Spin(7) Instantons and Hermitian Yang–Mills Connections for the Stenzel Metric

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Abstract: We use the highly symmetric Stenzel Calabi–Yau structure on T^*S^4 as a testing ground for the relationship between the Spin(7) instanton and Hermitian–Yang–Mills (HYM) equations. We reduce both problems to tractable ODEs and look for invariant solutions. In the abelian case, we establish local equivalence and prove a global nonexistence result. We analyze the nonabelian equations with structure group $SO(3)$ and construct the moduli space of invariant Spin(7) instantons in this setting. This is comprised of two 1-parameter families—one of them explicit—of irreducible Spin(7) instantons. Each carries a unique HYM connection. We thus negatively resolve the question regarding the equivalence of the two gauge theoretic PDEs. The HYM connections play a role in the compactification of this moduli space, exhibiting a *removable singularity* phenomenon that we aim to further examine in future work.

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1. Introduction

1.1. Summary and main results. The inclusion of $\mathrm{SU}(4)$ in $\mathrm{Spin}(7)$ demonstrates that a Calabi–Yau (CY) fourfold is –in a natural way– a $\mathrm{Spin}(7)$ manifold. The $\mathrm{SU}(4)$ and $\mathrm{Spin}(7)$ structures give rise to associated generalised instanton equations: the *Hermitian Yang–Mills (HYM) equations* and the *$\mathrm{Spin}(7)$ instanton equations* respectively. It is natural to inquire about the relationship of these two gauge theoretic problems. One immediately observes that HYM is a stronger condition. In the compact case, it is known that as long as an HYM connection exists, the two types of instantons coincide ([9], p. 25 Theorem 3.1). Consequently, if one hopes to display a compact counterexample to equivalence, there must not be any HYM connections at all. Furthermore, we have a general existence theorem for HYM connections over stable holomorphic bundles [17]. This restricts the choices of bundles one could look at. Finally, one could attempt to simplify the prohibitively complicated analysis by dimensional reduction. Unfortunately, compact, irreducible special holonomy manifolds admit no continuous symmetries [7]. This precludes the use of symmetry techniques. We are thus motivated to look for a noncompact counterexample. Since Lewis’s argument is essentially an energy estimate, it does not apply to the noncompact setting.

We work over a noncompact CY fourfold: the cotangent bundle of the 4-sphere equipped with the Stenzel metric [16]. We use the cohomogeneity one isometric $\mathrm{SO}(5)$ -action to reduce the instanton equations to tractable ODEs and proceed to classify the $\mathrm{SO}(5)$ -invariant solutions. In Sect. 2, we study the abelian equations. The two problems turn out to be locally equivalent (Theorem 2.1). We classify all solutions (Theorem 2.2) and prove a global nonexistence result (Theorem 2.3). In Sect. 3, we initiate our study of the nonabelian equations corresponding to structure group $\mathrm{SO}(3)$. We classify the relevant cohomogeneity one bundles and their invariant connections and solve the equations on the trivial bundle (Theorem 3.2). On this topology, all nontrivial instantons have holonomy $\mathrm{U}(1)$ and have thus already been encountered in Sect. 2. Section 4 is concerned with the construction of the moduli space of $\mathrm{SO}(5)$ -invariant $\mathrm{Spin}(7)$ instantons with structure group $\mathrm{SO}(3)$ over the Stenzel space. This contains solutions living on two distinct bundles which agree outside of a codimension 4 Cayley submanifold. On one of them, we are able to solve the equations explicitly (Theorem 4.8). We find a

noncompact 1-parameter family of Spin(7) instantons with a unique HYM connection embedded in its interior. This negatively resolves the question regarding the equivalence of the two gauge theoretic PDEs (Theorem 4.9). The equations on the other bundle cannot be integrated explicitly. We study them using dynamical systems analysis. We find a compact 1-parameter family of solutions carrying a unique HYM connection in its interior. This HYM connection is identified as the missing endpoint of the explicit family from before. Attaching it to the noncompact end, we obtain a natural compactification of the moduli space. This *removable singularity* phenomenon suggests that the HYM connections might play a role in the compactification of general Spin(7) instanton moduli spaces (over noncompact CY fourfolds), an issue we intend to further examine in future work.

1.2. The Stenzel manifold. In this section we provide a brief introduction to the Stenzel CY fourfold X^8 . For details, we refer to the articles [13, 16]. The latter carries out the corresponding calculations in complex dimension 3. The overall technique for studying invariant objects in cohomogeneity one is essentially the same as in the article [10].

1.2.1. The underlying manifold and the SO(5)-action The underlying space of X^8 is the cotangent bundle of the 4-sphere. The natural SO(5) action on S^4 extends to T^*S^4 by pullback. The singular orbit is the zero section S^4 . Its stabiliser is the group SO(4). The principal orbits are the positive radius sphere bundles. Their stabiliser is the group SO(3). They are 7-dimensional Stiefel manifolds.

The cotangent bundle of $\mathbb{R}^5$ can be identified with the following space:

$$T^*\mathbb{R}^5 = \left\{ (x, y^T) \mid x, y \in \mathbb{R}^5 \right\}. \quad (1.1)$$

Here, we adopt the convention that -given a choice of basis- elements of a vector space are represented by column vectors, whereas linear forms lying in its dual are represented by row vectors. This allows us to identify the natural duality pairing with matrix multiplication of the coordinate representations.

We obtain a model of T^*S^4 as an embedded submanifold of $T^*\mathbb{R}^5$ by including $S^4 \hookrightarrow \mathbb{R}^5$ as the unit radius round sphere. In this model, the SO(5) action takes the following form:

$$g \begin{pmatrix} x, y^T \end{pmatrix} = \begin{pmatrix} gx, y^T g^{-1} \end{pmatrix}. \quad (1.2)$$

An alternative model can be obtained by embedding T^*S^4 in $\mathbb{C}^5$ as a complex quadric. It thus inherits a natural complex structure. This is done as follows. Consider the degree 2 homogeneous polynomial:

$$F \stackrel{\text{def}}{=} z_1^2 + \cdots + z_5^2.$$

We split the complex coordinates of $\mathbb{C}^5$ into their real and imaginary parts $z_j = x_j + iy_j$ and introduce the functions:

$$\begin{aligned} r^2 &\stackrel{\text{def}}{=} |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2, \\ R_+^2 &\stackrel{\text{def}}{=} x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2, \quad R_-^2 \stackrel{\text{def}}{=} y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2. \end{aligned}$$

Define the map:

$$\begin{aligned} \Psi : \mathbb{C}^5 &\rightarrow T^*\mathbb{R}^5 \\ (z_1, \dots, z_5) &\mapsto \left(\frac{x}{R_+}, y^\top \right). \end{aligned} \quad (1.3)$$

It may be easily seen that this restricts to a diffeomorphism:

$$\Psi : F^{-1}(1) \xrightarrow{\sim} T^*S^4. \quad (1.4)$$

The following relations hold over the level set $X^8 \cong F^{-1}(1) \subset \mathbb{C}^5$:

$$R_+^2 = \frac{r^2 + 1}{2}, \quad R_-^2 = \frac{r^2 - 1}{2}, \quad r^2 = R_+^2 + R_-^2.$$

The minimum value of r on X^8 is $r = 1$ and the associated level set corresponds to the singular orbit. The latter sits inside X^8 as an embedded totally real submanifold [14]. We will denote the principal orbit at radius $r > 1$ as $\mathcal{O}_r$. We define the reference points:

$$p_r \stackrel{\text{def}}{=} (R_+, iR_-, 0, 0, 0)^\top \in \mathcal{O}_r, \quad p_1 \stackrel{\text{def}}{=} (1, 0, 0, 0, 0)^\top \in S^4. \quad (1.5)$$

They form a ray from $p_1 \in S^4$ to infinity. This choice fixes the embeddings of the principal and singular stabiliser groups in $\text{SO}(5)$. They are the copies of $\text{SO}(4)$ and $\text{SO}(3)$ occupying the lower right 4×4 and 3×3 blocks respectively. Furthermore, the choice (1.5) induces a projection map exhibiting $\mathcal{O}_r$ as a coset manifold:

$$\begin{aligned} \pi : \text{SO}(5) &\rightarrow \mathcal{O}_r, \\ g &\mapsto gp_r. \end{aligned} \quad (1.6)$$

The complement of the singular orbit splits as:

$$T^*S^4 \setminus S_4 \cong (0, \infty) \times \frac{\text{SO}(5)}{\text{SO}(3)}. \quad (1.7)$$

We study the adjoint action of $\text{SO}(5)$ to obtain a natural frame on $T_{p_r}\mathcal{O}_r$. The Lie algebra $\mathfrak{so}(5)$ consists of all 5×5 antisymmetric matrices under the commutator bracket. It is given by:

$$\mathfrak{so}(5) = \text{Span} \{ C_{ij} \mid 1 \leq i < j \leq 5 \},$$

where $C_{ij} = e_{ij} - e_{ji}$ and e_{ij} is the matrix with ij entry equal to 1 and all other entries vanishing. The bracket is characterized by the relations:

$$[C_{ij}, C_{ik}] = -C_{jk}, \quad (1.8)$$

$$[C_{ij}, C_{kl}] = 0 \text{ for } i \neq j \neq k \neq l. \quad (1.9)$$

We introduce the following notation:

$$\begin{aligned} X_1 &\stackrel{\text{def}}{=} C_{12}, \quad X_2 \stackrel{\text{def}}{=} C_{13}, \quad X_3 \stackrel{\text{def}}{=} C_{14}, \quad X_4 \stackrel{\text{def}}{=} C_{15}, \quad X_5 \stackrel{\text{def}}{=} C_{23} \\ X_6 &\stackrel{\text{def}}{=} C_{24}, \quad X_7 \stackrel{\text{def}}{=} C_{25}, \quad X_8 \stackrel{\text{def}}{=} C_{34}, \quad X_9 \stackrel{\text{def}}{=} C_{35}, \quad X_{10} \stackrel{\text{def}}{=} C_{45}. \end{aligned}$$

Let ρ be the restriction of $\text{Ad}_{\text{SO}(5)}$ to $\text{Stab}(p_r)$. An element $g \in \text{SO}(3)$ acts on $A \in \mathfrak{so}(5)$ by conjugation. We split this representation into irreducibles:

$$\mathfrak{so}(5) = \langle X_1 \rangle \oplus \langle X_2, X_3, X_4 \rangle \oplus \langle X_5, X_6, X_7 \rangle \oplus \langle X_{10}, -X_9, X_8 \rangle. \quad (1.10)$$

The first summand is trivial. The other three are isomorphic to the vector representation of $\text{SO}(3)$. The order in which the X_i appear corresponds to the standard ordered basis of $\mathbb{R}^3$. The Lie algebra of the stabiliser is given by the last summand. We define the natural reductive complement:

$$\mathfrak{m} = \langle X_1 \rangle \oplus \langle X_2, X_3, X_4 \rangle \oplus \langle X_5, X_6, X_7 \rangle. \quad (1.11)$$

Owing to (1.10), it is stable under ρ yielding the isotropy representation. Using (1.2), (1.3) and (1.6) we find that:

$$\begin{aligned} d\pi|_{\text{Id}} : \mathfrak{m} &\xrightarrow{\sim} T_{p_r}\mathcal{O}_r, \\ A &\mapsto (R + c_1(A), -R - r_2(A)). \end{aligned} \quad (1.12)$$

Here $c_1(\cdot)$ denotes the operation of taking the first column and $r_2(\cdot)$ denotes the operation of taking the second row. Using (1.12) we obtain the equations:

$$d\pi|_{\text{Id}} X_1 = -R + \partial_{x|_{p_r}^2} + R - \partial_{y|_{p_r}^1}, \quad (1.13)$$

$$d\pi|_{\text{Id}} X_2 = -R + \partial_{x|_{p_r}^3}, \quad (1.14)$$

$$d\pi|_{\text{Id}} X_3 = -R + \partial_{x|_{p_r}^4}, \quad (1.15)$$

$$d\pi|_{\text{Id}} X_4 = -R + \partial_{x|_{p_r}^5}, \quad (1.16)$$

$$d\pi|_{\text{Id}} X_5 = -R - \partial_{y|_{p_r}^3}, \quad (1.17)$$

$$d\pi|_{\text{Id}} X_6 = -R - \partial_{y|_{p_r}^4}, \quad (1.18)$$

$$d\pi|_{\text{Id}} X_7 = -R - \partial_{y|_{p_r}^5}. \quad (1.19)$$

It is evident from (1.13)–(1.15) that (for $r = 1$) X_1, X_2, X_3, X_4 correspond to infinitesimal motions in the horizontal direction along the base S^4 . Similarly, (1.17)–(1.19) demonstrate that (for $r > 1$) X_5, X_6, X_7 correspond to infinitesimal vertical motions along the fiber S^3 of the sphere bundle.

The vectors $(X_1, \dots, X_{10})$ provide a basis for $\mathfrak{so}(5)$. We denote the dual basis by $(\theta^1, \dots, \theta^{10})$ so that:

$$\theta^i(X_j) = \delta_j^i, \quad 1 \leq i, j \leq 10. \quad (1.20)$$

We have constructed appropriate frames (and dual coframes) for the tangent spaces to the orbits $\mathcal{O}_r$ along the reference ray (1.5). We now wish to complete these frames so that they span the full tangent space $T_{p_r}X^8$:

Proposition 1.1. *There exists a unique smooth vector field ∂_r on $X^8 \setminus S^4$ characterised by the following properties:*

1. The vector field ∂_r is tangent to $(0, \infty)$ in the splitting (1.7).
2. $dr(\partial_r) = 1$.

Let $(x, y) \in X^8 \subset \mathbb{C}^5$. The vector field ∂_r can be expressed as follows in terms of the standard coordinate vector fields on $\mathbb{C}^5$:

$$\partial_{r|(x,y)} = \frac{r}{2R_+^2} \left(\sum_{j=1}^5 x^j \partial_{x_{|(x,y)}^j} \right) + \frac{r}{2R_-^2} \left(\sum_{j=1}^5 y^j \partial_{y_{|(x,y)}^j} \right). \quad (1.21)$$

Evaluating the expression (1.21) at p_r we obtain:

$$\partial_r = \frac{r}{2R_+} \partial_{x^1} + \frac{r}{2R_-} \partial_{y^2}. \quad (1.22)$$

Over p_r , tensors can be written as linear combinations of tensor products of X_i , ∂_r and the dual coframe θ^i , dr . The tensor in question is invariant if and only if it is stabilised by the isotropy action. In that case, the basis expansion at p_r is well defined over $X^8 \setminus S^4$. Of all the vectors in our frame, only ∂_r and X_1 satisfy this condition. From here on, we will suppress application of $d\pi$ and evaluation at p_r . Using (1.13)–(1.22) we conclude that:

$$dx^1 = \frac{r}{2R_+} dr, \quad dy^1 = R_- \theta^1, \quad (1.23)$$

$$dx^2 = -R_+ \theta^1, \quad dy^2 = \frac{r}{2R_-} dr, \quad (1.24)$$

$$dx^3 = -R_+ \theta^2, \quad dy^3 = -R_- \theta^5, \quad (1.25)$$

$$dx^4 = -R_+ \theta^3, \quad dy^4 = -R_- \theta^6, \quad (1.26)$$

$$dx^5 = -R_+ \theta^4, \quad dy^5 = -R_- \theta^7. \quad (1.27)$$

Using equations (1.23)–(1.27) we obtain:

$$dz^1 = \frac{r}{2R_+} dr + i R_- \theta^1, \quad (1.28)$$

$$dz^2 = -R_+ \theta^1 + i \frac{r}{2R_-} dr, \quad (1.29)$$

$$dz^3 = -R_+ \theta^2 - i R_- \theta^5, \quad (1.30)$$

$$dz^4 = -R_+ \theta^3 - i R_- \theta^6, \quad (1.31)$$

$$dz^5 = -R_+ \theta^4 - i R_- \theta^7. \quad (1.32)$$

1.2.2. The Stenzel metric Since the second cohomology group of T^*S^4 vanishes, any Kähler structure comes from a global Kähler potential $\mathcal{F}(r^2)$. The associated Kähler form will then be:

$$\omega = \frac{i}{2} \partial \bar{\partial} \mathcal{F}(r^2).$$

We introduce the functions:

$$P(r) \stackrel{\text{def}}{=} \frac{r}{2} \left(\frac{R_+}{R_-} + \frac{R_-}{R_+} \right) \mathcal{F}'(r^2) + 2r R_+ R_- \mathcal{F}''(r^2), \quad (1.33)$$

$$Q(r) \stackrel{\text{def}}{=} R_+ R_- \mathcal{F}'(r^2). \quad (1.34)$$

Differentiating in the ambient $\mathbb{C}^5$ and changing frame using (1.28)–(1.32), we obtain:

$$\omega = P(r)dr \wedge \theta^1 + Q(r) \left(\theta^{25} + \theta^{36} + \theta^{47} \right). \quad (1.35)$$

The volume form of the resulting metric looks like:

$$\text{Vol}_\omega = \frac{\omega^4}{4!} = -PQ^3 dr \wedge \theta^{1234567}. \quad (1.36)$$

The complex structure J can be described by the following equations:

$$JX_1 = -\frac{2R_+R_-}{r}\partial_r, \quad J\partial_r = \frac{r}{2R_+R_-}X_1, \quad (1.37)$$

$$JX_2 = \frac{R_+}{R_-}X_5, \quad JX_5 = -\frac{R_-}{R_+}X_2, \quad (1.38)$$

$$JX_3 = \frac{R_+}{R_-}X_6, \quad JX_6 = -\frac{R_-}{R_+}X_3, \quad (1.39)$$

$$JX_4 = \frac{R_+}{R_-}X_7, \quad JX_7 = -\frac{R_-}{R_+}X_4. \quad (1.40)$$

Formulae (1.35), (1.37)–(1.40) can be used to obtain the associated Riemannian metric:

$$\begin{aligned} g = & \frac{rP}{2R_+R_-}dr \otimes dr + \frac{2R_+R_-P}{r}\theta^1 \otimes \theta^1 \\ & + \frac{R_+Q}{R_-} \left(\theta^2 \otimes \theta^2 + \theta^3 \otimes \theta^3 + \theta^4 \otimes \theta^4 \right) \\ & + \frac{R_-Q}{R_+} \left(\theta^5 \otimes \theta^5 + \theta^6 \otimes \theta^6 + \theta^7 \otimes \theta^7 \right). \end{aligned} \quad (1.41)$$

We wish to refine the invariant Kähler structures discussed so far by imposing the Calabi–Yau condition. This requires the canonical bundle of the underlying complex manifold to be trivial:

Proposition 1.2. *The canonical bundle K_{X^8} is holomorphically trivial.*

Proof. Let $S_i \subset \mathbb{C}^5$ be the open subset where $z_i \neq 0$. Introduce the following $(4, 0)$ -form on S_i :

$$\Omega_i \stackrel{\text{def}}{=} \frac{1}{z_i} \partial_{z_i} \lrcorner dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4 \wedge dz^5 \quad (1.42)$$

It is easily checked that the forms $\iota_{X^8}^* \Omega_i$ glue to a global holomorphic volume form on X^8 . $\square$

Using (1.42) and (1.28)–(1.32), the trivialization Ω constructed in the preceding proposition is easily written in terms of invariant forms:

$$\begin{aligned} \Re(\Omega) = & R_+^3 \theta^{1234} - R_+ R_-^2 \left(\theta^{1267} + \theta^{1537} + \theta^{1564} \right) \\ & + \frac{r}{2} dr \wedge \left(R_+ \left(\theta^{237} + \theta^{264} + \theta^{534} \right) - \frac{R_-^2}{R_+} \theta^{567} \right), \end{aligned} \quad (1.43)$$

$$\begin{aligned} \Im(\Omega) = & -R_-^3 \theta^{1567} + R_+^2 R_- \left(\theta^{1237} + \theta^{1264} + \theta^{1534} \right) \\ & + \frac{r}{2} dr \wedge \left(R_- \left(\theta^{267} + \theta^{537} + \theta^{564} \right) - \frac{R_+^2}{R_-} \theta^{234} \right). \end{aligned} \quad (1.44)$$

Writing $n = \dim_{\mathbb{C}}(X^8) = 4$, we have that the associated volume form is given by:

$$\text{Vol}_{\Omega} = (-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2} \right)^n \Omega \wedge \overline{\Omega} = -\frac{r}{2} R_+^2 R_-^2 dr \wedge \theta^{1234567}. \quad (1.45)$$

The Calabi–Yau equation is equivalent to volume compatibility:

$$\text{Vol}_{\omega} = \text{Vol}_{\Omega}. \quad (1.46)$$

Using (1.36) and (1.45), we discover that it amounts to an ODE for $\mathcal{F}'(r^2)$:

$$P Q^3 = \frac{r}{2} R_+^2 R_-^2. \quad (1.47)$$

Equation (1.47) can be solved explicitly. It admits a unique solution extending smoothly across the singular orbit S^4 . The resulting Riemannian metric has holonomy equal to $\text{SU}(4)$. It is known as the *Stenzel metric*. We obtain:

$$P(r) = \left(\frac{3}{4} \right)^{\frac{3}{4}} \frac{r(r^2 + 1)}{(r^2 + 2)^{\frac{3}{4}} (r + 1)^{\frac{1}{2}} (r - 1)^{\frac{1}{2}}}, \quad (1.48)$$

$$Q(r) = \frac{1}{2} \left(\frac{4}{3} \right)^{\frac{1}{4}} (r^2 + 2)^{\frac{1}{4}} (r + 1)^{\frac{1}{2}} (r - 1)^{\frac{1}{2}}. \quad (1.49)$$

Using (1.41), (1.48) and (1.49) we compute the pointwise norms of the vectors in the standard framing:

$$|X_1|^2 = \left(\frac{3}{4} \right)^{\frac{3}{4}} \frac{(r^2 + 1)^{\frac{3}{2}}}{(r^2 + 2)^{\frac{3}{4}}}, \quad (1.50)$$

$$|\partial_r|^2 = \left(\frac{3}{4} \right)^{\frac{3}{4}} \frac{r^2 (r^2 + 1)^{\frac{1}{2}}}{(r^2 + 2)^{\frac{3}{4}} (r + 1)(r - 1)}, \quad (1.51)$$

$$|X_2|^2 = |X_3|^2 = |X_4|^2 = \frac{1}{2} \left(\frac{4}{3} \right)^{\frac{1}{4}} (r^2 + 1)^{\frac{1}{2}} (r^2 + 2)^{\frac{1}{4}}, \quad (1.52)$$

$$|X_5|^2 = |X_6|^2 = |X_7|^2 = \frac{1}{2} \left(\frac{4}{3} \right)^{\frac{1}{4}} \frac{(r^2 + 2)^{\frac{1}{4}} (r + 1)(r - 1)}{(r^2 + 1)^{\frac{1}{2}}}. \quad (1.53)$$

As $r \rightarrow 1$, $|\partial_r|^2$ blows up monotonically, $|X_1|^2$, $|X_2|^2$, $|X_3|^2$, $|X_4|^2$ approach 1 and $|X_5|^2$, $|X_6|^2$, $|X_7|^2$ tend to 0. We can pull g back to the singular orbit S^4 to find that it is round of unit radius.

The Cayley calibration of a CY fourfold is given by ([15], p. 81):

$$\Phi = \frac{\omega^2}{2} + \Re(\Omega). \quad (1.54)$$

Using (1.54), we obtain:

$$\begin{aligned} \Phi = dr \wedge & \left[P Q \left(\theta^{125} + \theta^{136} + \theta^{147} \right) + \frac{r R_+}{2} \left(\theta^{237} + \theta^{264} + \theta^{534} \right) - \frac{r R_-^2}{2 R_+} \theta^{567} \right] \\ & + R_+^3 \theta^{1234} - R_+ R_-^2 \left(\theta^{1267} + \theta^{1537} + \theta^{1564} \right) + Q^2 \left(\theta^{2536} + \theta^{2547} + \theta^{3647} \right). \end{aligned} \quad (1.55)$$

When we pull Φ back to the singular orbit S^4 , only the θ^{1234} term survives. On S^4 we have $r = 1$. We therefore find that:

$$\iota_{S^4}^* \Phi = \theta^{1234}. \quad (1.56)$$

We conclude that the singular orbit is calibrated for Φ and is therefore a Cayley submanifold of the Spin(7) manifold (X^8, Φ) . As such, it is volume minimizing in its homology class [7].

Modifying the radial coordinate so that the corresponding radial vector is asymptotically of unit length and expanding the Stenzel metric at infinity, one finds that it is asymptotically conical of rate $\mu = -\frac{8}{3}$. In fact, this is the optimal rate [1]. In particular, on setting:

$$s \stackrel{\text{def}}{=} \left(\frac{4}{3} \right)^{\frac{5}{8}} r^{\frac{3}{4}}, \quad (1.57)$$

one finds that:

$$|g - h_{\text{cone}}|_{h_{\text{cone}}} = O(s^{-\frac{8}{3}}) \quad \text{as } s \rightarrow \infty. \quad (1.58)$$

Here, the conical metric:

$$h_{\text{cone}} = ds \otimes ds + s^2 \bar{h} \quad (1.59)$$

is defined over $X^8 \setminus S^4$ and it is built on the homogeneous link $\left(\frac{\text{SO}(5)}{\text{SO}(3)}, \bar{h} \right)$, where:

$$\bar{h} = \frac{9}{16} \theta^1 \otimes \theta^1 + \frac{3}{8} \sum_{j=2}^7 \theta^j \otimes \theta^j.$$

We briefly consider uniqueness issues for the above construction. The form Ω is not the only possible choice of holomorphic trivialization for $\mathcal{K}_{X^8}$. Others may be constructed through multiplication by a nowhere-vanishing holomorphic function. Scaling by nontrivial SO(5)-invariant functions will yield new SO(5)-invariant holomorphic trivializations. Changing Ω can affect the resulting metric and Cayley calibration through (1.46) and (1.54). An obvious modification is to scale by non-zero complex numbers. Scaling by non-negative reals modifies the Calabi–Yau equation, which ultimately results in rescaling the metric. Scaling by elements of U(1) does not affect (1.46) and hence leaves the metric unaltered. It does however alter the Cayley calibration. The trivialization Ω has been chosen in hindsight for the special geometric features it entails: it makes the radius of the round S^4 equal to 1 (which serves as a natural normalization condition for the rescaling freedom) and furthermore guarantees that the latter is calibrated for the induced Spin(7) structure.

2. SO(5)-Invariant Instantons with Structure Group U(1)

2.1. Cohomogeneity one bundles with structure group U(1). Let $r > 1$. The homogeneous U(1) bundles over the orbit $\mathcal{O}_r$ correspond to element-conjugacy (i.e. conjugation by a fixed element in the target) classes of Lie group homomorphisms:

$$\lambda : \mathrm{SO}(3) \rightarrow \mathrm{U}(1). \quad (2.1)$$

For the classification of homogeneous bundles and invariant connections see ([18], [13] Section 3.1, [10] Section 2.4). Our notation and conventions agree with the latter.

Since the target is abelian, the element-conjugacy relation is trivial: the classes are singletons. The only map of type (2.1) is $\lambda = 1$. Consequently, the only homogeneous U(1) bundle over $\mathcal{O}_r$ -up to equivariant principal bundle isomorphism- is the trivial one:

$$P_1 = \mathcal{O}_r \times \mathrm{U}(1) = \frac{\mathrm{SO}(5)}{\mathrm{SO}(3)} \times \mathrm{U}(1). \quad (2.2)$$

SO(5)-invariant U(1)-connections on P_1 are parameterised by representation morphisms:

$$\Lambda : \left(\mathfrak{m}, \mathrm{Ad}_{\mathrm{SO}(5)|_{\mathrm{SO}(3)}} \right) \rightarrow (\mathfrak{u}(1), \mathrm{Ad}_{\mathrm{U}(1)} \circ \lambda) = (i\mathbb{R}, 1). \quad (2.3)$$

Recalling the decomposition (1.11) and applying Schur's lemma, we obtain that:

$$\mathrm{Hom}_{\mathrm{SO}(3)}(\mathfrak{m}, \mathfrak{u}(1)) = i\mathbb{R}. \quad (2.4)$$

Here, the imaginary number $i\alpha$ corresponds to:

$$\Lambda_\alpha \stackrel{\mathrm{def}}{=} i\alpha\theta^1. \quad (2.5)$$

The cohomogeneity one bundle over $X^8 \setminus S^4$ associated to P_1 is obtained by pulling back along the map:

$$X^8 \setminus S^4 \xrightarrow{\sim} (1, \infty) \times \frac{\mathrm{SO}(5)}{\mathrm{SO}(3)} \rightarrow \frac{\mathrm{SO}(5)}{\mathrm{SO}(3)}.$$

We slightly abuse notation by suppressing the pullback symbol and denoting the resulting bundle by P_1 . It is trivial and it admits a unique extension across the singular orbit given by $X^8 \times \mathrm{U}(1)$.

Connections over $X^8 \setminus S^4$ can be put in temporal gauge (vanishing dr component) through an equivariant gauge transformation ([10], p. 973, Remark 3). Consequently, each invariant connection on P_1 is equivariantly gauge equivalent to one lying in the space:

$$\mathcal{A}_{\mathrm{inv}}(P_1) = \left\{ i\alpha(r)\theta^1 \mid \alpha \in C^\infty(1, \infty) \right\} \subset \mathcal{A}(P_1). \quad (2.6)$$

Such connections can only be related by an r -independent gauge transformation. If such a gauge transformation is equivariant, it is given by a fixed element of U(1) and it stabilises all connections. It follows that no two distinct elements of $\mathcal{A}_{\mathrm{inv}}(P_1)$ are equivariantly gauge equivalent.

We compute the curvature of $A \in \mathcal{A}(P_1)$:

$$F_A = dA$$

$$= i \frac{d\alpha}{dr} dr \wedge \theta^1 + i\alpha(r) d\theta^1. \quad (2.7)$$

To simplify the second term we use the Maurer-Cartan relations ([8], p. 41). For this calculation we require the structure constants of $\mathfrak{so}(5)$. They can be computed using (1.8) and (1.9). Carrying out the calculation gives:

$$d\theta^1 = \theta^{25} + \theta^{36} + \theta^{47}.$$

Incorporating this into (2.7), we obtain:

$$F_A = i \frac{d\alpha}{dr} dr \wedge \theta^1 + i\alpha(r) (\theta^{25} + \theta^{36} + \theta^{47}). \quad (2.8)$$

The Ambrose-Singer holonomy theorem implies that any non-flat $U(1)$ connection is irreducible. Consequently, all elements of $\mathcal{A}_{\text{inv}}(P_1)$ -excluding the trivial connection- are irreducible.

2.2. The $SO(5)$ -invariant ODEs. The Spin(7) instanton equation reads:

$$\star_g F_A = -\Phi \wedge F_A. \quad (2.9)$$

Since the metric diagonalises we have:

$$\star_g \theta^{i_0} \wedge \cdots \wedge \theta^{i_k} = \pm \frac{\sqrt{\det(g)}}{g_{i_0 i_0} \cdots g_{i_k i_k}} \theta^{i_{k+1}} \wedge \cdots \wedge \theta^{i_7}, \quad (2.10)$$

where -just for this equation- we have denoted dr by θ^0 . The sign in (2.10) is positive if and only if $(\theta^{i_0}, \dots, \theta^{i_7})$ forms an oriented coframe. Using (2.10) and (1.41) we compute:

$$\begin{aligned} \star_g (dr \wedge \theta^1) &= -\frac{Q^3}{P} \theta^{234567}, \\ \star_g \theta^{25} &= -P Q dr \wedge \theta^{13467}, \\ \star_g \theta^{36} &= -P Q dr \wedge \theta^{12457}, \\ \star_g \theta^{47} &= -P Q dr \wedge \theta^{12356}. \end{aligned}$$

Incorporating these in (2.8), we obtain:

$$\star_g F_A = -i \frac{Q^3}{P} \frac{d\alpha}{dr} \theta^{234567} - i P Q \alpha dr \wedge (\theta^{13467} + \theta^{12457} + \theta^{12356}). \quad (2.11)$$

We now use (1.55) and (2.8) to compute:

$$\begin{aligned} \Phi \wedge F_A &= -3i Q^2 \alpha(r) \theta^{234567} - i \left(Q^2 \frac{d\alpha}{dr} + 2P Q \alpha(r) \right) dr \\ &\quad \wedge (\theta^{13467} + \theta^{12457} + \theta^{12356}). \end{aligned} \quad (2.12)$$

Imposing (2.9) and comparing coefficients gives two equations. These are the same and read:

$$\frac{d\alpha}{dr} = -3\frac{P}{Q}\alpha. \quad (2.13)$$

The Hermitian Yang–Mills equations are:

$$F_A \wedge \star\omega = 0, \quad (2.14)$$

$$F_A \wedge \Omega = 0. \quad (2.15)$$

We find that (2.15) holds identically. This can be seen by direct computation using (1.43), (1.44) and (2.8). It follows that an $\text{SO}(5)$ -invariant $\text{U}(1)$ -connection A is HYM if and only if (2.14) holds.

Over a Hermitian manifold of complex dimension n , we have:

$$\star_g \omega = \frac{\omega^{n-1}}{(n-1)!}. \quad (2.16)$$

Using (1.35), we compute:

$$\omega^3 = 6PQ^2 dr \wedge (\theta^{12536} + \theta^{12547} + \theta^{13647}) + 6Q^3 \theta^{253647}. \quad (2.17)$$

Using (2.16), (2.17) and (2.8) we calculate:

$$\begin{aligned} F_A \wedge \star\omega &= F_A \wedge \frac{\omega^3}{3!} \\ &= -i \left(Q^3 \frac{d\alpha}{dr} + 3PQ^2 \alpha(r) \right) dr \wedge \theta^{1234567}. \end{aligned}$$

It follows that an $\text{SO}(5)$ -invariant $\text{U}(1)$ -connection is HYM if and only if:

$$\frac{d\alpha}{dr} = -3\frac{P}{Q}\alpha. \quad (2.18)$$

We observe that this equation is the same as (2.13).

Using the uniqueness part of the standard Picard theorem, we obtain:

Theorem 2.1. *An $\text{SO}(5)$ -invariant $\text{U}(1)$ -connection over $X^8 \setminus S^4$ is a $\text{Spin}(7)$ instanton if and only if it is Hermitian–Yang–Mills.*

2.3. Explicit solution and breakdown near the singular orbit. We study the ODE (2.18). Using (1.48) and (1.49) we write it as:

$$\frac{d\alpha}{dr} = -\frac{9}{2} \frac{r(r^2+1)}{(r^2+2)(r+1)(r-1)} \alpha(r). \quad (2.19)$$

We integrate (2.19) directly to see that the solution takes the following form for some $K \in \mathbb{R}$:

$$\alpha(r) = \frac{K}{(r^2+2)^{\frac{3}{4}}(r+1)^{\frac{3}{2}}(r-1)^{\frac{3}{2}}}. \quad (2.20)$$

An elementary calculation yields:

$$\frac{d\alpha}{dr} = -\frac{9K}{2} \frac{r(r^2+1)}{(r^2+2)^{\frac{7}{4}}(r+1)^{\frac{5}{2}}(r-1)^{\frac{5}{2}}}. \quad (2.21)$$

Recalling the formulae (2.6) and (2.8) and incorporating (2.20) and (2.21), we obtain:

Theorem 2.2. *Let P_1 be the unique cohomogeneity one $U(1)$ -bundle over $X^8 \setminus S^4$ (i.e. the trivial bundle). There exists a one-parameter family of smooth $SO(5)$ -invariant Spin(7) instantons $A_K \in \mathcal{A}_{inv}(P_1)$:*

$$A_K = \frac{iK}{(r^2+2)^{\frac{3}{4}}(r+1)^{\frac{3}{2}}(r-1)^{\frac{3}{2}}} \theta^1 \quad \text{where } K \in \mathbb{R}. \quad (2.22)$$

All elements of this family are HYM. They are all irreducible -apart from the trivial connection- and no two of them are gauge equivalent. Furthermore, these are all the invariant Spin(7) instantons on P_1 .

The curvature of A_K is given by:

$$F_{A_K} = iK \left(-\frac{9}{2} \frac{r(r^2+1)}{(r^2+2)^{\frac{7}{4}}(r+1)^{\frac{5}{2}}(r-1)^{\frac{5}{2}}} dr \wedge \theta^1 + \frac{\theta^{25} + \theta^{36} + \theta^{47}}{(r^2+2)^{\frac{3}{4}}(r+1)^{\frac{3}{2}}(r-1)^{\frac{3}{2}}} \right). \quad (2.23)$$

We need to justify our claim that these solutions lie in distinct gauge equivalence classes. We have seen that this holds when we quotient by the group of equivariant gauge transformations. The claim follows from the fact that gauge equivalent, irreducible, invariant connections are automatically equivariantly gauge equivalent ([12], Corollary 4.5).

Using (1.50)–(1.53), we find that:

$$|F_A|_g^2 = O(|r-1|^{-4}) \quad \text{as } r \rightarrow 1. \quad (2.24)$$

In particular:

$$\lim_{r \rightarrow 1} |F_A|_g^2 = +\infty.$$

Since the metric extends smoothly to the singular orbit, this behaviour is precluded for connections that are smooth over the whole space. We therefore obtain the following global nonexistence result:

Theorem 2.3. *There exist no global, abelian, $SO(5)$ -invariant Spin(7) instantons/HYM connections on the Stenzel manifold X^8 apart from the trivial connection $A = 0$ (corresponding to $K = 0$).*

As a closing remark, we note that breakdown around Cayley fourfolds is an interesting feature of the Spin(7)-instanton equation. It is related to the non-compactness of the moduli space. In Donaldson theory, noncompactness occurs in the form of a sequence of ASD instantons failing to have a limit due to finitely many point singularities. In the 8-dimensional Spin(7) setting, points are typically replaced by four-dimensional Cayley submanifolds [3, 4].

The non-existence result we encountered has to do with abelian gauge theory being too coarse to capture the behaviour we would like to see. In the following section we study the nonabelian equations associated to the structure group $\mathrm{SO}(3)$. The nonlinearity induced by the noncommutativity of the group *smooths* the equations and we are able to obtain solutions that extend over the singular orbit S^4 .

3. $\mathrm{SO}(5)$ -Invariant Instantons with Structure Group $\mathrm{SO}(3)$: Preliminaries and Analysis on the Trivial Bundle

3.1. Cohomogeneity one bundles with structure group $\mathrm{SO}(3)$.

3.1.1. Bundles and bundle extensions Let $r > 1$. The homogeneous $\mathrm{SO}(3)$ bundles over the orbit $\mathcal{O}_r$ correspond to element-conjugacy classes of Lie group homomorphisms:

$$\lambda : \mathrm{SO}(3) \rightarrow \mathrm{SO}(3). \quad (3.1)$$

There are two such classes. They are represented by the trivial map and the identity respectively. Consequently, there are precisely two homogeneous principal $\mathrm{SO}(3)$ bundles over $\mathcal{O}_r$ -up to equivariant principal bundle isomorphism. We denote these by P_1 and P_{Id} . Slightly abusing notation, we also denote by P_1 and P_{Id} the pullbacks of the respective bundles along the map:

$$X^8 \setminus S^4 \xrightarrow{\sim} (1, \infty) \times \frac{\mathrm{SO}(5)}{\mathrm{SO}(3)} \rightarrow \frac{\mathrm{SO}(5)}{\mathrm{SO}(3)}.$$

We now classify smooth homogeneous extensions of P_1 and P_{Id} across the singular orbit S^4 . These correspond to element-conjugacy classes of Lie group homomorphisms:

$$\mu : \mathrm{SO}(4) \rightarrow \mathrm{SO}(3). \quad (3.2)$$

Once such a map is chosen, one uses it to form the associated homogeneous bundle P_μ over S^4 . The extension is then determined by pulling P_μ back over X^8 through the natural projection:

$$X^8 \cong T^*S^4 \rightarrow S^4.$$

The element-conjugacy class of the restriction of μ to the lower right copy of $\mathrm{SO}(3)$ determines which bundle is being extended.

We are therefore required to classify element-conjugacy classes of homomorphisms of type (3.2). Natural representatives are described by passing through the respective universal covers. We have the two-sheeted covering maps:

$$\begin{aligned} \pi_{\mathrm{Spin}(4)} : \mathrm{Sp}(1)^2 &\twoheadrightarrow \mathrm{SO}(4), \\ \pi_{\mathrm{Spin}(3)} : \mathrm{Sp}(1) &\rightarrow \mathrm{SO}(3); \end{aligned}$$

where:

$$\begin{aligned} \pi_{\mathrm{Spin}(4)}(x, y) : \mathbb{H} &\rightarrow \mathbb{H}, \\ q &\mapsto xqy^{-1}, \end{aligned} \quad (3.3)$$

$$\pi_{\mathrm{Spin}(3)}(x) = \pi_{\mathrm{Spin}(4)}(x, x)|_{\mathbb{J}\mathrm{m}(\mathbb{H})}. \quad (3.4)$$

Considering (3.3) and (3.4), we obtain:

$$\mathrm{SO}(4) = \frac{\mathrm{Sp}(1)^2}{\{(1, 1), (-1, -1)\}}, \quad \mathrm{SO}(3) = \frac{\mathrm{Sp}(1)}{\pm 1}.$$

There are precisely three element-conjugacy classes of homomorphisms of type (3.2), one of them being that of the trivial map. The two nontrivial classes are represented by the two projections

$$\pi_1, \pi_2 : \mathrm{SO}(4) = \frac{\mathrm{Sp}(1)^2}{\{(1, 1), (-1, -1)\}} \rightarrow \frac{\mathrm{Sp}(1)}{\pm 1} \times \frac{\mathrm{Sp}(1)}{\pm 1} \rightarrow \frac{\mathrm{Sp}(1)}{\pm 1} = \mathrm{SO}(3).$$

Hence, there are precisely three principal $\mathrm{SO}(3)$ -bundles of cohomogeneity-one over X^8 . We denote these as P_1 , P_{π_1} and P_{π_2} . The first is the trivial bundle. It extends the trivial bundle on $X^8 \setminus S^4$. The other two bundles are nontrivial (see Sect. 4.1.2). They provide distinct extensions of P_{Id} .

3.1.2. Invariant connections on the complement of the singular orbit We now classify the invariant connections on the bundles P_1 and P_{Id} over $X^8 \setminus S^4$. For each of these connections we compute the associated curvature tensor in terms of the standard framing.

We introduce the following basis for $\mathfrak{so}(3)$:

$$e_1 \stackrel{\mathrm{def}}{=} C_{12}, \quad e_2 \stackrel{\mathrm{def}}{=} C_{13}, \quad e_3 \stackrel{\mathrm{def}}{=} C_{23}. \quad (3.5)$$

Here the matrices C_{ij} are defined as in section 1.2.1 and obey the commutation relations (1.8) and (1.9). The adjoint representation of $\mathrm{SO}(3)$ takes the form:

$$\mathrm{Ad}_{\mathrm{SO}(3)} = \mathfrak{so}(3) = \langle e_3, -e_2, e_1 \rangle. \quad (3.6)$$

Here $\mathrm{SO}(3)$ acts by its natural irreducible vector representation on $\mathbb{R}^3$ and we maintain the convention that the ordering in the bracket reflects the associated identification $\mathfrak{so}(3) \cong \mathbb{R}^3$.

In general, an invariant connection $A \in \mathcal{A}_{\mathrm{inv}}(P_\lambda)$ corresponds to a map of representations:

$$\Lambda : \left(\mathfrak{m}, \mathrm{Ad}_{\mathrm{SO}(5)|_{\mathrm{SO}(3)}} \right) \rightarrow \left(\mathfrak{so}(3), \mathrm{Ad}_{\mathrm{SO}(3)} \circ \lambda \right). \quad (3.7)$$

Given such a map, we use the canonical invariant connection $d\lambda$ as a reference and write:

$$A = d\lambda + \Lambda. \quad (3.8)$$

We first deal with P_1 . In this case $\lambda = 1$ and the target representation is trivial. Recalling the splitting (1.11) and applying Schur's lemma, we see that Λ must take the form:

$$\Lambda = \theta^1 \otimes \left(a^1 e_1 + a^2 e_2 + a^3 e_3 \right). \quad (3.9)$$

The canonical invariant connection A_1^{can} is represented by $d \, 1 = 0$. Evidently, it is flat.

As in Sect. 2.1, any connection over $X^8 \setminus S^4$ can be brought to temporal gauge by an equivariant gauge transformation. It follows that any invariant connection on P_1 is equivariantly gauge equivalent to one lying in the space:

$$\mathcal{A}_{\text{inv}}(P_1) = \left\{ \theta^1 \otimes \left(a^1(r) e_1 + a^2(r) e_2 + a^3(r) e_3 \right) \mid a^1, a^2, a^3 \in C^\infty(1, \infty) \right\}. \quad (3.10)$$

A gauge transformation relating two elements of $\mathcal{A}_{\text{inv}}(P_1)$ must be r -independent. If it is equivariant, it is given by a fixed element of $\text{SO}(3)$ acting on $\mathcal{A}_{\text{inv}}(P_1)$ by conjugation. It follows that the elements of $\mathcal{A}_{\text{inv}}(P_1)$ need not lie in distinct equivariant gauge equivalence classes.

A calculation analogous to the one in section 2.1 yields:

$$F_A = \left(\frac{da^i}{dr} dr \wedge \theta^1 + a^i \left(\theta^{25} + \theta^{36} + \theta^{47} \right) \right) \otimes e_i. \quad (3.11)$$

The Ambrose-Singer holonomy theorem implies that the elements of $\mathcal{A}(P_1)$ need not be irreducible. This happens -for instance- if one of the components a^i vanishes identically, in which case e_i does not lie in the holonomy algebra.

We now work on P_{Id} . In this case, the target representation is the adjoint representation of $\text{SO}(3)$. Recalling the decomposition (1.11) and applying Schur's lemma, we see that equivariant maps of type (3.7) always vanish on the first summand and either restrict to isomorphisms or the zero map on the second and third summands. The automorphisms of $\text{Ad}_{\text{SO}(3)}$ are given by multiplication by fixed scalars. We conclude that for $\lambda = \text{Id}$, maps of type (3.7) look like:

$$\Lambda = a \left(\theta^2 \otimes e_3 - \theta^3 \otimes e_2 + \theta^4 \otimes e_1 \right) + b \left(\theta^5 \otimes e_3 - \theta^6 \otimes e_2 + \theta^7 \otimes e_1 \right), \quad a, b \in \mathbb{R}.$$

Written over the symmetry group $\text{SO}(5)$, the canonical invariant connection $A_{\text{Id}}^{\text{can}} = d \text{Id}_{\text{SO}(3)}$ on P_{Id} takes the form:

$$A_{\text{Id}}^{\text{can}} = \theta^8 \otimes e_1 + \theta^9 \otimes e_2 + \theta^{10} \otimes e_3.$$

It is not flat. Its curvature is given by:

$$\begin{aligned} F_{A_{\text{Id}}^{\text{can}}} &= dA_{\text{Id}}^{\text{can}} + \frac{1}{2} [A_{\text{Id}}^{\text{can}} \wedge A_{\text{Id}}^{\text{can}}] \\ &= \left(\theta^{23} + \theta^{56} \right) \otimes e_1 + \left(\theta^{24} + \theta^{57} \right) \otimes e_2 + \left(\theta^{34} + \theta^{67} \right) \otimes e_3. \end{aligned}$$

The radial component of an invariant tensorial 1-form is an invariant section of the adjoint bundle. In this context, these objects correspond to fixed points of $\text{Ad}_{\text{SO}(3)}$. This representation has no fixed points, implying that all invariant connections are already in temporal gauge. Consequently, the space of invariant connections is given by:

$$\begin{aligned} \mathcal{A}_{\text{inv}}(P_{\text{Id}}) &= \left\{ A_{\text{Id}}^{\text{can}} + a(r) \left(\theta^2 e_3 - \theta^3 e_2 + \theta^4 e_1 \right) \right. \\ &\quad \left. + b(r) \left(\theta^5 e_3 - \theta^6 e_2 + \theta^7 e_1 \right) \mid a, b \in C^\infty(1, \infty) \right\}. \end{aligned} \quad (3.12)$$

Equivariant gauge transformations correspond to central elements of $\text{SO}(3)$. Since $\text{SO}(3)$ is centerless, the only possibility is the identity. Consequently, each invariant connection constitutes its own equivariant gauge equivalence class.

To compute the curvature of a general element $A = A_{\text{Id}}^{\text{can}} + \Lambda \in \mathcal{A}_{\text{inv}}(P_{\text{Id}})$, we use the formula:

$$F_A = F_{A_{\text{Id}}^{\text{can}}} + d_{A_{\text{Id}}^{\text{can}}} \Lambda + \frac{1}{2} [\Lambda \wedge \Lambda].$$

Routine calculation yields:

$$\begin{aligned} d_{A_{\text{Id}}^{\text{can}}} \Lambda &= d\Lambda + [A_{\text{Id}}^{\text{can}} \wedge \Lambda] \\ &= (b \theta^{14} - a \theta^{17}) \otimes e_1 + (a \theta^{16} - b \theta^{13}) \otimes e_2 + (b \theta^{12} - a \theta^{15}) \otimes e_3 \\ &\quad + \frac{da}{dr} dr \wedge (\theta^2 \otimes e_3 - \theta^3 \otimes e_2 + \theta^4 \otimes e_1) \\ &\quad + \frac{db}{dr} dr \wedge (\theta^5 \otimes e_3 - \theta^6 \otimes e_2 + \theta^7 \otimes e_1). \end{aligned} \quad (3.13)$$

The final summand is also easily seen to take the form:

$$\begin{aligned} \frac{1}{2} [\Lambda \wedge \Lambda] &= (-a^2 \theta^{23} - ab \theta^{26} + ab \theta^{35} - b^2 \theta^{56}) \otimes e_1 \\ &\quad (-a^2 \theta^{24} - ab \theta^{27} + ab \theta^{45} - b^2 \theta^{57}) \otimes e_2 \\ &\quad (-a^2 \theta^{34} - ab \theta^{37} + ab \theta^{46} - b^2 \theta^{67}) \otimes e_3. \end{aligned} \quad (3.14)$$

Overall, we obtain the following expression for the curvature:

$$\begin{aligned} F_A &= \left((1 - a^2) \theta^{23} + (1 - b^2) \theta^{56} - ab \theta^{26} + ab \theta^{35} + b \theta^{14} - a \theta^{17} \right. \\ &\quad \left. + \frac{da}{dr} dr \wedge \theta^4 + \frac{db}{dr} dr \wedge \theta^7 \right) \otimes e_1 \\ &\quad + \left((1 - a^2) \theta^{24} + (1 - b^2) \theta^{57} - ab \theta^{27} + ab \theta^{45} - b \theta^{13} \right. \\ &\quad \left. + a \theta^{16} - \frac{da}{dr} dr \wedge \theta^3 - \frac{db}{dr} dr \wedge \theta^6 \right) \otimes e_2 \\ &\quad + \left((1 - a^2) \theta^{34} + (1 - b^2) \theta^{67} - ab \theta^{37} + ab \theta^{46} + b \theta^{12} - a \theta^{15} \right. \\ &\quad \left. + \frac{da}{dr} dr \wedge \theta^2 + \frac{db}{dr} dr \wedge \theta^5 \right) \otimes e_3. \end{aligned} \quad (3.15)$$

The Ambrose-Singer holonomy theorem implies that all elements of $\mathcal{A}_{\text{inv}}(P_{\text{Id}})$ are irreducible. Since gauge equivalent, irreducible, invariant connections are equivariantly gauge equivalent ([12], Corollary 4.5), the elements of $\mathcal{A}_{\text{inv}}(P_{\text{Id}})$ all lie in distinct gauge equivalence classes.

3.1.3. Invariant connections on the extended bundles It remains to understand how to describe invariant connections on the extensions of P_1 and P_{Id} over S^4 . For P_1 this is easy. The unique extension is given by the trivial bundle. The canonical invariant reference connection is still equal to zero. It follows that the $\text{ad}(P_1)$ -valued forms (3.10) are still meaningful over the extended bundle and describe the relevant invariant connections with this choice of reference.

The situation is slightly more subtle for P_{Id} . The canonical invariant connection of P_{Id} disagrees with those of P_{π_1} and P_{π_2} . In fact, $A_{\text{Id}}^{\text{can}}$ does not smoothly extend on either bundle extension. To see this, we compute the canonical invariant connections of P_{π_1} , P_{π_2} . These are given by:

$$\begin{aligned} A_{\pi_1}^{\text{can}} &= d\pi_1 \\ &= (\theta^8 + \theta^7) \otimes e_1 + (\theta^9 - \theta^6) \otimes e_2 + (\theta^{10} + \theta^5) \otimes e_3 \\ &= A_{\text{Id}}^{\text{can}} + (\theta^5 \otimes e_3 - \theta^6 \otimes e_2 + \theta^7 \otimes e_1), \end{aligned} \quad (3.16)$$

$$\begin{aligned} A_{\pi_2}^{\text{can}} &= d\pi_2 \\ &= (\theta^8 - \theta^7) \otimes e_1 + (\theta^9 + \theta^6) \otimes e_2 + (\theta^{10} - \theta^5) \otimes e_3 \\ &= A_{\text{Id}}^{\text{can}} - (\theta^5 \otimes e_3 - \theta^6 \otimes e_2 + \theta^7 \otimes e_1). \end{aligned} \quad (3.17)$$

Now, an invariant connection $A \in \mathcal{A}_{\text{inv}}(P_{\text{Id}})$ over $X^8 \setminus S^4$ takes the form:

$$\begin{aligned} A &= A_{\text{Id}}^{\text{can}} + a(r) (\theta^2 \otimes e_3 - \theta^3 \otimes e_2 + \theta^4 \otimes e_1) \\ &\quad + b(r) (\theta^5 \otimes e_3 - \theta^6 \otimes e_2 + \theta^7 \otimes e_1). \end{aligned} \quad (3.18)$$

We rewrite it using $A_{\pi_1}^{\text{can}}$ and $A_{\pi_2}^{\text{can}}$ as the reference. This yields:

$$\begin{aligned} A &= A_{\pi_1}^{\text{can}} + a(r) (\theta^2 \otimes e_3 - \theta^3 \otimes e_2 + \theta^4 \otimes e_1) \\ &\quad + (b(r) - 1) (\theta^5 \otimes e_3 - \theta^6 \otimes e_2 + \theta^7 \otimes e_1), \end{aligned} \quad (3.19)$$

$$\begin{aligned} A &= A_{\pi_2}^{\text{can}} + a(r) (\theta^2 \otimes e_3 - \theta^3 \otimes e_2 + \theta^4 \otimes e_1) \\ &\quad + (b(r) + 1) (\theta^5 \otimes e_3 - \theta^6 \otimes e_2 + \theta^7 \otimes e_1). \end{aligned} \quad (3.20)$$

The forms θ^5 , θ^6 and θ^7 blow up as $r \rightarrow 1$. We conclude that a necessary condition for A to extend to P_{π_1} is:

$$\lim_{r \rightarrow 1} b(r) = 1. \quad (3.21)$$

Similarly, if A extends to P_{π_2} we have:

$$\lim_{r \rightarrow 1} b(r) = -1. \quad (3.22)$$

The connection $A_{\text{Id}}^{\text{can}}$ corresponds to $a = b = 0$. Both conditions (3.21) and (3.22) fail. Consequently $A_{\text{Id}}^{\text{can}}$ does not extend to either P_{π_1} or P_{π_2} .

3.2. The $SO(5)$ -invariant ODEs on P_1 : derivation and explicit solution. We proceed to study the Spin(7) instanton and Hermitian–Yang–Mills equations on the bundle P_1 . A general invariant connection A is equivariantly gauge equivalent to one of the form (3.10). The associated curvature tensor F_A is given by (3.11). These expressions are manifestly similar to (2.6) and (2.8). An identical computation to the one carried out in the abelian case gives:

$$F_A \wedge \Omega = 0,$$

$$F_A \wedge \star \omega = - \left(Q^3 \frac{da^i}{dr} + 3PQ^2 a^i(r) \right) dr \wedge \theta^{1234567} \otimes e_i.$$

Consequently, the invariant Hermitian Yang–Mills equations take the form:

$$\frac{da^i}{dr} = -3 \frac{P}{Q} a^i. \quad (3.23)$$

Using (1.54) and computing as in the abelian case we obtain:

$$\begin{aligned} \Phi \wedge F_A &= - \left[3Q^2 a^i(r) \theta^{234567} + \left(Q^2 \frac{da^i}{dr} + 2PQ a^i(r) \right) dr \right. \\ &\quad \left. \wedge \left(\theta^{13467} + \theta^{12457} + \theta^{12356} \right) \right] \otimes e_i, \\ \star_g F_A &= - \left[\frac{Q^3}{P} \frac{da^i}{dr} \theta^{234567} + PQ a^i dr \wedge \left(\theta^{13467} + \theta^{12457} + \theta^{12356} \right) \right] \otimes e_i. \end{aligned}$$

Equating these expressions yields the invariant Spin(7) instanton ODEs. They are identical to (3.23). We thus obtain the following local equivalence result:

Theorem 3.1. *An $SO(5)$ -invariant $SO(3)$ -connection $A \in \mathcal{A}_{\text{inv}}(P_1)$ over $X^8 \setminus S^4$ is a Spin(7) instanton if and only if it is Hermitian–Yang–Mills.*

The ODE (3.23) has already been studied in the context of the abelian equations. We immediately obtain an analogous existence/classification result:

Theorem 3.2. *Let P_1 be the trivial cohomogeneity one $SO(3)$ -bundle over $X^8 \setminus S^4$. There is a 3-parameter family of invariant Spin(7) instantons $A_{K^1, K^2, K^3} \in \mathcal{A}_{\text{inv}}(P_1)$:*

$$A_{K^1, K^2, K^3} = \frac{\theta^1}{(r^2 + 2)^{\frac{3}{4}} (r + 1)^{\frac{3}{2}} (r - 1)^{\frac{3}{2}}} \otimes K^i e_i \text{ where } K^i \in \mathbb{R}.$$

They are all HYM. Any invariant Spin(7) instanton on P_1 is equivariantly gauge equivalent to some element of this family.

The curvature of A_{K^1, K^2, K^3} is given by:

$$F_{A_{K^1, K^2, K^3}} = \left(-\frac{9}{2} \frac{r(r^2 + 1)}{(r^2 + 2)^{\frac{7}{4}} (r + 1)^{\frac{5}{2}} (r - 1)^{\frac{5}{2}}} dr \wedge \theta^1 + \frac{\theta^{25} + \theta^{36} + \theta^{47}}{(r^2 + 2)^{\frac{3}{4}} (r + 1)^{\frac{3}{2}} (r - 1)^{\frac{3}{2}}} \right) \otimes K^i e_i.$$

The Ambrose-Singer holonomy theorem implies that all the instantons of theorem 3.2 are reducible. In particular, the holonomy algebra of A_{K_1, K_2, K_3} is given by:

$$\mathfrak{hol}(A_{K_1, K_2, K_3}) = \langle K^i e_i \rangle < \mathfrak{so}(3).$$

When $A_{K_1, K_2, K_3} \neq 0$, at least one of the parameters K^i does not vanish and $\mathfrak{hol}(A_{K_1, K_2, K_3})$ is one-dimensional. The holonomy group $\text{Hol}(A_{K_1, K_2, K_3})$ is the associated one-parameter subgroup. Recalling the elementary fact that the one-parameter subgroups of $\text{SO}(3)$ are embedded circles, we conclude that (when $A_{K^1, K^2, K^3} \neq 0$) the holonomy group is a copy of $\text{U}(1)$ in $\text{SO}(3)$. Let $Q \subset P_1$ denote the trivial $\text{U}(1)$ -subbundle with fiber $\text{Hol}(A_{K_1, K_2, K_3})$. The reduction theorem implies that A_{K^1, K^2, K^3} restricts to an irreducible connection on $Q|_{X^8 \setminus S^4}$. The resulting instanton is one of those promised by Theorem 2.2. This makes rigorous the apparent similarities with the situation in section 2.

Unless $K^1 = K^2 = K^3 = 0$, the curvature norm of A_{K^1, K^2, K^3} is unbounded as $r \rightarrow 1$. We thus obtain the following global non-existence result:

Theorem 3.3. *There are no global $\text{SO}(5)$ -invariant $\text{Spin}(7)$ instantons/ HYM connections on the trivial $\text{SO}(3)$ -bundle P_1 over X^8 -apart from the trivial connection $A = 0$.*

3.3. Extendibility of connections across the singular orbit. It remains to study the equations on P_{Id} . This is the content of Sect. 4. As we shall see, P_{Id} admits solutions that are well behaved near S^4 . As a preliminary step, we need to understand when a general $A \in \mathcal{A}_{\text{inv}}(P_{\text{Id}})$ arises as the restriction of a global connection (either on P_{π_1} or P_{π_2}). Our task is to formulate the criterion of Eschenburg and Wang ([6], Lemma 1.1, p. 113) in the context of gauge theory on X^8 . A similar analysis is carried out in the foundational article of Lotay–Oliveira ([10], Appendix A.2).

3.3.1. Extendibility of tensorial forms: the criterion of Eschenburg and Wang Let S be a Lie group and let μ be a Lie group homomorphism:

$$\mu : \text{SO}(4) \rightarrow S.$$

Denote by λ the restriction of μ to the bottom right copy of $\text{SO}(3)$. Let P be the cohomogeneity one principal S -bundle over X^8 whose restrictions over each orbit $\mathcal{O}_r$ ($r \geq 1$) are given by:

$$P|_{\mathcal{O}_r} = \begin{cases} \text{SO}(5) \times_{(\text{SO}(4), \mu)} S & \text{if } r = 1 \\ \text{SO}(5) \times_{(\text{SO}(3), \lambda)} S & \text{if } r > 1. \end{cases}$$

Let (V, ρ) be a representation of the group S . We can form the associated vector bundle over X^8 :

$$\rho(P) \stackrel{\text{def}}{=} P \times_{\rho} V.$$

We consider the problem of extending $\text{SO}(5)$ -invariant $\rho(P)$ -valued k -forms across the singular orbit S^4 .

Eschenburg and Wang give necessary and sufficient conditions for extending invariant linear tensors across the singular orbit of a cohomogeneity one space. Since we are interested in bundle-valued forms, their technique does not apply directly. We resolve

this issue by passing to the total space P and working with V -valued forms instead. In this section we set up the requisite framework to implement this idea.

The manifold P is a cohomogeneity one space for the group $\mathrm{SO}(5) \times S$. Its principal orbits are isomorphic to P_λ and its singular orbit is the bundle P_μ .

Define the reference points:

$$x_r \stackrel{\text{def}}{=} \begin{cases} [1, 1] \in P_\mu & \text{if } r = 1 \\ [1, 1] \in P_\lambda & \text{if } r > 1. \end{cases}$$

With this definition, the point x_r lies in the fiber above $p_r \in \mathcal{O}_r$ for all $r \geq 1$.

Using these reference points, the isotropy subgroups corresponding to the principal and singular orbits are respectively given by:

$$\mathrm{Stab}(x_r) = \{(h, \lambda(h)) \in \mathrm{SO}(5) \times S \text{ such that } h \in \mathrm{SO}(3)\} \cong \mathrm{SO}(3), \quad (3.24)$$

$$\mathrm{Stab}(x_1) = \{(h, \mu(h)) \in \mathrm{SO}(5) \times S \text{ such that } h \in \mathrm{SO}(4)\} \cong \mathrm{SO}(4). \quad (3.25)$$

In formulae (3.24) and (3.25), $\mathrm{SO}(4)$ and $\mathrm{SO}(3)$ denote the bottom right inclusions of these groups in $\mathrm{SO}(5)$. In what follows, when we consider the action of $\mathrm{SO}(4)$ on P , it will be through its embedding in $\mathrm{SO}(5) \times S$ as the singular isotropy group (3.25).

Let ω be an invariant, tensorial form of type ρ . Its extendibility can be decided by studying the restriction $\omega|_W$ along a particularly simple embedded submanifold $W \subset P$. This will make the problem tractable. Let W be the union of the $\mathrm{SO}(4)$ -orbits of all points x_r in P :

$$W \stackrel{\text{def}}{=} \bigcup_{r \geq 1} \mathrm{Stab}(x_1) \cdot x_r. \quad (3.26)$$

This is a 4-dimensional linear $\mathrm{SO}(4)$ -representation. The $\mathrm{SO}(4)$ -action is obvious. The linear structure is inherited from $T^*S^4_{p_1}$ through the projection map:

$$\pi : P \rightarrow X^8.$$

In particular, the inverse function theorem implies that π restricts to a diffeomorphism:

$$\pi : W \xrightarrow{\sim} T^*_{p_1} S^4 \subset X^8. \quad (3.27)$$

The latter is a smoothly embedded submanifold of X^8 stable under the action of $\mathrm{SO}(4)$. The equivariance of π implies that W and $T^*_{p_1} S^4$ are isomorphic $\mathrm{SO}(4)$ -representations. Since $T^*_{p_1} S^4$ is a vector space, it can be naturally identified with the tangent space at its origin (e.g. by the exponential map of the underlying additive group). Endowing the latter with the isotropy action, this identification becomes equivariant. These considerations allow us to view W as the vector representation of $\mathrm{SO}(4)$:

$$W \cong \langle \partial_{y^2}, \partial_{y^3}, \partial_{y^4}, \partial_{y^5} \rangle. \quad (3.28)$$

The extendibility problem for invariant tensors is addressed by examining their restrictions along W . We are thus interested in finding a useful way to describe these restrictions. Pull the bundle

$$\Lambda^k T^*P \otimes \underline{V}$$

back to W using the inclusion map (here we denote by $\underline{V}$ the trivial vector bundle with fiber V). Since W is linear, the pullback is trivial. We will now give a particular trivialization that elucidates the action of $\mathrm{SO}(4)$. Using the canonical invariant connection to decompose TP into vertical and horizontal distributions, we obtain an equivariant identification:

$$TP \cong \pi^*TX^8 \oplus \underline{\mathfrak{s}}. \quad (3.29)$$

Furthermore, there is an obvious $\mathrm{SO}(4)$ -equivariant trivialization:

$$TX^8_{|\pi(W)} \cong \pi(W) \times (\langle X_1, \dots, X_4 \rangle \oplus \langle \partial_{y^2}, \partial_{y^3}, \partial_{y^4}, \partial_{y^5} \rangle). \quad (3.30)$$

Putting these together we have:

$$\begin{aligned} & \left(\Lambda^k T^*P \otimes \underline{V} \right)_{|W} \\ & \cong W \times \left(\Lambda^k \langle X_1, \dots, X_4 \rangle^* \otimes V \oplus \Lambda^k \langle \partial_{y^2}, \partial_{y^3}, \partial_{y^4}, \partial_{y^5} \rangle^* \otimes V \oplus \Lambda^k \mathfrak{s}^* \otimes V \right). \end{aligned}$$

Here, the action of $\mathrm{SO}(4)$ is as follows: The action on $\mathfrak{s}$ is trivial. The action on V is obtained by composing μ and ρ . Finally, the spaces $\langle X_1, \dots, X_4 \rangle^*$ and $\langle \partial_{y^2}, \partial_{y^3}, \partial_{y^4}, \partial_{y^5} \rangle^*$ are vector representations.

We study the restriction of ω along W_0 : the vector space W punctured at its origin

$$W_0 \stackrel{\text{def}}{=} W - \{x_1\}.$$

Since tensorial forms vanish on vertical vectors, $\omega|_{W_0}$ is a section of the trivial bundle with fiber equal to:

$$E \stackrel{\text{def}}{=} \Lambda^k \langle X_1, \dots, X_4 \rangle^* \otimes V \oplus \Lambda^k \langle \partial_{y^2}, \partial_{y^3}, \partial_{y^4}, \partial_{y^5} \rangle^* \otimes V.$$

Due to the triviality of the bundle, the form $\omega|_{W_0}$ amounts to an $\mathrm{SO}(4)$ -equivariant function:

$$f : W_0 \rightarrow E.$$

The invariance of ω implies that no information is lost in passing from ω to f . In fact, ω is determined by the values of f on the reference points x_r forming a ray from the origin of W to infinity. This recovers our usual description of invariant forms as curves in a group representation:

$$\omega_r : (1, \infty) \rightarrow E. \quad (3.31)$$

Eschenburg and Wang prove that the extendibility of ω is contingent to a representation-theoretic condition on the formal Taylor series expansion of an appropriate reparameterization of ω_r . This condition reflects the behaviour of f near x_1 .

The requisite reparameterization is obtained as follows. Using (3.28), the Euclidean metric on $\mathbb{R}^{10}$ induces an inner product on W . We consider the radial function of the associated norm. Concretely, we set:

$$t \stackrel{\text{def}}{=} R_- = \left(\frac{r^2 - 1}{2} \right)^{\frac{1}{2}}, \quad r(t) = \left(2t^2 + 1 \right)^{\frac{1}{2}}. \quad (3.32)$$

We thus obtain a curve:

$$\gamma(t) \stackrel{\text{def}}{=} \omega_{r(t)}.$$

It is clear that f determines γ and vice-versa:

$$\gamma(t) = f(x_{r(t)}).$$

The result of Eschenburg and Wang ([6], Lemma 1.1, p. 113) asserts that ω extends smoothly over the singular orbit if and only if the following hold:

- The curve γ is smooth from the right at $t = 0$
- The formal Taylor series of γ at $t = 0$ can be written as:

$$\gamma \sim \sum_{k \geq 0} u_k(x_{|_{t=1}}) t^k$$

where:

$$u_k : W \rightarrow E$$

is a homogeneous equivariant polynomial of degree k .

Note that we have provided explicit descriptions of the $\text{SO}(4)$ -actions on W and E . These descriptions facilitate the computations required for applications.

3.3.2. Application: extendibility of connections We are interested in studying the extendibility of tensorial forms ω describing connections on P (relative to the canonical invariant connection). Therefore -in the context of our application- we have:

$$S = \text{SO}(3), \quad V = \mathfrak{so}(3), \quad \rho = \text{Ad}_{\text{SO}(3)}, \quad k = 1.$$

Given our setup, ω will usually be available in the form (3.31). Given this data, we need to pass to the associated curve $\gamma(t)$ and express it in a basis of E coming from evaluation of homogeneous equivariant polynomials at $x_{|_{t=1}} \in W$. To achieve this, we need to be able to find appropriate equivariant polynomials. This task can be simplified if we understand the relevant representations in terms of quaternions. To this end, we identify the spaces W and $\langle X_1, \dots, X_4 \rangle$ with $\mathbb{H}$ by:

$$\langle X^1, X^2, X^3, X^4 \rangle \cong \langle 1, i, j, k \rangle \cong \langle \partial_{y^2}, \partial_{y^3}, \partial_{y^4}, \partial_{y^5} \rangle.$$

Furthermore, we lift the action of $\text{SO}(4)$ to $\text{Sp}(1)^2$ using the covering map $\pi_{\text{Spin}(4)}$. Under these identifications, the $\text{SO}(4)$ -action is captured by the usual spinor representation of $\text{Sp}(1)^2$ on $\mathbb{H}$.

General points $p \in W$ and $q \in \langle X_1, \dots, X_4 \rangle$ can be written as:

$$\begin{aligned} p &= p^0 X_1 + p^1 X_2 + p^2 X_3 + p^3 X_4, & q &= q^0 \partial_{y^2} + q^1 \partial_{y^3} + q^2 \partial_{y^4} + q^3 \partial_{y^5} \\ &= p^0 + p^1 i + p^2 j + p^3 k & &= q^0 + q^1 i + q^2 j + q^3 k. \end{aligned}$$

With this choice of coordinates we have:

$$x_{|_{t=1}} = 1 \in \text{Sp}(1) \subset \mathbb{H}.$$

The Lie algebra $\mathfrak{so}(3)$ can be naturally identified with $\mathfrak{sp}(1) = \mathfrak{Im}(\mathbb{H})$ using the differential of the covering map $\pi_{\text{Spin}(3)}$ (defined in (3.4)). This identification is Ad-equivariant. Explicitly, it takes the following form:

$$d\pi_{\text{Spin}(3)|_1} : i \mapsto -2e_3, \quad j \mapsto 2e_2, \quad k \mapsto -2e_1. \quad (3.33)$$

These considerations demonstrate that we require homogeneous $\text{Sp}(1)^2$ -equivariant polynomials:

$$u : \mathbb{H} \rightarrow \mathbb{H}^* \otimes \mathfrak{Im}(\mathbb{H}) \oplus \mathbb{H}^* \otimes \mathfrak{Im}(\mathbb{H})$$

with prescribed value at $x = 1$. Separating the components in the target, such maps take the form:

$$u(x)(p, q) = u_1(x)(p) + u_2(x)(q).$$

The $\text{Sp}(1)^2$ -equivariance condition for $u : W \rightarrow E$ translates to the following:

$$u_1(ax\bar{b})(p) = \text{Ad}_{\mu \circ \pi_{\text{Spin}(4)}(a,b)} u_1(x)(\bar{a}pb) \text{ for all } (a, b) \in \text{Sp}(1)^2, \quad (3.34)$$

$$u_2(ax\bar{b})(q) = \text{Ad}_{\mu \circ \pi_{\text{Spin}(4)}(a,b)} u_2(x)(\bar{a}qb) \text{ for all } (a, b) \in \text{Sp}(1)^2. \quad (3.35)$$

The Case of P_{π_1} In this case $\mu = \pi_1$. The action of $\text{Sp}(1)^2$ on $\mathfrak{Im}(\mathbb{H})$ is given by projecting the group element to the first factor and conjugating by the result. Conditions (3.34), (3.35) become:

$$u_1(ax\bar{b})(p) = au_1(x)(\bar{a}pb)\bar{a} \text{ for all } (a, b) \in \text{Sp}(1)^2, \quad (3.36)$$

$$u_2(ax\bar{b})(q) = au_2(x)(\bar{a}qb)\bar{a} \text{ for all } (a, b) \in \text{Sp}(1)^2. \quad (3.37)$$

Using (3.19), we write a general invariant connection over $X^8 \setminus S^4$ as $A_{\pi_1}^{\text{can}} + \omega$, where:

$$\omega = a(r(t))(\theta^2 \otimes e_3 - \theta^3 \otimes e_2 + \theta^4 \otimes e_1) + (b(r(t)) - 1)(\theta^5 \otimes e_3 - \theta^6 \otimes e_2 + \theta^7 \otimes e_1).$$

Using (1.24)–(1.27) and (3.33) we find that the form $\theta^2 \otimes e_3 - \theta^3 \otimes e_2 + \theta^4 \otimes e_1$ corresponds to:

$$\begin{aligned} (p, q) &\mapsto -\frac{1}{2}(p^1 i + p^2 j + p^3 k) \\ &= -\frac{1}{2}\mathfrak{Im}(p) \\ &= \frac{\langle p, 1 \rangle - p}{2}. \end{aligned} \quad (3.38)$$

Similarly, using (1.24)–(1.27), (3.32) and (3.33), we find that the form $\theta^5 \otimes e_3 - \theta^6 \otimes e_2 + \theta^7 \otimes e_1$ corresponds to:

$$\begin{aligned}
 (p, q) &\mapsto \frac{1}{2t} \left(q^1 i + q^2 j + q^3 k \right) \\
 &= \frac{1}{2t} \Im(q) \\
 &= \frac{q - \langle q, 1 \rangle}{2t}.
 \end{aligned} \tag{3.39}$$

Any equivariant polynomial u satisfying:

$$u(1)(p, q) = \frac{\langle p, 1 \rangle - p}{2} \tag{3.40}$$

has the following restriction on $S^3 \subset \mathbb{H}$:

$$u(x)(p) = \frac{\langle x, p \rangle - p\bar{x}}{2}. \tag{3.41}$$

Similarly, any equivariant polynomial v satisfying:

$$v(1)(p, q) = \frac{q - \langle q, 1 \rangle}{2} \tag{3.42}$$

has the following restriction on $S^3 \subset \mathbb{H}$:

$$v(x)(q) = \frac{q\bar{x} - \langle x, q \rangle}{2}. \tag{3.43}$$

As soon as u and v are specified on the unit sphere, they are extended to $\mathbb{H}$ by homogeneity. The extensions depend on the degree d , which is yet unspecified. Given d , we define:

$$\begin{aligned}
 u_d(x)(p) &\stackrel{\text{def}}{=} \begin{cases} |x|^d u\left(\frac{x}{|x|}\right)(p) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases} \\
 v_d(x)(p) &\stackrel{\text{def}}{=} \begin{cases} |x|^d v\left(\frac{x}{|x|}\right)(p) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}
 \end{aligned}$$

The admissible values of d are constrained: not all choices yield polynomial extensions. Equations (3.41) and (3.43) demonstrate that u_0 and v_0 are not constant functions. If they were polynomials they would have homogeneous degree $d = 0$ and would thus be constant. We conclude that the choice $d = 0$ is not admissible. However, it is clear that we could choose $d = 1$. This would correspond to defining the extensions by the formulae (3.41) and (3.43) on the whole of $\mathbb{H}$. Now, if two homogeneous polynomials agree on the unit sphere, they are related through multiplication by the homogeneous degree $2k$ polynomial $|x|^{2k}$. Consequently, all other admissible choices of d are obtained by adding even integers to $d = 1$. We conclude that for each odd positive integer:

$$d = 1 + 2k \tag{3.44}$$

we have precisely one homogeneous equivariant polynomial u_d of degree d satisfying (3.40) and precisely one homogeneous equivariant polynomial v_d of degree d satisfying (3.42).

We rewrite the form ω as:

$$\omega = a(r(t)) u_d(1) + \frac{b(r(t)) - 1}{t} v_d(1). \quad (3.45)$$

Applying the criterion of Eschenburg and Wang we obtain:

Proposition 3.4. *Let $A \in \mathcal{A}_{\text{inv}}(P_{\text{Id}})$ be an invariant connection defined over $X^8 \setminus S^4$. Let*

$$\omega = a(r(t)) (\theta^2 \otimes e_3 - \theta^3 \otimes e_2 + \theta^4 \otimes e_1) + (b(r(t)) - 1) (\theta^5 \otimes e_3 - \theta^6 \otimes e_2 + \theta^7 \otimes e_1)$$

be the tensorial form expressing A with respect to the canonical invariant connection of P_{π_1} . Then A extends over the singular orbit on P_{π_1} if and only if the following hold:

- *The function $a(r(t))$ is smooth from the right at $t = 0$, odd and $O(t)$.*
- *The function $b(r(t)) - 1$ is smooth from the right at $t = 0$, even and $O(t^2)$.*

The Case of P_{π_2} In this case $\mu = \pi_2$. The action of $\text{Sp}(1)^2$ on $\mathfrak{Im}(\mathbb{H})$ is given by projecting the group element to the second factor and conjugating by the result. Conditions (3.34), (3.35) become:

$$u_1(ax\bar{b})(p) = bu_1(x)(\bar{a}pb)\bar{b} \text{ for all } (a, b) \in \text{Sp}(1)^2, \quad (3.46)$$

$$u_2(ax\bar{b})(q) = bu_2(x)(\bar{a}qb)\bar{b} \text{ for all } (a, b) \in \text{Sp}(1)^2. \quad (3.47)$$

Using (3.20), we write a general invariant connection over $X^8 \setminus S^4$ as $A_{\pi_2}^{\text{can}} + \omega$, where:

$$\omega = a(r(t)) (\theta^2 \otimes e_3 - \theta^3 \otimes e_2 + \theta^4 \otimes e_1) + (b(r(t)) + 1) (\theta^5 \otimes e_3 - \theta^6 \otimes e_2 + \theta^7 \otimes e_1).$$

Recalling (3.38) and (3.39) we seek homogeneous equivariant polynomials u and v satisfying (3.40) and (3.42) respectively. Equivariance specifies their restrictions on $S^3 \subset \mathbb{H}$:

$$u(x)(p) = \frac{\langle x, p \rangle - \bar{x}p}{2}, \quad v(x)(q) = \frac{\bar{x}q - \langle x, q \rangle}{2}. \quad (3.48)$$

Arguing as in the case of P_{π_1} we find that there is precisely one homogeneous equivariant polynomial u_d and precisely one homogeneous equivariant polynomial v_d satisfying the requisite conditions in each odd degree $d = 1 + 2k$.

We rewrite the form ω as:

$$\omega = a(r(t)) u_d(1) + \frac{b(r(t)) + 1}{t} v_d(1). \quad (3.49)$$

Applying the criterion of Eschenburg and Wang we obtain:

Proposition 3.5. *Let $A \in \mathcal{A}_{\text{inv}}(P_{\text{Id}})$ be an invariant connection defined over $X^8 \setminus S^4$. Let*

$$\omega = a(r(t)) (\theta^2 \otimes e_3 - \theta^3 \otimes e_2 + \theta^4 \otimes e_1) + (b(r(t)) + 1) (\theta^5 \otimes e_3 - \theta^6 \otimes e_2 + \theta^7 \otimes e_1)$$

be the tensorial form expressing A with respect to the canonical invariant connection of P_{π_2} . Then A extends over the singular orbit on P_{π_2} if and only if the following hold:

- The function $a(r(t))$ is smooth from the right at $t = 0$, odd and $O(t)$.
- The function $b(r(t)) + 1$ is smooth from the right at $t = 0$, even and $O(t^2)$.

4. $SO(5)$ -Invariant Instantons with Structure Group $SO(3)$: Analysis on P_{Id}

In the remainder, we study the invariant equations on the bundle P_{Id} . The relevant analysis constitutes the heart of the present article.

4.1. The $SO(5)$ -invariant HYM ODEs on P_{Id} . We begin with the HYM equations for which the situation is significantly simpler.

4.1.1. Derivation A general invariant connection $A \in \mathcal{A}_{\text{inv}}(P_{Id})$ takes the form (3.12). The associated curvature tensor $F_A = F_A^j \otimes e_j$ is given in (3.15). The Hodge dual of the Kähler form has been computed in (2.17). We observe that:

$$F_A \wedge \star_g \omega = 0. \quad (4.1)$$

Consequently, the HYM equations reduce to (2.15). We write:

$$F_A \wedge \Omega = \left[F_A^j \wedge \Re(\Omega) + i F_A^j \wedge \Im(\Omega) \right] \otimes e_j.$$

Using (1.43) and (1.44) we obtain the results:

$$\begin{aligned} F_A^1 \wedge \Re(\Omega) &= (R_+^3(1-b^2) - R_+R_-^2(1-a^2))\theta^{123456} - 2R_+R_-^2ab\theta^{123567} \\ &\quad + \frac{r}{2} \left(R_+(1-b^2) - \frac{R_-^2}{R_+}(1-a^2) \right) dr \wedge \theta^{23567} + rR_+ab\,dr \wedge \theta^{23456} \\ &\quad + \left(\frac{rR_+}{2}b + R_+^3\frac{db}{dr} \right) dr \wedge \theta^{12347} - \left(\frac{rR_-^2}{2R_+}b + R_+R_-^2\frac{db}{dr} \right) dr \wedge \theta^{14567} \\ &\quad - \left(\frac{rR_+}{2}a + R_+R_-^2\frac{da}{dr} \right) dr \wedge \theta^{12467} + \left(\frac{rR_+}{2}a + R_+R_-^2\frac{da}{dr} \right) dr \wedge \theta^{13457}, \end{aligned} \quad (4.2)$$

$$\begin{aligned} F_A^2 \wedge \Re(\Omega) &= (R_+^3(1-b^2) - R_+R_-^2(1-a^2))\theta^{123457} - 2R_+R_-^2ab\theta^{124567} \\ &\quad + \frac{r}{2} \left(R_+(1-b^2) - \frac{R_-^2}{R_+}(1-a^2) \right) dr \wedge \theta^{24567} + rR_+ab\,dr \wedge \theta^{23457} \\ &\quad - \left(\frac{rR_+}{2}b + R_+^3\frac{db}{dr} \right) dr \wedge \theta^{12346} + \left(\frac{rR_-^2}{2R_+}b + R_+R_-^2\frac{db}{dr} \right) dr \wedge \theta^{13567} \\ &\quad + \left(\frac{rR_+}{2}a + R_+R_-^2\frac{da}{dr} \right) dr \wedge \theta^{12367} - \left(\frac{rR_+}{2}a + R_+R_-^2\frac{da}{dr} \right) dr \wedge \theta^{13456}, \end{aligned} \quad (4.3)$$

$$\begin{aligned} F_A^3 \wedge \Re(\Omega) &= (R_+^3(1-b^2) - R_+R_-^2(1-a^2))\theta^{123467} - 2R_+R_-^2ab\theta^{134567} \\ &\quad + \frac{r}{2} \left(R_+(1-b^2) - \frac{R_-^2}{R_+}(1-a^2) \right) dr \wedge \theta^{34567} + rR_+ab\,dr \wedge \theta^{23467} \end{aligned}$$

$$\begin{aligned}
& + \left(\frac{rR_+}{2}b + R_+^3 \frac{db}{dr} \right) dr \wedge \theta^{12345} - \left(\frac{rR_-^2}{2R_+}b + R_+R_-^2 \frac{db}{dr} \right) dr \wedge \theta^{12567} \\
& - \left(\frac{rR_+}{2}a + R_+R_-^2 \frac{da}{dr} \right) dr \wedge \theta^{12357} + \left(\frac{rR_+}{2}a + R_+R_-^2 \frac{da}{dr} \right) dr \wedge \theta^{12456},
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
F_A^1 \wedge \mathfrak{Im}(\Omega) &= (R_+^2R_-(1-b^2) - R_-^3(1-a^2))\theta^{123567} + 2R_+^2R_-ab\theta^{123456} \\
&- \frac{r}{2} \left(\frac{R_+^2}{R_-}(1-b^2) - R_-(1-a^2) \right) dr \wedge \theta^{23456} + rR_-ab\,dr \wedge \theta^{23567} \\
&- \left(\frac{rR_-}{2}b + R_+^2R_- \frac{db}{dr} \right) dr \wedge \theta^{12467} + \left(\frac{rR_-}{2}b + R_+^2R_- \frac{db}{dr} \right) dr \wedge \theta^{13457} \\
&+ \left(\frac{rR_-}{2}a + R_-^3 \frac{da}{dr} \right) dr \wedge \theta^{14567} - \left(\frac{rR_+^2}{2R_-}a + R_+^2R_- \frac{da}{dr} \right) dr \wedge \theta^{12347},
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
F_A^2 \wedge \mathfrak{Im}(\Omega) &= (R_+^2R_-(1-b^2) - R_-^3(1-a^2))\theta^{124567} + 2R_+^2R_-ab\theta^{123457} \\
&- \frac{r}{2} \left(\frac{R_+^2}{R_-}(1-b^2) - R_-(1-a^2) \right) dr \wedge \theta^{23457} + rR_-ab\,dr \wedge \theta^{24567} \\
&+ \left(\frac{rR_-}{2}b + R_+^2R_- \frac{db}{dr} \right) dr \wedge \theta^{12367} - \left(\frac{rR_-}{2}b + R_+^2R_- \frac{db}{dr} \right) dr \wedge \theta^{13456} \\
&- \left(\frac{rR_-}{2}a + R_-^3 \frac{da}{dr} \right) dr \wedge \theta^{13567} + \left(\frac{rR_+^2}{2R_-}a + R_+^2R_- \frac{da}{dr} \right) dr \wedge \theta^{12346},
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
F_A^3 \wedge \mathfrak{Im}(\Omega) &= (R_+^2R_-(1-b^2) - R_-^3(1-a^2))\theta^{134567} + 2R_+^2R_-ab\theta^{123467} \\
&- \frac{r}{2} \left(\frac{R_+^2}{R_-}(1-b^2) - R_-(1-a^2) \right) dr \wedge \theta^{23467} + rR_-ab\,dr \wedge \theta^{34567} \\
&- \left(\frac{rR_-}{2}b + R_+^2R_- \frac{db}{dr} \right) dr \wedge \theta^{12357} + \left(\frac{rR_-}{2}b + R_+^2R_- \frac{db}{dr} \right) dr \wedge \theta^{12456} \\
&+ \left(\frac{rR_-}{2}a + R_-^3 \frac{da}{dr} \right) dr \wedge \theta^{12567} - \left(\frac{rR_+^2}{2R_-}a + R_+^2R_- \frac{da}{dr} \right) dr \wedge \theta^{12345}.
\end{aligned} \tag{4.7}$$

Observe that there are similarities among the various components. In particular the vanishing of any one of them is equivalent to the full HYM system. Setting any one of the components to be zero gives the invariant equations. They are as follows:

$$\frac{da}{dr} = -\frac{ra}{2R_-^2}, \tag{4.8}$$

$$\frac{db}{dr} = -\frac{rb}{2R_+^2}, \tag{4.9}$$

$$R_+^2(1-b^2) = R_-^2(1-a^2), \tag{4.10}$$

$$ab = 0. \tag{4.11}$$

Consequently, the invariant HYM connections over P_{Id} obey the differential equations (4.8), (4.9) and satisfy the order 0 constraints (4.10), (4.11). Observe that (2.15) involves

only the holomorphic volume form of X^8 . As a result, the coefficients of the Stenzel metric do not appear in (4.8)–(4.11).

4.1.2. Explicit solution and asymptotics of the curvature tensor The Eqs. (4.8)–(4.11) can be solved explicitly. We obtain precisely two solutions:

$$A_{\text{HYM}\pi_1} \stackrel{\text{def}}{=} A_{\text{Id}}^{\text{can}} + \frac{1}{R_+} \left(\theta^5 \otimes e_3 - \theta^6 \otimes e_2 + \theta^7 \otimes e_1 \right), \quad (4.12)$$

$$A_{\text{HYM}\pi_2} \stackrel{\text{def}}{=} A_{\text{Id}}^{\text{can}} - \frac{1}{R_+} \left(\theta^5 \otimes e_3 - \theta^6 \otimes e_2 + \theta^7 \otimes e_1 \right). \quad (4.13)$$

The notation has been chosen in hindsight to reflect the bundle on which these connections extend. In particular, using Propositions 3.4 and 3.5, we find that $A_{\text{HYM}\pi_1}$ extends to P_{π_1} , while $A_{\text{HYM}\pi_2}$ extends to P_{π_2} . We treat $A_{\text{HYM}\pi_1}$ in detail. Recalling (3.32) we find that in this case:

$$a(r(t)) = 0, \quad b(r(t)) - 1 = \frac{1 - \sqrt{t^2 + 1}}{\sqrt{t^2 + 1}}.$$

The conditions on $a(r(t))$ are trivially satisfied. The function $b(r(t)) - 1$ is obviously smooth from the right at $t = 0$. It is even, since t only appears in power 2. One easily computes that both $b(r(t)) - 1$ and its first derivative vanish at $t = 0$. Consequently, $b(r(t)) - 1 = O(t^2)$.

Using (3.15) we compute the associated curvature tensors:

$$\begin{aligned} F_{A_{\text{HYM}\pi_1}} &= \left(\theta^{23} + \frac{R_-^2}{R_+^2} \theta^{56} + \frac{1}{R_+} \theta^{14} - \frac{r}{2R_+^3} dr \wedge \theta^7 \right) \otimes e_1 \\ &\quad + \left(\theta^{24} + \frac{R_-^2}{R_+^2} \theta^{57} - \frac{1}{R_+} \theta^{13} + \frac{r}{2R_+^3} dr \wedge \theta^6 \right) \otimes e_2 \\ &\quad + \left(\theta^{34} + \frac{R_-^2}{R_+^2} \theta^{67} + \frac{1}{R_+} \theta^{12} - \frac{r}{2R_+^3} dr \wedge \theta^5 \right) \otimes e_3, \end{aligned} \quad (4.14)$$

$$\begin{aligned} F_{A_{\text{HYM}\pi_2}} &= \left(\theta^{23} + \frac{R_-^2}{R_+^2} \theta^{56} - \frac{1}{R_+} \theta^{14} + \frac{r}{2R_+^3} dr \wedge \theta^7 \right) \otimes e_1 \\ &\quad + \left(\theta^{24} + \frac{R_-^2}{R_+^2} \theta^{57} + \frac{1}{R_+} \theta^{13} - \frac{r}{2R_+^3} dr \wedge \theta^6 \right) \otimes e_2 \\ &\quad + \left(\theta^{34} + \frac{R_-^2}{R_+^2} \theta^{67} - \frac{1}{R_+} \theta^{12} + \frac{r}{2R_+^3} dr \wedge \theta^5 \right) \otimes e_3. \end{aligned} \quad (4.15)$$

Since both solutions smoothly extend to the singular orbit, we can study their pullbacks. The connection $A_{\text{HYM}\pi_1}$ pulls back to the canonical invariant connection of P_{π_1} over S^4 :

$$\begin{aligned} A_{\text{SD}} &\stackrel{\text{def}}{=} A_{\text{HYM}\pi_1}|_{S^4} \\ &= \left(\theta^8 + \theta^7 \right) \otimes e_1 + \left(\theta^9 - \theta^6 \right) \otimes e_2 + \left(\theta^{10} + \theta^5 \right) \otimes e_3. \end{aligned} \quad (4.16)$$

Its curvature is given by:

$$F_{A_{SD}} = (\theta^{23} + \theta^{14}) \otimes e_1 + (\theta^{24} - \theta^{13}) \otimes e_2 + (\theta^{34} + \theta^{12}) \otimes e_3. \quad (4.17)$$

An explicit calculation demonstrates that A_{SD} is a self-dual instanton [2] on S^4 . This justifies our choice of notation.

The connection $A_{HYM_{\pi_2}}$ pulls back to the canonical invariant connection of P_{π_2} :

$$\begin{aligned} A_{A_{SD}} &\stackrel{\text{def}}{=} A_{HYM_{\pi_2}|_{S^4}} \\ &= (\theta^8 - \theta^7) \otimes e_1 + (\theta^9 + \theta^6) \otimes e_2 + (\theta^{10} - \theta^5) \otimes e_3. \end{aligned} \quad (4.18)$$

Its curvature is given by:

$$F_{A_{A_{SD}}} = (\theta^{23} - \theta^{14}) \otimes e_1 + (\theta^{24} + \theta^{13}) \otimes e_2 + (\theta^{34} - \theta^{12}) \otimes e_3. \quad (4.19)$$

An explicit calculation demonstrates that $A_{A_{SD}}$ is an anti-self-dual instanton [2] on S^4 . This justifies our choice of notation.

We wish to endow $\text{ad}(P_{\pi_1})$ with a fiber metric. To this end, it suffices to choose an Ad-invariant inner product on $\mathfrak{so}(3)$. In general, the choice of such an inner product is free. However, we shall choose $\langle \cdot, \cdot \rangle$ so that:

$$e_i \perp e_j \text{ if } i \neq j, \quad \|e_i\|^2 = 2. \quad (4.20)$$

This is the unique inner product on $\mathfrak{so}(3)$ satisfying:

$$\|\xi\|^2 = -\text{Tr}(\xi^2). \quad (4.21)$$

This identity is required to relate the Yang–Mills energy of instantons to characteristic classes of the underlying bundle. Using the Stenzel metric and the fiber metric (4.20) we see that the curvature norms of the two connections have the same constant value on all points of S^4 :

$$|F_{A_{SD}}|^2 = |F_{A_{A_{SD}}}|^2 = 12.$$

Since the restriction of the Stenzel metric on the singular orbit is round of unit radius, we have:

$$\begin{aligned} \mathcal{YM}(F_{A_{SD}}) &= \mathcal{YM}(F_{A_{A_{SD}}}) \\ &= \int_{S^4} |F_{A_{SD}}|^2 dV_g \\ &= 12 \text{Vol}(S^4) \\ &= 32\pi^2. \end{aligned} \quad (4.22)$$

Owing to (4.21), any $\text{SO}(3)$ -connection satisfies:

$$\text{Tr}(F_A^2) = -(|F_A^+|^2 - |F_A^-|^2) dV_g. \quad (4.23)$$

By Chern–Weil theory, the first Pontryagin class of the underlying $\mathrm{SO}(3)$ -bundle is the cohomology class of the 4-form:

$$p_1 = -\frac{1}{8\pi^2} \mathrm{Tr} \left(F_A^2 \right) \quad (4.24)$$

Using (4.23) and (4.24), we conclude that the self-dual $\mathrm{SO}(3)$ -instantons on S^4 have Yang–Mills energy equal to $8\pi^2$ times the integral of the first Pontryagin class of the underlying bundle. Similarly, the anti-self-dual instantons have Yang–Mills energy equal to $-8\pi^2$ times the integral of the first Pontryagin class. Using (4.22) we obtain:

$$p_1(P_{\pi_1}) = 4, \quad (4.25)$$

$$p_1(P_{\pi_2}) = -4. \quad (4.26)$$

We already knew that P_{π_1} and P_{π_2} are not equivariantly trivial nor equivariantly isomorphic to each other. The above calculation demonstrates that they are genuinely nontrivial and non-isomorphic (even if we drop the requirement that the identification be equivariant).

Over a manifold with vanishing second cohomology, each $\mathrm{SO}(3)$ -bundle is associated to a unique $\mathrm{SU}(2)$ -bundle through the natural two sheeted covering projection. The second Chern class of the lift is related to the first Pontryagin class of the original bundle by the equation ([2], pp. 41–42):

$$p_1 = -4c_2 \quad (4.27)$$

Consequently, the $\mathrm{SU}(2)$ -lift of A_{ASD} has topological charge equal to 1. The compactified moduli space of charge-one anti-self-dual instantons with structure group $\mathrm{SU}(2)$ over S^4 can be identified with the closed 5-ball B^5 ([2], p. 126). The lift of A_{ASD} coincides with the center of B^5 . It is the only $\mathrm{SO}(5)$ -invariant instanton in this moduli space. Analogous remarks hold for A_{SD} . This can be seen by reversing orientation to make it anti-self-dual.

Finally, we study the asymptotic behaviour of the curvature tensors corresponding to the two HYM solutions. Since (4.14) and (4.15) only differ by certain signs, it suffices to treat $A_{\mathrm{HYM}_{\pi_1}}$. Using (4.20), (1.50)–(1.53) and (4.14), we find that:

$$|F_{A_{\mathrm{HYM}_{\pi_1}}}|^2 = 8\sqrt{3} \frac{3r^4 + 10r^2 + 11}{(r^2 + 1)^3 (r^2 + 2)^{\frac{1}{2}}}.$$

In particular, we see that as $r \rightarrow \infty$:

$$|F_{A_{\mathrm{HYM}_{\pi_1}}}| = O(r^{-\frac{3}{2}}) = O(s^{-2}).$$

Here we have used the radial coordinate $s(r)$ introduced in (1.57) to exhibit the asymptotically conical growth of the Stenzel metric. This decay rate is not sufficient for the Yang–Mills energy to be finite: on an asymptotically conical n -manifold, a function is integrable when it decays faster than $O(s^{-n})$.

In fact, the decay rate of the curvature norm could have been inferred directly from (4.12) and (4.13). Both HYM solutions decay to the translation invariant limit $A_{\mathrm{Id}}^{\mathrm{can}}$. When a connection defined over an asymptotically conical manifold decays to a translation invariant limit, it is itself termed *asymptotically conical* ([5], p. 38). Such connections always have curvature tensors decaying like $O(s^{-2})$.

An important (gauge-invariant) attribute of an asymptotically conical connection is the rate at which the associated curvature tensor approaches its limit. In this case, we have:

$$|F_{A_{\text{HYM}_{\pi_1}}} - F_{A_{\text{Id}}^{\text{can}}}| = O(s^{-\frac{10}{3}}) \quad (4.28)$$

with the same being true of $A_{\text{HYM}_{\pi_2}}$.

4.2. The $SO(5)$ -invariant $Spin(7)$ instanton ODEs on P_{Id} .

4.2.1. Derivation We consider a general invariant connection $A \in \mathcal{A}_{\text{inv}}(P_{\text{Id}})$ with associated curvature field:

$$F_A = F_A^j \otimes e_j.$$

We remind the reader that these take the form (3.12), (3.15). Using (1.54) we compute:

$$\begin{aligned} \Phi \wedge F_A^1 &= \left(PQ(1-a^2) + \frac{rR_+}{2}b + R_+^3 \frac{db}{dr} \right) dr \wedge \theta^{12347} \\ &\quad + \left(PQ(1-b^2) - \frac{rR_-^2}{2R_+}b - R_+R_-^2 \frac{db}{dr} \right) dr \wedge \theta^{14567} \\ &\quad + \left(PQab - \frac{rR_+}{2}a - R_+R_-^2 \frac{da}{dr} \right) dr \wedge \theta^{12467} \\ &\quad + \left(-PQab + \frac{rR_+}{2}a + R_+R_-^2 \frac{da}{dr} \right) dr \wedge \theta^{13457} \\ &\quad + \left(\frac{rR_+}{2}(1-b^2) - \frac{rR_-^2}{2R_+}(1-a^2) - Q^2 \frac{db}{dr} \right) dr \wedge \theta^{23567} \\ &\quad + \left(rR_+ab - Q^2 \frac{da}{dr} \right) dr \wedge \theta^{23456} \\ &\quad + \left(R_+^3(1-b^2) - R_+R_-^2(1-a^2) - Q^2b \right) \theta^{123456} \\ &\quad + \left(-2R_+R_-^2ab + Q^2a \right) \theta^{123567}, \quad (4.29) \\ \Phi \wedge F_A^2 &= - \left(PQ(1-a^2) + \frac{rR_+}{2}b + R_+^3 \frac{db}{dr} \right) dr \wedge \theta^{12346} \\ &\quad + \left(-PQ(1-b^2) + \frac{rR_-^2}{2R_+}b + R_+R_-^2 \frac{db}{dr} \right) dr \wedge \theta^{13567} \\ &\quad + \left(-PQab + \frac{rR_+}{2}a + R_+R_-^2 \frac{da}{dr} \right) dr \wedge \theta^{12367} \\ &\quad + \left(PQab - \frac{rR_+}{2}a - R_+R_-^2 \frac{da}{dr} \right) dr \wedge \theta^{13456} \\ &\quad + \left(\frac{rR_+}{2}(1-b^2) - \frac{rR_-^2}{2R_+}(1-a^2) - Q^2 \frac{db}{dr} \right) dr \wedge \theta^{24567} \end{aligned}$$

$$\begin{aligned}
& + \left(r R_+ ab - Q^2 \frac{da}{dr} \right) dr \wedge \theta^{23457} \\
& + \left(R_+^3 (1 - b^2) - R_+ R_-^2 (1 - a^2) - Q^2 b \right) \theta^{123457} \\
& + \left(-2R_+ R_-^2 ab + Q^2 a \right) \theta^{124567}, \tag{4.30}
\end{aligned}$$

$$\begin{aligned}
\Phi \wedge F_A^3 = & \left(P Q (1 - a^2) + \frac{r R_+}{2} b + R_+^3 \frac{db}{dr} \right) dr \wedge \theta^{12345} \\
& + \left(P Q (1 - b^2) - \frac{r R_-^2}{2 R_+} b - R_+ R_-^2 \frac{db}{dr} \right) dr \wedge \theta^{12567} \\
& + \left(P Q ab - \frac{r R_+}{2} a - R_+ R_-^2 \frac{da}{dr} \right) dr \wedge \theta^{12357} \\
& + \left(-P Q ab + \frac{r R_+}{2} a + R_+ R_-^2 \frac{da}{dr} \right) dr \wedge \theta^{12456} \\
& + \left(\frac{r R_+}{2} (1 - b^2) - \frac{r R_-^2}{2 R_+} (1 - a^2) - Q^2 \frac{db}{dr} \right) dr \wedge \theta^{34567} \\
& + \left(r R_+ ab - Q^2 \frac{da}{dr} \right) dr \wedge \theta^{23467} \\
& + \left(R_+^3 (1 - b^2) - R_+ R_-^2 (1 - a^2) - Q^2 b \right) \theta^{123467} \\
& + \left(-2R_+ R_-^2 ab + Q^2 a \right) \theta^{134567}. \tag{4.31}
\end{aligned}$$

We now wish to calculate the Hodge dual of the curvature. We will require the Hodge duals of all 2-forms θ^{ij} . These can be computed using (1.41) and (2.10). Carrying out this calculation yields:

$$\begin{aligned}
\star_g F_A^1 = & - (1 - a^2) \frac{P Q R_-^2}{R_+^2} dr \wedge \theta^{14567} - (1 - b^2) \frac{P Q R_+^2}{R_-^2} dr \wedge \theta^{12347} \\
& + ab P Q dr \wedge \theta^{12467} - ab P Q dr \wedge \theta^{13457} \\
& - \frac{r Q^2}{2 R_+^2} b dr \wedge \theta^{23567} - a \frac{r Q^2}{2 R_-^2} dr \wedge \theta^{23456} \\
& + \frac{da}{dr} \frac{2 R_-^2 Q^2}{r} \theta^{123567} - \frac{db}{dr} \frac{2 R_+^2 Q^2}{r} \theta^{123456}, \tag{4.32}
\end{aligned}$$

$$\begin{aligned}
\star_g F_A^2 = & (1 - a^2) \frac{P Q R_-^2}{R_+^2} dr \wedge \theta^{13567} + (1 - b^2) \frac{P Q R_+^2}{R_-^2} dr \wedge \theta^{12346} \\
& - ab P Q dr \wedge \theta^{12367} + ab P Q dr \wedge \theta^{13456} \\
& - \frac{r Q^2}{2 R_+^2} b dr \wedge \theta^{24567} - a \frac{r Q^2}{2 R_-^2} dr \wedge \theta^{23457} \\
& + \frac{da}{dr} \frac{2 R_-^2 Q^2}{r} \theta^{124567} - \frac{db}{dr} \frac{2 R_+^2 Q^2}{r} \theta^{123457}, \tag{4.33}
\end{aligned}$$

$$\begin{aligned}
\star_g F_A^1 = & -(1-a^2) \frac{PQR_-^2}{R_+^2} dr \wedge \theta^{12567} - (1-b^2) \frac{PQR_+^2}{R_-^2} dr \wedge \theta^{12345} \\
& + abPQdr \wedge \theta^{12357} - abPQdr \wedge \theta^{12456} \\
& - \frac{rQ^2}{2R_+^2} b dr \wedge \theta^{34567} - a \frac{rQ^2}{2R_-^2} dr \wedge \theta^{23467} \\
& + \frac{da}{dr} \frac{2R_-^2 Q^2}{r} \theta^{134567} - \frac{db}{dr} \frac{2R_+^2 Q^2}{r} \theta^{123467}.
\end{aligned} \tag{4.34}$$

The Spin(7) instanton equations are given by:

$$\star_g F_A^i = -\Phi \wedge F_A^i. \tag{4.35}$$

The set of equations obtained by imposing (4.35) is the same for each $i = 1, 2, 3$. It is as follows:

$$\frac{da}{dr} = \frac{2PQ}{R_+ R_-^2} ab - \frac{r}{2R_-^2} a, \tag{4.36}$$

$$\frac{da}{dr} = \frac{rR_+}{Q^2} ab - \frac{r}{2R_-^2} a, \tag{4.37}$$

$$\frac{db}{dr} = \frac{PQ}{R_-^2 R_+} (1-b^2) - \frac{PQ}{R_+^3} (1-a^2) - \frac{r}{2R_+^2} b, \tag{4.38}$$

$$\frac{db}{dr} = \frac{rR_+}{2Q^2} (1-b^2) - \frac{R_-^2 r}{2R_+ Q^2} (1-a^2) - \frac{r}{2R_+^2} b. \tag{4.39}$$

This system is overdetermined unless the metric coefficients satisfy the condition:

$$PQ^3 = \frac{r}{2} R_+^2 R_-^2. \tag{4.40}$$

We recognize this as the SO(5)-invariant Monge–Ampère equation (1.47) distinguishing the Stenzel metric among the Kähler metrics induced from SO(5)-invariant potentials.

It is useful to work in coordinates compatible with the Eschenburg–Wang analysis. We therefore switch to the variable $t = R_-$. An elementary calculation shows that the system takes the form:

$$\begin{aligned}
\frac{da}{dt} &= \frac{\mathcal{P}a}{t} \left(b - \frac{1}{\mathcal{P}} \right), \\
\frac{db}{dt} &= \frac{\mathcal{P}}{2t} (1-b^2) - \frac{\mathcal{P}Q}{2} (1-a^2) - Qb.
\end{aligned} \tag{4.41}$$

where we have introduced the functions $\mathcal{P}, Q \in C^\infty [0, \infty)$ defined by:

$$\mathcal{P}(t) \stackrel{\text{def}}{=} \frac{\sqrt{6}\sqrt{2t^2+2}}{\sqrt{2t^2+3}}, \tag{4.42}$$

$$Q(t) \stackrel{\text{def}}{=} \frac{t}{t^2+1}. \tag{4.43}$$

4.2.2. Elementary observations We begin our analysis of the system (4.41). In this section we make a few elementary observations about the dynamics. First, we have the following:

Proposition 4.1. *The dynamics (4.41) preserve the vanishing of a and correspondingly if $a(t) \neq 0$ for some $t > 0$, then $a(t) \neq 0$ for all $t > 0$.*

Proof. The first statement is trivial. The second follows by the uniqueness part of the standard Picard theorem. $\square$

Next we observe a symmetry in the solution space:

Proposition 4.2. *Suppose that the pair (a, b) solves the system (4.41). Then so does $(-a, b)$.*

Proof. This follows from a trivial calculation. $\square$

We conclude the following: either $a = 0$ for all time, or a has a fixed sign throughout its lifespan. Furthermore it suffices to study the case $a > 0$ as -owing to the above observation- all solutions (a, b) with $a < 0$ can be obtained by considering a solution where $a > 0$ and reversing its sign.

The next proposition establishes that if one solution lies above another at some instant t^* , the inequality persists for all time. Here, ‘lying above’ is interpreted componentwise.

Proposition 4.3. *Suppose that $(a, b), (\tilde{a}, \tilde{b})$ are two solutions to the system (4.41) such that $a(t) > 0$ and $\tilde{a}(t) > 0$ for all $t > 0$. Suppose further that for some time $t^* > 0$ we have $a(t^*) > \tilde{a}(t^*)$ and $b(t^*) > \tilde{b}(t^*)$. These inequalities remain true for all $t \geq t^*$ for which both solutions exist.*

Proof. Suppose not. Let t_f be the first time for which the inequality fails. There are three cases:

1. $\tilde{a}(t_f) = a(t_f)$ and $\tilde{b}(t_f) = b(t_f)$,
2. $\tilde{a}(t_f) = a(t_f)$ and $\tilde{b}(t_f) < b(t_f)$,
3. $\tilde{a}(t_f) < a(t_f)$ and $\tilde{b}(t_f) = b(t_f)$.

Case 1 contradicts the uniqueness part of the standard Picard theorem.

Suppose case 2 holds. Consider the evolution of $a - \tilde{a}$:

$$\frac{d}{dt}(a - \tilde{a}) = \frac{\mathcal{P}ab}{t} - \frac{a}{t} - \left(\frac{\mathcal{P}\tilde{a}\tilde{b}}{t} - \frac{\tilde{a}}{t} \right).$$

At $t = t_f$ we have $\tilde{a}(t_f) = a(t_f) = s > 0$ and $\tilde{b}(t_f) < b(t_f)$. Consequently:

$$\frac{d}{dt}(a - \tilde{a})|_{t_f} = \frac{\mathcal{P}s}{t} (b(t_f) - \tilde{b}(t_f)) > 0.$$

It follows that $a(t) < \tilde{a}(t)$ for some time $t < t_f$ and the intermediate value theorem contradicts the fact that t_f is the first time for which the inequalities fail.

Suppose case 3 holds. Consider the evolution of $b - \tilde{b}$:

$$\begin{aligned} \frac{d}{dt}(b - \tilde{b}) &= \frac{\mathcal{P}}{2t} (1 - b^2) \\ &\quad - \frac{\mathcal{P}\mathcal{Q}}{2} (1 - a^2) - \mathcal{Q}b - \left(\frac{\mathcal{P}}{2t} (1 - \tilde{b}^2) - \frac{\mathcal{P}\mathcal{Q}}{2} (1 - \tilde{a}^2) - \mathcal{Q}\tilde{b} \right). \end{aligned}$$

At $t = t_f$ we have $\tilde{b}(t_f) = b(t_f)$ and $\tilde{a}(t_f) < a(t_f)$. Consequently:

$$\frac{d}{dt} (b - \tilde{b})|_{t_f} = \frac{\mathcal{P}\mathcal{Q}}{2} (a(t_f) - \tilde{a}(t_f)) (a(t_f) + \tilde{a}(t_f)) > 0. \quad (4.44)$$

which leads to a contradiction as above. $\square$

Proposition 4.4. *Suppose that (a, b) is a solution of (4.41) defined in a neighbourhood of $t_0 > 0$. Take initial data at t_0 satisfying $a(t_0) > 0$, $b(t_0) < 0$ and flow backwards. Then $a \rightarrow +\infty$ as $t \rightarrow t_{\text{blowup}} \geq 0$.*

Proof. We will bound a from below by a function v satisfying $v \rightarrow +\infty$ as $t \rightarrow 0$.

Consider the evolution of the product ab . Using the Eq. (4.41), compute:

$$\begin{aligned} \frac{d}{dt} (ab)|_t &= \dot{a}b + a\dot{b} \\ &= \frac{P}{2t} ab^2 + \frac{\mathcal{P}}{2t(t^2+1)} a + \frac{\mathcal{P}\mathcal{Q}}{2} a^3 - \left(\mathcal{Q} + \frac{1}{t} \right) ab > - \left(\mathcal{Q} + \frac{1}{t} \right) ab, \end{aligned} \quad (4.45)$$

where in the last line we used the fact that $a > 0$ for all time. By comparison, flowing backwards in time, ab stays below the solution of the I.V.P:

$$\begin{cases} \dot{u}(t) = - \left(\mathcal{Q} + \frac{1}{t} \right) u, \\ u(t_0) = a(t_0)b(t_0). \end{cases}$$

By assumption, the initial data satisfy:

$$a(t_0)b(t_0) < 0.$$

Consequently, $u < 0$ for all $0 < t < t_0$ and we conclude that the same is true of ab .

This allows us to estimate:

$$\dot{a}(t) = \frac{\mathcal{P}ab}{t} - \frac{a}{t} < -\frac{a}{t}.$$

Consequently, a lies above the solution to the following I.V.P backwards of t_0 :

$$\begin{cases} \dot{v}(t) = -\frac{v}{t}, \\ v(t_0) = a(t_0). \end{cases}$$

This is easily solved explicitly and we obtain the inequality:

$$a(t) \geq \frac{a(t_0)t_0}{t} \text{ for all } 0 < t \leq t_0.$$

$\square$

Corollary 4.5. *Let $T > 0$ and let $(a, b) \in C^1[0, T]$ be a solution of (4.41) satisfying $a \neq 0$. We have that $b(t) > 0$ for all $t \geq 0$ for which the solution exists.*

Proof. Trivially, $b(0) = \pm 1$. If not, then $\dot{b}(t)$ blows up as $t \rightarrow 0$. Hence it suffices to prove the result for $t > 0$. If we achieve this, the possibility that $b(0) = -1$ is excluded by continuity and thus we have that $b(0) = 1$.

Suppose that for some $t_0 > 0$, $b(t_0) < 0$. We have that $a(t_0) \neq 0$ by assumption. If $a(t_0) > 0$, the above proposition implies that a blows up to $+\infty$ near $t = 0$ contradicting the boundedness of the solution. If $a(t_0) < 0$, then $-a(t_0) > 0$. Since $(-a, b)$ is a solution, $-a$ blows up to $+\infty$ near $t = 0$. Hence, a blows up to $-\infty$ near $t = 0$.

Suppose that $b(t_0) = 0$ for some $t_0 > 0$. At such a point we have:

$$\dot{b}(t_0) = \frac{\mathcal{P}(t_0)}{2t_0(t_0^2 + 1)} + \frac{\mathcal{P}(t_0)\mathcal{Q}(t_0)}{2}a^2(t_0) > 0.$$

It follows that $b(t) < 0$ for some $0 < t < t_0$ and this brings us to the previous case. $\square$

Putting the above together: if (a, b) is a global solution of (4.41), either $a = 0$ identically or the sign of a is fixed and $b > 0$.

The final proposition in this section asserts that -when $a \neq 0$ - the long-term behaviour of (4.41) is essentially driven by a .

Proposition 4.6. *Let $(a, b) \in C^1[0, T]$ be a solution of (4.41). Suppose that $a \neq 0$. Either (a, b) survives for all $t > 0$, or there exists a finite blowup time $0 < t_{\text{blowup}} < \infty$ such that:*

$$|a| \rightarrow +\infty \text{ as } t \rightarrow t_{\text{blowup}}^-.$$

Proof. By the preceding results of this section, we have that $b > 0$ and the sign of a is fixed. Proposition 4.2 permits us to assume $a > 0$ without loss of generality. By the standard escape lemma, either the solution is global, or its phase space norm blows up to $+\infty$ at some finite time $0 < t_* < \infty$.

The task is to prove that when the latter occurs, we have that $a \rightarrow +\infty$ as $t \rightarrow t_*^-$. Suppose not. Then $b \rightarrow +\infty$ as $t \rightarrow t_*^-$. Let $\epsilon_1 > 0$ be such that:

$$b(t) > \frac{1}{\mathcal{P}(t)} \text{ for } t_* - \epsilon_1 < t < t_*. \quad (4.46)$$

The equation governing a implies that it is monotonically increasing past $t_* - \epsilon_1$. Since it doesn't converge to $+\infty$, it must be bounded above. We then have that:

$$M := \sup_{t_* - \epsilon_1 < t < t_*} \left(\frac{\mathcal{P}}{2\mathcal{Q}t} + \frac{\mathcal{P}}{2}a^2 \right) < \infty.$$

Let $0 < \epsilon_2 < \epsilon_1$ be such that:

$$b(t) > M \text{ for } t_* - \epsilon_2 < t < t_*. \quad (4.47)$$

Using the equation governing b , we find that for times past $t_* - \epsilon_2$:

$$\begin{aligned} \frac{db}{dt} &= \frac{\mathcal{P}}{2t} (1 - b^2) - \frac{\mathcal{P}\mathcal{Q}}{2} (1 - a^2) - \mathcal{Q}b. \\ &< \mathcal{Q} \left(\frac{\mathcal{P}}{2\mathcal{Q}t} + \frac{\mathcal{P}}{2}a^2 - b \right) \\ &< \mathcal{Q}(M - b) \\ &< 0. \end{aligned}$$

It follows that b is bounded above. This yields the requisite contradiction, establishing the claim with $t_{\text{blowup}} = t_*$. $\square$

4.2.3. Solutions extending on P_{π_2} : an explicit family of $\text{Spin}(7)$ -instantons containing a unique HYM connection We classify the solutions that extend on P_{π_2} . In the division suggested in the preceding subsection, this will be seen to correspond to the case $a = 0$. The system (4.41) reduces to a single nonlinear ODE that we can solve explicitly. We thus exhibit an explicit 1-parameter family of $\text{Spin}(7)$ instantons only one of which is HYM. This resolves (negatively) the question regarding the equivalence of the two gauge theoretic problems.

Owing to Proposition 3.5, solutions extending to P_{π_2} must satisfy $b(0) = -1$. Due to Corollary 4.5, the a -component of such a solution must vanish identically. The system (4.41) reduces to the following ODE:

$$\frac{db}{dt} = -\frac{\mathcal{P}\mathcal{Q}}{2} + \frac{\mathcal{P}}{2t} (1 - b^2) - \mathcal{Q}b. \quad (4.48)$$

This can be solved explicitly. We fix a positive reference time and parameterize solutions by their value at that time. We choose to work with $t_{\text{ref}} = \frac{\sqrt{6}}{2}$ (corresponding to $r_{\text{ref}} = 2$). This choice is arbitrary. Note that our approach excludes solutions blowing up at t_{ref} . This is not an issue as we are only interested in global instantons. Writing: $v = b(t_{\text{ref}})$, the associated solution to (4.48) takes the form:

$$b_v(t) = \frac{\sqrt{2}}{2} \left(1 + \frac{\sqrt{6} - v\sqrt{10t^2 + 15}}{\sqrt{30}v + \sqrt{6} - (\sqrt{5}v + 2)\sqrt{2t^2 + 3}} \right) \frac{1}{\sqrt{t^2 + 1}}. \quad (4.49)$$

Corresponding to b_v there is a local $\text{Spin}(7)$ instanton (3.20) on the restriction of P_{Id} over an open submanifold of the form:

$$(t_{\text{ref}} - \delta, t_{\text{ref}} + \delta) \times \frac{\text{SO}(5)}{\text{SO}(3)} \subset X^8. \quad (4.50)$$

An elementary calculation yields the values of v for which there exists a finite blowup time:

Proposition 4.7. *Let $v \in (-\infty, -\frac{2\sqrt{5}}{5}) \cup (\frac{\sqrt{10}}{5}, \infty)$. The connection A_v blows up (as witnessed -for instance- by a blowup of the pointwise curvature norm) at time $t_{\text{blowup}}(v)$ given by:*

$$t_{\text{blowup}}(v) = \frac{\sqrt{6}\sqrt{5v^2 - 2}}{2|\sqrt{5}v + 2|}. \quad (4.51)$$

For v outside of the range considered in the proposition, the solutions stay bounded for all time. These considerations lead to the following existence/classification result:

Theorem 4.8. *Let $v \in [-\frac{2\sqrt{5}}{5}, \frac{\sqrt{10}}{5})$. The connection A_v is a smooth $\text{Spin}(7)$ instanton on the extended bundle P_{π_2} . Furthermore, these are all the invariant $\text{Spin}(7)$ instantons on P_{π_2} .*

Proof. For $v \in [-\frac{2\sqrt{5}}{5}, \frac{\sqrt{10}}{5}]$, the function b_v is of class $C^\infty[0, \infty)$. We need to verify the extension conditions of Proposition 3.5. In particular we need to prove that $b_v(t) + 1$ is even and $O(t^2)$ at $t = 0$. We immediately exclude $v = \frac{\sqrt{10}}{5}$ as the associated solution satisfies $b(0) = 1$ and consequently fails the second extension condition. Now work for

$\nu \in [-\frac{2\sqrt{5}}{5}, \frac{\sqrt{10}}{5})$. The first condition is clear as t always appears squared in (4.49). The second condition is easily established by computing that:

$$b_\nu(0) + 1 = \dot{b}_\nu(0) = 0 \quad \text{for} \quad -\frac{2\sqrt{5}}{5} \leq \nu < \frac{\sqrt{10}}{5}. \quad (4.52)$$

For uniqueness, we note that any invariant Spin(7) instanton on P_{π_2} obeys (4.48) and all other solutions of this equation blow up. $\square$

The HYM connection $A_{\text{HYM}_{\pi_2}}$ lies in the interior of this family and corresponds to the choice $\nu = -\frac{\sqrt{10}}{5}$. It is the only HYM connection in the family. The boundary point $\nu_\partial = -\frac{2\sqrt{5}}{5}$ corresponds to the solution:

$$b_{\nu_\partial}(t) = -\frac{\sqrt{3}}{3} \frac{\sqrt{2t^2 + 3}}{\sqrt{t^2 + 1}}. \quad (4.53)$$

The associated Spin(7) instanton A_{ν_∂} differs from the others in that it yields a different limiting connection on P_{Id} over the Stiefel manifold $\frac{\text{SO}(5)}{\text{SO}(3)}$ at infinity. In particular, for $\nu \in (-\frac{2\sqrt{5}}{5}, \frac{\sqrt{10}}{5})$ it is easily seen that:

$$\lim_{t \rightarrow \infty} b_\nu(t) = 0. \quad (4.54)$$

The associated limiting connection is therefore equal to $A_{P_{\text{Id}}}^{\text{can}}$. However, for $\nu = \nu_\partial$, we have:

$$\lim_{t \rightarrow \infty} b_{\nu_\partial}(t) = -\frac{\sqrt{6}}{3} \quad (4.55)$$

and the associated connection at infinity is given by:

$$A_{\nu_\partial}^\infty = A_{P_{\text{Id}}}^{\text{can}} - \frac{\sqrt{6}}{3} \left(\theta^5 \otimes e_3 - \theta^6 \otimes e_2 + \theta^7 \otimes e_1 \right).$$

In the study of gauge theoretic moduli problems over noncompact manifolds, it is common practice to fix the limiting connection at infinity. With this perspective in mind, we could view the family of Theorem 4.8 as having a missing endpoint in both directions. The noncompactness at the left endpoint is then mended by adding in the instanton A_{ν_∂} (the latter lying in another moduli space because of its different asymptotic behaviour). Perhaps, similar noncompactness phenomena occur in general Spin(7) instanton moduli spaces.

When a principal G-bundle over a compact CY fourfold admits HYM connections, the HYM equations determine the minima of the Yang–Mills energy functional. We may thus (naively) expect the HYM solutions to be distinguished by a “smaller” Yang–Mills energy density in an appropriately defined sense. Over an asymptotically conical CY fourfold, a natural (yet crude) formulation of such a statement would be in terms of the radial decay rate of the pointwise curvature norm. Our examples suggest that this attribute fails to distinguish the HYM solutions among the Spin(7) instantons. We have already computed the pointwise curvature norm of $A_{\text{HYM}_{\pi_2}}$. It is of order $O(s^{-2})$ at infinity. It is clear from (4.54) and (4.55) that all elements of the family are asymptotically conical (i.e. they decay to translation invariant limits). Consequently, all the associated curvature tensors decay like $O(s^{-2})$.

We collect our observations regarding the relationship of the two equation systems in the following result:

Theorem 4.9. *Let (M^8, g, J, Ω) be a geodesically complete Calabi–Yau fourfold (i.e. $\text{Hol}(g) = \text{SU}(4)$). It is possible for $\text{Spin}(7)$ instantons on M^8 not to satisfy the HYM equation. We shall refer to such connections as pure $\text{Spin}(7)$ instantons. When M^8 is noncompact -in direct contrast to the compact case- pure $\text{Spin}(7)$ instantons can live on a principal G -bundle admitting HYM solutions. When M^8 is asymptotically conical, the HYM solutions cannot -in general- be distinguished by a faster decay rate. That is, the pointwise curvature norm of a pure $\text{Spin}(7)$ instanton can decay with the same rate as that of an HYM connection on the same bundle.*

4.2.4. Solutions extending on P_{π_1} We now wish to classify solutions that smoothly extend over P_{π_1} . In the previous section we found all solutions where $a = 0$. The only one satisfying $b(0) = 1$ corresponds to $v = \frac{\sqrt{10}}{5}$. The associated instanton is $A_{\text{HYM}_{\pi_1}}$. Any other solution would have nonvanishing a -component. Consequently, we have to deal with the full system (4.41). The first step is to obtain short time existence and uniqueness near the pole of the ODE. In doing so, we impose the Eschenburg–Wang extension conditions of Proposition 3.4. Subsequently, the task is to characterize which of these local solutions survive for all time to yield global $\text{Spin}(7)$ instantons.

Short Time Existence and Uniqueness. The analysis in this section relies on the method of Eschenburg and Wang ([6], section 6). We have adapted their existence result to our equation system and refined it to include continuous dependence on initial data. This does not follow from the standard Grönwall estimate as the I.V.P under consideration is singular. The continuity proof is based on the technique employed by McLeod et al. ([11], p. 147).

Theorem 4.10. *Let a_0 be a fixed real number. There exists a unique solution:*

$$(a, b)_{a_0} \in C^\infty[0, t_{\max}(a_0))$$

to the system (4.41) such that a is odd at $t = 0$, b is even at $t = 0$ and furthermore:

$$a(0) = 0, \quad (4.56)$$

$$\dot{a}(0) = a_0, \quad (4.57)$$

$$b(0) = 1. \quad (4.58)$$

This solution satisfies the extension conditions of Proposition 3.4 and thus yields a $\text{Spin}(7)$ instanton on the restriction of P_{π_1} over the open submanifold defined by $0 \leq t < t_{\max}(a_0)$.

Furthermore, we have that for any $K > 0$:

$$T_K \stackrel{\text{def}}{=} \inf \{t_{\max}(a_0) \mid a_0 \in [-K, K]\} > 0 \quad (4.59)$$

and the following mapping is continuous:

$$\begin{aligned} [-K, K] &\rightarrow C^0([0, T_K], \mathbb{R}^2), \\ a_0 &\mapsto (a, b)_{a_0}. \end{aligned} \quad (4.60)$$

We will prove this result in four stages. The first step is to study the formal Taylor series of smooth solutions at $t = 0$. The second step is to derive and analyze ODEs governing perturbations of high order polynomial truncations of the series. The idea is to show that, if the order is high enough, the resulting ODEs are uniquely soluble for sufficiently short time in suitable Banach spaces. The third step is to argue that the solutions so obtained are smooth and have the correct formal series at $t = 0$. The final step is to understand how this existence/ uniqueness argument behaves under change of initial data. This involves proving that the estimates can be made to be uniform in a_0 for a_0 in compact sets and establishing the desired continuity result.

Proposition 4.11. *Fix $a_0 \in \mathbb{R}$. There exists a unique $(a, b)_{a_0} \in \mathbb{R}[[t]]^2$ solving the system (4.41) and satisfying the conditions (4.56)–(4.58). Here, differentiation is understood in the formal sense (as a derivation of the formal power series ring).*

Proof. Considering our assumptions on a and b , we introduce their formal Taylor series at $t = 0$:

$$a = \sum_{k=0}^{\infty} \frac{a_k}{(2k+1)!} t^{2k+1}, \quad b = \sum_{k=0}^{\infty} \frac{b_k}{(2k)!} t^{2k}, \quad \text{where } b_0 = 1. \quad (4.61)$$

Using the parity of a, b and the coefficient functions, we introduce the series:

$$\begin{aligned} a(\mathcal{P}b - 1) &= \sum_{k=0}^{\infty} \frac{c_k}{(2k+1)!} t^{2k+1}, & \frac{\mathcal{P}\mathcal{Q}}{2} (1 - a^2) &= \sum_{k=0}^{\infty} \frac{e_k}{(2k+1)!} t^{2k+1}, \\ \frac{\mathcal{P}}{2} (1 - b^2) &= \sum_{k=0}^{\infty} \frac{d_k}{(2k)!} t^{2k} \quad \text{where } d_0 = 0, & \mathcal{Q}b &= \sum_{k=0}^{\infty} \frac{f_k}{(2k+1)!} t^{2k+1}. \end{aligned}$$

The ODE for a translates to the condition:

$$a_k = \frac{c_k}{2k+1} \quad \text{for all } k \geq 0. \quad (4.62)$$

We compute c_k in terms of $a_0, \dots, a_k, b_0, \dots, b_k$. This yields:

$$\begin{aligned} c_k &= \frac{d^{2k+1}}{dt^{2k+1}} \Big|_{t=0} (a(\mathcal{P}b - 1)), \\ &= a_k + \mathcal{G}(a_0, \dots, a_{k-1}, b_0, \dots, b_k). \end{aligned}$$

Here $\mathcal{G}$ denotes some function of coefficients of lower order. We will slightly abuse notation and maintain use of the symbol $\mathcal{G}$ in subsequent calculations -even though the particular function may not be the same. Using (4.62) we obtain:

$$\frac{2k}{2k+1} a_k = \mathcal{G}(a_0, \dots, a_{k-1}, b_0, \dots, b_k). \quad (4.63)$$

This determines a_k in terms of coefficients of lower order provided that $k \neq 0$. We conclude that we are allowed to choose a_0 freely.

We perform a similar calculation for b . The second equation in (4.41) translates to the relation:

$$b_{k+1} = \frac{d_{k+1}}{2k+2} - e_k - f_k \quad \text{for all } k \geq 0. \quad (4.64)$$

We note that e_k and f_k only involve terms depending on $a_0, \dots, a_k, b_0, \dots, b_k$ and it is thus unnecessary to compute them. We compute d_{k+1} in terms of $a_0, \dots, a_k, b_0, \dots, b_{k+1}$:

$$\begin{aligned} d_{k+1} &= \frac{d^{2k+2}}{dt^{2k+2}} \Big|_{t=0} \left(\frac{\mathcal{P}}{2} (1 - b^2) \right), \\ &= -2b_{k+1} + \mathcal{G}(b_0, \dots, b_k). \end{aligned}$$

Using (4.64), we obtain:

$$\frac{k+2}{k+1} b_{k+1} = \mathcal{G}(a_0, \dots, a_k, b_0, \dots, b_k).$$

It follows that b_{k+1} is determined by lower order coefficients for each $k \geq 0$.

The above calculations demonstrate that the formal Taylor series at 0 is uniquely determined by induction given a choice of $a_0 \in \mathbb{R}$. $\square$

Although the content of the preceding proposition is enough for the purposes of our existence theorem, continuity requires more refined knowledge of the formal Taylor series. In particular, we are interested in the dependence of its coefficients on a_0 . We explicitly calculate the first few terms of the series associated to some fixed choice of a_0 :

$$a(t) = a_0 t - \frac{a_0}{3} t^3 + \left(\frac{5a_0}{36} + \frac{a_0^3}{12} \right) t^5 + O(t^7), \quad (4.65)$$

$$b(t) = 1 - \frac{t^2}{2} + \left(\frac{3}{8} + \frac{a_0^2}{6} \right) t^4 + O(t^6). \quad (4.66)$$

In fact, we are able to obtain the following:

Proposition 4.12. *The coefficients of the formal Taylor series $(a, b)_{a_0}$ are polynomials (possibly of order 0) in a_0 .*

Proof. This is certainly true for a_0 , b_0 and b_1 . Repeating the calculations of the preceding proposition, but keeping track of the lower order terms yields the following recurrence relations for the coefficients:

$$\begin{aligned} a_k &= \frac{1}{2k} \sum_{m=1}^k \sum_{l=0}^m \binom{2k+1}{2m} \binom{2m}{2l} \mathcal{P}_{|t=0}^{(2(m-l))} a_{k-m} b_l, \\ b_{k+1} &= -\frac{1}{4k+8} \sum_{m=1}^k \sum_{l=0}^m \binom{2k+2}{2m} \binom{2m}{2l} \mathcal{P}_{|t=0}^{(2(k-m)+2)} b_{m-l} b_l \\ &\quad - \frac{1}{2k+4} \sum_{l=1}^k \binom{2k+2}{2l} b_{k+1-l} b_l \\ &\quad - \frac{k+1}{2k+4} (\mathcal{P}\mathcal{Q})_{|t=0}^{(2k+1)} + \frac{k+1}{2k+4} \sum_{m=1}^k \sum_{l=0}^{m-1} \binom{2k+1}{2m} \\ &\quad \left(\binom{2m}{2l+1} (\mathcal{P}\mathcal{Q})_{|t=0}^{(2(k-m)+1)} a_{m-l-1} a_l \right. \\ &\quad \left. - \frac{k+1}{k+2} \sum_{m=0}^k \binom{2k+1}{2m} \mathcal{Q}_{|t=0}^{(2(k-m)+1)} b_m \right). \end{aligned}$$

The result follows by induction. $\square$

We now discuss how to use this formal series in order to obtain an honest solution of the system (4.41). For ease of exposition, we introduce the following functions:

$$\begin{aligned} F_1(t, u, v) &\stackrel{\text{def}}{=} u (\mathcal{P}(t) v - 1), \\ F_2(t, v) &\stackrel{\text{def}}{=} \frac{\mathcal{P}(t)}{2} (1 - v^2), \\ F_3(t, u, v) &\stackrel{\text{def}}{=} -\frac{\mathcal{P}(t) \mathcal{Q}(t)}{2} (1 - u^2) - \mathcal{Q}(t) v. \end{aligned}$$

We rewrite the ODE system (4.41) as:

$$\begin{aligned} \frac{da}{dt} &= \frac{F_1(t, a, b)}{t}, \\ \frac{db}{dt} &= \frac{F_2(t, b)}{t} + F_3(t, a, b). \end{aligned}$$

Further, we let $p_m^a(t, a_0)$, $p_m^b(t, a_0)$ denote the order m Taylor polynomials corresponding to the initial data a_0 . These are obtained by truncating the respective series. We also introduce the following error functions capturing the failure of the Taylor polynomials to solve (4.41):

$$\begin{aligned} E_m^a(t, a_0) &\stackrel{\text{def}}{=} \frac{d}{dt} p_m^a(t, a_0) - \frac{F_1(t, p_m^a(t, a_0), p_m^b(t, a_0))}{t}, \\ E_m^b(t, a_0) &\stackrel{\text{def}}{=} \frac{d}{dt} p_m^b(t, a_0) - \frac{F_2(t, p_m^b(t, a_0))}{t} - F_3(t, p_m^a(t, a_0), p_m^b(t, a_0)). \end{aligned}$$

They are smooth and $O(t^m)$ at $t = 0$. To see this, recall that the full formal series was constructed by matching derivatives at the origin. Consequently, the first $m - 1$ derivatives of the error functions vanish at $t = 0$.

We now introduce the Banach spaces we will be working with. For any real $T > 0$ and integer $m \geq 0$ we define:

$$\mathcal{O}_T(m) \stackrel{\text{def}}{=} \left\{ f \in C^0[0, T] \text{ s.t. } \sup_{t \in [0, T]} \frac{|f(t)|}{t^m} < \infty \right\}, \quad (4.67)$$

$$\|f\|_{\mathcal{O}_T(m)} \stackrel{\text{def}}{=} \sup_{t \in [0, T]} \frac{|f(t)|}{t^m}. \quad (4.68)$$

Fix $T > 0$. We immediately observe that the error functions E_m^a , E_m^b lie in $\mathcal{O}_T(m)$ (they are $O(t^m)$ at $t = 0$). Furthermore, the functions p_m^a , $p_m^b - 1$ lie in $\mathcal{O}_T(1)$. In fact -in light of Proposition 4.12- we have:

Corollary 4.13. $E_m^a(t, \cdot)$, $E_m^b(t, \cdot)$ define continuous mappings from the space of initial data (the real numbers) into $\mathcal{O}_T(m)$. Similarly, $p_m^a(t, \cdot)$, $p_m^b(t, \cdot) - 1$ define continuous mappings from the space of initial data into $\mathcal{O}_T(1)$.

We finally recast the problem as an integral equation for a perturbation of the polynomials (p_m^a, p_m^b) . Given a pair of functions $(u, v) \in \mathcal{O}_T^{\oplus 2}(m)$ we define:

$$\Theta_{m, a_0}^1(u, v)(s) \stackrel{\text{def}}{=} \int_0^s \left(\frac{F_1(t, p_m^a(a_0, t) + u(t), p_m^b(a_0, t) + v(t))}{t} - \dot{p}_m^a(a_0, t) \right) dt,$$

$$\Theta_{m,a_0}^2(u,v)(s) \stackrel{\text{def}}{=} \int_0^s \left(\frac{F_2(t, p_m^b(a_0, t) + v(t))}{t} + F_3(t, p_m^a(a_0, t) + u(t), p_m^b(a_0, t) + v(t)) - \dot{p}_m^b(a_0, t) \right) dt.$$

It can be easily checked (by expanding out the integrands, counting order of vanishing and noting that integration raises this by one) that we obtain a nonlinear integral operator:

$$\Theta_{m,a_0} \stackrel{\text{def}}{=} \Theta_{m,a_0}^1 \times \Theta_{m,a_0}^2 : \mathcal{O}_T^{\oplus 2}(m) \rightarrow \mathcal{O}_T^{\oplus 2}(m). \quad (4.69)$$

The following proposition is the heart of the matter:

Proposition 4.14. *Let a_0 be fixed. Fix $R > 0$. For sufficiently large m (depending on F_1, F_2, F_3, R) and sufficiently small T (depending on m and a_0), the operator Θ_{m,a_0} has a unique fixed point (u, v) in $\overline{B}_R(0) \subset \mathcal{O}_T^{\oplus 2}(m)$. Furthermore, this fixed point is smooth on $[0, T]$ and the associated solution*

$$(a, b) \stackrel{\text{def}}{=} (p_m^a + u, p_m^b + v)$$

to the system (4.41) satisfies (4.56)–(4.58).

Proof. In what follows, our notation suppresses dependence on a_0 . Fix $R > 0$. We will select m and T such that Θ_m is a contraction on $\overline{B}_R(0) \subset \mathcal{O}_T^{\oplus 2}(m)$.

Consider the domain:

$$D_R \stackrel{\text{def}}{=} [0, 1] \times \overline{B}_{2R}(0, 1) \subset \mathbb{R}^3. \quad (4.70)$$

Let $L > 0$ be a Lipschitz constant in the (u, v) variables for the restrictions of F_1, F_2, F_3 on D_R . The constant L is controlled by L^∞ bounds on the restrictions of the derivatives of the F_i on D_R . Choose:

$$m > \max \{2L, 1\}.$$

Pick T such that:

$$T < \min \left\{ 1, \frac{(m+1)R}{2(\|E_m^a\|_{\mathcal{O}_1(m)} + \|E_m^b\|_{\mathcal{O}_1(m)})}, \frac{R}{\|p_m^a\|_{\mathcal{O}_1(m)}}, \frac{R}{\|p_m^b - 1\|_{\mathcal{O}_1(m)}} \right\}.$$

Clearly, for $t \in [0, T]$ we have:

$$|p_m^a(t)| \leq R, \quad |p_m^b(t) - 1| \leq R. \quad (4.71)$$

We claim that we also have:

$$\|\Theta_m(0, 0)\|_{\mathcal{O}_T(m)} \leq \frac{R}{2}. \quad (4.72)$$

To see this, we estimate as follows:

$$\begin{aligned} \|\Theta_m(0, 0)\|_{\mathcal{O}_T(m)} &= \|\Theta_m^1(0, 0)\|_{\mathcal{O}_T(m)} + \|\Theta_m^2(0, 0)\|_{\mathcal{O}_T(m)} \\ &= \left\| \int_0^r E_m^a(t) dt \right\|_{\mathcal{O}_T(m)} + \left\| \int_0^r E_m^b(t) dt \right\|_{\mathcal{O}_T(m)} \end{aligned}$$

$$\begin{aligned}
&\leq \sup_{r \in [0, T]} \frac{1}{r^m} \int_0^r |E_m^a(t)| dt + \sup_{r \in [0, T]} \frac{1}{r^m} \int_0^r |E_m^b(t)| dt \\
&\leq \sup_{r \in [0, T]} \frac{\|E_m^a\|_{\mathcal{O}_T(m)}}{r^m} \int_0^r t^m dt + \sup_{r \in [0, T]} \frac{\|E_m^b\|_{\mathcal{O}_T(m)}}{r^m} \int_0^r t^m dt \\
&\leq \sup_{r \in [0, T]} \frac{\|E_m^a\|_{\mathcal{O}_T(m)}}{m+1} r + \sup_{r \in [0, T]} \frac{\|E_m^b\|_{\mathcal{O}_T(m)}}{m+1} r \\
&\leq \frac{\|E_m^a\|_{\mathcal{O}_T(m)} + \|E_m^b\|_{\mathcal{O}_T(m)}}{(m+1)} T \leq \frac{R}{2}.
\end{aligned} \tag{4.73}$$

We now prove contraction estimates for Θ_m^1 and Θ_m^2 . Fix $0 \leq r \leq T$ and compute:

$$\begin{aligned}
\left| \Theta_m^1(u, v)(r) - \Theta_m^1(\tilde{u}, \tilde{v})(r) \right| &\leq \int_0^r \frac{1}{t} \left| F_1 \left(t, p_m^a + u, p_m^b + v \right) - F_1 \left(t, p_m^a + \tilde{u}, p_m^b + \tilde{v} \right) \right| dt \\
&\leq \int_0^r \frac{L}{t} (|u - \tilde{u}| + |v - \tilde{v}|) dt \\
&\leq L (\|u - \tilde{u}\|_{\mathcal{O}_T(m)} + \|v - \tilde{v}\|_{\mathcal{O}_T(m)}) \int_0^r t^{m-1} dt \\
&\leq \frac{Lr^m}{m} (\|u - \tilde{u}\|_{\mathcal{O}_T(m)} + \|v - \tilde{v}\|_{\mathcal{O}_T(m)}).
\end{aligned}$$

In this calculation, the L -Lipschitz estimate is valid due to (4.71) and the fact that the uniform norm is controlled by the $\mathcal{O}_T(m)$ norm when $0 < T < 1$. We conclude that:

$$\|\Theta_m^1(u, v) - \Theta_m^1(\tilde{u}, \tilde{v})\|_{\mathcal{O}_T(m)} \leq \frac{L}{m} (\|u - \tilde{u}\|_{\mathcal{O}_T(m)} + \|v - \tilde{v}\|_{\mathcal{O}_T(m)}). \tag{4.74}$$

A similar calculation yields:

$$\begin{aligned}
\|\Theta_m^2(u, v) - \Theta_m^2(\tilde{u}, \tilde{v})\|_{\mathcal{O}_T(m)} &\leq \left(\frac{L}{m} + \frac{LT}{m+1} \right) \|v - \tilde{v}\|_{\mathcal{O}_T(m)} + \frac{LT}{m+1} \|u - \tilde{u}\|_{\mathcal{O}_T(m)} \\
&\leq \frac{1}{2} (\|u - \tilde{u}\|_{\mathcal{O}_T(m)} + \|v - \tilde{v}\|_{\mathcal{O}_T(m)}).
\end{aligned} \tag{4.75}$$

Due to (4.72), (4.74) and (4.75), the closed R -ball in $\mathcal{O}_T^{\oplus 2}(m)$ is stable under Θ . The contraction mapping theorem (CMT) yields a unique fixed point (u, v) in this ball.

This fixed point is necessarily of class $C^1[0, T]$ (by the fundamental theorem of calculus). Consequently (a, b) is C^1 and it therefore constitutes an honest solution of (4.41) on $[0, T]$. Considering the order of vanishing of u at 0 and looking at the equations, we observe that $\dot{u}(t) = O(t^{m-1})$. Conditions (4.56)–(4.58) follow.

Full regularity follows by a simple bootstrap procedure. Since flows of smooth (non-autonomous) vector fields are smooth, (u, v) is smooth on $(0, T]$. The task is to establish smoothness at 0. Smoothness on $(0, T]$ legitimizes differentiation of the equations for $t > 0$. This gives an expression for the second derivatives of u and v involving terms in $\frac{u}{t^2}$, $\frac{v}{t^2}$, $\frac{\dot{u}}{t}$ and $\frac{\dot{v}}{t}$. It is thus clear that $u^{(2)}(t)$, $v^{(2)}(t) \rightarrow 0$ as $t \rightarrow 0$. Hence u, v are of class $C^2[0, T]$ with vanishing second derivative at 0. We can iterate this argument to conclude that u, v are of class $C^{m-1}[0, T]$ with vanishing derivatives at 0 up to order $m - 1$. The only constraint on m required for the contraction argument to run is $m > \max \{2L, 1\}$. It follows that the operator Θ_l is a contraction for arbitrarily large

$l > m$ (perhaps for shorter time T). Fixing $l > m$, we let (u_l, v_l) be the associated fixed point. Repeating the argument above, it lies in $C^{l-1}[0, T]$ with vanishing derivatives up to order $l - 1$. It is thus $O(t^{m+1})$. It follows that $(u_l + p_l^a - p_m^a, v_l + p_l^b - p_m^b)$ is also $O(t^{m+1})$. Consequently -by further decreasing T as necessary- we can arrange that the latter has as small $\mathcal{O}_T^{\oplus 2}(m)$ norm as we like. In particular, we take this to be less than R . Furthermore, $(u_l + p_l^a - p_m^a, v_l + p_l^b - p_m^b)$ is a fixed point of Θ_m . But Θ_m has a unique fixed point in the closed R -ball. It follows that:

$$(u, v) = (u_l + p_l^a - p_m^a, v_l + p_l^b - p_m^b) \quad (4.76)$$

and hence that u, v lie in C^{l-1} . Since l was arbitrary, the proof is complete. $\square$

We now have enough for the first part of Theorem 4.10. The preceding proposition guarantees the existence of a smooth solution (a, b) satisfying (4.56)–(4.58). The algebraic calculation in the start of this section uniquely specifies its full formal Taylor series at $t = 0$ so that it passes the extension criterion in Proposition 3.4. Finally, suppose that there is another smooth solution $(\tilde{a}, \tilde{b})$ satisfying (4.56)–(4.58). Arguing as above, we find that the two solutions share the same formal Taylor series at 0 (the series discovered in Proposition 4.11). Let m be as in Proposition 4.14. We have that $(a - p_m^a, b - p_m^b)$, $(\tilde{a} - p_m^a, \tilde{b} - p_m^b)$ are $O(t^{m+1})$. For short enough time T , the $\mathcal{O}_T^{\oplus 2}(m)$ norms of these functions are less than R . Since both functions are fixed points of Θ_m and lie in the closed R -ball, they are equal. Hence $(a, b) = (\tilde{a}, \tilde{b})$.

It remains to study the dependence of solutions on variations of the initial data a_0 . We immediately obtain:

Proposition 4.15. *Fix $K > 0$. We have:*

$$T_K = \inf \{t_{\max}(a_0) \mid a_0 \in [-K, K]\} > 0.$$

Proof. In our existence proof, once, R, L, m are fixed, T needs to be controlled from above by quantities decreasing with the $\mathcal{O}_1(m)$ norms of the error functions and the $\mathcal{O}_1(1)$ norms of $p_m^a, p_m^b - 1$. By Corollary 4.13, these norms depend continuously on a_0 and are hence bounded for a_0 in a compact set. It follows that we can choose T small enough so that the contraction argument works for all $a_0 \in [-K, K]$. $\square$

Note that the contraction constant can be taken to be the same across all $a_0 \in [-K, K]$. This is vital for the continuity proof, which we now discuss.

Proposition 4.16. *The mapping defined by:*

$$\begin{aligned} [-K, K] &\rightarrow C^0([0, T_K], \mathbb{R}^2) \\ a_0 &\mapsto (a, b)_{a_0} \end{aligned} \quad (4.77)$$

is continuous.

Proof. Consider the trivial (infinite-rank) vector bundle over $[-K, K]$:

$$E \stackrel{\text{def}}{=} [-K, K] \times \mathcal{O}_{T_K}^{\oplus 2}(m).$$

The following map is fiber-preserving and continuous:

$$S : E \rightarrow E,$$

$$(a_0, (u, v)) \mapsto (a_0, \Theta_{m,a_0}(u, v)).$$

There is a unique section s of E that is fixed by S (the one assigning to each choice of initial data the associated fixed point of $\Theta_{a_0,m}$). The task is to prove that s is continuous. To this end, we fix $x \in [-K, K]$ and prove that s is continuous at x . Fix $\epsilon > 0$ and define the following (continuous) section of E :

$$u_x(a_0) \stackrel{\text{def}}{=} (a_0, s(x)).$$

We will run the CMT iteration on each fiber with initial condition determined by u_x . Letting $0 < C < 1$ be the contraction constant of $\Theta_{a_0,m}$ and using the convergence rate estimate of the CMT we have:

$$\begin{aligned} \|\Theta_{m,a_0}^N(u_x(a_0)) - s(a_0)\|_{\mathcal{O}_{TK}^{\oplus 2}(m)} &\leq \frac{\|\Theta_{m,a_0}(u_x(a_0)) - u_x(a_0)\|_{\mathcal{O}_{TK}^{\oplus 2}(m)}}{1 - C} C^N \\ &\leq \frac{2R}{1 - C} C^N. \end{aligned} \quad (4.78)$$

Fix N large enough so that this quantity is controlled by $\frac{\epsilon}{2}$. Since S and u are continuous, we have:

$$\lim_{a_0 \rightarrow x} S^N u_x(a_0) = S^N u_x(x) = (x, s(x)).$$

Consequently, for a_0 sufficiently close to x , we can achieve:

$$\|\Theta_{m,a_0}^N(u_x(a_0)) - s(x)\|_{\mathcal{O}_{TK}^{\oplus 2}(m)} < \frac{\epsilon}{2}. \quad (4.79)$$

Using (4.78), (4.79) and the triangle inequality completes the proof. $\square$

Uniqueness implies that the solution associated to $a_0 = 0$ corresponds to A_{HYM, π_1} . This instanton will play a central role in the analysis of the global properties of the system.

Global Existence for Small Initial Data. The previous section yields a characterization of short-time solutions near the pole. We are now tasked with understanding which of these solutions are global. In this section we establish that:

Theorem 4.17. *There exists an $\epsilon > 0$ such that for $|a_0| < \epsilon$, the short-time solutions of Theorem 4.10 are global.*

The heart of the argument lies in the following proposition. Its conditions are subsequently easily verified (for small initial data) by a continuity argument.

Proposition 4.18. *Suppose that $a_0 > 0$ and let $(a, b)_{a_0}$ be a solution to the system (4.41) such that a attains a critical point in the spacetime region:*

$$t > \frac{\sqrt{6}}{2\sqrt{1-2a^2}}, \quad 0 < a < \frac{\sqrt{2}}{2}. \quad (4.80)$$

Then $t_{\max}(a_0) = +\infty$.

Proof. By Proposition 4.1, $a(t) > 0$ for all $0 < t < t_{\max}(a_0)$. Looking at the ODE for a , we conclude that the critical points of a are precisely the points where $b = \mathcal{P}^{-1}$. We seek an expression for the second derivative of a at a critical point occurring at time $t = t_{\text{crit}} > 0$. Differentiating the ODE for a and setting $b = \mathcal{P}^{-1}$, we obtain:

$$\frac{d^2 a}{dt^2} \Big|_{t=t_{\text{crit}}} = \frac{3 a(t_{\text{crit}})}{2 t_{\text{crit}}^2 (2 t_{\text{crit}}^2 + 3)} \left[\left(4 a(t_{\text{crit}})^2 - 2 \right) t_{\text{crit}}^2 + 3 \right]. \quad (4.81)$$

The first factor is strictly positive. Consequently, the nature of the critical point depends on the sign of:

$$F(t, a) \stackrel{\text{def}}{=} \left(4a^2 - 2 \right) t^2 + 3 \quad (4.82)$$

at $(t_{\text{crit}}, a(t_{\text{crit}}))$. For (t, a) in the spacetime region (4.80), we have $F(t, a) < 0$. Hence, any critical point occurring in the region is a maximum.

Suppose that a maximum does occur inside the region (4.80). For a short amount of time thereafter a is decreasing. The only way that a can ever increase again is if it reaches a minimum. A minimum can only occur if $(t, a(t))$ exits the spacetime region (4.80). For this to occur, a has to increase. It follows that a decreases for as long as the solution survives. Consequently a is bounded from above. Since $a > 0$, it follows that a is also bounded from below. Since a consistently decreases after the maximum point, we have that $b(t) < \mathcal{P}(t)^{-1}$ for $t > t_{\text{crit}}$. By Corollary 4.5, $b > 0$ for all time. Hence both a and b are bounded and thus survive for all time $t \geq 0$. $\square$

Proposition 4.18 applies provided that the initial data is small enough:

Proposition 4.19. *There exists $\epsilon > 0$ such that if $0 \leq a_0 < \epsilon$, then a_{a_0} attains a critical point in the spacetime region (4.80).*

Proof. The idea is to use a continuity argument and compare with the solution corresponding to $a_0 = 0$:

$$a_{\text{HYM}}(t) = 0, \quad b_{\text{HYM}}(t) = \frac{1}{\sqrt{t^2 + 1}}. \quad (4.83)$$

Note that $\mathcal{P}(0) = 2$ and $b(0) = 1$ (independently of the choice of a_0). Hence b always starts above $\mathcal{P}^{-1}$. For $a_0 = 0$, the solution b_{HYM} crosses $\mathcal{P}^{-1}$ at the time: $t = \frac{3\sqrt{2}}{2}$. For $a_0 > 0$, formulae (4.65) and (4.66) show that -at least for a very short time- to the right of $t = 0$ we have

$$(a, b) > (a_{\text{HYM}}, b_{\text{HYM}}) \quad (4.84)$$

where the inequality is understood componentwise. By Proposition 4.3, this inequality persists for as long as the solutions exist. Consequently, if $a_0 > 0$, b_{a_0} can only cross $\mathcal{P}^{-1}$ strictly after $t = \frac{3\sqrt{2}}{2}$.

Consider only $|a_0| \leq 1$. By the second assertion of Theorem 4.10, the maximal existence time of the resulting solutions is bounded below by a positive number T_1 . Furthermore, these solutions depend continuously on a_0 (in the $C^0[0, T_1]$ norm). Composing with the local flow associated to taking initial conditions at $t = T_1$, we see that the maximal existence time is lower semicontinuous in a_0 . Furthermore, we see that if

a particular choice of a_0 yields a solution surviving past some time $t = T$, the mapping sending initial conditions to their associated solutions is continuous from an open neighbourhood of a_0 into $C^0[0, T]$.

Since $(a_{\text{HYM}}, b_{\text{HYM}})$ (associated to $a_0 = 0$) is global, initial data close to 0 lead to solutions that survive arbitrarily long. In particular, we can choose $\epsilon > 0$ to be small enough so that solutions associated to $0 < a_0 < \epsilon$ survive past $t = 4$. Furthermore -at the expense of taking ϵ to be even smaller- we can appeal to continuity to arrange that:

$$\sup_{t \in [0, 4]} |a_{a_0}(t)| < \frac{1}{2}, \quad (4.85)$$

$$\sup_{t \in [0, 4]} \left| b_{a_0}(t) - \frac{1}{\sqrt{t^2 + 1}} \right| < \frac{1}{2} \inf_{t \in [3, 4]} \left| \frac{1}{\sqrt{t^2 + 1}} - \frac{1}{\mathcal{P}(t)} \right|. \quad (4.86)$$

Condition (4.85) implies that for any $\sqrt{3} < t \leq 4$ the point $(t, a(t))$ lies in the spacetime region (4.80). Condition (4.86) implies that for any $3 \leq t \leq 4$ we have:

$$b_{a_0}(t) < \frac{1}{\mathcal{P}(t)}.$$

By the intermediate value theorem, there exists a $0 < t_{\text{crit}} < 3$ where b_{a_0} crosses $\mathcal{P}^{-1}$. But we have seen that this time must be after $t = \frac{3\sqrt{2}}{2}$ and consequently after $t = \sqrt{3}$. Hence, the critical point at $t = t_{\text{crit}}$ occurs in the spacetime region (4.80). $\square$

Theorem 4.17 easily follows from the preceding two propositions and the symmetry of the system (4.41) -as formulated in Proposition 4.2.

Proof of Theorem 4.17. Proposition 4.19 yields a threshold $\epsilon > 0$ such that for any $0 \leq a_0 < \epsilon$, the a component of the associated solution attains a critical point in the region (4.80). Proposition 4.18 then implies that $(a, b)_{a_0}$ is global. Finally, Proposition 4.2 proves that solutions associated to $-\epsilon < a_0 \leq 0$ are global too. $\square$

Finite Time Blowup for Large Initial Data. We now wish to study the development of large initial data. We will obtain the following:

Theorem 4.20. *Suppose that:*

$$|a_0| > \frac{1}{2 \operatorname{arctanh}\left(\frac{1}{2}\right)}.$$

Then $(a, b)_{a_0}$ blows up in finite time at most equal to:

$$t_{\text{blowup}}(a_0) \stackrel{\text{def}}{=} \frac{3\sqrt{2}}{2} \frac{\left(1 - \tanh^2\left(\frac{1}{2|a_0|}\right)\right)^{\frac{1}{2}}}{1 - 2 \tanh\left(\frac{1}{2|a_0|}\right)}. \quad (4.87)$$

Furthermore, the blowup set:

$$\mathcal{S}_{\text{blowup}} \stackrel{\text{def}}{=} \{a_0 \in \mathbb{R} \text{ s.t. } (a, b)_{a_0} \text{ blows up in finite time}\} \quad (4.88)$$

is of the form:

$$\mathcal{S}_{\text{blowup}} = (-\infty, -x) \cup (x, \infty) \quad (4.89)$$

for some $0 < x \leq \frac{1}{2 \operatorname{arctanh}\left(\frac{1}{2}\right)}$.

Our analysis relies on an apriori bound for $\frac{b}{a}$:

Proposition 4.21. *For any $a_0 \geq 0$, the solution $(a, b)_{a_0}$ satisfies the following inequality for all $0 \leq t < t_{\max}(a_0)$:*

$$b(t) > \frac{t}{2\sqrt{t^2+1}} a(t). \quad (4.90)$$

Proof. Estimate (4.90) is clearly satisfied at $t = 0$. To show that it persists for as long as solutions survive, we let $t_\star > 0$ be any time such that:

$$b(t_\star) = \frac{t_\star}{2\sqrt{t_\star^2+1}} a(t_\star)$$

and we compute:

$$\frac{d}{dt} \Big|_{t=t_\star} \left(b(t) - \frac{t}{2\sqrt{t^2+1}} a(t) \right) = \sqrt{6} \frac{b(t_\star)^2 t_\star^2 + b(t_\star)^2 + 1}{t_\star \sqrt{2t_\star^2+2} \sqrt{2t_\star^2+3}} > 0. \quad (4.91)$$

□

Proposition 4.21 allows us to estimate:

$$\dot{a} = \frac{\mathcal{P}a}{t} \left(b - \frac{1}{\mathcal{P}} \right) > \frac{\mathcal{P}a}{t} \left(\frac{ta}{2\sqrt{t^2+1}} - \frac{1}{\mathcal{P}} \right). \quad (4.92)$$

Fix a positive reference time $t_0 > 0$. Estimate (4.92) implies that -past t_0 - a is bounded below by the solution of the following I.V.P. of Riccati type:

$$\begin{cases} \dot{u}(t) = u(t) \left(\frac{\sqrt{3}}{\sqrt{2t^2+3}} u(t) - \frac{1}{t} \right), \\ u(t_0) = a(t_0). \end{cases} \quad (4.93)$$

Setting $x_0 \stackrel{\text{def}}{=} a(t_0) > 0$, Eq. (4.93) can be solved explicitly to give:

$$u_{t_0, x_0}(t) = \frac{t_0 x_0}{t \left(t_0 x_0 \operatorname{arctanh} \left(\frac{\sqrt{3}}{\sqrt{2t^2+3}} \right) - t_0 x_0 \operatorname{arctanh} \left(\frac{\sqrt{3}}{\sqrt{2t_0^2+3}} \right) + 1 \right)}. \quad (4.94)$$

The task is now to determine conditions on $t_0 > 0$, $x_0 > 0$ such that the lower bound u_{t_0, x_0} blows up to $+\infty$ in finite time past t_0 .

We introduce the *threshold function*:

$$\mathcal{R}(t) \stackrel{\text{def}}{=} \frac{1}{t \operatorname{arctanh} \left(\frac{\sqrt{3}}{\sqrt{2t^2+3}} \right)}. \quad (4.95)$$

An elementary calculation demonstrates that the denominator of (4.94) has a zero in the non-negative real line if and only if:

$$x_0 > \mathcal{R}(t_0). \quad (4.96)$$

Furthermore, when (4.96) is satisfied, this zero is unique and located at the time given by:

$$\mathcal{T}(t_0, x_0) \stackrel{\text{def}}{=} \frac{\sqrt{6}}{2} \left(1 - \frac{3}{2t_0^2 + 3} \right)^{\frac{1}{2}} \frac{\left(1 - \tanh^2 \left(\frac{1}{t_0 x_0} \right) \right)^{\frac{1}{2}}}{\frac{\sqrt{3}}{\sqrt{2t_0^2 + 3}} - \tanh \left(\frac{1}{t_0 x_0} \right)}. \quad (4.97)$$

Consequently, the lower bound u_{t_0, x_0} will exhibit the requisite behaviour provided that:

$$x_0 > \mathcal{R}(t_0) \quad \text{and} \quad \mathcal{T}(t_0, x_0) > t_0. \quad (4.98)$$

As it turns out, the second condition is vacuous:

Proposition 4.22. *Fix $t_0 > 0$. We have:*

$$\mathcal{T}(t_0, \mathcal{R}(t_0)) = +\infty$$

and $\mathcal{T}(t_0, x_0)$ decreases monotonically to t_0 (as a function of x_0) for $x_0 > \mathcal{R}(t_0)$. In particular, we have the following pointwise limit:

$$\lim_{x_0 \rightarrow \infty} \mathcal{T}(t_0, x_0) = t_0.$$

Proof. The proof is an elementary explicit calculation which we omit. $\square$

Incorporating the content of Proposition 4.22 to our earlier discussion, we arrive at the following conclusion. For each time $t_0 > 0$, the value $\mathcal{R}(t_0)$ provides a threshold (justifying our terminology), such that if u solves (4.93) and satisfies

$$u(t_0) > \mathcal{R}(t_0)$$

then it blows up in finite time equal to $\mathcal{T}(t_0, u(t_0)) > t_0$. Fix $t_0 > 0$. For $u(t_0)$ close to (but above) the threshold, the blowup time can be arbitrarily large. As $u(t_0) \rightarrow \infty$, the blowup time approaches t_0 from above. Consequently, for very large initial data, the solution survives for arbitrarily short time past t_0 .

Since u_{t_0, x_0} bounds a from below, we obtain:

Proposition 4.23. *Suppose that $(a, b)_{a_0}$ is a solution of the system (4.41) satisfying:*

$$a(t_0) > \mathcal{R}(t_0) \text{ for some } t_0 > 0.$$

Then a blows up to $+\infty$ in finite time at most equal to $\mathcal{T}(t_0, a(t_0))$.

The task is then to verify that for large initial data a_0 , the a -component of the solution eventually crosses the threshold $\mathcal{R}$, depicted below (Fig. 1).

We will use the reference time $t_0 = \frac{3\sqrt{2}}{2}$. This is the time where b_{HYM} -introduced in (4.83)- crosses $\mathcal{P}^{-1}$ -introduced in (4.42). We are able to obtain the following bound:

Proposition 4.24. *Fix $a_0 > 0$. Let $(a, b)_{a_0}$ be the development of the initial data a_0 . We have:*

$$a \left(\frac{3\sqrt{2}}{2} \right) > \frac{2\sqrt{2}}{3} a_0. \quad (4.99)$$

Proof. Fix $a_0 > 0$. Formula (4.66) demonstrates that -at least for a short time-, b_{a_0} exceeds b_{HYM} to the right of $t = 0$. Proposition 4.3 establishes that $b_{a_0} > b_{\text{HYM}}$ until $t = t_{\max}(a_0)$. Incorporating this bound with the ODE governing a , we estimate (for $t > 0$):

$$\begin{aligned}\dot{a}(t) &= \frac{\mathcal{P}a}{t} \left(b - \frac{1}{\mathcal{P}} \right) \\ &> \frac{\mathcal{P}a}{t} \left(\frac{1}{\sqrt{t^2 + 1}} - \frac{1}{\mathcal{P}} \right) \\ &= \frac{a(t)}{t} \left(\frac{2\sqrt{3}}{\sqrt{2t^2 + 3}} - 1 \right).\end{aligned}\quad (4.100)$$

Motivated by this computation, we introduce the following (singular) I.V.P:

$$\begin{cases} \dot{v}(t) = \frac{v(t)}{t} \left(\frac{2\sqrt{3}}{\sqrt{2t^2 + 3}} - 1 \right), \\ v(0) = 0, \\ \dot{v}(0) = a_0. \end{cases}$$

The problem is well-posed (solutions exist and are uniquely determined by the prescribed initial data) and v takes the form:

$$v_{a_0}(t) = \frac{36 a_0 t}{(3 + \sqrt{6t^2 + 9})^2}. \quad (4.101)$$

The function v_{a_0} has a global maximum at time $t = t_0 = \frac{3\sqrt{2}}{2}$ with value $\frac{2\sqrt{2}}{3}a_0$.

If the I.V.P. (4.101) were not singular, we could invoke estimate (4.100) to conclude that v_{a_0} bounds a_{a_0} from below for as long as the latter survives. By virtue of the above remarks, this would complete the proof. However, the standard ODE comparison argument (for nonsingular I.V.Ps) does not apply directly.

Fig. 1. Graph of the threshold function $\mathcal{R}$

We sketch this well-known argument to identify where things go wrong. Suppose that we have two initial value problems on the real line associated to two smooth (potentially non-autonomous) vector fields F_1 and F_2 such that:

$$F_1(t, x) > F_2(t, x) \quad \text{for all } t \geq 0, x \in \mathbb{R}. \quad (4.102)$$

Let $s_1(t)$ and $s_2(t)$ be solutions satisfying the same initial condition:

$$x_0 = s_1(0) = s_2(0). \quad (4.103)$$

Since $F_1(0) > F_2(0)$ there is an open interval to the right of $t = 0$ where $s_1 > s_2$. Suppose, for a contradiction that this inequality does not persist for all time. At the first future crossing the difference $s_1 - s_2$ is increasing, which yields the requisite contradiction.

Attempting to run this argument for I.V.P's with a first order singularity at $t = 0$, we observe that the initial derivatives of s_1 and s_2 match. We thus run into trouble getting s_1 to exceed s_2 in the immediate future of the initial time. The rest of the argument carries through successfully.

Fortunately, the situation can be mended by comparing higher order data at the pole. The function v_{a_0} has the following formal Taylor series at $t = 0$:

$$v_{a_0} = a_0 t - \frac{a_0}{3} t^3 + \frac{5a_0}{36} t^5 + O(t^7). \quad (4.104)$$

Comparing this with (4.65), and recalling that $a_0 > 0$ we conclude that a_{a_0} exceeds v_{a_0} in the immediate future of the initial time. The rest of the standard comparison argument carries through, ultimately establishing the requisite estimate. $\square$

The upshot is that by choosing a_0 to be sufficiently large, we can arrange that $a(t_0)$ exceeds any number we like. In particular, we can arrange that $a(t_0)$ exceeds the threshold $\mathcal{R}(t_0)$.

We now have enough to complete the proof of Theorem 4.20.

Proof of Theorem 4.20. Evaluating (4.95) and (4.97) at the reference time $t_0 = \frac{3\sqrt{2}}{2}$ we obtain:

$$\mathcal{R}(t_0) = \frac{\sqrt{2}}{3 \operatorname{arctanh}\left(\frac{1}{2}\right)}, \quad \mathcal{T}(t_0, x) = \frac{3\sqrt{2}}{2} \frac{\left(1 - \tanh^2\left(\frac{\sqrt{2}}{3x}\right)\right)^{\frac{1}{2}}}{1 - 2 \tanh\left(\frac{\sqrt{2}}{3x}\right)}. \quad (4.105)$$

Let (a, b) be a solution of the system (4.41) satisfying:

$$a(t_0) > \frac{\sqrt{2}}{3 \operatorname{arctanh}\left(\frac{1}{2}\right)}. \quad (4.106)$$

Using (4.105) and Proposition 4.23, we conclude that the solution blows up to $+\infty$ in finite time at most equal to $\mathcal{T}(t_0, a(t_0))$. Proposition 4.24 guarantees that (4.106) is satisfied provided that we take:

$$a_0 > \frac{1}{2 \operatorname{arctanh}\left(\frac{1}{2}\right)}. \quad (4.107)$$

By Proposition 4.22, when $x > \mathcal{R}(t_0)$ we have that the function $\mathcal{T}(t_0, x)$ is monotonic in x . Condition (4.107) guarantees that the right hand side of (4.99) exceeds $\mathcal{R}(t_0)$ and is thus large enough for the monotonicity statement to apply. We obtain:

$$\mathcal{T}(t_0, a(t_0)) < \mathcal{T}\left(t_0, \frac{2\sqrt{2}}{3}a_0\right) = \frac{3\sqrt{2}}{2} \frac{\left(1 - \tanh^2\left(\frac{1}{2|a_0|}\right)\right)^{\frac{1}{2}}}{1 - 2 \tanh\left(\frac{1}{2|a_0|}\right)}.$$

Defining $t_{\text{blowup}}(a_0)$ to be equal to the right hand side of this inequality, we have established the first assertion of Theorem 4.20.

Define the positive and negative blowup sets as:

$$\begin{aligned} \mathcal{S}_{\text{blowup}}^+ &\stackrel{\text{def}}{=} \{a_0 \in \mathbb{R} \text{ s.t. } a_{a_0} \text{ blows up to } +\infty \text{ in finite time}\}, \\ \mathcal{S}_{\text{blowup}}^- &\stackrel{\text{def}}{=} \{a_0 \in \mathbb{R} \text{ s.t. } a_{a_0} \text{ blows up to } -\infty \text{ in finite time}\}. \end{aligned}$$

Proposition 4.6 implies that:

$$\mathcal{S}_{\text{blowup}} = \mathcal{S}_{\text{blowup}}^+ \cup \mathcal{S}_{\text{blowup}}^-.$$

The last assertion of Theorem 4.20 will follow from Proposition 4.2 if we establish the existence of $x > 0$ such that:

$$\mathcal{S}_{\text{blowup}}^+ = (x, \infty). \quad (4.108)$$

We first prove that the positive blowup set is open. Let $a_0 \in \mathcal{S}_{\text{blowup}}^+$ and let $t_\star$ be the blowup time of the associated solution. By definition:

$$\lim_{t \rightarrow t_\star} a(t) = +\infty.$$

Consequently, there is a time $T \in [\frac{t_\star}{2}, t_\star)$ such that:

$$a(T) > 2 \sup_{t \in [\frac{t_\star}{2}, t_\star]} \mathcal{R}(t).$$

By continuity with respect to variation of the initial data, we obtain a $\delta > 0$ such that for $\tilde{a}_0 \in (a_0 - \delta, a_0 + \delta)$:

$$\frac{a(T)}{2} < \tilde{a}(T) < \frac{3}{2}a(T).$$

Consequently:

$$\tilde{a}(T) > \sup_{t \in [\frac{t_\star}{2}, t_\star]} \mathcal{R}(t) \geq \mathcal{R}(T). \quad (4.109)$$

By Proposition 4.22, the initial data $\tilde{a}_0$ lead to finite-time blowup and $\mathcal{S}_{\text{blowup}}^+$ is indeed open.

Finally, by Proposition 4.3, if a certain choice of $a_0 > 0$ leads to finite-time blowup, so do all $\tilde{a}_0 > a_0$. Together with openness, this property yields (4.108) for some $x \geq 0$. Theorem 4.17 implies that $x > 0$. $\square$

4.2.5. The Moduli space The results of the preceding sections are sufficient to obtain a complete description of the moduli space of $\mathrm{SO}(5)$ invariant $\mathrm{Spin}(7)$ instantons with structure group $\mathrm{SO}(3)$ on the Stenzel manifold. We denote this object as $\mathcal{M}_{\mathrm{inv}}^{\mathrm{Spin}(7)}(X^8)$. The trivial bundle P_1 doesn't contribute to this moduli space. This is due to the nonexistence Theorem 3.3. Here we ignore the trivial solution $A = 0$.

Let P be a G -homogeneous (or cohomogeneity one) principal S -bundle. There are two natural ways to set up a moduli space of G -invariant solutions to a gauge-theoretic problem on P . One is to quotient the set of invariant solutions by the group of equivariant gauge transformations. The other is carried out in two steps. Initially one quotients the set of all (not necessarily invariant) solutions by the set of all (not necessarily equivariant) gauge transformations. The action of G on the total space P induces an action on the set of all connections. This action restricts to the set of solutions and passes to the quotient. The moduli space is then defined to be the G -invariant locus. There is an obvious map from the first construction to the second construction. If the fiber S is semisimple and we restrict attention to irreducible connections, this map is an isomorphism ([12], Corollary 4.5).

In our setting, the structure group is $\mathrm{SO}(3)$ (which is indeed semisimple) and furthermore, all solutions are irreducible. It follows that the two constructions coincide. We will follow the first. Recall that each invariant connection constitutes its own equivariant gauge equivalence class. Consequently, the moduli spaces on the individual bundles are:

$$\begin{aligned}\mathcal{M}(P_{\pi_1}) &\stackrel{\mathrm{def}}{=} \{A \in \mathcal{A}_{\mathrm{inv}}(P_{\pi_1}) \text{ s.t. } \star_g F_A = -\Phi \wedge F_A\}, \\ \mathcal{M}(P_{\pi_2}) &\stackrel{\mathrm{def}}{=} \{A \in \mathcal{A}_{\mathrm{inv}}(P_{\pi_2}) \text{ s.t. } \star_g F_A = -\Phi \wedge F_A\}.\end{aligned}$$

Due to the results of Sect. 4.2.4, we have that $\mathcal{M}(P_{\pi_1})$ is a compact interval. It can be parameterized by initial conditions $a_0 = \dot{a}(0)$ leading to global solutions. Using this parameterization, Theorem 4.20 gives us a number $x > 0$ such that:

$$\mathcal{M}(P_{\pi_1}) \cong [-x, x]. \quad (4.110)$$

The space $\mathcal{M}(P_{\pi_1})$ contains a unique HYM connection $A_{\mathrm{HYM}\pi_1}$ corresponding to $a_0 = 0$. It is represented by the red dot in the following diagram (Fig. 2). The black dots represent the boundary points $\pm x$.

Due to the results of Sect. 4.2.3, we have that $\mathcal{M}(P_{\pi_2})$ is a half-open half-closed interval. We can parameterize it by the value $v = b(t_0)$ at time $t_0 = \frac{\sqrt{6}}{2}$. Using this

Fig. 2. The Moduli space $\mathcal{M}(P_{\pi_1})$

parameterization and setting:

$$\nu_{\partial} \stackrel{\text{def}}{=} -\frac{2\sqrt{5}}{5}, \quad \nu_* \stackrel{\text{def}}{=} \frac{\sqrt{10}}{5},$$

we have that:

$$\mathcal{M}(P_{\pi_2}) = \{A_\nu \text{ s.t. } \nu \in [\nu_{\partial}, \nu_*]\} \cong [\nu_{\partial}, \nu_*]. \tag{4.111}$$

The space $\mathcal{M}(P_{\pi_2})$ contains a unique HYM connection $A_{\text{HYM}_{\pi_2}}$ corresponding to $\nu = -\frac{\sqrt{10}}{5}$. It is represented by the green dot in the following diagram (Fig. 3). The black dot represents the boundary point ν_{∂} .

We observe that $\mathcal{M}(P_{\pi_2})$ is not compact. Interestingly, it admits a natural compactification. To understand the noncompactness phenomenon, we study the (missing) limit $\nu \rightarrow \frac{\sqrt{10}}{5}$. To identify what the limit should be we work on $X^8 \setminus S^4$. Using the explicit formula (4.49) with $\nu = \frac{\sqrt{10}}{5}$ yields the HYM connection $A_{\text{HYM}_{\pi_1}}$. We conclude that (over $X^8 \setminus S^4$):

$$\lim_{\nu \rightarrow \nu_*} A_\nu = A_{\text{HYM}_{\pi_1}}. \tag{4.112}$$

This can be understood pointwise -with a choice of some background reference connection- or in a suitable weighted norm.

We conclude that the $\text{Spin}(7)$ instantons in $\mathcal{M}(P_{\pi_2})$ are trying to converge to the (unique) HYM connection of $\mathcal{M}(P_{\pi_1})$, but fail to do so as this connection does not smoothly extend to the bundle on which they live. Notably the singularity happens around a codimension 4 Cayley submanifold (the singular orbit S^4). This reasoning motivates us to glue in $\mathcal{M}(P_{\pi_1})$, by forcing the point $a_0 = 0$ to be the missing endpoint of $\mathcal{M}(P_{\pi_2})$. This leads to the following picture (Fig. 4) of $\mathcal{M}_{\text{inv}}^{\text{Spin}(7)}(X^8)$:

The notation in this diagram is consistent with Figs. 2 and 3. Crucially, the space $\mathcal{M}_{\text{inv}}^{\text{Spin}(7)}(X^8)$ is compact.

Fig. 3. The Moduli space $\mathcal{M}(P_{\pi_2})$

Fig. 4. The Moduli space $\mathcal{M}_{\text{inv}}^{\text{Spin}(7)}(X^8)$

This suggests a potential relationship between Spin(7) instantons and HYM connections. Indeed, they are not equivalent in general; but furthermore, the structure of $\mathcal{M}_{\text{inv}}^{\text{Spin}(7)}(X^8)$ hints that the latter might play a role in the compactification of Spin(7) instanton moduli spaces (over noncompact Calabi–Yau fourfolds). It may be a general phenomenon that certain sequences of Spin(7) instantons fail to converge because the limit lives on a different bundle. Furthermore this new bundle should agree with the original one outside of a (codimension 4) Cayley submanifold. The missing limit should be HYM.

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References

1. Conlon, Ronan J., Hein, Hans Joachim: Asymptotically Conical Calabi–Yau Manifolds I. *Duke Math. J.* **162**(15), 2855–2902 (2013)
2. Donaldson, Simon, Kronheimer, Peter B.: *The Geometry of Four-Manifolds*. Oxford University Press, Oxford (1991)
3. Donaldson, Simon, Segal, Ed.: *Gauge Theory in Higher Dimensions, II*. *Surv. Differ. Geom.* **16**, 1–42 (2011)
4. Donaldson, Simon, Thomas, Richard: *Gauge Theory in Higher Dimensions*. In *The Geometric Universe: Science, Geometry and the Work of Roger Penrose*, pages 31–47. Oxford University Press, 1998
5. Driscoll, Joseph: *Deformations of Asymptotically Conical G_2 Instantons*. PhD thesis, University of Leeds, (2020) Available at: <http://etheses.whiterose.ac.uk/27540/>
6. Eschenburg, J.H., Wang, McKenzie Y.: The Initial Value Problem for Cohomogeneity One Einstein Metrics. *The Journal of Geometric Analysis* **10**(1), 109–137 (2000)
7. Joyce, Dominic: *Riemannian Holonomy Groups and Calibrated Geometry*. Oxford University Press, Oxford (2007)
8. Kobayashi, Shoshichi, Nomizu, Katsumi: *Foundations of Differential Geometry*, vol. 1. Wiley (1963)
9. Lewis, C.: *Spin(7) Instantons*. PhD thesis, The University of Oxford, 1998. Available at: <http://people.maths.ox.ac.uk/~joyce/theses/theses.html>
10. Lotay, Jason D., Oliveira, Gonalo: $SU(2)^2$ -Invariant G_2 -Instantons. *Mathematische Annalen* **371**, 961–1011 (2018)
11. McLeod, J.B., Smoller, Joel A., Wasserman, Arthur G., Yau, S. T.: Smooth Static Solutions of the Einstein–Yang–Mills Equations. *Communications in Mathematical Physics* **143**(S1), 115–147 (1991)
12. Oliveira, Gonalo: Some Useful Facts on Invariant Connections. *To appear in ISAAC 2019 Conference Proceedings*. Available at the author’s webpage: <https://sites.google.com/view/goncalo-oliveira-math-webpage/home>
13. Oliveira, Gonalo: Calabi–Yau Monopoles for the Stenzel Metric. *Communications in Mathematical Physics* **341**, 699–728 (2016)
14. Patrizio, Giorgio, Wong, Pit-Mann.: Stein Manifolds with Compact Symmetric Center. *Mathematische Annalen* **289**(1), 355–382 (1991)

15. Salamon, Dietmar A., Walpuski, Thomas: Notes on the Octonions. 2017. Proceedings of the 23rd Gokova Geometry-Topology Conference, pp. 1-85
16. Stenzel, Matthew B.: Ricci-Flat Metrics on the Complexification of a Compact Rank One Symmetric Space. *Manuscripta Mathematica* **80**(1), 151–163 (1993)
17. Uhlenbeck, K., Yau, S.T.: On the Existence of Hermitian–Yang–Mills Connections in Stable Vector Bundles. *Communications on Pure and Applied Mathematics* **39**(S1), 257–293 (1986)
18. Wang, Hsien-Chung.: On Invariant Connections Over a Principal Fibre Bundle. *Nagoya Mathematical Journal* **13**, 1–19 (1958)

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