Superintegrable Systems on Moduli Spaces of Flat Connections

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Abstract: The main result of this paper is the construction of a family of superintegrable Hamiltonian systems on moduli spaces of flat connections on a principal G -bundle on a surface. The moduli space is a Poisson variety with Atiyah–Bott Poisson structure. Among particular cases of such systems are spin generalizations of Ruijsenaars–Schneider models.

Introduction

Let G be a connected simple linear algebraic group over $\mathbb{C}$. In this paper we construct superintegrable Hamiltonian systems on moduli spaces of flat G -connections over any oriented surface with nonempty boundary. Hamiltonians of such systems are traces of holonomies along non intersecting non self-intersecting curves. The construction naturally works in the same way for various real forms of G , for example for compact simple Lie groups or for split real forms.

The fact that Hamiltonian integrable systems appear in the context of gauge theories is not new, with Hitchin systems [H87] being one of the well known examples. Calogero–Moser type systems were put in the context of gauge theory in early 1990’s, see [GN94, GN95] and references therein.¹ At about the same time, Jeffrey and Weitsman [JW92, JW94] have constructed an integrable system on the $SU(2)$ moduli space of flat connections on a closed genus g surface. Hamiltonians of this system were given by traces of holonomies along the maximal collection of nonintersecting curves. In the higher rank case, the total number of hamiltonians which can be constructed from class functions of such holonomies is insufficient for Liouville integrability.

The main result of this paper involves the notion of a superintegrable Hamiltonian system which we review in Sect. 1, and of graph functions which we review in Sect. 2.2. Denote the moduli space of flat G -connections on Σ as $\mathcal{M}_{\Sigma}^G$. Our main result is the following theorem.

¹ See also [FK13, AO19] for more recent developments and references.

Theorem 1. *Let Σ be an oriented surface of genus g with $b > 0$ boundary components and G be a connected simple linear algebraic group over $\mathbb{C}$. For each disjoint union of pairwise nonhomotopic simple closed curves $C = C_1 \sqcup \cdots \sqcup C_k$ in Σ , none of which is homotopic to a boundary component, the following inclusion of Poisson algebras defines an affine superintegrable system*

$$Z_{\partial\Sigma} \subset B_C \subset J_{\Sigma \setminus C} \subset \mathcal{O}[\mathcal{M}_\Sigma^G].$$

Here $Z_{\partial\Sigma}$ is the algebra of G -invariant functions on holonomies around boundary components, B_C is a subalgebra of the coordinate ring $\mathcal{O}[\mathcal{M}_\Sigma^G]$ generated by graph functions $F_{\Gamma, V, C}$ with Γ being contractible to $C \sqcup \partial\Sigma$. Similarly, $J_{\Sigma \setminus C}$ is a subalgebra generated by graph functions $F_{\Gamma, V, C}$ with $\Gamma \subset (\Sigma \setminus C)$.

We also introduce the notion of a *refinement* of one superintegrable systems by the other. The refinement defines a partial order on the space of superintegrable systems.

For a torus with one puncture these systems are spin Ruijsenaars systems (also known as relativistic spin Calogero–Moser systems) [FGNR00, Res16]. For a torus with two punctures they are relativistic deformations of two sided spin Calogero–Moser systems [Res18] corresponding to symmetric pair $G \subset G \times G$ where G is embedded diagonally. A closely related example of systems on a torus with n punctures and rank one conjugation orbits of SL_n is closely related to the systems studied in [CF18]. In this case the systems are actually Liouville integrable and they admit a superintegrable refinement. The maximal integrability in this case is achieved by a superintegrable system with $r = \text{rk}(G)$ independent Poisson commuting integrals. The minimal integrability is a Liouville integrability with $b \text{rk}(G)$ Poisson commuting integrals. In our framework these correspond to different choices of cycles C .

The mapping class group acts on the moduli space of flat connections by Poisson automorphisms. If C and C' are collections of cycles on surface Σ which belong to the same orbit of the Mapping Class Group action, the corresponding superintegrable systems are isomorphic. An example of such isomorphism in the genus one case for involution exchanging the equator and meridian of the torus gives rise to a self-duality of the Ruijsenaars system [FGNR00, FR99, FK11]. For a spin version of this duality see [Res16].

The plan of this paper is as follows. In the first section we recall the definition of superintegrable (degenerately integrable) systems. There we also define the notion of an affine superintegrable system in the algebro-geometrical setting. The second section is an overview of basic notions about moduli spaces of flat connections on a surface. In this section we also recall the definition of graph functions and the description of Poisson brackets between two such functions. In the third section we describe the main result, the construction of a family of Hamiltonian systems defined by a choice of non intersecting, non self intersecting cycles and prove their superintegrability. At the end of this section we introduce the notion of a partial order on such systems. In section 4 we explain how solutions to equations of motion of these superintegrable systems can be solved using the projection method. Sect. 5 has some genus one examples. In the conclusion we define a conjecturally superintegrable system on the space of chord diagrams, discuss the case of non-generic conjugation orbits and some further directions.

1. Superintegrable Systems

1.1. Superintegrable systems. For an overview of superintegrable systems see for example [Sint, Res16]. Here we recall briefly basic definitions and introduce the notion of an affine superintegrable system.

The notion of a superintegrable Hamiltonian systems was introduced in [Nek72] (where he called them degenerate integrable systems) as a generalization of the Liouville integrable systems. Superintegrable system on a $2n$ dimensional smooth symplectic manifold has $m > n$ independent² first integrals, however only k of them are in involution and $m + k = 2n$. A particular case when $m = k = n$ corresponds to Liouville integrable systems. First examples of such systems appeared earlier [Pa26, F35, FMSW65]. A family of examples of superintegrable systems associated to Lie groups was given in [MF78]. More recent examples include characteristic systems on simple Poisson–Lie groups [Res03a], spin Calogero–Moser and Ruijsenaars systems [Res03b] and their relativistic counterparts [Res16].

Geometrically, a superintegrable system of rank k on a symplectic manifold $\mathcal{M}_{2n}$ consists of a Poisson manifold $\mathcal{P}_{2n-k}$, a manifold $\mathcal{B}_k$ ³ (considered as a Poisson manifold with trivial symplectic structure) and Poisson projections

$$\mathcal{M}_{2n} \xrightarrow{p_1} \mathcal{P}_{2n-k} \xrightarrow{p_2} \mathcal{B}_k \quad (1)$$

Denote by $A = C^\infty(\mathcal{M}_{2n})$ the algebra of smooth functions on $\mathcal{M}_{2n}$. It is equipped with a Poisson bracket $\{, \}$ determined by the symplectic structure. In terms of functions on $\mathcal{M}_{2n}$ superintegrable system is determined by two Poisson subalgebras of A :

$$B \subset J \subset A \quad (2)$$

where $J = C^\infty(\mathcal{P}_{2n-k})$ and $B = C^\infty(\mathcal{B}_k)$. The subalgebra B is a Poisson commutative subalgebra and J is the Poisson centralizer of B , i.e. J is the maximal subalgebra in A such every element from J Poisson commutes with every element from B .

Elements of subalgebra B define derivations of A

$$D_b : A \rightarrow A, \quad D_b(a) = \{b, a\}, \quad D_b(a_1 a_2) = a_1 D_b(a_2) + D_b(a_1) a_2, \quad b \in B, \quad a, a_1, a_2 \in A. \quad (3)$$

Because B is Poisson commutative, these derivations commute

$$D_{b_1} D_{b_2} - D_{b_2} D_{b_1} = 0$$

Derivation D_b is the Lie derivative along the Hamiltonian vector field on $\mathcal{M}_{2n}$ generated by b . We refer to elements of B as *Hamiltonians* of a superintegrable system (2). On the other hand, elements of subalgebra J are precisely the first integrals of the Hamilton flow generated by b , i.e. they are constant on flow lines of the Hamiltonian vector field generated by b .

Fix $b_1, \dots, b_k \in B$, a choice of independent Hamiltonians. Corresponding vector fields $D_{b_1}, \dots, D_{b_k} \in T\mathcal{M}_{2n}$ are independent at every point $x \in \mathcal{M}_{2n}$ and hence form a basis on the k -dimensional subspace of $T_x \mathcal{M}_{2n}$ defined by the level set of J . Algebraically, these vector fields $D_{b_1}, \dots, D_{b_k} \in \text{Der}_J(A)$ define derivations of A relative to J . Balance of dimensions in (1) is equivalent to the fact that $\text{Der}_J(A)$ is generated as an A -module by Hamiltonian vector fields of the form D_b for $b \in B$.

² On a smooth manifold $\mathcal{M}$ (complex or real) we call functions $f_1, \dots, f_k \in C^\infty(\mathcal{M})$ *independent* if the corresponding differentials $df_1, \dots, df_k \in \Gamma(T^*\mathcal{M})$ are linearly independent at every point in $\mathcal{M}$.

³ Typically $\mathcal{B}_k$ is not smooth but stratified by smooth strata with one smooth higher dimensional stratum. It is frequently an orbifold.

Fig. 1. One choice of arcs corresponding to generators of $\pi_1(\Sigma_{1,2}, p)$

1.3. Refinement of a superintegrable system. Here we will introduce the notion of a *refinement* of a superintegrable system. It gives a partial order on all superintegrable systems on a given Poisson algebra A . A minimal object in this partial order is known as a maximally superintegrable system. Maximal superintegrability means that all invariant tori are one dimensional [SInt].

Let

$$Z \subset B_1 \subset J_1 \subset A, \quad Z \subset B_2 \subset J_2 \subset A \quad (6)$$

be two superintegrable systems on a Poisson algebra A with the Poisson center Z .

Definition 3. The first superintegrable system is a refinement of the second if the following chain of Poisson inclusions hold:

$$Z \subset B_1 \subset B_2 \subset J_2 \subset J_1 \subset A$$

Clearly a refinement defines a partial order on the set of integrable systems.

Another important relation is an equivalence of integrable systems

Definition 4. Two integrable systems (6) are *equivalent* if there is a Poisson automorphism $\varphi : A \rightarrow A$ such that $J_2 = \varphi(J_1)$ and $B_1 = \varphi(B_2)$.

2. Moduli Spaces of Flat Connections

2.1. Character variety of the fundamental group. Let $\Sigma_{g,b}$ be an oriented surface of genus g with b boundary components. We will consider only surfaces with $b > 0$. The fundamental group $\pi_1(\Sigma, p)$ with the base point p of such surface is generated by elements $x_1, \dots, x_g, y_1, \dots, y_g, z_1, \dots, z_b$ with one defining relation:

$$\pi_1(\Sigma_{g,b}) = \langle x_1, \dots, x_g, y_1, \dots, y_g, z_1, \dots, z_b \mid [x_1, y_1] \dots [x_g, y_g] z_1 \dots z_b = \text{Id}_p \rangle.$$

Here x_i and y_i are fundamental cycles for Σ and z_i is the cycle for i -th boundary component.⁶

⁶ Throughout the text we always assume that the composition of paths is read from right to left. For example, $[x_1, y_1]z_1z_2$ corresponds to the contractible path along the boundary of a disc in $\Sigma_{1,2}$ on Fig. 1.

The *character variety* of $\pi_1(\Sigma_{g,b})$ in a simple complex linear algebraic group G is the categorical quotient of the space of group homomorphisms from $\pi_1(\Sigma_{g,b})$ to G with respect to conjugations by G^7 :

$$\mathcal{M}_{\Sigma}^G = (\pi_1(\Sigma_{g,b}) \rightarrow G) // G \quad := \quad \text{Spec} \left(\mathcal{O}[\text{Hom}(\pi_1(\Sigma_{g,b}), G)]^G \right)$$

For $b > 0$ we have an obvious isomorphism:

$$\mathcal{M}_{\Sigma}^G \simeq G^{\times(2g+b-1)} // G$$

We can choose this isomorphism as the mapping which assigns to a group homomorphism $\rho : \pi_1(\Sigma_{g,b}) \rightarrow G$ an element $(\rho(x_1), \dots, \rho(x_g), \rho(y_1), \dots, \rho(y_g), \rho(z_1), \dots, \rho(z_{b-1})) \in G^{\times(2g+b-1)}$ and then projects it to the conjugacy class in $G^{\times(2g+b-1)} // G$.

2.2. Moduli space of surface graph connections. Denote by $V(\Gamma)$ and $E(\Gamma)$ the set of vertices and edges of an oriented graph Γ . Graph connection is an assignment of a parallel transport $g_e \in G$ to each oriented edge $e \in E(\Gamma)$.

The gauge group $G(V) \simeq G^{V(\Gamma)} = \{V(\Gamma) \rightarrow G, \quad v \mapsto h_v\}$ acts on a graph connection g as

$$g_e \mapsto h_{t(e)} g_e h_{s(e)}^{-1}.$$

where $t(e)$ is the target vertex for an oriented edge e and $s(e)$ is its source vertex.

Assume that $\Gamma \subset \Sigma$ is an oriented embedded graph such that $\Sigma \setminus \Gamma$ is a disjoint collection of disks and b annuli corresponding to the boundary components of $\Sigma = \Sigma_{g,b}$. We will call such graphs *simple*. Hereinafter we allow graphs to have multiple edges and loops.

Let $F(\Gamma)$ be the set of contractible faces, i.e. the set of disks in $\Sigma_{g,b} \setminus \Gamma$. For every disk $D \in F(\Gamma)$ choose an arbitrary vertex $w_D \in \partial D$ and let

$$g_{\partial D} = g_{e_k}^{\epsilon_k} \dots g_{e_1}^{\epsilon_1}$$

be the holonomy of graph connection along the boundary of a disk, based at $w_D \in \partial D$. The product of elements $g_{e_j}^{\epsilon_j} \in G$ associated to each edge $e_j \in \partial D$ is given according to the total order induced by the orientation of the surface and choice of $w_D \in \partial D$. The relative orientation ϵ_j is +1 when the orientations of e_j and ∂D coincide and -1 otherwise.

A graph connection over $\Gamma \subset \Sigma$ is called *flat* if $g_{\partial D} = 1$ for all $D \in F(\Gamma)$. Note that this condition does not depend on the choice of the base vertex $w_D \in \partial D$. Define the space of *flat graph G -connections* $\mathcal{A}_{(\Sigma, \Gamma)}$ on $\Gamma \subset \Sigma$ as the space of such flat connections.

When G is a linear algebraic group, the space of flat graph connections is an algebraic subset of $G^{|E(\Gamma)|}$ equipped with a regular action of $G(V)$. The *moduli space of flat graph connections* is the categorical quotient

$$\mathcal{M}_{(\Sigma, \Gamma)}^G = \mathcal{A}_{(\Sigma, \Gamma)} // G(V) \quad := \quad \text{Spec} \left(\mathcal{O}[\mathcal{A}_{(\Sigma, \Gamma)}]^{G(V)} \right).$$

⁷ Note that despite its name, $\mathcal{M}_{\Sigma}^G$ is not a variety but rather an affine scheme. Throughout the text we will always think of $\mathcal{M}_{\Sigma}^G$ in terms of its coordinate ring, which is precisely the G -invariant subring of the coordinate ring of representation variety.

Fig. 2. Contraction of an edge

Let $e \in E(\Gamma)$ be an edge which connects two distinct vertices, denote as Γ_e an embedded graph obtained from Γ by contracting an edge e . Similarly, for any contractible face $D \in F(\Gamma)$ which has an edge $e \in \partial D$ that appears only once in ∂D we can define an embedded graph $\Gamma^e = \Gamma \setminus e$. Recall that Γ was simple, i.e. its complement in Σ is the union of disks and annuli. It is clear that $(\Sigma \setminus \Gamma^e)$ is also simple. The flatness condition $g_{\partial D} = 1$ guarantees the isomorphism of schemes.

Theorem 5. *The following schemes are isomorphic:*

$$\mathcal{M}_{(\Sigma, \Gamma)}^G \simeq \mathcal{M}_{(\Sigma, \Gamma_e)}^G \simeq \mathcal{M}_{(\Sigma, \Gamma^e)}^G \quad (7)$$

Proof. Resolving one of the relations $g_{\partial D} = 1$ we obtain an isomorphism of $G(V)$ -modules $\mathcal{O}[\mathcal{A}_{(\Sigma, \Gamma)}] \simeq \mathcal{O}[\mathcal{A}_{(\Sigma, \Gamma^e)}]$. Taking the $G(V)$ -invariant part we conclude

$$\mathcal{O}[\mathcal{A}_{(\Sigma, \Gamma)}]^{G(V)} \simeq \mathcal{O}[\mathcal{A}_{(\Sigma, \Gamma^e)}]^{G(V)},$$

and therefore $\mathcal{M}_{(\Sigma, \Gamma)}^G \simeq \mathcal{M}_{(\Sigma, \Gamma^e)}^G$.

Now let us prove the first isomorphism in (7). Let e_0 be an edge of Γ with adjacent vertices $v = s(e_0)$ to $w = t(e_0)$. Here $s(e)$, $t(e)$ stand for the source and target of e respectively. Contracting of e_0 maps vertices v and w to one vertex w_0 of the new graph Γ_{e_0} which is the contraction of Γ . Denote this mapping by π_{e_0} .

The contraction of Γ to Γ_{e_0} defines a functor between the fundamental groupoids which we will denote by the same letter $\pi_{e_0} : \Pi(\Gamma) \rightarrow \Pi(\Gamma_{e_0})$. It is clear that this is a projection. Choose a section $\phi : \Pi(\Gamma_{e_0}) \rightarrow \Pi(\Gamma)$ of π_{e_0} , so that $\pi_{e_0}\phi = id$. It is clear that ϕ does not have to be unique, but it always exists. This is illustrated on Fig. 2.

The functors π_{e_0} and ϕ define the projection $p_{e_0} : \mathcal{A}_{(\Sigma, \Gamma)} \rightarrow \mathcal{A}_{(\Sigma, \Gamma_{e_0})}$ and its section $f : \mathcal{A}_{(\Sigma, \Gamma_{e_0})} \rightarrow \mathcal{A}_{(\Sigma, \Gamma)}$. Taking quotients with respect to the gauge group we obtain the projection $[p_{e_0}] : \mathcal{M}_{(\Sigma, \Gamma)} \rightarrow \mathcal{M}_{(\Sigma, \Gamma_{e_0})}$ and its section $[f] : \mathcal{M}_{(\Sigma, \Gamma_{e_0})} \rightarrow \mathcal{M}_{(\Sigma, \Gamma)}$. Comparing dimensions we conclude that these mappings are isomorphisms. The section f depends on the choice of ϕ . It is easy to see that the corresponding mapping $[f]$ between moduli spaces does not. $\square$

Corollary 6. *For any oriented embedded graph $\Gamma \in \Sigma_{g,b}$ such that $\Sigma_{g,b} \setminus \Gamma$ is a disjoint union of discs and b annuli, the moduli space of graph connections is isomorphic to the G -character variety*

$$\mathcal{M}_{\Sigma}^G \simeq \mathcal{M}_{(\Sigma, \Gamma)}^G.$$

Corollary 7. *The dimensions of moduli spaces of graph connections on surface $\Sigma_{g,b}$ with $b > 0$ are:*

$$\dim \left(\mathcal{M}_{(\Sigma, \Gamma)}^G \right) = \begin{cases} 0 & g = 0, b = 1, \\ \text{rk}(G) & g = 0, b = 2, \\ (2g + b - 2) \dim G & \text{otherwise.} \end{cases} \quad (8)$$

2.3. *Colored immersed graphs.* Let Γ be an oriented graph.

Definition 8. An edge coloring of Γ is an assignment of a finite-dimensional G -representation V_e to each edge $e \in \Gamma$ of the oriented graph. We will denote edge colored graph as (Γ, V) .

We say that an oriented graph Γ is *immersed* in a surface $\Sigma_{g,b}$ if the inclusion mapping $I : \Gamma \rightarrow \Sigma_{g,b}$ is locally an embedding.⁸ If the intersection of Γ on $\Sigma_{g,b}$ are transversal, we will call it *transversally immersed*.

Orientation of the surface induces a cyclic order on edges adjacent to a given vertex. For each vertex $v \in V(\Gamma) \subset \Sigma$ let us choose a total ordering $e_1, \dots, e_n$ of the adjacent edges which agrees with the cyclic ordering induced by the orientation of Σ . We will refer to this as *edge ordering* of $\Gamma \subset \Sigma$. For a finite dimensional vector space V define $V^+ = V$ and $V^- = V^*$. For each vertex of edge ordered graph define the space

$$V(v) = V_{e_1}^{\epsilon_1} \otimes \dots \otimes V_{e_n}^{\epsilon_n} \quad (9)$$

Here $\epsilon_i = +1$ is oriented outward from v , i.e. $v = s(e_i)$ and $\epsilon = -1$ otherwise, i.e. when $v = t(e)$ ⁹.

Definition 9. A vertex coloring of (Γ, V) is an assignment of a vector $c_v \in V(v)^G$ to each vertex v of an edge ordered oriented graph Γ .

Here $V(v)^G$ is the space of G -invariant vectors in a representation $V(v)$.

We define *colored graph* $\Gamma = (\Gamma, V, c)$ as an edge ordered oriented immersed graph Γ equipped with an edge coloring V and a vertex coloring c .

2.4. *Invariant functions on connections associated to immersed graphs.*

2.4.1. . For each colored graph (Γ, V, c) embedded into Σ one can associate a function $F_{\Gamma, V, c}$ on connections on the principal G -bundle over $\Sigma_{g,b}$ [AMR96] as follows.

For a graph Γ with an edge coloring V define the following G -module

$$\mathbf{V}(\Gamma) = \bigotimes_{v \in V(\Gamma)} V(v) \quad (10)$$

Here we assume a choice of a linear ordering on vertices. The space $V(v)$ are defined in (9). Note that $V(v)$ can also be written as

$$V(v) = \bigotimes_{e \in S(v)} V(e)^{\epsilon(e, v)}$$

Here $S(v)$ is the star of vertex v ¹⁰ and $\epsilon(e, v) = +$ if e is outgoing and $\epsilon(e, v) = -$ if it is incoming.

⁸ Immersed graphs may intersect on a surface and intersections does not have to be transversal.

⁹ Note that because the category of G -modules is symmetric, there exists a canonical isomorphism between $V(v)$ for different choices of edge ordering. Still, it will be convenient for us to assume that we have chosen a total order of edges adjoint to the given vertex.

¹⁰ In our case this is the set of edges adjacent to v .

By changing the order of factors in tensor product we obtain a natural isomorphism of G -modules

$$\mathbf{V}(\Gamma) \simeq \bigotimes_{e \in E(\Gamma)} (V_e \otimes V_e^*) \quad (11)$$

The space $\mathbf{V}(\Gamma)$ comes equipped with a symmetric bilinear form $\langle \cdot, \cdot \rangle$ defined on pure tensors $\mathbf{u}, \mathbf{w} \in \mathbf{V}$ by the following rule

$$\langle \mathbf{u}, \mathbf{w} \rangle = \prod_{e \in E(\Gamma)} u_{e,t(e)}(w_{e,s(e)}) w_{e,t(e)}(u_{e,s(e)}), \quad (12)$$

where

$$\begin{aligned} \mathbf{u} &= \bigotimes_{v \in V(\Gamma)} \left(\bigotimes_{e \in S(v)} u_{e,v} \right), \\ \mathbf{w} &= \bigotimes_{v \in V(\Gamma)} \left(\bigotimes_{e \in S(v)} w_{e,v} \right). \end{aligned}$$

Here $u_{e,s(e)}, w_{e,s(e)} \in V_e$ and $u_{e,t(e)}, w_{e,t(e)} \in V_e^*$

2.4.2. . It is clear that a principal G -bundle E_Σ over Σ defines a principal G -bundle E_Γ over vertices of $\Gamma \subset \Sigma$. A connection A on E_Σ defines a graph connection on E_Γ with the parallel transport along the edge e being the parallel transport along $e \subset \Sigma$ with respect to connection A . The evaluation of the parallel transport along e in the representation V_e we will denote by $P_e^A : V_e \rightarrow V_e$.

Define the vector $\pi^A \in \mathbf{V}$ in the G -module (10) as

$$\pi^A = \bigotimes_{e \in E(\Gamma)} P_e^A \quad (13)$$

Here we assume a choice of total ordering on the set of edges and the identification with the tensor product with the reordered tensor product as in (11). The holonomy map P_e we regard as a vector in $V_e \otimes V_e^*$.

From the coloring of vertices we get another vector $\mathbf{c}^\Gamma \in \mathbf{V}(\Gamma)$

$$\mathbf{c}^\Gamma = \bigotimes_{v \in V(\Gamma)} c_v^\Gamma.$$

Here c_v^Γ is the coloring of the vertex v i.e. a vector in $V(v)^G \subset V(v)$.

Definition 10. Define graph functions using the symmetric bilinear form (12) as

$$F_{\Gamma,V,c}(A) = \langle \pi^A, \mathbf{c}^\Gamma \rangle. \quad (14)$$

Proposition 11. Graph functions have the following properties:

$$F_{\Gamma,V,c}(A^g) = F_{\Gamma,V,c}(A)$$

where A^g is a flat connection A after gauge transformation g . If the connection is flat, the function $F_{\Gamma,V,c}(A)$ depends only on the isotopy class $[\Gamma]$ of Γ .

Proof. By Definition 9, $c_p^\Gamma \in V(p)^G$ is an invariant vector for each vertex p of the graph. Hence, each function $F_{\Gamma,V,c}$ is invariant under gauge transformations of A . Moreover, when A is flat connection, $F_{\Gamma_1,V,c}(A) = F_{\Gamma_2,V,c}(A)$ whenever an immersed graph Γ_1 can be deformed to Γ_2 by the regular homotopy (i.e. homotopy through graph immersions).

As a result, for each triple $([\Gamma], V, c)$ where $[\Gamma]$ is the isotopy class of an immersed oriented ordered graph Γ , V is an edge coloring and c is a vertex coloring we have a function $F_{[\Gamma],V,c} = F_{\Gamma,V,c}$ on the moduli space of flat connections on $\Sigma_{g,b}$. As a consequence, F is a function of the moduli space of flat connections. $\square$

Theorem 12 [AMR96]. *Functions $F_{[\Gamma],V,c}$ span the coordinate ring (7) of the moduli space of G -representations of the fundamental group $\pi_1(\Sigma_{g,b}, p)$.*

Proof. By an algebraic analogue of Peter–Weyl theorem, the coordinate ring $R = \mathcal{O}[G^N]$ of G -representations of the fundamental group can be decomposed as a $G^N \times (G^N)^{op}$ -module

$$\mathcal{O}[G^N] \simeq \bigoplus_{\lambda_1, \dots, \lambda_N \in \Lambda} \left(\bigotimes_{j=1}^N W_{\lambda_j} \otimes W_{\lambda_j}^* \right), \quad (15)$$

where W_λ stands for a finite-dimensional irreducible representation with highest weight λ .

In particular, (15) is an isomorphism of G -modules, where G acts by conjugation. Now let $Q \in \mathcal{O}[G^N]^G$ be a G -invariant polynomial, by (15) it can be decomposed as

$$Q = \sum_{j=1}^m Q_j, \quad Q_j \in \left(W_{\lambda_{j,1}} \otimes W_{\lambda_{j,1}}^* \otimes \dots \otimes W_{\lambda_{j,N}} \otimes W_{\lambda_{j,N}}^* \right)^G$$

where each Q_j is a graph function associated to the ribbon graph with a single vertex and $N = 2g + b - 1$ morphisms corresponding to free generators of $\pi_1(\Sigma_{g,b}, p)$. The edge coloring of edges corresponding to generators of the fundamental group is given by $\lambda_{j,1}, \dots, \lambda_{j,N}$, while Q_j is an invariant vector which defines the coloring of the single vertex p . $\square$

2.5. Equivalence of graph functions. In this subsection we list elementary operations on colored immersed graphs which leave the associated graph function (14) invariant. These operations include collapsing a pair of neighbouring vertices (Fig. 3c) and thinning a pair of neighbouring edges (Fig. 3d). In each case shown on Fig. 3 we refer to the left graph as Γ_1 and to the right graph as Γ_2 .

Lemma 13 [FR99, KRWY]. *Let Γ_1, Γ_2 be a pair of colored immersed oriented graphs which coincide away from a disc shown on any of Fig. 3a–d. Then there exists a vertex coloring of Γ_2 , such that the two colored immersed graph give rise to the same invariant function on graph connections:*

$$F_{[\Gamma_1]} = F_{[\Gamma_2]}.$$

Proof. Consider a flat connection A on a principle G -bundle over $\Sigma_{g,b}$ and let

$$\pi_{\Gamma_1}^A = \bigotimes_{e \in E(\Gamma_1)} P_e \in \mathbf{V}(\Gamma_1), \quad \pi_{\Gamma_2}^A = \bigotimes_{e \in E(\Gamma_2)} Q_e \in \mathbf{V}(\Gamma_2). \quad (16)$$

Fig. 3. Graphs defining equivalent functions (Edges are oriented outwards unless specified)

be a pair of vectors in and as in (13) defined by the parallel transport along edges of the two immersed graphs. In each of the four parts we are going to prove that

$$F_{[\Gamma_1]}(A) = \langle \pi_{\Gamma_1}^A, \mathbf{c}^{\Gamma_1} \rangle = \langle \pi_{\Gamma_1}^A, \mathbf{c}^{\Gamma_2} \rangle = F_{[\Gamma_2]}(A) \quad (17)$$

for an appropriate choice of vertex coloring.

(A) Let

$$\text{Id}_{V_1} = \sum_i v_i \otimes v_i^*, \quad \text{Id}_{V_2} = \sum_j w_j \otimes w_j^*$$

where $\{v_i\}$, $\{v_i^*\}$ and $\{w_i\}$, $\{w_i^*\}$ be any two pairs of dual bases on V_1 , V_1^* and V_2 , V_2^* respectively, i.e. $\langle v_i, v_j^* \rangle = \delta_{i,j}$ and $\langle w_i, w_j^* \rangle = \delta_{i,j}$. Assign a vertex coloring of Γ_2 at v as

$$c_v^{\Gamma_2} = \sum_{i,j} v_i \otimes w_j^* \otimes v_i^* \otimes w_j \simeq \text{Id}_{V_1} \otimes \text{Id}_{V_2}$$

(B) Let Γ_1 and Γ_2 be a pair of colored oriented immersed graphs which coincide away from a disc shown on Fig. 3b. Suppose that the vertex coloring of Γ_1 at v is given by

$$c_v^{\Gamma_1} = \sum_j v_j^0 \otimes w_j^{(0)} \otimes v_j^{(1)} \otimes \cdots \otimes v_j^{(1)}.$$

Define $c_v^{\Gamma_2}$ as follows

$$c_v^{\Gamma_1} = \sum_j \langle v_j^0, w_j^{(0)} \rangle v_j^{(1)} \otimes \cdots \otimes v_j^{(1)}$$

Because the holonomy of flat connection along contractible loop e_0 is trivial, we get the following contribution to holonomy vector (16)

$$P_{e_0} = \text{Id}_{V_0} = \sum_{i=1}^M u_i \otimes u_i^* \quad (18)$$

where $\{u_i\}$ and $\{u_i^*\}$ is an arbitrary pair of dual bases in V_0 and V_0^* , i.e. $\langle u_i, u_l^* \rangle = \delta_{i,l}$. Then (17) follows immediately by

$$\sum_{j=1}^M \langle w_k^{(0)}, u_j \rangle \langle u_j^*, v_j^{(0)} \rangle = \langle w_k^{(0)}, v_j^{(0)} \rangle. \quad (19)$$

(C) Assume further that the vertex coloring of vertices a and b is given by a pair of G -invariant vectors

$$\begin{aligned} c_a^{\Gamma_1} &= \sum_j v_j^{(0)} \otimes v_j^{(1)} \otimes \cdots \otimes v_j^{(m)} \in (V_0 \otimes \cdots \otimes V_m)^G, \\ c_b^{\Gamma_1} &= \sum_k w_k^{(0)} \otimes w_k^{(m+1)} \otimes \cdots \otimes w_k^{(n)} \in (V_0^* \otimes V_{m+1} \otimes \cdots \otimes V_n)^G. \end{aligned}$$

Let c_v be a G -invariant vector defined as follows:

$$c_v^{\Gamma_2} = \sum_{j,k} \langle w_k^{(0)}, v_j^{(0)} \rangle v_j^{(1)} \otimes \cdots \otimes v_j^{(m)} \otimes w_k^{(m+1)} \otimes \cdots \otimes w_k^{(n)} \in (V_1 \otimes \cdots \otimes V_n)^G$$

Note that because both $F_{[\Gamma_1]}$ and $F_{[\Gamma_2]}$ are functions on gauge equivalence classes of flat connections, without loss of generality we can assume that holonomy of A in a contractible disc on Fig. 3c is trivial. Using the same reasoning as in (18) and (19) we conclude that (17) holds.

(D) For the remaining part, again without loss of generality we can assume that connection is trivial inside contractible disk shown on Fig. 3d. For $\varphi : V_0 \xrightarrow{\sim} V_1 \otimes V_2$, assign G -invariant vectors at both vertices induced by the isomorphism φ and its inverse, the statement (17) then follows by tautology.

□

Corollary 14. *Let $\Gamma \in \Sigma$ be an oriented immersed graph with a single vertex p and edges corresponding to free generators of $\pi_1(\Sigma, p)$. For any oriented immersed graph (Γ_1, W, d) we can present the associated graph function as*

$$F_{[\Gamma_1, W, d]} = F_{[\Gamma, V, c]}$$

for an appropriate choice of coloring of simple oriented surface graph $\Gamma \subset \Sigma$.

Proof. Because $F_{[\Gamma_1, W, d]}$ depends only on homotopy (through immersed graphs) equivalence type of Γ_1 , w.l.o.g. we can assume that $\Gamma \cup \Gamma_1$ has only double transverse intersection points. So by Lemma 13 we can turn $\Gamma \cup \Gamma_1$ first into an embedded graph, then collapse all vertices of Γ_1 into vertices of Γ_2 and, finally, by a sequence of moves Fig. 3b–d eliminate all vertices of Γ_1 . □

Fig. 4. Double intersection point at edges and associated graph $(\Gamma, \Gamma')_p$

Replace holonomies along edges by groups elements g_e given by a flat graph connection $g : E(\Gamma) \rightarrow G$ we identify the graph function $F_{[\Gamma, V, c]}$ with a function on the space flat graph connections for $\Gamma \subset \Sigma$. Taking into account isomorphisms (7) we can identify invariant graph functions with functions on the representation variety $\mathcal{M}_{\Sigma, b}^G$. Thus the corollary 14 establishes an isomorphism between the coordinate ring $\mathcal{O}[\mathcal{M}_{(\Sigma_{g, b}, \Gamma)}]^G$ of moduli space of graph connections and the coordinte ring $\mathcal{O}[\mathcal{M}_{\Sigma, b}^G]^G$.

2.6. Poisson structure on the moduli space of flat connections. The coordinate ring $\mathcal{O}[\mathcal{M}_{\Sigma}^G]$ of the G -character variety of $\pi_1(\Sigma_{g, b})$ comes equipped with a Poisson bracket equivariant under the action of the Mapping Class Group $Mod(\Sigma_{g, b})$ [AB83, Gol86, BG93]. Following [FR93, AMR96, FR99] we will define this bracket in terms of graph functions (14).

Let $([\Gamma], V, c)$ and $([\Gamma'], V', c')$ be a pair of isotopy equivalence classes of colored immersed graphs. Pick representatives Γ and Γ' which have only transverse double intersection points at edges.

Definition 15. Let $p \in \Gamma \cap \Gamma'$ be a transverse double intersection point of two colored immersed graphs $\Gamma = (\Gamma, V, c)$ and $\Gamma' = (\Gamma', V', c')$. Define (isotopy equivalence class of) colored immersed graph $[\Gamma \star_p \Gamma'] = ([\Gamma''], V'', c'')$ as follows:

Consider a small neighborhood of p and denote by $e \in \Gamma$ and $e' \in \Gamma'$ the corresponding edges of graphs containing the intersection point $p \in e \cap e'$.

Take union of immersed graphs $\Gamma \cup \Gamma'$ and add two extra vertices a, b which divide edges e, e' into e_1, e_2 and e'_1, e'_2 respectively as shown on Fig. 4b. Then add an edge e'' connecting a and b colored by adjoint representation $V_{e''}'' = \mathfrak{g}$ of G .¹¹ Define vertex coloring

$$\begin{aligned} c_a'' &= (\text{ad}_{V_e} \otimes Id_{\mathfrak{g}})(I) \in V_e \otimes V_e^* \otimes \mathfrak{g}, \\ c_b'' &= (\text{ad}_{V_{e'}} \otimes Id_{\mathfrak{g}})(I) \in V_{e'} \otimes V_{e'}^* \otimes \mathfrak{g}, \end{aligned} \quad (20)$$

where $I = \sum_i t^i \otimes t_i \in \mathfrak{g} \otimes \mathfrak{g}$ is a quadratic Casimir. Finally, assume that edge and vertex coloring of the remaining graph is inherited from Γ and Γ' .

Theorem 16. [FR93, AMR96] *The Poisson Bracket between $F_{[\Gamma]}$ and $F_{[\Gamma']}$ in terms of invariant graph functions reads*

$$\{F_{[\Gamma]}, F_{[\Gamma']}\} = \sum_{p \in \Gamma \cap \Gamma'} \epsilon(p) F_{[\Gamma \star_p \Gamma']}, \quad (21)$$

¹¹ Note that the adjoint representation is self-dual, so the orientation of e'' on Fig. 4b can be chosen arbitrarily.

where the sum is taken over the intersection points $p \in \Gamma \cap \Gamma'$ and $\epsilon(p) = \pm 1$ stands for the sing of an intersection point.

3. Main Construction

The following theorem describes superintegrable systems on moduli spaces of flat connections. Poisson commuting Hamiltonians of these systems are invariant functions of holonomies along systems of nonintersecting simple curves on the surface.

Denote by $Z_{\partial\Sigma}$ the subalgebra of the coordinate ring $A = \mathcal{O}(\mathcal{M}_{\Sigma}^G)$ spanned by G -invariant functions on holonomies around boundary components. It must be a subalgebra of the Poisson center of $\mathcal{A}$. Moreover, it follows from [BG93] that $Z_{\partial\Sigma}$ has the same Krull dimension as the Poisson center of A .

Let $C = C_1 \sqcup \dots \sqcup C_k$ in Σ be a disjoint union of pairwise nonhomotopic simple closed curves none of which is homotopic to a boundary component. Define $B_C \subset A$ as the subalgebra generated by graph functions $F_{\Gamma, V, C}$ with $\Gamma \subset C$. Similarly, define $J_{\Sigma \setminus C}$ as the subalgebra generated by graph functions $F_{\Gamma, V, C}$ with $\Gamma \subset (\Sigma \setminus C)$.

Theorem 17. *The system of subalgebras below defines an affine superintegrable system*

$$Z_{\partial\Sigma} \subset B_C \subset J_{\Sigma \setminus C} \subset A = \mathcal{O}[\mathcal{M}_{\Sigma}^G].$$

In the rest of this section we will prove this theorem. In order to prove it we should prove the balance of dimensions.

3.1. Trivial superintegrable system on a surface. Let Σ be an oriented surface of genus g with $b > 0$ boundary components and let $\mathcal{M}_{\Sigma}^G$ be the moduli space of G -representations of the fundamental group $\pi_1(\Sigma)$. Assume that we have at least one boundary component. Denote by $Z_{\partial\Sigma} \subset A = \mathcal{O}[\mathcal{M}_{\Sigma}^G]$ a subalgebra generated by graph functions $F_{[\Gamma]}$, where Γ is a colored graph with a single vertex and a single edge homotopic to one of the boundary components of Σ .

Lemma 18. *The subalgebra $Z_{\partial\Sigma} \subset \mathcal{O}[\mathcal{M}_{\Sigma}^G]$ belongs to the Poisson center of $\mathcal{O}[\mathcal{M}_{\Sigma}^G]$.*

Proof. Let $[\Gamma_1], [\Gamma_2]$ be a pair of isotopy equivalence classes of colored immersed graphs such that $[\Gamma_1]$ is homotopic to one of the boundary components. Then there exist members Γ_1 and Γ_2 with a trivial intersection. From (21) we get

$$\{F_{[\Gamma_1]}, F_{[\Gamma_2]}\} = 0. \quad (22)$$

By Theorem 12, invariant graph functions of the form $F_{[\Gamma_2]}$ span the entire coordinate ring, hence we conclude that $F_{[\Gamma_1]}$ belongs to the Poisson center of $\mathcal{O}[\mathcal{M}_{\Sigma}^G]$. $\square$

Note that $Z_{\partial\Sigma}$ is generated by b copies of $\mathcal{O}[G]^G$, one for each boundary component. In order to calculate Krull dimension of $Z_{\partial\Sigma}$ we will use the following pair of technical lemmas:

Lemma 19. *Let G be a connected simple linear algebraic group with a Lie Algebra $\mathfrak{g}$ and let*

$$\text{ch}_i \in \mathcal{O}[G]^G, \quad 1 \leq i \leq \text{rk}(G) \quad (23)$$

be a maximal algebraically independent subset of $\mathcal{O}[G]^G$. Fix an arbitrary element $M_0 \in G$, the following $3 \operatorname{rk}(G)$ functions on $G \times G$ are algebraically independent:

$$G_{1,i}(M_1, M_2) = \operatorname{ch}_i(M_1), \quad G_{2,i}(M_1, M_2) = \operatorname{ch}_i(M_2), \quad G_{3,i}(M_1, M_2) = \operatorname{ch}_i(M_0 M_1 M_2),$$

where $1 \leq i \leq \operatorname{rk}(G)$, $M_1, M_2 \in G$.

Proof. Note that collection of regular semisimple elements in G forms an open dense subset, so generically both M_2 and $M_2 M_0 M_1$ are regular semisimple. Consider a pair of Cartan subgroups $\mathbf{T}_{M_2} \subset G$ and $\mathbf{T}_{M_2 M_0 M_1} \subset G$ containing M_2 and $M_2 M_0 M_1$ respectively. Denote by $\mathfrak{h}_{M_2}$ and $\mathfrak{h}_{M_2 M_0 M_1}$ their Cartan subalgebras. The two Cartan decompositions of $\mathfrak{g}$ acquire the following form

$$\mathfrak{g} = \mathfrak{h}_{M_2} \oplus \mathfrak{h}_{M_2}^\perp = \mathfrak{h}_{M_2 M_0 M_1} \oplus \mathfrak{h}_{M_2 M_0 M_1}^\perp.$$

Now, for $X \in \mathfrak{g}$, consider a family of elements $(M_1, e^{\epsilon X} M_2 e^{-\epsilon X})$ on the common level set of $G_{1,i}$ and $G_{2,i}$. The remaining $\operatorname{rk}(G)$ class functions read

$$\begin{aligned} G_{3,i}(M_1, e^{\epsilon X} M_2 e^{-\epsilon X}) &= \operatorname{ch}_i(M_0 M_1 e^{\epsilon X} M_2 e^{-\epsilon X}) \\ &= \operatorname{ch}_i(M_0 M_1 (1 + \epsilon(X - M_2 X M_2^{-1})) M_2) + \mathbf{o}(\epsilon) \\ &= \operatorname{ch}_i(M_2 M_0 M_1 (1 + \epsilon \pi_{\mathfrak{h}_{M_2 M_0 M_1}}(X - M_2 X M_2^{-1}))) + \mathbf{o}(\epsilon), \end{aligned} \quad (24)$$

where $\pi_{\mathfrak{h}_{M_2 M_0 M_1}} : \mathfrak{g} \rightarrow \mathfrak{h}_{M_2 M_0 M_1}$ stands for the projection map. Note that in the last equality we have used the fact that $\mathfrak{h}_{M_2 M_0 M_1}^\perp$ -component of

$$X - \operatorname{Ad}(M_2)X \in \mathfrak{g} = \mathfrak{h}_{M_2 M_0 M_1} \oplus \mathfrak{h}_{M_2 M_0 M_1}^\perp$$

in the first order in ϵ makes no contribution to the variation of $G_{3,i}$.

Recall that M_2 is regular semisimple, so dimension of its centralizer is exactly $\operatorname{rk}(G)$. As a corollary, the following endomorphism of the Lie algebra

$$\Delta : \mathfrak{g} \rightarrow \mathfrak{g}, \quad \Delta(X) = X - \operatorname{Ad}_{M_2}(X)$$

is nondegenerate on $\mathfrak{h}_{M_2}^\perp$. On the other hand, because $\mathfrak{h}_{M_2}$ and $\mathfrak{h}_{M_2 M_0 M_1}$ are in generic relative position to each other, this would imply that

$$\pi_{\mathfrak{h}_{M_2 M_0 M_1}} \circ \Delta : \mathfrak{g} \rightarrow \mathfrak{h}_{M_2 M_0 M_1}$$

is surjective. Combining this fact with (24) we conclude the proof. $\square$

Lemma 20. As in the previous lemma, let G be a connected simple linear algebraic group with a Lie Algebra $\mathfrak{g}$ and let

$$\operatorname{ch}_i \in \mathcal{O}[G]^G, \quad 1 \leq i \leq \operatorname{rk}(G) \quad (25)$$

be a maximal algebraically independent subset of $\mathcal{O}[G]^G$. Now fix an arbitrary element $M_0 \in G$, the following $3 \operatorname{rk}(G)$ functions on $G \times G$ are algebraically independent:

$$G_{1,i}(M_1, M_2) = \operatorname{ch}_i(M_1), \quad G_{2,i}(M_1, M_2) = \operatorname{ch}_i(M_2), \quad G_{3,i}(M_1, M_2) = \operatorname{ch}_i(M_0[M_1, M_2]), \quad (26)$$

where $1 \leq i \leq \operatorname{rk}(G)$, $M_1, M_2 \in G$.

Proof. Note that there exists an open dense subset $U \in G \times G$ such that for every $(M_1, M_2) \in U$ the following three elements

$$\widetilde{M}_0 = M_2 M_1^{-1} M_2^{-1} M_0 M_1, \quad \widetilde{M}_1 = M_2 M_1^{-1} M_2^{-1}, \quad M_2$$

are regular semisimple. So without loss of generality we can assume $(M_1, M_2) \in U$ and also that M_1 and M_2 are in general relative position to each other. Denote by $\mathbf{T}_{\widetilde{M}_0}, \mathbf{T}_{\widetilde{M}_1}, \mathbf{T}_{M_2}$ their Cartan subgroups and by $\mathfrak{h}_{\widetilde{M}_0}, \mathfrak{h}_{\widetilde{M}_1}, \mathfrak{h}_{M_2}$ the corresponding Cartan subalgebras. We have

$$\mathfrak{g} = \mathfrak{h}_{\widetilde{M}_0} \oplus \mathfrak{h}_{\widetilde{M}_0}^\perp = \mathfrak{h}_{\widetilde{M}_1} \oplus \mathfrak{h}_{\widetilde{M}_1}^\perp = \mathfrak{h}_{M_2} \oplus \mathfrak{h}_{M_2}^\perp.$$

As in the previous lemma, for $\mathbf{X} \in \mathfrak{g}$, consider a family of elements $(M_1, e^{\epsilon X} M_2 e^{-\epsilon X})$ on the common level set of $G_{1,i}$ and $G_{2,i}$. For the remaining $\text{rk}(G)$ class functions we get

$$\begin{aligned} G_{3,i}(M_1, e^{\epsilon X} M_2 e^{-\epsilon X}) &= \text{ch}_i(M_0 M_1 e^{\epsilon X} M_2 e^{-\epsilon X} M_1^{-1} e^{\epsilon X} M_2^{-1} e^{-\epsilon X}) \\ &= \text{ch}_i\left(M_0 M_1 (1 + \epsilon(X - \text{Ad}_{M_2} X + \text{Ad}_{M_2 M_1^{-1}} X - \text{Ad}_{M_2 M_1^{-1} M_2^{-1}} X)) M_2 M_1^{-1} M_2^{-1}\right) + \mathfrak{o}(\epsilon) \\ &= \text{ch}_i\left(M_2 M_1^{-1} M_2^{-1} M_0 M_1 (1 + \epsilon(\text{Id} - \text{Ad}_{M_2 M_1^{-1} M_2^{-1}}) \circ (\text{Id} - \text{Ad}_{M_2})(X))\right) + \mathfrak{o}(\epsilon) \\ &= \text{ch}_i\left(\widetilde{M}_0 (1 + \epsilon \pi_{\mathfrak{h}_{\widetilde{M}_0}} \circ (\text{Id} - \text{Ad}_{M_2 M_1^{-1} M_2^{-1}}) \circ (\text{Id} - \text{Ad}_{M_2})(X))\right) \\ &= \text{ch}_i\left(\widetilde{M}_0 (1 + \epsilon \pi_{\mathfrak{h}_{\widetilde{M}_0}} \circ (\text{Id} - \text{Ad}_{\widetilde{M}_1}) \circ \pi_{\mathfrak{h}_{\widetilde{M}_1}^\perp} \circ (\text{Id} - \text{Ad}_{M_2})(X))\right) \end{aligned}$$

Here in the last equality we have used the fact that $\ker(\text{Id} - \text{Ad}_{\widetilde{M}_1}) = \mathfrak{h}_{\widetilde{M}_1}$, so

$$\text{Id} - \text{Ad}_{\widetilde{M}_1} = (\text{Id} - \text{Ad}_{\widetilde{M}_1}) \circ \pi_{\mathfrak{h}_{\widetilde{M}_1}^\perp}.$$

The statement of the Lemma will now follow from the fact that

$$\pi_{\mathfrak{h}_{\widetilde{M}_0}} \circ (\text{Id} - \text{Ad}_{\widetilde{M}_1}) \circ \pi_{\mathfrak{h}_{\widetilde{M}_1}^\perp} \circ (\text{Id} - \text{Ad}_{M_2}) : \mathfrak{g} \rightarrow \mathfrak{h}_{\widetilde{M}_0}$$

is surjective. To show that, recall that $\text{Id} - \text{Ad}_{M_2}$ is nondegenerate on $\mathfrak{h}_{M_2}^\perp$. Because $\mathfrak{h}_{M_2}^\perp$ and $\mathfrak{h}_{\widetilde{M}_1}^\perp$ are of the same dimension and are in general relative position, the restriction $\pi_{\mathfrak{h}_{M_2}^\perp} \big|_{\mathfrak{h}_{\widetilde{M}_1}^\perp}$ is a linear isomorphism between $\mathfrak{h}_{M_2}^\perp$ and $\mathfrak{h}_{\widetilde{M}_1}^\perp$. Next, $\text{Id} - \text{Ad}_{\widetilde{M}_1}$ is nondegenerate on $\mathfrak{h}_{\widetilde{M}_1}^\perp$. Combining this with the fact that $\mathfrak{h}_{\widetilde{M}_0}$ and $\mathfrak{h}_{\widetilde{M}_1}^\perp$ are in general relative position to each other we conclude the proof. $\square$

Corollary 21.

$$\dim \text{Spec } Z_{\partial \Sigma} = \begin{cases} (b-1) \text{rk}(G), & g=0, b \leq 2, \\ b \text{rk}(G), & \text{otherwise.} \end{cases} \quad (27)$$

Proof. Let $p \in \partial \Sigma$ be a point of the boundary. With the choice of generators of $\pi_1(\Sigma, p)$ as in Sect. 2.1 we consider elements of $Z_{\partial \Sigma} \subset \mathcal{O}[\mathcal{M}_\Sigma^G]$ as G -invariant functions on $G^{\times(2g+b-1)}$. In both cases the upper bound on dimension (27) follows by the fact that $Z_{\partial \Sigma}$ is generated by the appropriate number of copies of $\mathcal{O}[G]^G$, so the only thing we have to prove is that there exists a large enough algebraically independent subset of generators $Z_{\partial \Sigma}$.

Fix a maximal collection of algebraically independent class functions $\text{ch}_i \in \mathcal{O}[G]^G$ and consider the following cases:

- (i) $b \geq 3$: In this case $\{\text{ch}_i(\rho(z_j)), 1 \leq i \leq \text{rk}(G), 1 \leq j \leq b-3\}$ are algebraically independent as functions on $G^{\times(b-3)}$. At the same time, for fixed $\rho(z_j) \in G, 1 \leq j \leq b-3$, the remaining $3 \text{rk}(G)$ functions $\text{ch}_i(\rho(z_{b-2})), \text{ch}_i(\rho(z_{b-1})), \text{ch}_i(\rho(z_b))$ are independent as functions on the remaining $G \times G$ by Lemma 19. So, the entire collection of $3 \text{rk}(G)$ functions must be algebraically independent.
- (ii) $b = 2, g > 0$: In this case we have a set of $2 \text{rk}(G)$ algebraically independent functions $\text{ch}_i(\rho(b_1))$ and $\text{ch}_i(\rho(b_2)) = \text{ch}_i(\rho(b_1^{-1}[y_g, x_g] \dots [y_1, x_1]))$. To see it, recall that $\text{ch}_i(\rho(b_1))$ are algebraically independent as functions on G by construction. Next, apply Lemma 20 for

$$M_0 = \rho(b_1^{-1}[y_g, x_g] \dots [y_2, x_2]), \quad M_1 = y_g, \quad M_2 = x_g.$$

to conclude that $\text{ch}_i(\rho(b_1^{-1}[y_g, x_g] \dots [y_1, x_1]))$ are algebraically independent as functions on G^{2g} , a fixed level set of $\rho(b_1) = \text{const}$. Altogether this would imply that the entire collection of $2 \text{rk}(G)$ functions are independent as functions on G^{2g+1} .

- (iii) $b = 2, g = 0$: Here $\pi_1(\Sigma, p) = \langle b_1 \rangle$ and we have $\text{rk}(G)$ algebraically independent functions $\text{ch}_i(\rho(b_1))$.
- (iv) $b = 1, g > 0$: In this case we have $\text{rk}(G)$ algebraically independent functions $\text{ch}_i(\rho([x_1, y_1] \dots [x_g, y_g]))$. Algebraic independence follows from Lemma 20 applied for

$$M_0 = [x_1, y_2] \dots [x_{g-1}, y_{g-1}], \quad M_1 = x_g, \quad M_2 = y_g.$$

- (v) $b = 1, g = 0$: Statement holds trivially.

□

Proposition 22. *Inclusion $Z_{\partial\Sigma} \subset Z_{\Sigma} \subset A \subset A$ defines an affine superintegrable system.*

Proof. First, note that $Z_{\partial\Sigma}$ is finitely generated, indeed it is generated by b copies of $\mathcal{O}[G]^G$, one for each boundary component. Every $\mathcal{O}[G]^G$ is in turn finitely generated.

By Lemma 18 we know that $Z_{\partial\Sigma}$ is a subalgebra of the Poisson center of A . On the other hand, by results of [GHJW97, BG93] we know that the general fiber of $\mathcal{M}_{\Sigma}^G \rightarrow \text{Spec } Z_{\partial\Sigma}$ is equipped with Poisson bracket of rank $\dim \mathcal{M}_{\Sigma}^G - \dim \text{Spec } Z_{\partial\Sigma}$. □

3.2. Superintegrable system associated to non-separating curve. Let Σ be an oriented surface of genus g with $b \geq 1$ boundary components. There exists no nonseparating curves on genus zero surface, so throughout this section we can assume without loss of generality that $g > 0$. Let $C = C_1 \sqcup C_2 \sqcup \dots \sqcup C_r$ be a nonseparating curve which is a disjoint union of $r, 1 \leq r \leq g$ simple closed curves on Σ (see Fig. 5).

The fundamental group $\pi_1(\Sigma, p)$ is a free group with $2g + b - 1$ generators, it admits the following presentation

$$\pi_1(\Sigma, p) = \langle x_1, y_1, \dots, x_g, y_g, z_1, \dots, z_{b-1} \rangle, \quad (28)$$

where the first $2r$ generators $x_i, y_i, 1 \leq i \leq r$ are chosen for handles containing $C_1, \dots, C_r$ as shown on Fig. 6.

Fig. 5. An example of a nonseparating Curve

Fig. 6. Connected component of nonseparating curve

Lemma 23. Consider a subgroup of $\pi_1(\Sigma, p)$ which consists of (homotopy equivalence classes of) paths that do not cross C :

$$\pi_1(\Sigma, p | C) \subset \pi_1(\Sigma, p). \quad (29)$$

This subgroup is generated by the following elements

$$\pi_1(\Sigma, p | C) = \langle y_1, \dots, y_r, x_1 y_1 x_1^{-1}, \dots, x_r y_r x_r^{-1}, x_{r+1}, y_{r+1}, \dots, x_g, y_g, z_1, \dots, z_{b-1} \rangle. \quad (30)$$

Proof. Note that there is a natural surjective homomorphism

$$\psi : \pi_1(\Sigma \setminus C, p) \twoheadrightarrow \pi_1(\Sigma, p | C)$$

From the fundamental group of $\Sigma \setminus C$ to the subgroup (29). As a corollary, $\pi_1(\Sigma, p | C)$ is generated by the image of the generating set of $\pi_1(\Sigma \setminus C, p)$. Note that surface $\Sigma \setminus C$ has genus $g - r$ with $2r + b$ boundary components. In the standard presentation, $\pi_1(\Sigma \setminus C, p)$ is generated by $(g - r)$ pairs of A, B -cycles along with $2r + b - 1$ boundary cycles. Taking the image of this generating set under ψ we get (30). $\square$

The closed arcs homotopic to new boundary components are $y_1, \dots, y_r$ and $x_1 y_1 x_1^{-1}, \dots, x_r y_r x_r^{-1}$. As a result, $\pi_1(\Sigma \setminus C, p)$ is a free group generated by $2g + b - 1$ arcs.

A G -representation of $\pi_1(\Sigma, p)$ is determined by $2g + b - 1$ elements of G

$$x_i \mapsto X_i, \quad y_i \mapsto Y_i, \quad z_j \mapsto Z_j, \quad 1 \leq i \leq g, \quad 1 \leq j \leq b - 1. \quad (31)$$

As a corollary, $\text{Hom}(\pi_1(\Sigma, p), G)$ is an affine variety with the coordinate ring $\mathcal{O}[\text{Hom}(\pi_1(\Sigma, p), G)] \simeq \mathcal{O}[G]^{\otimes(2g+b-1)}$ isomorphic to the coordinate ring of $2g+b-1$ copies of G .

Inclusion (29) defines an inclusion of coordinate rings

$$\begin{aligned} \mathcal{O}[\text{Hom}(\pi_1(\Sigma, p | C), G)] &\subset \mathcal{O}[\text{Hom}(\pi_1(\Sigma, p), G)] \\ \cup &\cup \\ \mathcal{O}[\text{Hom}(\pi_1(\Sigma, p | C), G)]^G &\subset \mathcal{O}[\text{Hom}(\pi_1(\Sigma, p), G)]^G \end{aligned} \quad (32)$$

At the same time, $\pi_1(\Sigma, p | C)$ contains b infinite cyclic subgroups generated by boundary cycles. Let $Z_{\partial\Sigma}$ be a subalgebra of $\mathcal{O}[\mathcal{M}_\Sigma^G]$ generated by class functions of holonomies about the boundary components, or, equivalently, generated by $\mathcal{O}[\text{Hom}(\langle z_i \rangle, G)]^G$ for $1 \leq i \leq b$.

Similarly, let B_C be a subalgebra of $\mathcal{O}[\mathcal{M}_\Sigma^G]$ generated by $Z_{\partial\Sigma}$ together with traces of holonomies about the cycles C_i , in other words, by $\mathcal{O}[\text{Hom}(\langle y_i \rangle, G)]^G$ for $1 \leq i \leq r$.

Lemma 24.

$$\dim \text{Spec } B_C = (b+r) \text{rk}(G). \quad (33)$$

Proof. The upper bound on dimension follows from the fact that B_C is generated by $b+r$ copies of $\mathcal{O}[G]^G$. Fix a maximal algebraically independent collection of class functions $\text{ch}_i \in \mathcal{O}[G]^G$ and consider a family of $(b+r) \text{rk}(G)$ elements of B_C :

$$\text{ch}_i(\rho(y_j)), \quad \text{ch}_i(\rho(z_l)), \quad 1 \leq i \leq \text{rk}(G), \quad 1 \leq j \leq r, \quad 1 \leq l \leq b.$$

Here we imply $z_b = z_{b-1}^{-1} \dots z_1^{-1} [y_g, x_g] \dots [y_1, x_1]$.

The first $(b+r-1) \text{rk}(G)$ elements are algebraically independent as functions on G^{2g+b-1} because they involve different copies of G . As for the $\text{ch}_i(\rho(z_b))$, apply Lemma 20 for

$$M_0 = \rho(z_{b-1} \dots z_1 [y_g, x_g] \dots [y_2, x_2]), \quad M_1 = \rho(y_1), \quad M_2 = \rho(x_1)$$

to observe that restriction of $\text{ch}_i(\rho(z_{b-1}^{-1} \dots z_1^{-1} [y_g, x_g] \dots [y_1, x_1]))$ to generic common level set of the other $(b+r-1) \text{rk}(G)$ elements gives rise to $\text{rk}(G)$ independent functions.

□

Proposition 25. *The following chain of subalgebras*

$$Z_{\partial\Sigma} \subset B_C \subset J_{\Sigma \setminus C} \subset \mathcal{O}[\mathcal{M}_\Sigma^G],$$

defines an affine superintegrable system. Here $J = \mathcal{O}[\text{Hom}(\pi_1(\Sigma, p | C), G)]^G$ stands for the coordinate ring of the G -character variety of $\pi_1(\Sigma, p | C)$.

Proof. By construction, all algebras involved are finitely generated and from (21) it follows immediately that $\{Z, A\} = \{B, J\} = 0$. Hence Z is a subalgebra of the Poisson center of $\mathcal{O}[\mathcal{M}_\Sigma^G]$ and B is Poisson commutative and centralized by J .

Combining (33) with (27) we get

$$\dim \text{Spec } B_C - \dim \text{Spec } Z_{\partial\Sigma} = r \text{rk}(G). \quad (34)$$

On the other hand, any representation of $\pi_1(\Sigma, p | C)$ is determined by a $2g + b - 1$ elements of G

$$Y_i, \quad \tilde{Y}_j, \quad X_k, \quad Z_l, \quad 1 \leq i \leq g, \quad 1 \leq j \leq r, \quad r+1 \leq k \leq g, \quad 1 \leq l \leq b-1,$$

such that $\tilde{Y}_j \sim Y_j$ belong to the same conjugacy class, i.e. for all $1 \leq j \leq r$ we have $\tilde{Y}_j = X_j Y_j X_j^{-1}$ for some X_j . Note that for generic Y_j , dimension of its centralizer equals to $\text{rk}(G)$. As a corollary, we get

$$\dim \text{Hom}(\pi_1(\Sigma, p | C), G) = (2g + b - 1) \dim G - r \text{rk}(G).$$

Recall that for the purpose of this subsection we assume $g > 0$ and $b > 0$, so the action of G by conjugation on $\text{Hom}(\pi_1(\Sigma, p | C), G)$ has generically zero-dimensional stabilizer. As a corollary,

$$\dim \text{Spec } J_{\Sigma \setminus C} = (2g + b - 2) \dim G - r \text{rk}(G). \quad (35)$$

Combining (8) with (35) we get

$$\dim \mathcal{M}_{\Sigma}^G - \dim \text{Spec } J_{\Sigma \setminus C} = r \text{rk}(G). \quad (36)$$

Finally, comparing (34) with (36) we conclude that balance of dimensions (5) holds. $\square$

3.3. Product of surface superintegrable systems associated with gluing.

3.3.1. Choice of presentation of the fundamental groupoid Let Σ be an oriented surface with $b > 0$ boundary components and $C = C_1 \sqcup \dots \sqcup C_r$ be a disjoint union of r simple closed curves in Σ which cuts the surface into two parts $\Sigma = \Sigma_L \#_C \Sigma_R$. As in the statement of Theorem 1 we assume that none of the simple curves is homotopic to a boundary and all $C_1, \dots, C_r$ are pairwise nonhomotopic. As a corollary, neither Σ_L , nor Σ_R can be a cylinder. Further, because surface Σ has nontrivial boundary $\partial \Sigma \neq \emptyset$, at least one of the surfaces Σ_L, Σ_R has a boundary component not in C . Without loss of generality we can assume that $\partial \Sigma_L \supsetneq C$.

Because Σ_L and Σ_R both have at least one boundary component, their fundamental groups are free, say with m and n generators respectively. It will be convenient for us to choose the following presentation

$$\pi_1(\Sigma_L, p) = \langle a_1, \dots, a_m \rangle, \quad \pi_1(\Sigma_R, q) = \langle b_1, \dots, b_n, b_{n+1} \mid b_1 = F(b_2, \dots, b_{n+1}) \rangle. \quad (37)$$

Here $a_1, \dots, a_m$ are free generators of $\pi_1(\Sigma_L)$. Because $\partial \Sigma_L \supsetneq C$, we can choose the first r generators a_j , $1 \leq j \leq r$ to be homotopic to C_j as shown on Fig. 7. At the same time, for Σ_R we will choose the presentation as follows: Let $b_2, \dots, b_{n+1}$ correspond to closed paths from q to q which freely generate $\pi_1(\Sigma_R, q)$. Without loss of generality we can assume that $b_2, \dots, b_r$ are homotopic to $C_2, \dots, C_r$. Now, let b_1 be the closed path homotopic to C_1 , it can be expressed as a word in other generators $b_1 = F(b_2, \dots, b_{n+1})$.¹²

¹² Adding extra generator b_1 for $\pi_1(\Sigma_R, q)$ satisfying relation $b_1 = F(b_2, \dots, b_{n+1})$ allows us to treat both cases $\partial \Sigma_R = C$ and $\partial \Sigma_R \supsetneq C$ simultaneously without any changes in notation.

Fig. 7. Connected component of a separating curve

Denote by $\pi_1(\Sigma, p, q)$ the full subcategory of the fundamental groupoid $\pi_1(\Sigma)$ with two objects $p, q \in \Sigma$.¹³ By corollary of the Van Kampen's Theorem, groupoid $\pi_1(\Sigma, p, q)$ has the following presentation

$$\pi_1(\Sigma, p, q) = \left\langle a_1, \dots, a_m, e_1, \dots, e_r, b_1, \dots, b_n, b_{n+1} \mid \begin{array}{l} a_1 = e_1^{-1} b_1 e_1, \dots, a_r = e_r^{-1} b_r e_r, \\ b_1 = F(b_2, \dots, b_{n+1}) \end{array} \right\rangle, \quad (38)$$

Here open arcs $e_1, \dots, e_r$ stand for the paths connecting p and q which cross once exactly one of the separating cycles $C_1, \dots, C_r$, see Figs. 7 and 8.

Note that $\pi_1(\Sigma, p, q)$ is freely generated by $a_{r+1}, \dots, a_m, e_1, \dots, e_r, b_2, \dots, b_{n+1}$.

3.3.2. Coordinate rings of G -representations and relations between them Any G -representation of $\pi_1(\Sigma, p, q)$ is defined by $m + n + r + 1$ elements subject to $r + 1$ relations:

$$\begin{aligned} A_1, \dots, A_m, E_1, \dots, E_r, B_1, \dots, B_n, B_{n+1} &\in G, \\ B_j E_j &= E_j A_j, \quad 1 \leq j \leq r, \quad B_1 = F(B_2, \dots, B_{n+1}). \end{aligned} \quad (39)$$

This makes the set $\text{Hom}(\pi_1(\Sigma, p, q), G)$ of all G -representations of $\pi_1(\Sigma, p, q)$ into an algebraic subset of $G^{\times(m+n+r+1)}$. Moreover, $\text{Hom}(\pi_1(\Sigma, p, q), G)$ comes equipped with a $G \times G$ action:

$$\begin{aligned} A_j &\mapsto g A_j g^{-1}, & 1 \leq j \leq m, \\ E_k &\mapsto h E_k g^{-1}, & 1 \leq k \leq r, & (g, h) \in G \times G, \\ B_l &\mapsto h B_l h^{-1}, & 1 \leq l \leq n + 1, \end{aligned} \quad (40)$$

Of course, any G -representation of $\pi_1(\Sigma, p, q)$ also defines G -representations of $\pi_1(\Sigma_L, p)$ and $\pi_1(\Sigma_R, q)$. Explicitly, for our choice of generators we obtain a diagram of $G \times G$ -equivariant projections

$$\begin{array}{ccc} \text{Hom}(\pi_1(\Sigma, p, q), G) & \longrightarrow & \text{Hom}(\pi_1(\Sigma_R, q), G) \\ \downarrow & & \downarrow \\ \text{Hom}(\pi_1(\Sigma_L, p), G) & \longrightarrow & \text{Hom}(\pi_1(C), G) \parallel G \end{array} \quad (41)$$

¹³ We use two distinct points $p \in \Sigma_L$ and $q \in \Sigma_R$ in order to simplify the comparison of the formulas with Sect. 4.

$$\begin{array}{ccc}
 (A_1, \dots, A_m, E_1, \dots, E_r, B_1, \dots, B_{n+1}) & \xrightarrow{\quad\quad\quad} & (B_1, \dots, B_{n+1}) \\
 \downarrow & & \downarrow \\
 (A_1, \dots, A_m) & \xrightarrow{\quad\quad\quad} & ([A_1] = [B_1], \dots, [A_r] = [B_r])
 \end{array} \quad (42)$$

where $[A_j] = [B_j]$ stands for the common conjugacy class of A_j and B_j .

Let I be a $G \times G$ -invariant ideal in $\mathcal{O}[G^{\times(m+n+r+1)}]$ defined by equations (39), then

$$\mathcal{O}[\text{Hom}(\pi_1(\Sigma, p, q), G)] \simeq \mathcal{O}[G^{\times(m+n+r+1)}]/I. \quad (42)$$

In terms of the coordinate ring (42) of G -representations of $\pi_1(\Sigma, p, q)$ diagram (42) corresponds to the following diagram of inclusions of subalgebras

$$\begin{array}{ccc}
 \mathcal{O}[\text{Hom}(\pi_1(\Sigma, p, q), G)] & \supset & \mathcal{O}[\text{Hom}(\pi_1(\Sigma_R, q), G)] \\
 \cup & & \cup \\
 \mathcal{O}[\text{Hom}(\pi_1(\Sigma_L, p), G)] & \supset & H = \mathcal{O}[\text{Hom}(\pi_1(C), G)]^G
 \end{array} \quad (43)$$

Here $H = \mathcal{O}[\text{Hom}(\pi_1(C), G)]^G$ is the coordinate ring of a G -character variety of a disjoint union of r circles corresponding to connected components of $C = \partial \Sigma_L \cap \partial \Sigma_R = C_1 \sqcup \dots \sqcup C_r$. Hence $H \simeq (\mathcal{O}[G]^G)^{\otimes r}$ is generated by invariant functions on common conjugacy classes $[A_i] = [B_i]$, $1 \leq i \leq r$ of matrices A and B .

Lemma 26.

$$H = \mathcal{O}[\text{Hom}(\pi_1(\Sigma_L, p))] \cap \mathcal{O}[\text{Hom}(\pi_1(\Sigma_R, q))] \quad (44)$$

Proof. From (43) we already know that $H \subset \mathcal{O}[\text{Hom}(\pi_1(\Sigma_L, p))] \cap \mathcal{O}[\text{Hom}(\pi_1(\Sigma_R, q))]$, so we have to prove only the opposite inclusion. To this end, first recall that all algebras in (43) are $G \times G$ -modules w.r.t. the action (40). Moreover, $1 \times G$ is acting trivially on $\mathcal{O}[\text{Hom}(\pi_1(\Sigma_L, p))]$, while $G \times 1$ is acting trivially on $\mathcal{O}[\text{Hom}(\pi_1(\Sigma_R, q))]$. As a corollary, their intersection lies in the G -invariant part of both subalgebras $\mathcal{O}[\text{Hom}(\pi_1(\Sigma_L, p))] \cap \mathcal{O}[\text{Hom}(\pi_1(\Sigma_R, q))] \subset \mathcal{O}[\mathcal{M}_{\Sigma_L}^G] \cap \mathcal{O}[\mathcal{M}_{\Sigma_R}^G]$. By Theorem 12 combined with Corollary 14 we conclude that the right hand side of (44) is spanned by invariant graph functions on C , hence contained in H . $\square$

Now consider the following tensor product over H

$$J_0 = \mathcal{O}[\text{Hom}(\pi_1(\Sigma_L, p), G)] \otimes_H \mathcal{O}[\text{Hom}(\pi_1(\Sigma_R, q), G)]. \quad (45)$$

By Lemma 26, $J \subset \mathcal{O}[\text{Hom}(\pi_1(\Sigma, p, q), G)]$, this gives us another subalgebra of $\mathcal{O}[\text{Hom}(\pi_1(\Sigma, p, q), G)]$ which fits in the middle of diagram (43).

By construction, all rings involved in (43) and (45) are finitely generated integral domains. Hence, taking the spectrum of all rings in (43) and (45) we obtain a commutative diagram of $G \times G$ -equivariant dominant maps of affine schemes

$$\begin{array}{c}
 \text{Hom}(\pi_1(\Sigma, p, q), G) \xrightarrow{\quad \gamma_0 \quad} \text{Hom}(\pi_1(\Sigma_L, p), G) \times_{\text{Spec } H} \text{Hom}(\pi_1(\Sigma_R, q), G) \longrightarrow \text{Hom}(\pi_1(\Sigma_R, q), G) \\
 \downarrow \qquad \qquad \qquad \downarrow \beta_L \qquad \qquad \qquad \downarrow \beta_R \\
 \text{Hom}(\pi_1(\Sigma, p, q), G) \xrightarrow{\quad \beta_L \quad} \text{Spec } H
 \end{array} \quad (46)$$

where $\text{Spec}(H) \simeq (G // \text{Ad } G)^r$ and

$$\text{Hom}(\pi_1(\Sigma_L, p), G) \times_{\text{Spec } H} \text{Hom}(\pi_1(\Sigma_R, q), G) = \text{Spec } J_0$$

is the fiber product of maps β_L and β_R .

3.3.3. Product of superintegrable systems Taking the GIT quotient by $G \times G$ action of diagram (46) we obtain from the main diagonal the following chain of dominant maps of affine schemes

$$\begin{aligned}
 \text{Hom}(\pi_1(\Sigma, p, q), G) // (G \times G) &\xrightarrow{\gamma} \left(\text{Hom}(\pi_1(\Sigma_L, p), G) \times_{\text{Spec } H} \text{Hom}(\pi_1(\Sigma_R, q), G) \right) \\
 &\xrightarrow{\beta} \text{Spec } H
 \end{aligned} \quad (47)$$

Note that by (7), we have natural isomorphism

$$\text{Hom}(\pi_1(\Sigma, p, q), G) // (G \times G) \simeq \mathcal{M}_{\Sigma}^G$$

Remark 27. Key idea of the remaining part of this subsection is that projections (47) define an affine superintegrable system on $\mathcal{O}[\mathcal{M}_{\Sigma}^G]$ associated with the separating collection of cycles $C = \partial \Sigma_L \cap \partial \Sigma_R = C_1 \sqcup \dots \sqcup C_r$.

Moreover, we can actually prove a much more general statement, formulated in Proposition 29, that if each of the surfaces Σ_L, Σ_R was already equipped with an affine superintegrable system on $\mathcal{O}[\mathcal{M}_{\Sigma_L}]$ and $\mathcal{O}[\mathcal{M}_{\Sigma_R}]$ respectively, then gluing along C provides a way to construct a new superintegrable system on Σ which refines (47).

Lemma 28. Equation (47) defines a chain of Poisson projections.

Proof. Note that the neighbourhood of C is a disjoint union of r annuli. Recall that $H \simeq (\mathcal{O}[G]^G)^{\otimes r}$ is the tensor product of r copies of the coordinate ring of the moduli space of G -representations of an annulus. It is naturally equipped with the trivial Poisson structure and with two natural injective Poisson homomorphisms

$$H \xhookrightarrow{\beta_L^*} \mathcal{O}[\mathcal{M}_{\Sigma_L}^G] \qquad H \xhookrightarrow{\beta_R^*} \mathcal{O}[\mathcal{M}_{\Sigma_R}^G] \quad (48)$$

to the Poisson center of the coordinate rings of G -character varieties of $\pi_1(\Sigma_L, p)$ and $\pi_1(\Sigma_R, q)$ respectively. As a result,

$$H \xhookrightarrow{\beta^*} J = \mathcal{O}[\mathcal{M}_{\Sigma_L}^G] \otimes_H \mathcal{O}[\mathcal{M}_{\Sigma_R}^G]$$

is a Poisson embedding into the Poisson center of J . To finalize the proof, note that

$$J \hookrightarrow \gamma^* \operatorname{Hom}(\pi_1(\Sigma, p, q), G)^{G \times G} \simeq \mathcal{O}[\mathcal{M}_\Sigma^G]$$

is a subalgebra of the full coordinate ring of the moduli space of G -representations of $\pi_1(\Sigma, p, q)$. $\square$

Proposition 29. *Suppose that we have a pair of affine superintegrable systems on Σ_L and Σ_R defined by some choice of nonseparating cycles $C_L \subset \Sigma_L$ and $C_R \subset \Sigma_R$:*

$$Z_L \subset B_L \subset J_L \subset A_L = \mathcal{O}[\mathcal{M}_{\Sigma_L}^G], \quad Z_R \subset B_R \subset J_R \subset A_R = \mathcal{O}[\mathcal{M}_{\Sigma_R}^G], \quad (49)$$

where Z_L and Z_R denote Poisson centers of A_L and A_R respectively. Then

$$Z \subset B \subset J \subset A = \mathcal{O}[\mathcal{M}_\Sigma^G], \quad (50)$$

defines an affine superintegrable system on Σ . Here Z is the Poisson center of $\mathcal{O}[\mathcal{M}_\Sigma^G]$,

$$B = B_L \otimes_H B_R, \quad J = J_L \otimes_H J_R, \quad (51)$$

and $H = \mathcal{O}[\mathcal{M}_C^G]$ is the coordinate ring of the G -character variety of $\pi_1(C) \simeq \operatorname{Free}_r$.

Proof. First, note that

$$\{\mathcal{O}[\mathcal{M}_{\Sigma_L}^G], \mathcal{O}[\mathcal{M}_{\Sigma_R}^G]\} = 0,$$

where $\{, \}$ stands for the Poisson bracket on $\mathcal{M}_\Sigma^G$. Indeed, by Proposition 12 subalgebras $\mathcal{O}[\mathcal{M}_{\Sigma_L}^G]$ and $\mathcal{O}[\mathcal{M}_{\Sigma_R}^G]$ are spanned by invariant graph functions which have no intersections. Combining it with the fact that $\{B_L, J_L\} = 0 = \{B_R, J_R\}$ we conclude that $\{B, J\} = 0$, and, in particular, that B has a trivial Poisson bracket. So the only thing we have to prove is that Krull dimensions of the algebras satisfy (5).

Krull dimensions of algebras B and J are determined as follows¹⁴

$$\dim B = \dim B_L + \dim B_R - \dim H, \quad \dim J = \dim J_L + \dim J_R - \dim H. \quad (52)$$

Now denote by g_L and b_L the genus and number of boundary components of Σ_L . Similarly, let g_R and b_R denote the genus and number of boundary components of Σ_R . The resulting surface Σ then has a genus $g = g_L + g_R + (r - 1)$ and number of boundary components $b = b_L + b_R - 2r$. Note that we have assume that neither Σ_L nor Σ_R is a cylinder (i.e. $(g_L, b_L) \neq (0, 2) \neq (g_R, b_R)$). From (52) we have

$$\dim B + \dim J = \dim B_L + \dim B_R + \dim J_L + \dim J_R - 2 \dim H$$

On the other hand, because both systems (49) are superintegrable, by (5) we conclude that

$$\dim B + \dim J = \dim A_L + \dim Z_L + \dim A_R + \dim Z_R - 2 \dim H.$$

¹⁴ These two formulas is an algebraic version of a statement that for a pair of projections of irreducible affine varieties $X \rightarrow Z$ and $Y \rightarrow Z$ over $\mathbb{C}$, the dimension of the corresponding fiber product is $\dim(X \times_Z Y) = \dim X + \dim Y - \dim Z$.

Note that in order for (52) to be valid it is crucial that all algebras involved are finitely generated over $\mathbb{C}$ and have no zero divisors. Identities (52) then follow from the fact that transcendence degree of a field extension is additive w.r.t. the composition of field extensions.

Finally, using explicit formula (8) for Krull dimensions of A_L , A_R and (27) for Z_L , Z_R we obtain

$$\begin{aligned} \dim B + \dim J &= ((2g_L + b_L - 2) + (2g_R + b_R - 2)) \dim G + (b_L + b_R - 2r) \text{rank}(G) \\ &= (2g + b - 2) \dim G + b \text{rank}(G) \\ &= \dim A + \dim Z. \end{aligned}$$

Hence, condition (5) on Krull dimensions of subalgebras (50) is satisfied and the resulting system is superintegrable. $\square$

3.4. The proof of Theorem 17. Now we can prove our main result.

Proof. Let $\Sigma \setminus C = \Sigma_1 \sqcup \dots \sqcup \Sigma_m$, where $\Sigma_1, \dots, \Sigma_m$ stand for connected components. We will prove the statement of the theorem by induction in the number m of connected components.

The base case $m = 1$ follows by Proposition 25.

Now assume that the statement of the theorem holds for all (Σ', C') s.t. $\Sigma' \setminus C'$ has at most m connected components. Then for each pair (Σ, C) s.t. $\Sigma \setminus C$ has $m + 1$ connected components we can present $\Sigma = \Sigma_L \sqcup \Sigma_R$, where both $\Sigma_L \setminus C$ and $\Sigma_R \setminus C$ have at most m connected components. Define

$$C_L = C \setminus \partial \Sigma_R, \quad C_R = C \setminus \partial \Sigma_L, \quad C_{LR} = \partial \Sigma_L \cap \partial \Sigma_R.$$

Note that $C = C_L \sqcup C_{LR} \sqcup C_R$. By the inductive assumption we have two superintegrable systems

$$Z_{\partial \Sigma_L} \subset B_{C_L} \subset J_{\Sigma \setminus C_L} \subset \mathcal{O}[\mathcal{M}_{\Sigma_L}^G], \quad Z_{\partial \Sigma_R} \subset B_{C_R} \subset J_{\Sigma \setminus C_R} \subset \mathcal{O}[\mathcal{M}_{\Sigma_R}^G].$$

Now, applying Proposition 29 we finalize the proof. $\square$

The following proposition is an immediate consequence.

Proposition 30. *Let Σ be an oriented closed surface with $b > 0$ boundary components and $C = C_1 \sqcup \dots \sqcup C_k$ and $C' = C'_1 \sqcup \dots \sqcup C'_m$ be two collections of simple closed curves none of which is homotopic to a boundary component. Denote by $\mathcal{I}_C$ and $\mathcal{I}_{C'}$ the two superintegrable systems associated to C and C' as in Theorem 17.*

- *If $C \subset C'$ is a subcollection of C' , the integrable system $\mathcal{I}_C$ is a refinement of $\mathcal{I}_{C'}$.*
- *If an element of the Mapping Class Group of surface Σ bring the collection C to C' , it induces an equivalence of corresponding superintegrable systems.*

Remark 31. It is natural to expect that for an affine superintegrable system

$$Z_{\partial \Sigma} \subset B_C \subset J_{\Sigma \setminus C} \subset \mathcal{O}[\mathcal{M}_{\Sigma}^G],$$

subalgebra $Z_{\partial \Sigma}$ is the full Poisson center of $\mathcal{O}[\mathcal{M}_{\Sigma}^G]$, and this is true in the compact case. Note that this implies that $J_{\Sigma \setminus C}$ is the full Poisson centralizer of B_C .

3.5. Beyond generic orbits. The main result of the paper, formulated in Theorem 1, states that a collection of simple closed curves $C = C_1 \sqcup \dots \sqcup C_r$ on an oriented surface Σ defines a family of superintegrable systems. This family is parameterized by generic values of Casimir functions, in other words, by fixing holonomies around boundary components at regular conjugation orbits. We have left details of the case of nongeneric values of Casimir functions outside of the scope of the current paper.

We expect to prove the following stronger statement in a sequel publication. Consider a superintegrable system from Theorem 1 and let

$$\begin{array}{ccccc} A & \xrightarrow{p_1} & \mathcal{J} & \xrightarrow{p_2} & \mathcal{B} & \xrightarrow{p_3} & \mathcal{Z} \\ & & & \searrow p & & & \end{array}$$

be an associated chain of dominant maps of affine schemes preserving the Poisson bracket. Then for all $z \in \text{Spec } \mathcal{Z}$, every irreducible component of the fiber $p^{-1}(z)$ contains an open subset $\mathcal{M}_{2n}$ such that

$$\mathcal{M}_{2n} \xrightarrow{\pi_1} \mathcal{P}_{2n-k} \xrightarrow{\pi_2} \mathcal{B}_k$$

is a superintegrable system. Here π_1, π_2 are maps induced by p_1 and p_2 respectively.

Example of systems with lower rank orbits were studied in [CF18]. In that case $G = SL_m$, the surface Σ is a torus with n punctures, the conjugacy class of the monodromy around each puncture is fixed and is assumed to be of rank 1. It is easy to see that the dimension of the moduli space in this case is $2(m-1)n$. The choice of $C = C_1, \dots, C_n$ being separating cycles cutting the torus into n cylinders with one puncture in each cylinder produces a Liouville integrable system with $(m-1)n$ Poisson commuting integrals. For any C_i , the choice $C = C_i$ gives a superintegrable refinement of this Liouville integrable system.

4. Hamilton Flows on the Full Coordinate Ring

The Poisson bracket (21) on the coordinate ring $\mathcal{O}[\text{Hom}(\pi_1(\Sigma, p_1, \dots, p_m), G)]^G$ of the moduli space $\mathcal{M}_\Sigma^G$ can be obtained from the bivector field on $R = \mathcal{O}[\text{Hom}(\pi_1(\Sigma, p_1, \dots, p_m), G)]$ using two essentially different methods. One way is the “ r -matrix approach” developed in [FR93], the other way is the “Quasi Poisson” bracket approach developed in [AMM98, AK00, AKM02].

In the r -matrix approach one defines Poisson bracket on the full coordinate ring

$$\{, \}_{r\text{-matrix}} : R \otimes R \rightarrow R, \quad R = \mathcal{O}[\text{Hom}(\pi_1(\Sigma, p_1, \dots, p_m), G)], \quad (53)$$

which coincides with the Goldman bracket on the G -invariant part $R^G \simeq \mathcal{O}[\mathcal{M}_\Sigma^G]$.

In the Quasi Poisson approach the bivector field on $\text{Hom}(\pi_1(\Sigma, p_1, \dots, p_m), G)$ is not Poisson. The bracket $\{, \}_{quasi} : R \otimes R \rightarrow R$ satisfies the Jacobi identity only up to a fixed trivector field vanishing on R^G . As in the r -matrix approach, quasi Poisson bracket induces the Goldman bracket on R^G

$$\{, \}_{r\text{-matrix}} \Big|_{R^G \otimes R^G} = \{, \}_{quasi} \Big|_{R^G \otimes R^G} = \{, \}_{Goldman} : R^G \otimes R^G \rightarrow R^G.$$

Here we will focus on the r -matrix approach.¹⁵ The bracket (53) makes R into a left Lie module over R^G . In other words, the G -invariant part acts on the full coordinate ring by derivations

$$\{, \} : R^G \otimes R \rightarrow R, \quad (54)$$

satisfying

$$\begin{aligned} \{h, ab\} &= \{h, a\}b + a\{h, b\}, \quad \text{for all } h \in R^G, \quad a, b \in R, \\ \{h_1, \{h_2, a\}\} - \{h_2, \{h_1, a\}\} &= \{\{h_1, h_2\}_{\text{Goldman}}, a\}, \quad \text{for all } h_1, h_2 \in R^G, \quad a \in R. \end{aligned}$$

Action (54) allows one to integrate Hamilton flows described in Theorem 17. Moreover, with the appropriate choice of generators of the fundamental groupoid, this action acquires a remarkably simple form presented in Theorem 34 below. First, in Sect. 4.1, for a given collection C of pairwise nonintersecting simple closed curves we define a particular choice of marked points and generating arcs of $\pi_1(\Sigma, p_1, \dots, p_m)$. The proposed choice of generating arcs allows us to simultaneously separate variables for Hamilton flows corresponding to different curves in C . Next, in Sect. 4.2 we use r -matrix formalism to calculate the Hamiltonian flow (54) associated to a simple closed curve in C colored by a class function.

4.1. Choice of generators of π_1 . Let Σ be, as before, an oriented surface with nonempty boundary $\partial\Sigma = D_1 \sqcup \dots \sqcup D_b$ and let $C = C_1 \sqcup \dots \sqcup C_k$ be a disjoint collection of simple closed curves as in Theorem 17. Namely, we require that C_j are pairwise nonintersecting, pairwise nonhomotopic, and none of them are homotopic to the boundary. Let $\Sigma \setminus C = \Sigma_1 \sqcup \dots \sqcup \Sigma_m$, where $\Sigma_1, \dots, \Sigma_m$ stand for connected components. For the purpose of this section we further assume that $\partial\Sigma_j \cap \partial\Sigma \neq \emptyset$ for all $j \in \{1, \dots, m\}$. This assumption can be made with no loss generality when we describe Hamilton flows, because initial conditions with trivial monodromy about the boundary component effectively correspond to absence of the boundary component.

For any connected component Σ_j we choose a single marked point $p_j \in \partial\Sigma_j \cap \partial\Sigma$ on the common boundary with Σ . Each surface Σ_j is an oriented surface of genus g_j with b_j boundary components. Its boundary can be presented as a disjoint union

$$\partial\Sigma_j = D_{\beta_{j,1}} \sqcup \dots \sqcup D_{\beta_{j,r_j}} \sqcup C_{\alpha_{j,r_j+1}} \sqcup \dots \sqcup C_{\alpha_{j,b_j}}$$

of r_j connected components $D_{\beta_{j,1}}, \dots, D_{\beta_{j,r_j}} \subset \partial\Sigma$ of the boundary of the full surface Σ , as well as $b_j - r_j$ connected components $C_{\alpha_{j,r_j+1}}, \dots, C_{\alpha_{j,b_j}} \subset C$ of a curve C . Without loss of generality we can assume that $p_j \in D_{\beta_{j,1}}$.

Denote by

$$H_j = (C \cap \Sigma_j) \setminus \partial\Sigma_j = C_{\gamma_{j,1}} \sqcup \dots \sqcup C_{\gamma_{j,s_j}}, \quad (55)$$

the subcollection of simple closed curves in C which belong to Σ_j , but do not belong to $\partial\Sigma_j$. Such subcollection is nonseparating by definition.

We will choose a set of generators of $\pi_1(\Sigma, p_1, \dots, p_n)$ as follows:

¹⁵ For the analogue of the current section in terms of quasi Poisson bracket see [AR].

Fig. 8. Adjacent pair $\Sigma_j \cup \Sigma_k$

- (1) First, for each simple closed curve $C_i \in \partial \Sigma_j \cap \partial \Sigma_k$ which separates a pair of distinct connected components $\Sigma_j \neq \Sigma_k$ of $\Sigma \setminus C$ we choose a simple arc $e_i \in (\Sigma_j \setminus H_j) \cup (\Sigma_k \setminus H_k)$ from p_j to p_k passing once through C_j as shown on Fig. 8. Note that without loss of generality we can assume that different e_j intersect only at the endpoints. This can be achieved by sliding all intersection points between e_i and $e_{i'}$ in Σ_j to p_j followed by left multiplication by an element of $\pi_1(\Sigma_j, p_j)$. Similarly, one can get rid of the intersection points in Σ_k .
- (2) For each arc e_i chosen on the previous step we pick a simple closed curve a_i which starts and ends at p_j and homotopic to C_i as shown on Fig. 7. Again, we can assume that none of the a_i, e_l intersect away from the endpoints. This can be achieved by choosing a_i sufficiently close to $e_i \cup C_i$. Also, each a_i chosen on this step cuts Σ_j into two connected components, one of them is a cylinder given by the neighbourhood of C_i , the other is equivalent to the original surface. In what follows we assume that all further curves never enter into the cylinder.
- (3) For each nonseparating simple closed curve $C_{\gamma_{j,k}}$ from (55) we choose a pair of simple closed curves $x_{\gamma_{j,k}}$ and $y_{\gamma_{j,k}}$, where $y_{\gamma_{j,k}}$ is homotopic to $C_{\gamma_{j,k}}$, while $x_{\gamma_{j,k}}$ passes once through $C_{\gamma_{j,k}}$ as shown on Fig. 6.
- (4) At the last step we consider $\Sigma_j \setminus C$. This surface has genus $g_j - s_j$ and $b_j + 2s_j$ boundary components. Our choice of arcs already generates the paths around $r_j + 2s_j$ boundary components of $\Sigma_j \setminus C$. We can choose the remaining $2g_j + b_j - r_j$ free generators of $\pi_1(\Sigma_j \setminus C)$ arbitrarily.

Collection of marked points and free generators of $\pi_1(\Sigma, p_1, \dots, p_m)$ chosen above defines an ordered oriented ribbon graph with vertices $p_1, \dots, p_m$ and edges corresponding to generators. The total order of half-edges adjacent to the given vertex is defined by an orientation of the surface. An example of the total order of half-edges is given on Fig. 9.

Data of a ribbon graph can be encoded in m ordered sets of half-edges adjacent to p_j

$$S_j = \{h_{j,1}, \dots, h_{j,n_j}\}, \quad 1 \leq j \leq m. \quad (56)$$

Our ribbon graph is oriented, hence it will be convenient for us to label half-edges by generators of $\pi_1(\Sigma, p_1, \dots, p_m)$ and its inverses. We associate a generator for an outgoing edge and inverse of a generator for an ingoing edge. For example, ordered set

Fig. 9. An example of a total order of half-edges adjacent to $p_j \in \partial \Sigma_j$ (left to right)

of half-edges adjacent to p_j on Fig. 9 reads

$$S_j = \left\{ z_{j,2}^{-1}, z_{j,2}, \dots, x_{\gamma_{j,1}}^{-1}, y_{\gamma_{j,1}}, x_{\gamma_{j,1}}, y_{\gamma_{j,1}}^{-1}, \dots, a_{\alpha_{j,r_j+1}}, e_{\alpha_{j,r_j+1}}, a_{\alpha_{j,r_j+1}}^{-1}, \dots, e_{\alpha_{j,b_j}}^{-1} \right\}.$$

4.2. The r -matrix approach. Let $\mathfrak{g}$ be a Lie algebra of G and $r \in \mathfrak{g} \otimes \mathfrak{g}$ be a classical r -matrix, i.e. a solution to the Yang–Baxter equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0.$$

We will be interested in solutions corresponding to factorizable Lie bialgebras, i.e. classical r -matrices with the following symmetric part (with respect to the exchange of tensor components) given by the Killing form:

$$I = r + \sigma(r) = \sum_{J=1}^{\dim \mathfrak{g}} e_J \otimes e_J, \quad (57)$$

Here $\{e_J\}$ is a basis in $\mathfrak{g}$ which is orthonormal with respect to the Killing form and $\sigma(x \otimes y) = y \otimes x$. Such solutions are classified in [BD82].

The choice of free generators of $\pi_1(\Sigma, p_1, \dots, p_m)$ fixes a natural isomorphism $\text{Hom}(\pi_1(\Sigma, p_1, \dots, p_m), G) \simeq G^N$ where $N = 2g + b + m - 2$. For an ordered ribbon graph Fock and Rosly [FR93] introduced a Poisson structure on this space with the Poisson tensor defined in terms of factorizable classical r -matrices. Below we present the corresponding Poisson brackets between matrix element functions.

For two matrix valued functions M_{ij} and N_{ab} on a Poisson manifold we will use the following notation

$$\{M \otimes N\}_{(ia)(jb)} = \{M_{ij}, N_{ab}\}$$

for the matrix of Poisson brackets. Here the convention is $(M \otimes N)_{(ia)(jb)} = M_{ij} N_{ab}$.

Distinct arcs Basic building block of brackets corresponds to the case shown on Fig. 10a when the two oriented arcs share the source but have distinct targets different from their common source.

Fig. 10. Relative position of half edges

In this case the Fock and Rosly Poisson bracket between matrix elements¹⁶ of holonomies along these arcs reads

$$\{M_A \otimes M_B\} = (M_A \otimes M_B) r. \quad (58a)$$

One can easily calculate the Poisson bracket between matrix elements of powers and products of matrices. For example, by Leibnitz identity applied to $M_A M_A^{-1} = \text{Id} = M_B M_B^{-1}$ from (58a) we immediately obtain Poisson brackets for other three cases shown on Fig. 10b–d

$$\{N_A \otimes M_B\} = -(\text{Id} \otimes M_B) r (N_A \otimes \text{Id}), \quad (58b)$$

$$\{M_A \otimes N_B\} = -(M_A \otimes \text{Id}) r (\text{Id} \otimes N_B), \quad (58c)$$

$$\{N_A \otimes N_B\} = r (N_A \otimes N_B), \quad (58d)$$

where $N_A = M_A^{-1}$ and $N_B = M_B^{-1}$.

Similarly, brackets $\{M_B \otimes M_A\}, \dots, \{N_B \otimes N_A\}$ can be obtained by skew-symmetry, and the answer can be expressed in terms of r_{21} as follows

$$\{M_B \otimes M_A\} = -(M_B \otimes M_A) r_{21}, \quad (59a)$$

$$\{M_B \otimes N_A\} = (M_B \otimes \text{Id}) r_{21} (\text{Id} \otimes N_A), \quad (59b)$$

$$\{N_B \otimes M_A\} = (\text{Id} \otimes M_A) r_{21} (N_B \otimes \text{Id}), \quad (59c)$$

$$\{N_B \otimes N_A\} = -r_{21} (N_B \otimes N_A). \quad (59d)$$

Here $r_{21} = \sigma(r)$.

Bracket between matrix elements of a general distinct pair of arcs now can be computed by simply adding up to four terms of the form (58) and (59) corresponding to brackets between pairs of half edges whenever they are adjacent to the same vertex.

Self-brackets There are only three cases of relative position of half edges of the same arc as shown on Fig. 11.

Corresponding brackets are given by the following formulae

$$\{E \otimes E\} = r_a(E \otimes E) + (E \otimes E)r_a, \quad (60a)$$

$$\{A_R \otimes A_R\} = (A_R \otimes A_R)r_a + r_a(A_R \otimes A_R) + (\text{Id} \otimes A_R)r_{21}(A_R \otimes \text{Id}) - (A_R \otimes \text{Id})r_{12}(\text{Id} \otimes A_R), \quad (60b)$$

$$\{A_L \otimes A_L\} = r_a(A_L \otimes A_L) + (A_L \otimes A_L)r_a + (A_L \otimes \text{Id})r_{21}(\text{Id} \otimes A_L) - (\text{Id} \otimes A_L)r_{12}(A_L \otimes \text{Id}). \quad (60c)$$

where $r_a = \frac{1}{2}(r_{12} - r_{21})$ is the antisymmetric (with respect to the exchange of tensor components) part of r-matrix and we write r_{12} for r .

¹⁶ Here and below by matrix elements we always assume matrix elements in a finite dimensional representation.

Fig. 11. Relative position of half edges of the same arc

Fig. 12. Relative order of half-edges w.r.t to the closed arc

4.3. Hamiltonian flows generated by central functions. In this section we compute the evolution of matrix element functions on $\text{Hom}(\pi_1(\Sigma, p_1, \dots, p_m), G)$ with respect to the Hamiltonian flows generated by central functions on holonomies around simple closed arcs a_i or y_j defined in Sect. 4.1. We start by considering three basic cases shown on Fig. 12.

Proposition 32. Let $(a, e), (f, a), (a, g)$ be pairs of generators with relative order of half-edges as shown on Fig. 12, $H : G \rightarrow \mathbb{C}$ be a polynomial central function, $H(ghg^{-1}) = H(h)$ and $\pi : G \rightarrow \text{End}(V)$ be a finite dimensional representation of G . Denote by $A, E, F, G \in \text{Hom}(\pi_1(\Sigma, p_1, \dots, p_m), G)$ holonomies along the corresponding oriented arcs. We have

$$\{H(A), \pi_V(E)\} = \sum_J (e_J, \nabla H(A)) \pi_V(Ee_J), \quad (61a)$$

$$\{H(A), \pi_V(F)\} = \{H(A), \pi_V(G)\} = \{H(A), \pi_V(A)\} = 0. \quad (61b)$$

where e_J is a basis in $\mathfrak{g}$ orthonormal with respect to the Killing form and

$$(\nabla H(A), X) = \frac{d}{dt} H(e^{tX} A)|_{t=0}$$

Here $X \in \mathfrak{g}$ and $(\cdot, \cdot)$ is the Killing form.

Proof. It is enough to consider $H(A) = \text{Tr } A^k$ where trace is taken over a finite dimensional representation (which can be different from V). Poisson bracket between matrix elements of A and E can be calculated as a sum of two terms: first term of the form (58a) corresponds to the ordered pair of half edges (a, e) adjacent to the same vertex; second term of the form (59c) corresponds to the ordered pair of half edges (e, a^{-1}) . As a result, we have

$$\{A \otimes E\} = (A \otimes E) r_{12} + (\text{Id} \otimes E) r_{21} (A \otimes \text{Id}).$$

Hence, by Leibnitz Identity for all $k \in \mathbb{N}$ we get

$$\begin{aligned} \{A^k \otimes E\} &= \sum_{j=0}^{k-1} (A^j \otimes \text{Id}) \{A \otimes E\} (A^{k-j-1} \otimes \text{Id}) \\ &= \sum_{j=0}^{k-1} \left((A^{j+1} \otimes E) r_{12} (A^{k-j-1} \otimes \text{Id}) + (A^j \otimes E) r_{21} (A^{k-j} \otimes \text{Id}) \right). \end{aligned}$$

Taking partial trace with respect to the first component we obtain (61a) as

$$\begin{aligned} \{\text{Tr } A^k, E\} &= \text{Tr}_1 \{A^k \otimes E\} \\ &= \sum_{j=0}^{k-1} \text{Tr}_1 \left((A^{j+1} \otimes E) r_{12} (A^{k-j-1} \otimes \text{Id}) \right) + \sum_{j=0}^{k-1} \text{Tr}_1 \left((A^j \otimes E) r_{21} (A^{k-j} \otimes \text{Id}) \right) \\ &= \sum_{j=0}^{k-1} \text{Tr}_1 \left((A^k \otimes E) r_{12} \right) + \sum_{j=0}^{k-1} \text{Tr}_1 \left((A^k \otimes E) r_{21} \right) = k \text{Tr}_1 \left((A^k \otimes E) I \right). \end{aligned}$$

Poisson bracket between matrix elements of A and G can be calculated as a sum of two terms: first term of the form (58a) corresponds to ordered pair of half edges (a, g) ; second term of the form (58b) corresponds to ordered pair of half edges (a^{-1}, g) . Hence, we have

$$\{A \otimes G\} = (A \otimes G) r - (\text{Id} \otimes G) r (A \otimes \text{Id}).$$

By Leibnitz Identity we get that for all $k \in \mathbb{N}$

$$\{A^k \otimes G\} = \sum_{j=0}^{k-1} \left((A^{j+1} \otimes G) r (A^{k-j-1} \otimes \text{Id}) - (A^j \otimes G) r (A^{k-j} \otimes \text{Id}) \right).$$

Taking partial trace with respect to the first component we get

$$\begin{aligned} \{\text{Tr } A^k, G\} &= \sum_{j=0}^{k-1} \text{Tr}_1 \left((A^{j+1} \otimes G) r (A^{k-j-1} \otimes \text{Id}) \right) - \sum_{j=0}^{k-1} \text{Tr}_1 \left((A^j \otimes G) r (A^{k-j} \otimes \text{Id}) \right) \\ &= k \text{Tr}_1 \left((A^k \otimes G) r \right) - k \text{Tr}_1 \left((A^k \otimes G) r \right) = 0. \end{aligned}$$

Poisson bracket between matrix elements of A and F is determined by (59a) and (59c) and has the following form

$$\{A \otimes F\} = -(A \otimes F) r_{21} + (\text{Id} \otimes F) r_{21} (A \otimes \text{Id}) \quad (62)$$

Note that r_{21} is on the same side from F in both terms of (62), so as in the previous case we obtain for all $k \in \mathbb{N}$

$$\{\text{Tr } A^k, F\} = 0.$$

Finally, Poisson bracket between matrix elements of A is given by (60b)

$$\{A \otimes A\} = (A \otimes A) r_a + r_a (A \otimes A) + (\text{Id} \otimes A) r_{21} (A \otimes \text{Id}) - (A \otimes \text{Id}) r_{12} (\text{Id} \otimes A)$$

Fig. 13. Relative order of half edges for pairs of closed arcs

By Leibnitz Identity we get for all $k \in \mathbb{N}$

$$\begin{aligned} \{A^k \otimes A\} &= \sum_{j=0}^{k-1} (A^j \otimes \text{Id}) \{A \otimes A\} (A^{k-j-1} \otimes \text{Id}) \\ &= \sum_{j=0}^{k-1} \left((A^{j+1} \otimes A) r_a (A^{k-j-1} \otimes \text{Id}) + (A^j \otimes \text{Id}) r_a (A^{k-j} \otimes A) \right. \\ &\quad \left. + (A^j \otimes A) r_{21} (A^{k-j} \otimes \text{Id}) - (A^{j+1} \otimes \text{Id}) r_{12} (A^{k-j-1} \otimes A) \right) \end{aligned}$$

Taking the partial trace with respect to the first component we have

$$\begin{aligned} \{\text{Tr } A^k, A\} &= k \text{Tr}_1 \left((A^k \otimes A) r_a + r_a (A^k \otimes A) + (A^k \otimes A) r_{21} - r_{12} (A^k \otimes A) \right) \\ &= \frac{k}{2} \text{Tr}_1 [A^k \otimes A, I]. \end{aligned} \quad (63)$$

From the invariance of I and from the cyclic invariance of the trace we get

$$(\text{Tr} \otimes \text{Id})((A^k \otimes A)I) = (\text{Tr} \otimes \text{Id})((A^{k-1}I(A \otimes A)) = (\text{Tr} \otimes \text{Id})(I(A^k \otimes A))$$

This implies that the bracket $\{\text{Tr } A^k, A\}$ vanishes. $\square$

Corollary 33. Let (y, x) and (y, a) be pairs of generators with relative order of half edges as shown on Fig. 13 and H and π_V as in the previous proposition. Denote by $A, X, Y \in \text{Hom}(\pi_1(\Sigma, p_1, \dots, p_m), G)$ the holonomies along the corresponding closed oriented arcs. We have

$$\{H(Y), \pi_V(X)\} = \sum_J (e_J, \nabla H(A)) \pi_V(X e_J), \quad (64a)$$

$$\{H(Y), \pi_V(A)\} = \{H(Y), \pi_V(Y)\} = 0. \quad (64b)$$

Proof. Indeed, the Poisson bracket between $\text{Tr } Y^k$ and X_{ij} gets two contributions of the form (61a) and (61b) which correspond to the outgoing and the ingoing half edges of X respectively. As a result, we obtain (64a).

On the other hand, the Poisson bracket between $\text{Tr } Y^k$ and A_{ij} is trivial due to (61b) because none of the half edges of A appear in between half edges of Y . Similar logic applies to the bracket between $\text{Tr } A^k$ and Y_{ij} . $\square$

Note that the Hamilton flow given by $H(Y)$ preserves matrix element functions of the group commutator $\pi_V(XYX^{-1}Y^{-1})$.

Theorem 34. *Let Σ be an oriented surface with $b > 0$ boundary components and $C = C_1 \sqcup \dots \sqcup C_k$ be a disjoint union of pairwise nonintersecting simple closed curves in Σ as in Theorem 17, $H : G \rightarrow \mathbb{C}$ be a central function and $\pi_V : G \rightarrow \text{End}(V)$ be a finite dimensional representation. Fix a choice of marked points $p_1, \dots, p_m \in \partial\Sigma$ and generators of $\pi_1(\Sigma, p_1, \dots, p_m)$ as in Sect. 4.1 and denote by $A_i, E_i, X_l, Y_l, Z_o \in \text{Hom}(\pi_1(\Sigma, p_1, \dots, p_m), G)$ the holonomies along arcs a_i, e_i, x_l, y_l, z_o , then*

$$\begin{aligned} \{H(A_i), \pi_V(E_j)\} &= \begin{cases} \sum_J (e_J, \nabla H(A_i)) \pi_V(E_J e_J), & i = j, \\ 0, & i \neq j, \end{cases} \\ \{H(A_i), \pi_V(X_l)\} &= \{H(A_i), \pi_V(Y_l)\} = \{H(A_i), \pi_V(Z_o)\} = 0, \\ \{H(Y_l), \pi_V(X_s)\} &= \begin{cases} \sum_J (e_J, \nabla H(Y_l)) \pi_V(X_s e_J), & l = s, \\ 0, & l \neq s, \end{cases} \\ \{H(Y_l), \pi_V(A_i)\} &= \{H(Y_l), \pi_V(E_i)\} = \{H(Y_l), \pi_V(Z_o)\} = 0. \end{aligned}$$

Particular cases of Theorem 34 already appeared in the literature. For example, these brackets were computed in [FR99] for $G = SL(N, \mathbb{C})$ and a torus with one boundary component.

The following Corollary generalizes classical result from §4 of [Gol86] to the case of surfaces with several marked points on different boundary components and arbitrary number of separating cycles:

Corollary 35. *Fix $i, 1 \leq i \leq r$ and a central function $H : G \rightarrow \mathbb{C}$. Then the Hamiltonian flow generated by the function H evaluated on a cycle $C_i \in \Sigma$ is given by the following expressions*

- When $C_i \in \partial\Sigma_a \cap \partial\Sigma_b$ separates a pair of distinct connected components of $\Sigma \setminus C$ we have

$$\begin{aligned} E_j(t) &= \begin{cases} E_j \exp(t \nabla H(A_i)), & i = j, \\ E_j, & i \neq j, \end{cases} \\ X_l(t) &= X_l, \quad Y_l(t) = Y_l, \\ A_j(t) &= A_j, \quad Z_o(t) = Z_o. \end{aligned}$$

- Similarly, when C_l is a nonseparating cycle which belongs to the boundary of a single connected component of $\Sigma \setminus C$ the Hamiltonian flow generated by H evaluated at the holonomy along C_l is given by

$$\begin{aligned} X_s(t) &= \begin{cases} X_s \exp(t \nabla H(Y_l)), & l = s, \\ X_s, & l \neq s, \end{cases} \\ A_j(t) &= A_j, \quad E_j(t) = E_j \\ Y_l(t) &= Y_l, \quad Z_o(t) = Z_o. \end{aligned}$$

Remark 36. Let ψ_t is the evolution on the algebra of functions on $\text{Hom}(\pi_1(\Sigma, p_1, \dots, p_m), G)$ generated by H , i.e.

$$\psi_t^H(F) = F + \sum_{n \geq 1} \frac{t^n}{n!} \{H, \{H, \dots, \{H, F\} \dots\}\}.$$

If the functions H_1 and H_2 Poisson commute then $\psi_{t_1}^{H_1} \psi_{t_2}^{H_2} = \psi_{t_2}^{H_2} \psi_{t_1}^{H_1}$. If $H_1, \dots, H_k$ is a complete set of independent functions on holonomies along C , their joint flow lines generate angle variables on their level surfaces.

Remark 37. It is worth noting that Theorem 34 along with Corollary 35 can be used for computations of Hamilton flows on *decorated* moduli spaces, namely on the quotient of $\text{Hom}(\pi_1(\Sigma, p_1, \dots, p_m), G)$ by a proper subgroup of $G^{\times m}$. Examples of decorated character varieties include monodromy data of Painlevé equations [CMR17, Boa18].

5. Particular Cases

Here we will describe Poisson algebras $\mathcal{O}[\mathcal{M}_\Sigma^G]$ and corresponding superintegrable systems explicitly in few low dimensional examples when Σ is a torus with one or two boundary components. Writing out examples inevitably brings us to the problem of an explicit description of the invariant ring $\mathcal{O}[\mathcal{M}_\Sigma^G]$. For a simple linear algebraic group G such invariant rings are always finitely generated, yet the number of generators grows much faster than the dimension of the corresponding moduli space. Here we describe subalgebras $Z_{\partial\Sigma} \subset B_C \subset J_{\Sigma \setminus C} \subset \mathcal{O}[\mathcal{M}_\Sigma^G]$ and their Poisson brackets internally, in terms of generators of the invariant ring $\mathcal{O}[\mathcal{M}_\Sigma^G]$ for some simple examples.

5.1. Torus with one boundary component, $G = SL(2, \mathbb{C})$. Let $\Sigma_{1,1}$ denote the torus with one boundary component. Choose a marked point $p_1 \in \partial\Sigma_{1,1}$. The fundamental group $\pi_1(\Sigma_{1,1}, p_1)$ is freely generated by a pair of arcs X, Y along the equator and meridian of the torus. Let Γ be the ribbon graph associated to such choice of generators. The data of Γ is encoded by an ordered set

$$S_1 = \{X^{-1}, Y, X, Y^{-1}\} \quad (65)$$

of half-edges adjacent to p_1 as shown on Fig. 13a. In (65) we have labelled half-edges by generators X, Y of $\pi_1(\Sigma_{1,1}, p_1)$ and their inverses X^{-1}, Y^{-1} according to the convention introduced in the last paragraph of Sect. 4.1. Namely, we label an outgoing half-edge by the first power of the corresponding generator, while we label an ingoing edge by the inverse of the corresponding generator.

Moduli space The moduli space of flat $SL(2, \mathbb{C})$ -connections on a once punctured torus has dimension $n = 3$. We will use notation

$$\tau_A = \text{Tr}(A)$$

for the trace of a matrix A .

The coordinate ring of the moduli space in this example is a free commutative algebra with three generators

$$\mathcal{O}[\mathcal{M}_{\Sigma_{1,1}}^{SL(2, \mathbb{C})}] = \mathbb{C}[\tau_X, \tau_Y, \tau_{XY}]$$

where $X, Y \in SL(2, \mathbb{C})$ are elements representing standard a and b cycles. The Poisson bracket between these coordinate functions are:

$$\begin{aligned}\{\tau_X, \tau_Y\} &= -\tau_{XY} + \frac{1}{2}\tau_X\tau_Y, \\ \{\tau_X, \tau_{XY}\} &= \tau_Y - \frac{1}{2}\tau_X\tau_{XY}, \\ \{\tau_Y, \tau_{XY}\} &= -\tau_X + \frac{1}{2}\tau_Y\tau_{XY}.\end{aligned}$$

This is a rank 2 Poisson structure. The center of $\mathcal{O}[\mathcal{M}_{\Sigma_{1,1}}^{SL(2,\mathbb{C})}]$ is generated by a single Casimir function

$$Z = \mathbb{C}[z], \quad z = \tau_X^2 + \tau_Y^2 + \tau_{XY}^2 - \tau_X\tau_Y\tau_{XY} = \text{Tr}(XYX^{-1}Y^{-1}) \quad (66)$$

Superintegrable system Choose cycle C defining the system to be Y and the Hamiltonian $H = \tau_Y$. From (29) we obtain a chain of subgroups

$$\pi_1(C) = \langle XYX^{-1}Y^{-1} \rangle \subset \pi_1(\Sigma_{1,1} \setminus C, p) = \langle Y, XYX^{-1} \rangle \subset \pi_1(\Sigma_{1,1}, p) = \langle X, Y \rangle.$$

Here and below we write $\langle a_1, \dots, a_n \rangle$ for a free group generated by $a_1, \dots, a_n$.

The algebra of first integrals is then generated by three elements, two of which coincide

$$J = \mathcal{O}[\text{Hom}(\pi_1(\Sigma_{1,1} \setminus C, p))] = \mathbb{C}[\tau_{XYX^{-1}}, \tau_{Y^{-1}}, \tau_{XYX^{-1}Y^{-1}}] = \mathbb{C}[z, H].$$

As a result, we obtain the following chain of subalgebras of the coordinate ring

$$Z \subset B = \mathbb{C}[z, H] \subset J = \mathbb{C}[z, H] \subset \mathcal{O}[\mathcal{M}_{\Sigma_{1,1}}^{SL(2,\mathbb{C})}]. \quad (67)$$

In this case the algebra of Hamiltonians coincide with the algebra of first integrals and therefore the system is Liouville integrable.

Mapping class group action Consider the mapping class group $Mod(\Sigma_{1,1,0})$ (relative to the boundary) of the torus with one boundary component and no punctures. It contains two left Dehn twists along the X and Y cycles which satisfy the braid relation and are acting on our generators as follows:

$$D_X : \begin{cases} X \mapsto X, \\ Y \mapsto YX^{-1}, \end{cases} \quad D_Y : \begin{cases} X \mapsto XY, \\ Y \mapsto Y, \end{cases} \quad D_X D_Y D_X = D_Y D_X D_Y = \begin{cases} X \mapsto XYX^{-1}, \\ Y \mapsto X^{-1}. \end{cases}$$

This defines a pair of Poisson automorphisms $\mathcal{D}_X, \mathcal{D}_Y$ of the coordinate ring $\mathcal{O}[\mathcal{M}_{\Sigma_{1,1}}^{SL(2,\mathbb{C})}]$

$$\mathcal{D}_X = \begin{cases} \tau_X \mapsto \tau_X, \\ \tau_Y \mapsto -\tau_{XY} + \tau_X\tau_Y, \\ \tau_{XY} \mapsto \tau_Y, \end{cases} \quad \mathcal{D}_Y = \begin{cases} \tau_X \mapsto \tau_{XY}, \\ \tau_Y \mapsto \tau_Y, \\ \tau_{XY} \mapsto -\tau_X + \tau_Y\tau_{XY}, \end{cases} \quad (68)$$

Note that action of $Mod(\Sigma_{1,1,0})$ on the character variety factors through the action of the Mapping Class Group of a torus with one puncture $Mod(\Sigma_{1,0,1}) \simeq SL(2, \mathbb{Z})$.

Poisson automorphisms (68) define a family of isomorphic integrable systems associated to nonseparating cycles on $\Sigma_{1,1}$.

Table 1. Brackets between generators of $\mathcal{O}[\mathcal{M}_{\Sigma_{1,2}}^{SL(2)}]$

τ_X	τ_Y	τ_{XY}
τ_X 0	*	*
τ_Y $\frac{\tau_X \tau_Y}{2} - \tau_{XY}$	0	*
τ_{XY} $\tau_Y - \frac{1}{2} \tau_X \tau_{XY}$	$\frac{1}{2} \tau_Y \tau_{XY} - \tau_X$	0
τ_{XZ} 0	$\tau_{XYZ} - \frac{1}{2} \tau_Y \tau_{XZ} - \tau_Y \tau_Z + \frac{1}{2} \tau_{XY} \tau_{XZ} + \tau_{YZ}$	
τ_{YZ} $\tau_X \tau_Y \tau_Z - \tau_{XY} \tau_Z - \tau_Y \tau_{XZ} - \frac{1}{2} \tau_X \tau_{YZ} + \tau_{XYZ}$	0	$\tau_X \tau_Z - \tau_{XZ} - \frac{1}{2} \tau_{XY} \tau_{YZ}$
$\tau_{XYZ} \tau_Y \tau_Z - \tau_{XY} \tau_{XZ} - \tau_{YZ} + \frac{1}{2} \tau_X \tau_{XYZ}$	$\frac{1}{2} \tau_Y \tau_{XYZ} - \tau_{XZ}$	0
τ_{XZ}	τ_{YZ}	
τ_{YZ} $\tau_{XY} + \frac{1}{2} \tau_{XZ} \tau_{YZ} - \tau_Z \tau_{XYZ}$	0	
$\tau_{XYZ} \tau_Y - \frac{1}{2} \tau_{XZ} \tau_{XYZ}$	$-\tau_X + \tau_Y \tau_{XY} - \tau_Y \tau_Z \tau_{XYZ} + \frac{1}{2} \tau_{YZ} \tau_{XYZ} + \tau_Z \tau_{XZ}$	

5.2. *Torus with two boundary components*, $G = SL(2, \mathbb{C})$. The coordinate ring of the moduli space has dimension $n = 6$ and the coordinate ring is generated by 7 polynomials subject to the single relation

$$\mathcal{O}[\mathcal{M}_{\Sigma_{1,2}}^{SL(2, \mathbb{C})}] = \frac{\mathbb{C}[\tau_X, \tau_Y, \tau_Z, \tau_{XY}, \tau_{XZ}, \tau_{YZ}, \tau_{XYZ}]}{(\tau_{XYZ}^2 + \mu_1 \tau_{XYZ} + \mu_0)}.$$

Here τ_A stands for $\text{Tr}(A)$ and

$$\begin{aligned} \mu_1 &= \tau_Z \tau_{XY} + \tau_Y \tau_{XZ} + \tau_X \tau_{YZ} - \tau_X \tau_Y \tau_Z, \\ \mu_0 &= \tau_X \tau_Y \tau_{XY} + \tau_Y \tau_Z \tau_{YZ} + \tau_X \tau_Z \tau_{XZ} - \tau_{XY} \tau_{XZ} \tau_{YZ} \\ &\quad - \tau_{XY}^2 - \tau_{XZ}^2 - \tau_{YZ}^2 - \tau_X^2 - \tau_Y^2 - \tau_Z^2 + 4. \end{aligned}$$

Poisson brackets between generators are summarized in Table 1, where we have omitted generator τ_Z which belongs to the Poisson center of $\mathcal{O}[\mathcal{M}_{\Sigma_{1,2}}^{SL(2, \mathbb{C})}]$.

The Poisson center Z of the coordinate ring has Krull dimension $\dim Z = 2$ and has two algebraically independent Casimir elements corresponding to traces of monodromies around each of the boundary components

$$z_1 = \tau_Z, \quad z_2 = \text{Tr}(XYX^{-1}Y^{-1}Z) = -\tau_Y \tau_{XY} \tau_{XZ} + \tau_{XY} \tau_{XYZ} + \tau_X \tau_{XZ} - \tau_Y \tau_{YZ} + \tau_Y^2 \tau_Z - \tau_Z$$

5.2.1. *A superintegrable system associated to nonseparating cycle* Consider a superintegrable system given by a nonseparating cycle homotopic to Y , from (29) we get a chain of subgroups

$$\pi_1(C) = \langle Y \rangle \subset \pi_1(\Sigma_{1,2}, p | C) = \langle Y, XYX^{-1}, Z \rangle \subset \pi_1(\Sigma_{1,2}, p) = \langle X, Y, Z \rangle \quad (69)$$

Coordinate ring of the character variety of the smallest subgroup is generated by a single hamiltonian $H = \tau_Y$. At the same time, the coordinate ring of the character variety of

the middle subgroup in (69) has the following presentation¹⁷

$$J = \mathcal{O}[\text{Hom}(\pi_1(\Sigma_{1,1}, p | C), SL(2))]^{SL(2)} \\ = \frac{\mathbb{C}[\tau_Y, \tau_Z, \tau_{YZ}, \tau_{XYX^{-1}Z}, \tau_{YXYX^{-1}}, \tau_{XYX^{-1}YZ}]}{(\tau_{XYX^{-1}YZ}^2 + \lambda_1 \tau_{XYX^{-1}YZ} + \lambda_0)}, \quad (70)$$

where

$$\lambda_1 = \tau_Y \tau_{XYX^{-1}Z} + \tau_Z \tau_{YXYX^{-1}} + \tau_Y \tau_{YZ} - \tau_Y^2 \tau_Z, \\ \lambda_0 = \tau_Y^2 \tau_{YXYX^{-1}} + \tau_Y \tau_Z \tau_{XYX^{-1}Z} - \tau_{YZ} \tau_{YXYX^{-1}} \tau_{XYX^{-1}Z} + \tau_Y \tau_Z \tau_{YZ} \\ - \tau_{YXYX^{-1}}^2 - \tau_{XYX^{-1}Z}^2 - \tau_{YZ}^2 - 2\tau_Y^2 - \tau_Z^2.$$

Subalgebra $J \subset \mathcal{O}[\mathcal{M}_{\Sigma_{1,2}}^{SL(2)}]$ is closed under the Poisson bracket. Explicitly, all nonzero brackets between generators are given by

$$\{\tau_{XYX^{-1}Z}, \tau_{YXYX^{-1}}\} = \tau_Y \tau_Z - 2\tau_{YZ} - \tau_{XYX^{-1}Z} \tau_{YXYX^{-1}} + \tau_Y \tau_{XYX^{-1}YZ} \quad (71a)$$

$$\{\tau_{YZ}, \tau_{YXYX^{-1}}\} = \tau_Z \tau_Y^3 - \tau_{YZ} \tau_Y^2 - \tau_{XYX^{-1}Z} \tau_Y^2 - \tau_Z \tau_Y - \tau_Z \tau_{YXYX^{-1}} \tau_Y \quad (71b)$$

$$+ \tau_{XYX^{-1}YZ} \tau_Y + 2\tau_{XYX^{-1}Z} + \tau_{YZ} \tau_{YXYX^{-1}}$$

$$\{\tau_{YZ}, \tau_{XYX^{-1}Z}\} = \tau_Y^2 - \tau_{YZ} \tau_{XYX^{-1}Z} - 2\tau_{YXYX^{-1}} + \tau_Z \tau_{XYX^{-1}YZ} \quad (71c)$$

$$\{\tau_{XYX^{-1}YZ}, \tau_{YXYX^{-1}}\} = \tau_Z \tau_Y^2 + \tau_{XYX^{-1}YZ} \tau_Y^2 - 2\tau_{YZ} \tau_Y - \tau_{XYX^{-1}Z} \tau_{YXYX^{-1}} \tau_Y \quad (71d)$$

$$\{\tau_{XYX^{-1}YZ}, \tau_{YZ}\} = -\tau_Y^3 + \tau_{YZ} \tau_{XYX^{-1}Z} \tau_Y + 2\tau_{YXYX^{-1}} \tau_Y - \tau_Z \tau_{XYX^{-1}YZ} \tau_Y \quad (71e)$$

The algebra of first integrals (70) has Krull dimension 5. One can choose its maximal algebraically independent subset to contain two Casimir functions

$$z_1 = \tau_Z, \quad z_2 = \tau_Y \tau_{XYX^{-1}Z} - \tau_{XYX^{-1}YZ},$$

one Hamiltonian

$$H = \tau_Y,$$

and two more first integrals

$$g_1 = \tau_{YZ} = \tau_{YZ}, \quad g_2 = \tau_{XYX^{-1}Y} = \tau_{XY}^2 + \tau_X \tau_Y \tau_{XY} - \tau_X^2 + 2.$$

¹⁷ Finding a generating set together with transcendence basis for the corresponding field of fractions is always straightforward in the case of all rings involved in our paper. However, it is much more complicated to find a complete set of relations outside of the most elementary examples.

In this particular case, one can recall that J is isomorphic to a quotient ring of $\mathcal{O}[M_{\Sigma_{0,4}}^{SL(2)}]$ modulo the relation that traces of monodromies around the fixed pair of boundary components are equal to each other. This allows one to resolve one of the generators in $\mathcal{O}[M_{\Sigma_{0,4}}^{SL(2)}]$ and obtain presentation (70).

5.2.2. Separating cycle Now consider a superintegrable system given by a separating cycle homotopic to $XYX^{-1}Y^{-1}$. As in the previous case we have two Casimir elements z_1 and z_2 . Their generic level sets form a 2-parametric family of 4-dimensional symplectic manifolds. Choose the Hamiltonian on these phase spaces as

$$H = \text{Tr}(XYX^{-1}Y^{-1}) = \tau_X^2 + \tau_Y^2 + \tau_{XY}^2 - \tau_X \tau_Y \tau_{XY} - 2.$$

From (51) we have the following algebra of first integrals

$$\begin{aligned} J &= J_L \otimes_B J_R, & B &= \mathbb{C}[H] \simeq \mathcal{O}[SL(2)]^{SL(2)}, \\ J_L &= \mathbb{C}[\tau_X, \tau_Y, \tau_{XY}] \simeq \mathcal{O}[\mathcal{M}_{\Sigma_{1,1}}^{SL(2)}], & J_R &= \mathbb{C}[z_1, z_2, H] \simeq \mathcal{O}[\mathcal{M}_{\Sigma_{0,3}}^{SL(2)}]. \end{aligned}$$

Here J_R is Poisson commutative, while J_L is equipped with a Poisson bracket of generic rank 2 (see first three rows of Table 1). Algebra J has Krull dimension 5. An example of maximal algebraically independent subset is $z_1, z_2, H, g_1 = \tau_X, g_2 = \tau_Y$, i.e. generators of the Poisson center, the Hamiltonian and two additional first integrals.

5.3. Torus with one boundary component, $G = SL(3, \mathbb{C})$. The moduli space of flat $SL(3, \mathbb{C})$ -connections on a once punctured torus has dimension $n = 8$, the coordinate ring is generated by 9 polynomials subject to a single relation

$$\mathcal{O}[\mathcal{M}_{\Sigma_{1,1}}^{SL(3)}] = \frac{\mathbb{C}[\tau_X, \tau_Y, \tau_{X^{-1}}, \tau_{Y^{-1}}, \tau_{YX}, \tau_{YX^{-1}}, \tau_{Y^{-1}X}, \tau_{Y^{-1}X^{-1}}, \tau_{XYX^{-1}Y^{-1}}]}{(\tau_{XYX^{-1}Y^{-1}}^2 + \zeta_1 \tau_{XYX^{-1}Y^{-1}} + \zeta_0)}, \quad (72)$$

where ζ_1 and ζ_0 are polynomials in $\tau_X, \dots, \tau_{X^{-1}Y^{-1}}$:

$$\begin{aligned} \zeta_1 &= \text{Tr}(YXY^{-1}X^{-1}) \\ &= \tau_X \tau_{X^{-1}} + \tau_X \tau_Y \tau_{X^{-1}} \tau_{Y^{-1}} - \tau_{X^{-1}} \tau_{Y^{-1}} \tau_{YX} - \tau_Y \tau_{X^{-1}} \tau_{Y^{-1}X} - \tau_X \tau_{Y^{-1}} \tau_{YX^{-1}} \\ &\quad + \tau_{YX^{-1}} \tau_{Y^{-1}X} - \tau_X \tau_Y \tau_{Y^{-1}X^{-1}} + \tau_{YX} \tau_{Y^{-1}X^{-1}} - \tau_{XYX^{-1}Y^{-1}} + \tau_Y \tau_{Y^{-1}} - 3. \end{aligned} \quad (73)$$

and

$$\begin{aligned} \zeta_0 &= \tau_Y \tau_{Y^{-1}} \tau_X^3 - \tau_X^3 - \tau_Y \tau_{X^{-1}} \tau_{Y^{-1}} \tau_X^2 - \tau_Y^2 \tau_{YX} \tau_X^2 - \tau_{Y^{-1}} \tau_{YX} \tau_X^2 + \tau_{X^{-1}} \tau_{Y^{-1}} \tau_{YX^{-1}} \tau_X^2 \\ &\quad - \tau_{Y^{-1}}^2 \tau_{Y^{-1}X} \tau_X^2 - \tau_Y \tau_{Y^{-1}X} \tau_X^2 + \tau_Y \tau_{X^{-1}} \tau_{Y^{-1}X^{-1}} \tau_X^2 - \tau_{YX^{-1}} \tau_{Y^{-1}X^{-1}} \tau_X^2 + \tau_{X^{-1}} \tau_{Y^{-1}}^3 \tau_X \\ &\quad - \tau_Y^2 \tau_{X^{-1}} \tau_{Y^{-1}}^2 \tau_X + 2\tau_Y \tau_{YX}^2 \tau_X - \tau_{YX} \tau_{YX^{-1}}^2 \tau_X + 2\tau_{Y^{-1}} \tau_{Y^{-1}X}^2 \tau_X - \tau_{Y^{-1}X} \tau_{Y^{-1}}^2 \tau_{X^{-1}} \tau_X \\ &\quad + \tau_Y^3 \tau_{X^{-1}} \tau_X + 6\tau_{X^{-1}} \tau_X - \tau_Y \tau_{X^{-1}} \tau_{Y^{-1}} \tau_X + \tau_{X^{-1}}^2 \tau_{Y^{-1}} \tau_{YX} \tau_X - \tau_Y^2 \tau_{YX^{-1}} \tau_X \\ &\quad + \tau_Y \tau_{Y^{-1}}^2 \tau_{YX^{-1}} \tau_X - 3\tau_{Y^{-1}} \tau_{YX^{-1}} \tau_X + \tau_Y \tau_{X^{-1}} \tau_{YX} \tau_{YX^{-1}} \tau_X + \tau_Y \tau_{X^{-1}}^2 \tau_{Y^{-1}X} \tau_X \\ &\quad + \tau_Y \tau_{Y^{-1}} \tau_{YX} \tau_{Y^{-1}X} \tau_X + 3\tau_{YX} \tau_{Y^{-1}X} \tau_X - \tau_{X^{-1}} \tau_{YX^{-1}} \tau_{Y^{-1}X} \tau_X - \tau_{Y^{-1}}^2 \tau_{Y^{-1}X^{-1}} \tau_X \\ &\quad - 3\tau_Y \tau_{Y^{-1}X^{-1}} \tau_X + \tau_Y^2 \tau_{Y^{-1}} \tau_{Y^{-1}X^{-1}} \tau_X - \tau_{X^{-1}} \tau_{YX} \tau_{Y^{-1}X^{-1}} \tau_X + \tau_{X^{-1}} \tau_{Y^{-1}} \tau_{Y^{-1}X} \tau_{Y^{-1}X^{-1}} \tau_X \\ &\quad - \tau_Y^3 - \tau_{X^{-1}}^3 - \tau_{Y^{-1}}^3 - \tau_{YX}^3 - \tau_{YX^{-1}}^3 - \tau_{Y^{-1}X}^3 - \tau_{Y^{-1}X^{-1}}^3 + 2\tau_Y \tau_{X^{-1}} \tau_{YX^{-1}}^2 - \tau_Y \tau_{YX} \tau_{Y^{-1}X}^2 \\ &\quad + 2\tau_{X^{-1}} \tau_{Y^{-1}} \tau_{Y^{-1}X^{-1}}^2 - \tau_Y \tau_{YX^{-1}} \tau_{Y^{-1}X^{-1}}^2 + \tau_Y \tau_{X^{-1}}^3 \tau_{Y^{-1}} + 6\tau_Y \tau_{Y^{-1}} + \tau_Y \tau_{X^{-1}} \tau_{Y^{-1}}^2 \tau_{YX} \\ &\quad - \tau_Y^2 \tau_{X^{-1}} \tau_{YX} - 3\tau_{X^{-1}} \tau_{Y^{-1}} \tau_{YX} - \tau_Y^2 \tau_{X^{-1}}^2 \tau_{YX^{-1}} - \tau_{X^{-1}} \tau_{YX}^2 \tau_{YX^{-1}} - \tau_{X^{-1}}^2 \tau_{Y^{-1}} \tau_{YX^{-1}} \\ &\quad - \tau_{Y^{-1}}^2 \tau_{YX} \tau_{YX^{-1}} + 3\tau_Y \tau_{YX} \tau_{YX^{-1}} - \tau_{X^{-1}} \tau_{Y^{-1}}^2 \tau_{Y^{-1}X} - \tau_{Y^{-1}} \tau_Y^2 \tau_{Y^{-1}X} - 3\tau_Y \tau_{X^{-1}} \tau_{Y^{-1}X} \\ &\quad + \tau_Y^2 \tau_{X^{-1}} \tau_{Y^{-1}} \tau_{Y^{-1}X} - \tau_{X^{-1}}^2 \tau_{YX} \tau_{Y^{-1}X} - \tau_Y \tau_{Y^{-1}} \tau_{YX^{-1}} \tau_{Y^{-1}X} + 6\tau_{YX^{-1}} \tau_{Y^{-1}X} \\ &\quad - \tau_Y \tau_{X^{-1}}^2 \tau_{Y^{-1}X^{-1}} - \tau_{X^{-1}}^2 \tau_{Y^{-1}}^2 \tau_{Y^{-1}X^{-1}} - \tau_{Y^{-1}} \tau_{YX^{-1}}^2 \tau_{Y^{-1}X^{-1}} - \tau_{X^{-1}} \tau_{Y^{-1}}^2 \tau_{Y^{-1}X^{-1}} \end{aligned}$$

$$\begin{aligned} & -\tau_Y\tau_{Y^{-1}}\tau_{YX}\tau_{Y^{-1}X^{-1}}+6\tau_{YX}\tau_{Y^{-1}X^{-1}}+3\tau_{X^{-1}}\tau_{YX^{-1}}\tau_{Y^{-1}X^{-1}}+\tau_Y\tau_{X^{-1}}\tau_{Y^{-1}}\tau_{YX^{-1}}\tau_{Y^{-1}X^{-1}} \\ & -\tau_Y^2\tau_{Y^{-1}X}\tau_{Y^{-1}X^{-1}}+3\tau_{Y^{-1}X}\tau_{Y^{-1}X^{-1}}-\tau_{YX}\tau_{YX^{-1}}\tau_{Y^{-1}X}\tau_{Y^{-1}X^{-1}}-9 \end{aligned}$$

For a proof, see for example Lemma 5 in [Law07].

Poisson brackets between generators are summarized in Table 2, where we have omitted generator $\tau_{XYX^{-1}Y^{-1}}$ because it belongs to the Poisson center of $\mathcal{O}[\mathcal{M}_{\Sigma_{1,1}}^{SL(3,\mathbb{C})}]$, see [Law09].

Coordinate ring $\mathcal{O}[\mathcal{M}_{\Sigma_{1,1}}^{SL(3)}]$ has a Poisson center Z of Krull dimension $\dim Z = 2$. Two algebraically independent Casimir elements: $z_1 = \tau_{YXY^{-1}X^{-1}}$ and $z_2 = \tau_{XYX^{-1}Y^{-1}}$. The formula expressing z_1 in terms of generators is given in (73)

5.3.1. A superintegrable system Here we consider a superintegrable system defined by a single cycle homotopic to X . By (29) we have a chain of inclusions of subgroups of the fundamental group

$$\pi_1(C) = \langle X \rangle \subset \pi_1(\Sigma_{1,1}, p | C) = \langle X, YXY^{-1} \rangle \subset \pi_1(\Sigma_{1,1}, p) = \langle X, Y \rangle. \quad (74)$$

Choose two independent Hamiltonians in $\mathcal{O}[\text{Hom}(\pi_1(C), SL(3))]^{SL(3)}$ as

$$H_1 = \tau_X, \quad H_2 = \tau_{X^{-1}}$$

These two Poisson commuting Hamiltonians define a superintegrable system with the Poisson algebra of first integrals J defined above. We can choose a maximal algebraically independent subset in J to be z_1, z_2, H_1, H_2 with two more integrals

$$\begin{aligned} g_1 &= \text{Tr}(YXY^{-1}X) = \tau_{X^{-1}} - \tau_Y\tau_{X^{-1}}\tau_{Y^{-1}} + \tau_{Y^{-1}}\tau_{YX^{-1}} + \tau_{YX}\tau_{Y^{-1}X} + \tau_Y\tau_{Y^{-1}X^{-1}}, \\ g_2 &= \text{Tr}(YX^{-1}Y^{-1}X^{-1}) = \tau_X - \tau_X\tau_Y\tau_{Y^{-1}} + \tau_{Y^{-1}}\tau_{YX} + \tau_Y\tau_{Y^{-1}X} + \tau_{YX^{-1}}\tau_{Y^{-1}X^{-1}}. \end{aligned}$$

6. Conclusion

6.1. The algebra of chord diagrams on a surface and “universal superintegrable systems”. The algebra of chord diagrams is a universal model for the Poisson algebra of functions on moduli spaces of flat connections [AMR96], see also [T91] for the case of loop algebras on a surface, i.e. skein modules. Chord diagrams first appeared in the setting of finite type invariants of knots [V90] and in perturbative Chern-Simons invariants [BN91, K93]. They became an important tool in the related theory of Vassiliev invariants.

Let us recall some basic definitions. A *chord diagram* is a homotopy class of a graph on a surface which consists of solid lines and chords. Solid lines are oriented, cords are not oriented. They satisfy the analogue of Reidemeister relations (see [AMR96] for details) together with the so-called 4T-relation shown on Fig. 14.

The algebra of chord diagrams $Ch(\Sigma)$ on a surface Σ is a vector space spanned by $\mathbb{Z}$ -linear combinations of chord diagrams with

- the commutative multiplication of $[D_1]$ and $[D_2]$ is defined as $[D_1][D_2] = [D_1 \sqcup D_2]$, i.e. as the homotopy class of their disjoint union. T

Table 2. Brackets between generators of $\mathcal{O}[\mathcal{M}_{\Sigma_{1,1}}^{SL(3)}]$

	τ_X	τ_Y	$\tau_{X^{-1}}$
τ_X	0	*	*
τ_Y	$\frac{\tau_X \tau_Y}{3} - \tau_Y X$	0	*
$\tau_{X^{-1}}$	0	$\frac{1}{3} \tau_Y \tau_{X^{-1}} - \tau_Y X^{-1}$	0
$\tau_{Y^{-1}}$	$\tau_{Y^{-1}} X - \frac{1}{3} \tau_X \tau_{Y^{-1}}$	0	$\frac{1}{3} \tau_{X^{-1}} \tau_{Y^{-1}} - \tau_{Y^{-1}} X^{-1}$
τ_{YX}	$\tau_Y \tau_{X^{-1}} - \frac{2}{3} \tau_X \tau_Y X - \tau_Y X^{-1}$	$-\tau_X \tau_{Y^{-1}} + \frac{2}{3} \tau_Y \tau_Y X + \tau_{Y^{-1}} X$	$\tau_Y - \frac{1}{3} \tau_{X^{-1}} \tau_Y X$
$\tau_{YX^{-1}}$	$\frac{2}{3} \tau_X \tau_Y X^{-1} - \tau_Y$	$\tau_{X^{-1}} \tau_{Y^{-1}} - \frac{2}{3} \tau_Y \tau_Y X^{-1} - \tau_{Y^{-1}} X^{-1}$	$-\tau_X \tau_Y + \tau_Y X + \frac{2}{3} \tau_{X^{-1}} \tau_Y X^{-1}$
$\tau_{Y^{-1}X}$	$-\tau_{X^{-1}} \tau_{Y^{-1}} + \frac{2}{3} \tau_X \tau_{Y^{-1}} X + \tau_{Y^{-1}} X^{-1}$	$\tau_X - \frac{2}{3} \tau_Y \tau_{Y^{-1}} X$	$\frac{1}{3} \tau_{X^{-1}} \tau_{Y^{-1}} X - \tau_{Y^{-1}}$
$\tau_{Y^{-1}X^{-1}}$	$\tau_{Y^{-1}} - \frac{2}{3} \tau_X \tau_{Y^{-1}} X^{-1}$	$\frac{1}{3} \tau_Y \tau_{Y^{-1}} X^{-1} - \tau_{X^{-1}}$	$\tau_X \tau_{Y^{-1}} - \tau_{Y^{-1}} X - \frac{2}{3} \tau_{X^{-1}} \tau_{Y^{-1}} X^{-1}$
τ_{YX}	$\tau_{Y^{-1}}$	$\tau_Y X$	
$\tau_{YX^{-1}}$	$\frac{2}{3} \tau_{Y^{-1}} \tau_Y X - \tau_X$	0	
$\tau_{Y^{-1}X}$	$\tau_{X^{-1}} - \frac{1}{3} \tau_{Y^{-1}} \tau_Y X^{-1}$	$-\tau_Y^2 + \tau_X \tau_{X^{-1}} \tau_{Y^{-1}} + \tau_{Y^{-1}} - \frac{1}{3} \tau_Y X \tau_Y X^{-1} - \tau_{X^{-1}} \tau_{Y^{-1}} X - \tau_X \tau_{Y^{-1}} X^{-1}$	
$\tau_{Y^{-1}X^{-1}}$	$\tau_X \tau_Y - \tau_Y X - \frac{2}{3} \tau_{Y^{-1}} \tau_{Y^{-1}} X$	$-\tau_{X^{-1}} - \tau_Y \tau_{X^{-1}} \tau_{Y^{-1}} + \tau_{Y^{-1}} \tau_Y X^{-1} + \frac{2}{3} \tau_Y X \tau_{Y^{-1}} X + \tau_Y \tau_{Y^{-1}} X^{-1} + \tau_X^2$	
	$-\tau_Y \tau_{X^{-1}} + \tau_Y X^{-1} + \frac{2}{3} \tau_{Y^{-1}} \tau_{Y^{-1}} X^{-1}$	0	
	$\tau_Y X^{-1}$	$\tau_{Y^{-1}} X$	$\tau_{Y^{-1}} X^{-1}$
$\tau_{YX^{-1}}$	0	0	$\tau_{X^{-1}}^2 - \tau_X \tau_Y \tau_{Y^{-1}} + \tau_Y \tau_{Y^{-1}} X + \tau_Y \tau_{Y^{-1}} X^{-1} - \tau_X$
$\tau_{Y^{-1}X}$	*	0	$\tau_X \tau_Y \tau_{X^{-1}} - \tau_{X^{-1}} \tau_Y X - \frac{1}{3} \tau_{Y^{-1}} X \tau_{Y^{-1}} X^{-1} - \tau_Y^2$
$\tau_{Y^{-1}X^{-1}}$	*	*	0

Fig. 14. 4T relation in chord diagrams

- The Poisson bracket is a “universal” version of the Atiyah and Bott Poisson bracket. For two chord diagrams D_1 and D_2 their Poisson bracket is

$$\{[D_1], [D_2]\} = \sum_{p \in D_1 \cap D_2} \sum_{p \in D_1 \cap D_2} \epsilon(p) [D_1 \#_p D_2]$$

where the sum is taken over the intersection points of representatives D_1 and D_2 and $D_1 \#_p D_2$ is the result of the insertion of a chord in a small vicinity of point p . It is clear that the right side does not depend on the choice of representatives and on where the extra cord is inserted.

The Poisson center of $Z(\Sigma) \subset Ch(\Sigma)$ is, conjecturally, generated by chord diagrams which are contractible to the boundary of Σ .

To define a conjecturally superintegrable system on $Ch(\Sigma)$ choose a system of simple curves on the surface. Define the subalgebra of Hamiltonians $B(C)$ and the subalgebra of chord diagrams which are contractible to C and the subalgebra of first integrals $J(C)$ and the subalgebra of chord diagrams which can be separated from C . We have a natural inclusion of Poisson subalgebras:

$$Z(\Sigma) \subset B(C) \otimes Z(\Sigma) \subset J(C) \subset Ch(\Sigma)$$

We conjecture that this chain of inclusions define a superintegrable systems. In this infinite dimensional setting the superintegrability means that the space of all nontrivial Poisson derivations $Der_{J(C)}(Ch(\Sigma))$ of $Ch(\Sigma)$ which act trivially on $J(C)$ is generated as an $Ch(\Sigma)$ -module by Hamilton derivations corresponding to elements from $B(C)$.

6.2. Quantization and the algebra of links in a cylinder. The natural question about any integrable system is how to quantize it. In the case of superintegrable systems on chord diagrams there is a natural quantization.

There is a natural associative algebra that can be naturally associated to a surface. This is the algebra of links, or linked graphs, in $\Sigma \times I$ where I is an interval. This algebra, which we will denote $A(\Sigma)$ is the space of $\mathbb{Z}$ -linear combinations of links in $\Sigma \times I$. The natural associative multiplication on this space is the “placing of one link on the top of the other”:

$$[L_1] * [L_2] = [L_1 \sqcup L_2]$$

where in the right side of this equation we assume that $L_1 \subset \Sigma[1, 1/2]$ and $L_2 \subset \Sigma \times [1/2, 0]$. here $[L]$ is the topological link which the isotopy class of a geometrical link $L : S^{1 \times n} \rightarrow \Sigma \times I$.

The algebra $A(\Sigma)$ has a natural filtration with $Ch(\Sigma)$ being isomorphic to its associated graded algebra. In this sense $A(\Sigma)$ is the quantization of $Ch(\Sigma)$.

Finite dimensional representations of the algebra $A(\Sigma)$ can be easily constructed using [RT90]. Such representations use representation theory of $U_q(\mathfrak{g})$ at roots of unity. We will present details in a separate publication.

For generic q , the quantization can be done by quantizing Fock and Rosly brackets by R -matrices, see for example [AGS95, AS96]. It is worth noting that a closely related question for the extended moduli space with the choice of flags at marked points on the boundary can be studied by means of Cluster Algebras [FG06], see for example [SS17, GSV18, CMR17] and references therein.

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