 233] since $C^\infty(M)$ is Fréchet, hence barrelled. The corresponding continuity properties therefore also hold for the tensor operations introduced above. The isomorphisms (2.2), (2.3) are not topological, but still bornological by [36, Thm. 15] (which remains valid for spaces of C^k -sections with finite).

Smooth, and in fact even L_{loc}^1 -tensor fields are continuously and densely embedded via

$$\begin{aligned} T_s^r(M) &\hookrightarrow \mathcal{D}'^{(k)}\mathcal{T}_s^r(M) \\ t &\mapsto [(\theta_1, \dots, \theta_r, X_1, \dots, X_s) \mapsto [\omega \mapsto \int_M t(\theta_1, \dots, \theta_r, X_1, \dots, X_s)\omega]]. \end{aligned}$$

Here, ω is a one-density. If M is orientable, densities can be canonically identified with n -forms. The fact that $T_s^r(M)$ is dense in $\mathcal{D}'^{(k)}\mathcal{T}_s^r(M)$ uniquely fixes all the operations on distributional tensor fields to be introduced below in a way compatible with smooth pseudo-Riemannian geometry.

For any $t \in T_s^r(M)$ there is a unique extension that accepts one distributional argument. E.g., if $\tilde{\theta}_1 \in \mathcal{D}'\mathcal{T}_1^0(M)$, then since $t(\cdot, \theta_2, \dots, X_s) \in \mathfrak{X}(M)$ we may set

$$t(\tilde{\theta}_1, \theta_2, \dots, X_s) := \tilde{\theta}_1(t(\cdot, \theta_2, \dots, X_s)) \in \mathcal{D}'(M), \tag{2.4}$$

and analogously for the other slots.

Definition 2.1. A *distributional connection* is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathcal{D}'\mathcal{T}_0^1(M)$ satisfying for $X, X', Y, Y' \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$ the usual computational rules: $\nabla_{fX+X'}Y = f\nabla_X Y + \nabla_{X'}Y$, $\nabla_X(Y+Y') = \nabla_X Y + \nabla_X Y'$, $\nabla_X(fY) = X(f)Y + f\nabla_X Y$.

Using (2.4) any distributional connection can be extended to the entire tensor algebra, i.e., to a map $\nabla : \mathfrak{X}(M) \times T_s^r(M) \rightarrow \mathcal{D}'T_s^r(M)$, e.g. for $\Theta \in \Omega^1(M)$ we set

$$(\nabla_X \Theta)(Y) := X(\Theta(Y)) - \Theta(\nabla_X Y) \quad (X, Y \in \mathfrak{X}(M)). \tag{2.5}$$

Let \mathcal{G} be any of the spaces C^k ($0 \leq k$) or L_{loc}^p ($1 \leq p$), then we call a distributional connection a \mathcal{G} -connection if $\nabla_X Y$ is a \mathcal{G} -vector field for any $X, Y \in \mathfrak{X}(M)$. A particularly important case are L_{loc}^2 -connections since they form the largest class that allows for a stable definition of the curvature tensor in distributions, cf. [11, 28, 49].

Using a local frame one easily sees that by virtue of the respective computational rules any \mathcal{G} -connection can be (uniquely) extended to a map $\nabla : \mathfrak{X}_{\mathcal{F}}(M) \times \mathfrak{X}_{\mathcal{H}}(M) \rightarrow \mathfrak{X}_{\mathcal{G}}(M)$, provided that \mathcal{F} and \mathcal{H} are function spaces such that $\mathcal{F} \cdot \mathcal{G} \subseteq \mathcal{G}$, $D\mathcal{H} \subseteq \mathcal{G}$, and $\mathcal{H} \cdot \mathcal{G} \subseteq \mathcal{G}$. Hence, in particular, each L_{loc}^2 -connection extends in this way to a map

$$\nabla : \mathfrak{X}_{C^0}(M) \times \mathfrak{X}_{C^1}(M) \rightarrow \mathfrak{X}_{L_{\text{loc}}^2}(M). \tag{2.6}$$

Moreover, by a similar reasoning, \mathcal{G} -connections allow one to insert even less regular vector fields in the first and second slot at the price of a less regular outcome. In particular, any L_{loc}^2 -connection can be extended to a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}_{L_{\text{loc}}^2}(M) \rightarrow \mathcal{D}'T_0^1(M). \tag{2.7}$$

Using this extension, we have [28, Def. 3.3]:

Definition 2.2. The distributional Riemann tensor of an L_{loc}^2 -connection ∇ is the map $R : \mathfrak{X}(M)^3 \rightarrow \mathcal{D}'T_0^1(M)$,

$$R(X, Y, Z)(\theta) \equiv (R(X, Y)Z)(\theta) := (\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z)(\theta)$$

for $X, Y, Z \in \mathfrak{X}(M)$ and $\theta \in \Omega^1(M)$,

If F_i is a local frame in $\mathfrak{X}(U)$ and $F^j \in \Omega^1(U)$ its dual frame, then the Ricci tensor corresponding to ∇ is given by

$$\text{Ric}(X, Y) := (R(X, F_i)Y)(F^i) \in \mathcal{D}'(U) \quad (X, Y \in \mathfrak{X}(U)). \tag{2.8}$$

Recall the Koszul formula for the Levi-Civita connection of a smooth metric g on M :

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &\quad - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]) =: F(X, Y, Z) \end{aligned} \tag{2.9}$$

Focusing now on the case at hand in this work, suppose that g is merely C^1 (still allowing arbitrary signature). Then $F(X, Y, Z)$ is a well-defined element of $C^0(M)$, and

$$\nabla_X^b Y := Z \mapsto \frac{1}{2} F(X, Y, Z) \in \Omega_{C^0}^1(M) \subseteq \mathcal{D}'T_1^0(M) \tag{2.10}$$

defines the *distributional Levi-Civita connection* of g [28, Def. 4.2]. Note that this is not (yet) a distributional connection in the sense of Definition 2.1 since it is of order $(0, 1)$ instead of $(1, 0)$. In addition to the standard product rules it also possesses the properties (corresponding to [39, Thm. 3.11])

$$\begin{aligned} \nabla_X^b Y - \nabla_Y^b X &= [X, Y]^b, \text{ i.e., } (\nabla_X^b Y - \nabla_Y^b X)(Z) = g([X, Y], Z), \\ X(g(Y, Z)) &= (\nabla_X^b Y)(Z) + (\nabla_X^b Z)(Y) \end{aligned} \tag{2.11}$$

for all $X, Y, Z \in \mathfrak{X}(M)$. These equalities hold distributionally, hence also in C^0 .

In order to obtain an L^2_{loc} -connection from ∇^b (which in turn will allow us to define the curvature tensors we require) we need to raise the index via g , setting

$$g(\nabla_X Y, Z) := (\nabla^b_X Y)(Z) \quad (X, Y, Z \in \mathfrak{X}(M)). \tag{2.12}$$

This defines a continuous vector field $\nabla_X Y$, hence even a C^0 -connection. More generally, by [28, Sec. 4] this procedure yields an L^2_{loc} -connection even for the significantly larger Geroch-Traschen class of metrics ($g \in H^1_{\text{loc}} \cap L^\infty_{\text{loc}}$ and locally uniformly non-degenerate).

The Riemann tensor of $g \in C^1$ is now given by Definition 2.2. Furthermore, since in this case ∇ is a C^0 -connection, it follows that $(R(X, Y)Z)(\theta) \in \mathcal{D}'^{(1)}(M)$, and consequently, $R \in \mathcal{D}'^{(1)}\mathcal{T}_3^1(M)$. Indeed by the same reasoning that led to (2.6), (2.7) we may extend the C^0 -Levi-Civita connection of g to a map

$$\nabla : \mathfrak{X}_{C^0}(M) \times \mathfrak{X}_{C^1}(M) \rightarrow \mathfrak{X}_{C^0}(M), \text{ and } \nabla : \mathfrak{X}_{C^1}(M) \times \mathfrak{X}_{C^0}(M) \rightarrow \mathcal{D}'^{(1)}\mathcal{T}_0^1(M), \tag{2.13}$$

respectively. Here, the second equation shows that the term $([\nabla_X, \nabla_Y]Z)(\theta)$ yields an order one distribution, while $(\nabla_{[X,Y]}Z)(\theta) \in C^0$ since ∇ is a C^0 -connection.

For $W, X, Y, Z \in \mathfrak{X}(M)$ we define $R(W, X, Y, Z) \in \mathcal{D}'^{(1)}(M)$ by

$$\begin{aligned} R(W, X, Y, Z) &:= X(g(W, \nabla_Y Z)) - Y(g(W, \nabla_X Z)) \\ &\quad - g(\nabla_X W, \nabla_Y Z) + g(\nabla_Y W, \nabla_X Z) - g(W, \nabla_{[X,Y]}Z). \end{aligned}$$

Using (2.4), (2.7) and (2.11) it is straightforward to check that cf. [28, Rem. 4.5]

$$R(W, Z, X, Y) = g(W, R(X, Y)Z).$$

Alternatively, one may verify this identity in local coordinates using the fact that there is a well-defined multiplication of distributions of first order with C^1 -functions.

The Ricci tensor of a C^1 metric g is given by (2.8), hence, in particular, is again a distributional tensor field of order 1. Note that by (2.3) this allows us to apply it to C^1 -vector fields, which will be essential in the context of the genericity condition below. Similarly, since the Riemann tensor, again by (2.3), acts on C^1 -vector fields, we may in particular use g -orthonormal frames (which generically are only C^1) to calculate the Ricci tensor, which is also vital below. Explicitly, (2.8) holds for F_i a g -orthonormal frame and F^i its g -dual. All these facts can alternatively be derived directly from the extension of ∇ in (2.13).

Again, by the same observations, we have that the scalar curvature S is a distribution of order 1 (this answers a concern from [7, Sec. 3.2]) and that the standard local formulae hold in $\mathcal{D}'^{(1)}$:

$$\begin{aligned} R^m_{ijk} &= \partial_j \Gamma^m_{ik} - \partial_k \Gamma^m_{ij} + \Gamma^m_{js} \Gamma^s_{ik} - \Gamma^m_{ks} \Gamma^s_{ij}, \\ \text{Ric}_{ij} &= R^m_{imj}, \\ S &= \text{Ric}^i_i. \end{aligned}$$

This shows that the local definitions given in [7, (3.2),(3.3)] are compatible with the global approach developed here.

2.2. *Regularisation.* As in previous works on the generalisation of singularity theorems to metrics of differentiability below C^2 [7, 24–26], an essential tool in our approach will be suitable regularisations, both of the metrics themselves and of the derived distributional curvature quantities. A general regularisation scheme based on chart-wise convolution with a mollifier $\rho \in \mathcal{D}(B_1(0))$, $\int \rho = 1$, $\rho \geq 0$ (cf. [13, 3.2.10], [24, Sec. 2], [7, Sec. 3.3]) is as follows: Cover M by a countable and locally finite family of relatively compact chart neighbourhoods (U_i, ψ_i) ($i \in \mathbb{N}$). Let $(\zeta_i)_i$ be a subordinate partition of unity with $\text{supp}(\zeta_i) \subseteq U_i$ for all i and choose a family of cut-off functions $\chi_i \in \mathcal{D}(U_i)$ with $\chi_i \equiv 1$ on a neighbourhood of $\text{supp}(\zeta_i)$. Finally, for $\varepsilon \in (0, 1]$ set $\rho_\varepsilon(x) := \varepsilon^{-n} \rho(\frac{x}{\varepsilon})$. Then denoting by f_* (respectively f^*) push-forward (resp. pull-back) of distributions under a diffeomorphism f , for any $\mathcal{T} \in \mathcal{D}'T'_s(M)$ consider the expression

$$\mathcal{T} \star_M \rho_\varepsilon(x) := \sum_i \chi_i(x) \psi_i^* \left((\psi_i \circ \zeta_i \cdot \mathcal{T}) * \rho_\varepsilon \right)(x). \tag{2.14}$$

Here, $\psi_i \circ \zeta_i \cdot \mathcal{T}$ is viewed as a compactly supported distributional tensor field on \mathbb{R}^n , so componentwise convolution with ρ_ε yields a smooth field on \mathbb{R}^n . The cut-off functions χ_i secure that $(\varepsilon, x) \mapsto \mathcal{T} \star_M \rho_\varepsilon(x)$ is a smooth map on $(0, 1] \times M$. For any compact set $K \Subset M$ there is an ε_K such that for all $\varepsilon < \varepsilon_K$ and all $x \in K$ (2.14) reduces to a finite sum with all $\chi_i \equiv 1$ (hence to be omitted from the formula), namely when ε_K is less than the distance between the support of $\zeta_i \circ \psi_i^{-1}$ and the boundary of $\psi_i(U_i)$ for all i with $U_i \cap K \neq \emptyset$.

Just as is the case for smoothing via convolution in the local setting, $\mathcal{T} \star_M \rho_\varepsilon$ converges to \mathcal{T} as $\varepsilon \rightarrow 0$ in $\mathcal{D}'T'_s$, and indeed in C^k_{loc} or $W^{k,p}_{\text{loc}}$ ($p < \infty$) if \mathcal{T} is contained in these spaces [7, Prop. 3.5].

For a given C^1 Lorentzian metric g we will in addition require concrete regularisations that are adapted to the causal structure induced by g . This construction goes back to Chrusciel and Grant [5, Prop. 1.2], cf. also [24, Prop. 2.5]. The version we will use is [7, Lem. 4.2, Cor. 4.3]:

Lemma 2.3. *Let (M, g) be a C^1 -spacetime. Then for any $\varepsilon > 0$ there exist smooth Lorentzian metrics $\check{g}_\varepsilon, \hat{g}_\varepsilon$ on M , time orientable by the same timelike vector field as g , and satisfying:*

- (i) $\check{g}_\varepsilon < g < \hat{g}_\varepsilon$ for all ε .
- (ii) \check{g}_ε and \hat{g}_ε converge to g in C^1_{loc} as $\varepsilon \rightarrow 0$.
- (iii) $\check{g}_\varepsilon - g \star_M \rho_\varepsilon \rightarrow 0$ in C^∞_{loc} and for any $K \Subset M$ there exist $c_K > 0$ and $\varepsilon_0(K)$ such that $\|\check{g}_\varepsilon - g \star_M \rho_\varepsilon\|_{\infty, K} \leq c_K \varepsilon$ and $\|g - g \star_M \rho_\varepsilon\|_{\infty, K} \leq c_K \varepsilon$ for all $\varepsilon < \varepsilon_0$. An analogous statement holds for \hat{g}_ε as well as for the inverse metrics $g^{-1}, (\check{g}_\varepsilon)^{-1}$ and $(\hat{g}_\varepsilon)^{-1}$.

In this work we will generally use $\check{g}_\varepsilon, \hat{g}_\varepsilon$ to denote the regularisations of Lemma 2.3, which are of utmost importance throughout.

Next we recall the following essential result on convergence of the curvature under regularisations, which is a special case of several so-called stability results, see [11, Thm. 2], [28, Thm. 4.6], [50, Thm. 5.1].

Lemma 2.4. *Let (M, g) be a C^1 -spacetime, and let g_ε be either \check{g}_ε or \hat{g}_ε . Then we have for the Riemann and the Ricci tensor*

$$R[g_\varepsilon] \rightarrow R[g] \text{ and } \text{Ric}[g_\varepsilon] \rightarrow \text{Ric}[g] \text{ distributionally.} \tag{2.15}$$

In particular, the Riemann and the Ricci tensor possess the usual symmetries.

We will frequently need the following result from ODE theory (cf. [15, Ch.2, Thm. 3.2]), which we will mainly use in the context of regularisation given below in Corollary 2.6.

Proposition 2.5. *Let f and f_m , $m \in \mathbb{N}$ be continuous functions on an open set $E \subseteq \mathbb{R} \times \mathbb{R}^n$ with $f_m \rightarrow f$ in C_{loc}^0 . Let $y_m = y_m(t)$ be a solution to the initial value problem*

$$y'_m(t) = f_m(t, y_m(t)), \quad y'_m(t_m) = y_{m_0}$$

with data $(t_m, y_{m_0}) \in E$ and maximal interval of existence (ω_m^-, ω_m^+) . Suppose that the sequence of initial data converges,

$$(t_m, y_{m_0}) \rightarrow (t_0, y_0) \in E.$$

Then there exist a solution $y = y(t)$ for the initial value problem

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0,$$

defined on a maximal interval (ω^-, ω^+) , and a subsequence m_l with the property that for any $s_1 < s_2$ with $(s_1, s_2) \subseteq (\omega^-, \omega^+)$ we have $(s_1, s_2) \subseteq (\omega_{m_l}^-, \omega_{m_l}^+)$ for all large l , and $y_{m_l} \rightarrow y$ in $C_{\text{loc}}^1(\omega^-, \omega^+)$.⁴ In particular,

$$\limsup_m \omega_m^- \leq \omega^- < \omega^+ \leq \liminf_m \omega_m^+.$$

We will mostly use the previous result to conclude that g_ε -geodesics, where $g_\varepsilon \in \{\check{g}_\varepsilon, \hat{g}_\varepsilon\}$, converge uniformly on compact intervals to a g -geodesic:

Corollary 2.6. *Let (M, g) be a C^1 -spacetime and let $g_m = \check{g}_{\varepsilon_m}$ or $g_m = \hat{g}_{\varepsilon_m}$ with $\varepsilon_m \rightarrow 0$. Let $\gamma_m : (a_m, b_m) \rightarrow M$ be inextendible g_m -geodesics. Suppose that $\gamma_m(t_0) \rightarrow p \in M$ and $\dot{\gamma}_m(t_0) \rightarrow v \in T_p M$, where $t_0 \in \bigcap (a_m, b_m)$. Then there exist an inextendible g -geodesic $\gamma : (a, b) \rightarrow M$ with $\gamma(t_0) = p$, $\dot{\gamma}(t_0) = v$ and a subsequence m_l such that whenever $(s_1, s_2) \subseteq (a, b)$, we have $(s_1, s_2) \subseteq (a_{m_l}, b_{m_l})$ for all large l and γ_{m_l} converges in $C_{\text{loc}}^2(a, b)$ to γ . Moreover,*

$$\limsup_m a_m \leq a < b \leq \liminf_m b_m.$$

Proof. As this is a local statement, we may assume $M = \mathbb{R}^n$. We rewrite the geodesic equations for g_m as a first order initial value problem (with Γ_m the Christoffel symbols of g_m) and $1 \leq i \leq n$:

$$\begin{aligned} \dot{\gamma}_m^i(t) &= \eta_m^i(t), \\ \dot{\eta}_m^i &= -(\Gamma_m)^i_{jk}(\gamma_m(t))\eta_m^j(t)\eta_m^k(t), \\ \gamma_m^i(t_0) &= p_m^i, \\ \eta_m^i(t_0) &= v_m^i. \end{aligned}$$

We define the continuous functions $f_m : \mathbb{R}^{2n} \rightarrow \mathbb{R}$, $f_m(x, y) := (y, -(\Gamma_m)^i_{jk}(x)y^j y^k)$, and similarly $f : \mathbb{R}^{2n} \rightarrow \mathbb{R}$, $f(x, y) := (y, -\Gamma_{jk}(x)y^j y^k)$. Then $f_m \rightarrow f$ uniformly on compact subsets of \mathbb{R}^{2n} , and by assumption $(p_m, v_m) \rightarrow (p, v) \in \mathbb{R}^{2n}$. The existence of an inextendible g -geodesic sublimit γ of the sequence γ_m now follows from Proposition 2.5, and the C_{loc}^2 -convergence follows suit. \square

⁴ In [15, Ch.2, Thm. 3.2], convergence is only stated in C_{loc}^0 , but inserting this into the equivalent integral equations immediately yields the claim as given here.

In the case of only future or past inextendibility, an obvious analogue of Corollary 2.6 holds.

Remark 2.7. (Parallel transport) Under the assumptions of Corollary 2.6, let $w_m \in T_{\gamma_m(t_0)}M$, $w_m \rightarrow w \in T_pM$. Then since the ODEs governing parallel transport are linear, there are unique smooth vector fields W_m along γ_m that are global solutions of the corresponding initial value problem, and by the standard results on continuous dependence of solutions on the right hand side and the data, they converge in C^1_{loc} to the unique C^1 -vector field W along γ resulting from g -parallel transport of w along γ .

2.3. Energy conditions. Let us briefly recall the timelike and null energy conditions for C^1 -spacetimes. They are formulated in a way so that they imply corresponding conditions for the curvatures of approximating metrics. If the metric is $C^{1,1}$ or better, then the energy conditions presented here are equivalent to the usual pointwise (or pointwise a.e.) energy conditions. We follow [7, Sec. 3-5], to which we also refer for proofs. We start with the timelike energy condition, whose distributional generalisation is straightforward, cf. [7, Def. 3.3].

Definition 2.8 (*Distributional timelike energy condition*). Let (M, g) be a C^1 -spacetime. We say that (M, g) satisfies the *distributional timelike energy condition* if for any timelike $X \in \mathfrak{X}(M)$, $\text{Ric}(X, X) \in \mathcal{D}'^{(1)}(M)$ is a nonnegative scalar distribution on M .

Recall that a scalar distribution $u \in \mathcal{D}'(M)$ is nonnegative, $u \geq 0$, if $u(\omega) \equiv \langle u, \omega \rangle \geq 0$ for all nonnegative test densities $\omega \in \Gamma_c(M, \text{Vol}(M))$. A nonnegative distribution is always a measure [19, Thm. 2.1.7] and hence a distribution of order 0. Moreover, non-negativity is stable with respect to regularisation [19, Thm. 2.1.9]. Finally, for $u, v \in \mathcal{D}'(M)$ we write $u \geq v$ if $u - v \geq 0$.

Remark 2.9. We note that an equivalent formulation of the distributional timelike energy condition consists in imposing that, for any $U \subseteq M$ open and any $X \in \mathfrak{X}(U)$ timelike, $\text{Ric}(X, X) \geq 0$ in $\mathcal{D}'^{(1)}(U)$. Indeed, given a timelike $X \in \mathfrak{X}(U)$, let $V \Subset U$ be open and choose a cut-off function $\chi \in \mathcal{D}(U)$ that equals 1 on a neighbourhood of \bar{V} . By time-orientability of M there exists some timelike $T \in \mathfrak{X}(M)$ of the same time-orientation as X , so $Y := \chi X + (1 - \chi)T$ is a global timelike vector field on M . Then for any non-negative $\omega \in \Gamma_c(M, \text{Vol}(M))$ with support in V we have $0 \leq \langle \text{Ric}(Y, Y), \omega \rangle = \langle \text{Ric}(X, X), \omega \rangle$, and by the sheaf property of $\mathcal{D}'^{(1)}$, we obtain $\text{Ric}(X, X) \geq 0$ in $\mathcal{D}'^{(1)}(U)$.

The usefulness of Definition 2.8 is highlighted by its implications for the curvatures of approximating metrics, cf. [7, Lem. 4.1, Lem. 4.6].

Lemma 2.10. *Let (M, g) be a C^1 -spacetime satisfying the distributional timelike energy condition and let K be compact in M . Then*

$$\forall C > 0 \forall \delta > 0 \forall \kappa < 0 \exists \varepsilon_0 > 0 \forall \varepsilon < \varepsilon_0 \forall X \in TM|_K$$

$$\text{with } g(X, X) \leq \kappa \text{ and } \|X\|_h \leq C : \text{Ric}[\check{g}_\varepsilon](X, X) > -\delta.$$

The following is an immediate consequence of the previous result.

Corollary 2.11. *Let (M, g) be a C^1 -spacetime satisfying the distributional timelike energy condition and let K be compact in M . Then*

$$\forall \delta > 0 \exists \varepsilon_0 > 0 \forall \varepsilon < \varepsilon_0 \forall X \in TM|_K \text{ with } \check{g}_\varepsilon(X, X) = -1 : \text{Ric}[\check{g}_\varepsilon](X, X) > -\delta.$$

Let us now turn to the proper distributional formulation of the null energy condition. The naive approach of defining it in analogy to Definition 2.8 does not yield analogues of Lemma 2.10 and Corollary 2.11 because vector fields that are \check{g}_ε -null are only almost g -null. This necessitates a definition of the following form.

Definition 2.12 (*Distributional null energy condition*, [7, Def. 5.1]). Let (M, g) be a C^1 -spacetime. We say (M, g) satisfies the *distributional null energy condition* if for any compact set K and any $\delta > 0$ there exists $\varepsilon = \varepsilon(\delta, K) > 0$ such that $\text{Ric}(X, X) > -\delta$ (as distributions) for any local smooth vector field $X \in \mathfrak{X}(U)$ ($U \subseteq K$ open) with $\|X\|_h = 1$ and $|g(X, X)| < \varepsilon$ on U .

The above requirement $|g(X, X)| < \varepsilon$ may equivalently be replaced by $\|X - N\|_h < \varepsilon$, where N is a C^1 g -null vector field on U , cf. [7, Lem. 5.2]. Yet another useful equivalent reformulation is the following rescaled version, cf. [7, Def. 5.3].

Lemma 2.13. *A C^1 -spacetime (M, g) satisfies the distributional null energy condition if and only if for any compact $K \subseteq M$, any $c_1, c_2 > 0$ and any $\delta > 0$ there is $\varepsilon = \varepsilon(\delta, K, c_1, c_2) > 0$ such that $\text{Ric}(X, X) > -\delta$ (as distributions) for any local smooth vector field $X \in \mathfrak{X}(U)$, $U \subseteq K$, with $0 < c_1 \leq \|X\|_h \leq c_2$ and $|g(X, X)| < \varepsilon$ on U .*

The following analogue of Lemma 2.10 and Corollary 2.11 shows that the above definition of the null energy condition is the correct one in the C^1 -case (see [7, Lem. 5.5]).

Lemma 2.14. *Let (M, g) be a C^1 -spacetime satisfying the distributional null energy condition. Let $K \subseteq M$ be compact and let $c_1, c_2 > 0$. Then for all $\delta > 0$ there is $\varepsilon_0 = \varepsilon_0(\delta, K, c_1, c_2) > 0$ such that*

$$\forall \varepsilon < \varepsilon_0 \forall X \in TM|_K \text{ with } 0 < c_1 \leq \|X\|_h \leq c_2 \text{ and } \check{g}_\varepsilon(X, X) = 0 : \text{Ric}[\check{g}_\varepsilon](X, X) > -\delta.$$

The above distributional versions of the timelike and null energy conditions were successfully used in [7] to prove the singularity theorems of Hawking and of Penrose for C^1 -spacetimes. For our C^1 -proof of the Hawking-Penrose singularity theorem, we need a distributional version of the genericity condition along geodesics.

Definition 2.15 (*Distributional genericity condition*). Let (M, g) be a C^1 -spacetime and let $\gamma : I \rightarrow M$ be a causal geodesic. We say that the *distributional genericity condition* holds at $\gamma(t_0)$, $t_0 \in I$, if there is a neighbourhood U of $\gamma(t_0)$, C^1 -vector fields X, V on U with the property that $(X \circ \gamma)(t) = \dot{\gamma}(t)$, $(V \circ \gamma)(t) \perp \dot{\gamma}(t)$ for all $t \in I$ with $\gamma(t) \in U$, and there exist $c > 0$ and $\delta > 0$ such that for all C^1 -vector fields \tilde{X}, \tilde{V} on U with $\|X - \tilde{X}\|_h < \delta$ and $\|V - \tilde{V}\|_h < \delta$ we have

$$g(R(\tilde{X}, \tilde{V})\tilde{V}, \tilde{X}) > c \text{ in } \mathcal{D}'^{(1)}(U). \tag{2.16}$$

This condition extends the one used in the $C^{1,1}$ -case in [8, Def. 2.2] as we shall discuss in Remark 2.19, below. Again it allows us to derive an approximate genericity condition for regularisations. Note, however, that it will turn out to be essential to use C^1 -vector fields (and their approximations), which can be inserted into $R \in \mathcal{D}'^{(1)}\mathcal{T}_3^1(M)$, by (2.3).

Also the distributional genericity condition appropriately restricts to smaller sets, a technical point which we clarify first.

Remark 2.16. In Definition 2.15 we may shrink the neighbourhood U while retaining the constants c and δ . More precisely, let $U, X, V, c,$ and δ be as in the definition and let $W \Subset U$, then (2.16) holds for all C^1 -vector fields \tilde{X}_W, \tilde{V}_W on W which satisfy $\|X|_W - \tilde{X}_W\|_h < \delta$ and $\|V|_W - \tilde{V}_W\|_h < \delta$. Indeed, by the sheaf property of distributions, it suffices to establish that (2.16) then holds locally around each point $x \in W$. To establish this let $\chi \in \mathcal{D}(W)$ be a plateau function that equals 1 in a neighbourhood W' of x and set $\tilde{X} = X + \chi(\tilde{X}_W - X)$ and likewise for \tilde{V} . Then \tilde{X} and \tilde{V} are C^1 -vector fields on U with $\|X - \tilde{X}\|_h < \delta$ and $\|V - \tilde{V}\|_h < \delta$ (2.16). So we obtain on W'

$$c < g(R(\tilde{X}, \tilde{V})\tilde{V}, \tilde{X}) = g(R(\tilde{X}_W, \tilde{V}_W)\tilde{V}_W, \tilde{X}_W). \tag{2.17}$$

Lemma 2.17. *Let (M, g) be a C^1 -spacetime and let g_ε be either \check{g}_ε or \hat{g}_ε . Suppose γ is a g -causal g -geodesic and assume that the distributional genericity condition holds at $\gamma(t_0)$ (with neighbourhood U , vector fields X, V and constants c, δ). Suppose that $X_\varepsilon, V_\varepsilon$ are C^1 -vector fields on U with $X_\varepsilon, V_\varepsilon \rightarrow X, V$ in $C^1_{\text{loc}}(U)$. Then for any compact subset $K \Subset U$ there is $\tilde{\varepsilon} > 0$ such that for all $\varepsilon < \tilde{\varepsilon}$:*

$$g_\varepsilon(R[g_\varepsilon](X_\varepsilon, V_\varepsilon)V_\varepsilon, X_\varepsilon) > \frac{c}{2}$$

on K .

Proof. We may take U to be relatively compact. By assumption, for all C^1 -vector fields \tilde{X}, \tilde{V} on U with $\|X - \tilde{X}\|_h < \delta$ and $\|V - \tilde{V}\|_h < \delta$, we have

$$g(R(\tilde{X}, \tilde{V})\tilde{V}, \tilde{X}) > c \text{ in } \mathcal{D}'^{(1)}(U).$$

Since $X_\varepsilon \rightarrow X$ and $V_\varepsilon \rightarrow V$ in C^1_{loc} , there is $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, $\|X_\varepsilon - X\|_h < \delta$ and $\|V_\varepsilon - V\|_h < \delta$. Let from now on $\varepsilon < \varepsilon_0$. Hence, the assumption gives

$$g(R(X_\varepsilon, V_\varepsilon)V_\varepsilon, X_\varepsilon) > c \text{ in } \mathcal{D}'^{(1)}(U).$$

Since \star_M -convolution with a non-negative mollifier ρ_ε respects positivity, we have

$$g(R(X_\varepsilon, V_\varepsilon)V_\varepsilon, X_\varepsilon) \star_M \rho_\varepsilon > c.$$

We claim that

$$g(R(X_\varepsilon, V_\varepsilon)V_\varepsilon, X_\varepsilon) \star_M \rho_\varepsilon - (g \star_M \rho_\varepsilon)(R[g \star_M \rho_\varepsilon](X_\varepsilon, V_\varepsilon)V_\varepsilon, X_\varepsilon) \rightarrow 0 \text{ in } C^0_{\text{loc}}(U).$$

This is seen by first showing that

$$g(R(X_\varepsilon, V_\varepsilon)V_\varepsilon, X_\varepsilon) \star_M \rho_\varepsilon - g((R[g \star_M \rho_\varepsilon](X_\varepsilon, V_\varepsilon)V_\varepsilon, X_\varepsilon)) \rightarrow 0 \text{ in } C^0_{\text{loc}}(U).$$

The latter locally corresponds to the question whether given a net h_ε of C^1 -functions that converges in C^1_{loc} to $h \in C^1$ and a first order distribution T we can show $(h_\varepsilon T) \star \rho_\varepsilon - h_\varepsilon(T \star \rho_\varepsilon) \rightarrow 0$ in C^0_{loc} . This is shown by arguing along the lines of [7, Lem. 4.8] and using that T can locally be written as a first order distributional derivative of a C^0 -function. By a similar argument (noting the fast convergence speed of $g \star_M \rho_\varepsilon \rightarrow g$), the right hand side above is C^0_{loc} -equivalent (i.e. the difference goes to 0 in C^0_{loc}) to

$$(g \star_M \rho_\varepsilon)(R[g \star_M \rho_\varepsilon](X_\varepsilon, V_\varepsilon)V_\varepsilon, X_\varepsilon).$$

This is C_{loc}^0 -equivalent to

$$(g \star_M \rho_\varepsilon)(R[g \star_M \rho_\varepsilon](X_\varepsilon, V_\varepsilon)V_\varepsilon, X_\varepsilon)$$

by the same arguments as in [7, Lem. 4.6] and uniform boundedness of $X_\varepsilon, V_\varepsilon$.

Next, we show that

$$(g \star_M \rho_\varepsilon)(R[g \star_M \rho_\varepsilon](X_\varepsilon, V_\varepsilon)V_\varepsilon, X_\varepsilon) - g_\varepsilon(R[g_\varepsilon](X_\varepsilon, V_\varepsilon)V_\varepsilon, X_\varepsilon) \rightarrow 0$$

in $C_{\text{loc}}^0(U)$. Upon writing everything out in coordinates, we see that only terms of the form

$$(g \star_M \rho_\varepsilon)\partial\Gamma[g \star_M \rho_\varepsilon]X_\varepsilon V_\varepsilon V_\varepsilon X_\varepsilon - g_\varepsilon\partial\Gamma[g_\varepsilon]X_\varepsilon V_\varepsilon V_\varepsilon X_\varepsilon$$

are of interest, because for any of the other terms in the difference, both summands converge in $C_{\text{loc}}^0(U)$ to the respective expressions for g . Since the $X_\varepsilon, V_\varepsilon$ are uniformly bounded, we may omit them in our estimates without loss of information. But for terms as above, we only need to consider those expressions where we have second derivatives of the metrics, since $\partial_j(g \star_M \rho_\varepsilon)^{mk} \rightarrow \partial_j g^{mk}$ in $C_{\text{loc}}^0(U)$ and similarly for $\partial_j(g_\varepsilon)^{mk}$. Hence, we work in a chart, fix a compact set K and are left with estimating the following expression on K :

$$\begin{aligned} & |(g \star_M \rho_\varepsilon)_{bc}(g \star_M \rho_\varepsilon)^{mk} \partial_s \partial_r (g \star_M \rho_\varepsilon)_{ij} - (g_\varepsilon)_{bc}(g_\varepsilon)^{mk} \partial_s \partial_r (g_\varepsilon)_{ij}| \\ & \leq |(g \star_M \rho_\varepsilon)_{bc}(g \star_M \rho_\varepsilon)^{mk}| \cdot |\partial_s \partial_r (g \star_M \rho_\varepsilon)_{ij} - \partial_s \partial_r (g_\varepsilon)_{ij}| + II. \end{aligned}$$

The first term goes to 0 uniformly on K by Lemma 2.3(ii). We continue estimating the remaining term II :

$$\begin{aligned} II & = |(g \star_M \rho_\varepsilon)_{bc}(g \star_M \rho_\varepsilon)^{mk} \partial_s \partial_r (g_\varepsilon)_{ij} - (g_\varepsilon)_{bc}(g_\varepsilon)^{mk} \partial_s \partial_r (g_\varepsilon)_{ij}| \\ & \leq |\partial_s \partial_r (g_\varepsilon)_{ij}| \cdot |(g \star_M \rho_\varepsilon)_{bc}(g \star_M \rho_\varepsilon)^{mk} - (g_\varepsilon)_{bc}(g_\varepsilon)^{mk}|. \end{aligned}$$

To show that this goes to 0 uniformly on K , it suffices to show that the second factor can be estimated by a constant times ε by the same argument as in the proof of [7, Lem. 4.5]. Since $\|g_\varepsilon - (g \star_M \rho_\varepsilon)\|_{\infty, K}$ and $\|g_\varepsilon^{-1} - (g \star_M \rho_\varepsilon)^{-1}\|_{\infty, K}$ can both be estimated by a constant times ε again by Lemma 2.3(iii), this is easily seen to hold.

Hence, there is $\varepsilon_1 > 0$ such that for all $\varepsilon < \min(\varepsilon_0, \varepsilon_1)$,

$$g_\varepsilon(R[g_\varepsilon](X_\varepsilon, V_\varepsilon)V_\varepsilon, X_\varepsilon) > \frac{c}{2} \text{ on } K.$$

□

Next we derive from the distributional genericity condition an estimate on the tidal force operator for the regularised metrics. It is this result which allows us to enter into the Riccati comparison techniques in Sect. 4.

Lemma 2.18. *Let the assumptions of Lemma 2.17 hold and let γ_ε be g_ε -causal g_ε -geodesics, whose g_ε -causal character is the same as the g -causal character of γ and which converge in C_{loc}^1 to γ . Then there are $\tilde{c} > 0, r > 0, \varepsilon_0 > 0$ and a neighbourhood U of $\gamma(t_0)$ such that for any $0 < \varepsilon < \varepsilon_0$ there exist vector fields E_i^ε on U such that $E_i^\varepsilon \circ \gamma_\varepsilon$ constitutes a g_ε -orthonormal frame along γ_ε and such that $E_i^\varepsilon \rightarrow E_i$ in $C_{\text{loc}}^1(U)$, where $E_i \circ \gamma$ is an orthonormal frame along γ . Further, there exists some $C = C(\varepsilon) > 0$ such*

that along γ_ε the ε -tidal force operator $[R_\varepsilon](t) := [R_\varepsilon(\cdot, \dot{\gamma}_\varepsilon(t))\dot{\gamma}_\varepsilon(t)] : [\dot{\gamma}_\varepsilon(t)]^\perp \rightarrow [\dot{\gamma}_\varepsilon(t)]^\perp$ fulfills

$$[R_\varepsilon](t) > \text{diag}(\tilde{c}, -C, \dots, -C) \text{ on } [t_0 - r, t_0 + r] \tag{2.18}$$

in terms of the frame E_i^ε and where we have used the shorthand R_ε for $R[g_\varepsilon]$.

Proof. Let w.l.o.g. $t_0 = 0$. We deal with the case of γ being null or timelike simultaneously and construct a g -orthonormal frame for $[\dot{\gamma}]^\perp$ along γ . First note that, if γ is timelike, $V \circ \gamma$ is nowhere proportional to $\dot{\gamma}$, but the same may not be true in the null case. First we will show that we can replace V by a vector field whose restriction to γ is parallel in $[\dot{\gamma}]^\perp$ and still retain (2.16).

There must exist some t such that $V \circ \gamma(t) \notin \text{span}(\dot{\gamma}(t))$. Indeed, suppose to the contrary that $V(\gamma(t)) = f(t) \cdot X(\gamma(t))$ for some real-valued C^1 function f and all t . Then extending both $X \circ \gamma$ and $V \circ \gamma$ in a cylindrically constant fashion as given by Lemma B.1, we obtain vector fields \tilde{V}, \tilde{X} on a neighbourhood W of $\gamma(0)$ that are proportional and can be made arbitrarily close to V resp. X in C^0 when shrinking W since they restrict to the same vector fields on γ . According to Definition 2.15 they therefore satisfy (2.16) close to $\gamma(0)$. However, by the symmetry properties of R (cf. Lemma 2.4) and the proportionality of \tilde{V} and \tilde{X} we must have $R(\tilde{X}, \tilde{V})\tilde{V} = 0$ in $\mathcal{D}'(W)$, a contradiction.

Hence there exists some $v := V(t_1)$ with $v \in [\dot{\gamma}(t_1)]^\perp$. Defining V_1 as the parallel translate of v along γ we obtain $V_1 \in [\dot{\gamma}]^\perp$ along all of γ . By the same continuity argument as above, (2.16) then still holds for the vector fields V_1 and X (note that a t_1 as above must exist arbitrarily close to 0). In the timelike case we may assume that γ is parametrised to unit speed, and we set $e_n := \dot{\gamma}(0)$. If γ is null we can write $\dot{\gamma}(0) = e_{n-1} + e_n$ for orthonormal vectors e_{n-1}, e_n with e_n timelike.

In the null case, extend $e_1 := V_1(0) \in [\dot{\gamma}(0)]^\perp$ to a g -orthonormal basis $\{e_i\}_{i=1}^n$ (with e_{n-1}, e_n as above) and define E_i to be the g -parallel translate of e_i along γ (so that $E_1 = V_1$ and $X = E_{n-1} + E_n$ along γ). For γ timelike we also construct an orthonormal frame $E_1 = V_1, E_2, \dots, E_n = X$ along γ via parallel transport. As $\gamma_\varepsilon \rightarrow \gamma$ in C^1_{loc} , we can find a g_ε orthonormal basis $\{e_i^\varepsilon\}$ at $\gamma_\varepsilon(0)$ with $e_i^\varepsilon \rightarrow e_i$ in TM , and such that $\dot{\gamma}_\varepsilon(0) = e_n^\varepsilon$ if γ_ε is g_ε -timelike, while $\dot{\gamma}_\varepsilon(0) = e_{n-1}^\varepsilon + e_n^\varepsilon$ with e_n^ε timelike if γ_ε is g_ε -null. For every ε let E_i^ε be the g_ε -parallel translate of e_i^ε along γ_ε . By Remark 2.7, $E_i^\varepsilon \rightarrow E_i$ in C^1_{loc} (i.e. in $T(TM)$, uniformly on compact time intervals) and we can use Lemma B.2 to extend E_i^ε to vector fields on U , converging in $C^1_{\text{loc}}(U)$ to the extensions of E_i to U .

As we have now found vector fields E_i^ε that restrict to a g_ε -frame along γ_ε and converge to the g -frame E_i along γ , we are in a position to apply Lemma 2.17 and hence estimate the g_ε -tidal force operator along γ_ε .

For the remainder of the proof we roughly follow the layout of the proof of [8, Prop. 3.6]. There, for a $C^{1,1}$ -metric g the statement is first shown for the g -tidal force operator along γ and then carried over for small ε by an approximation argument. However, in the C^1 -setting this is no longer possible, as R is now merely distributional and for a matrix with distributional entries eigenvalues are not defined. Also, a direct approximation procedure of that form is no longer feasible. Instead we use Lemma 2.17, and argue separately for any ε , making the procedure (and hence the constant C) dependent on ε^5 . More precisely:

Let c be the constant given by the genericity condition and let $\tilde{c} := \frac{c}{2}$ and $0 < c_1 < \tilde{c}$. Set $R_{ij}^\varepsilon := \langle R_\varepsilon(E_i^\varepsilon, X_\varepsilon)X_\varepsilon, E_j^\varepsilon \rangle$, $i, j = 1, \dots, d$, where $X_\varepsilon = E_n^\varepsilon$ in the timelike case

⁵ This is, however, sufficient in our proof of the main result.

and $X_\varepsilon = E_{n-1}^\varepsilon + E_n^\varepsilon$ in the null case. As in the proof of (3.5) in [8, Lemma 3.7], we choose $C(\varepsilon) > 0$ such that $\lambda_{\min}^\varepsilon(x) \geq \frac{\|(R_{ij}^\varepsilon)_{j=2}^d\|_\varepsilon}{\tilde{c} - c_1}$ for all $x \in U$, where $\lambda_{\min}^\varepsilon(x)$ is the smallest eigenvalue of $(R_{ij}^\varepsilon)_{i,j=2}^d + C(\varepsilon)\text{Id}_{d-1}$ at $x \in U$. By Lemma 2.17 there exists some $\tilde{\varepsilon} > 0$ such that $R_{11}^\varepsilon > \tilde{c}$ for all $0 < \varepsilon < \tilde{\varepsilon}$ on U (shrinking U once more if necessary). For these ε it then follows that $R_{ij}^\varepsilon - \text{diag}(c_1, -C(\varepsilon), \dots, -C(\varepsilon))$ is positive definite on U . Then picking $r > 0$ such that $\gamma_\varepsilon([-r, r]) \subseteq U$ for all ε small, evaluation along γ_ε gives the claim. \square

Remark 2.19. Note that while Definition 2.15 may at first sight appear to be stronger than a straightforward generalisation of the genericity condition given for $C^{1,1}$ -metrics in [8], due to the local boundedness of R in that regularity it is actually equivalent:

For $C^{1,1}$ -metrics Definition 2.15 clearly implies Definition 2.2 of [8]. For the reverse implication, let X, V be continuous vector fields in a neighbourhood of $\gamma(t_0)$ such that $X(\gamma(t)) = \dot{\gamma}(t)$ and $V(\gamma(t)) \in (\dot{\gamma}(t))^\perp$ for all t and such that $\langle R(V, X)X, V \rangle > c > 0$. Arguing exactly as in the proof of Lemma 2.18 we can then use parallel transport and cylindrically constant extension to find C^1 -vector fields that are uniformly close to X and V . It therefore suffices to show that the genericity condition from [8] is stable under locally uniform perturbation. Thus let \tilde{V}, \tilde{X} be C^1 vector fields that are uniformly close to V, X . Then one can locally in a coordinate system estimate $|\tilde{V}_i - V_i| \leq \delta, |\tilde{X}_i - X_i| \leq \delta$ for some small δ . Consequently,

$$\langle R(\tilde{V}, \tilde{X})\tilde{X}, \tilde{V} \rangle = g_{ij}R_{klm}^i \tilde{V}^l \tilde{X}^m \tilde{X}^k \tilde{V}^j \geq g_{ij}R_{klm}^i V^l X^m X^k V^j + \mathcal{O}(\delta)$$

where the last term collects all δ -contributions and takes into account the local boundedness of R . Consequently, choosing δ small enough (and appropriately shrinking the neighbourhood of $\gamma(t_0)$) we can secure that the sum still remains $> c/2$.

3. Branching

In metric geometry, where geodesics are defined as (local) minimisers of the length functional, local uniqueness of geodesics is expressed in the form of a *non-branching* condition. Here, a branch point is defined as an element of a minimiser at which the curve splits into two minimisers that on some positive parameter interval do not have another point in common, cf., e.g., [44,51]. Similarly, in the synthetic Lorentzian setting [22], the role of causal geodesics is taken on by maximising causal curves, and non-branching is formulated analogously. In both cases, lower synthetic sectional curvature bounds (formulated via triangle comparison in constant curvature model spaces) imply non-branching [22,44]. Synthetic Ricci curvature bounds in metric measure spaces (curvature-dimension conditions), on the other hand, do not imply non-branching of geodesics [38]. Similarly, in recent work on synthetic Ricci curvature bounds in Lorentzian pre-length spaces [4], a timelike non-branching condition is required in addition to a timelike-curvature dimension condition to obtain a version of the Hawking singularity theorem.

In the C^1 -setting we are concerned with in this paper, the coincidence between causal local maximisers and geodesics that is familiar from smooth Lorentzian geometry ceases to hold (cf. Example 3.2 below), although it is still true that causal maximisers are geodesics (Lemma A.4). In addition, one generically cannot expect unique local solvability of the geodesic initial value problem. Nevertheless, on physical grounds, it seems reasonable to assign a privileged role to causal geodesics that *are* locally maximising.

We therefore introduce a (weak notion of a) non-branching condition that is intended to preclude locally maximising geodesics from branching (where we do not require the second branch to be maximising as well):

Definition 3.1 (*Non-branching conditions*). Let (M, g) be a C^1 -spacetime. A geodesic $\gamma : [a, b] \rightarrow M$ branches at $t_0 \in (a, b)$ if there exist $\varepsilon > 0$ and some geodesic σ with $\gamma|_{[t_0-\varepsilon, t_0]} \subseteq \sigma$, but $\gamma|_{(t_0, t_0+\varepsilon)} \cap \sigma = \emptyset$. (M, g) is called maximally causally (resp. timelike, resp. null) non-branching (MCNB, MTNB, MNNB), if no maximal causal (resp. timelike, resp. null) geodesic branches in the above sense.

Example 3.2. A C^1 -spacetime in which maximal causal branching occurs can be constructed from the second Riemannian example given in [16]. There, a $C^{1,\alpha}$ -Riemannian metric ($\alpha < 1$) in the (u, v) -plane is given with non-unique solutions to the geodesic initial value problem starting at $\{v = 0\}$. Further it is shown that the geodesic boundary value problem for geodesics γ starting at the surface $\{v = 0\}$ is uniquely solvable and hence such geodesics are at least initially minimizing: If they were not, by properties of any C^1 -Riemannian manifold as a locally compact length space, there would exist a minimiser σ between two points on γ . However in C^1 -Riemannian manifolds minimisers are geodesics and by uniqueness (of the boundary value problem) $\gamma = \sigma$ (cf. [42] for proofs and references).

In [18] it is shown that causal geodesics in static spacetimes are maximising if and only if their Riemannian parts are minimizing and so one can easily construct both maximising timelike and null geodesics that branch.

Lemma 3.3. *If the C^1 -spacetime (M, g) is MCNB (resp. MTNB, MNNB), then any two maximal causal (resp. timelike, null) geodesics that meet tangentially at an interior point must coincide on their maximal domain of definition.*

Proof. We only show this for MCNB spacetimes, the other cases are proven in exactly the same way by replacing all occurrences of “causal” with “timelike” resp. “null”. Let γ, σ be two maximising, causal geodesics meeting tangentially at p , which is not an endpoint of either curve. W.l.o.g. we may assume that $\gamma(0) = \sigma(0) = p$. It suffices to show that there is $\varepsilon > 0$ such that $\gamma|_{[0, \varepsilon)} = \sigma|_{[0, \varepsilon)}$. Indeed since then γ and σ meet tangentially at $t = \varepsilon$, the maximal such interval coincides with the maximal domain of definition (to the right and analogously to the left).

Assume to the contrary that $\gamma|_{[0, \varepsilon)} \neq \sigma|_{[0, \varepsilon)}$ for any ε . As the curves both satisfy the geodesic equation at p and their tangents agree, so do their second derivatives. Thus γ and $\gamma|_{(-\varepsilon, 0]} \cup \sigma|_{[0, \varepsilon)}$ are both (unbroken) geodesics and by MCNB γ cannot branch at $t = 0$. So for every $\varepsilon > 0$ we have $\gamma|_{(0, \varepsilon)} \cap \sigma|_{(0, \varepsilon)} \neq \emptyset$. Hence there exist $t_k \downarrow 0$ such that $\gamma(t_k) = \sigma(t_k)$, and by our indirect assumption there exists $s_k \downarrow 0$ such that $\gamma(s_k) \neq \sigma(s_k)$. For any k set

$$\eta_k := \sup\{a \in [0, s_k] : \gamma|_{(s_k-a, s_k]} \cap \sigma|_{(s_k-a, s_k]} = \emptyset\}.$$

Further choose some $\tilde{\eta}_k > 0$ such that for $I_k := (s_k - \eta_k, s_k + \tilde{\eta}_k)$ we have $\gamma|_{I_k} \cap \sigma|_{I_k} = \emptyset$. Since $t_k \downarrow 0$ and $\gamma(t_k) = \sigma(t_k)$, we know that $s_k > \eta_k$. However, $\gamma(s_k - \eta_k) = \sigma(s_k - \eta_k)$ by our choice of η_k . Set $s_k - \eta_k = r_k$ and for simplicity of notation assume $r_k < t_k$. If $\dot{\gamma}(r_k) \neq \dot{\sigma}(r_k)$, the curve $\alpha := \gamma|_{[0, r_k]} \cup \sigma|_{[r_k, t_k]}$ is a broken geodesic from p to $\gamma(t_k)$ and hence not maximising by [43, Lem. 3.2], i.e., $d(p, \gamma(t_k)) > L(\alpha)$. Since γ and σ are maximisers we have $d(p, \gamma(r_k)) = L(\gamma|_{[0, r_k]}) = L(\sigma|_{[0, r_k]})$. Consequently,

$$d(p, \sigma(t_k)) = L(\sigma|_{[0, t_k]}) = L(\sigma|_{[0, r_k]}) + L(\sigma|_{[r_k, t_k]}) = L(\alpha) < d(p, \sigma(t_k)),$$

a contradiction, and hence $\dot{\gamma}(r_k) = \dot{\sigma}(r_k)$.

Again employing the geodesic equation, also the second derivatives of γ and σ coincide at r_k , and so the curves $\gamma|_{[0, s_k]}$ and $\gamma|_{[0, r_k]} \cup \sigma|_{[r_k, s_k]}$ display maximal causal branching, a contradiction. \square

The following result shows that in maximally causally non-branching spacetimes, maximal causal geodesics can always be approximated by geodesics for the regularised spacetimes \hat{g}_ε and \check{g}_ε . Whether such a result is true in general is questionable and very likely regularisation dependent.

Proposition 3.4. *Let (M, g) be a C^1 -spacetime that is MCNB.*

- (i) *Suppose that M is globally hyperbolic and let $\varepsilon_k \searrow 0$ ($k \rightarrow \infty$). Set $g_k := \hat{g}_{\varepsilon_k}$ or $g_k := \check{g}_{\varepsilon_k}$ for all k . If $\gamma : [0, a] \rightarrow M$ is a maximising, timelike g -geodesic then for any small $\delta > 0$ there exists a subsequence g_{k_l} of g_k and g_{k_l} -maximising, timelike geodesics γ_l converging in C^1 to $\gamma|_{[0, a-\delta]}$.*
- (ii) *Let $g_k = \check{g}_{\varepsilon_k}$ for all k . If M is causal and $\gamma : [0, a] \rightarrow M$ is a g -maximising null geodesic, then for any small $\delta > 0$, there exists a subsequence g_{k_l} of g_k , $t_l \downarrow 0$ and g_{k_l} -null geodesics $\gamma_l : [t_l, a - \delta] \rightarrow M$ contained in $\partial I^+_{g_{k_l}}(\gamma(0))$ (hence in particular g_{k_l} -maximising), which converge in C^1_{loc} to $\gamma|_{[0, a-\delta]}$.
If, furthermore, M is strongly causal and S is an acausal set such that $\gamma \subseteq E^+(S)$, then even $\gamma_l : [t_l, a - \delta] \rightarrow E^+(S)$ and $\gamma_l(t_l) \in S$ with $\gamma_l(t_l) \rightarrow \gamma(0)$.*

Proof. Fix $\delta \in (0, a)$ and set $p := \gamma(0)$, $q := \gamma(a)$, and $q_\delta := \gamma(a - \delta)$.

- (i) Since γ is timelike, $q_\delta \in I^+(p)$ and without loss of generality we may assume that also $q_\delta \in I^+_k(p)$ for all k . Since g is globally hyperbolic we can also assume w.l.o.g. that g_k are globally hyperbolic as well (see [26, Prop. 2.3 (iv)], which remains valid for $g \in C^1$). Thus there exist g_k -maximising, timelike geodesics γ_k from p to q_δ [41, Prop. 6.5]. By Corollary 2.6 an affinely reparametrised subsequence, without loss of generality γ_k itself, converges in C^1_{loc} to a g -causal geodesic σ . By Lemma A.13 σ is a g -maximising timelike geodesic from p to q_δ , in particular $d(p, q_\delta) = L(\gamma|_{[0, a-\delta]}) = L(\sigma)$. We cannot yet apply Lemma 3.3 since it is not clear whether σ is maximising beyond q_δ . However denote by $\eta := \sigma \cup \gamma|_{[a-\delta, a]}$ the concatenation of σ and γ . Since $d(p, q) = L(\gamma) = L(\gamma|_{[0, a-\delta]}) + L(\gamma|_{[a-\delta, a]}) = L(\sigma) + L(\gamma|_{[a-\delta, a]}) = L(\eta)$, it is a maximising, timelike geodesic from p to q . We can now apply Lemma 3.3 to the curves η and γ , to obtain $\eta = \gamma$ and hence $\sigma = \gamma|_{[0, a-\delta]}$.
- (ii) A similar statement was shown in [43, Cor. 3.5], but we need to adapt some of the main points of the argument to our assumptions.
Choose a sequence of points $q_\delta^k \in \partial I^+_{g_k}(p)$ converging to q_δ . There exist future directed, g_k -null maximising geodesics γ_k ending at q_δ^k and contained in $\partial I^+_{g_k}(p)$. W.l.o.g we may assume their terminal velocities to be h -normalized, so we can apply Corollary 2.6 to obtain a subsequence of γ_k (denoted in the same way) converging to a g -null geodesic σ . Since $\partial I^+_{g_k}(p) \subseteq \overline{I^+(p)}$ (recall that $g_k = \check{g}_{\varepsilon_k}$) we know that $\sigma \subseteq \partial I^+(p)$, since otherwise $q_\delta \in I^+(p)$, hence σ is maximising (cf. Lemma A.15). Note that γ and σ meet tangentially at q_δ , because otherwise $\eta := \sigma \cup \gamma|_{[a-\delta, a]}$ would not be maximising by Lemma A.4 and so $q \in I^+(\sigma) \subseteq I^+(\overline{I^+(p)}) \subseteq I^+(p)$, a contradiction. Lemma 3.3 now shows that $\gamma = \eta$ and so $\sigma = \gamma$, wherever both curves are defined. There are two possibilities for σ , either it is past inextendible or it reaches p . However, if it is past inextendible, we must

have $\sigma \supseteq \gamma|_{[0, a-\delta]}$. Hence in both cases the γ_k converge to γ in C^1_{loc} . Further this means that there are $t_k \in [0, a - \delta]$ with $\gamma_k(t_k) \rightarrow p$, and we have $t_k \rightarrow t_0 = 0$ since otherwise $\gamma(t_0) = p = \gamma(0)$, contradicting the causality of M . Now assume M to be strongly causal and $\gamma \subseteq E^+(S)$ for S acausal, then there exists a g -causally convex, g -globally hyperbolic neighbourhood U of p by [32, Lem. 3.21] (this result holds also for C^1 -spacetimes). In such neighbourhoods it holds that $I^+_U(S) = I^+(S) \cap U$, $J^+_U(S) = J^+(S) \cap U$ and hence also $E^+_U(S) = \partial I^+_U(S) = \partial I^+(S) \cap U$. Clearly U is also g_k -causally convex and g_k -globally hyperbolic and so these equalities also hold for g_k . Note that in the same fashion as for the case of a single point p one can show that $\gamma_k \rightarrow \gamma|_{[0, a-\delta]}$. If γ_k reaches S , we immediately obtain $\gamma_k \subseteq E^+_k(S)$. In the other case $\gamma_k \subseteq \partial I^+_k(S) \setminus J^+_k(S)$ is past inextendible. However, as they converge to $\gamma|_{[0, a-\delta]}$, there must be $p_k \in \gamma_k$, with $p_k \rightarrow p$. Hence, w.l.o.g. $p_k \in \partial I^+_k(S) \cap U = E^+_{U, k}(S)$ and so $p_k \in J^+_k(S)$, a contradiction. We have shown that γ_k reach S , and it only remains to show that they do so close to 0. Again let $\gamma_k(t_k) \in S$ with $\gamma_k(t_k) \rightarrow p$. If $t_k \rightarrow t_0$, then $\gamma(t_0) \in S$ and by acausality of S we must have $t_0 = 0$. \square

Remark 3.5. (i) As the proof of Proposition 3.4 shows, for (i) it suffices to assume (M, g) to be MTNB, while for (ii) MNNB would suffice.
 (ii) We will make use of Proposition 3.4(ii) in two different scenarios: Thm. 4.3 and Proposition 5.5 below. In the first case (M causal) all we need to ensure is that γ_l are maximizing for long enough, i.e. $a - \delta - t_l$ is close to a . In the second (M strongly causal) it is important to also obtain that γ_l reaches S , in order to be able to use the initial conditions assumed on S .

4. No Lines

In this section we will prove that, under suitable causality and energy conditions, complete causal geodesics stop being maximising also in (MCNB) C^1 -spacetimes. To this end we first prove the existence of conjugate points for causal geodesics in smooth spacetimes under weakened versions of the energy conditions. More precisely we will invoke the energy conditions derived for regularisations in Lemma 2.10 and in Lemma 2.14 from the distributional energy conditions, as well as in Lemma 2.18 from the distributional genericity condition. The following is a strengthened version of [8, Prop. 4.2]:

Lemma 4.1. *Let (M, g) be a smooth spacetime. Given some $c > 0$ and $0 < r < \frac{\pi}{4\sqrt{c}}$, there exist $\delta(c, r) > 0$ and $T(c, r) > 0$ such that any causal geodesic γ defined on $[-T, T]$ for which the following hold*

- (i) $\text{Ric}(\dot{\gamma}, \dot{\gamma}) \geq -\delta$ on $[-T, T]$ and
- (ii) *there exists a smooth parallel orthonormal frame for $[\dot{\gamma}]^\perp$ and some $C > 0$ such that w.r.t. this frame the tidal force operator satisfies $[R](t) > \text{diag}(c, -C, \dots, -C)$ on $[-r, r]$,*

possesses a pair of conjugate points on $[-T, T]$.

Proof. This actually follows by a more careful choice of constants in the proof of [8, Prop. 4.2]. To see this we briefly recall the main steps of that proof. The argument proceeds indirectly, assuming that for any $\delta > 0$ and $T > 0$ there is some γ satisfying (i) and (ii) without conjugate points in $[-T, T]$. Denote by $[A]$ the unique Jacobi tensor class along γ with $[A](-T) = 0$ and $[A](0) = \text{id}$. Taking $[E_1](t), \dots, [E_d](t)$ as

in (ii), linear endomorphisms of $[\dot{\gamma}]^\perp$ are written as matrices in this basis, and we set $[\tilde{R}](t) := \text{diag}(c, -C, \dots, -C)$ with C as in (ii). Then by (ii), $[\tilde{R}](t) < [R](t)$ on $[-r, r]$.

The self-adjoint operator $[B] := [\dot{A}] \cdot [A]^{-1}$ satisfies the matrix Riccati equation

$$[\dot{B}] + [B]^2 + [R] = 0, \tag{4.1}$$

and we denote by $[\tilde{B}]$ the solution to (4.1), with $[\tilde{R}]$ instead of $[R]$ and initial value prescribed at some $t_1 \in [-r, r]$. We may even choose $t_1 \in [-r, 0]$ and $[\tilde{B}](t_1) := \tilde{\beta}(t_1) \cdot \text{id}$, where $\tilde{\beta}(t_1)$ is greater or equal than the largest eigenvalue of $[B](t_1)$. More precisely, examining the Raychaudhuri equation

$$\dot{\theta} + \frac{1}{d}\theta^2 + \text{tr}(\sigma^2) + \text{tr}([R]) = 0, \tag{4.2}$$

for the expansion $\theta = \text{tr}([B])$ (with $\sigma = [B] - \frac{1}{d}\theta \cdot \text{id}$), it follows that one can pick $\tilde{\beta}(t_1) := f(v, \delta, r)$ and $[\tilde{B}](t_1) := f(v, \delta, r) \cdot \text{id}$ to indeed achieve that $[B](t) \leq [\tilde{B}](t)$ on $[t_1, r]$. Here, $f(v, \delta, r) = \sqrt{\frac{2v}{r} + \delta + \frac{v}{d}}$ with $v = 4d/T$.

Moreover, due to the fact that both $[\tilde{R}]$ and $[\tilde{B}](t_1)$ are diagonal, the Riccati equation for $[\tilde{B}]$ decouples. Indeed it can be explicitly solved by

$$[\tilde{B}](t) = \frac{1}{d} \text{diag}(H_{c,f}(t), H_{-C,f}(t), \dots, H_{-C,f}(t)),$$

where

$$H_{c,f}(t) = d\sqrt{c} \cot(\sqrt{c}(t - t_1) + \text{arccot}(f/\sqrt{c})),$$

and

$$H_{-C,f}(t) = d\sqrt{C} \tanh(\sqrt{C}(t - t_1) + \text{artanh}(f/\sqrt{C})).$$

From this point on, the proof of [8, Prop. 4.2] proceeds by exclusively analysing the function $H_{c,f}$, and the only requirement on r turns out to be $4r\sqrt{c} < \pi$. Once this is granted, δ and T can be chosen depending only on c to arrive at the desired contradiction. □

Recall that a line is an inextendible causal geodesic maximising the Lorentzian distance between any of its points.

Theorem 4.2. (No timelike lines). *Let (M, g) be a globally hyperbolic, MTNB C^1 -spacetime satisfying the distributional timelike energy condition and the distributional genericity condition along any inextendible timelike geodesic. Then there is no complete timelike line in M .*

Proof. Suppose $\gamma : \mathbb{R} \rightarrow M$ is a complete timelike line. We may assume that distributional genericity holds at $t_0 = 0$ along γ . We approximate g by $\check{g}_\varepsilon < g$, which are hence also globally hyperbolic. By Lemma 2.18 there exist $c > 0, 0 < r < \frac{\pi}{4\sqrt{c}}$ with the following property: For any C^1 -approximation γ_ε of γ by \check{g}_ε -timelike geodesics, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, there is $C = C(\varepsilon) > 0$ so that (ii) in Lemma 4.1 is satisfied for R_ε . Now for the above choice of c and r , pick $\delta > 0$ and $T > 0$ as in Lemma 4.1, and let $\tilde{T} > T$. By Proposition 3.4(i), for the curve $\gamma|_{[-T, \tilde{T}]}$

and $\delta < \tilde{T} - T$, we obtain a subsequence $\check{g}_{\varepsilon_k}$ and $\check{g}_{\varepsilon_k}$ -maximal timelike geodesics γ_k from $\gamma(-T)$ to $\gamma(\tilde{T})$ converging to $\gamma|_{[-T, \tilde{T}]}$ in C^1 .

Let K be a compact neighbourhood of $\gamma([-T, T])$. Since $\gamma_{\varepsilon_k} \rightarrow \gamma$ in $C^1([-T, T])$, there are $k_0 \in \mathbb{N}$, $\tilde{C} > 0$ and $\kappa < 0$ such that for all $k \geq k_0$, we have $\gamma_{\varepsilon_k}([-T, T]) \subseteq K$, $\|\dot{\gamma}_{\varepsilon_k}\|_h \leq \tilde{C}$ and $\check{g}_{\varepsilon_k}(\dot{\gamma}_{\varepsilon_k}, \dot{\gamma}_{\varepsilon_k}) < \kappa$ on $[-T, T]$. Hence by [7, Lem. 4.6] (and the remark preceding it) we have $\text{Ric}_{\varepsilon_k}(\dot{\gamma}_{\varepsilon_k}, \dot{\gamma}_{\varepsilon_k}) \geq -\delta$ on $[-T, T]$ for large k . Therefore, also (i) in Lemma 4.1 is satisfied for γ_{ε_k} , yielding the existence of conjugate points for any γ_{ε_k} (k large) in $[-T, T]$, a contradiction since they are maximising even in the strictly larger interval $[-T, \tilde{T}]$. \square

Theorem 4.3. (No null lines) *Let (M, g) be a causal, MNNB C^1 -spacetime that satisfies the distributional null energy condition and the distributional genericity condition along any inextendible null geodesic. Then there is no complete null line in M .*

Proof. Suppose $\gamma : \mathbb{R} \rightarrow M$ is a complete null line. We will again make use of the approximation \check{g}_ε . Suppose without loss of generality that the distributional genericity condition holds along γ at $t_0 = 0$. By Lemma 2.18 there exist $c > 0$ and $0 < r < \frac{\pi}{4\sqrt{c}}$ satisfying: For any C^1 -approximation γ_ε of γ by \check{g}_ε -null geodesics, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, there is $C = C(\varepsilon) > 0$ so that (ii) in Lemma 4.1 is satisfied for R_ε . Choose $\delta > 0$ and $T > 0$ as in Lemma 4.1 for this pair (c, r) and let $\tilde{T} > T$. Using Proposition 3.4(ii) for the curve $\gamma|_{[-T, \tilde{T}]}$ and $S = \gamma(-T)$ (cf. Remark 3.5(ii)), there exists a subsequence $\check{g}_{\varepsilon_k}$ and $\check{g}_{\varepsilon_k}$ -maximal null geodesics $\gamma_k : [-T, \tilde{T}] \rightarrow M$ converging to $\gamma|_{[-T, \tilde{T}]}$ in C^1 .

Since $\gamma_k \rightarrow \gamma$ in $C^1([-T, T])$, once we choose a compact neighbourhood K of $\gamma([-T, T])$, there are $k_0 \in \mathbb{N}$ and $\tilde{C}_2 > \tilde{C}_1 > 0$ such that for all $k \geq k_0$, we have $\gamma_{\varepsilon_k}([-T, T]) \subseteq K$, $\tilde{C}_1 < \|\dot{\gamma}_k\|_h < \tilde{C}_2$ in $[-T, T]$. Hence, Lemma 2.14 implies that $\text{Ric}_{\varepsilon_k}(\dot{\gamma}_k, \dot{\gamma}_k) > -\delta$ in $[-T, T]$. But then Lemma 4.1 may be invoked to give the existence of conjugate points on $\gamma_k|_{[-T, T]}$ for large k , a contradiction since they were supposed to be maximising even on $[-T, \tilde{T}]$. \square

5. Initial Conditions

The classical Hawking–Penrose theorem for spacetimes of dimension 4 assumes three alternative initial conditions: the existence of a compact, spacelike hypersurface, a trapped (2-)surface, or a “trapped point”, i.e., a point from where all future (or past) null geodesics converge. The second condition was later generalised in [10] to trapped submanifolds of arbitrary codimension m with $1 < m < n = \dim(M)$.

While the first condition does not need any special attention here we will start by generalising the trapped submanifold-case to C^1 -spacetimes. As for $C^{1,1}$ -spacetimes (see [8, Sec. 6.2]), trapped submanifolds are defined in the support sense. However, to show that normal null geodesics emanating from them stop being maximising is more delicate now due to the lack of an exponential map, cf. (the proof of) Proposition 5.5. At the end of this section we will deal with the case of a “trapped point”.

Definition 5.1 (Support submanifolds). Let (M, g) be a C^1 -spacetime and let $S, \tilde{S} \subseteq M$ be submanifolds. We say that \tilde{S} is a future support submanifold for S at $q \in S$ if $\dim S = \dim \tilde{S}$, $q \in \tilde{S}$, and there is a neighbourhood U of q in M such that $\tilde{S} \cap U \subseteq J_U^+(S)$.

Definition 5.2 (Trapped C^0 -submanifolds). Let (M, g) be a C^1 -spacetime of dimension n . We say that a C^0 -submanifold $S \subseteq M$ of codimension $1 \leq m < n$ is a *future trapped submanifold* if it is compact without boundary and for any $p \in S$ there exists a neighbourhood U of p such that $U \cap S$ is achronal in U and moreover S has past pointing timelike mean curvature in the sense of support submanifolds, i.e. for any $q \in S$ there exists a future C^2 -support submanifold \tilde{S} for S at q whose mean curvature vector at q is past-pointing timelike.

Now, to show that lightrays from trapped submanifolds stop maximising, we reduce the problem to a question about smooth approximating metrics, which are understood much better. The essential results that deal with the corresponding situations for smooth metrics are [8, Lem. 6.4] and [7, Lem. 5.6]. To give a precise formulation we first introduce some notation.

Suppose $S \subseteq M$ is a spacelike C^2 -submanifold of codimension $1 < m < n$, $p \in S$ and $\nu \in T_p M$ is a future null normal to S . Let γ be a geodesic with $\gamma(0) = p$, $\dot{\gamma}(0) = \nu$. Let e_1, \dots, e_{n-m} be an ON-basis on S around p . Further let E_1, \dots, E_{n-m} be the parallel translates of $e_1(p), \dots, e_{n-m}(p)$ along γ , which are C^1 since they satisfy the parallel transport equation. We will use this notation for the following results. It will be clear from the context which surface the E_i refer to.

Lemma 5.3. *Let (M, g) be a smooth spacetime and let S be a codimension- m ($1 < m < n$) spacelike C^2 submanifold of M . Let γ be a geodesic starting in S such that $\nu := \dot{\gamma}(0) \in TM|_S$ is a future null normal to S . Suppose that $c := \mathbf{k}_S(\nu) > 0$ and let $b > 1/c$. Then there is $\delta = \delta(b, c) > 0$ such that, if for E_i as described above*

$$\sum_{i=1}^{n-m} g(R(E_i, \dot{\gamma})\dot{\gamma}, E_i) \geq -\delta$$

along γ , then $\gamma|_{[0,b]}$ is not maximising from S , provided that γ exists up to $t = b$.

Lemma 5.4. *Let (M, g) be a smooth spacetime and let S be a codimension-2 spacelike C^2 submanifold of M . Let γ be a geodesic starting in S such that $\nu := \dot{\gamma}(0)$ is a future null normal to S . Suppose that $c := \mathbf{k}_S(\nu) > 0$ and let $b > 1/c$. Then there is $\delta = \delta(b, c) > 0$ such that, if*

$$\text{Ric}(\dot{\gamma}, \dot{\gamma}) \geq -\delta$$

along γ , then $\gamma|_{[0,b]}$ is not maximising from S , provided that γ exists up to $t = b$.

The following result is the C^1 -analogue of [8, Prop. 6.5]. As was the case for the distributional genericity condition, cf. Definition 2.15, also here we have to assume that the distributional curvature condition at hand is stable under C^0 -perturbations of the vector fields involved. This ensures that we can derive the necessary curvature conditions for the smooth approximating metrics in order to be able to use Lemmas 5.3 and 5.4.

Proposition 5.5 (Light rays from a submanifold). *Let (M, g) be a strongly causal, MNNB C^1 -spacetime and let $\tilde{S} \subseteq M$ be a C^2 -spacelike submanifold of codimension $1 < m < n$. Suppose $\mathbf{k}_{\tilde{S}}(\nu) = g(\nu, H_p) > c > 0$ and let $b > 1/c$. Suppose there is a null geodesic γ with $\dot{\gamma}(0) = \nu$, a neighbourhood U of $\gamma|_{[0,b]}$ and C^1 -extensions \bar{E}_i of E_i and \bar{N} of $\dot{\gamma}$ to U such that for each $\delta > 0$ there exists $\eta > 0$ such that for all collections of C^1 -vector*

fields $\{\tilde{E}_1, \dots, \tilde{E}_{n-m}, \tilde{N}\}$ on U with $\|\tilde{E}_i - \bar{E}_i\|_h < \eta$ for all i and $\|\tilde{N} - \bar{N}\|_h < \eta$, we have

$$\sum_{i=1}^{n-m} g(R(\tilde{E}_i, \tilde{N})\tilde{N}, \tilde{E}_i) \geq -\delta \text{ in } \mathcal{D}'^{(1)}(U). \tag{5.1}$$

Then $\gamma|_{[0,b]}$ is not maximising from \tilde{S} .

Note that condition (5.1) localises in a similar manner as the genericity condition. For the latter see Remark 2.16.

Proof. Let $g_\varepsilon = \check{g}_\varepsilon$, $R_\varepsilon = R[\check{g}_\varepsilon]$. Clearly, $\mathbf{k}_{\check{S}}$ is continuous and $\mathbf{k}_{\check{S},\varepsilon} \rightarrow \mathbf{k}_{\check{S}}$ uniformly on compact sets. Hence, there is a neighbourhood V of v in $TM|_{\check{S}}$ and ε_0 such that $\forall \varepsilon \leq \varepsilon_0$ and $\forall v \in V$: $\mathbf{k}_{\check{S},\varepsilon}(v) > c$. Note that g -spacelikeness implies g_ε -spacelikeness. Hence $U \cap \tilde{S}$ is g_ε -spacelike and we may assume that $W := \pi(V)$ is contained in $U \cap \tilde{S}$.

Suppose now to the contrary that $\gamma|_{[0,b]}$ maximises the distance from $\bar{U} \cap \tilde{S}$. Let $1/c < b' < b'' < b$ and let $q = \gamma(b'')$. Find $q_k \in \partial J_k^+(\bar{U} \cap \tilde{S})$ (where $\partial J_k := \partial J_{g_{\varepsilon_k}}$ and $\varepsilon_k \rightarrow 0$) with $q_k \rightarrow q$. By Corollary A.12 there are g_{ε_k} -null geodesics $\gamma_k : I_k \rightarrow M$ (in future directed parametrisation) with $\gamma_k(b'') = q_k$ either intersecting $\bar{U} \cap \tilde{S}$ or past inextendible. We may assume that $\|\dot{\gamma}_k(b'')\|_h$ are all equal to $\|\dot{\gamma}(b'')\|_h$, hence we may assume that the $\dot{\gamma}_k(b'')$ converge to a g -null vector v and the geodesics converge to a g -null geodesic γ_v in C_{loc}^2 by Corollary 2.6. Strong causality allows us to invoke Lemma A.30, so we can find a neighbourhood W of $\gamma(0)$ with $W \subseteq U$ such that $W \cap \tilde{S}$ is acausal in M . By (the proof of) Proposition 3.4(ii) (see also Remark 3.5(i)) we obtain that $\gamma = \gamma_v$, and that (up to picking a subsequence) γ_k reaches \tilde{S} for k large at $t_k \downarrow 0$.

Since any of the g_{ε_k} -geodesics γ_k reaches \tilde{S} for all $\varepsilon_k < \varepsilon_0$ (for some suitable $\varepsilon_0 > 0$) at points $\gamma_k(t_k) \in \tilde{S}$, by C_{loc}^1 convergence of γ_k to γ we also have $\dot{\gamma}_k(t_k) \in V$. Note that, due to $\gamma_k(t_k) \rightarrow \gamma(0)$, we can pick g_{ε_k} -orthonormal bases for $T_{\gamma_k(t_k)}\tilde{S}$ converging to a g -orthonormal basis for $T_{\gamma(0)}\tilde{S}$. Denote by $E_i^{\varepsilon_k}$ and E_i the g_{ε_k} - and g -parallel transports of these bases along γ_k and γ , respectively. By Lemma B.2 there exist C^1 -extensions $\tilde{E}_i^{\varepsilon_k}$ and \tilde{E}_i to the (possibly shrunk) neighbourhood U . By a similar argument one can extend the velocity vector fields along γ_k and γ to all of U , they will be denoted by \tilde{N}^ε and \tilde{N} , respectively.

Assume also without loss of generality that U is relatively compact and that $\gamma_k([t_k, b]) \subseteq U$ for large k . Pick $\delta = \delta(b', c)$ according to Lemma 5.3. Since they restrict to the same vector fields along γ , by shrinking U and by continuity of \bar{E}_i and \tilde{E}_i resp. \bar{N} and \tilde{N} , we can assume these vector fields to be arbitrarily close in $\|\cdot\|_h$ on U . Moreover, by the convergence of $\tilde{E}_i^{\varepsilon_k}$ and $\tilde{N}^{\varepsilon_k}$ to \tilde{E}_i and \tilde{N} , respectively, also these vector fields can be made arbitrarily $\|\cdot\|_h$ -close to \bar{E}_i and \bar{N} , respectively, on U for large k . The assumption of the Proposition therefore implies that

$$\sum_{i=1}^{n-m} g(R(\tilde{E}_i^{\varepsilon_k}, \tilde{N}^{\varepsilon_k})\tilde{N}^{\varepsilon_k}, \tilde{E}_i^{\varepsilon_k}) \geq -\delta/2$$

in $\mathcal{D}'(U)$ for k large. Since $\star_M \rho_{\varepsilon_k}$ respects inequalities, we conclude that also

$$\sum_{i=1}^{n-m} g(R(\tilde{E}_i^{\varepsilon_k}, \tilde{N}^{\varepsilon_k})\tilde{N}^{\varepsilon_k}, \tilde{E}_i^{\varepsilon_k}) \star_M \rho_{\varepsilon_k} \geq -\delta/2$$

on U for large k .

By the same reasoning as in Lemma 2.17 we obtain (once again writing R_{ε_k} for $R[g_{\varepsilon_k}]$)

$$g(R(\tilde{E}_i^{\varepsilon_k}, \tilde{N}^{\varepsilon_k})\tilde{N}^{\varepsilon_k}, \tilde{E}_i^{\varepsilon_k}) \star_M \rho_{\varepsilon_k} - g_{\varepsilon_k}(R_{\varepsilon_k}(\tilde{E}_i^{\varepsilon_k}, \tilde{N}^{\varepsilon_k})\tilde{N}^{\varepsilon_k}, \tilde{E}_i^{\varepsilon_k}) \rightarrow 0$$

uniformly on U as $\varepsilon_k \rightarrow 0, i = 1, \dots, n - m$. But then, for large k ,

$$\sum_{i=1}^{n-m} g_{\varepsilon_k}(R_{\varepsilon_k}(\tilde{E}_i^{\varepsilon_k}, \tilde{N}^{\varepsilon_k})\tilde{N}^{\varepsilon_k}, \tilde{E}_i^{\varepsilon_k}) \geq -\delta.$$

In particular, the above holds along γ_k , so

$$\sum_{i=1}^{n-m} g_{\varepsilon_k}(R_{\varepsilon_k}(E_i^{\varepsilon_k}(t), \dot{\gamma}_k(t))\dot{\gamma}_k(t), E_i^{\varepsilon_k}(t)) \geq -\delta$$

for all $t \in [t_k, b']$, where we may assume that k is so large that all γ_k are defined on $[t_k, b']$.

Due to our choice of δ , Lemma 5.3 implies that, for k sufficiently large, each γ_k stops maximising at parameter $b' + t_k$ at the latest (if γ_k is not g_ε -normal to \tilde{S} , then γ_k stops maximising the distance immediately, see Remark 5.9 below). By construction the γ_k maximize from t_k to b' , hence if k is so large that $b' + t_k < b''$ we obtain a contradiction. □

Remark 5.6. By arguments analogous to those given for the genericity condition, cf. Remark 2.19, namely essentially by boundedness of R on compact sets, one can see that if $g \in C^{1,1}$, then the curvature condition (5.1) in Proposition 5.5 is equivalent to the one given in [8, Prop. 6.5].

Proposition 5.7 (*The case of trapped surfaces*). *Let (M, g) be a strongly causal, MNNB C^1 -spacetime satisfying the distributional null energy condition. Let $\tilde{S} \subseteq M$ be a codimension-2 C^2 -spacelike submanifold such that $\mathbf{k}_{\tilde{S}}(v) > c > 0$, where v is a null normal to \tilde{S} , and let $b > 1/c$. If γ is a null geodesic with $\dot{\gamma}(0) = v$, then $\gamma|_{[0,b]}$ is not maximising from \tilde{S} .*

Proof. The proof is completely analogous to that of Proposition 5.5, with the sole difference that one needs to use Lemma 5.4 instead of Lemma 5.3. Let us give a brief outline: Approximate g again by $g_{\varepsilon_k} \equiv \check{g}_{\varepsilon_k}$ and the compact segment $\gamma|_{[0,b]}$ as before by g_{ε_k} -null geodesics γ_k . Then by Lemma 2.14 (see also the proof of [7, Thm. 5.7]), for all $\delta > 0$ and large k we have $\text{Ric}_{\varepsilon_k}(\dot{\gamma}_k, \dot{\gamma}_k) > -\delta$. Using this, combined with Lemma 5.4 and proceeding as in Proposition 5.5, one arrives at a contradiction as before. □

Remark 5.8. The corresponding $C^{1,1}$ -version of Proposition 5.7 was dealt with in [8, Remark 6.6(i)], however, a similar line of reasoning is not available in our case of C^1 -metrics: While it is still possible to extend an ON-frame along a curve to a neighbourhood, this does not suffice to derive condition (5.1) of Proposition 5.5, as the needed rigidity of the condition cannot be guaranteed. More precisely: For any ON-frame $\{\bar{E}_i, \bar{N}\}$ on U the equality (as before \bar{N} denotes the extension of γ' in the frame)

$$\sum_{i=1}^{n-m} g(R(\bar{E}_i, \bar{N})\bar{N}, \bar{E}_i) = \text{Ric}[g](\bar{N}, \bar{N}) \geq -\delta$$

still holds and the final inequality is due to the distributional null energy condition. However, to carry out a proof similar to the one of Proposition 5.5, this estimate would need to hold for all collections of vector fields close to the frame in the sense given in the Proposition. This cannot be guaranteed for C^1 -metrics anymore as R is, in general, unbounded. So instead, the slightly different focusing result used in Proposition 5.7 is required.

Remark 5.9. If, in the situation of Proposition 5.5, ν is null but not a null normal, then geodesics in that direction immediately enter I^+ : This is shown in the same way as in the smooth case, e.g. by using ideas from [39, Lem. 10.45, Lem. 10.50]: Let c be a geodesic in direction ν , hence in particular C^2 . By [43, Lem. 3.1] for a (piecewise) C^1 -vector field X along c with $g(X', \dot{c}) < 0$, any variation c_s with variation field X is timelike and longer than c (for small s).

Now for any vector v at p not a null normal to a spacelike surface S , w.l.o.g. there exists $y \in T_p S$ with $g(y, v) > 0$. Define $V(t) = (1 - \frac{t}{b})Y(t)$, where Y is the parallel translate of y along c . A straightforward calculation shows $g(V', \dot{c}) < 0$. Now choose a variation c_s of c with variational vector field V such that $c_s(0) \in S$. We obtain that c_s is timelike from S to $c(b)$ and as b was arbitrary the statement follows.

The main result on trapped submanifolds in C^1 -spacetimes now is the following:

Proposition 5.10. *Let (M, g) be a strongly causal, MNNB C^1 -spacetime and let $S \subseteq M$ be a trapped C^0 -submanifold of codimension m , $1 < m < \dim(M) = n$. If $m = 2$, suppose that (M, g) satisfies the distributional null energy condition, and if $m \neq 2$, suppose that any support submanifold \tilde{S} of S satisfies the condition in Proposition 5.5 in any null normal direction ν and for any null geodesic γ with $\dot{\gamma}(0) = \nu$. Then $E^+(S)$ is compact or (M, g) is null geodesically incomplete.*

Proof. Suppose (M, g) is null geodesically complete.

$E^+(S)$ is relatively compact: Assume, to the contrary, that there are $q_k \in E^+(S)$ with $q_k \rightarrow \infty$. Lemma A.5 guarantees the existence of future directed maximising null geodesics $\gamma_k : [0, t_k] \rightarrow M$ connecting $p_k \in S$ to q_k , which we may assume to be parametrised by h -unit speed. By compactness we may further assume that $\gamma_k(0) = p_k \rightarrow p \in S$, and we write $\gamma_k(t_k) = q_k$. Due to $q_k \rightarrow \infty$, only finitely many γ_k are contained in a fixed neighbourhood U of p . So we may apply Theorem A.10, implying that (a subsequence, not relabeled, of) γ_k converges to a g -null curve $\gamma : [0, \infty) \rightarrow M$ (in h -unit speed parametrisation) from p . Since the γ_k are S -maximising, so is γ . In particular, γ is a g -null pre-geodesic. Reparametrising γ as a geodesic, it is still defined on $[0, \infty)$ due to null geodesic completeness and so $\gamma : [0, \infty) \rightarrow M$ is a future complete g -null S -ray. Let \tilde{S} be a future support submanifold of S at p , then γ maximises the distance from \tilde{S} everywhere, because otherwise there would be some T such that $\gamma(T) \in I^+(\tilde{S}) \subseteq I^+(S)$, contradicting the fact that γ is an S -ray. As this cannot happen due to Proposition 5.5 resp. Proposition 5.7, $E^+(S)$ is relatively compact.

$E^+(S)$ is closed: Let $q_k \in E^+(S), q_k \rightarrow q$. Let $\gamma_k : [0, t_k] \rightarrow M$ be null pre-geodesics (in h -unit speed parametrisation) from some $p_k \in S$ to q_k . By compactness of S , we may assume that $p_k \rightarrow p \in S$. If $q = p$, then we are done. Otherwise, there are two possibilities: If the t_k are bounded, then by going over to subsequences, $t_k \rightarrow t \in (0, \infty)$. Since $q \neq p$, we are in a position to invoke Theorem A.11 to get a maximising causal limit geodesic $\gamma : [0, t] \rightarrow M$ connecting p to q . γ cannot be timelike, as this would imply $q \in I^+(S)$ and hence $q_k \in I^+(S)$ for large k . Hence $q \in J^+(S) \setminus I^+(S) = E^+(S)$. The other possibility is that the t_k are unbounded, i.e. w.l.o.g. $t_k \rightarrow \infty$. Since (M, g) is

strongly causal, this means that the γ_k leave every compact set, i.e., $\gamma_k(t_k) \rightarrow \infty$ (after possibly relabelling t_k). However, as $E^+(S)$ is contained in some compact set, this is not possible. \square

Corollary 5.11. *Let the assumptions be as in Proposition 5.10. Then $E^+(S) \cap S$ is achronal, and $E^+(E^+(S) \cap S)$ is compact or M is null geodesically incomplete.*

Proof. This is shown as in the smooth case, see [10, Prop. 4] or [46, Prop. 4.3] and using that for any $p \in S$ there is some neighbourhood U such that $S \cap U$ is achronal in U . \square

We now start with our discussion of “trapped points”. In [8, Sec. 6.3], a faithful way to generalise the classical condition, which uses Jacobi tensor classes—a tool no longer available already for $C^{1,1}$ -metrics—is presented: It uses the mean curvature of spacelike 2-surfaces given as the level sets of the exponential map that generate the light cone. While the latter tool presently is not at our disposal, the very formulation of the corresponding condition again uses support manifolds and can be adopted verbatim. However, it is now more delicate to derive that the horismos of a trapped point is a trapped set.

Definition 5.12 (*Trapped points*).

A point $p \in M$ is *future trapped* if for any future pointing null vector $v \in T_pM$ and for any null geodesic γ with $\gamma(0) = p, \dot{\gamma}(0) = v$, there exists a parameter t and a spacelike C^2 -submanifold of codimension $m = 2$ with $\tilde{S} \subseteq J^+(p), \gamma(t) \in \tilde{S}$ and $\mathbf{k}_{\tilde{S}}(\dot{\gamma}(t)) > 0$.

Proposition 5.13. *Let (M, g) be a strongly causal, MNNB C^1 -spacetime satisfying the distributional null energy condition. If $p \in M$ is a future trapped point and (M, g) is null geodesically complete then $E^+(p)$ is compact.*

Proof. Since trapped points are defined by means of codimension-2 submanifolds \tilde{S} , we will use Proposition 5.7 for which (only) the distributional null energy condition is needed. The proof is analogous to that of Proposition 5.10.

$E^+(p)$ is relatively compact: Suppose $q_j \in E^+(p), q_j \rightarrow \infty$. Let $\gamma_j : [0, t_j] \rightarrow M$ be maximising null geodesics connecting p to q_j , cf. Lemma A.5. Then, up to a subsequence, the γ_j converge to a g -null ray $\gamma : [0, \infty) \rightarrow M$ from p (the domain is $[0, \infty)$ even in g -affine parametrisation since (M, g) is null geodesically complete). By assumption, there is a C^2 -spacelike submanifold \tilde{S} of codimension 2 and a parameter t such that $\mathbf{k}_{\tilde{S}}(\dot{\gamma}(t)) > c > 0$ and we let $b > 1/c$. γ is indeed an \tilde{S} -ray: If not, it would enter $I^+(\tilde{S}) \subseteq I^+(p)$, a contradiction. But by the previous results in this section, γ cannot maximise from \tilde{S} past b , a contradiction since it is a ray.

$E^+(p)$ is closed: This is shown in exactly the same way as closedness was shown in the proof of Proposition 5.10. \square

6. The Main Result

Given the results established so far, the general mechanics of the proof of the Hawking–Penrose theorem remains the same as in the smooth case. There do, however, remain notable differences in the details in this lower regularity (e.g. the use of [7, Prop. 2.13] in the proof of Theorem 6.2), so we include the full argument.

Lemma 6.1. *Let (M, g) be a strongly causal C^1 -spacetime such that no inextendible null geodesic in M is maximising. Let $A \subseteq M$ be achronal such that $E^+(A)$ is compact, and let γ be a future inextendible timelike curve contained in $D^+(E^+(A))^\circ$. Then $F := E^+(A) \cap J^-(\gamma)$ is achronal and $E^-(F)$ is compact.*

Proof. Due to Lemma A.16 we may assume that $A = \bar{A}$. Since $E^+(A)$ is always achronal and $F \subseteq E^+(A)$, it is also achronal. By compactness of $E^+(A)$, F is compact. Note that $E^-(F) \subseteq F \cup \partial J^-(\gamma)$ by similar arguments as in [21, Lem. 9.3.4], so to show compactness of $E^-(F)$, it suffices to show compactness of $E^-(F) \cap \partial J^-(\gamma)$.

Suppose now that $v \in TM|_F$ is past pointing causal and let $c_v : [0, a) \rightarrow M$ be any past inextendible, past directed geodesic with $\dot{c}_v(0) = v$ (recall that in C^1 -spacetimes we no longer have unique solvability of the geodesic initial value problem). Then $c_v \subseteq \bar{J}^-(\gamma)$, and $\bar{J}^-(\gamma) = I^-(\gamma) \cup \partial J^-(\gamma)$. We show that c_v meets $I^-(\gamma)$: Suppose, to the contrary, that c_v is a null geodesic entirely contained in $\partial J^-(\gamma)$. Since γ is a future inextendible timelike curve, we have $J^-(\gamma) = I^-(\gamma)$, hence c_v lies entirely in $\partial J^-(\gamma) \setminus J^-(\gamma)$. By Corollary A.12, there is a future directed, future inextendible null geodesic λ starting at $c_v(0)$ entirely in $\partial J^-(\gamma)$ (γ is closed in M by Corollary A.33 and Lemma A.32). This means that either $c_v\lambda$ is an inextendible broken null geodesic, hence not maximising by Lemma A.4, or an inextendible unbroken maximising null geodesic. By assumption such a geodesic cannot exist, hence in either case $c_v\lambda$ cannot lie entirely in $\partial J^-(\gamma)$ and so c_v must enter $I^-(\gamma)$.

Thus we have shown that for any v and any geodesic c_v as above there is $t_v = t_v(v, c_v)$ (for which c_v is still defined) such that $c_v(t_v) \in I^-(\gamma)$. It remains to show that $E^-(F) \cap \partial J^-(\gamma)$ is compact.

$E^-(F) \cap \partial J^-(\gamma)$ is relatively compact: Suppose there are $q_k \in E^-(F) \cap \partial J^-(\gamma)$ with $q_k \rightarrow \infty$. Then there are past-directed maximising null pre-geodesics $c_k : [0, t_k] \rightarrow M$ in h -unit speed parametrisation with $c_k(0) = p_k \in F$ and $c_k(t_k) = q_k$. Since $q_k \rightarrow \infty$, also $t_k \rightarrow \infty$. By compactness, we may assume $p_k \rightarrow p \in F$. Since $q_k \rightarrow \infty$, there is a neighbourhood U which almost all c_k leave and we are in a position to apply the Theorem A.10. We get a past-directed inextendible null limit curve $c : [0, \infty) \rightarrow M$ from p and a subsequence of the c_k (w.l.o.g. c_k itself) which converges to c locally uniformly. It is easy to see that all c_k have to lie entirely in $\partial J^-(\gamma)$, hence so does the limit c . In particular, c is everywhere maximising and may be reparametrised as an inextendible geodesic $c : [0, a) \rightarrow M$ entirely contained in $\partial J^-(\gamma)$. This is a contradiction since we showed before that each past directed null geodesic with initial velocity in $TM|_F$ has to enter $I^-(\gamma)$.

$E^-(F) \cap \partial J^-(\gamma)$ is closed: Let $q_j \in E^-(F) \cap \partial J^-(\gamma)$, $q_j \rightarrow q$, where we may assume $q \notin F$, otherwise there is nothing to prove. Connect $F \ni p_j \rightarrow q_j$ via maximising null geodesics. Up to a subsequence, their limit has to be a maximising null geodesic from p to q , hence $q \in E^-(F) \cap \partial J^-(\gamma)$, because otherwise one would again obtain an inextendible maximising null geodesic entirely in $\partial J^-(\gamma)$, which would lead to the same contradiction as before. \square

Now we are ready to give the C^1 -version of the causal Hawking-Penrose theorem. It is the direct analog of [17, p. 538, Thm.] and [8, Thm. 7.4], respectively.

Theorem 6.2. *Let (M, g) be a C^1 -spacetime. Then the following conditions cannot all hold:*

- (i) (M, g) is causal.
- (ii) No inextendible timelike geodesic in an open globally hyperbolic subset is everywhere maximising.
- (iii) No inextendible null geodesic is everywhere maximising.
- (iv) There is an achronal set $A \subseteq M$ such that $E^+(A)$ or $E^-(A)$ is compact.

Proof. Suppose all of the above conditions hold. Then by Lemma A.31, (i) and (iii) imply that (M, g) is strongly causal. W.l.o.g. we assume that $E^+(A)$ is compact. By

Lemma A.35 there exists a future inextendible (in M) timelike curve $\gamma \subseteq D^+(E^+(A))^\circ$. Set $F := E^+(A) \cap J^-(\gamma)$, which by Lemma 6.1 is achronal and $E^-(F)$ is compact. Once more by Lemma A.35 there is some past-inextendible timelike curve $\lambda \subseteq D^-(E^-(F))^\circ$. Exactly as in the smooth case, see e.g. the second paragraph of [8, Proof of Thm. 7.4], one can show $\gamma \subseteq D^+(E^-(F))^\circ$, by invoking Proposition A.20 and Lemma A.24. So $\gamma, \lambda \subseteq D(E^-(F))^\circ$ which by Proposition A.28 is globally hyperbolic. Now we pick sequences $p_k = \lambda(s_k)$ and $q_k = \gamma(t_k)$ with $t_k, s_k \nearrow \infty$, leaving every compact set, which is possible because γ and λ are inextendible curves in M contained in a strongly causal set (Lemma A.32). Also because $\lambda \subseteq J^-(E^-(F)) \subseteq J^-(F) \subseteq J^-(J^-(\gamma))$ and because γ is timelike, we can choose $p_k \in I^-(q_k)$.

By [7, Prop. 2.13] there exist maximising timelike geodesics $\tilde{\gamma}_k$ in $D(E^-(F))^\circ$ from p_k to q_k , which must intersect $E^-(F)$, say at r_k . By compactness of $E^-(F)$ we can assume $r_k \rightarrow r$ and by Corollary 2.6 there exists an inextendible, causal limit geodesic $\tilde{\gamma}$, which is also maximising (as a limit of maximisers).

By (iii), $\tilde{\gamma}$ cannot be null, so it must be timelike. As a limit of curves in $D(E^-(F))^\circ$ it is contained in $\overline{D(E^-(F))}$. Again using Proposition A.20 and Lemma A.24 one can now proceed as in the last step in [8, Proof of Thm. 7.4] to arrive at the desired contradiction (to (ii)). \square

Finally, we formulate and prove the main result of this work, the analytical Hawking-Penrose theorem for maximally causally non-branching C^1 -spacetimes.

Theorem 6.3. *(The Hawking-Penrose singularity theorem for C^1 -metrics)*

Let (M, g) be a C^1 -spacetime of dimension n with the following properties.

- (i) (M, g) is causal.
- (ii) (M, g) satisfies the distributional timelike and null energy conditions.
- (iii) (M, g) satisfies the distributional genericity condition along any inextendible causal geodesic.
- (iv) (M, g) is maximally causally non-branching.
- (v) (M, g) contains one of the following.
 - (a) a compact, achronal, edgeless set.
 - (b) a future trapped point.
 - (c) a future trapped C^0 -submanifold of codimension 2.
 - (d) a future trapped C^0 -submanifold of codimension $2 < m < n$ such that its support submanifolds satisfy the condition in Proposition 5.5.

Then (M, g) is causally geodesically incomplete.

Proof. Assuming causal geodesic completeness, we would like to deduce a contradiction by using Theorem 6.2.

Note that due to Theorems 4.2 and 4.3 the assumptions (ii) and (iii) of Theorem 6.2 are satisfied and it remains to establish the existence of a future or past trapped set, i.e. an achronal set with compact future or past horismos. This will be implied by (v), but we have to distinguish the different cases:

By Proposition A.36, (a) implies the existence of a trapped set. Note that by Lemma A.31 (which can be applied by Theorem 4.3) (M, g) is strongly causal and hence Proposition 5.13 covers the case (b), whereas Proposition 5.10 covers cases (c) and (d). \square

Funding Open access funding provided by University of Vienna.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

A. Appendix: Causality Results in C^1 -Spacetimes

In this section, we collect various results on the causality of C^1 -Lorentzian metrics that will be needed in the proof of the main result. Unless stated otherwise, throughout this section (M, g) consists of a connected smooth (Hausdorff and second countable) manifold M , and a C^1 -Lorentzian metric g that is time-orientable (i.e. there exists a smooth timelike vector field on M). Many of the results in this section hold in even greater generality, namely either for Lorentzian metrics that are merely Lipschitz or even in the general setting of closed cone structures [35]. Although we will occasionally note this, in proving the results in this section we will stay close to smooth causality theory (mainly based on the comprehensive reference work [34]) and only make those changes from classical proofs that are required due to the absence of certain tools (e.g., convex normal neighbourhoods) in the C^1 -setting.

We begin with two elementary results, namely the openness of timelike futures and pasts and the push-up property. Both of these hold for causally plain spacetimes, which include spacetimes with Lipschitz continuous metrics [5, Cor. 1.17].

Lemma A.1. *For any $A \subseteq M$, the sets $I^\pm(A)$ are open in M .*

Proof. See [5, Prop. 1.21]. \square

Lemma A.2. *Let $p, q, r \in M$ such that $p \leq q \ll r$ or $p \ll q \leq r$. Then $p \ll r$.*

Proof. See [5, Lem. 1.22]. \square

Remark A.3. Note that from this one can show as in the smooth case, e.g. [39, Lem. 14.03]⁶, that the timelike relation is open, i.e., if $p \ll q$ then there are neighbourhoods U_p, U_q of p and q respectively, such that for all $x \in U_p$ and all $y \in U_q$, we have $x \ll y$.

Lemma A.4. *Let (M, g) be a C^1 -spacetime and let $\gamma : [a, b] \rightarrow M$ be a maximising causal curve. Then γ is a (reparametrisation of a) causal geodesic. In particular, it is a C^2 -curve and has a causal character.*

Proof. See [27, Thm. 1.1] or [43, Thm. 3.3]. \square

For any point $q \in M$, we denote by $E^+(q) := J^+(q) \setminus I^+(q)$ its future horismos. Its past horismos $E^-(q)$ is defined analogously.

Lemma A.5. *If $p \in E^+(q)$ then there exists a maximising null geodesic segment from q to p .*

⁶ the use of convex sets is not necessary once openness of I^+ is established

Proof. Let c be a future causal curve from q to p . Since $p \notin I^+(q)$, c is null and maximising from q to p (because of push-up), hence it is (a reparametrisation of) a maximising null geodesic by Lemma A.4. \square

Lemma A.6. *For any subset $A \subseteq M$, we have $I^\pm(A) = J^\pm(A)^\circ$ and $\partial I^\pm(A) = \partial J^\pm(A)$. Moreover, the sets $E^\pm(A)$ and $\partial J^\pm(A)$ are achronal.*

Proof. This follows by the same proofs as in the smooth case (cf. [34, Thm. 2.27], [39, Cor. 14.27]). \square

Definition A.7. (Edge)

Let $A \subseteq M$ be achronal. The *edge* of A , denoted by $\text{edge}(A)$, is the set of all $x \in \bar{A}$ such that for any neighbourhood U of x , there is a timelike curve from $I_U^-(x)$ to $I_U^+(x)$ that does not meet A .

Lemma A.8. *Let (M, g) be a C^1 -spacetime. An achronal set $A \subseteq M$ is a topological hypersurface if and only if $A \cap \text{edge}(A) = \emptyset$. Moreover, A is a closed topological hypersurface if and only if $\text{edge}(A) = \emptyset$.*

Proof. The proofs for smooth metrics (cf. [39, Prop. 14.25, Cor. 14.26]) still hold in C^1 . \square

Lemma A.9. *For any $A \subseteq M$, $\partial J^+(A)$ is a (topologically) closed, achronal topological hypersurface.*

Proof. This follows from the fact that $\partial J^+(A)$ is achronal and $\text{edge}(\partial J^+(A)) = \emptyset$, which is proven in precisely the same way as in the smooth case. \square

Recall the notion of a limit maximising sequence of causal curves: A sequence $\gamma_k : [a_k, b_k] \rightarrow M$ of future directed causal curves is called *limit maximising* if there is $\varepsilon_k \rightarrow 0$ such that

$$L(\gamma_k) \geq d_g(\gamma(a_k), \gamma(b_k)) - \varepsilon_k,$$

where d_g is the time separation function induced by g . In particular a sequence of maximising curves is limit maximising. We shall employ the following version of the limit curve theorem:

Theorem A.10. (Limit Curve Theorem)

Let h be a complete Riemannian metric on M and let $\gamma_k : [a_k, b_k] \rightarrow M$ be future directed, h -arc length parametrised causal curves with $0 \in [a_k, b_k]$ such that the sequence $\{\gamma_k(0)\}$ has an accumulation point y . Suppose there are $a \leq 0$ and $b \geq 0$ such that $a_k \rightarrow a$, $b_k \rightarrow b$. If there is a neighbourhood U of y such that almost all γ_k leave U , then a subsequence of γ_k converges h -uniformly on compact sets to a future directed causal Lipschitz curve $\gamma : [a, b] \rightarrow M$. γ is future resp. past inextendible iff $b = \infty$ resp. $a = -\infty$. If $\{\gamma_k\}$ is limit maximising, then γ is maximising.

Proof. This follows from [30, Thm. 3.1] and [5, Thm. 1.6] by the same proof as in [8, Thm. A.6]. \square

For a much more general version of the limit curve theorem, valid in closed cone structures, we refer to [35, Thm. 2.14]. We shall also require the following variant of the limit curve theorem for a sequence of curves with converging past and future endpoints, cf. [30, Thm. 3.1(2)].

Theorem A.11 (*Limit Curve Theorem: two converging endpoints*).

Let $\gamma_k : [0, b_k] \rightarrow M$ be a sequence of future-directed, causal curves in h -arc length parametrisation connecting x_k to z_k , with $x_k \rightarrow x$ and $z_k \rightarrow z$. Suppose that $b_k \rightarrow b \in (0, \infty)$. If there is a neighbourhood U of x such that only a finite number of γ_k are entirely contained in U , then there is a Lipschitz future-directed, causal curve $\gamma : [0, b] \rightarrow M$ connecting x to z such that a subsequence of the γ_k converges to γ h -uniformly on compact sets. Moreover, if $\{\gamma_k\}$ is limit maximising, then γ is maximising.

Proof. This is contained in the statement of [30, Thm. 3.1(2)], whose proof is easily seen to hold for C^1 -metrics once Theorem A.10 is established. \square

Corollary A.12. Let $A \subseteq M$. For any point $x \in \partial J^+(A) \setminus \bar{A}$, there is a causal curve $\gamma \subseteq \partial J^+(A)$ with future endpoint x that is either past inextendible and does not meet \bar{A} or has a past endpoint in A . It is a (reparametrisation of a) maximising null geodesic. If A is closed and $x \notin J^+(A)$, then $\gamma \subseteq \partial J^+(A) \setminus J^+(A)$ and it is past inextendible.

Proof. Using Theorem A.10 and Lemma A.5, this follows as in [8, Prop. A.7]. \square

Lemma A.13. Let (M, g) be a C^1 -spacetime and let $\{g_k\}$ be either the sequence $\{\hat{g}_{\varepsilon_k}\}$ or $\{\check{g}_{\varepsilon_k}\}$ with $\varepsilon_k \downarrow 0$. Let $p, q \in M$ and let γ_k be a g_k -maximising curve from p to q , so γ_k is in particular a g_k -geodesic. Suppose that γ_k converge to a curve γ in C^1_{loc} . If $\{g_k\} = \{\check{g}_{\varepsilon_k}\}$, suppose in addition that (M, g) is globally hyperbolic and $p \ll_g q$. Then γ is a g -maximising geodesic from p to q .

Proof. The case $\{g_k\} = \{\hat{g}_{\varepsilon_k}\}$ follows immediately from [41, Prop. 6.5], so we will only consider the case $\{g_k\} = \{\check{g}_{\varepsilon_k}\}$ together with the additional assumptions of (M, g) being globally hyperbolic and $p \ll_g q$ in this case. Due to global hyperbolicity, the space $C(p, q)$ of future directed causal curves from p to q (considered up to orientation preserving reparametrisations) is compact with respect to its natural topology (see [41, Thm. 3.2]). Consequently, the h -speeds of curves in $C(p, q)$ are uniformly bounded by some constant $C_1 > 0$, where h is some complete Riemannian metric on M .

Since (M, g) is globally hyperbolic, there is an L_g -maximising g -geodesic $c : [0, 1] \rightarrow M$ with $c(0) = p, c(1) = q$ [41, Prop. 6.4]. Also, $p \ll_g q$ implies that c is g -timelike. By compactness, there is a constant $C_2 > 0$ such that $g(\dot{c}, \dot{c}) < -C_2$ on $[0, 1]$, and hence $g_k(\dot{c}, \dot{c}) < 0$ for k large, implying that c is g_k -timelike for such k . If we set $\delta_k := d_h(g, g_k)$ (the h -distance of g, g_k as in [5, (1.6)]), then

$$\begin{aligned} L_g(c) &= \int_0^1 \sqrt{-g(\dot{c}, \dot{c})} dt \leq \int_0^1 \sqrt{-g_k(\dot{c}, \dot{c}) + C_1^2 \delta_k} dt \leq L_{g_k}(c) + C_1 \sqrt{\delta_k} \\ &\leq d_{g_k}(p, q) + C_1 \sqrt{\delta_k} = L_{g_k}(\gamma_k) + C_1 \sqrt{\delta_k} \rightarrow L_g(\gamma) \quad (k \rightarrow \infty), \end{aligned}$$

so $L_g(\gamma) \geq L_g(c)$, which proves that also γ is maximising from p to q . \square

Lemma A.14. Let $\gamma_k \subseteq \overline{J^+(p)}$ be causal curves converging locally uniformly to a causal curve γ with future endpoint q . If $d_g(p, q) = 0$, then γ is a maximising null geodesic.

Proof. Clearly $\gamma \subseteq \overline{J^+(p)}$. If γ would meet $I^+(p)$, then there would exist some $r \in I^+(p) \cap J^-(q)$ and using push-up we could conclude that $q \in I^+(p)$, contradicting $d_g(p, q) = 0$. Hence $\gamma \subseteq \partial J^+(p)$, which is achronal and γ is a maximising null geodesic by Lemma A.4. \square

Lemma A.15. Let $g_k = \check{g}_{\varepsilon_k}$ and let $\gamma_k \subseteq \overline{J^+(p)}$ be g_k -null geodesics converging in C^1_{loc} to a g -geodesic γ that ends in q . Assume that $d_g(p, q) = 0$. Then γ is a maximising g -null geodesic.

Proof. This follows immediately from Lemma A.14 upon noting that any g_k -null geodesic is g causal. \square

Lemma A.16. *Let A be achronal. Then so is \overline{A} . Moreover, if $E^+(A)$ is closed, then $E^\pm(A) = E^\pm(\overline{A})$.*

Proof. This follows by the same proof as in [8, Lem. A.8]. \square

Lemma A.17. *Let (M, g) be a C^1 -spacetime. Then every point in M has a neighbourhood base of globally hyperbolic neighbourhoods.*

Proof. Fix any smooth, time-orientable Lorentzian metric \hat{g} with $g \prec \hat{g}$. For $x \in M$, [34, Thm. 2.7] gives a neighbourhood base V_m of neighbourhoods of x that are globally hyperbolic with respect to $\hat{g}|_{V_m}$. Any Cauchy hypersurface for \hat{g} in V_m is also a Cauchy hypersurface for g , so the claim follows from [41, Thm. 5.7]. \square

Lemma A.18. *Let S be a spacelike (C^2 -) hypersurface and let $p \in S$. Then there exists a neighbourhood V of p in M such that $V \cap S$ is a Cauchy hypersurface in V .*

Proof. The proof of the analogous result [25, Lemma A.25] also works in the C^1 -setting. \square

Definition A.19. (Cauchy development and Cauchy horizon)

Let A be an achronal set. The *future Cauchy development* of A is the set

$$D^+(A) := \{x \in M : \text{every past inextendible causal curve through } x \text{ meets } A\}.$$

The *future Cauchy horizon* of A is defined as

$$H^+(A) := \overline{D^+(A)} \setminus I^-(D^+(A)).$$

The *past Cauchy development and past Cauchy horizon* are defined analogously.

The first part of the following result is partly a C^1 -analogue of [39, Lem. 14.51], where it is proved using convex normal neighbourhoods, a tool that we do not have at our disposal in the C^1 -setting. The use of such neighbourhoods can, however, be avoided, as our arguments below illustrate.

Proposition A.20. *Let A be a closed, achronal set. Then*

$$\overline{D^+(A)} = \{x \in M : \text{every p.i. t.l. curve through } x \text{ meets } A\}$$

and

$$\partial D^+(A) = A \cup H^+(A).$$

Proof. Let T be the set on the right hand side in the first equation.

$\overline{D^+(A)} \subseteq T$: Suppose there is a point $p \in \overline{D^+(A)} \setminus T$. Then there is some past inextendible timelike (future directed) curve $\alpha : (a, 0] \rightarrow M$ with $\alpha(0) = p$ not meeting A . Let $p_k \in D^+(A)$, $p_k \rightarrow p$ and let U_k be open neighbourhoods of p with $U_k \rightarrow \{p\}$. By choosing subsequences, we may assume that $p_k \in I_{U_k}^+(\alpha(-1/k))$ and $\alpha|_{[-1/k, 0]} \subseteq U_k$ for all k .

Construct now past inextendible causal curves α_k by connecting $\alpha(-1/k)$ to p_k in a timelike way such that these curves stay in U_k , and let the rest of α_k (i.e. the part to the past of $\alpha(-1/k)$) be the original curve α . Then the α_k are all past inextendible timelike, and since $p_k \in D^+(A)$ they must meet A at some a_k . Since α does not meet A , the

a_k have to lie on the replaced piece. But these pieces are entirely in U_k , which go to $\{p\}$, so we see that $a_k \rightarrow p$. Since A is closed, we have $p \in A$, which is absurd since $p \notin T \supseteq A$ by assumption.

$D^+(A) \supseteq T$: Suppose $q \notin \overline{D^+(A)}$, and let $r \in I^-_{M \setminus \overline{D^+(A)}}(q)$. Since in particular $r \notin D^+(A)$, there is a past inextendible causal curve α from r missing A . Precisely as in the smooth case [39, Lem. 14.30(1)], noting that pushup still holds in C^1 , there is a past inextendible timelike curve from q not meeting A . Thus $q \notin T$.

Finally, $\partial D^+(A) = A \cup H^+(A)$ follows exactly as in the smooth case (cf. [39, Lem. 14.52]). \square

We note that the previous result in fact remains true even for locally Lipschitz proper cone structures [35, Thm. 2.35, 2.36].

Lemma A.21. *Let A be achronal. Then $H^+(A)$ is a closed, achronal set.*

Proof. The proof of [34, Prop. 3.15] carries over unchanged to C^1 metrics. \square

Lemma A.22. *Let A be closed and achronal. If $x \in D^+(A) \setminus H^+(A)$, then every past inextendible causal curve through x meets $I^-(A)$. Moreover, if $x \in D(A)^\circ$, then every future (resp. past) inextendible future (resp. past) directed causal curve emanating from x intersects $I^+(A)$ (resp. $I^-(A)$).*

Proof. This follows from [34, Prop. 3.27] and [34, Prop. 3.42], which still hold for C^1 -metrics. \square

Lemma A.23. *Let A be closed, achronal. Let $x \in J^+(A) \setminus D^+(A)$ or $x \in I^+(A) \setminus D^+(A)^\circ$. Then every causal curve from x to A meets $H^+(A)$.*

Proof. The proof for $C^{1,1}$ -metrics [8, Lem. A.12] still holds in C^1 because of Lemma A.1, Lemma A.2, and Proposition A.20. \square

Using these results, one establishes the following formula for the timelike future of the Cauchy horizon of a closed, achronal set precisely as in the $C^{1,1}$ -case, cf. [8, Lem. A.13].

Lemma A.24. *Let A be closed and achronal. Then $I^+(H^+(A)) = I^+(A) \setminus \overline{D^+(A)}$.*

Lemma A.25. *Let A be closed and achronal. Then $\text{edge}(H^+(A)) \subseteq \text{edge}(A)$.*

Proof. Based on Lemmas A.23 and A.24, the proof of [8, Lem. A.14] carries over to the C^1 -setting. \square

Lemma A.26. *Let A be an achronal set, then $H^+(\partial J^+(A))$ is a closed, achronal, topological hypersurface.*

Proof. $H^+(\partial J^+(A))$ is closed and achronal by Lemma A.21, and by the previous result, combined with Lemmas A.8 and A.9, $\text{edge}(H^+(\partial J^+(A))) \subseteq \text{edge}(\partial J^+(A)) = \emptyset$, so the claim follows from Lemma A.8. \square

Lemma A.27. *Let A be closed, achronal. Then $H^+(\overline{E^+(A)}) \subseteq H^+(\partial J^+(A))$.*

Proof. The proof of [8, Lem. A.16] carries over, using Corollary A.12, Proposition A.20 and Lemmas A.23, A.24. \square

Proposition A.28. *Let (M, g) be a C^1 -spacetime and let $A \subseteq M$ be a closed, achronal set. Then the interior of its Cauchy development, i.e. the set $D(A)^\circ = (D^+(A) \cup D^-(A))^\circ$, if nonempty, is a globally hyperbolic C^1 -spacetime when considered with the induced metric.*

Proof. Based on Theorem A.10 and Lemma A.22, this follows from the same proof as in [34, Thm. 3.45]. \square

Definition A.29. (Strong causality)

A C^1 -spacetime (M, g) is called *strongly causal at* $p \in M$ if for any neighbourhood U of p there is a neighbourhood V of p with $V \subseteq U$ such that any causal curve starting and ending in V is contained in U . (M, g) is called *strongly causal* if it is strongly causal at every point.

Lemma A.30. (M, g) is strongly causal at p if and only if for any neighbourhood U of p there is a neighbourhood V of p with $V \subseteq U$ such that no causal curve leaving V ever returns.

Proof. This follows from [33, Lem. 3.21], which continues to hold in the C^1 -case. \square

Lemma A.31. Let (M, g) be chronological and suppose there are no inextendible maximising null geodesics in M . Then (M, g) is strongly causal.

Proof. Using Theorem A.10, this follows by the same argument as [8, Lem. A.19]. \square

Lemma A.32. A strongly causal C^1 -spacetime (M, g) is non-totally and non-partially imprisoning: No future or past inextendible causal curve is contained in a compact set and no future or past inextendible causal curve returns to a compact set infinitely often.

Proof. The classical proof for the smooth case (see [39, Lem. 14.13]) holds even for continuous metrics. \square

Corollary A.33. Let (M, g) be a non-totally and non-partially imprisoning C^1 -spacetime. Then the image of every inextendible causal curve is closed in M .

Proof. Let γ be any inextendible causal curve. Since M is a manifold, the topology on M is compactly generated. Let $K \subseteq M$ be any compact subset. By non-total and non-partial imprisonment, $\gamma \cap K$ is a finite union of closed segments of γ , hence $\gamma \cap K$ is closed in K . Since K was arbitrary, γ is closed in M . \square

Lemma A.34. Let (M, g) be a strongly causal C^1 -spacetime and let $A \subseteq M$ be a closed, achronal subset. If $H^+(E^+(A))$ is nonempty, then it is noncompact.

Proof. The proof of the smooth case goes through, because all the necessary preliminary results continue to hold in C^1 (see the outline in [8, Lem. 7.1]). \square

Lemma A.35. Let (M, g) be a strongly causal C^1 -spacetime. Let $A \subseteq M$ be an achronal subset such that $E^+(A)$ is compact. Then there is a future inextendible timelike curve γ contained in $D^+(E^+(A))^\circ$.

Proof. The proof of the smooth case goes through in C^1 , see e.g. [17, Lem. 2.12], or also [21, Lem. 9.3.3], combined with Lemma A.16. \square

Proposition A.36. Let (M, g) be a C^1 -spacetime and let $A \subseteq M$ be achronal and edgeless. Then $E^+(A) = A$, in particular if A is compact so is $E^+(A)$.

Proof. This was shown e.g. in [34, Cor. 2.145] and the proof carries over to C^1 spacetimes. \square

B. Appendix: Extending Vector Fields in C^1 -Spacetimes Uniformly

When applying the genericity condition (Definition 2.15) we need to extend vectors or vector fields along curves to neighbourhoods, due to the fact that distributional (or also L^∞) curvature quantities are not well defined on points or along curves (as these are sets of measure zero). It is rather straightforward to extend a vector field along a curve to a neighbourhood. For C^1 -metrics one can still use parallel transport along certain curves spanning an open set. Alternatively, one can also use “cylindrically constant” (see below) extensions in coordinates around a (part of a) curve.

In our case, we need to simultaneously extend vector fields along a sequence of converging curves. To be more precise, let γ_k be a sequence converging in C^1 to some γ and let V_k be vector fields along γ_k and V along γ such that $V_k \rightarrow V$ in C^1 as $k \rightarrow \infty$. Then given a point on γ , does it possess a neighbourhood U such that all V_k can be extended to vector fields on U and such that their extensions converge in C^1 to the extension of V ?

To begin with, we consider the situation of extending vector fields around a point of a curve to a neighbourhood by cylindrically constant extension in coordinates.

Lemma B.1 (*Cylindrically constant extension of vector fields on curves*). *Let $\gamma : I \rightarrow M$ be a C^1 -curve which is regular at 0, i.e. $\dot{\gamma}(0) \neq 0$ and set $p := \gamma(0)$. Then there exists a chart $(U, (x^1, \dots, x^n))$ around p with the following property: For any C^1 vector field V along γ there exists a C^1 extension \tilde{V} of V to U such that, in these coordinates, \tilde{V} is independent of x^2, \dots, x^n .*

Proof. Since the claim is local we may assume that $M = \mathbb{R}^n$, that $p = 0$, and that $\gamma = (\gamma^1, \dots, \gamma^n)$ with $\gamma_1'(t) > c > 0$ for some c and all $t \in (a, b)$, where $a < 0 < b$. Thus γ^1 is a C^1 -diffeomorphism from (a, b) onto some interval J . Then we set $U := J \times B_R^{n-1}(0)$, where $R > 0$ is such that $\gamma((a, b)) \subseteq U$. Given a C^1 -vector field V along γ , i.e., a C^1 -map $V : (a, b) \ni t \mapsto (\gamma(t), v(t))$ with $v \in C^1((a, b), \mathbb{R}^n)$, set $\tilde{V} : U \rightarrow U \times \mathbb{R}^n$,

$$\tilde{V}(x) := (x, v((\gamma^1)^{-1}(x_1)))$$

to obtain the desired extension. \square

Now that we have set up a way of extending vector fields in a cylindrically constant fashion, we can implement an analogous procedure uniformly on a sequence of curves.

Lemma B.2. *Let $\gamma_k : [-1, 1] \rightarrow M$ be a sequence of C^1 -curves converging in $C^1([-1, 1])$ to the regular C^1 -curve $\gamma : [-1, 1] \rightarrow M$ and let V_k and V be C^1 vector fields along γ_k and γ , respectively. Further, let $V_k \rightarrow V$ in C^1 , i.e., both $V_k \rightarrow V$ and $V_k' \rightarrow V'$ in TM , uniformly on $[-1, 1]$. Then there exists some open neighbourhood U of $\gamma(0)$ and C^1 extensions \tilde{V}_k of V_k and \tilde{V} of V to U such that $\tilde{V}_k \rightarrow \tilde{V}$ in $C^1_{\text{loc}}(U)$.*

Proof. Again we may assume that $M = \mathbb{R}^n$. As in the previous Lemma we write $\gamma = (\gamma^1, \dots, \gamma^n)$ and can assume without loss of generality that $\gamma(0) = 0$, and that $(\gamma^1)'(t) > c > 0$ for some c and all $t \in (a, b)$, where $a < 0 < b$. Due to the assumption on the C^1 -convergence of the γ_k , we may additionally suppose that the same inequality holds on (a, b) for each $(\gamma_k^1)'$. Thus each γ_k^1 and γ^1 itself are C^1 -diffeomorphisms from (a, b) onto their respective image. In addition (restricting to n large if necessary), there exists a nontrivial interval J around 0 that is contained in $\bigcap_k \gamma_k^1((a, b)) \cap \gamma((a, b))$. Then we set $U := J \times B_R^{n-1}(0)$, where $R > 0$ is chosen

such that $\bigcup_k \gamma_k((a, b)) \cup \gamma((a, b)) \subseteq U$. We can write V in the form $t \mapsto (\gamma(t), v(t))$ with $v \in C^1((a, b), \mathbb{R}^n)$, and analogously $V_k(t) = (\gamma_k(t), v_k(t))$, with $v_k \rightarrow v$ in C^1 . For $x \in U$ we set

$$\tilde{V}(x) := (x, v((\gamma^1)^{-1}(x^1))) \quad \tilde{V}_k(x) := (x, v_k((\gamma_k^1)^{-1}(x^1))),$$

giving C^1 -extensions of V resp. V_k to U . By [2, Cor. 1] we have that $(\gamma_k^1)^{-1}$ converges to $(\gamma^1)^{-1}$ locally uniformly, and that the same is true for the first derivatives. Consequently, $\tilde{V}_k \rightarrow \tilde{V}$ in $C_{\text{loc}}^1(U)$. \square

As the proof shows, if the V_k converge to V merely in C_{loc}^0 then one can still find C^1 -extensions to U that also converge in C_{loc}^0 .

References

- Alexander, S., Graf, M., Kunzinger, M., Sämann, C.: Generalized cones as Lorentzian length spaces: causality, curvature, and singularity theorems, *Comm. Anal. Geom.* (to appear)
- Barvinek, E., Daler, I., Francocircu, J.: Convergence of sequences of inverse functions. *Arch. Math. (Brno)* **27B**, 201–204 (1991)
- Bourbaki, N.: *Elements of Mathematics. Algebra, Part I: Chapters 1–3*. Hermann, Paris (1974)
- Cavalletti, F., Mondino, A.: Optimal transport in Lorentzian synthetic spaces, synthetic timelike Ricci curvature lower bounds and applications. [arXiv: 2004.08934](https://arxiv.org/abs/2004.08934)
- Chruściel, P.T., Grant, J.D.E.: On Lorentzian causality with continuous metrics. *Classical Quantum Gravity* **29**(14), 145001, 32 (2012)
- Chruściel, P.T., Grant, J.D.E., Kunzinger M., Minguzzi, E.: Non-regular spacetime geometry. *J. Phys. Conf. Ser.* 968 (2018)
- Graf, M.: Singularity theorems for C^1 -Lorentzian metrics. *Commun. Math. Phys.* **378**(2), 1417–1450 (2020)
- Graf, M., Grant, J.D.E., Kunzinger, M., Steinbauer, R.: The Hawking–Penrose singularity theorem for $C^{1,1}$ -Lorentzian metrics. *Commun. Math. Phys.* **360**, 1009–1042 (2018)
- Galloway, G., Ling, E., Sbierski, J.: Timelike completeness as an obstruction to C^0 -extensions. *Commun. Math. Phys.* **359**(3), 937–949 (2018)
- Galloway, G., Senovilla, J.: Singularity theorems based on trapped submanifolds of arbitrary codimension. *Classical Quantum Gravity* **27**(15), 152002,10 (2010)
- Geroch, R., Traschen, J.: Strings and other distributional sources in general relativity. *Phys. Rev. D* **36**, 1017–1031 (1987)
- Grant, J.D.E., Kunzinger, M., Sämann, C., Steinbauer, R.: The future is not always open. *Lett. Math. Phys.* **110**(1), 83–103 (2020)
- Grosser, M., Kunzinger, M., Oberguggenberger, M., Steinbauer, R.: *Geometric Theory of Generalized Functions*. Kluwer, Dordrecht (2001)
- Hawking, S.W., Ellis, G.F.R.: *The Large Scale Structure of Space-Time*. Cambridge University Press, Cambridge (1973)
- Hartman, P.: *Ordinary Differential Equations*, Volume 38 of *Classics in Applied Mathematics*. SIAM (2002)
- Hartman, P., Wintner, A.: On the problems of geodesics in the small. *Am. J. Math.* **73**, 132–148 (1951)
- Hawking, S.W., Penrose, R.: The singularities of gravitational collapse and cosmology. *Proc. Roy. Soc. Lond. A* **314**, 529–548 (1970)
- Hedicke, J., Minguzzi, E., Schinnerl, B., Steinbauer, R., Suhr, S.: Causal simplicity and (maximal) null pseudoconvexity, submitted. [arXiv:2105.08998](https://arxiv.org/abs/2105.08998) [gr-qc]
- Hörmander, L.: *The Analysis of Linear Partial Differential Operators I*. Springer, Berlin (1990)
- Horváth, J.: *Topological Vector Spaces and Distributions*, vol. I. Addison-Wesley Publishing Co. Reading, Boston (1966)
- Kriele, M.: *Spacetime*. Springer, Berlin (2001)
- Kunzinger, M., Sämann, C.: Lorentzian length spaces. *Ann. Global Anal. Geom.* **54**(3), 399–447 (2018)
- Kunzinger, M., Steinbauer, R., Stojković, M.: The exponential map of a $C^{1,1}$ -metric. *Differ. Geom. Appl.* **34**, 14–24 (2014)
- Kunzinger, M., Steinbauer, R., Stojković, M., Vickers, J.A.: A regularisation approach to causality theory for $C^{1,1}$ -Lorentzian metrics. *Gen. Relativ. Gravit.* **46**, 1738, 18 (2014)

25. Kunzinger, M., Steinbauer, R., Stojković, Vickers, J.A.: Hawking’s singularity theorem for $C^{1,1}$ -metrics. *Classical Quantum Gravity* **32**, 075012, 19 (2015)
26. Kunzinger, M., Steinbauer, R., Vickers, J.A.: The Penrose singularity theorem in regularity $C^{1,1}$. *Classical Quantum Gravity* **32**, 155010 (2015)
27. Lange, C., Lytchak, A., Sämann C.: Lorentz meets Lipschitz. *Adv. Theor. Math. Phys.* (to appear)
28. LeFloch, P.G., Mardare, C.: Definition and stability of Lorentzian manifolds with distributional curvature. *Port. Math. (N.S.)* **64**(4), 535–573 (2007)
29. Marsden, J.E.: Generalized Hamiltonian mechanics: a mathematical exposition of non-smooth dynamical systems and classical Hamiltonian mechanics. *Arch. Rational Mech. Anal.* **28**, 323–361 (1967/68)
30. Minguzzi, E.: Limit curve theorems in Lorentzian geometry. *J. Math. Phys.* **49**(9), 092501 (2008)
31. Minguzzi, E.: Convex neighborhoods for Lipschitz connections and sprays. *Monatsh. Math.* **177**(4), 569–625 (2015)
32. Minguzzi, E., Sanchez, M.: The causal hierarchy of spacetimes, recent developments in pseudo Riemannian geometry. *ESI Lect. Math. Phys.* **299–358**, 0609119 (2008)
33. Minguzzi, E., Sánchez, M.: The causal hierarchy of spacetimes. In: *Recent Developments in Pseudo-Riemannian Geometry. ESI Lectures in Mathematics and Physics*, pp. 299–358. European Mathematical Society, Zurich (2008)
34. Minguzzi, E.: Lorentzian causality theory. In: *Living Reviews in Relativity*, vol. 22:3, pp. 1–202 Springer (2019)
35. Minguzzi, E.: Causality theory for closed cone structures with applications. *Rev. Math. Phys.* **31**(5), 1930001, 139 (2019)
36. Nigsch, E.A.: Bornologically isomorphic representations of distributions on manifolds. *Monatsh. Math.* **170**, 49–63 (2013)
37. Oberguggenberger, M.: Über Folgenkonvergenz in lokalkonvexen Räumen. *Math. Nachr.* **129**, 219–234 (1986)
38. Ohta, S.: Examples of spaces with branching geodesics satisfying the curvature-dimension condition. *Bull. Lond. Math. Soc.* **46**(1), 19–25 (2014)
39. O’Neill, B.: *Semi-Riemannian Geometry with Applications to Relativity*. Pure and Applied Mathematics, vol. 103. Academic Press, New York (1983)
40. Penrose, R.: Gravitational collapse and space-time singularities. *Phys. Rev. Lett.* **14**, 57–59 (1965)
41. Sämann, C.: Global hyperbolicity for spacetimes with continuous metrics. *Ann. Henri Poincaré* **17**(6), 1429–1455 (2016)
42. Sämann, C., Steinbauer, R.: On geodesics in low regularity. *J. Phys. Conf. Ser.* **968**, 012010, 14 (2018)
43. Schinnerl, B., Steinbauer, R.: A Note on the Gannon-Lee theorem. *Lett. Math. Phys.* **111**, 142 (2021)
44. Shiohama, K.: *An Introduction to the Geometry of Alexandrov Spaces*. Lecture Notes Series, vol. 8. Seoul National University (1993)
45. Sbierski, J.: The C^0 -inextendibility of the Schwarzschild spacetime and the spacelike diameter in Lorentzian geometry. *J. Differ. Geom.* **108**, 319–378 (2018)
46. Senovilla, J.M.M.: Singularity theorems and their consequences. *Gen. Relat. Gravit.* **30**(5), 701–848 (1998)
47. Senovilla, J.M.M.: A critical appraisal of the singularity theorems. [arXiv:2108.07296](https://arxiv.org/abs/2108.07296) [gr-qc] (2021)
48. Senovilla, J.M.M., Garfinkle, D.: The 1965 Penrose singularity theorem. *Classical Quantum Gravity* **32**(12), 124008, 45 (2015)
49. Steinbauer, R.: A note on distributional semi-Riemannian geometry. *Novi Sad. J. Math.* **38**(3), 189–199 (2008)
50. Steinbauer, R., Vickers, J.: On the Geroch-Traschen class of metrics. *Classical Quantum Gravity* **26**(6), 065001, 19 (2009)
51. Villani, C.: *Optimal Transport. Old and New*, Grundlehren der Mathematischen Wissenschaften. Springer, Berlin (2009)