Reaction-Driven Relaxation in Three-Dimensional Keller–Segel–Navier–Stokes Interaction

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Abstract: The Keller–Segel–Navier–Stokes system

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \chi \nabla \cdot (n \nabla c) + \rho n - \mu n^2, \\ c_t + u \cdot \nabla c = \Delta c - c + n, \\ u_t + (u \cdot \nabla) u = \Delta u + \nabla P + n \nabla \phi + f(x, t), \end{cases} \quad \nabla \cdot u = 0, \quad (\star)$$

is considered in a smoothly bounded convex domain $\Omega \subset \mathbb{R}^3$, with $\phi \in W^{2,\infty}(\Omega)$ and $f \in C^1(\bar{\Omega} \times [0, \infty); \mathbb{R}^3)$, and with $\chi > 0$, $\rho \in \mathbb{R}$ and $\mu > 0$. As recent literature has shown, for all reasonably mild initial data a corresponding no-flux/no-flux/Dirichlet initial-boundary value problem possesses a global generalized solution, but the knowledge on its regularity properties has not yet exceeded some information on fairly basic integrability features. The present study reveals that whenever $\omega > 0$, requiring that

$$\frac{\rho}{\min\{\mu, \mu^{\frac{3}{2}+\omega}\}} < \eta$$

with some $\eta = \eta(\omega) > 0$, and that f satisfies a suitable assumption on ultimate smallness, is sufficient to ensure that each of these generalized solutions becomes eventually smooth and classical. Furthermore, under these hypotheses $(\star)$ is seen to admit an absorbing set with respect to the topology in $L^\infty(\Omega)$. By trivially applying to the case when $\mu > 0$ is arbitrary and $\rho \leq 0$, these results especially assert essentially unconditional statements on eventual regularity in taxis-reaction systems interacting with liquid environments, such as arising in contexts of models for broadcast spawning discussed in recent literature.

1. Introduction

Regularity issues form a central aspect in the literature concerned with the analysis of evolution systems accounting for taxis mechanisms. In fact, well-known findings

on the occurrence of spontaneous singularity formation already in simple Keller–Segel type systems form quite unambiguous caveats which indicate that including chemotactic cross-diffusion as a model element may go along with substantial limitations of solution regularity ([22,39,48,54]; cf. also the survey [34]). Corresponding mathematical questions naturally become yet more sophisticated when taxis processes are embedded into more intricate models, and understanding the singularity-supporting potential of chemotactic cross-diffusion in complex frameworks has accordingly attracted considerable attention in the past years ([1,42]).

The present work addresses this problem area in the context of a model for the interaction of a chemotactically active population with a liquid environment, as found to be of relevance not only in experimental setups involving populations of swimming bacteria ([8,37,52]), but moreover also in descriptions of spatio-temporal evolution in processes of broadcast spawning during coral fertilization ([7,25,26,38]). Specifically, we shall be concerned with the Keller–Segel–Navier–Stokes system

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \chi \nabla \cdot (n \nabla c) + \rho n - \mu n^2, & x \in \Omega, \ t > 0, \\ c_t + u \cdot \nabla c = \Delta c - c + n, & x \in \Omega, \ t > 0, \\ u_t + (u \cdot \nabla)u = \Delta u + \nabla P + n \nabla \phi + f(x, t), \quad \nabla \cdot u = 0, & x \in \Omega, \ t > 0, \end{cases} \quad (1.1)$$

for an unknown population density n in an N -dimensional domain Ω , and for a signal concentration c and an incompressible fluid represented through its velocity field u and an associated pressure P . By requiring χ to be positive, (1.1) models attractive tactic motion of individuals toward increasing signal concentrations, additionally affected by transport of both these quantities through the surrounding fluid which, in turn, is influenced not only by an external force f but also by cells through buoyancy. Under the assumptions $\mu > 0$ and $\rho \in \mathbb{R}$ considered here, (1.1) moreover accounts for quadratic degradation in the population density, and hence both addresses chemotaxis-growth processes in which populations undergo natural logistic-type proliferation and death ([18,42]), and also covers situations in which quadratic absorption, then mainly accompanied by the choice $\rho = 0$ or even $\rho < 0$, is due to the inclusion of reaction mechanisms ([26]).

With regard to solution regularity, the interplay of chemotactic cross-diffusion with such zero-order dissipation seems quite delicate, though yet far from completely understood, already in contexts of corresponding fluid-free Keller–Segel systems. Indeed, in the resulting version of (1.1) with $u \equiv 0$ any choice of $\mu > 0$ is sufficient to suppress any blow-up phenomenon in two-dimensional initial value problems in the sense that for widely arbitrary initial data, global bounded solutions always exist ([41,45]); in associated three- and higher-dimensional counterparts, however, similar findings on exclusion of explosions to date seem to rely on the stronger hypothesis that $\mu > \mu_0$ with some $\mu_0 = \mu_0(\Omega) > 0$ ([53,61]), while for small values of $\mu > 0$ only some weak solutions are known to exist globally ([32]). Although some studies concerned with simplified model variants have revealed some considerably strong singularity-counteracting effects of logistic damping in the sense of immediate regularization of strongly singular distributions ([33,60]), not only results on possibly transient emergence of high population densities ([24,56]), but moreover especially some detections of genuine blow-up both in high-dimensional systems with quadratic zero-order dissipation ([12]), and in three-dimensional models involving some subquadratic but yet superlinear absorption ([12]), indicate some persistence of taxis-driven destabilization also in the presence of such degradation mechanisms.

In light of these prerequisites, it seems far from surprising that the knowledge on corresponding issues in coupled chemotaxis-fluid systems of the form (1.1) is yet quite

thin. After all, results on smooth global solvability could be established for the two-dimensional version of (1.1) whenever μ is positive ([10,50]), while a similar statement could be derived when $N = 3$ and $\mu \geq 23$ at least for a Stokes simplification of (1.1) in which the nonlinear convective term $(u \cdot \nabla)u$ is neglected ([49]). For the fully coupled three-dimensional Keller–Segel–Navier–Stokes system (1.1) with arbitrary $\mu > 0$, however, merely a statement on global existence of certain generalized solutions seems available, asserting quite poor regularity properties only (see Proposition 1.1 and Definition 9.2 below). In this sense, (1.1) seems much less understood than its well-studied relative

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \chi \nabla \cdot (n \nabla c), & x \in \Omega, \ t > 0, \\ c_t + u \cdot \nabla c = \Delta c - nc, & x \in \Omega, \ t > 0, \\ u_t + (u \cdot \nabla)u = \Delta u + \nabla P + n \nabla \phi, \quad \nabla \cdot u = 0, & x \in \Omega, \ t > 0, \end{cases} \quad (1.2)$$

and logistic variants thereof, in which the decisive difference in signal evolution, here accounting for signal consumption through individuals rather than production as in (1.1), has facilitated energy-based analytical approaches to establish results not only on global existence of solutions in fairly natural frameworks of weak solvability ([9,31,36,57]), but also on qualitative aspects such as eventual regularization and large-time stabilization toward homogeneous states ([6,31,55,58]; see also [28] and [4] for an analysis of small-data solutions).

In comparison to (1.2), the system (1.1) apparently lacks any similarly meaningful energy-like global dissipative features; in fact, the well-known gradient structure of the classical proliferation-free Keller–Segel system ([40]) seems to disappear already when only one of the extra model elements in (1.1) is added to the latter, that is when either logistic contributions are included without any fluid interplay, or when, alternatively, a coupling to the (Navier-)Stokes equations is considered in the absence of such zero-order terms. This can be viewed as indicating the possibility of dynamics considerably far from spatial homogeneity, and hence remarkably different from that in (1.2): Indeed, the combined action of self-enhancing chemoattraction with logistic proliferation is not only known to generate spatially structured equilibria ([29]), but may beyond this bring about some quite colorful dynamical facets, as detected partially in the course of numerical simulations ([19]), and partially even by means of rigorous analysis ([24,56]). Apart from this, several findings in the recent analytical literature have revealed some nontrivial qualitative and even quantitative effects that fluid interaction may have on the solution behavior in various types of chemotaxis systems ([20,25–27]).

Main results. The purpose of the present study now consists in developing an approach that, despite the challenges accordingly resulting from a lack of favorable structural properties, is capable of identifying situations in which solutions to the fully coupled system (1.1) exhibit regular behavior at least eventually. To formulate this more precisely, let us assume henceforth that $\Omega \subset \mathbb{R}^3$ be a bounded convex domain with smooth boundary, that $\chi > 0$, $\rho \in \mathbb{R}$ and $\mu > 0$, and that

$$\phi \in W^{2,\infty}(\Omega) \quad \text{and} \quad f \in C^1(\bar{\Omega} \times [0, \infty); \mathbb{R}^3). \quad (1.3)$$

We shall then consider (1.1) along with the initial conditions

$$n(x, 0) = n_0(x), \quad c(x, 0) = c_0(x) \quad \text{and} \quad u(x, 0) = u_0(x), \quad x \in \Omega, \quad (1.4)$$

and under the boundary conditions

$$\frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = 0 \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega, \quad (1.5)$$

where our standing assumptions are that

$$\begin{cases} n_0 \in C^0(\bar{\Omega}) \text{ is nonnegative with } n_0 \not\equiv 0, & \text{that} \\ c_0 \in W^{1,\infty}(\Omega) \text{ is nonnegative,} & \text{and that} \\ u_0 \in W^{2,2}(\Omega; \mathbb{R}^3) \cap W_0^{1,2}(\Omega; \mathbb{R}^3) \cap L_\sigma^2(\Omega), \end{cases} \quad (1.6)$$

with $L_\sigma^p(\Omega) := \{\varphi \in L^p(\Omega; \mathbb{R}^3) \mid \nabla \cdot \varphi = 0\}$ denoting the space of all solenoidal vector fields in $L^p(\Omega; \mathbb{R}^3)$ for $p > 1$.

In order to appropriately recall from [59] a basic result on existence and approximation, for $\varepsilon \in (0, 1)$ we furthermore introduce the regularized variant of (1.1), (1.4), (1.5) given by

$$\begin{cases} n_{\varepsilon t} + u_\varepsilon \cdot \nabla n_\varepsilon = \Delta n_\varepsilon - \chi \nabla \cdot \left(\frac{n_\varepsilon}{1 + \varepsilon n_\varepsilon} \nabla c_\varepsilon \right) + \rho n_\varepsilon - \mu n_\varepsilon^2, & x \in \Omega, \ t > 0, \\ c_{\varepsilon t} + u_\varepsilon \cdot \nabla c_\varepsilon = \Delta c_\varepsilon - c_\varepsilon + n_\varepsilon, & x \in \Omega, \ t > 0, \\ u_{\varepsilon t} + (Y_\varepsilon u_\varepsilon \cdot \nabla) u_\varepsilon = \Delta u_\varepsilon + \nabla P_\varepsilon + n_\varepsilon \nabla \phi + f(x, t), \quad \nabla \cdot u_\varepsilon = 0, & x \in \Omega, \ t > 0, \\ \frac{\partial n_\varepsilon}{\partial \nu} = 0, \quad \frac{\partial c_\varepsilon}{\partial \nu} = 0, \quad u_\varepsilon = 0, & x \in \partial\Omega, \ t > 0, \\ n_\varepsilon(x, 0) = n_0(x), \quad c_\varepsilon(x, 0) = c_0(x), \quad u_\varepsilon(x, 0) = u_0(x), & x \in \Omega, \end{cases} \quad (1.7)$$

with the family $(Y_\varepsilon)_{\varepsilon \in (0,1)}$ of Yosida approximations determined by

$$Y_\varepsilon v := (1 + \varepsilon A)^{-1} v, \quad v \in L_\sigma^2(\Omega), \ \varepsilon \in (0, 1), \quad (1.8)$$

where here and below, A represents the Stokes operator under homogeneous Dirichlet boundary conditions in Ω , with its respective realization in $L^p(\Omega; \mathbb{R}^3)$ for $p > 1$ defined in its domain $D(A_p) := W^{2,p}(\Omega; \mathbb{R}^3) \cap W_0^{1,p}(\Omega; \mathbb{R}^3) \cap L_\sigma^p(\Omega)$, and with the corresponding fractional powers thereof denoted by $A^\beta = A_p^\beta$, $\beta \in \mathbb{R}$, in the sequel.

Within this setting, the following result on global existence and approximation has been obtained in [59].

Proposition 1.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded convex domain with smooth boundary, assume that $\chi > 0$, $\rho \in \mathbb{R}$ and $\mu > 0$, and suppose that ϕ , f and (n_0, c_0, u_0) comply with (1.3) and (1.6). Then there exist functions*

$$\begin{cases} n \in L^\infty((0, \infty); L^1(\Omega)) \cap L_{loc}^2(\bar{\Omega} \times [0, \infty)) \cap L_{loc}^{\frac{16}{13}}([0, \infty); W^{1, \frac{16}{13}}(\Omega)), \\ c \in L^\infty((0, \infty); L^6(\Omega)) \cap L_{loc}^{\frac{5}{3}}([0, \infty); W^{2, \frac{8}{3}}(\Omega)) \quad \text{and} \\ u \in L^\infty((0, \infty); L_\sigma^2(\Omega)) \cap L_{loc}^{\frac{3}{2}}(\bar{\Omega} \times [0, \infty)) \cap L_{loc}^2([0, \infty); W_0^{1,2}(\Omega)), \end{cases} \quad (1.9)$$

such that (n, c, u) forms a global generalized solution of (1.1), (1.4), (1.5) in terms of Definition 9.2 below. Furthermore, this solution can be obtained by approximation through the regularized problems (1.7) in the sense that for each $\varepsilon \in (0, 1)$ one can find functions

$$\begin{cases} n_\varepsilon \in C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)), \\ c_\varepsilon \in \bigcap_{q \geq 1} C^0([0, \infty); W^{1,q}(\Omega)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)), \\ u_\varepsilon \in \bigcap_{\beta \in (0,1)} C^0([0, \infty); D(A_2^\beta)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty); \mathbb{R}^3) \quad \text{and} \\ P_\varepsilon \in C^{1,0}(\Omega \times (0, \infty)) \end{cases}$$

such that $n_\varepsilon > 0$ and $c_\varepsilon \geq 0$ in $\bar{\Omega} \times (0, \infty)$, that $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ solves (1.7) in the classical sense, and that with some $(\varepsilon_j)_{j \in \mathbb{N}} \subset (0, 1)$ fulfilling $\varepsilon_j \searrow 0$ as $j \rightarrow \infty$, we have

$$n_\varepsilon \rightarrow n, \quad c_\varepsilon \rightarrow c \quad \text{and} \quad u_\varepsilon \rightarrow u \quad \text{a.e. in } \Omega \times (0, \infty)$$

as $\varepsilon = \varepsilon_j \searrow 0$.

In line with the above discussion, in general it seems unclear how far regularity properties beyond those documented in (1.9) can be expected, especially in view of the fact that the considered coupling to the full three-dimensional Navier-Stokes system apparently limits possible extensions of knowledge on boundedness features in the fluid-related part. The main results of the present manuscript now make sure that at least when the reproduction parameter ρ lies below a certain positive number, increasing with the degradation coefficient μ , all the solutions obtained in Proposition 1.1 become smooth and bounded after some individual relaxation time, and that in this case even a bounded absorbing set within a convenient topology can be identified:

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded convex domain with smooth boundary, and let $\chi > 0$ and $\phi \in W^{2,\infty}(\Omega)$. Then for all $\omega > 0$ there exist $\eta = \eta(\omega) > 0$ and $\kappa = \kappa(\omega) > 0$ with the following property: Suppose that $\rho \in \mathbb{R}$, $\mu > 0$ and $f \in C^1(\bar{\Omega} \times [0, \infty); \mathbb{R}^3)$ are such that*

$$\rho < \eta \cdot \min \left\{ \mu, \mu^{\frac{3}{2}+\omega} \right\} \quad (1.10)$$

and

$$\limsup_{t \rightarrow \infty} \int_t^{t+1} \|f(\cdot, s)\|_{L^{\frac{3}{2}+\omega}(\Omega)}^2 ds < \kappa, \quad (1.11)$$

as well as

$$\sup_{t>0} \int_t^{t+1} \|f(\cdot, s)\|_{L^{\mathbf{p}}(\Omega)}^{\mathbf{q}} ds < \infty \quad (1.12)$$

hold with some $\mathbf{p} > \frac{3}{2}$ and $\mathbf{q} > \frac{2\mathbf{p}}{2\mathbf{p}-3}$, that n_0, c_0 and u_0 satisfy (1.6), and that (n, c, u) denotes the corresponding global generalized solution of (1.1), (1.4), (1.5) from Proposition 1.1. Then one can find $t_0 = t_0(\omega, \eta, f, n_0, c_0, u_0) > 0$ such that

$$n \in C^{2,1}(\bar{\Omega} \times [t_0, \infty)), \quad c \in C^{2,1}(\bar{\Omega} \times [t_0, \infty)) \quad \text{and} \quad u \in C^{2,1}(\bar{\Omega} \times [t_0, \infty); \mathbb{R}^3), \quad (1.13)$$

and such that with some $P \in C^{1,0}(\bar{\Omega} \times [t_0, \infty))$, the quadruple (n, c, u, P) is a classical solution of (1.1), (1.5) in $\bar{\Omega} \times [t_0, \infty)$. Moreover, under these assumptions on f the problem (1.1) possesses a bounded absorbing set in $(L^\infty(\Omega))^5$ in the sense that there exists $C = C(\omega, f) > 0$ such that any such solution satisfies

$$\limsup_{t \rightarrow \infty} \left\{ \|n(\cdot, t)\|_{L^\infty(\Omega)} + \|c(\cdot, t)\|_{L^\infty(\Omega)} + \|u(\cdot, t)\|_{L^\infty(\Omega)} \right\} \leq C. \quad (1.14)$$

Remark. i) We emphasize that since (1.10) is trivially satisfied whenever $\rho \leq 0$, the conclusion of Theorem 1.2 fully covers situations in which the considered population does not spontaneously proliferate, as naturally present in contexts merely involving reaction-like quadratic degradation in the zero-order part, such as in the modeling framework addressed in [25] and [26] to describe broadcast spawning.

ii) Apart from those specified above, further possible dependencies on Ω , Φ and χ influence the choice of the quantities η , κ , t_0 and C in Theorem 1.2, as would, in a natural manner, the additional inclusion of non-normalized further system parameters such as diffusivities or rates of signal production and decay. In order to avoid further expansion of the already considerable technicalities in the arguments to be developed in the sequel, here and below we refrain from precisely tracking these explicitly.

iii) Already in the fluid-free case $u \equiv 0$ trivially included, Theorem 1.2 provides some progress in comparison with the existing literature: In contrast to the precedent finding on eventual smoothness in the corresponding taxis-only version of (1.1) stated in [32, Theorem 1.1], namely, the above result reveals ultimate regularity under an assumption which, besides being independent of the initial data, relates the required smallness condition on ρ to μ through (1.10) in an essentially explicit manner.

iv) A natural problem arising in the interpretation of Theorem 1.2 appears to consist in providing information about the large time behavior of solutions which is more detailed than that contained in (1.14). In fact, a preliminary result in this direction asserts that under the explicit assumption that $\mu > \frac{\chi\sqrt{\rho_+}}{4}$, the solution from Proposition 1.1 satisfies $\text{ess} \lim_{t \rightarrow \infty} \left\{ \|n(\cdot, t) - \frac{\rho_+}{\mu}\|_{L^1(\Omega)} + \|c(\cdot, t) - \frac{\rho_+}{\mu}\|_{L^p(\Omega)} \right\} = 0$ for all $p \in [1, 6)$, and that if furthermore we have $\int_t^{t+1} \|f(\cdot, s)\|_{L^{\frac{6}{5}}(\Omega)}^2 ds \rightarrow 0$ as $t \rightarrow \infty$, then also $\text{ess} \lim_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^2(\Omega)} = 0$ ([59]). Upon a straightforward interpolation, it evidently follows from Theorem 1.2 that if these requirements are fulfilled beyond those in (1.10)-(1.12), then actually for all $p \in [1, \infty)$ we have

$$\left\| n(\cdot, t) - \frac{\rho_+}{\mu} \right\|_{L^p(\Omega)} + \left\| c(\cdot, t) - \frac{\rho_+}{\mu} \right\|_{L^p(\Omega)} + \|u(\cdot, t)\|_{L^p(\Omega)} \rightarrow 0 \quad \text{as } t \rightarrow \infty; \quad (1.15)$$

apart from that, if even $\rho = 0$ then the results from [23] and [5] become directly applicable so as to assert the quantitative estimate

$$\frac{1}{C(t+1)} \leq \frac{1}{|\Omega|} \|n(\cdot, t)\|_{L^1(\Omega)} \leq \|n(\cdot, t)\|_{L^\infty(\Omega)} \leq \frac{C}{t+1} \quad \text{for all } t > t_0 \quad (1.16)$$

with some appropriately large $C > 0$, and with $t_0 > 0$ as in Theorem 1.2 (cf. also [23] for a bound on $\sup_{t > t_0} (t+1) \|\nabla c(\cdot, t)\|_{L^p(\Omega)}$ for arbitrary $p \geq 1$). In more general parameter frameworks, and especially for positive ρ and small values of μ , however, in view of known analytical results on multiplicity in corresponding steady state problems ([29]) and of simulation-based indications for the possibility of quite chaotic solution behavior ([19]) we do not expect simple asymptotics as in (1.15) to prevail the dynamics in (1.1).

v) Our approach will make essential use of the boundedness assumption on the physical domain Ω ; in fact, inter alia due to a lack of comparison principles for (1.1) this requirement appears to be crucial already in the derivation of very basic boundedness features especially – but not exclusively – when $\rho > 0$ (cf. Sect. 2). We therefore have to leave open here the interesting question how far conclusions similar to those

from Theorem 1.2 can be drawn in cases of unbounded domains, and particularly when $\Omega = \mathbb{R}^3$.

Strategy. Our analysis can be viewed as being composed of a first part in which the action of chemotaxis is yet essentially faded out, and a second level in which full tribute is paid to the whole complexity of (1.1). In particular, basic eventual smallness properties of n_ε , as obtained for suitably small ρ from mere integration and subsequent interpolation, will be summarized in Sect. 2 and thereafter used in Sect. 3 to derive corresponding smallness features of $A^\beta u_\varepsilon$ with respect to the norm in $L^2(\Omega)$ for some β in the range $(\frac{1}{4}, \infty)$ throughout which $D(A_2^\beta)$ continuously embeds into $L^3(\Omega; \mathbb{R}^3)$. By means of a standard zero-order testing procedure and an application of maximal Sobolev regularity theory to the second Eq. in (1.7), in Sect. 4 this will in provide some basic knowledge on eventual smallness of ∇c_ε in an appropriately integrated sense.

The second stage of our argument will then be entered in Sect. 5, where by inter alia explicitly using the first Eq. in (1.7) for a second time, a coupled quantity of the form $\int_\Omega \psi(n_\varepsilon - \frac{\rho_+}{\mu}) + \int_\Omega |\nabla c_\varepsilon|^{2p}$ will be seen to enjoy some quasi-Lyapunov property for some $p > \frac{3}{2}$ and a suitably designed function ψ on $\mathbb{R}$ satisfying $s^{-p}\psi(s) \rightarrow 1$ as $s \rightarrow +\infty$; we remark already here that only from this point on, dependencies of the obtained estimates on χ appear. In Sects. 6 and 7, the improved eventual smallness properties of ∇c_ε thereby implied will be developed into L^∞ bounds for n_ε and a Hölder estimate for u_ε , whereupon standard regularity theories for parabolic and Stokes evolution problems become applicable so as to confirm the statement from Theorem 1.2 in Sect. 8.

Without explicit further mentioning, throughout the sequel we shall suppose that $\phi \in W^{2,\infty}(\Omega)$ is fixed, and given $\chi > 0$, $\rho \in \mathbb{R}$, $\mu > 0$, $f \in C^1(\overline{\Omega} \times [0, \infty); \mathbb{R}^3)$ and initial data fulfilling (1.6), for $\varepsilon \in (0, 1)$ we let $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ denote the corresponding solution of (1.7) obtained in Proposition 1.1. Apart from that, let us announce already here that within each of our proofs, constants will be labeled as $C_1, C_2, \dots$, and that in order to avoid an abundant globally consecutive numbering involving high indices, constants such as C_1 may attain different values in different proofs.

2. Basic Eventual Bounds for n_ε

A starting point for our asymptotic analysis is formed by the following basic information on global and eventual bounds for the first solution component, as resulting from a simple integration of the first Eq. in (1.7) in a standard manner.

Lemma 2.1. *i) We have*

$$\int_\Omega n_\varepsilon(x, t) dx \leq m := \max \left\{ \int_\Omega n_0, \frac{\rho_+ |\Omega|}{\mu} \right\} \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1) \quad (2.1)$$

as well as

$$\int_t^{t+1} \int_\Omega n_\varepsilon^2(x, s) dx ds \leq \frac{(\rho_+ + 1)m}{\mu} \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1). \quad (2.2)$$

ii) If $\rho_0 > 0$ is such that $\rho_0 \geq \rho$, then

$$\int_\Omega n_\varepsilon(x, t) dx \leq 2|\Omega| \cdot \frac{\rho_0}{\mu} \quad \text{for all } t \geq \frac{\ln 2}{\rho_0} \text{ and } \varepsilon \in (0, 1) \quad (2.3)$$

and

$$\int_t^{t+1} \int_{\Omega} n_{\varepsilon}^2(x, s) dx ds \leq 2|\Omega| \cdot \frac{\rho_0(\rho_0 + 1)}{\mu^2} \quad \text{for all } t \geq \frac{\ln 2}{\rho_0} \text{ and } \varepsilon \in (0, 1). \quad (2.4)$$

Proof. We fix any $\rho_1 \geq \rho_+$ and integrate the first Eq. in (1.7) over Ω to see that

$$\frac{d}{dt} \int_{\Omega} n_{\varepsilon} = \rho \int_{\Omega} n_{\varepsilon} - \mu \int_{\Omega} n_{\varepsilon}^2 \leq \rho_1 \int_{\Omega} n_{\varepsilon} - \mu \int_{\Omega} n_{\varepsilon}^2 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1). \quad (2.5)$$

Since $(\int_{\Omega} n_{\varepsilon})^2 \leq |\Omega| \cdot \int_{\Omega} n_{\varepsilon}^2$ for all $t > 0$ and $\varepsilon \in (0, 1)$ by the Cauchy-Schwarz inequality, this implies that $y_{\varepsilon}(t) := \int_{\Omega} n_{\varepsilon}(x, t) dx$, $t \geq 0$, $\varepsilon \in (0, 1)$, satisfies

$$y'_{\varepsilon}(t) \leq \rho_1 y_{\varepsilon}(t) - \frac{\mu}{|\Omega|} y_{\varepsilon}^2(t) \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1) \quad (2.6)$$

and hence, by a comparison argument,

$$y_{\varepsilon}(t) \leq \max \left\{ y_{\varepsilon}(0), \frac{\rho_1 |\Omega|}{\mu} \right\} \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1). \quad (2.7)$$

i) Choosing $\rho_1 := \rho_+$ in (2.7), we immediately obtain (2.1), whereupon integrating (2.5) in time shows that

$$\begin{aligned} \int_{\Omega} n_{\varepsilon}(\cdot, t+1) + \mu \int_t^{t+1} \int_{\Omega} n_{\varepsilon}^2 &\leq \int_{\Omega} n_{\varepsilon}(\cdot, t) + \rho_+ \int_t^{t+1} \int_{\Omega} n_{\varepsilon} \\ &\leq m + \rho_+ m \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1) \end{aligned}$$

and thereby proves (2.2).

ii) Taking $\rho_1 := \rho_0$ now, an explicit solution of the Bernoulli-type ODE associated with (2.6) yields the inequality

$$\begin{aligned} y_{\varepsilon}(t) &\leq \left\{ \frac{\mu}{\rho_0 |\Omega|} \cdot (1 - e^{-\rho_0 t}) + \frac{1}{y_{\varepsilon}(0)} \cdot e^{-\rho_0 t} \right\}^{-1} \\ &\leq \left\{ \frac{\mu}{\rho_0 |\Omega|} \cdot (1 - e^{-\rho_0 t}) \right\}^{-1} \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1), \end{aligned}$$

which directly yields (2.3), because $1 - e^{-\rho_0 t} \geq \frac{1}{2}$ for all $t \geq \frac{\ln 2}{\rho_0}$. Thereafter, once again integrating (2.5) we see that

$$\begin{aligned} \int_{\Omega} n_{\varepsilon}(\cdot, t+1) + \mu \int_t^{t+1} \int_{\Omega} n_{\varepsilon}^2 &\leq 2|\Omega| \cdot \frac{\rho_0}{\mu} + \rho_0 \cdot \left(2|\Omega| \cdot \frac{\rho_0}{\mu} \right) \\ &= 2|\Omega| \cdot \frac{\rho_0(\rho_0 + 1)}{\mu} \quad \text{for all } t \geq \frac{\ln 2}{\rho_0} \text{ and } \varepsilon \in (0, 1), \end{aligned}$$

and hence obtain (2.4). $\square$

Through appropriate interpolation, the latter shows how the particular form of the assumption in (1.10) enters a further eventual boundedness property of n_ε which involves topological information somewhat weaker than that in (2.4), but which on the other hand apparently yields genuine ultimate smallness, and which does so also in some cases when in Lemma 2.1 the expression $\frac{\rho_0}{\mu^2}$ is large.

Lemma 2.2. *Let $\omega > 0$. Then there exists $\theta_1 = \theta_1(\omega) \in (0, \frac{1}{2})$ with the following property: For all $\delta > 0$ one can find $\eta_1 = \eta_1(\omega, \delta) > 0$ such that whenever $\rho \in \mathbb{R}$ and $\mu > 0$ are such that (1.10) holds with some $\eta < \eta_1(\omega, \delta)$, then there exists $t_0 = t_0(\eta) > 0$ such that for each $p \in [\frac{3}{2}, \frac{3}{2} + \theta_1]$ and any $\varepsilon \in (0, 1)$ we have*

$$\int_t^{t+1} \|n_\varepsilon(\cdot, s)\|_{L^p(\Omega)}^2 ds < \delta \quad \text{for all } t > t_0. \quad (2.8)$$

Proof. Given $\omega > 0$, we let $\theta = \theta_1(\omega) > 0$ be suitably small such that $\theta < \frac{1}{2}$ and $\theta < \omega$, noting that the latter ensures the inequality

$$\left(\frac{3}{2} + \omega\right) \cdot \frac{4}{3 + 2\theta} = 2 \cdot \frac{3 + 2\omega}{3 + 2\theta} > 2, \quad (2.9)$$

and that the former restriction warrants that $a = a(\omega) := \frac{2+4\theta}{3+2\theta}$ satisfies $a < 1$. Now writing

$$C_1 = C_1(\omega) := \max \left\{ 1, |\Omega|^{\frac{8\theta}{9}} \right\}, \quad (2.10)$$

for arbitrary $\delta > 0$ we choose $\eta_1 = \eta_1(\omega, \delta) > 0$ small enough fulfilling $\eta_1 \leq 1$ as well as

$$4|\Omega|^{2-a} \eta_1^{2-a} < \frac{\delta}{C_1}, \quad (2.11)$$

and henceforth assume that $\rho \in \mathbb{R}$ and $\mu > 0$ are such that (1.10) holds with some positive $\eta < \eta_1$. Then with

$$\rho_0 = \rho_0(\eta) := \eta \cdot \min \left\{ \mu, \mu^{\frac{3}{2} + \omega} \right\} \quad (2.12)$$

we clearly have $\rho_0 > \rho$ and moreover also $\rho_0 > 0$, whence Lemma 2.1 applies to say that for any $\varepsilon \in (0, 1)$, both

$$\|n_\varepsilon(\cdot, t)\|_{L^1(\Omega)} \leq 2|\Omega| \cdot \frac{\rho_0}{\mu} \quad \text{for all } t \geq t_0 \quad (2.13)$$

and

$$\int_t^{t+1} \|n_\varepsilon(\cdot, s)\|_{L^2(\Omega)}^2 ds \leq 2|\Omega| \cdot \frac{\rho_0(\rho_0 + 1)}{\mu^2} \quad \text{for all } t \geq t_0 \quad (2.14)$$

are valid with $t_0 = t_0(\eta) := \frac{\ln 2}{\rho_0}$. As $\frac{3}{2} + \theta < 2$, we may interpolate here by means of the Hölder inequality to infer from (2.13) and (2.14) that according to our definition of a we have

$$\int_t^{t+1} \|n_\varepsilon(\cdot, s)\|_{L^{\frac{3}{2} + \theta}(\Omega)}^2 ds$$

$$\begin{aligned}
&\leq \int_t^{t+1} \|n_\varepsilon(\cdot, s)\|_{L^2(\Omega)}^{2a} \cdot \|n_\varepsilon(\cdot, s)\|_{L^1(\Omega)}^{2(1-a)} ds \\
&\leq \left(2|\Omega| \cdot \frac{\rho_0}{\mu}\right)^{2(1-a)} \int_t^{t+1} \|n_\varepsilon(\cdot, s)\|_{L^2(\Omega)}^{2a} ds \\
&\leq \left(2|\Omega| \cdot \frac{\rho_0}{\mu}\right)^{2(1-a)} \left(\int_t^{t+1} \|n_\varepsilon(\cdot, s)\|_{L^2(\Omega)}^2 ds\right)^a \\
&\leq \left(2|\Omega| \cdot \frac{\rho_0}{\mu}\right)^{2(1-a)} \cdot \left(2|\Omega| \cdot \frac{\rho_0(\rho_0 + 1)}{\mu^2}\right)^a \\
&= (2|\Omega|)^{2-a} \rho_0^{2-a} (\rho_0 + 1)^a \mu^{-2} \quad \text{for all } t \geq t_0 \text{ and } \varepsilon \in (0, 1).
\end{aligned} \tag{2.15}$$

Here if $\mu \leq 1$, then (2.12) along with the restriction $\eta \leq \eta_1 \leq 1$ warrants that $\rho_0 + 1 = \eta\mu^{\frac{3}{2}+\omega} + 1 \leq \eta + 1 \leq 2$, and that hence

$$\begin{aligned}
(2|\Omega|)^{2-a} \rho_0^{2-a} (\rho_0 + 1)^a \mu^{-2} &\leq (2|\Omega|)^{2-a} \rho_0^{2-a} \cdot 2^a \cdot \mu^{-2} \\
&= 4|\Omega|^{2-a} \cdot (\eta\mu^{\frac{3}{2}+\omega})^{2-a} \mu^{-2} \\
&= 4|\Omega|^{2-a} \eta^{2-a} \mu^{(\frac{3}{2}+\omega)(2-a)-2},
\end{aligned}$$

so that computing $(\frac{3}{2} + \omega)(2 - a) = (\frac{3}{2} + \omega) \cdot \frac{4}{3+2\theta}$ and recalling (2.9) we find that in this case,

$$(2|\Omega|)^{2-a} \rho_0^{2-a} (\rho_0 + 1)^a \mu^{-2} \leq 4|\Omega|^{2-a} \eta^{2-a} < \frac{\delta}{C_1}$$

by (2.11).

Next, if $\mu > 1$ then (2.12) means that $\rho_0 = \eta\mu$, and thus in the case $\rho_0 \leq 1$ we can estimate

$$\begin{aligned}
(2|\Omega|)^{2-a} \rho_0^{2-a} (\rho_0 + 1)^a \mu^{-2} &\leq (2|\Omega|)^{2-a} \rho_0^{2-a} \cdot 2^a \cdot \mu^{-2} \\
&= 4|\Omega|^{2-a} (\eta\mu)^{2-a} \mu^{-2} \\
&= 4|\Omega|^{2-a} \eta^{2-a} \mu^{-a},
\end{aligned}$$

using that then we still have $\rho_0 + 1 \leq 2$. As now $\mu^{-a} \leq 1$, again invoking (2.11) we see that

$$(2|\Omega|)^{2-a} \rho_0^{2-a} (\rho_0 + 1)^a \mu^{-2} < \frac{\delta}{C_1}$$

also in this case.

Finally, if yet $\mu > 1$ but now $\rho_0 > 1$, then estimating $\rho_0 + 1 \leq 2\rho_0$ leads to the inequality

$$\begin{aligned}
(2|\Omega|)^{2-a} \rho_0^{2-a} (\rho_0 + 1)^a \mu^{-2} &\leq (2|\Omega|)^{2-a} \rho_0^{2-a} \cdot (2\rho_0)^a \cdot \mu^{-2} \\
&= 4|\Omega|^{2-a} \eta^2 \\
&\leq 4|\Omega|^{2-a} \eta^{2-a} \\
&< \frac{\delta}{C_1},
\end{aligned}$$

once more because of (2.11), and again due to the restriction $\eta_1 \leq 1$. In summary, in both cases $\mu \leq 1$ and $\mu > 1$ we infer from (2.15) that

$$\int_t^{t+1} \|n_\varepsilon(\cdot, s)\|_{L^{\frac{3}{2}+\theta}(\Omega)}^2 ds < \frac{\delta}{C_1} \quad \text{for all } t \geq t_0 \text{ and } \varepsilon \in (0, 1).$$

Thus, if $p \in [\frac{3}{2}, \frac{3}{2} + \theta]$ is arbitrary, then by the Hölder inequality we find that

$$\begin{aligned} \int_t^{t+1} \|n_\varepsilon(\cdot, s)\|_{L^p(\Omega)}^2 ds &\leq |\Omega|^{\frac{2(p_1-p)}{p_1 p}} \int_t^{t+1} \|n_\varepsilon(\cdot, s)\|_{L^{\frac{3}{2}+\theta}(\Omega)}^2 ds \\ &\leq |\Omega|^{\frac{2(p_1-p)}{p_1 p}} \cdot \frac{\delta}{C_1} \quad \text{for all } t \geq t_0 \text{ and } \varepsilon \in (0, 1) \end{aligned} \quad (2.16)$$

with $p_1 := \frac{3}{2} + \theta$, where since

$$0 \leq \frac{2(p_1 - p)}{p_1 p} \leq \frac{2\theta}{(\frac{3}{2})^2} = \frac{8\theta}{9},$$

we can estimate

$$|\Omega|^{\frac{2(p_1-p)}{p_1 p}} \leq \max \left\{ 1, |\Omega|^{\frac{8\theta}{9}} \right\}.$$

Recalling (2.10), we thereby see that (2.16) entails (2.8). $\square$

3. Smallness of u_ε in $D(A_2^\beta)$ for Some $\beta > \frac{1}{4}$

The purpose of this key section is to make sure that when applied to suitably small $\delta > 0$, the bounds obtained in Lemma 2.2 provide sufficient eventual smallness properties of the coupling-induced contribution $n_\varepsilon \nabla \phi$ to the forcing term in the Navier-Stokes subsystem of (1.7), so as to warrant ultimate estimates for u_ε with respect to the norm in $D(A_2^\beta)$ for some β exceeding the number $\frac{1}{4}$ quite commonly encountered in regularity analysis of three-dimensional Navier-Stokes problems ([46]).

As a means to appropriately derive upper estimates for functions satisfying linearly damped ODIs involving sources for which certain averaged bounds are known, from [59, Lemma 3.4] let us recall the following observation that will be referred to not only in this section, but also in Lemma 4.1 below.

Lemma 3.1. *Let $t_0 \in \mathbb{R}$, $T \in (t_0, \infty]$ and $a > 0$, and suppose that $y \in C^0([t_0, T)) \cap C^1((t_0, T))$ has the property that*

$$y'(t) + ay(t) \leq h(t) \quad \text{for all } t \in (t_0, T) \quad (3.1)$$

with some nonnegative $h \in L^1(\mathbb{R})$ for which there exist $\tau > 0$ and $b > 0$ such that

$$\frac{1}{\tau} \int_t^{t+\tau} h(s) ds \leq b \quad \text{for all } t \in (t_0, T). \quad (3.2)$$

Then

$$y(t) \leq e^{-a(t-t_0)} y(t_0) + \frac{b\tau}{1 - e^{-a\tau}} \quad \text{for all } t \in [t_0, T), \quad (3.3)$$

and in particular

$$y(t) \leq y(t_0) + \frac{b\tau}{1 - e^{-a\tau}} \quad \text{for all } t \in [t_0, T). \quad (3.4)$$

Through an analysis of the energy inequality associated with the approximate Navier-Stokes Eq. in (1.7), a first conclusion of Lemma 3.1 now asserts eventual smallness of a temporally averaged Dirichlet integral associated with the fluid velocity field, provided that the external force f satisfies a smallness assumption slightly weaker than that in (1.11).

Lemma 3.2. *Let $\omega > 0$. Then for all $\delta > 0$ there exist $\eta_2 = \eta_2(\omega, \delta) > 0$ and $\kappa_2 = \kappa_2(\delta) > 0$ with the property that if $\rho \in \mathbb{R}$, $\mu > 0$ and $f \in C^1(\bar{\Omega} \times [0, \infty); \mathbb{R}^3)$ are such that (1.10) as well as*

$$\limsup_{t \rightarrow \infty} \int_t^{t+1} \|f(\cdot, s)\|_{L^{\frac{6}{5}}(\Omega)}^2 ds < \kappa \quad (3.5)$$

hold with some $\eta < \eta_2(\omega, \delta)$ and $\kappa < \kappa_2(\delta)$, then one can find $t_0 = t_0(\omega, \eta, \kappa, n_0, u_0) > 0$ such that for all $\varepsilon \in (0, 1)$ we have

$$\int_t^{t+1} \|\nabla u_\varepsilon(\cdot, s)\|_{L^2(\Omega)}^2 < \delta \quad \text{for all } t \geq t_0. \quad (3.6)$$

Proof. Given $\omega > 0$, we let $\theta = \theta_1(\omega) \in (0, \frac{1}{2})$ be as provided by Lemma 2.2 and abbreviate $p = p(\omega) := \frac{3}{2} + \theta$ and $C_1 = C_1(\omega) := |\Omega|^{\frac{5p-6}{6p}}$. Moreover, we let $C_2 := \|\nabla \phi\|_{L^\infty(\Omega)}$ and invoke a Poincaré inequality and a Sobolev inequality to find $C_3 > 0$ and $C_4 > 0$ such that

$$C_3 \int_\Omega |\varphi|^2 \leq \int_\Omega |\nabla \varphi|^2 \quad \text{for all } \varphi \in W_0^{1,2}(\Omega; \mathbb{R}^3) \quad (3.7)$$

and

$$\|\varphi\|_{L^6(\Omega)} \leq C_4 \|\nabla \varphi\|_{L^2(\Omega)} \quad \text{for all } \varphi \in W_0^{1,2}(\Omega; \mathbb{R}^3). \quad (3.8)$$

Writing $C_5 := 1 + (1 - e^{-\frac{C_3}{2}})^{-1}$, for fixed $\delta > 0$ we thereupon pick $\kappa_2 = \kappa_2(\delta) > 0$ and $\delta_1 = \delta_1(\omega, \delta) > 0$ such that

$$2C_4^2 C_5 \kappa_2 < \frac{\delta}{8} \quad (3.9)$$

as well as

$$2C_1^2 C_2^2 C_4^2 C_5 \delta_1 < \frac{\delta}{8}. \quad (3.10)$$

We then suppose that $f \in C^1(\bar{\Omega} \times [0, \infty); \mathbb{R}^3)$ has the property (3.5) with some $\kappa < \kappa_2$, and that $\rho \in \mathbb{R}$ and $\mu > 0$ are such that (1.10) holds with some $\eta < \eta_2 = \eta_2(\omega, \delta) := \eta_1(\omega, \delta_1)$, where $\eta_1(\cdot, \cdot)$ is as provided by Lemma 2.2. Thus, (3.5) implies that there exists $t_1 = t_1(\kappa) > 0$ such that

$$\int_t^{t+1} \|f(\cdot, s)\|_{L^{\frac{6}{5}}(\Omega)}^2 ds \leq \kappa \quad \text{for all } t \geq t_1, \quad (3.11)$$

whereas Lemma 2.2 yields $t_2 = t_2(\eta, \kappa) > t_1$ such that for all $\varepsilon \in (0, 1)$, the corresponding solution has the property that that

$$\int_t^{t+1} \|n_\varepsilon(\cdot, s)\|_{L^p(\Omega)}^2 ds \leq \delta_1 \quad \text{for all } t \geq t_2. \quad (3.12)$$

Furthermore, using Lemma 2.1 and the Hölder inequality, we see that with $m := \max \left\{ \int_\Omega n_0, \frac{\rho_+ |\Omega|}{\mu} \right\}$ we have

$$\begin{aligned} & \int_t^{t+1} \|n_\varepsilon(\cdot, s)\|_{L^p(\Omega)}^2 ds \\ & \leq C_6 = C_6(\omega, n_0) := |\Omega|^{\frac{2-p}{p}} \cdot \frac{(1+\rho_+)m}{\mu} \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1), \end{aligned} \quad (3.13)$$

while by continuity of f on $\bar{\Omega} \times [0, t_2]$ we can fix $C_7 = C_7(\eta, \kappa) > 0$ fulfilling

$$\int_t^{t+1} \|f(\cdot, s)\|_{L^{\frac{6}{5}}(\Omega)}^2 ds \leq C_7 \quad \text{for all } t > 0. \quad (3.14)$$

We next abbreviate

$$C_8 = C_8(\omega, \eta, \kappa, n_0, u_0) := \int_\Omega |u_0|^2 + \frac{C_9}{1 - e^{-\frac{C_3}{2}}}, \quad (3.15)$$

where

$$C_9 = C_9(\omega, \eta, \kappa, n_0) := 2C_2^2 C_4^2 C_6 + 2C_4^2 C_7. \quad (3.16)$$

and choose $t_0 = t_0(\omega, \eta, \kappa, n_0, u_0) > t_2$ large enough fulfilling

$$C_8 e^{-\frac{C_3}{2}(t_0 - t_2)} \leq \frac{\delta}{4}. \quad (3.17)$$

Then testing the third Eq. in (1.7) by u_ε we obtain

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 = \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla \phi + \int_\Omega f \cdot u_\varepsilon \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1),$$

where using the Hölder inequality, (3.8) as well as Young's inequality show that by definition on C_2 we have

$$\begin{aligned} & \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla \phi + \int_\Omega f \cdot u_\varepsilon \\ & \leq \|u_\varepsilon\|_{L^6(\Omega)} \cdot \left\{ C_2 \|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)} + \|f\|_{L^{\frac{6}{5}}(\Omega)} \right\} \\ & \leq C_4 \|\nabla u_\varepsilon\|_{L^2(\Omega)} \cdot \left\{ C_2 \|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)} + \|f\|_{L^{\frac{6}{5}}(\Omega)} \right\} \\ & \leq \frac{1}{2} \int_\Omega |\nabla u_\varepsilon|^2 + \frac{C_4^2}{2} \cdot \left\{ C_2 \|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)} + \|f\|_{L^{\frac{6}{5}}(\Omega)} \right\}^2 \\ & \leq \frac{1}{2} \int_\Omega |\nabla u_\varepsilon|^2 + C_2^2 C_4^2 \|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)}^2 + C_4^2 \|f\|_{L^{\frac{6}{5}}(\Omega)}^2 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} |u_{\varepsilon}|^2 + \int_{\Omega} |\nabla u_{\varepsilon}|^2 &\leq 2C_2^2 C_4^2 \|n_{\varepsilon}\|_{L^{\frac{6}{5}}(\Omega)}^2 \\ &\quad + 2C_4^2 \|f\|_{L^{\frac{6}{5}}(\Omega)}^2 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1), \end{aligned}$$

so that since according to (3.7) we have

$$\int_{\Omega} |\nabla u_{\varepsilon}|^2 \geq \frac{C_3}{2} \int_{\Omega} |u_{\varepsilon}|^2 + \frac{1}{2} \int_{\Omega} |\nabla u_{\varepsilon}|^2 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1),$$

the functions $y_{\varepsilon}(t) := \int_{\Omega} |u_{\varepsilon}(x, t)|^2 dx$, $t \geq 0$, and $h_{\varepsilon}(t) := 2C_2^2 C_4^2 \|n_{\varepsilon}(\cdot, t)\|_{L^{\frac{6}{5}}(\Omega)}^2 + 2C_4^2 \|f(\cdot, t)\|_{L^{\frac{6}{5}}(\Omega)}^2$, $t > 0$, satisfy

$$y'_{\varepsilon}(t) + \frac{C_3}{2} y_{\varepsilon}(t) + \frac{1}{2} \int_{\Omega} |\nabla u_{\varepsilon}|^2 \leq h_{\varepsilon}(t) \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1). \quad (3.18)$$

In order to firstly derive a temporally global bound for y from this, we recall (3.13), (3.14) and (3.16) to see that

$$\int_t^{t+1} h_{\varepsilon}(s) ds \leq 2C_2^2 C_4^2 \cdot C_6 + 2C_4^2 \cdot C_7 = C_9 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1),$$

so that Lemma 3.1 turns (3.18) into the uniform estimate

$$y_{\varepsilon}(t) \leq e^{-\frac{C_3}{2}t} y_{\varepsilon}(0) + \frac{C_9}{1 - e^{-\frac{C_3}{2}}} \leq C_8 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1) \quad (3.19)$$

by (3.15). In order to obtain a sharper bound for large times, we next use (3.12) and apply the Hölder inequality to see that by definition of C_1 ,

$$\begin{aligned} \int_t^{t+1} \|n_{\varepsilon}(\cdot, s)\|_{L^{\frac{6}{5}}(\Omega)}^2 ds &\leq \int_t^{t+1} \left\{ \|n_{\varepsilon}(\cdot, s)\|_{L^p(\Omega)} \cdot |\Omega|^{\frac{5p-6}{6p}} \right\}^2 ds \\ &\leq C_1^2 \delta_1 \quad \text{for all } t \geq t_2 \text{ and } \varepsilon \in (0, 1), \end{aligned}$$

which combined with (3.11) and the fact that $t_2 > t_1$ implies that

$$\int_t^{t+1} h_{\varepsilon}(s) ds \leq C_{10} := 2C_2^2 C_4^2 \cdot C_1^2 \delta_1 + 2C_4^2 \kappa \quad \text{for all } t \geq t_2 \text{ and } \varepsilon \in (0, 1). \quad (3.20)$$

Consequently, Lemma 3.1 shows that (3.18) firstly entails the inequality

$$y_{\varepsilon}(t) \leq e^{-\frac{C_3}{2}(t-t_2)} y_{\varepsilon}(t_2) + \frac{C_{10}}{1 - e^{-\frac{C_3}{2}}} \quad \text{for all } t \geq t_2 \text{ and } \varepsilon \in (0, 1),$$

whence thanks to (3.19) and (3.17),

$$y_{\varepsilon}(t) \leq e^{-\frac{C_3}{2}(t_0-t_2)} \cdot C_8 + \frac{C_{10}}{1 - e^{-\frac{C_3}{2}}}$$

$$\leq \frac{\delta}{4} + \frac{C_{10}}{1 - e^{-\frac{C_3}{2}}} \quad \text{for all } t \geq t_0 \text{ and } \varepsilon \in (0, 1).$$

Thereafter, an integration of (3.18) using (3.6) yields

$$\begin{aligned} \frac{1}{2} \int_t^{t+1} \int_{\Omega} |\nabla u_{\varepsilon}|^2 &\leq y_{\varepsilon}(t) + \int_t^{t+1} h_{\varepsilon}(s) ds \\ &\leq \frac{\delta}{4} + \frac{C_{10}}{1 - e^{-\frac{C_3}{2}}} + C_{10} \\ &< \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2} \quad \text{for all } t \geq t_0 \text{ and } \varepsilon \in (0, 1), \end{aligned}$$

because (3.9) and (3.10) assert that

$$\frac{C_{10}}{1 - e^{-\frac{C_3}{2}}} + C_{10} = 2C_1^2 C_2^2 C_4^2 C_5 \delta_1 + 2C_4^2 C_5 \kappa < \frac{\delta}{8} + \frac{\delta}{8} = \frac{\delta}{4}.$$

The proof is thereby complete. $\square$

In a next step we apply the latter to conveniently small δ to see by an analysis based on classical smoothing properties of the Stokes semigroup that if now we make use of (1.11) in its full strength with regard to the topological framework therein, then we can achieve the announced goal of this section in the following flavor:

Lemma 3.3. *Let $\omega > 0$. Then one can find $\theta_3 = \theta_3(\omega) > 0$ such that for each $\delta > 0$ there exist $\eta_3 = \eta_3(\omega, \delta) > 0$ and $\kappa_3 = \kappa_3(\omega, \delta) > 0$ such that if $f \in C^1(\bar{\Omega} \times [0, \infty); \mathbb{R}^3)$, $\rho \in \mathbb{R}$ and $\mu > 0$ have the property that (1.10) as well as (1.11) are valid with some $\eta < \eta_3$ and $\kappa < \kappa_3$, then one can choose $t_0 = t_0(\omega, \eta, \kappa, n_0, u_0) > 0$ such that for all $\beta \in [\frac{1}{4}, \frac{1}{4} + \theta_3]$, each $r \in [3, 3 + \theta_3]$ and any $\varepsilon \in (0, 1)$ we have*

$$\|A^{\beta} u_{\varepsilon}(\cdot, t)\|_{L^2(\Omega)} < \delta \quad \text{for all } t \geq t_0 \quad (3.21)$$

and

$$\|u_{\varepsilon}(\cdot, t)\|_{L^r(\Omega)} < \delta \quad \text{for all } t \geq t_0. \quad (3.22)$$

Proof. A substantial part of the proof will consist in creating an adequate setup for a testing procedure involving the third Eq. in (1.7), and properly preparing the choice of the constants θ_3 , η_3 , κ_3 and t_0 . To achieve this, for fixed $\omega > 0$ we take $\theta_1 = \theta_1(\omega) \in (0, \frac{1}{2})$ from Lemma 2.2 and let $p = p(\omega) := \frac{3}{2} + \min\{\theta_1, \omega\}$. Then since $p > \frac{3}{2}$, we have $\frac{5}{2} - \frac{3}{p} > \frac{1}{2}$ and $\frac{5}{2} - \frac{3}{p} > \frac{3}{2} - \frac{3}{2p}$, and since $p < 2$ we furthermore know that $\frac{3}{2} - \frac{3}{2p} < \frac{3}{4} < 1$, which implies that it is possible to pick $\beta_0 = \beta_0(\omega) \in (\frac{1}{4}, \frac{1}{2})$ fulfilling

$$\frac{3}{2} - \frac{3}{2p} < 2\beta_0 < \frac{5}{2} - \frac{3}{p}. \quad (3.23)$$

Here the right inequality warrants that $2 \cdot \frac{1-2\beta_0}{2} - \frac{3}{p} > -\frac{3}{2}$ and that accordingly

$D(A_p^{\frac{1-2\beta_0}{2}}) \hookrightarrow L^2(\Omega)$ ([15, 21]), so that we can find $C_1 = C_1(\omega) > 0$ such that

$$\|\varphi\|_{L^2(\Omega)} \leq C_1 \|A^{\frac{1-2\beta_0}{2}} \varphi\|_{L^p(\Omega)} \quad \text{for all } \varphi \in D(A_p^{\frac{1-2\beta_0}{2}}),$$

in particular implying that

$$\|A^{-\frac{1-2\beta_0}{2}}\varphi\|_{L^2(\Omega)} \leq C_1\|\varphi\|_{L^p(\Omega)} \quad \text{for all } \varphi \in L^p_\sigma(\Omega). \quad (3.24)$$

On the other hand, from the left inequality in (3.23) we obtain that $2\beta_0 - \frac{3}{2} > -\frac{3}{2p}$ and that $2 \cdot \frac{1+2\beta_0}{2} - \frac{3}{2} > 1 - \frac{3}{2p}$, respectively implying that $D(A_2^{\beta_0}) \hookrightarrow L^{2p}(\Omega)$ and $D(A_2^{\frac{1+2\beta_0}{2}}) \hookrightarrow W^{1,2p}(\Omega)$; as a consequence, we can fix $C_2 = C_2(\omega) > 0$ and $C_3 = C_3(\omega) > 0$ fulfilling

$$\|\varphi\|_{L^{2p}(\Omega)} \leq C_2\|A^{\beta_0}\varphi\|_{L^2(\Omega)} \quad \text{for all } \varphi \in D(A_2^{\beta_0}) \quad (3.25)$$

and

$$\|\nabla\varphi\|_{L^{2p}(\Omega)} \leq C_3\|A^{\frac{1+2\beta_0}{2}}\varphi\|_{L^2(\Omega)} \quad \text{for all } \varphi \in D(A_2^{\frac{1+2\beta_0}{2}}). \quad (3.26)$$

Apart from this, the fact that $\beta_0 > \frac{1}{4}$ along with our restriction $p > \frac{3}{2}$ enables us to pick $\theta_3 = \theta_3(\omega) > 0$ such that

$$\frac{1}{4} + \theta_3 \leq \beta_0 \quad (3.27)$$

and

$$3 + \theta_3 \leq 2p. \quad (3.28)$$

Here, (3.28) implies that $L^{2p}(\Omega) \subset L^{3+\theta_3}(\Omega)$, whence by (3.25), for later convenience we can also pick $C_4 = C_4(\omega) > 0$ such that

$$\|\varphi\|_{L^{3+\theta_3}(\Omega)} \leq C_4\|A^{\beta_0}\varphi\|_{L^2(\Omega)} \quad \text{for all } \varphi \in D(A_2^{\beta_0}). \quad (3.29)$$

Next, recalling that $\beta_0 < \frac{1}{2}$, that $\beta \leq \beta_0$ by (3.27), and that the family $(A^{-\lambda})_{\lambda \in (0,1)}$ is bounded in the space of bounded linear operators on $L^2(\Omega; \mathbb{R}^3)$ ([43, Lemma 2.6.3]), it is clear that we can find $C_5 = C_5(\omega) > 0$, $C_6 = C_6(\omega) > 0$ and $C_7 = C_7(\omega) > 0$ such that

$$C_5\|A^{\beta_0}\varphi\|_{L^2(\Omega)} \leq \|A^{\frac{1+2\beta_0}{2}}\varphi\|_{L^2(\Omega)} \quad \text{for all } \varphi \in D(A_2^{\frac{1+2\beta_0}{2}}) \quad (3.30)$$

and

$$\|A^{\beta_0}\varphi\|_{L^2(\Omega)} \leq C_6\|\nabla\varphi\|_{L^2(\Omega)} \quad \text{for all } \varphi \in D(A_2^{\frac{1}{2}}) \quad (3.31)$$

as well as

$$\|A^{\beta}\varphi\|_{L^2(\Omega)} \leq C_7\|A^{\beta_0}\varphi\|_{L^2(\Omega)} \quad \text{for all } \varphi \in D(A^{\beta_0}), \quad (3.32)$$

and that letting $\mathcal{P}$ denote the Helmholtz projection on $L^2(\Omega; \mathbb{R}^3)$ ([46]) and recalling that $\mathcal{P}$ actually is bounded on $L^p(\Omega; \mathbb{R}^3)$ ([14]), we can choose $C_8 = C_8(\omega) > 0$ fulfilling

$$\|\mathcal{P}\varphi\|_{L^p(\Omega)} \leq C_8\|\varphi\|_{L^p(\Omega)} \quad \text{for all } \varphi \in L^p(\Omega), \quad (3.33)$$

whereupon we abbreviate

$$C_9 = C_9(\omega) := 2C_1^2 C_2^2 C_3^2 C_8^2 \quad (3.34)$$

and

$$C_{10} := \|\nabla \phi\|_{L^\infty(\Omega)}. \quad (3.35)$$

Now given $\delta > 0$, we fix $\delta_1 = \delta_1(\omega, \delta) > 0$ small enough such that with

$$C_{11} := \max \left\{ 1, |\Omega|^{\frac{\theta_3}{9}} \right\}, \quad (3.36)$$

we have

$$C_4 \sqrt{\delta_1} \leq \frac{\delta}{C_{11}} \quad (3.37)$$

and

$$\delta_1 \leq \frac{1}{8C_9} \quad (3.38)$$

as well as

$$C_7 \sqrt{\delta_1} \leq \delta, \quad (3.39)$$

then pick $\delta_2 = \delta_2(\omega, \delta) > 0$ and $\delta_3 = \delta_3(\omega, \delta) > 0$ satisfying

$$\frac{1}{1 - e^{-\frac{C_5^2}{4}}} \cdot \delta_2 < \frac{\delta_1}{2} \quad (3.40)$$

and

$$C_6^2 \delta_3 < \frac{\delta_1}{2}, \quad (3.41)$$

and thereafter choose $\delta_4 = \delta_4(\omega, \delta) > 0$ and $\tilde{\kappa} = \tilde{\kappa}(\omega, \delta) > 0$ with the properties that

$$2C_1^2 C_8^2 C_{10}^2 \delta_4 < \frac{\delta_2}{2} \quad (3.42)$$

as well as

$$2C_1^2 C_8^2 C_{12} \tilde{\kappa} < \frac{\delta_2}{2}, \quad (3.43)$$

where $C_{12} = C_{12}(\omega) := |\Omega|^{\frac{2(q-p)}{pq}}$ with $q := \frac{3}{2} + \omega$.

We now let $\eta_1 = \eta_1(\omega, \delta_4)$ be as given by Lemma 2.2 and take $\eta_2 = \eta_2(\omega, \delta_3)$ as well as $\kappa_2 = \kappa_2(\delta_3)$ from Lemma 3.2 to finally define $\eta_3 = \eta_3(\omega, \delta) := \min\{\eta_1, \eta_2\}$ and $\kappa_3 = \kappa_3(\omega, \delta) := \min\{\tilde{\kappa}, \frac{\kappa_2}{C_{13}}\}$ with $C_{13} = C_{13}(\omega) := |\Omega|^{\frac{5q-6}{5q}}$.

Henceforth assuming that $\rho \in \mathbb{R}$, $\mu > 0$ and $f \in C^1(\bar{\Omega} \times [0, \infty); \mathbb{R}^3)$ are such that (1.10) and (1.11) hold with some $\eta < \eta_3$ and $\kappa < \kappa_3$, we then obtain from Lemma 2.2

and the fact that $p \leq \frac{3}{2} + \theta_1$ that there exists $t_1 = t_1(\omega, \eta, \kappa, n_0, u_0) > 0$ such that for all $\varepsilon \in (0, 1)$ we have

$$\int_t^{t+1} \|n_\varepsilon(\cdot, s)\|_{L^p(\Omega)}^2 ds \leq \delta_4 \quad \text{for all } t \geq t_1 \text{ and } \varepsilon \in (0, 1), \quad (3.44)$$

whereas since $p \leq \frac{3}{2} + \omega$, (1.11) in combination with the Hölder inequality implies the existence of $t_2 = t_2(\omega, \eta, \kappa, n_0, u_0) > t_1$ such that

$$\begin{aligned} & \int_t^{t+1} \|f(\cdot, s)\|_{L^p(\Omega)}^2 ds \\ & \leq C_{12} \int_t^{t+1} \|f(\cdot, s)\|_{L^{\frac{3}{2}+\omega}(\Omega)}^2 ds \leq C_{12}\kappa \quad \text{for all } t \geq t_2 \text{ and } \varepsilon \in (0, 1) \end{aligned} \quad (3.45)$$

and

$$\begin{aligned} & \int_t^{t+1} \|f(\cdot, s)\|_{L^{\frac{6}{5}}(\Omega)}^2 ds \\ & \leq C_{13} \int_t^{t+1} \|f(\cdot, s)\|_{L^{\frac{3}{2}+\omega}(\Omega)}^2 ds \leq C_{13}\kappa \leq \kappa_2 \quad \text{for all } t \geq t_2 \text{ and } \varepsilon \in (0, 1). \end{aligned} \quad (3.46)$$

As $\eta \leq \eta_2$, the latter enables us to infer from Lemma 3.2 that there exists $t_3 = t_3(\omega, \eta, \kappa, n_0, u_0) > t_2$ fulfilling

$$\int_{t_3}^{t_3+1} \|\nabla u_\varepsilon(\cdot, s)\|_{L^2(\Omega)}^2 ds < \delta_3 \quad \text{for all } \varepsilon \in (0, 1), \quad (3.47)$$

and we claim that this entails (3.22) if we let $t_0 = t_0(\omega, \eta, \kappa, n_0, u_0) := t_3 + 1$.

To verify this, we first observe that (3.47) implies that for each $\varepsilon \in (0, 1)$ we can pick $t_\varepsilon = t_\varepsilon(\omega, \eta, \kappa, n_0, u_0) \in (t_3, t_3 + 1)$ such that $\|\nabla u_\varepsilon(\cdot, t_\varepsilon)\|_{L^2(\Omega)}^2 < \delta_3$, from which thanks to (3.31), (3.41) and (3.38) we obtain that

$$\int_{\Omega} |A^{\beta_0} u_\varepsilon(\cdot, t_\varepsilon)|^2 \leq C_6^2 \delta_3 < \frac{\delta_1}{2} \leq \frac{1}{4C_9}. \quad (3.48)$$

In particular, writing $y(t) := \int_{\Omega} |A^{\beta_0} u_\varepsilon(x, t)|^2 dx$, $t \geq t_\varepsilon$, we see that

$$S := \left\{ T > t_\varepsilon \mid y(t) \leq \frac{1}{4C_9} \text{ for all } t \in [t_\varepsilon, T] \right\}$$

is not empty and hence $T := \sup S \in (t_\varepsilon, \infty]$ well-defined. In order to make sure that actually $T = \infty$, we use the third Eq. in (1.7) to compute

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} |A^{\beta_0} u_\varepsilon|^2 + \int_{\Omega} |A^{\frac{1+2\beta_0}{2}} u_\varepsilon|^2 \\ & = - \int_{\Omega} A^{2\beta_0} u_\varepsilon \cdot \mathcal{P}[(Y_\varepsilon u_\varepsilon \cdot \nabla) u_\varepsilon] + \int_{\Omega} A^{2\beta_0} u_\varepsilon \cdot \mathcal{P}[n_\varepsilon \nabla \phi] \\ & \quad + \int_{\Omega} A^{2\beta_0} u_\varepsilon \cdot \mathcal{P} f \quad \text{for all } t > 0. \end{aligned} \quad (3.49)$$

Here by self-adjointness of A and its fractional powers, employing Young's inequality, (3.24) and (3.33) we can estimate

$$\begin{aligned}
 \int_{\Omega} A^{2\beta_0} u_{\varepsilon} \cdot \mathcal{P} f &= \int_{\Omega} A^{\frac{1+2\beta_0}{2}} u_{\varepsilon} \cdot A^{-\frac{1-2\beta_0}{2}} \mathcal{P} f \\
 &\leq \frac{1}{4} \int_{\Omega} |A^{\frac{1+2\beta_0}{2}} u_{\varepsilon}|^2 + \int_{\Omega} |A^{-\frac{1-2\beta_0}{2}} \mathcal{P} f|^2 \\
 &\leq \frac{1}{4} \int_{\Omega} |A^{\frac{1+2\beta_0}{2}} u_{\varepsilon}|^2 + C_1^2 \|\mathcal{P} f\|_{L^p(\Omega)}^2 \\
 &\leq \frac{1}{4} \int_{\Omega} |A^{\frac{1+2\beta_0}{2}} u_{\varepsilon}|^2 + C_1^2 C_8^2 \|f\|_{L^p(\Omega)}^2 \quad \text{for all } t > 0, \quad (3.50)
 \end{aligned}$$

and likewise we obtain

$$\begin{aligned}
 \int_{\Omega} A^{2\beta_0} u_{\varepsilon} \cdot \mathcal{P}[n_{\varepsilon} \nabla \phi] &\leq \frac{1}{4} \int_{\Omega} |A^{\frac{1+2\beta_0}{2}} u_{\varepsilon}|^2 + C_1^2 C_8^2 \|n_{\varepsilon} \nabla \phi\|_{L^p(\Omega)}^2 \\
 &\leq \frac{1}{4} \int_{\Omega} |A^{\frac{1+2\beta_0}{2}} u_{\varepsilon}|^2 + C_1^2 C_8^2 C_{10}^2 \|n_{\varepsilon}\|_{L^p(\Omega)}^2 \quad \text{for all } t > 0 \\
 &\quad (3.51)
 \end{aligned}$$

by definition of C_{10} .

Proceeding similarly and then using the Cauchy-Schwarz inequality, (3.25) and (3.26), we see that the convective term can be controlled according to

$$\begin{aligned}
 & - \int_{\Omega} A^{2\beta_0} u_{\varepsilon} \cdot \mathcal{P}[(Y_{\varepsilon} u_{\varepsilon} \cdot \nabla) u_{\varepsilon}] \\
 & \leq \frac{1}{4} \int_{\Omega} |A^{\frac{1+2\beta_0}{2}} u_{\varepsilon}|^2 + C_1^2 C_8^2 \|(Y_{\varepsilon} u_{\varepsilon} \cdot \nabla) u_{\varepsilon}\|_{L^p(\Omega)}^2 \\
 & \leq \frac{1}{4} \int_{\Omega} |A^{\frac{1+2\beta_0}{2}} u_{\varepsilon}|^2 + C_1^2 C_8^2 \|Y_{\varepsilon} u_{\varepsilon}\|_{L^{2p}(\Omega)}^2 \|\nabla u_{\varepsilon}\|_{L^{2p}(\Omega)}^2 \\
 & \leq \frac{1}{4} \int_{\Omega} |A^{\frac{1+2\beta_0}{2}} u_{\varepsilon}|^2 + C_1^2 C_8^2 \cdot C_2^2 \|A^{\beta_0} Y_{\varepsilon} u_{\varepsilon}\|_{L^2(\Omega)}^2 \cdot C_3^2 \|A^{\frac{1+2\beta_0}{2}} u_{\varepsilon}\|_{L^2(\Omega)}^2 \quad (3.52)
 \end{aligned}$$

for all $t > 0$. Since A^{β_0} and Y_{ε} commute on e.g. $D(A_2)$, and since it can easily be checked that $\|Y_{\varepsilon} \varphi\|_{L^2(\Omega)} \leq \|\varphi\|_{L^2(\Omega)}$ for all $\varphi \in L_{\sigma}^2(\Omega)$, by definition of T we herein have

$$\|A^{\beta_0} Y_{\varepsilon} u_{\varepsilon}\|_{L^2(\Omega)}^2 \leq \|A^{\beta_0} u_{\varepsilon}\|_{L^2(\Omega)}^2 \leq \frac{1}{4C_9} \quad \text{for all } t \in (t_{\varepsilon}, T),$$

so that recalling the definition of C_9 , from (3.52) we obtain

$$- \int_{\Omega} A^{2\beta_0} u_{\varepsilon} \cdot \mathcal{P}[(Y_{\varepsilon} u_{\varepsilon} \cdot \nabla) u_{\varepsilon}] \leq \frac{3}{8} \int_{\Omega} |A^{\frac{1+2\beta_0}{2}} u_{\varepsilon}|^2 \quad \text{for all } t \in (t_{\varepsilon}, T).$$

In conjunction with (3.49), (3.50) and (3.51), this shows that

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \int_{\Omega} |A^{\beta_0} u_{\varepsilon}|^2 + \frac{1}{8} \int_{\Omega} |A^{\frac{1+2\beta_0}{2}} u_{\varepsilon}|^2 \\
 & \leq C_1^2 C_8^2 C_{10}^2 \|n_{\varepsilon}\|_{L^p(\Omega)}^2 + C_1^2 C_8^2 \|f\|_{L^p(\Omega)}^2 \quad \text{for all } t \in (t_{\varepsilon}, T),
 \end{aligned}$$

which in light of (3.30) implies that

$$y'(t) + \frac{C_5^2}{4} y(t) \leq h(t) \quad \text{for all } t \in (t_\varepsilon, T) \quad (3.53)$$

with $h(t) := 2C_1^2 C_8^2 C_{10}^2 \|n_\varepsilon(\cdot, t)\|_{L^p(\Omega)}^2 + 2C_1^2 C_8^2 \|f(\cdot, t)\|_{L^p(\Omega)}^2$, $t > 0$.

Since (3.44) and (3.45) combined with (3.42), (3.43) and the fact that $\kappa \leq \kappa_1$ guarantee that

$$\begin{aligned} \int_t^{t+1} h(s) ds &\leq 2C_1^2 C_8^2 C_{10}^2 \delta_4 + 2C_1^2 C_8^2 C_{12} \kappa \\ &< \frac{\delta_2}{2} + \frac{\delta_2}{2} = \delta_2 \quad \text{for all } t > t_\varepsilon, \end{aligned}$$

Lemma 3.1 says that (3.53) implies the inequality

$$y(t) \leq e^{-\frac{C_5^2}{4}(t-t_\varepsilon)} y(t_\varepsilon) + \frac{1}{1 - e^{-\frac{C_5^2}{4}}} \cdot \delta_2 \quad \text{for all } t \in [t_\varepsilon, T),$$

which according to (3.48) and (3.40) in particular warrants that

$$\begin{aligned} y(t) &\leq \int_\Omega |A^{\beta_0} u_\varepsilon(\cdot, t_\varepsilon)|^2 + \frac{1}{1 - e^{-\frac{C_5^2}{4}}} \cdot \delta_2 \\ &< \frac{\delta_1}{2} + \frac{\delta_1}{2} = \delta_1 \quad \text{for all } t \in [t_\varepsilon, T). \end{aligned} \quad (3.54)$$

As therefore $y(t) \leq \frac{1}{8C_9}$ for all $t \in (t_\varepsilon, T)$ due to (3.38), by continuity of y this firstly shows that indeed T cannot be finite, and thereupon we secondly conclude from (3.54) that by (3.32) and (3.39),

$$\|A^\beta u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \leq C_7 \|A^{\beta_0} u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} < C_7 \sqrt{\delta_1} \leq \delta \quad \text{for all } t \geq t_\varepsilon,$$

and that by (3.29) and (3.37),

$$\|u_\varepsilon(\cdot, t)\|_{L^{3+\theta_3}(\Omega)} \leq C_4 \sqrt{y(t)} < C_4 \sqrt{\delta_1} \leq \frac{\delta}{C_{11}} \quad \text{for all } t \geq t_\varepsilon.$$

By means of the Hölder inequality, for arbitrary $r \in [3, 3 + \theta_3]$ we hence obtain from the definition (3.36) of C_{11} that

$$\|u_\varepsilon(\cdot, t)\|_{L^r(\Omega)} \leq |\Omega|^{\frac{3+\theta_3-r}{(3+\theta_3)r}} \cdot \|u_\varepsilon(\cdot, t)\|_{L^{3+\theta_3}(\Omega)} < C_{11} \cdot \frac{\delta}{C_{11}} = \delta \quad \text{for all } t \geq t_\varepsilon,$$

because $0 \leq \frac{3+\theta_3-r}{(3+\theta_3)r} \leq \frac{\theta_3}{9}$ for any such r . Since $t_\varepsilon < t_3 + 1 = t_0$, this completes the proof. $\square$

4. Smallness of $\int_t^{t+1} \|\nabla c_\varepsilon(\cdot, s)\|_{L^q(\Omega)}^2 ds$ for Some $q > 3$

We next address the taxis gradient as the quantity of apparently most crucial influence on regularity in the cross-diffusion interplay in (1.7), in this section aiming at the derivation of a spatio-temporal boundedness feature thereof.

In a preliminary step toward this, we pursue a standard L^2 testing strategy for the equation determining c_ε , hence obtaining some basic result on ultimate smallness which, apart from mere solenoidality, does not rely on any quantitative information about fluid regularity:

Lemma 4.1. *Let $\omega > 0$. Then for all $\delta > 0$ there exists $\eta_4 = \eta_4(\omega, \delta) > 0$ with the property that if $f \in C^1(\bar{\Omega} \times [0, \infty); \mathbb{R}^3)$, $\rho \in \mathbb{R}$ and $\mu > 0$ are such that (1.10) holds with some $\eta < \eta_4$, then one can pick $t_0 = t_0(\delta, \eta, n_0, c_0) > 0$ such that*

$$\int_{\Omega} c_\varepsilon^2(x, t) dx < \delta \quad \text{for all } t \geq t_0 \quad (4.1)$$

and any $\varepsilon \in (0, 1)$.

Proof. Given $\omega > 0$ and $\delta > 0$, we let $\theta_1 = \theta_1(\omega) \in (0, \frac{1}{2})$ be as given by Lemma 2.2 and choose $\delta_1 = \delta_1(\omega, \delta) > 0$ small enough such that

$$\frac{C_1 C_2 \delta_1}{1 - e^{-1}} < \frac{\delta}{2} \quad (4.2)$$

holds with $C_1 = C_1(\omega) := |\Omega|^{\frac{5p-6}{3p}}$, where $p = p(\omega) := \frac{3}{2} + \theta_1$, and where $C_2 > 0$ is a constant satisfying

$$\|\varphi\|_{L^6(\Omega)}^2 \leq C_2 \cdot \left\{ \int_{\Omega} |\nabla \varphi|^2 + \int_{\Omega} \varphi^2 \right\} \quad \text{for all } \varphi \in W^{1,2}(\Omega). \quad (4.3)$$

Then with $\eta_1 := \eta_1(\omega, \delta_1)$ taken from Lemma 2.2, we claim that the desired conclusion holds if we let $\eta_4 = \eta_4(\omega, \delta) := \eta_1$.

To verify this, given $\rho \in \mathbb{R}$ and $\mu > 0$ such that (1.10) is valid with some $\eta < \eta_4$, we first apply Lemma 2.2 and the Hölder inequality to find $t_1 = t_1(\eta) > 0$ such that

$$\begin{aligned} \int_t^{t+1} \|n_\varepsilon(\cdot, s)\|_{L^{\frac{6}{5}}(\Omega)}^2 ds &\leq C_1 \int_t^{t+1} \|n_\varepsilon(\cdot, s)\|_{L^p(\Omega)}^2 ds \\ &< C_1 \delta_1 \quad \text{for all } t \geq t_1 \text{ and } \varepsilon \in (0, 1). \end{aligned} \quad (4.4)$$

We next recall that according to Lemma 2.1, writing $m := \max \left\{ \int_{\Omega} n_0, \frac{\rho_+ |\Omega|}{\mu} \right\}$ we have

$$\int_t^{t+1} \|n_\varepsilon(\cdot, s)\|_{L^2(\Omega)}^2 ds \leq C_3 = C_3(n_0) := \frac{(1 + \rho_+)m}{\mu} \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1),$$

which again by the Hölder inequality implies that

$$\int_t^{t+1} \|n_\varepsilon(\cdot, s)\|_{L^{\frac{6}{5}}(\Omega)}^2 ds \leq C_4 = C_4(n_0) := |\Omega|^{\frac{2}{3}} C_3 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1). \quad (4.5)$$

We now test the second Eq. in (1.7) against c_ε and apply the Hölder inequality along with (4.3) and Young's inequality to find that

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \int_{\Omega} c_\varepsilon^2 + \int_{\Omega} |\nabla c_\varepsilon|^2 + \int_{\Omega} c_\varepsilon^2 = \int_{\Omega} n_\varepsilon c_\varepsilon \\
 & \leq \|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)} \|c_\varepsilon\|_{L^6(\Omega)} \\
 & \leq \|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)} \cdot \sqrt{C_2} \cdot \left\{ \int_{\Omega} |\nabla c_\varepsilon|^2 + \int_{\Omega} c_\varepsilon^2 \right\}^{\frac{1}{2}} \\
 & \leq \frac{1}{2} \left\{ \int_{\Omega} |\nabla c_\varepsilon|^2 + \int_{\Omega} c_\varepsilon^2 \right\} + \frac{C_2}{2} \|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)}^2 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1),
 \end{aligned}$$

and that hence $y_\varepsilon(t) := \int_{\Omega} c_\varepsilon^2(x, t) dx$ and $h_\varepsilon(t) := C_2 \|n_\varepsilon(\cdot, t)\|_{L^{\frac{6}{5}}(\Omega)}^2$, $t \geq 0$, $\varepsilon \in (0, 1)$, satisfy

$$y'_\varepsilon(t) + y_\varepsilon(t) \leq h_\varepsilon(t) \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1). \quad (4.6)$$

As

$$\int_t^{t+1} h_\varepsilon(s) ds \leq C_2 C_4 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1)$$

by (4.5), in view of Lemma 3.1 this firstly implies that

$$\begin{aligned}
 y_\varepsilon(t) & \leq e^{-t} y_\varepsilon(0) + \frac{C_2 C_4}{1 - e^{-1}} \\
 & \leq C_5 = C_5(n_0, c_0) := \int_{\Omega} c_0^2 + \frac{C_2 C_4}{1 - e^{-1}} \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1). \quad (4.7)
 \end{aligned}$$

We thereupon fix $t_0 = t_0(\delta, \eta, n_0, c_0) > t_1$ large enough fulfilling

$$C_5 e^{-(t_0 - t_1)} < \frac{\delta}{2} \quad (4.8)$$

and again apply Lemma 3.1, where now using t_1 as a starting point enables us to rely on the possibly stronger information (4.4), rather than (4.5), to conclude that since

$$\int_t^{t+1} h_\varepsilon(s) ds \leq C_1 C_2 \delta_1 \quad \text{for all } t > t_1 \text{ and } \varepsilon \in (0, 1),$$

we have

$$y_\varepsilon(t) \leq e^{-(t-t_1)} y_\varepsilon(t_1) + \frac{C_1 C_2 \delta_1}{1 - e^{-1}} \quad \text{for all } t \geq t_1 \text{ and } \varepsilon \in (0, 1).$$

Thanks to (4.7), (4.8) and (4.2), this entails that

$$\begin{aligned}
 y_\varepsilon(t) & \leq e^{-(t-t_1)} \cdot C_5 + \frac{C_1 C_2 \delta_1}{1 - e^{-1}} \\
 & \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta \quad \text{for all } t \geq t_0 \text{ and } \varepsilon \in (0, 1),
 \end{aligned}$$

and thereby proves (4.1). $\square$

With this information at hand, we can appropriately control the lower-order contributions to the linear inhomogeneous heat equation, as satisfied by c_ε , in the course of an estimation procedure based on maximal Sobolev regularity theory for the latter, thereby obtaining higher order and especially gradient estimates.

Lemma 4.2. *Let $\omega > 0$. Then there exists $\theta_5 = \theta_5(\omega) > 0$ such that for any $\delta > 0$ one can find $\eta_5 = \eta_5(\omega, \delta) > 0$ and $\kappa_5 = \kappa_5(\omega, \delta) > 0$ with the property that if $\rho \in \mathbb{R}$, $\mu > 0$ and $f \in C^1(\bar{\Omega} \times [0, \infty); \mathbb{R}^3)$ satisfy (1.10) and (1.11) with some $\eta < \eta_5$ and $\kappa < \kappa_5$, then there exists $t_0 = t_0(\omega, \delta, \eta, n_0, c_0, u_0) > 0$ such that for any choice of $q \in [3, 3 + \theta_5]$ and $\varepsilon \in (0, 1)$ we have*

$$\int_t^{t+1} \|\nabla c_\varepsilon(\cdot, s)\|_{L^q(\Omega)}^2 ds < \delta \quad \text{for all } t \geq t_0. \quad (4.9)$$

Proof. With $\theta_1 = \theta_1(\omega) \in (0, \frac{1}{2})$ taken from Lemma 2.2, we let $\theta = \theta(\omega) := \min\{\theta_1, \omega\}$, $p = p(\omega) := \frac{3}{2} + \theta$ and $\theta_5 = \theta_5(\omega) := \frac{3p}{3-p} - 3 \equiv \frac{12\theta}{3-2\theta}$, so that $W^{2,p}(\Omega) \hookrightarrow W^{1, \frac{3p}{3-p}}(\Omega)$, and hence by using the Hölder inequality one can readily find $C_1 = C_1(\omega) > 0$ such that for any $q \in [3, 3 + \theta_5]$ we have

$$\|\nabla \varphi\|_{L^q(\Omega)} \leq C_1 \|\varphi\|_{W^{2,p}(\Omega)} \quad \text{for all } \varphi \in W^{2,p}(\Omega). \quad (4.10)$$

As $p > 1$, well-known results on maximal Sobolev regularity properties of the Neumann heat semigroup $(e^{t(\Delta-1)})_{t \geq 0}$ ([17]) become applicable to provide $C_2 = C_2(\omega) > 0$ such that whenever $h \in C^0(\bar{\Omega} \times [0, 2])$ and $\varphi \in C^{2,1}(\bar{\Omega} \times [0, 2])$ are such that

$$\begin{cases} \varphi_t = \Delta \varphi - \varphi + h(x, t), & x \in \Omega, \quad t \in (0, 2), \\ \frac{\partial \varphi}{\partial \nu} = 0, & x \in \partial \Omega, \quad t \in (0, 2), \\ \varphi(x, 0) = 0, & x \in \Omega, \end{cases}$$

then

$$\int_0^2 \|\varphi(\cdot, s)\|_{W^{2,p}(\Omega)}^2 ds \leq C_2 \int_0^2 \|h(\cdot, s)\|_{L^p(\Omega)}^2 ds. \quad (4.11)$$

Moreover abbreviating $C_3 = C_3(\omega) := |\Omega|^{\frac{2-p}{p}}$, given $\delta > 0$ we can pick positive constants $\delta_i = \delta_i(\omega, \delta) > 0$, $i \in \{1, 2, 3\}$, small enough such that

$$6C_2\delta_1 < \frac{\delta}{4C_1^2} \quad (4.12)$$

and

$$3C_1^2 C_2 \delta_2^2 \leq \frac{1}{2} \quad (4.13)$$

as well as

$$24C_2 C_3 \delta_3 < \frac{\delta}{4C_1^2}. \quad (4.14)$$

We thereafter define $\eta_5 = \eta_5(\omega, \delta) := \min\{\eta_1, \eta_3, \eta_4\}$ and $\kappa_5 = \kappa_5(\omega, \delta) := \kappa_3$, where $\eta_1 := \eta_1(\omega, \delta_1)$ is as provided by Lemma 2.2, $\eta_3 := \eta_3(\omega, \delta_2)$ and $\kappa_3 := \kappa_3(\omega, \delta_2)$ are taken from Lemma 3.3 and $\eta_4 := \eta_4(\omega, \delta_3)$ is obtained by an application of Lemma 4.1.

Supposing henceforth that $\rho \in \mathbb{R}$, $\mu > 0$ and $f \in C^1(\bar{\Omega} \times [0, \infty); \mathbb{R}^3)$ are such that (1.10) and (1.11) hold with some $\eta < \eta_5$ and $\kappa < \kappa_5$, we then infer from Lemma 2.2 that since $\eta < \eta_1$ and $p = \frac{3}{2} + \theta \leq \frac{3}{2} + \theta_1$, there exists $t_1 = t_1(\eta) > 0$ such that for all $\varepsilon \in (0, 1)$,

$$\int_t^{t+1} \|n_\varepsilon(\cdot, s)\|_{L^p(\Omega)}^2 ds < \delta_1 \quad \text{for all } t \geq t_1 \text{ and } \varepsilon \in (0, 1), \quad (4.15)$$

and then use (1.11) to find $t_2 = t_2(\eta) > t_1$ fulfilling

$$\int_t^{t+1} \|f(\cdot, s)\|_{L^{\frac{3}{2}+\omega}(\Omega)}^2 ds < \kappa \quad \text{for all } t \geq t_2 \text{ and } \varepsilon \in (0, 1).$$

As $\kappa < \kappa_3(\omega, \delta_2)$ and $\eta < \eta_3(\omega, \delta_2)$, this allows us to invoke Lemma 3.3 which shows that with some $t_3 = t_3(\omega, \eta, \kappa, n_0, u_0) > t_2$ we have

$$\|u_\varepsilon(\cdot, t)\|_{L^3(\Omega)} < \delta_2 \quad \text{for all } t \geq t_3 \text{ and } \varepsilon \in (0, 1). \quad (4.16)$$

Furthermore, the restriction $\eta < \eta_4(\omega, \delta_3)$ warrants that Lemma 4.1 becomes applicable so as to yield $t_4 = t_4(\omega, \delta, \eta, n_0, c_0, u_0) > t_3$ such that for all $\varepsilon \in (0, 1)$,

$$\begin{aligned} & \int_t^{t+1} \|c_\varepsilon(\cdot, s)\|_{L^p(\Omega)}^2 \\ & \leq C_3 \sup_{s \in (t, t+1)} \|c_\varepsilon(\cdot, s)\|_{L^2(\Omega)}^2 < C_3 \delta_3 \quad \text{for all } t \geq t_4 \text{ and } \varepsilon \in (0, 1), \end{aligned} \quad (4.17)$$

where we have used the Hölder inequality and the fact that $p \leq \frac{3}{2} + \theta_1 < 2$.

Now given $t > t_0 = t_0(\omega, \delta, \eta, n_0, c_0, u_0) := t_4 + 1$, we fix a cut-off function $\zeta \in C^\infty([t-1, t+1])$ such that $\text{supp } \zeta \subset (t-1, t+1]$, $\zeta \equiv 1$ in $[t, t+1]$ and $0 \leq \zeta' \leq 2$ in $[t-1, t+1]$, and let

$$z(x, t) := \zeta(t) \cdot c_\varepsilon(x, t), \quad x \in \bar{\Omega}, \quad t \in [t-1, t+1],$$

for fixed $\varepsilon \in (0, 1)$. Then z is a solution of

$$\begin{cases} z_t = \Delta z - z + \zeta n_\varepsilon - u_\varepsilon \cdot \nabla z + \zeta' c_\varepsilon & \text{in } \Omega \times (t-1, t+1), \\ \frac{\partial z}{\partial \nu} = 0 & \text{on } \partial\Omega \times (t-1, t+1), \\ z(\cdot, t-1) = 0 & \text{in } \Omega, \end{cases}$$

whence applying (4.11) to $\varphi(x, s) := z(x, t-1+s)$, $(x, s) \in \bar{\Omega} \times [0, 2]$, shows that

$$\begin{aligned} \int_{t-1}^{t+1} \|z(\cdot, s)\|_{W^{2,p}(\Omega)}^2 ds & \leq C_2 \int_{t-1}^{t+1} \left\| \zeta(s) n_\varepsilon(\cdot, s) - u_\varepsilon(\cdot, s) \cdot \nabla z(\cdot, s) \right. \\ & \quad \left. + \zeta'(s) c_\varepsilon(\cdot, s) \right\|_{L^p(\Omega)}^2 ds \\ & \leq 3C_2 \int_{t-1}^{t+1} \|n_\varepsilon(\cdot, s)\|_{L^p(\Omega)}^2 ds \\ & \quad + 3C_2 \int_{t-1}^{t+1} \|u_\varepsilon(\cdot, s) \cdot \nabla z(\cdot, s)\|_{L^p(\Omega)}^2 ds \end{aligned}$$

$$+12C_2 \int_{t-1}^{t+1} \|c_\varepsilon(\cdot, s)\|_{L^p(\Omega)}^2 ds, \quad (4.18)$$

because $0 \leq \zeta^2 \leq 1$ and $\zeta'^2 \leq 4$. Here using (4.15) and (4.12) we can estimate

$$3C_2 \int_{t-1}^{t+1} \|n_\varepsilon(\cdot, s)\|_{L^p(\Omega)}^2 ds \leq 3C_2 \cdot 2\delta_1 < \frac{\delta}{4C_1^2}, \quad (4.19)$$

while combining (4.17) with (4.14) yields

$$12C_2 \int_{t-1}^{t+1} \|c_\varepsilon(\cdot, s)\|_{L^p(\Omega)}^2 ds \leq 12C_2 \cdot 2C_3\delta_3 < \frac{\delta}{4C_1^2}. \quad (4.20)$$

In treating the second last integral in (4.18), we invoke the Hölder inequality along with (4.16), (4.10) and (4.13) to obtain

$$\begin{aligned} 3C_2 \int_{t-1}^{t+1} \|u_\varepsilon(\cdot, s) \cdot \nabla z(\cdot, s)\|_{L^p(\Omega)}^2 ds &\leq 3C_2 \int_{t-1}^{t+1} \|u_\varepsilon(\cdot, s)\|_{L^3(\Omega)}^2 \|\nabla z(\cdot, s)\|_{L^{\frac{3p}{3-p}}(\Omega)}^2 ds \\ &\leq 3C_2 \cdot \delta_2^2 \int_{t-1}^{t+1} \|\nabla z(\cdot, s)\|_{L^{\frac{3p}{3-p}}(\Omega)}^2 ds \\ &\leq 3C_2 \delta_2^2 \cdot C_1^2 \int_{t-1}^{t+1} \|z(\cdot, s)\|_{W^{2,p}(\Omega)}^2 ds \\ &\leq \frac{1}{2} \int_{t-1}^{t+1} \|z(\cdot, s)\|_{W^{2,p}(\Omega)}^2 ds. \end{aligned}$$

In conjunction with (4.18), (4.19) and (4.20), this shows that

$$\begin{aligned} &\frac{1}{2C_1^2} \int_{t-1}^{t+1} \|\nabla z(\cdot, s)\|_{L^q(\Omega)}^2 ds \\ &\leq \frac{1}{2} \int_{t-1}^{t+1} \|z(\cdot, s)\|_{W^{2,p}(\Omega)}^2 ds < \frac{\delta}{4C_1^2} + \frac{\delta}{4C_1^2} = \frac{\delta}{2C_1^2} \quad \text{for all } t \geq t_0 \end{aligned}$$

and thereby proves the lemma, because $z \equiv c_\varepsilon$ in $\Omega \times (t, t+1)$. $\square$

5. Smallness of ∇c_ε in $L^{2p}(\Omega)$ for Some $p > \frac{3}{2}$

In this section of key importance we shall next aim at deriving appropriate temporally uniform eventual smallness properties of n_ε , and especially of ∇c_ε , with respect to norms which can be viewed supercritical in the sense that their control will quite directly imply L^∞ bounds for n_ε . Indeed, as seen in Lemma 6.1 below, the space $L^3(\Omega)$ will retain some threshold character with regard to taxis gradient regularity, quite elaborately analyzed in contexts of fluid-free Keller-Segel systems ([3]), at least to a certain extent also in the present setting, and accordingly the main objective of this section will be to ultimately bound ∇c_ε with respect to the norm in $L^{2p}(\Omega)$ for some $p > \frac{3}{2}$. This will be accomplished on the basis of the observation that if, in dependence of the parameter ω in (1.10) and (1.11) the number $p > \frac{3}{2}$ is chosen suitably close to $\frac{3}{2}$, then for some

appropriately constructed function $\psi = \psi(s)$ on $\mathbb{R}$ vanishing at $s = 0$ and essentially growing like s^p as $s \rightarrow +\infty$, the quantity

$$\int_{\Omega} \psi\left(n_{\varepsilon} - \frac{\rho_+}{\mu}\right) + \int_{\Omega} |\nabla c_{\varepsilon}|^{2p} \quad (5.1)$$

plays the role of a quasi-entropy functional in the sense of satisfying a superlinearly forced ODI with an eventually small source (cf. (5.52)), and hence remaining conveniently controllable beyond times at which this functional is small. This conclusion, to be drawn in Lemma 5.4, will be prepared by Lemma 5.1 and Lemma 5.3 which separately describe the time evolution of the summands appearing in (5.1).

5.1. Construction of a quasi-entropy functional coupling n_{ε} to ∇c_{ε} . In the following, given $\rho \in \mathbb{R}$ and $\mu > 0$ we write

$$\gamma := \frac{\rho_+}{\mu}, \quad (5.2)$$

and for $p \in (1, 2)$ we introduce $\psi = \psi_{p,\gamma} \in W_{loc}^{2,\infty}((-\gamma, \infty)) \cap C^2(\mathbb{R} \setminus \{0, \gamma\})$ by defining, in the case $\gamma > 0$,

$$\psi_{p,\gamma}(s) := \begin{cases} 0 & \text{if } s \leq 0, \\ \frac{p}{2} \gamma^{p-2} s^2 & \text{if } s \in (0, \gamma), \\ s^p - \frac{2-p}{2} \gamma^p & \text{if } s \geq \gamma, \end{cases} \quad (5.3)$$

and, in the case $\gamma = 0$,

$$\psi_{p,\gamma}(s) := s_+^p \quad \text{for all } s \in \mathbb{R}. \quad (5.4)$$

Moreover, for $\varepsilon \in (0, 1)$ we let

$$N_{\varepsilon}(x, t) := n_{\varepsilon}(x, t) - \gamma, \quad x \in \bar{\Omega}, \quad t \geq 0, \quad (5.5)$$

and then obtain upon straightforward computation that in both cases $\rho > 0$ and $\rho \leq 0$ we have

$$N_{\varepsilon t} = \Delta N_{\varepsilon} - \chi \nabla \cdot \left((N_{\varepsilon} + \gamma) \nabla c_{\varepsilon} \right) - |\rho| N_{\varepsilon} - \mu N_{\varepsilon}^2 - u_{\varepsilon} \cdot \nabla N_{\varepsilon} \quad \text{in } \Omega \times (0, \infty). \quad (5.6)$$

The following lemma, explicitly requiring convexity in order to avoid yet further technicalities, then relates the temporal growth of $\int_{\Omega} |\nabla c_{\varepsilon}|^{2p}$ to quantities that contain N_{ε} in their essential part:

Lemma 5.1. *Let $p \in (\frac{3}{2}, 2)$ and $r > 3$. Then for all $\delta > 0$ there exists $K_1 = K_1(\delta, p, r) > 0$ such that for any choice of $\rho \in \mathbb{R}$, $\mu > 0$ and $\varepsilon \in (0, 1)$, with γ , N_{ε} and $\psi = \psi_{p,\gamma}$ as defined by (5.2), (5.5) as well as (5.3) and (5.4), whenever $t_0 \geq 0$ and $T \in (t_0, \infty]$ we have*

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} |\nabla c_{\varepsilon}|^{2p} + \frac{2(p-1)}{p} \int_{\Omega} \left| \nabla |\nabla c_{\varepsilon}|^p \right|^2 \\ & + \left\{ p - K_1 \cdot \left(M_2^{\frac{2r}{r-3}}(\varepsilon, T, r) + M_2^2(\varepsilon, T, r) \right) \right\} \cdot \int_{\Omega} |\nabla c_{\varepsilon}|^{2p} \end{aligned}$$

$$\begin{aligned}
&\leq \delta \int_{\{N_\varepsilon > \gamma\}} N_\varepsilon^{p-2} |\nabla N_\varepsilon|^2 \\
&\quad + K_1 \cdot \left\{ \int_{\Omega} \psi(N_\varepsilon) + \int_{\Omega} |\nabla c_\varepsilon|^{2p} \right\}^{\frac{2p-1}{2p-3}} \\
&\quad + K_1 \cdot \left\{ M_1^{\frac{2p(2p-1)}{4p-3}} + \gamma^{\frac{2p(2p-1)}{4p-3}} + \gamma^{2p} + \gamma^{\frac{p(2p-1)}{2p-3}} \right\} \quad \text{for all } t \in (t_0, T), \quad (5.7)
\end{aligned}$$

where

$$M_1(\varepsilon, T) := \sup_{t \in (t_0, T)} \|n_\varepsilon(\cdot, t)\|_{L^1(\Omega)} \quad (5.8)$$

and

$$M_2(\varepsilon, T, r) := \sup_{t \in (t_0, T)} \|u_\varepsilon(\cdot, t)\|_{L^r(\Omega)}. \quad (5.9)$$

Proof. By direct computation using the identity $\nabla c_\varepsilon \cdot \nabla \Delta c_\varepsilon = \frac{1}{2} \Delta |\nabla c_\varepsilon|^2 - |D^2 c_\varepsilon|^2$, from the second Eq. in (1.7) we obtain that for all $t > 0$ and $\varepsilon \in (0, 1)$,

$$\begin{aligned}
\frac{1}{2p} \frac{d}{dt} \int_{\Omega} |\nabla c_\varepsilon|^{2p} &= \int_{\Omega} |\nabla c_\varepsilon|^{2p-2} \nabla c_\varepsilon \cdot \nabla \left\{ \Delta c_\varepsilon - c_\varepsilon + n_\varepsilon - u_\varepsilon \cdot \nabla c_\varepsilon \right\} \\
&= \frac{1}{2} \int_{\Omega} |\nabla c_\varepsilon|^{2p-2} \Delta |\nabla c_\varepsilon|^2 - \int_{\Omega} |\nabla c_\varepsilon|^{2p-2} |D^2 c_\varepsilon|^2 - \int_{\Omega} |\nabla c_\varepsilon|^{2p} \\
&\quad + \int_{\Omega} |\nabla c_\varepsilon|^{2p-2} \nabla c_\varepsilon \cdot \nabla n_\varepsilon - \int_{\Omega} |\nabla c_\varepsilon|^{2p-2} \nabla c_\varepsilon \cdot \nabla (u_\varepsilon \cdot \nabla c_\varepsilon), \quad (5.10)
\end{aligned}$$

where integrating by parts yields

$$\begin{aligned}
&\frac{1}{2} \int_{\Omega} |\nabla c_\varepsilon|^{2p-2} \Delta |\nabla c_\varepsilon|^2 \\
&= -\frac{p-1}{2} \int_{\Omega} |\nabla c_\varepsilon|^{2p-4} \left| \nabla |\nabla c_\varepsilon|^2 \right|^2 + \frac{1}{2} \int_{\partial\Omega} |\nabla c_\varepsilon|^{2p-2} \frac{\partial |\nabla c_\varepsilon|^2}{\partial \nu} \\
&\leq -\frac{p-1}{2} \int_{\Omega} |\nabla c_\varepsilon|^{2p-4} \left| \nabla |\nabla c_\varepsilon|^2 \right|^2 \\
&= -\frac{2(p-1)}{p^2} \int_{\Omega} \left| \nabla |\nabla c_\varepsilon|^p \right|^2 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1), \quad (5.11)
\end{aligned}$$

because for any such ε , $\frac{\partial |\nabla c_\varepsilon|^2}{\partial \nu} \leq 0$ on $\partial\Omega$ due to the convexity of Ω and the fact that $\frac{\partial c_\varepsilon}{\partial \nu} = 0$ on $\partial\Omega$ ([35]).

In the rightmost integral in (5.10), we also integrate by parts and then apply the pointwise inequality $|\Delta c_\varepsilon| \leq \sqrt{3} |D^2 c_\varepsilon|$ as well as Young's inequality to estimate

$$\begin{aligned}
&-\int_{\Omega} |\nabla c_\varepsilon|^{2p-2} \nabla c_\varepsilon \cdot \nabla (u_\varepsilon \cdot \nabla c_\varepsilon) \\
&= \int_{\Omega} |\nabla c_\varepsilon|^{2p-2} \Delta c_\varepsilon (u_\varepsilon \cdot \nabla c_\varepsilon)
\end{aligned}$$

$$\begin{aligned}
& + (p-1) \int_{\Omega} |\nabla c_{\varepsilon}|^{2p-4} (\nabla |\nabla c_{\varepsilon}|^2 \cdot \nabla c_{\varepsilon}) (u_{\varepsilon} \cdot \nabla c_{\varepsilon}) \\
& \leq \sqrt{3} \int_{\Omega} |u_{\varepsilon}| \cdot |\nabla c_{\varepsilon}|^{2p-1} |D^2 c_{\varepsilon}| \\
& \quad + (p-1) \int_{\Omega} |u_{\varepsilon}| \cdot |\nabla c_{\varepsilon}|^{2p-2} \left| \nabla |\nabla c_{\varepsilon}|^2 \right| \\
& \leq \frac{1}{2} \int_{\Omega} |\nabla c_{\varepsilon}|^{2p-2} |D^2 c_{\varepsilon}|^2 + \frac{3}{2} \int_{\Omega} |u_{\varepsilon}|^2 |\nabla c_{\varepsilon}|^{2p} \\
& + \frac{p-1}{16} \int_{\Omega} |\nabla c_{\varepsilon}|^{2p-4} \left| \nabla |\nabla c_{\varepsilon}|^2 \right|^2 + 4(p-1) \int_{\Omega} |u_{\varepsilon}|^2 |\nabla c_{\varepsilon}|^{2p} \\
& = \frac{1}{2} \int_{\Omega} |\nabla c_{\varepsilon}|^{2p-2} |D^2 c_{\varepsilon}|^2 + \frac{p-1}{4p^2} \int_{\Omega} \left| \nabla |\nabla c_{\varepsilon}|^p \right|^2 \\
& \quad + \left(4p - \frac{5}{2} \right) \int_{\Omega} |u_{\varepsilon}|^2 |\nabla c_{\varepsilon}|^{2p} \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1). \quad (5.12)
\end{aligned}$$

Here we use the Hölder inequality and (5.9) to see that whenever $t_0 \geq 0$ and $T \in (t_0, \infty]$,

$$\begin{aligned}
\left(4p - \frac{5}{2} \right) \int_{\Omega} |u_{\varepsilon}|^2 |\nabla c_{\varepsilon}|^{2p} & \leq \left(4p - \frac{5}{2} \right) \left\{ \int_{\Omega} |u_{\varepsilon}|^r \right\}^{\frac{2}{r}} \left\{ \int_{\Omega} |\nabla c_{\varepsilon}|^{\frac{2pr}{r-2}} \right\}^{\frac{r-2}{r}} \\
& \leq \left(4p - \frac{5}{2} \right) M_2^2(\varepsilon, T, r) \left\| |\nabla c_{\varepsilon}|^p \right\|_{L^{\frac{2r}{r-2}}(\Omega)}^2 \quad \text{for all } t \in (t_0, T) \text{ and } \varepsilon \in (0, 1),
\end{aligned}$$

and since $r > 3$ implies that $\frac{2r}{r-2} < 6$, we may invoke the Gagliardo-Nirenberg inequality and Young's inequality to find $C_1 = C_1(p, r) > 0$ and $C_2 = C_2(p, r) > 0$ satisfying

$$\begin{aligned}
& \left(4p - \frac{5}{2} \right) M_2^2(\varepsilon, T, r) \left\| |\nabla c_{\varepsilon}|^p \right\|_{L^{\frac{2r}{r-2}}(\Omega)}^2 \\
& \leq C_1 M_2^2(\varepsilon, T, r) \left\| \nabla |\nabla c_{\varepsilon}|^p \right\|_{L^2(\Omega)}^{\frac{6}{r}} \left\| |\nabla c_{\varepsilon}|^p \right\|_{L^2(\Omega)}^{\frac{2(r-3)}{r}} + C_1 M_2^2(\varepsilon, T, r) \left\| |\nabla c_{\varepsilon}|^p \right\|_{L^2(\Omega)}^2 \\
& \leq \frac{p-1}{4p^2} \left\| \nabla |\nabla c_{\varepsilon}|^p \right\|_{L^2(\Omega)}^2 + C_2 M_2^{\frac{2r}{r-3}}(\varepsilon, T, r) \left\| |\nabla c_{\varepsilon}|^p \right\|_{L^2(\Omega)}^2 \\
& \quad + C_1 M_2^2(\varepsilon, T, r) \left\| |\nabla c_{\varepsilon}|^p \right\|_{L^2(\Omega)}^2, \quad \text{for all } t \in (t_0, T) \text{ and } \varepsilon \in (0, 1),
\end{aligned}$$

so that (5.12) all in all shows that

$$\begin{aligned}
& - \int_{\Omega} |\nabla c_{\varepsilon}|^{2p-2} \nabla c_{\varepsilon} \cdot \nabla (u_{\varepsilon} \cdot \nabla c_{\varepsilon}) \\
& \leq \frac{1}{2} \int_{\Omega} |\nabla c_{\varepsilon}|^{2p-2} |D^2 c_{\varepsilon}|^2 + \frac{p-1}{2p^2} \int_{\Omega} \left| \nabla |\nabla c_{\varepsilon}|^p \right|^2 \\
& \quad + \left\{ C_2 M_2^{\frac{2r}{r-3}}(\varepsilon, T, r) + C_1 M_2^2(\varepsilon, T, r) \right\} \\
& \quad \cdot \int_{\Omega} |\nabla c_{\varepsilon}|^{2p} \quad \text{for all } t \in (t_0, T) \text{ and } \varepsilon \in (0, 1). \quad (5.13)
\end{aligned}$$

We next address the second last integral in (5.10), in which we first partially proceed as above in integrating by parts and applying Young's inequality to see that

$$\begin{aligned}
& \int_{\Omega} |\nabla c_{\varepsilon}|^{2p-2} \nabla c_{\varepsilon} \cdot \nabla n_{\varepsilon} \\
&= \int_{\Omega} |\nabla c_{\varepsilon}|^{2p-2} \nabla c_{\varepsilon} \cdot \nabla N_{\varepsilon} \\
&= - \int_{\Omega} N_{\varepsilon} |\nabla c_{\varepsilon}|^{2p-2} \Delta c_{\varepsilon} - (p-1) \int_{\Omega} N_{\varepsilon} |\nabla c_{\varepsilon}|^{2p-4} \nabla c_{\varepsilon} \cdot \nabla |\nabla c_{\varepsilon}|^2 \\
&\leq \sqrt{3} \int_{\Omega} |N_{\varepsilon}| \cdot |\nabla c_{\varepsilon}|^{2p-2} |D^2 c_{\varepsilon}| + (p-1) \int_{\Omega} |N_{\varepsilon}| \cdot |\nabla c_{\varepsilon}|^{2p-3} |\nabla |\nabla c_{\varepsilon}|^2| \\
&\leq \frac{1}{2} \int_{\Omega} |\nabla c_{\varepsilon}|^{2p-2} |D^2 c_{\varepsilon}|^2 + \frac{3}{2} \int_{\Omega} N_{\varepsilon}^2 |\nabla c_{\varepsilon}|^{2p-2} \\
&\quad + \frac{p-1}{8} \int_{\Omega} |\nabla c_{\varepsilon}|^{2p-4} |\nabla |\nabla c_{\varepsilon}|^2|^2 + 2(p-1) \int_{\Omega} N_{\varepsilon}^2 |\nabla c_{\varepsilon}|^{2p-2} \\
&= \frac{1}{4} \int_{\Omega} |\nabla c_{\varepsilon}|^{2p-2} |D^2 c_{\varepsilon}|^2 + \frac{p-1}{2p^2} \int_{\Omega} |\nabla |\nabla c_{\varepsilon}|^p|^2 \\
&\quad + \left(2p - \frac{1}{2}\right) \int_{\Omega} N_{\varepsilon}^2 |\nabla c_{\varepsilon}|^{2p-2} \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1). \tag{5.14}
\end{aligned}$$

Here in view of the definitions (5.3) and (5.4) of ψ , for $t > 0$ and $\varepsilon \in (0, 1)$ we split the last integral according to

$$\begin{aligned}
\left(2p - \frac{1}{2}\right) \int_{\Omega} N_{\varepsilon}^2 |\nabla c_{\varepsilon}|^{2p-2} &= \left(2p - \frac{1}{2}\right) \int_{\{N_{\varepsilon} \leq \gamma\}} N_{\varepsilon}^2 |\nabla c_{\varepsilon}|^{2p-2} \\
&\quad + \left(2p - \frac{1}{2}\right) \int_{\{N_{\varepsilon} > \gamma\}} N_{\varepsilon}^2 |\nabla c_{\varepsilon}|^{2p-2}, \tag{5.15}
\end{aligned}$$

where by Young's inequality

$$\begin{aligned}
& \left(2p - \frac{1}{2}\right) \int_{\{N_{\varepsilon} \leq \gamma\}} N_{\varepsilon}^2 |\nabla c_{\varepsilon}|^{2p-2} \\
&\leq \left(2p - \frac{1}{2}\right) \gamma^2 \int_{\Omega} |\nabla c_{\varepsilon}|^{2p-2} \\
&\leq \frac{1}{2} \int_{\Omega} |\nabla c_{\varepsilon}|^{2p} + C_4 \gamma^{2p} \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1)
\end{aligned} \tag{5.16}$$

with some $C_4 = C_4(p) > 0$, because $N_{\varepsilon} \geq -\gamma$ by nonnegativity of n_{ε} . Moreover, employing the Hölder, Young and Gagliardo-Nirenberg inequalities we obtain $C_5 = C_5(p) > 0$ such that writing

$$y_{\varepsilon}(t) := \int_{\Omega} \psi(N_{\varepsilon}(x, t)) dx + \int_{\Omega} |\nabla c_{\varepsilon}(x, t)|^{2p} dx, \quad t \geq 0, \varepsilon \in (0, 1), \tag{5.17}$$

we have

$$\left(2p + \frac{1}{2}\right) \int_{\{N_{\varepsilon} > \gamma\}} N_{\varepsilon}^2 |\nabla c_{\varepsilon}|^{2p-2}$$

$$\begin{aligned}
&\leq \left(2p + \frac{1}{2}\right) \left\{ \int_{\{N_\varepsilon > \gamma\}} N_\varepsilon^{2p} \right\}^{\frac{1}{p}} \left\{ \int_{\Omega} |\nabla c_\varepsilon|^{2p} \right\}^{\frac{p-1}{p}} \\
&\leq \left(2p + \frac{1}{2}\right) \left\{ \int_{\{N_\varepsilon > \gamma\}} [(N_\varepsilon^{\frac{p}{2}} - \gamma^{\frac{p}{2}}) + \gamma^{\frac{p}{2}}]^4 \right\}^{\frac{1}{p}} \cdot y_\varepsilon^{\frac{p-1}{p}}(t) \\
&\leq \left(2p + \frac{1}{2}\right) \left\{ 8 \int_{\{N_\varepsilon > \gamma\}} (N_\varepsilon^{\frac{p}{2}} - \gamma^{\frac{p}{2}})_+^4 + 8|\Omega|\gamma^{2p} \right\}^{\frac{1}{p}} \cdot y_\varepsilon^{\frac{p-1}{p}}(t) \\
&= \left(2p + \frac{1}{2}\right) \left\{ 8 \left\| (N_\varepsilon^{\frac{p}{2}} - \gamma^{\frac{p}{2}})_+ \right\|_{L^4(\Omega)}^4 + 8|\Omega|\gamma^{2p} \right\}^{\frac{1}{p}} \cdot y_\varepsilon^{\frac{p-1}{p}}(t) \\
&\leq C_5 \left\{ \left\| \nabla (N_\varepsilon^{\frac{p}{2}} - \gamma^{\frac{p}{2}})_+ \right\|_{L^2(\Omega)}^{\frac{3}{p}} \left\| (N_\varepsilon^{\frac{p}{2}} - \gamma^{\frac{p}{2}})_+ \right\|_{L^2(\Omega)}^{\frac{1}{p}} \right. \\
&\quad \left. + \left\| (N_\varepsilon^{\frac{p}{2}} - \gamma^{\frac{p}{2}})_+ \right\|_{L^{\frac{4}{p}}(\Omega)}^{\frac{4}{p}} + \gamma^2 \right\} \cdot y_\varepsilon^{\frac{p-1}{p}}(t)
\end{aligned}$$

for all $t > 0$ and $\varepsilon \in (0, 1)$. Observing that herein we have

$$\left\| \nabla (N_\varepsilon^{\frac{p}{2}} - \gamma^{\frac{p}{2}})_+ \right\|_{L^2(\Omega)}^2 = \frac{p^2}{4} \int_{\{N_\varepsilon > \gamma\}} N_\varepsilon^{p-2} |\nabla N_\varepsilon|^2 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1)$$

as well as

$$\begin{aligned}
\left\| (N_\varepsilon^{\frac{p}{2}} - \gamma^{\frac{p}{2}})_+ \right\|_{L^2(\Omega)}^2 &= \int_{\{N_\varepsilon > \gamma\}} (N_\varepsilon^{\frac{p}{2}} - \gamma^{\frac{p}{2}})^2 \\
&\leq \int_{\{N_\varepsilon > \gamma\}} N_\varepsilon^p \\
&= \int_{\{N_\varepsilon > \gamma\}} \left\{ \psi(N_\varepsilon) + \frac{2-p}{2} \gamma^p \right\} \\
&\leq \int_{\Omega} \psi(N_\varepsilon) + C_6 \gamma^p \\
&\leq y_\varepsilon(t) + C_6 \gamma^p \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1)
\end{aligned}$$

with $C_6 = C_6(p) := \frac{2-p}{2}|\Omega|$, and that given $t_0 \geq 0$ and $T \in (t_0, \infty]$, by (5.5) and (5.8) we have

$$\left\| (N_\varepsilon^{\frac{p}{2}} - \gamma^{\frac{p}{2}})_+ \right\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2}{p}} = \int_{\{N_\varepsilon > \gamma\}} (N_\varepsilon^{\frac{p}{2}} - \gamma^{\frac{p}{2}})^{\frac{2}{p}} \leq \int_{\{N_\varepsilon > \gamma\}} N_\varepsilon \leq \int_{\Omega} n_\varepsilon \leq M_1(\varepsilon, T)$$

for all $t \in (t_0, T)$ and $\varepsilon \in (0, 1)$, on using Young's inequality we see that there exists $C_7 = C_7(\delta, p) > 0$ such that for all $t \in (t_0, T)$ and $\varepsilon \in (0, 1)$,

$$\begin{aligned}
\left(2p + \frac{1}{2}\right) \int_{\{N_\varepsilon > \gamma\}} N_\varepsilon^2 |\nabla c_\varepsilon|^{2p-2} &\leq \frac{\delta}{2p} \int_{\{N_\varepsilon > \gamma\}} N_\varepsilon^{p-2} |\nabla N_\varepsilon|^2 \\
&\quad + C_7 \left\| (N_\varepsilon^{\frac{p}{2}} - \gamma^{\frac{p}{2}})_+ \right\|_{L^2(\Omega)}^{\frac{2}{2p-3}} y^{\frac{2p-2}{2p-3}}(t) \\
&\quad + C_5 \left\| (N_\varepsilon^{\frac{p}{2}} - \gamma^{\frac{p}{2}})_+ \right\|_{L^{\frac{4}{p}}(\Omega)}^{\frac{4}{p}} y_\varepsilon^{\frac{p-1}{p}}(t) + C_5 \gamma^2 y_\varepsilon^{\frac{p-1}{p}}(t)
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\delta}{2p} \int_{\{N_\varepsilon > \gamma\}} N_\varepsilon^{p-2} |\nabla N_\varepsilon|^2 \\
&\quad + C_7 \cdot \left\{ y_\varepsilon^{2p-1}(t) + C_6 \gamma^p y_\varepsilon^{2p-2}(t) \right\}^{\frac{1}{2p-3}} \\
&\quad + C_5 M_1^2(\varepsilon, T) y_\varepsilon^{\frac{p-1}{p}}(t) + C_5 \gamma^2 y_\varepsilon^{\frac{p-1}{p}}(t).
\end{aligned}$$

Three more applications of Young's inequality thereupon provide $C_8 = C_8(p) > 0$ and $C_9 = C_9(\delta, p) > 0$ fulfilling

$$\begin{aligned}
\left(2p + \frac{1}{2}\right) \int_{\{N_\varepsilon > \gamma\}} N_\varepsilon^2 |\nabla c_\varepsilon|^{2p-2} &\leq \frac{\delta}{2p} \int_{\{N_\varepsilon > \gamma\}} N_\varepsilon^{p-2} |\nabla N_\varepsilon|^2 \\
&\quad + C_7 \cdot \left\{ 2y_\varepsilon^{2p-1}(t) + C_8 \gamma^{p(2p-1)} \right\}^{\frac{1}{2p-3}} \\
&\quad + C_5 \left\{ M_1^{\frac{2p(2p-1)}{4p-3}}(\varepsilon, T) + y_\varepsilon^{\frac{2p-1}{2p-3}}(t) \right\} \\
&\quad + C_5 \left\{ \gamma^{\frac{2p(2p-1)}{4p-3}} + y_\varepsilon^{\frac{2p-1}{2p-3}}(t) \right\} \\
&\leq \frac{\delta}{2p} \int_{\{N_\varepsilon > \gamma\}} N_\varepsilon^{p-2} |\nabla N_\varepsilon|^2 \\
&\quad + C_9 y_\varepsilon^{\frac{2p-1}{2p-3}}(t) + C_9 \gamma^{\frac{p(2p-1)}{2p-3}} \\
&\quad + C_5 \gamma^{\frac{2p(2p-1)}{4p-3}} + C_5 M_1^{\frac{2p(2p-1)}{4p-3}}(\varepsilon, T)
\end{aligned}$$

for all $t \in (t_0, T)$ and $\varepsilon \in (0, 1)$. Combined with (5.15), (5.16) and (5.14), this shows that

$$\begin{aligned}
\int_{\Omega} |\nabla c_\varepsilon|^{2p-2} \nabla c_\varepsilon \cdot \nabla n_\varepsilon &\leq \frac{\delta}{2p} \int_{\{N_\varepsilon > \gamma\}} N_\varepsilon^{p-2} |\nabla N_\varepsilon|^2 \\
&\quad + \frac{1}{2} \int_{\Omega} |\nabla c_\varepsilon|^{2p-2} |D^2 c_\varepsilon|^2 + \frac{p-1}{2p^2} \int_{\Omega} |\nabla |\nabla c_\varepsilon|^p|^2 + \frac{1}{2} \int_{\Omega} |\nabla c_\varepsilon|^{2p} \\
&\quad + C_9 y_\varepsilon^{\frac{2p-1}{2p-3}}(t) + C_5 M_1^{\frac{2p(2p-1)}{4p-3}}(\varepsilon, T) + C_5 \gamma^{\frac{2p(2p-1)}{4p-3}} \\
&\quad + C_4 \gamma^{2p} + C_9 \gamma^{\frac{p(2p-1)}{2p-3}}
\end{aligned}$$

for all $t \in (t_0, T)$ and $\varepsilon \in (0, 1)$, which in conjunction with (5.10), (5.11) and (5.13) readily yields (5.7). $\square$

In order to prepare an appropriate control of the first summand on the right-hand side of (5.7) in the course of a testing procedure associated with the identity (5.5), let us note the following immediate consequence of the definitions in (5.3) and (5.4).

Lemma 5.2. *Let $p \in (1, 2)$, $\gamma \geq 0$ and $\psi = \psi_{p,\gamma}$ be defined by (5.3) and (5.4), respectively. Then*

$$(\psi'(s))^{\frac{2-p}{p-1}} \psi''(s) \leq p^{\frac{1}{p-1}} \quad \text{for all } s \in \mathbb{R} \setminus \{0, \gamma\}. \quad (5.18)$$

Proof. If $s < 0$, (5.18) is trivial. If $\gamma > 0$ and $s \in (0, \gamma)$, then by (5.3) we have $\psi'(s) = p\gamma^{p-2}s < p\gamma^{p-1}$ and $\psi''(s) = p\gamma^{p-2}$, so that since $p \leq 2$ we can estimate

$$(\psi'(s))^{\frac{2-p}{p-1}} \psi''(s) < (p\gamma^{p-1})^{\frac{2-p}{p-1}} \cdot p\gamma^{p-2} = p^{\frac{1}{p-1}}.$$

Finally, in the case $s > \gamma$ we see from (5.3) and (5.4) that $\psi'(s) = ps^{p-1}$ and $\psi''(s) = p(p-1)s^{p-2}$ and hence

$$(\psi'(s))^{\frac{2-p}{p-1}} \psi''(s) = (ps^{p-1})^{\frac{2-p}{p-1}} \cdot p(p-1)s^{p-2} = p^{\frac{1}{p-1}}(p-1) \leq p^{\frac{1}{p-1}},$$

again because $p \leq 2$. $\square$

We can thereby establish an ODI that limits the growth of $\int_{\Omega} \psi(N_{\varepsilon})$ and, yet more importantly, contains the first summand from the right of (5.7) in its dissipative part:

Lemma 5.3. *Let $p \in (\frac{3}{2}, 2]$. Then for all $\delta > 0$ there exists $K_2 = K_2(\delta, p) > 0$ such that for each $\rho \in \mathbb{R}$, $\mu > 0$ and $\varepsilon \in (0, 1)$, the quantities γ , N_{ε} and $\psi = \psi_{p,\gamma}$ introduced in (5.2), (5.5), (5.3) and (5.4) have the property that for any $t_0 \geq 0$ and $T \in (t_0, \infty]$,*

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \psi(N_{\varepsilon}) + \frac{p(p-1)}{2} \int_{\{N_{\varepsilon} > \gamma\}} N_{\varepsilon}^{p-2} |\nabla N_{\varepsilon}|^2 \\ & \leq \delta \int_{\Omega} |\nabla |\nabla c_{\varepsilon}|^p|^2 + \delta \int_{\Omega} |\nabla c_{\varepsilon}|^{2p} \\ & \quad + K_2 \cdot \left\{ \int_{\Omega} \psi(N_{\varepsilon}) \right\}^{\frac{2p-1}{2p-3}} \\ & \quad + K_2 \cdot \left\{ M_1^{\frac{p^2}{p-1}}(\varepsilon, T) + \gamma^{\frac{p(2p-1)}{2p-3}} + \gamma^{\frac{p^2}{p-1}} + \gamma^{\frac{1}{p-1}} \right\} \end{aligned} \quad (5.19)$$

for all $t \in (t_0, T)$ and $\varepsilon \in (0, 1)$, where $M_1(\cdot, \cdot)$ is as in (5.8).

Proof. On the basis of (5.6), using that $\psi \in W_{loc}^{2,\infty}((-\gamma, \infty))$, and that $N_{\varepsilon} > -\gamma$ in $\overline{\Omega} \times (0, \infty)$ by positivity of n_{ε} , we see that $0 \leq t \mapsto \int_{\Omega} \psi(N_{\varepsilon})$ belongs to $C^0([0, \infty)) \cap C^1((0, \infty))$ for each $\varepsilon \in (0, 1)$, and that we may integrate by parts in computing

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \psi(N_{\varepsilon}) \\ & = \int_{\Omega} \psi'(N_{\varepsilon}) \cdot \left\{ \Delta N_{\varepsilon} - \chi \nabla \cdot \left((N_{\varepsilon} + \gamma) \nabla c_{\varepsilon} \right) - |\rho| N_{\varepsilon} - \mu N_{\varepsilon}^2 - u_{\varepsilon} \cdot \nabla N_{\varepsilon} \right\} \\ & = - \int_{\Omega} \psi''(N_{\varepsilon}) |\nabla N_{\varepsilon}|^2 + \chi \int_{\Omega} (N_{\varepsilon} + \gamma) \psi''(N_{\varepsilon}) \nabla N_{\varepsilon} \cdot \nabla c_{\varepsilon} \\ & \quad - |\rho| \int_{\Omega} N_{\varepsilon} \psi'(N_{\varepsilon}) - \mu \int_{\Omega} N_{\varepsilon}^2 \psi'(N_{\varepsilon}) \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1), \end{aligned} \quad (5.20)$$

because $\nabla \cdot u_{\varepsilon} \equiv 0$. Since $\psi' \geq 0$ on $(0, \infty)$ and $\psi' \equiv 0$ on $(-\infty, 0)$, herein we can estimate

$$-|\rho| \int_{\Omega} N_{\varepsilon} \psi'(N_{\varepsilon}) \leq 0 \quad \text{and}$$

$$-\mu \int_{\Omega} N_{\varepsilon}^2 \psi'(N_{\varepsilon}) \leq 0 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1), \quad (5.21)$$

and since ψ'' is nonnegative, we may invoke Young's inequality to find that

$$\begin{aligned} & \chi \int_{\Omega} (N_{\varepsilon} + \gamma) \psi''(N_{\varepsilon}) \nabla N_{\varepsilon} \cdot \nabla c_{\varepsilon} \\ & \leq \frac{1}{4} \int_{\Omega} \psi''(N_{\varepsilon}) |\nabla N_{\varepsilon}|^2 + \chi^2 \int_{\Omega} (N_{\varepsilon} + \gamma)^2 \psi''(N_{\varepsilon}) |\nabla c_{\varepsilon}|^2 \\ & \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1). \end{aligned} \quad (5.22)$$

On splitting the latter integral and recalling the definition of ψ , we see that in the case $\gamma > 0$,

$$\begin{aligned} \chi^2 \int_{\Omega} (N_{\varepsilon} + \gamma)^2 \psi''(N_{\varepsilon}) |\nabla c_{\varepsilon}|^2 &= \chi^2 \int_{\{0 \leq N_{\varepsilon} \leq \gamma\}} (N_{\varepsilon} + \gamma)^2 \psi''(N_{\varepsilon}) |\nabla c_{\varepsilon}|^2 \\ & \quad + \chi^2 \int_{\{N_{\varepsilon} > \gamma\}} (N_{\varepsilon} + \gamma)^2 \psi''(N_{\varepsilon}) |\nabla c_{\varepsilon}|^2 \\ & \leq \chi^2 \int_{\{0 \leq N_{\varepsilon} \leq \gamma\}} (2\gamma)^2 \cdot p\gamma^{p-2} |\nabla c_{\varepsilon}|^2 \\ & \quad + \chi^2 \int_{\{N_{\varepsilon} > \gamma\}} (2N_{\varepsilon})^2 \cdot p(p-1)N_{\varepsilon}^{p-2} |\nabla c_{\varepsilon}|^2 \end{aligned}$$

for all $t > 0$ and $\varepsilon \in (0, 1)$, which shows that regardless of the sign of ρ we have

$$\begin{aligned} & \chi^2 \int_{\Omega} (N_{\varepsilon} + \gamma)^2 \psi''(N_{\varepsilon}) |\nabla c_{\varepsilon}|^2 \\ & \leq 4p\gamma^p \chi^2 \int_{\Omega} |\nabla c_{\varepsilon}|^2 + 4p(p-1) \chi^2 \int_{\{N_{\varepsilon} > \gamma\}} N_{\varepsilon}^2 |\nabla c_{\varepsilon}|^2 \\ & \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1), \end{aligned} \quad (5.23)$$

where by Young's inequality we obtain $C_1 = C_1(\delta, p) > 0$ such that

$$4p\gamma^p \chi^2 \int_{\Omega} |\nabla c_{\varepsilon}|^2 \leq \frac{\delta}{2} \int_{\Omega} |\nabla c_{\varepsilon}|^{2p} + C_1 \gamma^{\frac{p^2}{p-1}} \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1). \quad (5.24)$$

To estimate the second term on the right of (5.23), we first apply the Hölder inequality to obtain

$$\begin{aligned} \int_{\{N_{\varepsilon} > \gamma\}} N_{\varepsilon}^p |\nabla c_{\varepsilon}|^2 & \leq \left\{ \int_{\{N_{\varepsilon} > \gamma\}} N_{\varepsilon}^{\frac{3p^2}{3p-1}} \right\}^{\frac{3p-1}{3p}} \cdot \left\{ \int_{\Omega} |\nabla c_{\varepsilon}|^{6p} \right\}^{\frac{1}{3p}} \\ & = \|N_{\varepsilon}^{\frac{p}{2}}\|_{L^{\frac{6p}{3p-1}}(\{N_{\varepsilon} > \gamma\})}^2 \left\| |\nabla c_{\varepsilon}|^p \right\|_{L^6(\Omega)}^{\frac{2}{p}} \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1), \end{aligned} \quad (5.25)$$

where

$$\|N_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{6p}{3p-1}}(\{N_\varepsilon > \gamma\})}^2 \leq p^{-\frac{p}{p-1}} \left\| \left(\psi'(N_\varepsilon) \right)^{\frac{p}{2p-2}} \right\|_{L^{\frac{6p}{3p-1}}(\Omega)}^2 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1),$$

because according to (5.3) and (5.4) we have $\psi' \geq 0$ on $\mathbb{R}$ and

$$\left(\psi'(s) \right)^{\frac{p}{2p-2}} = (ps^{p-1})^{\frac{p}{2p-2}} = p^{\frac{p}{2p-2}} s^{\frac{p}{2}} \quad \text{for all } s > \gamma. \quad (5.26)$$

Noting that $\frac{6p}{3p-1} < 6$ due to the fact that $p > \frac{1}{2}$, in (5.25) we may thus employ the Gagliardo-Nirenberg inequality to find $C_2 = C_2(p) > 0$ such that

$$\begin{aligned} \|N_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{6p}{3p-1}}(\{N_\varepsilon > \gamma\})}^2 &\leq C_2 \left\| \nabla \left(\psi'(N_\varepsilon) \right)^{\frac{p}{2p-2}} \right\|_{L^2(\Omega)}^{\frac{1}{p}} \left\| \left(\psi'(N_\varepsilon) \right)^{\frac{p}{2p-2}} \right\|_{L^2(\Omega)}^{\frac{2p-1}{p}} \\ &\quad + C_2 \left\| \left(\psi'(N_\varepsilon) \right)^{\frac{p}{2p-2}} \right\|_{L^{\frac{2}{p}}(\Omega)}^2 \end{aligned}$$

for all $t > 0$ and $\varepsilon \in (0, 1)$. Moreover, thanks to the Sobolev inequality associated with the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ we can pick $C_3 > 0$ fulfilling

$$\begin{aligned} \left\| |\nabla c_\varepsilon|^p \right\|_{L^6(\Omega)}^2 &\leq C_3 \cdot \left\{ \left\| \nabla |\nabla c_\varepsilon|^p \right\|_{L^2(\Omega)}^2 + \left\| |\nabla c_\varepsilon|^p \right\|_{L^2(\Omega)}^2 \right\} \\ &\quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1), \end{aligned}$$

so that by (5.25), using Young's inequality we can find $C_4 = C_4(\delta, p) > 0$ such that the last summand in (5.23) can be estimated according to

$$\begin{aligned} &4p(p-1)\chi^2 \int_{\{N_\varepsilon > \gamma\}} N_\varepsilon^p |\nabla c_\varepsilon|^2 \\ &\leq 4p(p-1)\chi^2 C_2 C_3^{\frac{1}{p}} \cdot \left\{ \left\| \nabla \left(\psi'(N_\varepsilon) \right)^{\frac{p}{2p-2}} \right\|_{L^2(\Omega)}^{\frac{1}{p}} \left\| \left(\psi'(N_\varepsilon) \right)^{\frac{p}{2p-2}} \right\|_{L^2(\Omega)}^{\frac{2p-1}{p}} \right. \\ &\quad \left. + \left\| \left(\psi'(N_\varepsilon) \right)^{\frac{p}{2p-2}} \right\|_{L^{\frac{2}{p}}(\Omega)}^2 \right\} \times \\ &\quad \times \left\{ \left\| \nabla |\nabla c_\varepsilon|^p \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \left\| |\nabla c_\varepsilon|^p \right\|_{L^2(\Omega)}^2 \right\}^{\frac{1}{p}} \\ &\leq \delta \cdot \left\{ \left\| \nabla |\nabla c_\varepsilon|^p \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \left\| |\nabla c_\varepsilon|^p \right\|_{L^2(\Omega)}^2 \right\} \\ &\quad + C_4 \cdot \left\{ \left\| \nabla \left(\psi'(N_\varepsilon) \right)^{\frac{p}{2p-2}} \right\|_{L^2(\Omega)}^{\frac{1}{p}} \left\| \left(\psi'(N_\varepsilon) \right)^{\frac{p}{2p-2}} \right\|_{L^2(\Omega)}^{\frac{2p-1}{p}} \right. \\ &\quad \left. + \left\| \left(\psi'(N_\varepsilon) \right)^{\frac{p}{2p-2}} \right\|_{L^{\frac{2}{p}}(\Omega)}^2 \right\}^{\frac{p}{p-1}} \\ &\leq \delta \cdot \left\{ \left\| \nabla |\nabla c_\varepsilon|^p \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \left\| |\nabla c_\varepsilon|^p \right\|_{L^2(\Omega)}^2 \right\} \\ &\quad + 2^{\frac{1}{p-1}} C_4 \left\| \nabla \left(\psi'(N_\varepsilon) \right)^{\frac{p}{2p-2}} \right\|_{L^2(\Omega)}^{\frac{2}{p-1}} \left\| \left(\psi'(N_\varepsilon) \right)^{\frac{p}{2p-2}} \right\|_{L^2(\Omega)}^{\frac{2p-1}{p-1}} \end{aligned}$$

$$+2^{\frac{1}{p-1}} C_4 \left\| \left(\psi'(N_\varepsilon) \right)^{\frac{p}{2p-2}} \right\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2p}{p-1}} \quad (5.27)$$

for all $t > 0$ and $\varepsilon \in (0, 1)$. Here in order to relate the second last summand to the integral in (5.20) stemming from diffusion, we use Lemma 5.2 to estimate

$$\begin{aligned} \left\| \nabla \left(\psi'(N_\varepsilon) \right)^{\frac{p}{2p-2}} \right\|_{L^2(\Omega)}^2 &= \int_{\Omega} \left| \nabla \left(\psi'(N_\varepsilon) \right)^{\frac{p}{2p-2}} \right|^2 \\ &= \frac{p^2}{4(p-1)^2} \int_{\Omega} \left(\psi'(N_\varepsilon) \right)^{\frac{2-p}{p-1}} \left(\psi''(N_\varepsilon) \right)^2 |\nabla N_\varepsilon|^2 \\ &\leq C_5 \int_{\Omega} \psi''(N_\varepsilon) |\nabla N_\varepsilon|^2 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1) \end{aligned}$$

with $C_5 = C_5(p) := \frac{p^2}{4(p-1)^2} \cdot p^{\frac{1}{p-1}}$.

Since $\frac{1}{p-1} < 2$ thanks to our assumption $p > \frac{3}{2}$, we may thus again invoke Young's inequality to find $C_6 = C_6(\delta, p) > 0$ satisfying

$$\begin{aligned} 2^{\frac{1}{p-1}} C_4 \left\| \nabla \left(\psi'(N_\varepsilon) \right)^{\frac{p}{2p-2}} \right\|_{L^2(\Omega)}^{\frac{1}{p-1}} &\left\| \left(\psi'(N_\varepsilon) \right)^{\frac{p}{2p-2}} \right\|_{L^2(\Omega)}^{\frac{2p-1}{p-1}} \\ &\leq 2^{\frac{1}{p-1}} C_4 C_5^{\frac{1}{2p-2}} \left\{ \int_{\Omega} \psi''(N_\varepsilon) |\nabla N_\varepsilon|^2 \right\}^{\frac{1}{2p-2}} \left\| \left(\psi'(N_\varepsilon) \right)^{\frac{p}{2p-2}} \right\|_{L^2(\Omega)}^{\frac{2p-1}{p-1}} \\ &\leq \frac{1}{4} \int_{\Omega} \psi''(N_\varepsilon) |\nabla N_\varepsilon|^2 + C_6 \left\| \left(\psi'(N_\varepsilon) \right)^{\frac{p}{2p-2}} \right\|_{L^2(\Omega)}^{\frac{2(2p-1)}{2p-3}} \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1), \end{aligned} \quad (5.28)$$

where in the last term we recall (5.26) and (5.3) to estimate

$$\begin{aligned} \left\| \left(\psi'(N_\varepsilon) \right)^{\frac{p}{2p-2}} \right\|_{L^2(\Omega)}^2 &= \int_{\{0 \leq N_\varepsilon \leq \gamma\}} \left(p\gamma^{p-2} N_\varepsilon \right)^{\frac{p}{p-1}} + \int_{\{N_\varepsilon > \gamma\}} \left(pN_\varepsilon^{p-1} \right)^{\frac{p}{p-1}} \\ &\leq (p\gamma^{p-1})^{\frac{p}{p-1}} |\Omega| + p^{\frac{p}{p-1}} \int_{\{N_\varepsilon > \gamma\}} N_\varepsilon^p \\ &= (p\gamma^{p-1})^{\frac{p}{p-1}} |\Omega| + p^{\frac{p}{p-1}} \int_{\{N_\varepsilon > \gamma\}} \left(\psi(N_\varepsilon) + \frac{2-p}{2} \gamma^p \right) \\ &\leq p^{\frac{p}{p-1}} \int_{\Omega} \psi(N_\varepsilon) + \frac{4-p}{2} \cdot p^{\frac{p}{p-1}} \gamma^p |\Omega| \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1). \end{aligned}$$

As in the last term in (5.27), given $t_0 \geq 0$ and $T \in (t_0, \infty]$ we similarly find that

$$\begin{aligned} \left\| \left(\psi'(N_\varepsilon) \right)^{\frac{p}{2p-2}} \right\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2}{p}} &= \int_{\{0 \leq N_\varepsilon \leq \gamma\}} \left(p\gamma^{p-2} N_\varepsilon \right)^{\frac{1}{p-1}} + \int_{\{N_\varepsilon > \gamma\}} \left(pN_\varepsilon^{p-1} \right)^{\frac{1}{p-1}} \\ &\leq (p\gamma^{p-1})^{\frac{1}{p-1}} |\Omega| + p^{\frac{1}{p-1}} \int_{\{N_\varepsilon > \gamma\}} N_\varepsilon \\ &\leq p^{\frac{1}{p-1}} |\Omega| \gamma + p^{\frac{1}{p-1}} \int_{\Omega} n_\varepsilon \\ &\leq p^{\frac{1}{p-1}} |\Omega| \gamma + p^{\frac{1}{p-1}} M_1(\varepsilon, T) \quad \text{for all } t \in (t_0, T) \text{ and } \varepsilon \in (0, 1) \end{aligned}$$

by definition (5.8) of $M_1(\varepsilon, T)$, from (5.27), (5.28) and Young's inequality we conclude that with some $C_7 = C_7(\delta, p) > 0$ we have

$$\begin{aligned}
& 4p(p-1)\chi^2 \int_{\{N_\varepsilon > \gamma\}} N_\varepsilon^p |\nabla c_\varepsilon|^2 \\
& \leq \delta \cdot \left\{ \int_{\Omega} |\nabla |\nabla c_\varepsilon|^p|^2 + \frac{1}{2} \int_{\Omega} |\nabla c_\varepsilon|^{2p} \right\} \\
& \quad + \frac{1}{4} \int_{\Omega} \psi''(N_\varepsilon) |\nabla N_\varepsilon|^2 \\
& \quad + C_7 \cdot \left\{ \int_{\Omega} \psi(N_\varepsilon) \right\}^{\frac{2p-1}{2p-3}} + C_7 \gamma^{\frac{p(2p-1)}{2p-3}} \\
& \quad + C_7 \cdot \left\{ \gamma^{\frac{p^2}{p-1}} + M_1^{\frac{p^2}{p-1}}(\varepsilon, T) \right\} \quad \text{for all } t \in (t_0, T) \text{ and } \varepsilon \in (0, 1). \quad (5.29)
\end{aligned}$$

Finally observing that by (5.3) and (5.4),

$$\frac{1}{2} \int_{\Omega} \psi''(N_\varepsilon) |\nabla N_\varepsilon|^2 \geq \frac{p(p-1)}{2} \int_{\{N_\varepsilon > \gamma\}} N_\varepsilon^{p-2} |\nabla N_\varepsilon|^2 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1),$$

we only need to collect (5.21)-(5.24) and (5.29) to infer from (5.20) that indeed (5.19) holds if $K_2 = K_2(\delta, p) > 0$ is appropriately large. $\square$

5.2. Coupling $\int_{\Omega} \psi_{p,\gamma}(N_\varepsilon)$ to $\int_{\Omega} |\nabla c_\varepsilon|^{2p}$. By suitably combining Lemma 5.1 with Lemma 5.3, and by additionally referring to the preliminary bounds obtained in Lemma 2.2, Lemma 3.3 and Lemma 4.2, we can now achieve the main purpose of this section:

Lemma 5.4. *Let $\omega > 0$. Then there exists $\theta_6 = \theta_6(\omega) > 0$ such that for all $\delta > 0$ one can find $\eta_6 = \eta_6(\omega, \delta) > 0$ and $\kappa_6 = \kappa_6(\omega, \delta) > 0$ with the property that if $\rho \in \mathbb{R}$, $\mu > 0$ and $f \in C^1(\bar{\Omega} \times [0, \infty); \mathbb{R}^3)$ are such that (1.10) and (1.11) are valid with some $\eta < \eta_6$ and $\kappa < \kappa_6$, then there exists $t_0 = t_0(\omega, \delta, \eta, \kappa, n_0, c_0, u_0) > 0$ such that for any choice of $\varepsilon \in (0, 1)$, we have*

$$\int_{\Omega} n_\varepsilon^p(x, t) dx + \int_{\Omega} |\nabla c_\varepsilon(x, t)|^{2p} dx < \delta \quad \text{for all } t \geq t_0, \quad (5.30)$$

where $p = p(\omega) := \frac{3}{2} + \theta_6$.

Proof. For fixed $\omega > 0$, we let $\theta_1 := \theta_1(\omega) \in (0, \frac{1}{2})$, $\theta_3 := \theta_3(\omega) > 0$ and $\theta_5 := \theta_5(\omega) > 0$ be as given by Lemma 2.2, Lemma 3.3 and Lemma 4.2, respectively, and set $\theta_6 = \theta_6(\omega) := \min\{\theta_1, \frac{\theta_5}{2}\}$, $p = p(\omega) := \frac{3}{2} + \theta_6$ and $r = r(\omega) := 3 + \theta_3$. We then choose positive numbers $\delta_1 = \delta_1(\omega)$ and $\delta_2 = \delta_2(\omega)$ satisfying

$$\delta_1 \leq \frac{p(p-1)}{2} \quad (5.31)$$

as well as

$$\delta_2 \leq \frac{2(p-1)}{p} \quad \text{and} \quad \delta_2 \leq \frac{p}{2}, \quad (5.32)$$

and let $K_1 := K_1(\delta_1, p, r)$ and $K_2 := K_2(\delta_2, p)$ denote the constants provided by Lemma 5.1 and Lemma 5.3, respectively.

Now given $\delta > 0$, we fix $\delta_3 = \delta_3(\omega, \delta) > 0$ small enough such that writing $\alpha = \alpha(\omega) := \frac{2p-1}{2p-3} > 1$ we have

$$(K_1 + K_2)\delta_3^{\alpha-1} \leq \frac{1}{8}, \quad (5.33)$$

and such that

$$2^p \delta_3 < \frac{\delta}{2}, \quad (5.34)$$

and thereafter we pick $\gamma_0 = \gamma_0(\omega, \delta) > 0$ suitably small fulfilling

$$(4 - p) \cdot 2^{p-1} |\Omega| \cdot \gamma_0^p < \frac{\delta}{2} \quad (5.35)$$

and

$$K_1 \cdot \left\{ \left(2|\Omega| \gamma_0 \right)^{\frac{2p(2p-1)}{4p-3}} + \gamma_0^{\frac{2p(2p-1)}{4p-3}} + \gamma_0^{2p} + \gamma_0^{\frac{2(2p-1)}{2p-3}} \right\} < \frac{\delta_3}{16} \quad (5.36)$$

as well as

$$K_2 \cdot \left\{ \left(2|\Omega| \gamma_0 \right)^{\frac{p^2}{p-1}} + \gamma_0^{\frac{p(2p-1)}{2p-3}} + \gamma_0^{\frac{p^2}{p-1}} + \gamma_0^{\frac{1}{p-1}} \right\} < \frac{\delta_3}{16}. \quad (5.37)$$

We now pick $\delta_4 = \delta_4(\omega, \delta) > 0$ and $\delta_5 = \delta_5(\omega, \delta) > 0$ such that

$$2\delta_4 \leq \left(\frac{\delta_3}{4} \right)^{\frac{2}{p}} \quad (5.38)$$

and

$$K_1 \cdot \left\{ \delta_5^{\frac{2r}{r-3}} + \delta_5^2 \right\} \leq \frac{p}{2}, \quad (5.39)$$

and let $\eta_1 := \eta_1(\omega, \delta_4)$, $\eta_3 := \eta_3(\omega, \delta_5)$, $\kappa_3 := \kappa_3(\omega, \delta_5)$, $\eta_5 := \eta_5(\omega, \delta_4)$ and $\kappa_5 := \kappa_5(\omega, \delta_4)$ be as correspondingly provided by Lemma 2.2, Lemma 3.3 and Lemma 4.2, respectively.

We thereupon claim that the statement of the lemma holds if we pick $\eta_6 = \eta_6(\omega, \delta) > 0$ in such a way that

$$\eta_6 \leq \min\{\eta_1, \eta_3, \eta_5\} \quad \text{and} \quad \eta_6 < \gamma_0, \quad (5.40)$$

and define

$$\kappa_6 = \kappa_6(\omega, \delta) := \min\{\kappa_3, \kappa_5\}. \quad (5.41)$$

To see this, we suppose that $\rho \in \mathbb{R}$, $\mu > 0$ and $f \in C^1(\bar{\Omega} \times [0, \infty); \mathbb{R}^3)$ are such that (1.10) and (1.11) hold with some $\eta < \eta_6$ and $\kappa < \kappa_6$. On applying Lemma 2.2, we then

obtain that since $\eta < \eta_1$ and $p = \frac{3}{2} + \theta_6 \leq \frac{3}{2} + \theta_1$, there exists $t_1 = t_1(\eta) > 0$ such that for all $\varepsilon \in (0, 1)$ we have

$$\int_{t-1}^t \|n_\varepsilon(\cdot, s)\|_{L^p(\Omega)}^2 ds < \delta_4 \quad \text{for all } t \geq t_1. \quad (5.42)$$

Likewise, Lemma 3.3 asserts the existence of $t_2 = t_2(\omega, \eta, \kappa, n_0, u_0) > t_1$ such that for each $\varepsilon \in (0, 1)$,

$$\|u_\varepsilon(\cdot, t)\|_{L^r(\Omega)} < \delta_5 \quad \text{for all } t \geq t_2, \quad (5.43)$$

because $\eta < \eta_3$, $\kappa < \kappa_3$ and $r \leq 3 + \theta_3$, and using that $\eta < \eta_5$, $\kappa < \kappa_5$ and $2p \leq 3 + \theta_5$ we infer from Lemma 4.2 that with some $t_3 = t_3(\omega, \delta, \eta, \kappa, n_0, c_0, u_0) > t_2$ we have

$$\int_{t-1}^t \|\nabla c_\varepsilon(\cdot, s)\|_{L^{2p}(\Omega)}^2 ds < \delta_4 \quad \text{for all } t \geq t_3 \text{ and } \varepsilon \in (0, 1). \quad (5.44)$$

Finally, noting that $\gamma := \frac{\rho_+}{\mu}$ satisfies

$$\gamma < \gamma_0 \quad (5.45)$$

by (5.40), and that hence $\rho_0 := \mu\gamma_0$ is positive with $\rho_0 > \rho$, we may invoke Lemma 2.1 to find $t_0 = t_0(\omega, \delta, \eta, \kappa, n_0, c_0, u_0) > t_3$ such that

$$\int_{\Omega} n_\varepsilon(x, t) dx < 2|\Omega|\gamma_0 \quad \text{for all } t \geq t_0 \text{ and } \varepsilon \in (0, 1). \quad (5.46)$$

In order to prove that this choice ensures that (5.30) holds for each fixed $\varepsilon \in (0, 1)$, given $t \geq t_0$ we first apply the Hölder inequality to obtain from (5.42) and (5.44) that

$$\begin{aligned} & \int_{t-1}^t \left\{ \|n_\varepsilon(\cdot, s)\|_{L^p(\Omega)} + \|\nabla c_\varepsilon(\cdot, s)\|_{L^{2p}(\Omega)}^2 \right\} ds \\ & \leq \int_{t-1}^t \left\{ \|n_\varepsilon(\cdot, s)\|_{L^p(\Omega)}^2 + \|\nabla c_\varepsilon(\cdot, s)\|_{L^{2p}(\Omega)}^2 \right\} ds \\ & < 2\delta_4, \end{aligned}$$

whence we can find $t_\varepsilon \in (t-1, t)$ fulfilling

$$\|n_\varepsilon(\cdot, t_\varepsilon)\|_{L^p(\Omega)} + \|\nabla c_\varepsilon(\cdot, t_\varepsilon)\|_{L^{2p}(\Omega)}^2 < 2\delta_4.$$

Since $(\xi_1 + \xi_2)^{\frac{1}{p}} \leq \xi_1^{\frac{1}{p}} + \xi_2^{\frac{1}{p}}$ for all nonnegative ξ_1 and ξ_2 , in view of (5.38) this entails that

$$\|n_\varepsilon(\cdot, t_\varepsilon)\|_{L^p(\Omega)}^p + \|\nabla c_\varepsilon(\cdot, t_\varepsilon)\|_{L^{2p}(\Omega)}^{2p} < (2\delta_4)^p \leq \frac{\delta_3}{4},$$

and that therefore the function $y \in C^1([t_\varepsilon, \infty))$ defined by

$$y(s) := \int_{\Omega} \psi(N_\varepsilon(x, s)) dx + \int_{\Omega} |\nabla c_\varepsilon(x, s)|^{2p} dx, \quad s \geq t_\varepsilon,$$

where $\psi = \psi_{p,\gamma}$ is as in (5.3) and (5.4), satisfies

$$y(t_\varepsilon) < \frac{\delta_3}{4}, \quad (5.47)$$

because in both cases $\gamma > 0$ and $\gamma = 0$ we have $\psi(s) \leq s^p$ for all $s > 0$.

In particular, (5.47) implies that

$$S := \left\{ T_0 \in [t_\varepsilon, t] \mid y(s) < \delta_3 \text{ for all } s \in [t_\varepsilon, T_0] \right\}$$

is not empty and hence $T := \sup S$ well-defined with $T > t_\varepsilon$ by continuity of y . To see that actually $T = t$, we estimate y by analyzing an ODI which can be derived for this function from Lemma 5.1 and Lemma 5.3. To this end, we let

$$\overline{M}_1 := \sup_{s > t_\varepsilon} \|n_\varepsilon(\cdot, s)\|_{L^1(\Omega)} \quad \text{and} \quad \overline{M}_2 := \sup_{s > t_\varepsilon} \|u_\varepsilon(\cdot, s)\|_{L^r(\Omega)}$$

and then obtain from (5.46) that

$$\overline{M}_1 \leq 2|\Omega|\gamma_0, \quad (5.48)$$

whereas (5.43) combined with (5.39) entails that

$$K_1 \cdot \left\{ \overline{M}_2^{\frac{2r}{r-3}} + \overline{M}_2^2 \right\} \leq \frac{p}{2}. \quad (5.49)$$

In conjunction with the outcome of Lemma 5.1, (5.31), (5.45) and (5.36), the latter two inequalities say that

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} |\nabla c_\varepsilon|^{2p} + \frac{2(p-1)}{p} \int_{\Omega} \left| \nabla |\nabla c_\varepsilon|^p \right|^2 + \frac{p}{2} \int_{\Omega} |\nabla c_\varepsilon|^{2p} \\ & \leq \frac{p(p-1)}{2} \int_{\{N_\varepsilon > \gamma\}} N_\varepsilon^{p-2} |\nabla N_\varepsilon|^2 \\ & \quad + K_1 \cdot \left\{ \int_{\Omega} \psi(N_\varepsilon) + \int_{\Omega} |\nabla c_\varepsilon|^{2p} \right\}^\alpha \\ & \quad + K_1 \cdot \left\{ \left(2|\Omega|\gamma_0 \right)^{\frac{2p(2p-1)}{4p-3}} + \gamma_0^{\frac{2p(2p-1)}{4p-3}} + \gamma_0^{2p} + \gamma_0^{\frac{2(2p-1)}{2p-3}} \right\} \\ & \leq \frac{p(p-1)}{2} \int_{\{N_\varepsilon > \gamma\}} N_\varepsilon^{p-2} |\nabla N_\varepsilon|^2 + K_1 y^\alpha + \frac{\delta_3}{16} \quad \text{on } (t_\varepsilon, \infty), \end{aligned} \quad (5.50)$$

while using (5.48) along with Lemma 5.3, (5.32), (5.45) and (5.37) shows that

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \psi(N_\varepsilon) + \frac{p(p-1)}{2} \int_{\{N_\varepsilon > \gamma\}} N_\varepsilon^{p-2} |\nabla N_\varepsilon|^2 \\ & \leq \frac{2(p-1)}{p} \int_{\Omega} \left| \nabla |\nabla c_\varepsilon|^p \right|^2 + \frac{p}{2} \int_{\Omega} |\nabla c_\varepsilon|^{2p} \\ & \quad + K_2 \cdot \left\{ \int_{\Omega} \psi(N_\varepsilon) \right\}^\alpha \end{aligned}$$

$$\begin{aligned}
& + K_2 \cdot \left\{ \left(2|\Omega|\gamma_0 \right)^{\frac{p^2}{p-1}} + \gamma_0^{\frac{p(2p-1)}{2p-3}} + \gamma_0^{\frac{p^2}{p-1}} + \gamma_0^{\frac{1}{p-1}} \right\} \\
& \leq \frac{2(p-1)}{p} \int_{\Omega} |\nabla |\nabla c_{\varepsilon}|^p|^2 + \frac{p}{2} \int_{\Omega} |\nabla c_{\varepsilon}|^{2p} + K_2 y^{\alpha} + \frac{\delta_3}{16} \quad \text{on } (t_{\varepsilon}, \infty).
\end{aligned} \tag{5.51}$$

Adding (5.50) to (5.51) yields the inequality

$$y'(s) \leq (K_1 + K_2)y^{\alpha}(s) + \frac{\delta_3}{16} + \frac{\delta_3}{16} = (K_1 + K_2)y^{\alpha}(s) + \frac{\delta_3}{8} \quad \text{for all } s > t_{\varepsilon}, \tag{5.52}$$

so that by definition of T and (5.33) we obtain that

$$\begin{aligned}
y'(s) & \leq (K_1 + K_2)\delta_3^{\alpha} + \frac{\delta_3}{8} \\
& = \delta_3 \cdot \left\{ (K_1 + K_2)\delta_3^{\alpha-1} + \frac{1}{8} \right\} \\
& \leq \delta_3 \cdot \left\{ \frac{1}{8} + \frac{1}{8} \right\} = \frac{\delta_3}{4} \quad \text{for all } s \in (t_{\varepsilon}, T).
\end{aligned}$$

According to (5.47) and the fact that $T \leq t < t_{\varepsilon} + 1$, on integration this implies that

$$y(s) \leq y(t_{\varepsilon}) + \frac{\delta_3}{4} \cdot (t - t_{\varepsilon}) \leq y(t_{\varepsilon}) + \frac{\delta_3}{4} < \frac{\delta_3}{2} \quad \text{for all } s \in (t_{\varepsilon}, T).$$

Again by continuity of y , this rules out the possibility that $T < t$, meaning that indeed

$$y(s) < \delta_3 \quad \text{for all } s \in [t_{\varepsilon}, t)$$

and that hence

$$y(t) \leq \delta_3 \quad \text{for all } t \geq t_0. \tag{5.53}$$

Now since by (5.3) and (5.4), in both cases $\gamma > 0$ and $\gamma = 0$ we have

$$\begin{aligned}
\int_{\Omega} \psi(N_{\varepsilon}) & \geq \int_{\{N_{\varepsilon} > \gamma\}} \psi(N_{\varepsilon}) = \int_{\{N_{\varepsilon} > \gamma\}} \left\{ N_{\varepsilon}^p - \frac{2-p}{2} \gamma^p \right\} \\
& \geq \int_{\{N_{\varepsilon} > \gamma\}} N_{\varepsilon}^p - \frac{2-p}{2} |\Omega| \gamma^p \quad \text{for all } t > 0
\end{aligned}$$

and

$$\begin{aligned}
\int_{\{N_{\varepsilon} > \gamma\}} N_{\varepsilon}^p & = \int_{\{n_{\varepsilon} > 2\gamma\}} (n_{\varepsilon} - \gamma)^p > \int_{\{n_{\varepsilon} > 2\gamma\}} \left(\frac{n_{\varepsilon}}{2} \right)^p = \frac{1}{2^p} \cdot \left\{ \int_{\Omega} n_{\varepsilon}^p - \int_{\{n_{\varepsilon} \leq 2\gamma\}} n_{\varepsilon}^p \right\} \\
& \geq \frac{1}{2^p} \int_{\Omega} n_{\varepsilon}^p - \frac{1}{2^p} \cdot |\Omega| \cdot (2\gamma)^p = \frac{1}{2^p} \int_{\Omega} n_{\varepsilon}^p - |\Omega| \gamma^p \quad \text{for all } t > 0,
\end{aligned}$$

it follows that

$$\int_{\Omega} n_{\varepsilon}^p \leq 2^p \cdot \left\{ \int_{\Omega} \psi(N_{\varepsilon}) + |\Omega| \gamma^p + \frac{2-p}{2} |\Omega| \gamma^p \right\}$$

$$= 2^p \int_{\Omega} \psi(N_\varepsilon) + (4 - p) \cdot 2^{p-1} |\Omega| \gamma^p \quad \text{for all } t > 0.$$

In view of (5.45), (5.34) and (5.35), the inequality (5.53) therefore implies that

$$\begin{aligned} \int_{\Omega} n_\varepsilon^p(x, t) dx + \int_{\Omega} |\nabla c_\varepsilon(x, t)|^{2p} dx &\leq 2^p y(t) + (4 - p) \cdot 2^{p-1} |\Omega| \gamma^p \\ &\leq 2^p \delta_3 + (4 - p) \cdot 2^{p-1} |\Omega| \gamma_0^p \\ &< \frac{\delta}{2} + \frac{\delta}{2} = \delta \quad \text{for all } t \geq t_0, \end{aligned}$$

as claimed. $\square$

6. Boundedness of n_ε

Now thanks to the fact that in (5.30) we have $2p > 3$, we may rely on known smoothing properties of the heat semigroup to assert eventual L^∞ bounds for n_ε in the following sense.

Lemma 6.1. *Let $\omega > 0$. Then there exist $\eta_7 = \eta_7(\omega) > 0$ and $\kappa_7 = \kappa_7(\omega) > 0$ such that whenever $\rho \in \mathbb{R}$, $\mu > 0$ and $f \in C^1(\bar{\Omega} \times [0, \infty); \mathbb{R}^3)$ are such that (1.10) and (1.11) hold with some $\eta < \eta_7$ and $\kappa < \kappa_7$, one can find $t_0 = t_0(\omega, \eta, \kappa, n_0, c_0, u_0) > 0$ and $C = C(\omega) > 0$ such that*

$$\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t \geq t_0 \quad (6.1)$$

and each $\varepsilon \in (0, 1)$.

Proof. With $\theta_3 := \theta_3(\omega)$, $\eta_3 := \eta_3(\omega, 1)$ and $\kappa_3 := \kappa_3(\omega, 1)$ taken from Lemma 3.3 and $\theta_6(\omega) > 0$, $\eta_6(\omega, 1)$ and $\kappa_6(\omega, 1)$ as provided by Lemma 5.4, we let $\eta_7 \equiv \eta_7(\omega) := \min\{\eta_3, \eta_6\}$ and $\kappa_7 \equiv \kappa_7(\omega) := \min\{\kappa_3, \kappa_6\}$ and assume that (1.10) and (1.11) hold with some $\eta < \eta_7$ and $\kappa < \kappa_7$. Then applying Lemma 3.3 and Lemma 5.4, we see that writing $p = p(\omega) := 3 + \min\{\theta_3, 2\theta_6\}$, we can fix $t_1 = t_1(\omega, \eta, \kappa, n_0, c_0, u_0) > 0$ and $C_1 = C_1(\omega) > 0$ such that for all $\varepsilon \in (0, 1)$ we have

$$\|u_\varepsilon(\cdot, t)\|_{L^p(\Omega)} \leq C_1 \quad \text{and} \quad \|\nabla c_\varepsilon(\cdot, t)\|_{L^p(\Omega)} \leq C_1 \quad \text{for all } t \geq t_1 \text{ and } \varepsilon \in (0, 1), \quad (6.2)$$

where in view of Lemma 2.1 we may also assume on enlarging t_1 and C_1 if necessary that

$$\|n_\varepsilon(\cdot, t)\|_{L^1(\Omega)} \leq C_1 \quad \text{for all } t \geq t_1 \text{ and } \varepsilon \in (0, 1). \quad (6.3)$$

In order to see that (6.1) holds for $t_0 = t_0(\omega, \eta, \kappa, n_0, c_0, u_0) := t_1 + 1$, we estimate the finite numbers

$$M_\varepsilon(T) := \sup_{t \in (t_0, T)} \|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)}, \quad T > t_0, \quad \varepsilon \in (0, 1), \quad (6.4)$$

by decomposing

$$n_\varepsilon(\cdot, t) = e^{\Delta} n_\varepsilon(\cdot, t-1) - \int_{t-1}^t e^{(t-s)\Delta} \nabla \cdot \left\{ h_{1\varepsilon}(\cdot, s) n_\varepsilon(\cdot, s) \right\} ds$$

$$\begin{aligned}
& + \int_{t-1}^t e^{(t-s)\Delta} h_{2\varepsilon}(\cdot, s) ds \\
& =: n_{1\varepsilon}(\cdot, t) + n_{2\varepsilon}(\cdot, t) + n_{3\varepsilon}(\cdot, t), \quad t \geq t_0, \varepsilon \in (0, 1),
\end{aligned} \tag{6.5}$$

again with $(e^{\tau\Delta})_{\tau \geq 0}$ denoting the Neumann heat semigroup in Ω and

$$\begin{aligned}
h_{1\varepsilon}(x, t) &:= \chi \nabla c_\varepsilon(x, t) + u_\varepsilon(x, t) \quad \text{as well as} \\
h_{2\varepsilon}(x, t) &:= \rho n_\varepsilon(x, t) - \mu n_\varepsilon^2(x, t), \quad t \geq t_0, \varepsilon \in (0, 1).
\end{aligned}$$

Here thanks to a standard L^1 - L^∞ smoothing property of $(e^{\tau\Delta})_{\tau \geq 0}$, there exists $C_2 > 0$ such that due to (6.3) we have

$$\|n_{1\varepsilon}(\cdot, t)\|_{L^\infty(\Omega)} \leq C_2 \|n_\varepsilon(\cdot, t-1)\|_{L^1(\Omega)} \leq C_1 C_2 \quad \text{for all } t \geq t_0 \text{ and } \varepsilon \in (0, 1). \tag{6.6}$$

and since $\rho\xi - \mu\xi^2 \leq \frac{\rho_+^2}{4\mu}$ for all $\xi \geq 0$, we may invoke the maximum principle to obtain the one-sided estimate

$$n_{3\varepsilon}(\cdot, t) \leq \frac{\rho_+^2}{4\mu} \quad \text{in } \Omega \quad \text{for all } t \geq t_0 \text{ and } \varepsilon \in (0, 1). \tag{6.7}$$

Finally, fixing any $p_0 = p_0(\omega) \in (3, p)$ we can make use of a further known smoothing property of the Neumann heat semigroup (cf. [13, Lemma 3.3] for a version precisely covering the present situation) to find $C_3 = C_3(\omega) > 0$ fulfilling

$$\begin{aligned}
& \|n_{2\varepsilon}(\cdot, t)\|_{L^\infty(\Omega)} \\
& \leq C_3 \int_{t-1}^t (t-s)^{-\frac{1}{2}-\frac{3}{2p_0}} \|h_{1\varepsilon}(\cdot, s) n_\varepsilon(\cdot, s)\|_{L^{p_0}(\Omega)} ds \\
& \quad \text{for all } t \geq t_0 \text{ and } \varepsilon \in (0, 1),
\end{aligned} \tag{6.8}$$

where by the Hölder inequality, (6.2), (6.4) and (6.3), given $T > t_0$ we have

$$\begin{aligned}
& \|h_{1\varepsilon}(\cdot, s) n_\varepsilon(\cdot, s)\|_{L^{p_0}(\Omega)} \\
& \leq \|h_{1\varepsilon}(\cdot, s)\|_{L^p(\Omega)} \|n_\varepsilon(\cdot, s)\|_{L^{\frac{pp_0}{p-p_0}}(\Omega)} \\
& \leq \|h_{1\varepsilon}(\cdot, s)\|_{L^p(\Omega)} \|n_\varepsilon(\cdot, s)\|_{L^\infty(\Omega)}^a \|n_\varepsilon(\cdot, s)\|_{L^1(\Omega)}^{1-a} \\
& \leq (\chi + 1) C_1 \cdot M_\varepsilon^a(T) \cdot C_1^{1-a} \quad \text{for all } s \in (t_0, T) \text{ and } \varepsilon \in (0, 1)
\end{aligned} \tag{6.9}$$

with $a = a(\omega) := 1 - \frac{p-p_0}{pp_0} \in (0, 1)$. Since $p_0 > 3$ entails that $\int_0^1 \sigma^{-\frac{1}{2}-\frac{3}{2p_0}} d\sigma$ is finite, combining (6.5)-(6.9) we conclude that there exists $C_4 = C_4(\omega) > 0$ such that whenever $T > t_0$ and $\varepsilon \in (0, 1)$,

$$\sup_{x \in \Omega} n_\varepsilon(x, t) \leq C_4 + C_4 M_\varepsilon^a(T) \quad \text{for all } t \in (t_0, T).$$

This clearly implies that

$$M_\varepsilon(T) \leq \max \left\{ 1, (2C_4)^{\frac{1}{1-a}} \right\} \quad \text{for all } T > t_0 \text{ and each } \varepsilon \in (0, 1),$$

and thereby establishes (6.1) on taking $T \rightarrow \infty$. $\square$

7. A Hölder Bound for u_ε

Together with the hypothesis (1.12), which is now explicitly referred to for the first time, the outcome of Lemma 6.1 next provides regularity information on the forcing terms in the Navier-Stokes part of (1.7) that is sufficient to infer the following consequence concerning eventual Hölder regularity of the velocity field.

Lemma 7.1. *Let $\omega > 0$. Then there exist $\eta_8 = \eta_8(\omega) > 0$ and $\kappa_8 = \kappa_8(\omega) > 0$ with the following property: If $\rho \in \mathbb{R}$, $\mu > 0$ and $f \in C^1(\bar{\Omega} \times [0, \infty); \mathbb{R}^3)$ are such that (1.10) and (1.11) hold with some $\eta < \eta_8$ and $\kappa < \kappa_8$, and if furthermore there exist $\mathbf{p} > \frac{3}{2}$ and $\mathbf{q} > \frac{2\mathbf{p}}{2\mathbf{p}-3}$ such that (1.12) is valid, then one can find $\alpha = \alpha(\omega, f) \in (0, 1)$, $t_0 = t_0(\omega, \eta, f, n_0, c_0, u_0) > 0$ and $C = C(\omega, f) > 0$ such that for all $\varepsilon \in (0, 1)$,*

$$\|u_\varepsilon\|_{C^{\alpha, \frac{\alpha}{2}}(\bar{\Omega} \times [t, t+1])} \leq C \quad \text{for all } t \geq t_0. \quad (7.1)$$

Proof. With $\theta_3 := \theta_3(\omega)$, $\eta_3 := \eta_3(\omega, 1)$ and $\kappa_3 := \kappa_3(1)$ as provided by Lemma 3.3 and $\eta_7 := \eta_7(\omega)$ and $\kappa_7 := \kappa_7(\omega)$ as in Lemma 6.1, we let $\eta_8 = \eta_8(\omega) := \min\{\eta_3, \eta_7\}$ and $\kappa_8 = \kappa_8(\omega) := \min\{\kappa_3, \kappa_7\}$ and suppose that ρ, μ and f have the assumed properties. We then pick $\beta_0 = \beta_0(\omega) > \frac{1}{4}$ such that $\beta_0 \leq \frac{1}{4} + \theta_3$, so that in particular $2\beta_0 - \frac{3}{2} > -1$ and hence there exists $p_1 = p_1(\omega) > 3$ fulfilling

$$2\beta_0 - \frac{3}{2} > -\frac{3}{p_1}. \quad (7.2)$$

We next fix any $p_2 = p_2(\omega) > 3$ such that $p_2 < p_1$, choose $p_3 = p_3(\omega) > 3$ such that

$$p_3 \leq 3 + \theta_3 \quad \text{and} \quad p_3 < \frac{p_1 p_2}{p_1 - p_2} \quad (7.3)$$

and write $\lambda = \lambda(\omega, f) := \max\{\mathbf{p}, p_2\}$. Then according to the inequalities $p_2 > 3$ and $\mathbf{q} > \frac{2\mathbf{p}}{2\mathbf{p}-3}$ it is possible to fix $\beta = \beta(\omega) \in (0, 1)$ in such a way that

$$\beta > \frac{3}{2\lambda} \quad (7.4)$$

and

$$\beta < 1 - \frac{1}{\mathbf{q}} - \frac{3}{2\mathbf{p}} + \frac{3}{2\lambda} \quad (7.5)$$

as well as

$$\beta < \frac{1}{2} - \frac{3}{2p_2} + \frac{3}{2\lambda}, \quad (7.6)$$

where (7.4) guarantees that for some $\alpha_1 = \alpha_1(\omega, f) > 0$ we have $D(A_\lambda^\beta) \hookrightarrow C^{\alpha_1}(\bar{\Omega}; \mathbb{R}^3)$ ([15, 21]), implying that there exists $C_1 = C_1(\omega) > 0$ such that

$$\|\varphi\|_{L^\infty(\Omega)} \leq \|\varphi\|_{C^{\alpha_1}(\bar{\Omega})} \leq C_1 \|A^\beta \varphi\|_{L^\lambda(\Omega)} \quad \text{for all } \varphi \in D(A_\lambda^\beta). \quad (7.7)$$

Moreover, since $\beta_0 \leq \frac{1}{4} + \theta_3$ and $p_3 \leq 3 + \theta_3$, Lemma 3.3 and Lemma 6.1 yield positive constants $t_1 = t_1(\omega, \eta, \kappa, n_0, c_0, u_0)$ and $C_2 = C_2(\omega, f)$ such that for all $\varepsilon \in (0, 1)$ we have

$$\|A^{\beta_0} u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \leq 1, \quad \|u_\varepsilon(\cdot, t)\|_{L^{p_3}(\Omega)} \leq 1$$

$$\text{and } \|n_\varepsilon(\cdot, t)\|_{L^\lambda(\Omega)} \leq C_2 \quad \text{for all } t \geq t_1, \quad (7.8)$$

and we first claim that these inequalities imply a uniform bound, independent of $\varepsilon \in (0, 1)$ and $T > t_0$ with $t_0 = t_0(\omega, \eta, f, n_0, c_0, u_0) := t_1 + 1$, for the numbers

$$M_\varepsilon(T) := \sup_{t \in (t_0, T)} \|A^\beta u_\varepsilon(\cdot, t)\|_{L^\lambda(\Omega)}, \quad T > t_0, \quad \varepsilon \in (0, 1). \quad (7.9)$$

To achieve this, we recall that if for vectors $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ and $w = (w_1, w_2, w_3) \in \mathbb{R}^3$ we define $v \otimes w := (a_{ij})_{i,j \in \{1,2,3\}}$ by letting $a_{ij} := v_i w_j$ for $i, j \in \{1, 2, 3\}$, then since $Y_\varepsilon u_\varepsilon$ is solenoidal, we have $(Y_\varepsilon u_\varepsilon \cdot \nabla) u_\varepsilon \equiv \nabla \cdot (Y_\varepsilon u_\varepsilon \otimes u_\varepsilon)$ ([46, cf. p. 265]). We thereby obtain the representation

$$\begin{aligned} u_\varepsilon(\cdot, t) &= e^{-A} u_\varepsilon(\cdot, t-1) - \int_{t-1}^t e^{-(t-s)A} \mathcal{P} \left[\nabla \cdot \left(Y_\varepsilon u_\varepsilon(\cdot, s) \otimes u_\varepsilon(\cdot, s) \right) \right] ds \\ &\quad + \int_{t-1}^t e^{-(t-s)A} \mathcal{P} \left[n_\varepsilon(\cdot, s) \nabla \phi \right] ds + \int_{t-1}^t e^{-(t-s)A} \mathcal{P} f(\cdot, s) ds \\ &=: u_{1\varepsilon}(\cdot, t) + \dots + u_{4\varepsilon}(\cdot, t), \quad t \geq t_0, \end{aligned} \quad (7.10)$$

to which we apply the fractional power A^β . By (7.8) and standard smoothing properties of the Stokes semigroup ([46], [16, p. 201]), we see that with some $C_3 = C_3(\omega, f) > 0$,

$$\|A^\beta u_{1\varepsilon}(\cdot, t)\|_{L^\lambda(\Omega)} \leq C_3 \|u_\varepsilon(\cdot, t-1)\|_{L^{p_3}(\Omega)} \leq C_3 \quad \text{for all } t \geq t_0. \quad (7.11)$$

Likewise, the third inequality in (7.8) along with the boundedness of $\mathcal{P}$ in $L^\lambda(\Omega; \mathbb{R}^3)$ ([14]) and of $\nabla \phi$ in $L^\infty(\Omega)$ entails the existence of positive constants $C_4(\omega, f)$ and $C_5(\omega, f)$ such that

$$\begin{aligned} \|A^\beta u_{3\varepsilon}(\cdot, t)\|_{L^\lambda(\Omega)} &\leq C_4 \int_{t-1}^t (t-s)^{-\beta} \left\| \mathcal{P} \left[n_\varepsilon(\cdot, s) \nabla \phi \right] \right\|_{L^\lambda(\Omega)} ds \\ &\leq C_5 \int_{t-1}^t (t-s)^{-\beta} \|n_\varepsilon(\cdot, s)\|_{L^\lambda(\Omega)} ds \\ &\leq C_2 C_5 \int_{t-1}^t (t-s)^{-\beta} ds = \frac{C_2 C_5}{1-\beta} \quad \text{for all } t \geq t_0. \end{aligned} \quad (7.12)$$

In quite a similar manner, recalling (1.12) and, again, standard L^p - L^q estimates for the Stokes semigroup, we can find $C_6 = C_6(\omega, f) > 0$ and $C_7 = C_7(\omega, f) > 0$ such that

$$\begin{aligned} &\|A^\beta u_{\varepsilon 4}(\cdot, t)\|_{L^\lambda(\Omega)} \\ &\leq C_6 \int_{t-1}^t (t-s)^{-\beta - \frac{3}{2}(\frac{1}{p} - \frac{1}{\lambda})} \|f(\cdot, s)\|_{L^p(\Omega)} ds \\ &\leq C_6 \left\{ \int_{t-1}^t (t-s)^{-[\beta + \frac{3}{2p} - \frac{3}{2\lambda}] \cdot \frac{q}{q-1}} ds \right\}^{\frac{q-1}{q}} \cdot \left\{ \int_{t-1}^t \|f(\cdot, s)\|_{L^p(\Omega)}^q ds \right\}^{\frac{1}{q}} \\ &\leq C_7 \quad \text{for all } t \geq t_0, \end{aligned} \quad (7.13)$$

where we also have used the Hölder inequality and the fact that

$$\left[\beta + \frac{3}{2p} - \frac{3}{2\lambda} \right] \cdot \frac{q}{q-1} < \left(1 - \frac{1}{q} \right) \cdot \frac{q}{q-1} = 1,$$

as asserted by (7.5).

In order to estimate $A^\beta u_{2\varepsilon}$ appropriately, we once more invoke L^p - L^q estimates for the Stokes evolution operator to obtain $C_8 = C_8(\omega, f) > 0$ such that

$$\begin{aligned} & \|A^\beta u_{2\varepsilon}(\cdot, t)\|_{L^\lambda(\Omega)} \\ & \leq C_8 \int_{t-1}^t (t-s)^{-\beta-\frac{1}{2}-\frac{3}{2}(\frac{1}{p_2}-\frac{1}{\lambda})} \|Y_\varepsilon u_\varepsilon(\cdot, s) \otimes u_\varepsilon(\cdot, s)\|_{L^{p_2}(\Omega)} ds \quad \text{for all } t > 0. \end{aligned} \quad (7.14)$$

Here by the Hölder inequality and the fact that $D(A_2^{\beta_0}) \hookrightarrow L^{p_1}(\Omega; \mathbb{R}^3)$ by (7.2), we can find positive constants $C_9 = C_9(\omega, f)$ and $C_{10} = C_{10}(\omega, f)$ such that due to (7.8) we have

$$\begin{aligned} & \|Y_\varepsilon u_\varepsilon(\cdot, s) \otimes u_\varepsilon(\cdot, s)\|_{L^{p_2}(\Omega)} \\ & \leq C_9 \|Y_\varepsilon u_\varepsilon(\cdot, s)\|_{L^{p_1}(\Omega)} \|u_\varepsilon(\cdot, s)\|_{L^{\frac{p_1 p_2}{p_1 - p_2}}(\Omega)} \\ & \leq C_{10} \|A^{\beta_0} Y_\varepsilon u_\varepsilon(\cdot, s)\|_{L^2(\Omega)} \|u_\varepsilon(\cdot, s)\|_{L^\infty(\Omega)}^a \|u_\varepsilon(\cdot, s)\|_{L^{p_3}(\Omega)}^{1-a} \\ & \leq C_{10} \|A^{\beta_0} u_\varepsilon(\cdot, s)\|_{L^2(\Omega)} \|u_\varepsilon(\cdot, s)\|_{L^\infty(\Omega)}^a \|u_\varepsilon(\cdot, s)\|_{L^{p_3}(\Omega)}^{1-a} \\ & \leq C_{10} \|u_\varepsilon(\cdot, s)\|_{L^\infty(\Omega)}^a \quad \text{for all } s > 0 \end{aligned} \quad (7.15)$$

with $a := 1 - \frac{(p_1 - p_2)p_3}{p_1 p_2} \in (0, 1)$, where we have once more made use of the fact that Y_ε and A^{β_0} commute on $D(A_2^{\beta_0})$, and that Y_ε is nonexpansive on $L_\sigma^2(\Omega)$. In light of (7.7) and the definition (7.9) of $M_\varepsilon(T)$, thanks to the fact that

$$\beta + \frac{1}{2} + \frac{3}{2} \left(\frac{1}{p_2} - \frac{1}{\lambda} \right) < 1$$

guaranteed by (7.6), combining (7.14) with (7.15) yields $C_{11} = C_{11}(\omega, f) > 0$ such that

$$\|A^\beta u_{2\varepsilon}(\cdot, t)\|_{L^\lambda(\Omega)} \leq C_{11} M_\varepsilon^a(T) \quad \text{for all } t \in (t_0, T).$$

Together with (7.10)-(7.13), this shows that with some $C_{12} = C_{12}(\omega, f) > 0$ we have

$$\|A^\beta u_\varepsilon(\cdot, t)\|_{L^\lambda(\Omega)} \leq C_{12} + C_{12} M_\varepsilon^a(T) \quad \text{for all } t \in (t_0, T),$$

and that hence

$$M_\varepsilon(T) \leq C_{13} = C_{13}(\omega, f) := \max \left\{ 1, (2C_{12})^{\frac{1}{1-a}} \right\} \quad \text{for all } T > t_0,$$

in view of (7.7) implying that

$$\|u_\varepsilon(\cdot, t)\|_{C^{\alpha_1}(\bar{\Omega})} \leq C_1 C_{13} \quad \text{for all } t \geq t_0. \quad (7.16)$$

Now by means of a standard adaptation of the above reasoning ([11]), involving estimates quite similar to those used before, it is next possible to find $C_{14} = C_{14}(\omega, f) > 0$ and $\alpha_2 = \alpha_2(\omega, f) > 0$ such that for all $\varepsilon \in (0, 1)$,

$$\|A^\beta u_\varepsilon(\cdot, t) - A^\beta u_\varepsilon(\cdot, \tau)\|_{L^\lambda(\Omega)} \leq C_{14} (t - \tau)^{\alpha_2} \quad \text{for all } \tau \geq t_0 \text{ and } t \in [\tau, \tau + 1].$$

In conjunction with (7.16), this readily yields the claim for suitably small $\alpha = \alpha(\omega, f) \in (0, 1)$. $\square$

8. Eventual Regularity. Proof of Theorem 1.2

Straightforward applications of results on parabolic Hölder regularity, and of Schauder theories for the linear inhomogeneous heat and the Stokes evolution equations, finally turn the above into the following higher order estimates.

Lemma 8.1. *Let $\omega > 0$. Then there exist $\eta = \eta(\omega) > 0$ and $\kappa = \kappa(\omega) > 0$ such that if $\rho \in \mathbb{R}$, $\mu > 0$ and $f \in C^1(\bar{\Omega} \times [0, \infty); \mathbb{R}^3)$ are such that (1.10), (1.11) and (1.12) are valid with some $\mathbf{p} > \frac{3}{2}$ and $\mathbf{q} > \frac{2\mathbf{p}}{2\mathbf{p}-3}$, then there exist $\alpha = \alpha(\omega, f) \in (0, 1)$ and $t_0 = t_0(\omega, \eta, f, n_0, c_0, u_0) > 0$ such that for each $T > t_0$ and $\varepsilon \in (0, 1)$,*

$$\|n_\varepsilon\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{\Omega} \times [t_0, T])} + \|c_\varepsilon\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{\Omega} \times [t_0, T])} + \|u_\varepsilon\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{\Omega} \times [t_0, T])} \leq C \quad (8.1)$$

holds with some $C = C(\omega, f, T) > 0$.

Proof. For $i \in \{3, 6\}$ we let $\theta_i := \theta_i(\omega)$, $\eta_i := \eta_i(\omega, 1)$ and $\kappa_i := \kappa_i(\omega, 1)$ be as in Lemma 3.3 and Lemma 5.4, and for $i \in \{7, 8\}$ we take $\eta_i := \eta_i(\omega)$ and $\kappa_i := \kappa_i(\omega)$ from Lemma 6.1 and Lemma 7.1. Then assuming (1.10), (1.11) and (1.12) to be satisfied with some $\eta < \min\{\eta_3, \eta_6, \eta_7, \eta_8\}$, $\kappa < \min\{\kappa_3, \kappa_6, \kappa_7, \kappa_8\}$, $\mathbf{p} > \frac{3}{2}$ and $\mathbf{q} > \frac{2\mathbf{p}}{2\mathbf{p}-3}$, we once more write the first Eq. in (1.7) in the form

$$n_{\varepsilon t} = \Delta n_\varepsilon - \nabla \cdot (n_\varepsilon h_{1\varepsilon}(x, t)) + h_{2\varepsilon}(x, t), \quad x \in \Omega, \quad t > 0, \quad \varepsilon \in (0, 1), \quad (8.2)$$

with

$$\begin{aligned} h_{1\varepsilon}(x, t) &:= \chi \nabla c_\varepsilon(x, t) + u_\varepsilon(x, t) \\ \text{and } h_{2\varepsilon}(x, t) &:= \rho n_\varepsilon(x, t) - \mu n_\varepsilon^2(x, t), \quad x \in \Omega, \quad t > 0, \quad \varepsilon \in (0, 1). \end{aligned}$$

Here we see from Lemma 3.3, Lemma 5.4 and Lemma 6.1 that there exist $t_1 = t_1(\omega, \eta, f, n_0, c_0, u_0) > 0$ and $C_1 = C_1(\omega, f) > 0$ such that with $p_1 = p_1(\omega) := 3 + \min\{2\theta_3, 2\theta_6\}$, for all $\varepsilon \in (0, 1)$ we have

$$\begin{aligned} \|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} &\leq C_1, \quad \|h_{1\varepsilon}(\cdot, t)\|_{L^{p_1}(\Omega)} \leq C_1 \quad \text{and} \\ \|h_{2\varepsilon}(\cdot, t)\|_{L^\infty(\Omega)} &\leq C_1 \quad \text{for all } t \geq t_1. \end{aligned}$$

Since $p_1 > 3$, we can therefore apply standard parabolic Hölder regularity theory ([44, Theorem 1.3]) to see that there exists $\alpha_1 = \alpha_1(\omega) > 0$ with the property that for each $T > t_2 = t_2(\omega, \eta, f, n_0, c_0, u_0) := t_1 + 1$ we can find $C_2 = C_2(\omega, f, T) > 0$ fulfilling

$$\|n_\varepsilon\|_{C^{\alpha_1, \frac{\alpha_1}{2}}(\bar{\Omega} \times [t_2, T])} \leq C_2 \quad \text{for all } \varepsilon \in (0, 1). \quad (8.3)$$

Along with the outcome of Lemma 7.1, this shows that the coefficient functions n_ε and u_ε appearing in the second Eq. in (1.7) both satisfy ε -independent estimates in $C^{\alpha_2, \frac{\alpha_2}{2}}(\bar{\Omega} \times [t_3, T])$ for some $\alpha_2 = \alpha_2(\omega, f) \in (0, 1)$ and $t_3 > t_2$ and each $T > t_3$, which by parabolic Schauder theory ([30]) implies the existence of $\alpha_3 = \alpha_3(\omega, f) \in (0, 1)$ such that for all $T > t_4 = t_4(\omega, \eta, f, n_0, c_0, u_0) := t_3 + 1$ we have

$$\|c_\varepsilon\|_{C^{2+\alpha_3, 1+\frac{\alpha_3}{2}}(\bar{\Omega} \times [t_4, T])} \leq C_3 \quad \text{for all } \varepsilon \in (0, 1) \quad (8.4)$$

with some $C_3 = C_3(\omega, f, T) > 0$. This in turn suggests to re-interpret (8.2) as

$$n_{\varepsilon t} = \Delta n_\varepsilon - h_{1\varepsilon}(x, t) \cdot \nabla n_\varepsilon - h_{3\varepsilon}(x, t), \quad x \in \Omega, \quad t > 0, \quad \varepsilon \in (0, 1),$$

with $h_{1\varepsilon}$ as above and

$$h_{3\varepsilon}(x, t) := \chi n_\varepsilon(x, t) \Delta c_\varepsilon(x, t) + \rho n_\varepsilon(x, t) - \mu n_\varepsilon^2(x, t), \quad x \in \Omega, t > 0, \varepsilon \in (0, 1).$$

Now, namely, we know that for some $\alpha_4 = \alpha_4(\omega, f) \in (0, 1)$, all $T > t_4$ and any $\varepsilon \in (0, 1)$ we have

$$\|h_{i\varepsilon}\|_{C^{\alpha_4, \frac{\alpha_4}{2}}(\bar{\Omega} \times [t_4, T])} \leq C_4 \quad \text{for } i \in \{1, 3\}$$

with some $C_4 = C_4(\omega, f, T) > 0$, so that again by parabolic Schauder theory we obtain $\alpha_5 = \alpha_5(\omega, f) \in (0, 1)$ with the property that for all $T > t_5 = t_5(\omega, \eta, f, n_0, c_0, u_0) := t_4 + 1$ we can find $C_5 = C_5(\omega, f, T) > 0$ such that

$$\|n_\varepsilon\|_{C^{2+\alpha_5, 1+\frac{\alpha_5}{2}}(\bar{\Omega} \times [t_5, T])} \leq C_5 \quad \text{for all } \varepsilon \in (0, 1). \quad (8.5)$$

Finally, combining (8.3) with Lemma 7.1 and our overall assumption $f \in C^1(\bar{\Omega} \times [0, \infty); \mathbb{R}^3)$ allows us to follow a standard reasoning involving Schauder estimates for the linear inhomogeneous Stokes evolution Eq. ([47]) to see that

$$\|u_\varepsilon\|_{C^{2+\alpha_6, 1+\frac{\alpha_6}{2}}(\bar{\Omega} \times [t_6, T])} \leq C_6$$

is valid for some $\alpha_6 = \alpha_6(\omega, f) \in (0, 1)$ and $t_6 = t_6(\omega, \eta, f, n_0, c_0, u_0) > t_5$, each $T > t_6$ and $\varepsilon \in (0, 1)$ and some appropriately large $C_6 = C_6(\omega, f, T) > 0$. Together with (8.4) and (8.5), this proves (8.1). $\square$

In view of the Arzelà-Ascoli theorem, the following consequence of the latter is immediate:

Lemma 8.2. *Let $\omega > 0$. Then there exist $\eta = \eta(\omega) > 0$ and $\kappa = \kappa(\omega) > 0$ such that if $\rho \in \mathbb{R}$, $\mu > 0$ and $f \in C^1(\bar{\Omega} \times [0, \infty); \mathbb{R}^3)$ are such that (1.10), (1.11) and (1.12) hold with some $\mathbf{p} > \frac{3}{2}$ and $\mathbf{q} > \frac{2\mathbf{p}}{2\mathbf{p}-3}$, then there exists $t_0 = t_0(\omega, \eta, f, n_0, c_0, u_0) > 0$ with the property that given any $(\varepsilon_j)_{j \in \mathbb{N}} \subset (0, 1)$ satisfying $\varepsilon_j \rightarrow 0$ as $j \rightarrow \infty$ one can find $(n, c, u, P) \in (C^{2,1}(\bar{\Omega} \times [t_0, \infty)))^2 \times C^{2,1}(\bar{\Omega} \times [t_0, \infty); \mathbb{R}^3) \times C^{1,0}(\bar{\Omega} \times [t_0, \infty))$ and a subsequence $(\varepsilon_{j_k})_{k \in \mathbb{N}}$ such that*

$$\begin{cases} n_\varepsilon \rightarrow n & \text{in } C_{loc}^{2,1}(\bar{\Omega} \times [t_0, \infty)), \\ c_\varepsilon \rightarrow c & \text{in } C_{loc}^{2,1}(\bar{\Omega} \times [t_0, \infty)) \\ u_\varepsilon \rightarrow u & \text{in } C_{loc}^{2,1}(\bar{\Omega} \times [t_0, \infty)), \end{cases} \quad \text{and} \quad (8.6)$$

and such that (n, c, u, P) is a classical solution of (1.1), (1.5) in $\bar{\Omega} \times [t_0, \infty)$.

Proof. All statements directly result from Lemma 8.1 upon an application of the Arzelà-Ascoli theorem and thereafter taking $\varepsilon = \varepsilon_{j_k} \rightarrow 0$ in each of the expressions in the PDE system in (1.7) separately, finally constructing the associated pressure P by means of a standard procedure ([46, 51]). $\square$

The derivation of our main results hence reduces to suitably collecting the essence of the above:

Proof of Theorem 1.2. Taking $\eta = \eta(\omega)$ and $\kappa = \kappa(\omega)$ as in Lemma 8.2, from a combination of the latter lemma with Proposition 1.1 we immediately obtain that for some $t_0 = t_0(\omega, \eta, f, n_0, c_0, u_0) \geq 0$, the global generalized solution (n, c, u) from said proposition indeed enjoys the eventual smoothness properties listed in (1.13), and that with some $P \in C^{1,0}(\bar{\Omega} \times [t_0, \infty))$, (n, c, u, P) actually solves (1.1), (1.5) classically in $\bar{\Omega} \times [t_0, \infty)$. Finally, the boundedness properties claimed in (1.14) are implied by Lemma 6.1, Lemma 4.1, Lemma 5.4 and Lemma 7.1 upon noting that $W^{1,2p}(\Omega) \hookrightarrow L^\infty(\Omega)$ for each $p > \frac{3}{2}$. $\square$

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9. Appendix: The Underlying Solution Concept

For completeness, we finally import from [59] the framework of generalized solvability that underlies the existence statement from Proposition 1.1. Here with regard to the requirements imposed on candidates for solutions, the main relaxation consists in considering the first sub-problem in (1.1)-(1.4)-(1.5) fulfilled if merely certain transformed versions of n play roles of weak sub- and supersolutions thereof in a sense specified as follows:

Definition 9.1. Let $\Phi \in C^2([0, \infty))$ be nonnegative with $\Phi' > 0$ on $[0, \infty)$, and let $n_0 : \Omega \rightarrow [0, \infty)$, $c_0 : \Omega \rightarrow \mathbb{R}$ and $u_0 : \Omega \rightarrow \mathbb{R}^3$ are such that $\Phi(n_0) \in L^1(\Omega)$. Suppose that $n : \Omega \times (0, \infty) \rightarrow [0, \infty)$, $c : \Omega \times (0, \infty) \rightarrow \mathbb{R}$ and $u : \Omega \times (0, \infty) \rightarrow \mathbb{R}^3$ are such that ∇n , ∇c and Δc are measurable, that

$$\left\{ \Phi(n), \Phi''(n)|\nabla n|^2, \Phi(n)\Delta c, n\Phi'(n)\Delta c, n\Phi'(n), n^2\Phi'(n) \right\} \subset L^1_{loc}(\bar{\Omega} \times [0, \infty))$$

and

$$\left\{ \Phi'(n)\nabla n, \Phi(n)\nabla c, \Phi(n)u \right\} \subset L^1_{loc}(\bar{\Omega} \times [0, \infty); \mathbb{R}^3)$$

and that $\nabla \cdot u \equiv 0$ in $\mathcal{D}'(\Omega \times (0, \infty))$. Then we say that (n, c, u) is a *global weak Φ -subsolution* (resp., a *global weak Φ -supersolution*) of the first Eqs. in (1.1)-(1.4)-(1.5) if

$$-\int_0^\infty \int_\Omega \Phi(n)\varphi_t - \int_\Omega \Phi(n_0)\varphi(\cdot, 0) \stackrel{(\geq)}{\leq} -\int_0^\infty \int_\Omega \Phi''(n)|\nabla n|^2\varphi$$

$$\begin{aligned}
& - \int_0^\infty \int_\Omega \Phi'(n) \nabla n \cdot \nabla \varphi \\
& + \chi \int_0^\infty \int_\Omega \Phi(n) \nabla c \cdot \nabla \varphi + \chi \int_0^\infty \int_\Omega \Phi(n) \Delta c \varphi - \chi \int_0^\infty \int_\Omega n \Phi'(n) \Delta c \varphi \\
& + \rho \int_0^\infty \int_\Omega n \Phi'(n) \varphi - \mu \int_0^\infty \int_\Omega n^2 \Phi'(n) \varphi + \int_0^\infty \int_\Omega \Phi(n) u \cdot \nabla \varphi
\end{aligned}$$

holds for all nonnegative $\varphi \in C_0^\infty(\bar{\Omega} \times [0, \infty))$.

The solution concept under consideration, partially resembling that also used in related contexts (cf. e.g. [2]) can now be specified as follows, again letting $v \otimes w := (a_{ij})_{i,j \in \{1,2,3\}}$, with $a_{ij} := v_i w_j$, $i, j \in \{1, 2, 3\}$, for $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ and $w = (w_1, w_2, w_3) \in \mathbb{R}^3$.

Definition 9.2. Let $n_0 \in L^1(\Omega)$, $c_0 \in L^1(\Omega)$ and $u_0 \in L^1(\Omega; \mathbb{R}^3)$, and assume that

$$\begin{cases} n \in L^1_{loc}(\bar{\Omega} \times [0, \infty)), \\ c \in L^1_{loc}([0, \infty); W^{1,1}(\Omega)) \quad \text{and} \\ u \in L^1_{loc}([0, \infty); W^{1,1}_0(\Omega; \mathbb{R}^3)) \end{cases}$$

are such that

$$cu \in L^1_{loc}(\bar{\Omega} \times [0, \infty); \mathbb{R}^3) \quad \text{and} \quad u \otimes u \in L^1_{loc}(\Omega \times [0, \infty); \mathbb{R}^{3 \times 3}),$$

and that $n \geq 0$ a.e. in $\Omega \times (0, \infty)$. Then (n, c, u) is called a *global generalized solution* of (1.1)-(1.4)-(1.5) if $\nabla \cdot u = 0$ in $\mathcal{D}'(\Omega \times (0, \infty))$, if

$$\begin{aligned}
& - \int_0^\infty \int_\Omega c \varphi_t - \int_\Omega c_0 \varphi(\cdot, 0) \\
& = - \int_0^\infty \int_\Omega \nabla c \cdot \nabla \varphi - \int_0^\infty \int_\Omega c \varphi + \int_0^\infty \int_\Omega n \varphi + \int_0^\infty \int_\Omega cu \cdot \nabla \varphi
\end{aligned}$$

for all $\varphi \in C_0^\infty(\bar{\Omega} \times [0, \infty))$ and

$$\begin{aligned}
& - \int_0^\infty \int_\Omega u \cdot \varphi_t - \int_\Omega u_0 \cdot \varphi(\cdot, 0) \\
& = - \int_0^\infty \int_\Omega \nabla u \cdot \nabla \varphi + \int_0^\infty \int_\Omega (u \otimes u) \cdot \nabla \varphi + \int_0^\infty \int_\Omega n \nabla \phi \cdot \varphi + \int_0^\infty \int_\Omega f \cdot \varphi
\end{aligned}$$

for all $\varphi \in C_0^\infty(\Omega \times [0, \infty); \mathbb{R}^3)$ fulfilling $\nabla \cdot \varphi \equiv 0$, and if there exist $\Phi_1, \Phi_2 \in C^2([0, \infty))$ such that $\Phi'_1 > 0$ and $\Phi'_2 > 0$ on $[0, \infty)$, and such that (n, c, u) is a weak Φ_1 -subsolution and a weak Φ_2 -supersolution of the first Eqs. in (1.1)-(1.4)-(1.5) in the sense of Definition 9.1.

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