The Controlling L_∞ -Algebra, Cohomology and Homotopy of Embedding Tensors and Lie–Leibniz Triples

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Received: 6 October 2020 / Accepted: 3 February 2021
 Published online: 4 April 2021 – © The Author(s) 2021

Abstract: In this paper, we first construct the controlling algebras of embedding tensors and Lie–Leibniz triples, which turn out to be a graded Lie algebra and an L_∞ -algebra respectively. Then we introduce representations and cohomologies of embedding tensors and Lie–Leibniz triples, and show that there is a long exact sequence connecting various cohomologies. As applications, we classify infinitesimal deformations and central extensions using the second cohomology groups. Finally, we introduce the notion of a homotopy embedding tensor which will induce a Leibniz_∞ -algebra. We realize Kotov and Strobl’s construction of an L_∞ -algebra from an embedding tensor, as a functor from the category of homotopy embedding tensors to that of Leibniz_∞ -algebras, and a functor further to that of L_∞ -algebras.

Contents

1. Introduction	270
2. Embedding Tensors, omni-Lie Algebras and Leibniz Algebras	272
3. The Controlling Graded Lie Algebra and Cohomology of Embedding Tensors	276
3.1 The controlling graded Lie algebra of LieRep pairs	277
3.2 The controlling graded Lie algebra of embedding tensors	278
3.3 Cohomology of embedding tensors	280
4. The Controlling L_∞ -Algebra of Lie–Leibniz Triples	282
4.1 L_∞ -algebras and higher derived brackets	282
4.2 The controlling L_∞ -algebra of Lie–Leibniz triples	284
5. Cohomology of Lie–Leibniz Triples and Applications	285
5.1 Regular cohomology of Lie–Leibniz triples and infinitesimal deformations	286
5.2 Cohomology with arbitrary coefficients and central extensions	290
6. Homotopy Embedding Tensors and Higher Structures	294
6.1 Homotopy embedding tensors and Leibniz_∞ -algebras	295

6.2 Leibniz $_{\infty}$ -algebras and A_{∞} -algebras	298
6.3 Leibniz $_{\infty}$ -algebras and L_{∞} -algebras	300

1. Introduction

An embedding tensor on a Lie algebra representation $(\mathfrak{g}, V)$ is a linear map $T : V \rightarrow \mathfrak{g}$ satisfying a quadratic equivariancy constraint (see Definition 2.2). Leibniz algebras, embedding tensors and their associated tensor hierarchies provide a nice and efficient way to construct supergravity theories and further to higher gauge theories (see e.g. [6, 7, 18, 24, 51] and references therein for a rich physics literature on this subject, and see [27, 28] for a math-friendly introduction on this subject). Recently, this topic has attracted much attention of the mathematical physics world. First of all, a sharp and beautiful observation by Kotov and Strobl in a recent article [24] demonstrates for us a possible mathematical nature behind the various calculations from embedding tensors to their associated tensor hierarchies in the physics literature. An embedding tensor gives rise to a Leibniz algebra, which further gives rise to an L_{∞} -algebra, and this corresponds to tensor hierarchies in physics literature. We see later that both procedures are functorial (the first one in Sect. 2, and the second one in Sect. 6), moreover the functoriality can be extended with *homotopy* added in for all objects. In fact, the second procedure is a composition of several very classic results [35, 37]. This therefore guarantees us, from a category viewpoint, that the process from embedding tensors to tensor hierarchies, and its corresponding transition from supergravity theories to higher gauge theories, is natural. Then, almost at the same time, appeared several other approaches to encode this process to tensor hierarchies, involving also Leibniz field theory: [50, 52] are from the point of view of enhanced Leibniz structures; [29] builds an L_{∞} -algebra extension from a Leibniz algebra, which is apparently different from the second functor described above. The functoriality was shown in both procedures.

In our setting, we further conjecture that the above two functors are functorial in an ∞ -category sense. We will explore this direction in a future work [45]. Notice that the homotopy nature of L_{∞} -algebras suggests homotopy also in the category hosting these objects.

In this article, we provide a rich math tool box for embedding tensors and Lie–Leibniz triples, which seem not yet existing in the mathematical literature. Indeed, as a sort of algebra (or operad), embedding tensors involve not only binary but also unary operations. We develop the theory of controlling algebras, thus further the theory of cohomology and homotopy for embedding tensors and Lie–Leibniz triples.

To establish a good cohomology theory for an object, besides the standard homological algebraic method of projective resolutions, there is also another shorter way through its “controlling algebraic object”. Let us explain this idea in the case of a Lie algebra $\mathfrak{g}$. We start with a vector space $\mathfrak{g}$, then the graded vector space $\oplus_{k=0}^{+\infty} \text{Hom}(\wedge^k \mathfrak{g}, \mathfrak{g})$ equipped with the Nijenhuis–Richardson bracket $[\cdot, \cdot]_{\text{NR}}$ becomes a graded Lie algebra (g.l.a.), or a differential graded Lie algebra (d.g.l.a.) with 0 differential [38]. Then a Lie algebra structure on $\mathfrak{g}$ corresponds *exactly* to a Maurer–Cartan element $\pi \in \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g})$. We call this g.l.a. $(\oplus_{k=0}^{+\infty} \text{Hom}(\wedge^k \mathfrak{g}, \mathfrak{g}), [\cdot, \cdot]_{\text{NR}})$ the *controlling algebra of Lie algebra structures* on $\mathfrak{g}$. Furthermore, since $[\pi, \pi]_{\text{NR}} = 0$, $d_{\pi} := [\pi, \cdot]_{\text{NR}}$ satisfies $d_{\pi}^2 = 0$, thus d_{π} is a differential. The controlling g.l.a. $\oplus_{k=0}^{+\infty} \text{Hom}(\wedge^k \mathfrak{g}, \mathfrak{g})$ together with d_{π} becomes exactly the Chevalley–Eilenberg complex to calculate the cohomology of $\mathfrak{g}$ with coefficients in its adjoint representation. This is a general phenomenon and works not only for Lie algebras, but also for associative algebras, Leibniz algebras, n -Lie algebras, and

pre-Lie algebras. See the review [10, 17] for more details. Thus, we use this principal as a guide to develop cohomology theories for embedding tensors (Sect. 3) and Lie–Leibniz triples (Sect. 4). Here a Lie–Leibniz triple [28] consists of a Lie algebra representation $(\mathfrak{g}, V)$, and an embedding tensor $T : V \rightarrow \mathfrak{g}$. The subtle difference between these two concepts shows up, e.g. in deformation theory. To deform an embedding tensor, we fix $(\mathfrak{g}, V)$ and deform only the operator T , but to deform a Lie–Leibniz triple, we are allowed to deform also the Lie algebra and its representation $(\mathfrak{g}, V)$ simultaneously. It turns out that the controlling algebraic structure for embedding tensors is a g.l.a. and that for Lie–Leibniz triples is an L_∞ -algebra. Thus indeed the theory of Lie–Leibniz triples is more involved.

We give some immediate applications (also as verifications) of this cohomology theory of Lie–Leibniz triples in Sect. 5. It does behave as it should: Given a Lie–Leibniz triple,

- (1) its second cohomology classes in H^2 with coefficients in the adjoint representation correspond exactly to the equivalence classes of its infinitesimal deformations;
- (2) its second cohomology classes in H^2 with coefficients in the trivial representation correspond exactly to the equivalence classes of central extensions.

Here, we actually need a bit of additional luck in the second application: We need to develop a cohomology theory for Lie–Leibniz triples with arbitrary coefficients, but not just the one from adjoint representation. For this, we find that there is a natural Lie–Leibniz triple structure on the endomorphisms of a 2-term complex of vector spaces. This natural structure comes from the strict Lie 2-algebra structure on them [46], and a strict Lie 2-algebra is a special Lie–Leibniz triple (see Example 2.10 and Example 2.13).

Finally, in Sect. 6, we study how embedding tensors cooperate with homotopy. That is, what a homotopy embedding tensor should be, and how Kotov–Strobl’s functor **KS** behaves with respect to homotopy. Will **KS** still produces an L_∞ -algebra or something involving even more homotopy? This will test how stable the concept of embedding tensors and the procedure to topological hierarchies are. We still use the tool of controlling algebras to develop the homotopy theory. A standard approach [35, 54] to give a homotopy $\mathcal{P}$ -algebraic structure is to construct a minimal model $\mathcal{P}_\infty$ of the operad $\mathcal{P}$. Along this approach, L_∞ -algebras and A_∞ -algebras are well developed. Moreover, Leibniz_∞ -algebras are defined as the algebras over the minimal model [2, 31] of the *Leibniz* operad. However, apart from this approach, we can also use Maurer–Cartan elements of the aforementioned controlling algebra on a graded vector space to define a homotopy algebraic structure. For example, to define an L_∞ -algebra, one can start with a graded vector space $\mathfrak{g}^\bullet$ and define an L_∞ -algebra to be a Maurer–Cartan element of the g.l.a. $(\text{Hom}(\mathbf{S}(\mathfrak{g}^\bullet), \mathfrak{g}^\bullet), [\cdot, \cdot]_{\text{NR}})$. Using this method, we define a homotopy embedding tensor to be a Maurer–Cartan element of a graded version of the controlling algebra that we develop in Sect. 3. Then we show that a homotopy embedding tensor gives rise to a Leibniz_∞ -algebra, and a Leibniz_∞ -algebra further gives rise to an L_∞ -algebra. We further prove that these two processes are functorial. Thus the functor **KS** extends to a homotopic version.

We want to emphasize that embedding tensors and Lie–Leibniz triples have been already known in mathematics literature under the name of averaging operators and averaging algebras respectively for a long time. In the last century, many studies on averaging operators were done for various special algebras, such as function spaces, Banach algebras, and the topics and methods were largely analytic [5, 9, 19, 41]. See the well-written introduction in [40] for more details. More recently, people have begun to study averaging operators in double algebras, classical Yang–Baxter equation, conformal

algebras, and the procedure of replication in the operad theory [1, 15, 21, 39]. It is not yet clear to us how these aspects of embedding tensors and Lie–Leibniz triples are related. But we wish our article makes some first steps to understand these concepts more deeply.

2. Embedding Tensors, omni-Lie Algebras and Leibniz Algebras

In this section, first we establish relations between embedding tensors, omni-Lie algebras and Leibniz algebras. Then we give some interesting examples of embedding tensors.

Definition 2.1. A **LieRep pair** consists of a Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ and a representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ of $\mathfrak{g}$ on a vector space V .

We denote a **LieRep pair** by $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}), (V; \rho))$, or simply by $(\mathfrak{g}, V)$ if there is no confusion.

Definition 2.2. (i) An **embedding tensor** on a LieRep pair $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}), (V; \rho))$ is a linear map $T : V \longrightarrow \mathfrak{g}$ satisfying the following quadratic constraint:

$$[Tu, Tv]_{\mathfrak{g}} = T(\rho(Tu)(v)), \quad \forall u, v \in V. \quad (1)$$

(ii) A **Lie–Leibniz triple** is a triple $(\mathfrak{g}, V, T)$, where $(\mathfrak{g}, V)$ is a LieRep pair and $T : V \longrightarrow \mathfrak{g}$ is an embedding tensor on the LieRep pair $(\mathfrak{g}, V)$.

Definition 2.3. Let $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}), (V; \rho), T)$ and $((\mathfrak{g}', \{\cdot, \cdot\}_{\mathfrak{g}'}) , (V', \rho'), T')$ be two Lie–Leibniz triples. A **homomorphism** from $((\mathfrak{g}', \{\cdot, \cdot\}_{\mathfrak{g}'}) , (V', \rho'), T')$ to $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}), (V; \rho), T)$ consists of a Lie algebra homomorphism $\phi : \mathfrak{g}' \longrightarrow \mathfrak{g}$ and a linear map $\varphi : V' \longrightarrow V$ such that

$$T \circ \varphi = \phi \circ T', \quad (2)$$

$$\varphi \rho'(x)(u) = \rho(\phi(x))(\varphi(u)), \quad \forall x \in \mathfrak{g}', u \in V'. \quad (3)$$

In particular, if ϕ and φ are invertible, then (ϕ, φ) is called an **isomorphism**.

The algebraic structure underlying an embedding tensor is a **Leibniz algebra**, which is a vector space $\mathfrak{G}$ together with a bilinear operation $[\cdot, \cdot]_{\mathfrak{G}} : \mathfrak{G} \otimes \mathfrak{G} \rightarrow \mathfrak{G}$ such that

$$[x, [y, z]_{\mathfrak{G}}]_{\mathfrak{G}} = [[x, y]_{\mathfrak{G}}, z]_{\mathfrak{G}} + [y, [x, z]_{\mathfrak{G}}]_{\mathfrak{G}}, \quad \forall x, y, z \in \mathfrak{G}.$$

Proposition 2.4. ([1]) Let $T : V \rightarrow \mathfrak{g}$ be an embedding tensor on a LieRep pair $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}), (V; \rho))$. Then there exists a Leibniz algebra structure $[\cdot, \cdot]_T$ on V given by

$$[u, v]_T := \rho(Tu)v, \quad \forall u, v \in V. \quad (4)$$

Remark 2.5. This association gives rise to a functor $F : \text{ET} \rightarrow \text{Leibniz-Alg}$ from the category of embedding tensors to that of Leibniz algebras. The direction from Leibniz algebras to embedding tensors is less well behaved. It is easy to check that the association in [24] gives rise to a functor $G : \text{Leibniz-Alg} \rightarrow \text{ET}$, and $F \circ G = \text{Id}$ as also noticed therein. But $G \circ F \neq \text{Id}$, and these two functors do not differ even by a natural transformation. There is another natural association given in [28], namely for a Leibniz algebra $(\mathfrak{G}, [\cdot, \cdot]_{\mathfrak{G}})$, the left multiplication $L : \mathfrak{G} \rightarrow \mathfrak{gl}(\mathfrak{G})$ given by

$$L_x y = [x, y]_{\mathfrak{G}}, \quad \forall x, y \in \mathfrak{G}, \quad (5)$$

is an embedding tensor on the Lie algebra $\mathfrak{gl}(\mathfrak{G})$ with respect to the natural representation on the vector space $\mathfrak{G}$. Even though with this method, it is more likely to create a natural transformation, it does not give rise to even a functor $\text{Leibniz-Alg} \rightarrow \text{ET}$ in the first place.

In the sequel, we give an alternative explanation of Proposition 2.4 using integrable subspaces of omni-Lie algebras. For this purpose, we give an interesting example of embedding tensors.

Example 2.6. Let V be a vector space. Then the general linear Lie algebra $\mathfrak{gl}(V)$ represents on the direct sum $\mathfrak{gl}(V) \oplus V$ naturally via:

$$\rho(A)(B + v) = [A, B] + Av, \quad \forall A, B \in \mathfrak{gl}(V), v \in V.$$

Let $P : \mathfrak{gl}(V) \oplus V \rightarrow \mathfrak{gl}(V)$ be the projection to the first summand. Then we have

$$P\left(\rho(P(A + u))(B + v)\right) = P(\rho(A)(B + v)) = [A, B] = [P(A + u), P(B + v)],$$

for all $A, B \in \mathfrak{gl}(V)$, $u, v \in V$. Thus, P is an embedding tensor on $\mathfrak{gl}(V)$ with respect to the representation $(\mathfrak{gl}(V) \oplus V; \rho)$.

By Proposition 2.4, there is an induced Leibniz algebra structure on $\mathfrak{gl}(V) \oplus V$ given by

$$[A + u, B + v]_{\text{ol}} = \rho(P(A + u))(B + v) = [A, B] + Av. \quad (6)$$

The above bracket $[\cdot, \cdot]_{\text{ol}}$ is exactly the omni-Lie bracket on $\mathfrak{gl}(V) \oplus V$ introduced by Weinstein in [56]. Recall that an **omni-Lie algebra** is a triple $(\mathfrak{gl}(V) \oplus V, [\cdot, \cdot]_{\text{ol}}, (\cdot, \cdot)_+)$, where the omni-Lie bracket $[\cdot, \cdot]_{\text{ol}}$ is given by (6), and $(\cdot, \cdot)_+$ is a symmetric V -valued pairing given by

$$(A + u, B + v)_+ = Av + Bu. \quad (7)$$

Definition 2.7. A subspace $H \subset \mathfrak{gl}(V) \oplus V$ is said to be **integrable** if $[H, H]_{\text{ol}} \subset H$.

Now we are ready to characterize embedding tensors using integrable subspaces of the omni-Lie algebra.

Theorem 2.8. Let $T : V \rightarrow \mathfrak{gl}(V)$ be a linear map. Then T is an embedding tensor on the general linear Lie algebra $\mathfrak{gl}(V)$ with respect to the natural representation on V if and only if the graph of T , denoted by G_T , is an integrable subspace of the omni-Lie algebra $(\mathfrak{gl}(V) \oplus V, [\cdot, \cdot]_{\text{ol}}, (\cdot, \cdot)_+)$.

Proof. For all $u, v \in V$, we have

$$[Tu + u, Tv + v]_{\text{ol}} = [Tu, Tv] + (Tu)v,$$

which implies that the graph of T is integrable if and only if $[Tu, Tv] = T((Tu)v)$, i.e. T is an embedding tensor. $\square$

Remark 2.9. Since the omni-Lie bracket $[\cdot, \cdot]_{\text{ol}}$ is a Leibniz algebra structure, it follows that an integrable subspace is also a Leibniz algebra. Thus, if $T : V \rightarrow \mathfrak{gl}(V)$ is an embedding tensor, then G_T is a Leibniz algebra. Since G_T and V are isomorphic, so there is an induced Leibniz algebra structure on V . This Leibniz algebra structure on V is exactly the one given in Proposition 2.4.

In the rest of this section, we give various interesting examples.

Example 2.10 (differential Lie algebras). Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, d)$ be a differential Lie algebra, that is a Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ with a derivation d such that $d \circ d = 0$. Then we have

$$d[dx, y]_{\mathfrak{g}} = [d^2x, y]_{\mathfrak{g}} + [dx, dy]_{\mathfrak{g}} = [dx, dy]_{\mathfrak{g}}.$$

Thus, d is an embedding tensor on the **LieRep** pair $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}), (\mathfrak{g}; \text{ad}))$.

Example 2.11 (an example from supergravity). This example is taken from physics literature [24, 42] on supergravity in space-time dimension 4, which is one of the origins where the concept of embedding tensors appear. The vector space V is taken as the fundamental representation $\mathbf{56} = (\wedge^2 \mathbb{R}^8) \oplus (\wedge^2 \mathbb{R}^8)^*$, of $E_{7(7)}$, the non-compact real form of E_7 . We take $\mathfrak{g} = \mathfrak{so}(8)$, the Lie algebra of real skew-symmetric matrices. In fact $\text{SO}(8) \subset E_{7(7)}$ and the naturally induced representation ρ of $\mathfrak{so}(8)$ on V is simply the sum of a wedge product of the fundamental representation of $\mathfrak{so}(8)$ and its dual. More precisely, $\mathfrak{so}(8)$ naturally represents on $W := \wedge^2 \mathbb{R}^8$ via

$$\bar{\rho}(A)(u \wedge v) = (Au) \wedge v + u \wedge Av, \quad \forall u, v \in \mathbb{R}^8.$$

Let $\bar{\rho}^*$ be the dual representation of $\mathfrak{so}(8)$ on W^* . Then $\rho = \bar{\rho} + \bar{\rho}^*$ is a representation of $\mathfrak{so}(8)$ on V . Let $E_{ij} = R_{ij} - R_{ji}$ be a basis of $\mathfrak{g}$, where R_{ij} denotes the 8×8 matrix with the (i, j) -position being 1 and elsewhere being 0. Then we have

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{ik}E_{jl} + \delta_{jl}E_{ki} - \delta_{il}E_{kj}.$$

Let $\{e_1, \dots, e_8\}$ be the basis of $\mathbb{R}^8$ where e_i is the vector with the i -th position being 1 and elsewhere being 0. Then $\{e_i \wedge e_j\}_{i < j}$ forms a basis of W . Let $\{e_1^*, \dots, e_8^*\}$ be the dual basis. So $\{e_i^* \wedge e_j^*\}_{i < j}$ forms a basis of W^* .

Define $T : V \rightarrow \mathfrak{so}(8)$ by

$$T(e_i \wedge e_j) = E_{ij}, \quad T(e_i^* \wedge e_j^*) = 0.$$

Then we have

$$\begin{aligned} T(\rho(T(e_i \wedge e_j))(e_k \wedge e_l)) &= T((E_{ij}e_k) \wedge e_l + e_k \wedge E_{ij}e_l) \\ &= T(\delta_{jk}e_i \wedge e_l - \delta_{ik}e_j \wedge e_l + \delta_{jl}e_k \wedge e_i - \delta_{il}e_k \wedge e_j) \\ &= \delta_{jk}E_{il} - \delta_{ik}E_{jl} + \delta_{jl}E_{ki} - \delta_{il}E_{kj} \\ &= [E_{ij}, E_{kl}] \\ &= [T(e_i \wedge e_j), T(e_k \wedge e_l)], \end{aligned}$$

which implies that T is an embedding tensor on the **LieRep** pair $(\mathfrak{so}(8), V)$.

The induced Leibniz algebra structure on V is given by

$$\begin{aligned} [e_i \wedge e_j + e_p^* \wedge e_q^*, e_k \wedge e_l + e_m^* \wedge e_n^*]_T &= \rho(T(e_i \wedge e_j + e_p^* \wedge e_q^*))(e_k \wedge e_l + e_m^* \wedge e_n^*) \\ &= \bar{\rho}(E_{ij})(e_k \wedge e_l) + \bar{\rho}^*(E_{ij})(e_m^* \wedge e_n^*) \\ &= \delta_{jk}e_i \wedge e_l - \delta_{ik}e_j \wedge e_l \\ &\quad + \delta_{jl}e_k \wedge e_i - \delta_{il}e_k \wedge e_j \\ &\quad + \delta_{jm}e_i^* \wedge e_n^* - \delta_{im}e_j^* \wedge e_n^* \\ &\quad + \delta_{jn}e_m^* \wedge e_i^* - \delta_{in}e_m^* \wedge e_j^*. \end{aligned}$$

Notice that even though the first term $\bar{\rho}(E_{ij})(e_k \wedge e_l) = [E_{ij}, E_{kl}] = -\bar{\rho}(E_{kj})(e_i \wedge e_j)$ has antisymmetric property, the second term make the bracket $[\cdot, \cdot]_T$ not antisymmetric. Thus we end up really with a Leibniz algebra, not a Lie algebra. Mathematically, this example can be generalized to all Lie algebra $\mathfrak{g}$ acts on $V = \mathfrak{g} \oplus \mathfrak{g}^*$ via its adjoint and coadjoint representation. That is, the natural projection to the first factor $T : V \rightarrow \mathfrak{g}$ is an embedding tensor on $\mathfrak{g}$ with respect to the action on $V = \mathfrak{g} \oplus \mathfrak{g}^*$.

Example 2.12 (endomorphism algebra of a 2-term complex). Given a 2-term complex of vector spaces $W \xrightarrow{\mathfrak{T}} \mathfrak{h}$, we define $\text{End}(W \xrightarrow{\mathfrak{T}} \mathfrak{h})$ by

$$\text{End}(W \xrightarrow{\mathfrak{T}} \mathfrak{h}) \triangleq \{(A_0, A_1) \in \mathfrak{gl}(\mathfrak{h}) \oplus \mathfrak{gl}(W) \mid A_0 \circ \mathfrak{T} = \mathfrak{T} \circ A_1\}.$$

It is obvious that $\text{End}(W \xrightarrow{\mathfrak{T}} \mathfrak{h})$ with the commutator bracket $[\cdot, \cdot]_C$ is a Lie algebra. Moreover it represents on $\text{Hom}(\mathfrak{h}, W)$ via

$$\begin{aligned} \rho(A_0, A_1)(\Phi) &= [(A_0, A_1), \Phi]_C \triangleq A_1 \circ \Phi - \Phi \circ A_0, \\ \forall (A_0, A_1) \in \text{End}(W \xrightarrow{\mathfrak{T}} \mathfrak{h}), \quad \Phi \in \text{Hom}(\mathfrak{h}, W). \end{aligned}$$

Define $\bar{\mathfrak{T}} : \text{Hom}(\mathfrak{h}, W) \rightarrow \text{End}(W \xrightarrow{\mathfrak{T}} \mathfrak{h})$ by

$$\bar{\mathfrak{T}}(\Phi) \triangleq (\Phi \circ \mathfrak{T}, \mathfrak{T} \circ \Phi), \quad \forall \Phi \in \text{Hom}(\mathfrak{h}, W).$$

Then it is straightforward to deduce that $(\text{End}(W \xrightarrow{\mathfrak{T}} \mathfrak{h}), \text{Hom}(\mathfrak{h}, W), \bar{\mathfrak{T}})$ is a Lie–Leibniz triple.

This Lie–Leibniz triple plays an important role in the representation theory of Lie–Leibniz triples. See Definition 5.11 for more details. In fact, this embedding tensor comes from a strict 2-Lie algebra structure on the endomorphisms of a 2-term complex described in [46]. This can be generalized to any strict Lie 2-algebra as follows.

Example 2.13. (strict Lie 2-algebras) A strict Lie 2-algebra is a 2-term graded vector spaces $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0$ equipped with a linear map $d_{\mathfrak{g}} : \mathfrak{g}_1 \rightarrow \mathfrak{g}_0$ and a skew-symmetric bilinear map $[\cdot, \cdot]_{\mathfrak{g}} : \mathfrak{g}_i \wedge \mathfrak{g}_j \rightarrow \mathfrak{g}_{i+j}$, $0 \leq i+j \leq 1$, such that for all $x, y, z \in \mathfrak{g}_0$, $a, b \in \mathfrak{g}_1$ the following equalities are satisfied:

- (a) $d_{\mathfrak{g}}[x, a]_{\mathfrak{g}} = [x, d_{\mathfrak{g}}a]_{\mathfrak{g}}$, $[d_{\mathfrak{g}}a, b]_{\mathfrak{g}} = [a, d_{\mathfrak{g}}b]_{\mathfrak{g}}$,
- (b) $[[x, y]_{\mathfrak{g}}, z]_{\mathfrak{g}} + [[y, z]_{\mathfrak{g}}, x]_{\mathfrak{g}} + [[z, x]_{\mathfrak{g}}, y]_{\mathfrak{g}} = 0$, $[[x, y]_{\mathfrak{g}}, a]_{\mathfrak{g}} + [[y, a]_{\mathfrak{g}}, x]_{\mathfrak{g}} + [[a, x]_{\mathfrak{g}}, y]_{\mathfrak{g}} = 0$.

Define ρ from $\mathfrak{g}_0$ to $\mathfrak{gl}(\mathfrak{g}_1)$ by $\rho(x)(a) = [x, a]_{\mathfrak{g}}$. Then, $(\mathfrak{g}_1; \rho)$ is a representation of the Lie algebra $(\mathfrak{g}_0, [\cdot, \cdot]_{\mathfrak{g}})$ and $d_{\mathfrak{g}}$ is an embedding tensor on the LieRep pair $((\mathfrak{g}_0, [\cdot, \cdot]_{\mathfrak{g}}), (\mathfrak{g}_1; \rho))$.

As strict Lie 2-algebras are equivalent to crossed modules of Lie algebras, we naturally have the following example.

Example 2.14. (crossed modules of Lie algebras) A crossed module of Lie algebras is a quadruple $(\mathfrak{g}_0, \mathfrak{g}_1, d, \rho)$, where $\mathfrak{g}_0, \mathfrak{g}_1$ are Lie algebras, $d : \mathfrak{g}_1 \rightarrow \mathfrak{g}_0$ and $\rho : \mathfrak{g}_0 \rightarrow \text{Der}(\mathfrak{g}_1)$ are homomorphisms of Lie algebras such that for all $x \in \mathfrak{g}_0$ and $m, n \in \mathfrak{g}_1$, we have

$$d(\rho(x)m) = [x, d(m)]_{\mathfrak{g}_0}, \quad \rho(d(m))(n) = [m, n]_{\mathfrak{g}_1}.$$

Then d is an embedding tensor on the LieRep pair $((\mathfrak{g}_0, [\cdot, \cdot]_{\mathfrak{g}_0}), (\mathfrak{g}_1; \rho))$.

Examples 2.13 and 2.14 can be generalized to the following more general case.

Example 2.15. (Lie objects in the infinitesimal tensor category of linear maps) Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ be a Lie algebra and $(V; \rho)$ a representation. If a linear map $T : V \rightarrow \mathfrak{g}$ is $\mathfrak{g}$ -equivariant, that is,

$$T(\rho(x)v) = [x, Tv]_{\mathfrak{g}}, \quad \forall x \in \mathfrak{g}, v \in V,$$

then T is an embedding tensor on the **LieRep** pair $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}), (V; \rho))$. In fact $V \xrightarrow{T} \mathfrak{g}$ is a Lie object in the infinitesimal tensor category of linear maps if and only if $T : V \rightarrow \mathfrak{g}$ is $\mathfrak{g}$ -equivariant. Let $(\mathfrak{G}, [\cdot, \cdot]_{\mathfrak{G}})$ be a Leibniz algebra and $\mathfrak{G}^{\text{ann}}$ be the two-sided ideal of $\mathfrak{G}$ generated by $[x, x]_{\mathfrak{G}}$ for all $x \in \mathfrak{G}$. Then the natural projection π from $\mathfrak{G}$ to $\mathfrak{G}/\mathfrak{G}^{\text{ann}}$ gives a Lie object in the infinitesimal tensor category of linear maps and π is an embedding tensor. See [24, 33] for more details.

Example 2.16. The **Heisenberg algebra** $H_3(\mathbb{C})$ is a three-dimensional complex Lie algebra generated by elements e_1, e_2 and e_3 with Lie brackets $[e_1, e_2] = e_3, [e_1, e_3] = 0, [e_2, e_3] = 0$. $T = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$ is an embedding tensor on $H_3(\mathbb{C})$ with respect to the adjoint representation if and only if $[Te_i, Te_j] = T[Te_i, e_j], i, j \in \{1, 2, 3\}$. First by

$$\begin{aligned} 0 &= [Te_1, Te_1] = T[Te_1, e_1] = T[r_{11}e_1 + r_{21}e_2 + r_{31}e_3, e_1] \\ &= -r_{21}r_{13}e_1 - r_{21}r_{23}e_2 - r_{21}r_{33}e_3, \end{aligned}$$

we obtain $r_{21}r_{13} = 0, r_{21}r_{23} = 0, r_{21}r_{33} = 0$. Similarly, we can obtain

$$\begin{aligned} r_{12}r_{13} &= 0, & r_{12}r_{23} &= 0, & r_{12}r_{33} &= 0, \\ r_{11}r_{13} &= r_{22}r_{13} = 0, & r_{11}r_{23} &= r_{22}r_{23} = 0, & r_{11}r_{22} - r_{12}r_{21} &= r_{11}r_{33} = r_{22}r_{33}, \\ r_{23}r_{13} &= 0, & r_{23}^2 &= 0, & r_{23}r_{33} &= r_{11}r_{23} - r_{13}r_{21} = 0, \\ r_{13}^2 &= 0, & r_{13}r_{23} &= 0, & -r_{13}r_{33} &= r_{12}r_{23} - r_{13}r_{22} = 0. \end{aligned}$$

Therefore, we have

(i) If $r_{13} = r_{23} = r_{33} = 0$, then $T = \begin{pmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}$ is an embedding tensor on $H_3(\mathbb{C})$ if

and only if $r_{11}r_{22} = r_{12}r_{21}$.

(ii) If $r_{13} = r_{23} = 0$ and $r_{33} \neq 0$, then $r_{12} = r_{21} = 0$ and $T = \begin{pmatrix} r_{11} & 0 & 0 \\ 0 & r_{22} & 0 \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$ is an embedding tensor on $H_3(\mathbb{C})$ if and only if $r_{11}r_{22} = r_{22}r_{33} = r_{11}r_{33}$.

These two cases exhaust all the possibilities of embedding tensors on $H_3(\mathbb{C})$ with respect to the adjoint representation.

3. The Controlling Graded Lie Algebra and Cohomology of Embedding Tensors

In this section first we recall the controlling g.l.a. that characterize Leibniz algebras as Maurer–Cartan elements and the g.l.a. governing a **LieRep** pair that was originally given in [3]. Then we construct the controlling g.l.a. of embedding tensors. Finally we introduce the cohomologies of embedding tensors.

3.1. The controlling graded Lie algebra of LieRep pairs. A permutation $\sigma \in \mathbb{S}_n$ is called an $(i, n-i)$ -**shuffle** if $\sigma(1) < \dots < \sigma(i)$ and $\sigma(i+1) < \dots < \sigma(n)$. If $i = 0$ or n , we assume $\sigma = \text{Id}$. The set of all $(i, n-i)$ -shuffles will be denoted by $\mathbb{S}_{(i, n-i)}$. The notion of an $(i_1, \dots, i_k)$ -shuffle and the set $\mathbb{S}_{(i_1, \dots, i_k)}$ are defined analogously.

Let $\mathfrak{g}$ be a vector space. We consider the graded vector space

$$C^\bullet(\mathfrak{g}, \mathfrak{g}) = \bigoplus_{n=0}^{+\infty} C^n(\mathfrak{g}, \mathfrak{g}) = \bigoplus_{n=0}^{+\infty} \text{Hom}(\otimes^{n+1} \mathfrak{g}, \mathfrak{g}).$$

It is known that $C^\bullet(\mathfrak{g}, \mathfrak{g})$ equipped with the **Balavoine bracket** [4]:

$$[P, Q]_{\mathbf{B}} = P \bar{\circ} Q - (-1)^{pq} Q \bar{\circ} P, \quad \forall P \in C^p(\mathfrak{g}, \mathfrak{g}), Q \in C^q(\mathfrak{g}, \mathfrak{g}), \quad (8)$$

is a g.l.a. where $P \bar{\circ} Q \in C^{p+q}(\mathfrak{g}, \mathfrak{g})$ is defined by $P \bar{\circ} Q = \sum_{k=1}^{p+1} P \circ_k Q$, and $\circ_k$ is defined by

$$(P \circ_k Q)(x_1, \dots, x_{p+q+1}) = \sum_{\sigma \in \mathbb{S}_{(k-1, q)}} (-1)^{(k-1)q} (-1)^\sigma P(x_{\sigma(1)}, \dots, x_{\sigma(k-1)}, Q(x_{\sigma(k)}, \dots, x_{\sigma(k+q-1)}, x_{k+q}, \dots, x_{p+q+1})).$$

Remark 3.1. In fact, the Balavoine bracket is the commutator of coderivations on the cofree conilpotent coZinbiel coalgebra $\bar{\mathbf{T}}(\mathfrak{g})$. See [2, 53] for more details. Note that on the same graded vector space $C^\bullet(\mathfrak{g}, \mathfrak{g})$ there is the Gerstenhaber bracket [49] which is the commutator of coderivations on the cofree conilpotent coassociative coalgebra $\bar{\mathbf{T}}(\mathfrak{g})$.

The following conclusion is straightforward.

Lemma 3.2. For $\Omega \in C^1(\mathfrak{g}, \mathfrak{g}) = \text{Hom}(\otimes^2 \mathfrak{g}, \mathfrak{g})$, we have

$$\begin{aligned} \frac{1}{2} [\Omega, \Omega]_{\mathbf{B}}(x_1, x_2, x_3) &= \Omega \bar{\circ} \Omega(x_1, x_2, x_3) \\ &= \Omega(\Omega(x_1, x_2), x_3) - \Omega(x_1, \Omega(x_2, x_3)) + \Omega(x_2, \Omega(x_1, x_3)). \end{aligned}$$

Thus, Ω defines a Leibniz algebra structure if and only if $[\Omega, \Omega]_{\mathbf{B}} = 0$, i.e. Ω is a Maurer–Cartan element of the g.l.a. $(C^\bullet(\mathfrak{g}, \mathfrak{g}), [\cdot, \cdot]_{\mathbf{B}})$.

Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be vector spaces and elements in $\mathfrak{g}_1$ will be denoted by x, y, z, x_i and elements in $\mathfrak{g}_2$ will be denoted by u, v, v_i . Let $c : \mathfrak{g}_2^{\otimes n} \rightarrow \mathfrak{g}_1$ be a linear map. Define a linear map $\hat{c} \in C^{n-1}(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2)$ by

$$\hat{c}((x_1, v_1) \otimes \dots \otimes (x_n, v_n)) := (c(v_1, \dots, v_n), 0).$$

In general, for a given linear map $f : \mathfrak{g}_{i(1)} \otimes \mathfrak{g}_{i(2)} \otimes \dots \otimes \mathfrak{g}_{i(n)} \rightarrow \mathfrak{g}_j, i(1), \dots, i(n), j \in \{1, 2\}$, we define a linear map $\hat{f} \in C^{n-1}(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2)$ by

$$\hat{f} := \begin{cases} f & \text{on } \mathfrak{g}_{i(1)} \otimes \mathfrak{g}_{i(2)} \otimes \dots \otimes \mathfrak{g}_{i(n)}, \\ 0 & \text{all other cases.} \end{cases}$$

We call the linear map $\hat{f}$ a **horizontal lift** of f .

Let us write $\hat{C}^n(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V) := \text{Hom}(\wedge^{n+1} \mathfrak{g}, \mathfrak{g}) \oplus \text{Hom}(\wedge^n \mathfrak{g} \otimes V, V)$. Using the horizontal lift, we can regard $\hat{C}^n(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V)$ as a subspace of $C^n(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V)$.

Theorem 3.3. *The above defined $\oplus_{n=0}^{+\infty} \hat{C}^n(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V)$ is a subalgebra of the g.l.a. $(C^\bullet(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V), [\cdot, \cdot]_{\mathfrak{B}})$. Moreover, a Maurer–Cartan element of the g.l.a. $(\oplus_{n=0}^{+\infty} \hat{C}^n(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V), [\cdot, \cdot]_{\mathfrak{B}})$ is exactly a LieRep pair.*

Proof. It is straightforward to deduce that $\oplus_{n=0}^{+\infty} \hat{C}^n(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V)$ is a subalgebra of the graded Lie algebra $(\oplus_{n=0}^{+\infty} C^n(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V), [\cdot, \cdot]_{\mathfrak{B}})$. Moreover, let $(\mu, \rho) \in \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g}) \oplus \text{Hom}(\mathfrak{g} \otimes V, V)$ be a Maurer–Cartan element. Then $[\hat{\mu} + \hat{\rho}, \hat{\mu} + \hat{\rho}]_{\mathfrak{B}} = 0$ implies that μ defines a Lie algebra structure on $\mathfrak{g}$ and ρ is a representation of the Lie algebra $(\mathfrak{g}, \mu)$ on V . $\square$

3.2. The controlling graded Lie algebra of embedding tensors. Let $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}), (V; \rho))$ be a LieRep pair. Usually we will also use μ to indicate the Lie bracket $[\cdot, \cdot]_{\mathfrak{g}}$. We have a Leibniz algebra structure $\mu \boxplus \rho$ on $\mathfrak{g} \oplus V$, which is given by

$$(\mu \boxplus \rho)((x, u), (y, v)) = ([x, y]_{\mathfrak{g}}, \rho(x)v). \quad (9)$$

This Leibniz algebra is called the **hemisemidirect product** of the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ and the representation $(V; \rho)$. It first appeared in [20].

Theorem 3.4. *Let $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}), (V; \rho))$ be a LieRep pair. Then $(\oplus_{k=1}^{+\infty} \text{Hom}(\otimes^k V, \mathfrak{g}), \llbracket \cdot, \cdot \rrbracket)$ is a graded Lie algebra, where the graded Lie bracket $\llbracket \cdot, \cdot \rrbracket$ is given in a derived fashion*

$$\llbracket \theta, \phi \rrbracket = (-1)^{m-1} [[\mu \boxplus \rho, \theta]_{\mathfrak{B}}, \phi]_{\mathfrak{B}}, \quad \forall \theta \in \text{Hom}(\otimes^m V, \mathfrak{g}), \phi \in \text{Hom}(\otimes^n V, \mathfrak{g}). \quad (10)$$

More precisely, it is given by

$$\begin{aligned} & \llbracket \theta, \phi \rrbracket(v_1, \dots, v_{m+n}) \\ &= \sum_{k=1}^m \sum_{\sigma \in \mathbb{S}_{(k-1, n)}} (-1)^{(k-1)n+1} \\ & \quad (-1)^{\sigma} \theta(v_{\sigma(1)}, \dots, v_{\sigma(k-1)}, \rho(\phi(v_{\sigma(k)}, \dots, v_{\sigma(k+n-1)}))v_{k+n}, v_{k+n+1}, \dots, v_{m+n}) \\ & \quad + \sum_{\sigma \in \mathbb{S}_{(m, n)}} (-1)^{mn+1} \\ & \quad (-1)^{\sigma} [\theta(v_{\sigma(1)}, \dots, v_{\sigma(m)}, \phi(v_{\sigma(m+1)}, \dots, v_{\sigma(m+n-1)}, v_{\sigma(m+n)}))]_{\mathfrak{g}} \\ & \quad + \sum_{k=1}^n \sum_{\sigma \in \mathbb{S}_{(k-1, m)}} (-1)^{m(k+n-1)} \\ & \quad (-1)^{\sigma} \phi(v_{\sigma(1)}, \dots, v_{\sigma(k-1)}, \rho(\theta(v_{\sigma(k)}, \dots, v_{\sigma(k+m-1)}))v_{k+m}, v_{k+m+1}, \dots, v_{m+n}). \end{aligned}$$

Moreover, its Maurer–Cartan elements are precisely embedding tensors.

Proof. In short, the graded Lie algebra $(\oplus_{k=1}^{+\infty} \text{Hom}(\otimes^k V, \mathfrak{g}), \llbracket \cdot, \cdot \rrbracket)$ is obtained via the derived bracket [22, 23, 55]. In fact, the Balavoine bracket $[\cdot, \cdot]_{\mathfrak{B}}$ associated to the direct sum vector space $\mathfrak{g} \oplus V$ gives rise to a graded Lie algebra $(\oplus_{k=1}^{+\infty} \text{Hom}(\otimes^k(\mathfrak{g} \oplus V), \mathfrak{g} \oplus V), [\cdot, \cdot]_{\mathfrak{B}})$. Since $\mu : \wedge^2 \mathfrak{g} \rightarrow \mathfrak{g}$ is a Lie algebra structure and $\rho : \mathfrak{g} \otimes V \rightarrow V$ is a representation of $\mathfrak{g}$ on V , therefore the hemisemidirect product Leibniz algebra structure $\mu \boxplus \rho$ is a Maurer–Cartan element of the graded Lie algebra $(\oplus_{k=1}^{+\infty} \text{Hom}(\otimes^k(\mathfrak{g} \oplus V), \mathfrak{g} \oplus V), [\cdot, \cdot]_{\mathfrak{B}})$.

V), $[\cdot, \cdot]_{\mathbb{B}}$), defining a differential $d_{\mu \boxplus \rho}$ on $(\oplus_{k=1}^{+\infty} \text{Hom}(\otimes^k (\mathfrak{g} \oplus V), \mathfrak{g} \oplus V), [\cdot, \cdot]_{\mathbb{B}})$ via $d_{\mu \boxplus \rho} = [\mu \boxplus \rho, \cdot]_{\mathbb{B}}$. Since the subspace $\oplus_{k=1}^{+\infty} \text{Hom}(\otimes^k V, \mathfrak{g})$ is abelian under $[\cdot, \cdot]_{\mathbb{B}}$ by degree reasons, the differential $d_{\mu \boxplus \rho}$ gives rise to a graded Lie algebra structure on the graded vector space $\oplus_{k=1}^{+\infty} \text{Hom}(\otimes^k V, \mathfrak{g})$ via the derived bracket (10).

For $T \in \text{Hom}(V, \mathfrak{g})$, we have

$$\frac{1}{2} \llbracket T, T \rrbracket(v_1, v_2) = [Tv_1, Tv_2]_{\mathfrak{g}} - T(\rho(Tv_1)v_2), \quad \forall v_1, v_2 \in V,$$

which implies that Maurer–Cartan elements are precisely embedding tensors. $\square$

There is a close relationship between the graded Lie algebra $(\oplus_{k=1}^{+\infty} \text{Hom}(\otimes^k V, \mathfrak{g}), \llbracket \cdot, \cdot \rrbracket)$ and the graded Lie algebra $(C^\bullet(V, V), [\cdot, \cdot]_{\mathbb{B}})$, where $[\cdot, \cdot]_{\mathbb{B}}$ is the Balavoine bracket defined by (8).

Define a linear map $\Phi : \text{Hom}(\otimes^k V, \mathfrak{g}) \rightarrow \text{Hom}(\otimes^{k+1} V, V)$ for $k = 1, 2, \dots$, by

$$\begin{aligned} \Phi(f)(u_1, \dots, u_k, u_{k+1}) &= -\rho(f(u_1, \dots, u_k))u_{k+1}, \\ \forall f \in \text{Hom}(\otimes^k V, \mathfrak{g}), \quad u_1, \dots, u_{k+1} &\in V. \end{aligned} \quad (11)$$

Proposition 3.5. *The linear map Φ , defined by (11), is a homomorphism of graded Lie algebras from $(\oplus_{k=1}^{+\infty} \text{Hom}(\otimes^k V, \mathfrak{g}), \llbracket \cdot, \cdot \rrbracket)$ to $(C^\bullet(V, V), [\cdot, \cdot]_{\mathbb{B}})$.*

Proof. For $\theta \in \text{Hom}(\otimes^m V, \mathfrak{g})$ and $\phi \in \text{Hom}(\otimes^n V, \mathfrak{g})$, we have $[\Phi(\theta), \Phi(\phi)]_{\mathbb{B}} \in \text{Hom}(\otimes^{m+n+1} V, V)$. More precisely, for all $u_1, \dots, u_{m+n+1} \in V$, we have

$$\begin{aligned} &\Phi(\theta) \circ \Phi(\phi)(u_1, \dots, u_{m+n+1}) \\ &= \sum_{k=1}^{m+1} \sum_{\sigma \in \mathbb{S}_{(k-1, n)}} (-1)^{(k-1)n} \\ &\quad (-1)^\sigma \Phi(\theta)(u_{\sigma(1)}, \dots, u_{\sigma(k-1)}, \Phi(\phi)(u_{\sigma(k)}, \dots, u_{\sigma(k+n-1)}, u_{k+n}, u_{k+n+1}, \dots, u_{m+n+1})) \\ &= \sum_{k=1}^m \sum_{\sigma \in \mathbb{S}_{(k-1, n)}} (-1)^{(k-1)n} \\ &\quad (-1)^\sigma \rho(\theta(u_{\sigma(1)}, \dots, u_{\sigma(k-1)}, \rho(\phi(u_{\sigma(k)}, \dots, u_{\sigma(k+n-1)}))u_{k+n}, u_{k+n+1}, \dots, u_{m+n}))u_{m+n+1} \\ &\quad + \sum_{\sigma \in \mathbb{S}_{(m, n)}} (-1)^{mn} \\ &\quad (-1)^\sigma \rho(\theta(u_{\sigma(1)}, \dots, u_{\sigma(m)}))\rho(\phi(u_{\sigma(m+1)}, \dots, u_{\sigma(m+n)}))u_{m+n+1}. \end{aligned}$$

For any $\tau \in \mathbb{S}_{(n, m)}$, we can define $\sigma \in \mathbb{S}_{(m, n)}$ by

$$\sigma(i) = \begin{cases} \tau(n+i), & 1 \leq i \leq m; \\ \tau(i-m), & m+1 \leq i \leq m+n. \end{cases}$$

Thus, we have $(-1)^\sigma = (-1)^{mn}(-1)^\tau$. In fact, the elements of $\mathbb{S}_{(n, m)}$ are in one-to-one correspondence with the elements of $\mathbb{S}_{(m, n)}$. Then we have

$$\begin{aligned} &\Phi(\phi) \circ \Phi(\theta)(u_1, \dots, u_{m+n+1}) \\ &= \sum_{k=1}^n \sum_{\tau \in \mathbb{S}_{(k-1, m)}} (-1)^{(k-1)m} \\ &\quad (-1)^\tau \rho(\phi(u_{\tau(1)}, \dots, u_{\tau(k-1)}, \rho(\theta(u_{\tau(k)}, \dots, u_{\tau(k+m-1)}))u_{k+m}, u_{k+m+1}, \dots, u_{m+n}))u_{m+n+1} \end{aligned}$$

$$\begin{aligned}
& + \sum_{\tau \in \mathbb{S}_{(n,m)}} (-1)^{mn} \\
& (-1)^\tau \rho(\phi(u_\tau(1), \dots, u_\tau(n))) \rho(\theta(u_\tau(n+1), \dots, u_\tau(m+n))) u_{m+n+1} \\
& = \sum_{k=1}^n \sum_{\sigma \in \mathbb{S}_{(k-1,m)}} (-1)^{(k-1)m} \\
& (-1)^\sigma \rho(\phi(u_\sigma(1), \dots, u_\sigma(k-1), \rho(\theta(u_\sigma(k), \dots, u_\sigma(k+m-1))) u_{k+m}, u_{k+m+1}, \dots, u_{m+n})) u_{m+n+1} \\
& + \sum_{\sigma \in \mathbb{S}_{(m,n)}} \\
& (-1)^\sigma \rho(\phi(u_\sigma(m+1), \dots, u_\sigma(m+n))) \rho(\theta(u_\sigma(1), \dots, u_\sigma(m))) u_{m+n+1}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
& [\Phi(\theta), \Phi(\phi)]_{\mathbf{B}}(u_1, \dots, u_{m+n+1}) \\
& = \sum_{k=1}^m \sum_{\sigma \in \mathbb{S}_{(k-1,n)}} (-1)^{(k-1)n} (-1)^\sigma \\
& \rho(\theta(u_\sigma(1), \dots, u_\sigma(k-1), \\
& \rho(\phi(u_\sigma(k), \dots, u_\sigma(k+n-1))) u_{k+n}, u_{k+n+1}, \dots, u_{m+n})) u_{m+n+1} \\
& - (-1)^{mn} \sum_{k=1}^n \sum_{\sigma \in \mathbb{S}_{(k-1,m)}} (-1)^{(k-1)m} (-1)^\sigma \\
& \rho(\phi(u_\sigma(1), \dots, u_\sigma(k-1), \rho(\theta(u_\sigma(k), \dots, \\
& u_\sigma(k+m-1))) u_{k+m}, u_{k+m+1}, \dots, u_{m+n})) u_{m+n+1} \\
& + (-1)^{mn} \sum_{\sigma \in \mathbb{S}_{(m,n)}} (-1)^\sigma \rho([\theta(u_\sigma(1), \dots, u_\sigma(m)), \\
& \phi(u_\sigma(m+1), \dots, u_\sigma(m+n))]_{\mathbf{g}}) u_{m+n+1} \\
& = \Phi([\theta, \phi])(u_1, \dots, u_{m+n+1}).
\end{aligned}$$

Thus, Φ is a homomorphism from $(\oplus_{k=1}^{+\infty} \text{Hom}(\otimes^k V, \mathfrak{g}), [\cdot, \cdot])$ to $(C^\bullet(V, V), [\cdot, \cdot]_{\mathbf{B}})$. $\square$

Remark 3.6. The above result gives another intrinsic explanation of Proposition 2.4. By Lemma 3.2, Maurer–Cartan elements of the g.l.a $(C^\bullet(V, V), [\cdot, \cdot]_{\mathbf{B}})$ are Leibniz algebra structures on V . By Theorem 3.4, Maurer–Cartan elements of the g.l.a $(\oplus_{k=1}^{+\infty} \text{Hom}(\otimes^k V, \mathfrak{g}), [\cdot, \cdot])$ are embedding tensors. By Proposition 3.5, Φ sends Maurer–Cartan elements to Maurer–Cartan elements. Thus, an embedding tensor $T : V \rightarrow \mathfrak{g}$ induces a Leibniz algebra structure on V .

3.3. Cohomology of embedding tensors. First we review representations and cohomologies of Leibniz algebras. A **representation** of a Leibniz algebra $(\mathfrak{G}, [\cdot, \cdot]_{\mathfrak{G}})$ is a triple $(V; \rho^L, \rho^R)$, where V is a vector space, $\rho^L, \rho^R : \mathfrak{G} \rightarrow \mathfrak{gl}(V)$ are linear maps such that for all $x, y \in \mathfrak{G}$,

$$\begin{aligned}
\rho^L([x, y]_{\mathfrak{G}}) &= [\rho^L(x), \rho^L(y)], \\
\rho^R([x, y]_{\mathfrak{G}}) &= [\rho^L(x), \rho^R(y)], \\
\rho^R(y) \circ \rho^L(x) &= -\rho^R(y) \circ \rho^R(x).
\end{aligned}$$

Here $[\cdot, \cdot] : \wedge^2 \mathfrak{gl}(V) \rightarrow \mathfrak{gl}(V)$ is the commutator Lie bracket on $\mathfrak{gl}(V)$. It is straight-

forward to see that $(\mathfrak{G}; L, R)$, where the left multiplication $L : \mathfrak{G} \longrightarrow \mathfrak{gl}(\mathfrak{G})$ is given by (5) and the right multiplication $R : \mathfrak{G} \longrightarrow \mathfrak{gl}(\mathfrak{G})$ is defined by $R_x y = [y, x]_{\mathfrak{G}}$, is a representation of $(\mathfrak{G}, [\cdot, \cdot]_{\mathfrak{G}})$, which is called the **regular representation**.

Definition 3.7 ([34]). Let $(V; \rho^L, \rho^R)$ be a representation of a Leibniz algebra $(\mathfrak{G}, [\cdot, \cdot]_{\mathfrak{G}})$. The **Loday–Pirashvili cohomology** of $\mathfrak{G}$ with coefficients in V is the cohomology of the cochain complex $(C^\bullet(\mathfrak{G}, V) = \bigoplus_{k=0}^{+\infty} C^k(\mathfrak{G}, V), \partial)$, where $C^k(\mathfrak{G}, V) = \text{Hom}(\otimes^k \mathfrak{G}, V)$ and the coboundary operator $\partial : C^k(\mathfrak{G}, V) \longrightarrow C^{k+1}(\mathfrak{G}, V)$ is defined by

$$\begin{aligned} (\partial f)(x_1, \dots, x_{k+1}) &= \sum_{i=1}^k (-1)^{i+1} \rho^L(x_i) f(x_1, \dots, \hat{x}_i, \dots, x_{k+1}) \\ &\quad + (-1)^{k+1} \rho^R(x_{k+1}) f(x_1, \dots, x_k) \\ &\quad + \sum_{1 \leq i < j \leq k+1} (-1)^i f(x_1, \dots, \hat{x}_i, \dots, x_{j-1}, \\ &\quad \quad [x_i, x_j]_{\mathfrak{G}}, x_{j+1}, \dots, x_{k+1}), \end{aligned}$$

for all $x_1, \dots, x_{k+1} \in \mathfrak{G}$. The resulting cohomology is denoted by $H^*(\mathfrak{G}, V)$.

The regular representation $(\mathfrak{G}; L, R)$ is very important. We denote the corresponding cochain complex by $(C^\bullet \mathfrak{G}, \mathfrak{G}), \partial^{reg})$ and the resulting cohomology by $H_{\text{reg}}^*(\mathfrak{G})$.

Let $T : V \longrightarrow \mathfrak{g}$ be an embedding tensor on a LieRep pair $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}), (V; \rho))$. By Proposition 2.4, $[u, v]_T := \rho(Tu)v$ defines a Leibniz algebra structure on V . Furthermore, define $\rho^L : V \rightarrow \mathfrak{gl}(\mathfrak{g})$ and $\rho^R : V \rightarrow \mathfrak{gl}(\mathfrak{g})$ by

$$\rho^L(u)y := [Tu, y]_{\mathfrak{g}}, \quad \rho^R(v)x := [x, Tv]_{\mathfrak{g}} - T(\rho(x)v). \quad (12)$$

Lemma 3.8. *With above notations, $(\mathfrak{g}; \rho^L, \rho^R)$ is a representation of the Leibniz algebra $(V, [\cdot, \cdot]_T)$.*

Proof. It follows from direct verification. $\square$

Let $\partial_T : \text{Hom}(\otimes^k V, \mathfrak{g}) \longrightarrow \text{Hom}(\otimes^{k+1} V, \mathfrak{g})$ be the corresponding Loday–Pirashvili coboundary operator of the Leibniz algebra $(V, [\cdot, \cdot]_T)$ with coefficients in $(\mathfrak{g}; \rho^L, \rho^R)$. More precisely, $\partial_T : \text{Hom}(\otimes^k V, \mathfrak{g}) \longrightarrow \text{Hom}(\otimes^{k+1} V, \mathfrak{g})$ is given by

$$\begin{aligned} \partial_T \theta(u_1, \dots, u_{k+1}) &= \sum_{i=1}^k (-1)^{i+1} [Tu_i, \theta(u_1, \dots, \hat{u}_i, \dots, u_{k+1})]_{\mathfrak{g}} \\ &\quad + (-1)^{k+1} [\theta(u_1, \dots, u_k), Tu_{k+1}]_{\mathfrak{g}} \\ &\quad + (-1)^k T(\rho(\theta(u_1, \dots, u_k))u_{k+1}) \\ &\quad + \sum_{1 \leq i < j \leq k+1} (-1)^i \theta(u_1, \dots, \hat{u}_i, \dots, \\ &\quad \quad u_{j-1}, \rho(Tu_i)(u_j), u_{j+1}, \dots, u_{k+1}). \end{aligned} \quad (13)$$

The coboundary operator ∂_T can be alternatively described by the following formula.

Proposition 3.9. *Let $T : V \rightarrow \mathfrak{g}$ be an embedding tensor. Then we have*

$$\partial_T \theta = (-1)^{k-1} \llbracket T, \theta \rrbracket, \quad \forall f \in \text{Hom}(\otimes^k V, \mathfrak{g}), \quad k = 1, 2, \dots,$$

where the bracket $\llbracket \cdot, \cdot \rrbracket$ is given by (10).

Proof. It follows from straightforward verification. $\square$

Now we define a cohomology theory governing deformations of an embedding tensor $T : V \rightarrow \mathfrak{g}$. Define the space of 0-cochains $\mathfrak{C}^0(T)$ to be 0 and of 1-cochains $\mathfrak{C}^1(T)$ to be $\mathfrak{g}$. For $n \geq 2$, define the space of n -cochains $\mathfrak{C}^n(T)$ as $\mathfrak{C}^n(T) = \text{Hom}(\otimes^{n-1} V, \mathfrak{g})$.

Definition 3.10. Let T be an embedding tensor on a LieRep pair $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}), (V; \rho))$. We define the **cohomology of the embedding tensor** T to be the cohomology of the cochain complex $(\mathfrak{C}^\bullet(T) = \bigoplus_{k=0}^{+\infty} \mathfrak{C}^k(T), \partial_T)$. The corresponding k -th cohomology group is denoted by $\mathcal{H}^k(T)$.

At the end of this section, we study the relation between the cohomology of an embedding tensor $T : V \rightarrow \mathfrak{g}$ and the cohomology of the underlying Leibniz algebra $(V, [\cdot, \cdot]_T)$ given in Proposition 2.4.

Theorem 3.11. Let $T : V \rightarrow \mathfrak{g}$ be an embedding tensor on a LieRep pair $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}), (V; \rho))$. Then Φ , defined by (11), is a homomorphism from the cochain complex $(\mathfrak{C}^\bullet(T), \partial_T)$ of the embedding tensor T to the cochain complex $(\mathfrak{C}^\bullet(V, V), \partial^{reg})$ of the underlying Leibniz algebra $(V, [\cdot, \cdot]_T)$, that is, we have the following commutative diagram:

$$\begin{array}{ccc} \text{Hom}(\otimes^k V, \mathfrak{g}) & \xrightarrow{\Phi} & \text{Hom}(\otimes^{k+1} V, V) \\ \partial_T \downarrow & & \downarrow \partial^{reg} \\ \text{Hom}(\otimes^{k+1} V, \mathfrak{g}) & \xrightarrow{\Phi} & \text{Hom}(\otimes^{k+2} V, V). \end{array} \quad (14)$$

Consequently, Φ induces a homomorphism $\Phi_* : \mathcal{H}^k(T) \rightarrow H_{\text{reg}}^k(V, V)$ between the corresponding cohomology groups.

Proof. By Propositions 3.5 and 3.9, for all $\theta \in \text{Hom}(\otimes^k V, \mathfrak{g})$, we have

$$\Phi(\partial_T \theta) = (-1)^{k-1} \Phi \llbracket T, \theta \rrbracket = (-1)^{k-1} \llbracket \Phi(T), \Phi(\theta) \rrbracket = \partial^{reg}(\Phi(\theta)).$$

Thus, Φ is a homomorphism of cochain complexes from $(\mathfrak{C}^\bullet(T), \partial_T)$ to $(\mathfrak{C}^\bullet(V, V), \partial^{reg})$ and Φ_* is a homomorphism between the corresponding cohomology groups. $\square$

4. The Controlling L_∞ -Algebra of Lie–Leibniz Triples

In this section, we apply T. Voronov's higher derived bracket to construct an L_∞ -algebra that characterizes Lie–Leibniz triples as Maurer–Cartan elements.

4.1. L_∞ -algebras and higher derived brackets. Let $V^\bullet = \bigoplus_{k \in \mathbb{Z}} V^k$ be a $\mathbb{Z}$ -graded vector space. We will denote by $\mathbf{S}(V^\bullet)$ the symmetric algebra of $V^\bullet$. That is, $\mathbf{S}(V^\bullet) := \mathbf{T}(V^\bullet)/I$, where $\mathbf{T}(V^\bullet)$ is the tensor algebra and I is the 2-sided ideal of $\mathbf{T}(V^\bullet)$ generated by all homogeneous elements of the form $x \otimes y - (-1)^{xy} y \otimes x$. We will write $\mathbf{S}(V^\bullet) = \bigoplus_{i=0}^{+\infty} \mathbf{S}^i(V)$. Moreover, we denote the reduced tensor algebra and reduced symmetric algebra by $\bar{\mathbf{T}}V^\bullet := \bigoplus_{i=1}^{+\infty} \mathbf{T}^i(V^\bullet)$ and $\bar{\mathbf{S}}(V^\bullet) := \bigoplus_{i=1}^{+\infty} \mathbf{S}^i(V^\bullet)$ respectively. Denote the product of homogeneous elements $v_1, \dots, v_n \in V^\bullet$ in $\mathbf{S}^n(V^\bullet)$ by $v_1 \odot \dots \odot v_n$. The degree of $v_1 \odot \dots \odot v_n$ is by definition the sum of the degrees of v_i . For a permutation

$\sigma \in \mathbb{S}_n$ and $v_1, \dots, v_n \in V^\bullet$, the Koszul sign $\varepsilon(\sigma) = \varepsilon(\sigma; v_1, \dots, v_n) \in \{-1, 1\}$ is defined by

$$v_1 \odot \cdots \odot v_n = \varepsilon(\sigma; v_1, \dots, v_n) v_{\sigma(1)} \odot \cdots \odot v_{\sigma(n)}.$$

The desuspension operator s^{-1} changes the grading of $V^\bullet$ according to the rule $(s^{-1}V^\bullet)^i := V^{i+1}$. The degree -1 map $s^{-1} : V^\bullet \rightarrow s^{-1}V^\bullet$ is defined by sending $v \in V^\bullet$ to its copy $s^{-1}v \in s^{-1}V^\bullet$.

The notion of an L_∞ -algebra was introduced by Stasheff in [48]. See [25, 26] for more details.

Definition 4.1. An L_∞ -algebra is a $\mathbb{Z}$ -graded vector space $\mathfrak{g}^\bullet = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}^k$ equipped with a collection ($k \geq 1$) of linear maps $l_k : \bigotimes^k \mathfrak{g}^\bullet \rightarrow \mathfrak{g}^\bullet$ of degree 1 with the property that, for any homogeneous elements $x_1, \dots, x_n \in \mathfrak{g}^\bullet$, we have

- (i) (**graded symmetry**) for every $\sigma \in \mathbb{S}_n$,
 $l_n(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = \varepsilon(\sigma; x_1, \dots, x_n) l_n(x_1, \dots, x_n)$.
- (ii) (**generalized Jacobi identity**) for all $n \geq 1$,

$$\sum_{i=1}^n \sum_{\sigma \in \mathbb{S}_{(i, n-i)}} \varepsilon(\sigma; x_1, \dots, x_n) l_{n-i+1}(l_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(n)}) = 0.$$

Remark 4.2. An L_∞ -algebra structure on a graded vector space $\mathfrak{g}^\bullet$ is equivalent to a codifferential on the cofree conilpotent cocommutative coalgebra $\hat{\mathbb{S}}(\mathfrak{g}^\bullet)$.

Definition 4.3. The set of **Maurer–Cartan elements** of an L_∞ -algebra $(\mathfrak{g}^\bullet, \{l_i\}_{i=1}^{+\infty})$ is the set of those $\alpha \in \mathfrak{g}^0$ satisfying the Maurer–Cartan equation $\sum_{k=1}^{+\infty} \frac{1}{k!} l_k(\alpha, \dots, \alpha) = 0$.

Let α be a Maurer–Cartan element. Define $l_k^\alpha : \bigotimes^k \mathfrak{g}^\bullet \rightarrow \mathfrak{g}^\bullet$ ($k \geq 1$) by

$$l_k^\alpha(x_1, \dots, x_k) = \sum_{n=0}^{+\infty} \frac{1}{n!} l_{k+n}(\underbrace{\alpha, \dots, \alpha}_n, x_1, \dots, x_k). \quad (15)$$

Proposition 4.4. ([16]) *With the above notation, $(\mathfrak{g}^\bullet, \{l_k^\alpha\}_{k=1}^{+\infty})$ is an L_∞ -algebra. The L_∞ -algebra $(\mathfrak{g}^\bullet, \{l_k^\alpha\}_{k=1}^{+\infty})$ is called the **twisted L_∞ -algebra**.*

In the sequel, we recall T. Voronov’s derived brackets [55], which is a useful tool to construct explicit L_∞ -algebras.

Definition 4.5. V -data consist of a quadruple $(L, \mathfrak{h}, P, \Delta)$ where

- $(L, [\cdot, \cdot])$ is a graded Lie algebra,
- $\mathfrak{h}$ is an abelian graded Lie subalgebra of $(L, [\cdot, \cdot])$,
- $P : L \rightarrow L$ is a projection, that is $P \circ P = P$, whose image is $\mathfrak{h}$ and kernel is a graded Lie subalgebra of $(L, [\cdot, \cdot])$,
- Δ is an element of $\ker(P)^1$ such that $[\Delta, \Delta] = 0$.

Theorem 4.6 ([55]). *Let $(L, \mathfrak{h}, P, \Delta)$ be V -data. Then the graded vector space $s^{-1}L \oplus \mathfrak{h}$ is an L_∞ -algebra where*

$$\begin{aligned} l_1(s^{-1}x, a) &= (-s^{-1}[\Delta, x], P(x + [\Delta, a])), \\ l_2(s^{-1}x, s^{-1}y) &= (-1)^x s^{-1}[x, y], \end{aligned}$$

$$l_k(s^{-1}x, a_1, \dots, a_{k-1}) = P[\cdots [[x, a_1], a_2], \dots, a_{k-1}], \quad k \geq 2,$$

$$l_k(a_1, \dots, a_{k-1}, a_k) = P[\cdots [[\Delta, a_1], a_2], \dots, a_k], \quad k \geq 2.$$

Here $a, a_1, \dots, a_k$ are homogeneous elements of $\mathfrak{h}$ and x, y are homogeneous elements of L . All other L_∞ -algebra products that are not obtained from the ones written above by permutations of arguments, will vanish.

Let L' be a graded Lie subalgebra of L that satisfies $[\Delta, L'] \subset L'$. Then $s^{-1}L' \oplus \mathfrak{h}$ is an L_∞ -subalgebra of the above L_∞ -algebra. In particular, $(\mathfrak{h}, \{l_k\}_{k=1}^{+\infty})$ is an L_∞ -algebra, where

$$l_k(a_1, \dots, a_k) = P \underbrace{[\cdots [\Delta, a_1], a_2], \dots, a_k]}_k, \quad \text{for } a_1, \dots, a_k \in \mathfrak{h}. \quad (16)$$

L_∞ -algebras were constructed using the above method to study simultaneous deformations of morphisms between Lie algebras in [13, 14], and to study simultaneous deformations of relative Rota-Baxter Lie algebras in [30].

4.2. The controlling L_∞ -algebra of Lie–Leibniz triples. Let $\mathfrak{g}$ and V be two vector spaces. Then we have a graded Lie algebra $(\oplus_{n=0}^{+\infty} C^n(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V), [\cdot, \cdot]_{\mathbb{B}})$. This graded Lie algebra gives rise to V-data therefore also to an L_∞ -algebra.

Proposition 4.7. *We have V-data $(L, \mathfrak{h}, P, \Delta)$ as follows:*

- the graded Lie algebra $(L, [\cdot, \cdot])$ is given by $(\oplus_{n=0}^{+\infty} C^n(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V), [\cdot, \cdot]_{\mathbb{B}})$;
- the abelian graded Lie subalgebra $\mathfrak{h}$ is given by

$$\mathfrak{h} := \oplus_{n=0}^{+\infty} \text{Hom}(\otimes^{n+1} V, \mathfrak{g}); \quad (17)$$

- $P : L \rightarrow L$ is the projection onto the subspace $\mathfrak{h}$;
- $\Delta = 0$.

Consequently, we obtain an L_∞ -algebra $(s^{-1}L \oplus \mathfrak{h}, \{l_k\}_{k=1}^{+\infty})$, where l_i are given by

$$l_1(s^{-1}Q, \theta) = P(Q),$$

$$l_2(s^{-1}Q, s^{-1}Q') = (-1)^Q s^{-1}[Q, Q']_{\mathbb{B}},$$

$$l_k(s^{-1}Q, \theta_1, \dots, \theta_{k-1}) = P[\cdots [Q, \theta_1]_{\mathbb{B}}, \dots, \theta_{k-1}]_{\mathbb{B}},$$

for homogeneous elements $\theta, \theta_1, \dots, \theta_{k-1} \in \mathfrak{h}$, homogeneous elements $Q, Q' \in L$ and all the other possible combinations vanish.

Proof. Note that $\mathfrak{h} = \oplus_{n=0}^{+\infty} \text{Hom}(\otimes^{n+1} V, \mathfrak{g})$ is an abelian subalgebra of $(L, [\cdot, \cdot])$. Since P is the projection onto $\mathfrak{h}$, it is obvious that $P \circ P = P$. It is also straightforward to see that the kernel of P is a graded Lie subalgebra of $(L, [\cdot, \cdot])$. Thus $(L, \mathfrak{h}, P, \Delta = 0)$ are V-data.

The other conclusions follows immediately from Theorem 4.6. $\square$

By Theorem 3.3, we obtain that

$$L' = \oplus_{n=0}^{+\infty} \hat{C}^n(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V), \quad \text{where}$$

$$\hat{C}^n(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V) = \text{Hom}(\wedge^{n+1} \mathfrak{g}, \mathfrak{g}) \oplus \text{Hom}(\wedge^n \mathfrak{g} \otimes V, V) \quad (18)$$

is a graded Lie subalgebra of $(\oplus_{n=0}^{+\infty} C^n(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V), [\cdot, \cdot]_{\mathbb{B}})$. Then we have the following result.

Proposition 4.8. *With above notations, $(s^{-1}L' \oplus \mathfrak{h}, \{l_i\}_{i=1}^{+\infty})$ is an L_∞ -algebra, where l_i are given by*

$$\begin{aligned} l_2(s^{-1}Q, s^{-1}Q') &= (-1)^Q s^{-1}[Q, Q']_{\mathbb{B}}, \\ l_k(s^{-1}Q, \theta_1, \dots, \theta_{k-1}) &= P[\cdots [Q, \theta_1]_{\mathbb{B}}, \dots, \theta_{k-1}]_{\mathbb{B}}, \end{aligned}$$

for homogeneous elements $\theta_1, \dots, \theta_{k-1} \in \mathfrak{h}$, homogeneous elements $Q, Q' \in L'$, and all the other possible combinations vanish.

Proof. It follows from Theorem 4.6 and Proposition 4.7. $\square$

Now we are ready to give the controlling L_∞ -algebra of Lie–Leibniz triples, which is the main result in this subsection.

Theorem 4.9. *Let $\mathfrak{g}$ and V be two vector spaces, $\mu \in \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g})$, $\rho \in \text{Hom}(\mathfrak{g} \otimes V, V)$ and $T \in \text{Hom}(V, \mathfrak{g})$. Then $((\mathfrak{g}, \mu), (V; \rho), T)$ is a Lie–Leibniz triple if and only if $(s^{-1}(\mu \boxplus \rho), T)$ is a Maurer–Cartan element of the L_∞ -algebra $(s^{-1}L' \oplus \mathfrak{h}, \{l_i\}_{i=1}^{+\infty})$ given in Proposition 4.8.*

Proof. Let $(s^{-1}(\mu \boxplus \rho), T)$ be a Maurer–Cartan element of $(s^{-1}L' \oplus \mathfrak{h}, \{l_i\}_{i=1}^{+\infty})$. Then we have

$$\begin{aligned} & \sum_{k=1}^{+\infty} \frac{1}{k!} l_k \left((s^{-1}(\mu \boxplus \rho), T), \dots, (s^{-1}(\mu \boxplus \rho), T) \right) \\ &= \frac{1}{2!} l_2 \left((s^{-1}(\mu \boxplus \rho), T), (s^{-1}(\mu \boxplus \rho), T) \right) \\ & \quad + \frac{1}{3!} l_3 \left((s^{-1}(\mu \boxplus \rho), T), (s^{-1}(\mu \boxplus \rho), T), (s^{-1}(\mu \boxplus \rho), T) \right) \\ &= \left(-s^{-1} \frac{1}{2} [\mu \boxplus \rho, \mu \boxplus \rho]_{\mathbb{B}}, \frac{1}{2} [[\mu \boxplus \rho, T]_{\mathbb{B}}, T]_{\mathbb{B}} \right) \\ &= (0, 0). \end{aligned}$$

Thus, we obtain

$$[\mu \boxplus \rho, \mu \boxplus \rho]_{\mathbb{B}} = 0, \quad [[\mu \boxplus \rho, T]_{\mathbb{B}}, T]_{\mathbb{B}} = 0.$$

By Theorems 3.3 and 3.4, $(\mathfrak{g}, \mu)$ is a Lie algebra, $(V; \rho)$ is its representation and T is an embedding tensor on the LieRep pair $((\mathfrak{g}, \mu), (V; \rho))$. $\square$

5. Cohomology of Lie–Leibniz Triples and Applications

In this section, we introduce a cohomology theory of Lie–Leibniz triples and justify it by using it to classify infinitesimal deformations and central extensions.

5.1. Regular cohomology of Lie–Leibniz triples and infinitesimal deformations. In this subsection, first we recall the cohomology of a **LieRep** pair, and then we introduce a regular cohomology of a Lie–Leibniz triple. Finally, we use the second cohomology group to characterize infinitesimal deformations of a Lie–Leibniz triple.

Let $((\mathfrak{g}, \mu), (V; \rho))$ be a **LieRep** pair. Define the set of 0-cochains $\mathfrak{C}^0(\mathfrak{g}, \rho)$ to be 0. For $n \geq 1$, we define the set of n -cochains $\mathfrak{C}^n(\mathfrak{g}, \rho)$ by

$$\mathfrak{C}^n(\mathfrak{g}, \rho) = \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{g}) \oplus \text{Hom}(\wedge^{n-1} \mathfrak{g} \otimes V, V).$$

Define the coboundary operator $\delta: \mathfrak{C}^n(\mathfrak{g}, \rho) \rightarrow \mathfrak{C}^{n+1}(\mathfrak{g}, \rho)$ by

$$\delta f = (-1)^{n-1} [\mu \boxplus \rho, f]_{\mathfrak{B}}. \quad (19)$$

By Theorem 3.3, $\delta^2 = 0$. Thus we obtain a cochain complex $(\oplus_n \mathfrak{C}^n(\mathfrak{g}, \rho), \delta)$.

Definition 5.1. The cohomology of the cochain complex $(\oplus_n \mathfrak{C}^n(\mathfrak{g}, \rho), \delta)$ is called the **cohomology of the LieRep pair**.

Now we give the precise formula of the coboundary operator δ . Write $f = (f_{\mathfrak{g}}, f_V)$ and $\delta f = ((\delta f)_{\mathfrak{g}}, (\delta f)_V)$, where $f_{\mathfrak{g}} \in \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{g})$, $f_V \in \text{Hom}(\wedge^{n-1} \mathfrak{g} \otimes V, V)$, $(\delta f)_{\mathfrak{g}} \in \text{Hom}(\wedge^{n+1} \mathfrak{g}, \mathfrak{g})$ and $(\delta f)_V \in \text{Hom}(\wedge^n \mathfrak{g} \otimes V, V)$. Then we have

$$(\delta f)_{\mathfrak{g}} = (-1)^{n-1} [\mu, f_{\mathfrak{g}}]_{\mathfrak{B}} = d_{\text{CE}} f_{\mathfrak{g}},$$

where $d_{\text{CE}}: \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{g}) \rightarrow \text{Hom}(\wedge^{n+1} \mathfrak{g}, \mathfrak{g})$ is the **Chevalley–Eilenberg coboundary operator** of the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$, and $(\delta f)_V$ is given by

$$\begin{aligned} (\delta f)_V(x_1, \dots, x_n, v) &= (-1)^{n-1} [\mu \boxplus \rho, f]_{\mathfrak{B}}(x_1, \dots, x_n, v) \\ &= (-1)^{n-1} \left((\mu \boxplus \rho) \bar{\circ} f - (-1)^{n-1} f \bar{\circ} (\mu \boxplus \rho) \right)(x_1, \dots, x_n, v) \\ &= \sum_{1 \leq i < j \leq n} (-1)^i f_V(x_1, \dots, \hat{x}_i, \dots, x_{j-1}, [x_i, x_j]_{\mathfrak{g}}, x_{j+1}, \dots, x_n, v) \\ &\quad + (-1)^{n-1} \rho(f_{\mathfrak{g}}(x_1, \dots, x_n))v \\ &\quad + \sum_{i=1}^n (-1)^{i+1} \left(\rho(x_i) f_V(x_1, \dots, \hat{x}_i, \dots, x_n, v) \right. \\ &\quad \left. - f_V(x_1, \dots, \hat{x}_i, \dots, x_n, \rho(x_i)v) \right), \end{aligned}$$

for all $x_1, \dots, x_n \in \mathfrak{g}$ and $v \in V$.

Now we are ready to define the cohomology of a Lie–Leibniz triple. Let $((\mathfrak{g}, \mu), (V; \rho), T)$ be a Lie–Leibniz triple, i.e. $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is a representation of the Lie algebra $(\mathfrak{g}, \mu)$ and $T: V \rightarrow \mathfrak{g}$ is an embedding tensor. Define the set of 0-cochains $\mathfrak{C}^0(\mathfrak{g}, \rho, T)$ to be 0. For $n \geq 1$, define the space of n -cochains $\mathfrak{C}^n(\mathfrak{g}, \rho, T)$ by

$$\begin{aligned} \mathfrak{C}^n(\mathfrak{g}, \rho, T) &:= \mathfrak{C}^n(\mathfrak{g}, \rho) \oplus \mathfrak{C}^n(T) \\ &= \left(\text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{g}) \oplus \text{Hom}(\wedge^{n-1} \mathfrak{g} \otimes V, V) \right) \oplus \text{Hom}(\otimes^{n-1} V, \mathfrak{g}). \end{aligned}$$

Define the **coboundary operator** $\mathcal{D} : \mathfrak{C}^n(\mathfrak{g}, \rho, T) \rightarrow \mathfrak{C}^{n+1}(\mathfrak{g}, \rho, T)$ by

$$\begin{aligned} \mathcal{D}(f, \theta) &= (-1)^{n-2} \left(-[\mu \boxplus \rho, f]_{\mathbb{B}}, [[\mu \boxplus \rho, T]_{\mathbb{B}}, \theta]_{\mathbb{B}} \right. \\ &\quad \left. + \frac{1}{n!} \underbrace{[\cdots [f, T]_{\mathbb{B}}, T]_{\mathbb{B}}, \dots, T]_{\mathbb{B}}} \right) \end{aligned} \quad (20)$$

$$= (\delta f, \partial_T \theta + \Omega_T f), \quad \forall f \in \mathfrak{C}^n(\mathfrak{g}, \rho), \theta \in \mathfrak{C}^n(T), \quad (21)$$

where δ and ∂_T are given by (19) and (13), and $\Omega_T : \mathfrak{C}^n(\mathfrak{g}, \rho) \rightarrow \mathfrak{C}^{n+1}(T)$ is defined by

$$\Omega_T f := (-1)^{n-2} \frac{1}{n!} \underbrace{[\cdots [f, T]_{\mathbb{B}}, T]_{\mathbb{B}}, \dots, T]_{\mathbb{B}}}. \quad (22)$$

The precise formula for Ω_T is given as follows.

Lemma 5.2. *The operator $\Omega_T : \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{g}) \oplus \text{Hom}(\wedge^{n-1} \mathfrak{g} \otimes V, V) \rightarrow \text{Hom}(\otimes^n V, \mathfrak{g})$ is given by*

$$\Omega_T(f_{\mathfrak{g}}, f_V)(v_1, \dots, v_n) = (-1)^n \left(f_{\mathfrak{g}}(Tv_1, \dots, Tv_n) - Tf_V(Tv_1, \dots, Tv_{n-1}, v_n) \right), \quad (23)$$

where $f_{\mathfrak{g}} \in \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{g})$, $f_V \in \text{Hom}(\wedge^{n-1} \mathfrak{g} \otimes V, V)$ and $v_1, \dots, v_n \in V$.

Proof. By Remark 3.1, it is convenient to view the elements of $\oplus_{n=0}^{+\infty} \mathfrak{C}^n(\mathfrak{g} \oplus V; \mathfrak{g} \oplus V)$ as coderivations of $\bar{T}(\mathfrak{g} \oplus V)$. The coderivations corresponding to f and T will be denoted by $\bar{f}$ and $\bar{T}$ respectively. Then, by induction, we have

$$\begin{aligned} &\underbrace{[\cdots [f, T]_{\mathbb{B}}, T]_{\mathbb{B}}, \dots, T]_{\mathbb{B}}}_{n}((x_1, v_1), \dots, (x_n, v_n)) \\ &= \sum_{i=0}^n (-1)^i \binom{n}{i} \left(\underbrace{\bar{T} \circ \cdots \circ \bar{T}}_i \circ (\bar{f}_{\mathfrak{g}} + \bar{f}_V) \circ \underbrace{\bar{T} \circ \cdots \circ \bar{T}}_{n-i} \right)((x_1, v_1), \dots, (x_n, v_n)) \\ &= (n! f_{\mathfrak{g}}(Tv_1, \dots, Tv_n), 0) + \left((-1)^1 n(n-1)! Tf_V(Tv_1, \dots, Tv_{n-1}, v_n), 0 \right) \\ &= n! \left(f_{\mathfrak{g}}(Tv_1, \dots, Tv_n) - Tf_V(Tv_1, \dots, Tv_{n-1}, v_n), 0 \right), \end{aligned}$$

which implies that (23) holds. $\square$

Theorem 5.3. *With the above notations, $(\oplus_{n=0}^{+\infty} \mathfrak{C}^n(\mathfrak{g}, \rho, T), \mathcal{D})$ is a cochain complex, i.e. $\mathcal{D} \circ \mathcal{D} = 0$.*

Proof. By Theorem 4.9, $(s^{-1}(\mu \boxplus \rho), T)$ is a Maurer–Cartan element of the L_∞ -algebra $(s^{-1}L' \oplus \mathfrak{h}, \{l_i\}_{i=1}^{+\infty})$ given in Proposition 4.8. By Proposition 4.4, there is a twisted L_∞ -algebra $(s^{-1}L' \oplus \mathfrak{h}, \{l_i^{(s^{-1}(\mu \boxplus \rho), T)}\}_{i=1}^{+\infty})$. For any $(f, \theta) \in \mathfrak{C}^n(\mathfrak{g}, \rho, T)$, we have $(s^{-1}f, \theta) \in (s^{-1}L' \oplus \mathfrak{h})^{n-2}$ and

$$\begin{aligned}
l_1^{(s^{-1}(\mu \boxplus \rho), T)}(s^{-1}f, \theta) &= \sum_{n=0}^{+\infty} \frac{1}{n!} l_{k+n} \left(\underbrace{(s^{-1}(\mu \boxplus \rho), T), \dots, (s^{-1}(\mu \boxplus \rho), T)}_n, (s^{-1}f, \theta) \right) \\
&= l_2(s^{-1}(\mu \boxplus \rho), s^{-1}f) + l_3(s^{-1}(\mu \boxplus \rho), T, \theta) \\
&\quad + \frac{1}{n!} l_{n+1}(f, \underbrace{T, \dots, T}_n) \\
&= (-s^{-1}[\mu \boxplus \rho, f]_{\mathbb{B}}, [[\mu \boxplus \rho, T]_{\mathbb{B}}, \theta]_{\mathbb{B}} \\
&\quad + \frac{1}{n!} [\dots [f, T]_{\mathbb{B}}, T]_{\mathbb{B}}, \dots, T]_{\mathbb{B}}).
\end{aligned}$$

By (20), we deduce that

$$\mathcal{D}(f, \theta) = (-1)^{n-2} l_1^{(s^{-1}(\mu+\rho), T)}(s^{-1}f, \theta).$$

By $l_1^{(s^{-1}(\mu+\rho), T)} \circ l_1^{(s^{-1}(\mu+\rho), T)} = 0$, we obtain that $(\oplus_{n=0}^{+\infty} \mathfrak{C}^n(\mathfrak{g}, \rho, T), \mathcal{D})$ is a cochain complex. $\square$

About the relation between the operator δ , ∂_T and Ω_T , we have

Corollary 5.4. *With the above notations, we have $\Omega_T \circ \delta + \partial_T \circ \Omega_T = 0$.*

Proof. For all $(f, \theta) \in \mathfrak{C}^n(\mathfrak{g}, \rho, T)$, by the fact $\delta \circ \delta = \partial_T \circ \partial_T = 0$, we have

$$0 = (\mathcal{D} \circ \mathcal{D})(f, \theta) = \mathcal{D}(\delta f, \partial_T \theta + \Omega_T f) = (\delta(\delta f), \partial_T(\partial_T \theta + \Omega_T f) + \Omega_T(\delta f)),$$

which implies that $\Omega_T \circ \delta + \partial_T \circ \Omega_T = 0$. $\square$

Definition 5.5. The cohomology of the cochain complex $(\oplus_{n=0}^{+\infty} \mathfrak{C}^n(\mathfrak{g}, \rho, T), \mathcal{D})$ is called the **regular cohomology of the Lie–Leibniz triple** $((\mathfrak{g}, \mu), (V; \rho), T)$. We denote its n -th cohomology group by $\mathcal{H}_{\text{reg}}^n(\mathfrak{g}, \rho, T)$

The formula of the coboundary operator $\mathcal{D}$ can be well-explained by the following diagram:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \mathfrak{C}^n(\mathfrak{g}, \rho) & \xrightarrow{\delta} & \mathfrak{C}^{n+1}(\mathfrak{g}, \rho) & \xrightarrow{\delta} & \mathfrak{C}^{n+2}(\mathfrak{g}, \rho) \longrightarrow \cdots \\
& & \searrow \Omega_T & & \searrow \Omega_T & & \\
\cdots & \longrightarrow & \mathfrak{C}^n(T) & \xrightarrow{\partial_T} & \mathfrak{C}^{n+1}(T) & \xrightarrow{\partial_T} & \mathfrak{C}^{n+2}(T) \longrightarrow \cdots
\end{array}$$

Theorem 5.6. *Let $((\mathfrak{g}, \mu), (V; \rho), T)$ be a Lie–Leibniz triple. Then there is a short exact sequence of cochain complexes:*

$$0 \longrightarrow (\oplus_{n=0}^{+\infty} \mathfrak{C}^n(T), \partial_T) \xrightarrow{\iota} (\oplus_{n=0}^{+\infty} \mathfrak{C}^n(\mathfrak{g}, \rho, T), \mathcal{D}) \xrightarrow{p} (\oplus_{n=0}^{+\infty} \mathfrak{C}^n(\mathfrak{g}, \rho), \delta) \longrightarrow 0,$$

where ι and p are the inclusion map and the projection map.

Consequently, there is a long exact sequence of the cohomology groups:

$$\cdots \longrightarrow \mathcal{H}^n(T) \xrightarrow{\mathcal{H}^n(\iota)} \mathcal{H}_{\text{reg}}^n(\mathfrak{g}, \rho, T) \xrightarrow{\mathcal{H}^n(p)} \mathcal{H}^n(\mathfrak{g}, \rho) \xrightarrow{c^n} \mathcal{H}^{n+1}(T) \longrightarrow \cdots,$$

where the connecting map c^n is defined by $c^n([f]) = [\Omega_T f]$, for all $[f] \in \mathcal{H}^n(\mathfrak{g}, \rho)$.

Proof. By (21), we have the short exact sequence of cochain complexes which induces a long exact sequence of cohomology groups. Also by (21), c^n is given by $c^n([f]) = [\Omega_T f]$. $\square$

At the end of this section, we study infinitesimal deformations of Lie–Leibniz triples. Let $\mathbb{K}[[t]]$ be the ring of formal power series in one variable t . Let $((\mathfrak{g}, \mu), (V; \rho), T)$ be a Lie–Leibniz triple. Let $\mathfrak{g}[[t]]$ and $V[[t]]$ be the spaces of formal power series in t with coefficients in $\mathfrak{g}$ and V respectively.

Definition 5.7. If $[\cdot, \cdot]_t = [\cdot, \cdot]_{\mathfrak{g}} + t\omega$ defines a Lie algebra structure on $\mathfrak{g}[[t]]/(t^2)$, $\rho_t = \rho + t\varrho$ defines a representation of the Lie algebra $(\mathfrak{g}[[t]]/(t^2), [\cdot, \cdot]_t)$ on $V[[t]]/(t^2)$ and $T_t = T + tT : V[[t]]/(t^2) \rightarrow \mathfrak{g}[[t]]/(t^2)$ satisfies

$$[T_t u, T_t v]_t = T_t \left(\rho_t(T_t u)(v) \right), \quad \forall u, v \in V, \quad (24)$$

for $\omega \in \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g})$, $\varrho \in \text{Hom}(\mathfrak{g} \otimes V, V)$ and $T : V \rightarrow \mathfrak{g}$, we say that (ω, ϱ, T) generates an **infinitesimal deformation** of the Lie–Leibniz triple $((\mathfrak{g}, \mu), (V; \rho), T)$.

Let (ω, ϱ, T) generate an infinitesimal deformation. By the fact that $[\cdot, \cdot]_t = [\cdot, \cdot]_{\mathfrak{g}} + t\omega$ defines a Lie algebra structure on $\mathfrak{g}[[t]]/(t^2)$, we get

$$d_{\mathfrak{CE}}\omega = 0. \quad (25)$$

Then since $(V[[t]]/(t^2); \rho_t)$ is a representation of $(\mathfrak{g}[[t]]/(t^2), [\cdot, \cdot]_t)$, we obtain

$$\rho(\omega(x, y)) + \varrho([x, y]_{\mathfrak{g}}) = [\rho(x), \varrho(y)] + [\varrho(x), \rho(y)]. \quad (26)$$

By (24), we deduce that

$$[T u, T v]_{\mathfrak{g}} + [T u, T v]_{\mathfrak{g}} + \omega(T u, T v) = T \left(\rho(T u)v + \varrho(T u)v \right) + T(\rho(T u)v). \quad (27)$$

Proposition 5.8. *The triple (ω, ϱ, T) generates an infinitesimal deformation if and only if (ω, ϱ, T) is a 2-cocycle of the Lie–Leibniz triple $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}), (V; \rho), T)$.*

Proof. By (25), (26) and (27), we deduce that (ω, ϱ, T) generates an infinitesimal deformation of the Lie–Leibniz triple $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}), (V; \rho), T)$ if and only if (ω, ϱ, T) is a 2-cocycle. $\square$

In the sequel, we define equivalences between infinitesimal deformations of a Lie–Leibniz triple and show that infinitesimal deformations of a Lie–Leibniz triple are classified by its second cohomology group.

Definition 5.9. Two infinitesimal deformations of a Lie–Leibniz triple $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}), (V; \rho), T)$ generated by (ω, ϱ, T) and (ω', ϱ', T') are said to be **equivalent** if there exist $N \in \mathfrak{gl}(\mathfrak{g})$, $S \in \mathfrak{gl}(V)$ and $x \in \mathfrak{g}$ such that $(\text{Id}_{\mathfrak{g}} + tN + t\text{ad}_x, \text{Id}_V + tS + t\rho(x))$ is a homomorphism from $((\mathfrak{g}[[t]]/(t^2), [\cdot, \cdot]_{\mathfrak{g}} + t\omega'), (V[[t]]/(t^2); \rho + t\varrho'), T + tT')$ to $((\mathfrak{g}[[t]]/(t^2), [\cdot, \cdot]_{\mathfrak{g}} + t\omega), (V[[t]]/(t^2), \rho + t\varrho), T + tT)$.

Since $\text{Id}_{\mathfrak{g}} + tN + t\text{ad}_x$ is a Lie algebra morphism from $(\mathfrak{g}[[t]]/(t^2), [\cdot, \cdot]_{\mathfrak{g}} + t\omega')$ to $(\mathfrak{g}[[t]]/(t^2), [\cdot, \cdot]_{\mathfrak{g}} + t\omega)$, we get

$$\omega' - \omega = d_{\mathbb{C}\mathbb{E}}N. \quad (28)$$

By the equality $(\text{Id}_V + tS + t\rho(x))(\rho + t\varrho')(y)u = (\rho + t\varrho)((\text{Id}_{\mathfrak{g}} + tN + t\text{ad}_x)y)(\text{Id}_V + tS + t\rho(x))u$, we deduce that

$$\varrho'(y)u - \varrho(y)u = \rho(Ny)u + \rho(y)Su - S\rho(y)u, \quad \forall y \in \mathfrak{g}, u \in V. \quad (29)$$

By the equality $(\text{Id}_{\mathfrak{g}} + tN + t\text{ad}_x) \circ (T + tT') = (T + tT) \circ (\text{Id}_V + tS + t\rho(x))$, we obtain

$$T' - T = T \circ \rho(x) - \text{ad}_x \circ T - N \circ T + T \circ S. \quad (30)$$

Theorem 5.10. *Let $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}), \rho, T)$ be a Lie–Leibniz triple. If two infinitesimal deformations generated by (ω, ϱ, T) and (ω', ϱ', T') are equivalent, then (ω, ϱ, T) and (ω', ϱ', T') are in the same cohomology class in $\mathcal{H}_{\text{reg}}^2(\mathfrak{g}, \rho, T)$.*

Proof. By (28), (29) and (30), we deduce that

$$(\omega', \varrho', T') - (\omega, \varrho, T) = \mathcal{D}(N, S, x),$$

which implies that (ω, ϱ, T) and (ω', ϱ', T') are in the same cohomology class. $\square$

5.2. Cohomology with arbitrary coefficients and central extensions. In this subsection, we introduce the cohomology of a Lie–Leibniz triple with coefficients in an arbitrary representation and classify central extensions of a Lie–Leibniz triple using the second cohomology group.

With the help of the Lie–Leibniz triple given in Example 2.12, we give the notion of a representation of a Lie–Leibniz triple as follows.

Definition 5.11. A **representation of a Lie–Leibniz triple** $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}), (V; \rho), T)$ on a 2-term complex of vector spaces $W \xrightarrow{\overline{\mathfrak{T}}} \mathfrak{h}$ is a Lie–Leibniz triple homomorphism (ϕ, φ) from $(\mathfrak{g}, V, T)$ to $(\text{End}(W \xrightarrow{\overline{\mathfrak{T}}} \mathfrak{h}), \text{Hom}(\mathfrak{h}, W), \overline{\mathfrak{T}})$. More precisely, it consists of a Lie algebra homomorphism $\phi : \mathfrak{g} \longrightarrow \text{End}(W \xrightarrow{\overline{\mathfrak{T}}} \mathfrak{h})$ and a linear map $\varphi : V \longrightarrow \text{Hom}(\mathfrak{h}, W)$ such that

$$\overline{\mathfrak{T}} \circ \varphi = \phi \circ T, \quad (31)$$

$$\varphi\rho(x)(u) = [\phi(x), \varphi(u)]_C, \quad \forall x \in \mathfrak{g}, u \in V. \quad (32)$$

Usually we will denote a representation by $(W \xrightarrow{\overline{\mathfrak{T}}} \mathfrak{h}, \phi, \varphi)$. Since $\text{End}(W \xrightarrow{\overline{\mathfrak{T}}} \mathfrak{h}) \subset \mathfrak{gl}(\mathfrak{h}) \oplus \mathfrak{gl}(W)$, for any $x \in \mathfrak{g}$, we will always write $\phi(x) = (\phi_{\mathfrak{h}}(x), \phi_W(x))$ for $\phi_{\mathfrak{h}}(x) \in \mathfrak{gl}(\mathfrak{h})$ and $\phi_W(x) \in \mathfrak{gl}(W)$. The following result follows from straightforward verifications.

Proposition 5.12. *Let $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}), (V; \rho), T)$ be a Lie–Leibniz triple and $(W \xrightarrow{\overline{\mathfrak{T}}} \mathfrak{h}, \phi, \varphi)$ its representation. Then $((\mathfrak{g} \oplus \mathfrak{h}, [\cdot, \cdot]_{\phi_{\mathfrak{h}}}), (V \oplus W, \rho + \phi_W + \varphi), T + \overline{\mathfrak{T}})$ is a Lie–Leibniz triple, where $[\cdot, \cdot]_{\phi_{\mathfrak{h}}}$ is the semidirect product Lie bracket given by*

$$[x + \alpha, y + \beta]_{\phi_{\mathfrak{h}}} = [x, y]_{\mathfrak{g}} + \phi_{\mathfrak{h}}(x)\beta - \phi_{\mathfrak{h}}(y)\alpha, \quad \forall x, y \in \mathfrak{g}, \alpha, \beta \in \mathfrak{h}, \quad (33)$$

and the representation $\rho + \phi_W + \varphi$ of the Lie algebra $(\mathfrak{g} \oplus \mathfrak{h}, [\cdot, \cdot]_{\phi_{\mathfrak{h}}})$ on $V \oplus W$ is given by

$$\begin{aligned} & (\rho + \phi_W + \varphi)(x + \alpha)(u + \xi) \\ &= \rho(x)u + \phi_W(x)\xi - \varphi(u)\alpha, \quad \forall x \in \mathfrak{g}, \alpha \in \mathfrak{h}, u \in V, \xi \in W. \end{aligned} \quad (34)$$

This Lie–Leibniz triple is called the **semidirect product** of $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}), (V; \rho), T)$ and the representation $(W \xrightarrow{\mathfrak{T}} \mathfrak{h}, \phi, \varphi)$.

Let $(W \xrightarrow{\mathfrak{T}} \mathfrak{h}, \phi, \varphi)$ be a representation of a Lie–Leibniz triple $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}), (V; \rho), T)$. Define the set of 0-cochains $\mathfrak{C}^0(\mathfrak{g}, \rho, T, \phi, \varphi, \mathfrak{T})$ to be 0. For $n \geq 1$, define the space of n -cochains $\mathfrak{C}^n(\mathfrak{g}, \rho, T, \phi, \varphi, \mathfrak{T})$ by

$$\mathfrak{C}^n(\mathfrak{g}, \rho, T, \phi, \varphi, \mathfrak{T}) = \left(\text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{h}) \oplus \text{Hom}(\wedge^{n-1} \mathfrak{g} \otimes V, W) \right) \oplus \text{Hom}(\otimes^{n-1} V, \mathfrak{h}).$$

Define the **coboundary operator** $\mathcal{D}_R : \mathfrak{C}^n(\mathfrak{g}, \rho, T, \phi, \varphi, \mathfrak{T}) \rightarrow \mathfrak{C}^{n+1}(\mathfrak{g}, \rho, T, \phi, \varphi, \mathfrak{T})$ by

$$\mathcal{D}_R(f, \theta) = (\delta f, \partial \theta + \Omega f),$$

for all $f = (f_{\mathfrak{g}}, f_V) \in \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{h}) \oplus \text{Hom}(\wedge^{n-1} \mathfrak{g} \otimes V, W)$, $\theta \in \text{Hom}(\otimes^{n-1} V, \mathfrak{h})$. Here δ , ∂ and Ω are given as follows:

- Write $\delta f = ((\delta f)_{\mathfrak{g}}, (\delta f)_V) \in \text{Hom}(\wedge^{n+1} \mathfrak{g}, \mathfrak{h}) \oplus \text{Hom}(\wedge^n \mathfrak{g} \otimes V, W)$. Then $(\delta f)_{\mathfrak{g}} = d_{\text{CE}} f_{\mathfrak{g}}$, where $d_{\text{CE}} : \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{h}) \rightarrow \text{Hom}(\wedge^{n+1} \mathfrak{g}, \mathfrak{h})$ is the Chevalley–Eilenberg coboundary operator of the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ with coefficients in $(\mathfrak{h}, \phi_{\mathfrak{h}})$, and $(\delta f)_V$ is given by

$$\begin{aligned} & (\delta f)_V(x_1, \dots, x_n, v) \\ &= \sum_{1 \leq i < j \leq n} (-1)^i f_V(x_1, \dots, \hat{x}_i, \dots, x_{j-1}, [x_i, x_j]_{\mathfrak{g}}, x_{j+1}, \dots, x_n, v) \\ &+ (-1)^n \varphi(v) f_{\mathfrak{g}}(x_1, \dots, x_n) \\ &+ \sum_{i=1}^n (-1)^{i+1} \left(\phi_W(x_i) f_V(x_1, \dots, \hat{x}_i, \dots, x_n, v) \right. \\ &\quad \left. - f_V(x_1, \dots, \hat{x}_i, \dots, x_n, \rho(x_i)v) \right), \end{aligned}$$

for all $x_1, \dots, x_n \in \mathfrak{g}$ and $v \in V$.

- $\partial : \text{Hom}(\otimes^{n-1} V, \mathfrak{h}) \longrightarrow \text{Hom}(\otimes^n V, \mathfrak{h})$ is given by

$$\begin{aligned} \partial \theta(u_1, \dots, u_n) &= \sum_{i=1}^n (-1)^{i+1} \phi_{\mathfrak{h}}(T u_i) \theta(u_1, \dots, \hat{u}_i, \dots, u_n) \\ &+ (-1)^n \mathfrak{T}(\varphi(u_n)(\theta(u_1, \dots, u_{n-1}))) \\ &+ \sum_{1 \leq i < j \leq n} (-1)^i \theta(u_1, \dots, \hat{u}_i, \dots, u_{j-1}, \\ &\quad \rho(T u_i)(u_j), u_{j+1}, \dots, u_n). \end{aligned}$$

- $\Omega : \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{h}) \oplus \text{Hom}(\wedge^{n-1} \mathfrak{g} \otimes V, W) \rightarrow \text{Hom}(\otimes^n V, \mathfrak{h})$ is defined by

$$\begin{aligned} & \Omega(f_{\mathfrak{g}}, f_V)(u_1, \dots, u_n) \\ &= (-1)^n \left(f_{\mathfrak{g}}(Tu_1, \dots, Tu_n) - \mathfrak{T}f_V(Tu_1, \dots, Tu_{n-1}, u_n) \right), \quad \forall u_1, \dots, u_n \in V. \end{aligned}$$

Theorem 5.13. *With the above notations, $(\oplus_{n=0}^{+\infty} \mathcal{C}^n(\mathfrak{g}, \rho, T, \phi, \varphi, \mathfrak{T}), \mathcal{D}_R)$ is a cochain complex, i.e. $\mathcal{D}_R \circ \mathcal{D}_R = 0$.*

Proof. We only give a sketch of the proof and leave details to readers. Consider the semidirect product Lie–Leibniz triple $((\mathfrak{g} \oplus \mathfrak{h}, [\cdot, \cdot]_{\phi_{\mathfrak{h}}}), (V \oplus W, \rho + \phi_W + \varphi), T + \mathfrak{T})$ given in Proposition 5.12, and the associated cochain complex $(\oplus_{n=0}^{+\infty} \mathcal{C}^n(\mathfrak{g} \oplus \mathfrak{h}, \rho + \phi_W + \varphi, T + \mathfrak{T}), \mathcal{D})$ given in Theorem 5.3. It is straightforward to deduce that $(\oplus_{n=0}^{+\infty} \mathcal{C}^n(\mathfrak{g}, \rho, T, \phi, \varphi, \mathfrak{T}), \mathcal{D}_R)$ is a subcomplex of $(\oplus_{n=0}^{+\infty} \mathcal{C}^n(\mathfrak{g} \oplus \mathfrak{h}, \rho + \phi_W + \varphi, T + \mathfrak{T}), \mathcal{D})$. Thus, $\mathcal{D}_R \circ \mathcal{D}_R = 0$. $\square$

Definition 5.14. The cohomology of the cochain complex $(\oplus_{n=0}^{+\infty} \mathcal{C}^n(\mathfrak{g}, \rho, T, \phi, \varphi, \mathfrak{T}), \mathcal{D}_R)$ is called the **cohomology of the Lie–Leibniz triple** $((\mathfrak{g}, \mu), (V; \rho), T)$ with coefficients in the representation $(W \xrightarrow{\mathfrak{T}} \mathfrak{h}, \phi, \varphi)$. We denote its n -th cohomology group by $\mathcal{H}^n(\mathfrak{g}, \rho, T, \phi, \varphi, \mathfrak{T})$.

In particular, $(W \xrightarrow{\mathfrak{T}} \mathfrak{h}, \phi = 0, \varphi = 0)$ is naturally a representation of the Lie–Leibniz triple $((\mathfrak{g}, \mu), (V; \rho), T)$, which is called the **trivial representation**, and the corresponding n -th cohomology group is denoted by $\mathcal{H}_{\text{tri}}^n(\mathfrak{g}, \rho, T, \mathfrak{T})$.

At the end of this subsection we study central extensions of Lie–Leibniz triples and show that central extensions are classified by the second cohomology group $\mathcal{H}_{\text{tri}}^2(\mathfrak{g}, \rho, T, \mathfrak{T})$ as applications.

Definition 5.15. Let $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}), (V; \rho), T)$ and $((\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}}), (W, \varrho), \mathfrak{T})$ be two Lie–Leibniz triples. An **extension** of $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}), (V; \rho), T)$ by $((\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}}), (W, \varrho), \mathfrak{T})$ is a short exact sequence of Lie–Leibniz triple homomorphisms:

$$\begin{array}{ccccccc} 0 & \longrightarrow & W & \xrightarrow{i} & \hat{V} & \xrightarrow{p} & V \longrightarrow 0 \\ & & \mathfrak{T} \downarrow & & \hat{T} \downarrow & & T \downarrow \\ 0 & \longrightarrow & \mathfrak{h} & \xrightarrow{i} & \hat{\mathfrak{g}} & \xrightarrow{p} & \mathfrak{g} \longrightarrow 0, \end{array}$$

where $((\hat{\mathfrak{g}}, [\cdot, \cdot]_{\hat{\mathfrak{g}}}), (\hat{V}, \hat{\rho}), \hat{T})$ is a Lie–Leibniz triple.

It is called a **central extension** if $[\alpha, \hat{x}]_{\hat{\mathfrak{g}}} = 0$, $\hat{\rho}(\hat{x})\xi = 0$ and $\hat{\rho}(\alpha)\hat{u} = 0$ for all $\alpha \in \mathfrak{h}$, $\xi \in W$, $\hat{x} \in \hat{\mathfrak{g}}$ and $\hat{u} \in \hat{V}$.

In the sequel, we only consider central extensions.

Definition 5.16. A **section** of a central extension $((\hat{\mathfrak{g}}, [\cdot, \cdot]_{\hat{\mathfrak{g}}}), (\hat{V}, \hat{\rho}), \hat{T})$ of a Lie–Leibniz triple $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}), (V; \rho), T)$ by $(\mathfrak{h}, W, \mathfrak{T})$ consists of linear maps $\mathfrak{s} : \mathfrak{g} \longrightarrow \hat{\mathfrak{g}}$ and $s : V \longrightarrow \hat{V}$ such that

$$p \circ s = \text{Id}_V, \quad p \circ \mathfrak{s} = \text{Id}_{\mathfrak{g}}.$$

Let $(\mathfrak{s}, s)$ be a section of a central extension $((\hat{\mathfrak{g}}, [\cdot, \cdot]_{\hat{\mathfrak{g}}}), (\hat{V}, \hat{\rho}), \hat{T})$. We further define

$$\omega \in \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{h}), \quad \varpi \in \text{Hom}(\mathfrak{g} \otimes V, W), \quad \mathcal{T} \in \text{Hom}(V, \mathfrak{h})$$

by

$$\begin{aligned} \omega(x, y) &= [\mathfrak{s}(x), \mathfrak{s}(y)]_{\hat{\mathfrak{g}}} - \mathfrak{s}[x, y]_{\mathfrak{g}}, \quad \forall x, y \in \mathfrak{g}, \\ \varpi(x, u) &= \hat{\rho}(\mathfrak{s}(x))s(u) - s(\rho(x)u), \quad \forall x \in \mathfrak{g}, u \in V, \\ \mathcal{T}(u) &= \hat{T}(s(u)) - \mathfrak{s}(T(u)), \quad \forall u \in V. \end{aligned}$$

Via the section $(\mathfrak{s}, s)$, $\hat{\mathfrak{g}} \cong \mathfrak{g} \oplus \mathfrak{h}$ and $\hat{V} \cong V \oplus W$. Transfer the Lie–Leibniz triple structure on $\hat{\mathfrak{g}}$ and $\hat{V}$ to that on $\mathfrak{g} \oplus \mathfrak{h}$ and $V \oplus W$, we obtain

$$\begin{aligned} [x + \alpha, y + \beta]_{\hat{\mathfrak{g}}} &= [x, y]_{\mathfrak{g}} + \omega(x, y), \quad \forall x, y \in \mathfrak{g}, \alpha, \beta \in \mathfrak{h}, \\ \hat{\rho}(x + \alpha)(u + \xi) &= \rho(x)u + \varpi(x, u), \quad \forall x \in \mathfrak{g}, u \in V, \alpha \in \mathfrak{h}, \xi \in W, \\ \hat{T}(u + \xi) &= Tu + \mathfrak{T}\xi + \mathcal{T}u, \quad \forall u \in V, \xi \in W. \end{aligned}$$

Theorem 5.17. *With the above notations, $(\omega, \varpi, \mathcal{T})$ is a 2-cocycle of the Lie–Leibniz triple $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}), (V; \rho), T)$ with the trivial coefficients in $W \xrightarrow{\mathfrak{T}} \mathfrak{h}$. Moreover, its cohomological class does not depend on the choice of sections.*

Proof. First by the fact that $[\cdot, \cdot]_{\hat{\mathfrak{g}}}$ satisfies the Jacobi identity, we deduce that ω is 2-cocycle of the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$, i.e. $\text{d}_{\text{CE}}\omega = 0$. Then since $\hat{\rho}$ is a representation of the Lie algebra $(\hat{\mathfrak{g}}, [\cdot, \cdot]_{\hat{\mathfrak{g}}})$ on $V \oplus W$, we obtain

$$\begin{aligned} 0 &= \hat{\rho}([x, y]_{\hat{\mathfrak{g}}})u - [\hat{\rho}(x), \hat{\rho}(y)](u) \\ &= \rho([x, y]_{\mathfrak{g}})u + \varpi([x, y]_{\mathfrak{g}}, u) - \rho(x)\rho(y)u \\ &\quad - \varpi(x, \rho(y)u) + \rho(y)\rho(x)u + \varpi(y, \rho(x)u) \\ &= \varpi([x, y]_{\mathfrak{g}}, u) + \varpi(y, \rho(x)u) - \varpi(x, \rho(y)u). \end{aligned}$$

Therefore, $\delta(\omega, \varpi) = 0$.

Finally since $\hat{T}$ is an embedding tensor, we obtain

$$\begin{aligned} 0 &= [\hat{T}(u + \xi), \hat{T}(v + \eta)]_{\hat{\mathfrak{g}}} - \hat{T}(\hat{\rho}(\hat{T}(u + \xi))(v + \eta)) \\ &= [Tu, Tv]_{\mathfrak{g}} + \omega(Tu, Tv) - T(\rho(Tu)v) - \mathfrak{T}\varpi(Tu, v) - \mathcal{T}\rho(Tu)v \\ &= \omega(Tu, Tv) - \mathfrak{T}\varpi(Tu, v) - \mathcal{T}\rho(Tu)v, \end{aligned}$$

which implies that $\Omega(\omega, \varpi) + \partial\mathcal{T} = 0$. Therefore, $\mathcal{D}_R(\omega, \varpi, \mathcal{T}) = 0$, i.e. $(\omega, \varpi, \mathcal{T})$ is a 2-cocycle.

Let $(\mathfrak{s}', s')$ be another section and $(\omega', \varpi', \mathcal{T}')$ be the associated 2-cocycle. Assume that $\mathfrak{s}' = \mathfrak{s} + N$ and $s' = s + S$ for $N \in \text{Hom}(\mathfrak{g}, \mathfrak{h})$ and $S \in \text{Hom}(V, W)$. Then we have

$$\begin{aligned} (\omega' - \omega)(x, y) &= [\mathfrak{s}'(x), \mathfrak{s}'(y)]_{\hat{\mathfrak{g}}} - \mathfrak{s}'[x, y]_{\mathfrak{g}} - [\mathfrak{s}(x), \mathfrak{s}(y)]_{\hat{\mathfrak{g}}} \\ &\quad + \mathfrak{s}[x, y]_{\mathfrak{g}} = -N([x, y]_{\mathfrak{g}}) = \text{d}_{\text{CE}}N(x, y), \\ (\varpi' - \varpi)(x, u) &= \hat{\rho}(\mathfrak{s}'(x))s'(u) - s'(\rho(x)u) - \hat{\rho}(\mathfrak{s}(x))s(u) + s(\rho(x)u) = -S(\rho(x)u), \\ (\mathcal{T}' - \mathcal{T})u &= \hat{T}(s'(u)) - \mathfrak{s}'(T(u)) - \hat{T}(s(u)) + \mathfrak{s}(T(u)) = \mathfrak{T}Su - NTu, \end{aligned}$$

which implies that $(\omega', \varpi', \mathcal{T}') - (\omega, \varpi, \mathcal{T}) = \mathcal{D}_R(N, S, 0)$. Thus, $(\omega', \varpi', \mathcal{T}')$ and $(\omega, \varpi, \mathcal{T})$ are in the same cohomology class. $\square$

Isomorphisms between central extensions can be obviously defined as follows.

Definition 5.18. Let $((\hat{\mathfrak{g}}, [\cdot, \cdot]_{\hat{\mathfrak{g}}}), (\hat{V}, \hat{\rho}), \hat{T})$ and $((\tilde{\mathfrak{g}}, [\cdot, \cdot]_{\tilde{\mathfrak{g}}}), (\tilde{V}, \tilde{\rho}), \tilde{T})$ be two central extensions of a Lie–Leibniz triple $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}), (V; \rho), T)$ by $(\mathfrak{h}, W, \mathfrak{T})$. They are said to be **isomorphic** if there exists an isomorphism of Lie–Leibniz triples (κ, λ) such that the following diagram commutes:

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & W & \longrightarrow & \tilde{V} & \longrightarrow & V & \longrightarrow & 0 \\
 & & \parallel & & \searrow \lambda & & \parallel & & \\
 0 & \longrightarrow & W & \longrightarrow & \hat{V} & \longrightarrow & V & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \mathfrak{h} & \longrightarrow & \tilde{\mathfrak{g}} & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \\
 & & \parallel & & \searrow \kappa & & \parallel & & \\
 0 & \longrightarrow & \mathfrak{h} & \longrightarrow & \hat{\mathfrak{g}} & \longrightarrow & \mathfrak{g} & \longrightarrow & 0.
 \end{array}$$

Theorem 5.19. Central extensions of a Lie–Leibniz triple $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}), (V; \rho), T)$ by $(\mathfrak{h}, W, \mathfrak{T})$ are classified by the second cohomology group $\mathcal{H}_{\text{tri}}^2(\mathfrak{g}, \rho, T, \mathfrak{T})$.

Proof. Let $((\hat{\mathfrak{g}}, [\cdot, \cdot]_{\hat{\mathfrak{g}}}), (\hat{V}, \hat{\rho}), \hat{T})$ and $((\tilde{\mathfrak{g}}, [\cdot, \cdot]_{\tilde{\mathfrak{g}}}), (\tilde{V}, \tilde{\rho}), \tilde{T})$ be two isomorphic central extensions. Let $(\tilde{s}, \tilde{s})$ be a section of $((\tilde{\mathfrak{g}}, [\cdot, \cdot]_{\tilde{\mathfrak{g}}}), (\tilde{V}, \tilde{\rho}), \tilde{T})$, and $(\tilde{\omega}, \tilde{\omega}, \tilde{T})$ be the corresponding 2-cocycle. Define $(\hat{s}, \hat{s})$ by

$$\hat{s} = \kappa \circ \tilde{s}, \quad \hat{s} = \lambda \circ s.$$

Then it is obvious that $(\hat{s}, \hat{s})$ is a section of $((\hat{\mathfrak{g}}, [\cdot, \cdot]_{\hat{\mathfrak{g}}}), (\hat{V}, \hat{\rho}), \hat{T})$. Let $(\hat{\omega}, \hat{\omega}, \hat{T})$ be the corresponding 2-cocycle. Then we have

$$\begin{aligned}
 \hat{\omega}(x, y) &= [\hat{s}(x), \hat{s}(y)]_{\hat{\mathfrak{g}}} - \hat{s}[x, y]_{\mathfrak{g}} \\
 &= [\lambda \tilde{s}(x), \lambda \tilde{s}(y)]_{\hat{\mathfrak{g}}} - \lambda \tilde{s}[x, y]_{\mathfrak{g}} = \lambda([\tilde{s}(x), \tilde{s}(y)]_{\tilde{\mathfrak{g}}} - \tilde{s}[x, y]_{\mathfrak{g}}) = \tilde{\omega}(x, y).
 \end{aligned}$$

Similarly, we have

$$\tilde{\omega} = \hat{\omega}, \quad \tilde{T} = \hat{T}.$$

By Theorem 5.17, isomorphic central extensions gives rise to the same cohomological class in $\mathcal{H}_{\text{tri}}^2(\mathfrak{g}, \rho, T, \mathfrak{T})$.

The converse part can be easily checked and we omit details. $\square$

6. Homotopy Embedding Tensors and Higher Structures

In this section, we define homotopy embedding tensors and establish various relations between homotopy embedding tensors, Leibniz $_{\infty}$ -algebras, A_{∞} -algebras and L_{∞} -algebras. In particular, we construct L_{∞} -algebras from homotopy embedding tensors via Leibniz $_{\infty}$ -algebras, which generalizes the construction of L_{∞} -algebras from embedding tensors given by Kotov and Strobl in [24]. A hidden aim of us to build up homotopy theory for embedding tensors is to try to find possible equivalence between them to provide equivalence of the corresponding physical models. To build up weak equivalence between homotopy embedding tensors is our next aim which we postpone to the future. If the physical model is topological, then weak equivalence should definitely provide a suitable equivalence. But quite possibly, weak equivalence might not provide non-trivial equivalence between embedding tensors themselves, just like, weak equivalences for Lie algebras viewed as L_{∞} -algebras are simply isomorphisms.

6.1. Homotopy embedding tensors and Leibniz $_\infty$ -algebras. In this subsection, we introduce the notion of a homotopy embedding tensor on an L_∞ -algebra and show that it induces a Leibniz $_\infty$ -algebra generalizing Proposition 2.4.

Let $V^\bullet$ be a graded vector space. Denote by $\text{Hom}^n(\bar{T}(V^\bullet), V^\bullet)$ the space of degree n linear maps from the graded vector space $\bar{T}(V^\bullet)$ to the graded vector space $V^\bullet$. Obviously, an element $f \in \text{Hom}^n(\bar{T}(V^\bullet), V^\bullet)$ is the sum of $f_i : \otimes^i V^\bullet \rightarrow V^\bullet$. We will write $f = \sum_{i=1}^{+\infty} f_i$. Set $\mathbf{C}^n(V^\bullet, V^\bullet) := \text{Hom}^n(\bar{T}(V^\bullet), V^\bullet)$ and $\mathbf{C}^\bullet(V^\bullet, V^\bullet) := \bigoplus_{n \in \mathbb{Z}} \mathbf{C}^n(V^\bullet, V^\bullet)$. As the graded version of the Balavoine bracket given in [4], the **graded Balavoine bracket** $[\cdot, \cdot]_{\mathbf{B}}$ on the graded vector space $\mathbf{C}^\bullet(V^\bullet, V^\bullet)$ is given by:

$$[f, g]_{\mathbf{B}} := f \bar{\circ} g - (-1)^{mn} g \bar{\circ} f, \quad \forall f = \sum_{i=1}^{+\infty} f_i \in \mathbf{C}^m(V^\bullet, V^\bullet),$$

$$g = \sum_{j=1}^{+\infty} g_j \in \mathbf{C}^n(V^\bullet, V^\bullet), \quad (35)$$

where $f \bar{\circ} g \in \mathbf{C}^{m+n}(V^\bullet, V^\bullet)$ is defined by

$$f \bar{\circ} g = \left(\sum_{i=1}^{+\infty} f_i \right) \bar{\circ} \left(\sum_{j=1}^{+\infty} g_j \right) := \sum_{s=1}^{+\infty} \left(\sum_{i+j=s+1} f_i \bar{\circ} g_j \right), \quad (36)$$

while $f_i \bar{\circ} g_j \in \text{Hom}(\otimes^s V^\bullet, V^\bullet)$ is defined by $f_i \bar{\circ} g_j = \sum_{k=1}^i f_i \bar{\circ}_k g_j$ and $f_i \bar{\circ}_k g_j$ is defined by

$$(f_i \bar{\circ}_k g_j)(v_1, \dots, v_s) = \sum_{\sigma \in \mathbb{S}_{(k-1, j-1)}} (-1)^{\beta_k} \varepsilon(\sigma)$$

$$f_i(v_{\sigma(1)}, \dots, v_{\sigma(k-1)}, g_j(v_{\sigma(k)}, \dots, v_{\sigma(k+j-2)}, v_{k+j-1}, v_{k+j}, \dots, v_s),$$

where $\beta_k = n(v_{\sigma(1)} + v_{\sigma(2)} + \dots + v_{\sigma(k-1)})$.

Similar as the classical case, $(\mathbf{C}^\bullet(V^\bullet, V^\bullet), [\cdot, \cdot]_{\mathbf{B}})$ is a graded Lie algebra.

The notion of a Leibniz $_\infty$ -algebra was introduced in [31], and further studied in [2, 43, 53].

Definition 6.1. A **Leibniz $_\infty$ -algebra** is a $\mathbb{Z}$ -graded vector space $\mathfrak{G}^\bullet = \bigoplus_{k \in \mathbb{Z}} \mathfrak{G}^k$ equipped with a collection ($k \geq 1$) of linear maps $\theta_k : \otimes^k \mathfrak{G}^\bullet \rightarrow \mathfrak{G}^\bullet$ of degree 1 such that $\sum_{k=1}^{+\infty} \theta_k$ is a Maurer–Cartan element of the graded Lie algebra $(\mathbf{C}^\bullet(\mathfrak{G}^\bullet, \mathfrak{G}^\bullet), [\cdot, \cdot]_{\mathbf{B}})$. More precisely, for any homogeneous elements $x_1, \dots, x_n \in \mathfrak{G}^\bullet$, the following equality holds:

$$0 = \sum_{i=1}^n \sum_{k=1}^{n-i+1} \sum_{\sigma \in \mathbb{S}_{(k-1, i-1)}} (-1)^{\gamma_k} \varepsilon(\sigma)$$

$$\theta_{n-i+1}(x_{\sigma(1)}, \dots, x_{\sigma(k-1)}, \theta_i(x_{\sigma(k)}, \dots, x_{\sigma(k+i-2)}, x_{k+i-1}), x_{k+i}, \dots, x_n),$$

where $\gamma_k = x_{\sigma(1)} + \dots + x_{\sigma(k-1)}$.

It is obvious that an L_∞ -algebra is naturally a Leibniz $_\infty$ -algebra.

Definition 6.2. Let $(\mathfrak{G}^\bullet, \{\theta_k\}_{k=1}^{+\infty})$ and $(\mathfrak{G}'^\bullet, \{\theta'_k\}_{k=1}^{+\infty})$ be two Leibniz $_\infty$ -algebras. A Leibniz $_\infty$ -algebra **homomorphism** from $(\mathfrak{G}^\bullet, \{\theta_k\}_{k=1}^{+\infty})$ to $(\mathfrak{G}'^\bullet, \{\theta'_k\}_{k=1}^{+\infty})$ consists of a collection of degree 0 graded multilinear maps $f_k : \otimes^k \mathfrak{G}^\bullet \rightarrow \mathfrak{G}'^\bullet$, $k \geq 1$ with the property that, for any $n \geq 1$ and homogeneous elements $x_1, \dots, x_n \in \mathfrak{G}^\bullet$, the following equality holds:

$$\begin{aligned} & \sum_{i=1}^n \sum_{k=1}^{n-i+1} \sum_{\sigma \in \mathbb{S}_{(k-1, i-1)}} (-1)^{\gamma_k} \varepsilon(\sigma) \\ & f_{n-i+1}(x_{\sigma(1)}, \dots, x_{\sigma(k-1)}, \theta_i(x_{\sigma(k)}, \dots, x_{\sigma(k+i-2)}, x_{k+i-1}), x_{k+i}, \dots, x_n) \\ &= \sum_{p=1}^n \sum_{\substack{\sigma \in \mathbb{E}_{(k_1, \dots, k_p)} \\ k_1 + \dots + k_p = n}} \varepsilon(\sigma) \\ & \theta'_p(f_{k_1}(x_{\sigma(1)}, \dots, x_{\sigma(k_1)}), \dots, f_{k_p}(x_{\sigma(k_1+\dots+k_{p-1}+1)}, \dots, x_{\sigma(n)})), \end{aligned}$$

where $\gamma_k = x_{\sigma(1)} + \dots + x_{\sigma(k-1)}$ and $\mathbb{E}_{(k_1, \dots, k_p)}$ denotes the set of shuffles $\sigma \in \mathbb{S}_{(k_1, \dots, k_p)}$ such that $\sigma(k_1) < \sigma(k_1 + k_2) < \dots < \sigma(k_1 + k_2 + \dots + k_p)$.

Proposition-definition 6.3. Let $\mathfrak{g}^\bullet$ and $V^\bullet$ be graded vector spaces. Let $l_k : \mathbb{S}^k(\mathfrak{g}^\bullet) \rightarrow \mathfrak{g}^\bullet$ and $\rho_k : \mathbb{S}^{k-1}(\mathfrak{g}^\bullet) \otimes V^\bullet \rightarrow V^\bullet$, $k \geq 1$ be linear maps of degree 1. We define $l_k \boxplus \rho_k : \otimes^k(\mathfrak{g}^\bullet \oplus V^\bullet) \rightarrow \mathfrak{g}^\bullet \oplus V^\bullet$ as follows

$$(l_k \boxplus \rho_k)((x_1, v_1), \dots, (x_k, v_k)) = (l_k(x_1, \dots, x_k), \rho_k(x_1, \dots, x_{k-1}, v_k)). \quad (37)$$

Then $(\mathfrak{g}^\bullet \oplus V^\bullet, \{l_k \boxplus \rho_k\}_{k=1}^{+\infty})$ is a Leibniz $_\infty$ -algebra if and only if $(\mathfrak{g}^\bullet, \{l_k\}_{k=1}^{+\infty})$ is an L_∞ -algebra and $(V^\bullet, \{\rho_k\}_{k=1}^{+\infty})$ is its representation. This Leibniz $_\infty$ -algebra is called the **hemisemidirect product** of $(\mathfrak{g}^\bullet, \{l_k\}_{k=1}^{+\infty})$ and $(V^\bullet, \{\rho_k\}_{k=1}^{+\infty})$.

A representation of an L_∞ -algebra will give rise to V-data.

Proposition 6.4. Let $(\mathfrak{g}^\bullet, \{l_k\}_{k=1}^{+\infty})$ be an L_∞ -algebra and $(V^\bullet, \{\rho_k\}_{k=1}^{+\infty})$ a representation of $(\mathfrak{g}^\bullet, \{l_k\}_{k=1}^{+\infty})$. Then the following quadruple form V-data:

- the graded Lie algebra $(L, [\cdot, \cdot])$ is given by $(\mathbf{C}^\bullet(\mathfrak{g}^\bullet \oplus V^\bullet, \mathfrak{g}^\bullet \oplus V^\bullet), [\cdot, \cdot]_\mathbf{B})$;
- the abelian graded Lie subalgebra $\mathfrak{h}$ is given by $\mathfrak{h} := \oplus_{n \in \mathbb{Z}} \text{Hom}^n(\bar{\Gamma}(V^\bullet), \mathfrak{g}^\bullet)$;
- $P : L \rightarrow L$ is the projection onto the subspace $\mathfrak{h}$;
- $\Delta = \sum_{k=1}^{+\infty} (l_k \boxplus \rho_k)$.

Consequently, $(\mathfrak{h}, \{l_k\}_{k=1}^{+\infty})$ is an L_∞ -algebra, where l_k is given by (16).

Proof. It is obvious that $\text{Im } P = \mathfrak{h}$ is an abelian graded Lie subalgebra of the g.l.a. $(\mathbf{C}^\bullet(\mathfrak{g}^\bullet \oplus V^\bullet, \mathfrak{g}^\bullet \oplus V^\bullet), [\cdot, \cdot]_\mathbf{B})$. Moreover, $\ker P$ is also a graded Lie subalgebra. Since $\Delta = \sum_{k=1}^{+\infty} (l_k \boxplus \rho_k)$ is the hemisemidirect product Leibniz $_\infty$ -algebra structure on $\mathfrak{g}^\bullet \oplus V^\bullet$, we have $[\Delta, \Delta]_\mathbf{B} = 0$ and $P(\Delta) = 0$. Thus $(L, \mathfrak{h}, P, \Delta)$ are V-data. Hence by Theorem 4.6, we obtain the higher derived brackets $\{l_k\}_{k=1}^{+\infty}$ on the abelian graded Lie subalgebra $\mathfrak{h}$. $\square$

Now we are ready to define a homotopy embedding tensor, which is the main object in this section. A homotopy embedding tensor on an L_∞ -algebra is a generalization of an embedding tensor on a Lie algebra.

Definition 6.5. With above notations, a degree 0 element $\Theta = \sum_{k=1}^{+\infty} \Theta_k \in \text{Hom}(\bar{T}(V^\bullet), \mathfrak{g}^\bullet)$ is called a **homotopy embedding tensor** on $(\mathfrak{g}^\bullet, \{l_k\}_{k=1}^{+\infty})$ with respect to the representation $(V^\bullet, \{\rho_k\}_{k=1}^{+\infty})$ if $\Theta = \sum_{k=1}^{+\infty} \Theta_k$ is a Maurer–Cartan element of the L_∞ -algebra $(\mathfrak{h}, \{l_k\}_{k=1}^{+\infty})$ ¹, that is,

$$P\left(e^{[\cdot, \Theta]_{\mathfrak{B}}} \sum_{k=1}^{+\infty} (l_k \boxplus \rho_k)\right) = 0. \quad (38)$$

Definition 6.6. Let $\Theta = \sum_{k=1}^{+\infty} \Theta_k \in \text{Hom}(\bar{T}(V^\bullet), \mathfrak{g}^\bullet)$ and $\Theta' = \sum_{k=1}^{+\infty} \Theta'_k \in \text{Hom}(\bar{T}(V^\bullet), \mathfrak{g}^\bullet)$ be homotopy embedding tensors on $(\mathfrak{g}^\bullet, \{l_k\}_{k=1}^{+\infty})$ with respect to the representation $(V^\bullet, \{\rho_k\}_{k=1}^{+\infty})$. A **strict homomorphism** from Θ' to Θ consists of an L_∞ -algebra strict homomorphism $\phi_{\mathfrak{g}^\bullet} : \mathfrak{g}^\bullet \rightarrow \mathfrak{g}^\bullet$ and a graded linear map $\phi_{V^\bullet} : V^\bullet \rightarrow V^\bullet$ of degree 0 such that for any $n \geq 1$ and homogeneous elements $x_1, \dots, x_{n-1} \in \mathfrak{g}^\bullet$, $v_1, \dots, v_n, u \in V^\bullet$, the following equalities hold:

$$\phi_{\mathfrak{g}^\bullet}(\Theta'_n(v_1, \dots, v_n)) = \Theta_n(\phi_{V^\bullet}(v_1), \dots, \phi_{V^\bullet}(v_n)), \quad (39)$$

$$\phi_{V^\bullet}(\rho_n(x_1, \dots, x_{n-1}, u)) = \rho_n(\phi_{\mathfrak{g}^\bullet}(x_1), \dots, \phi_{\mathfrak{g}^\bullet}(x_{n-1}), \phi_{V^\bullet}(u)). \quad (40)$$

An embedding tensor induces a Leibniz algebra (Proposition 2.4). Similarly, we have the following result.

Theorem 6.7. Let $\Theta = \sum_{k=1}^{+\infty} \Theta_k \in \text{Hom}(\bar{T}(V^\bullet), \mathfrak{g}^\bullet)$ be a homotopy embedding tensor on $(\mathfrak{g}^\bullet, \{l_k\}_{k=1}^{+\infty})$ with respect to the representation $(V^\bullet, \{\rho_k\}_{k=1}^{+\infty})$.

- (i) $e^{[\cdot, \Theta]_{\mathfrak{B}}} \sum_{k=1}^{+\infty} (l_k \boxplus \rho_k)$ is a Maurer–Cartan element of the g.l.a. $(\mathbf{C}^\bullet(\mathfrak{g}^\bullet \oplus V^\bullet, \mathfrak{g}^\bullet \oplus V^\bullet), [\cdot, \cdot]_{\mathfrak{B}})$;
- (ii) there is a Leibniz_∞ -algebra structure on $V^\bullet$ given by

$$\theta_k(v_1, \dots, v_k) = \left(e^{[\cdot, \Theta]_{\mathfrak{B}}} \sum_{k=1}^{+\infty} (l_k \boxplus \rho_k)\right)(v_1, \dots, v_k). \quad (41)$$

- (iii) the association in (ii) gives rise to a functor S from the category of homotopy embedding tensors to that of Leibniz_∞ -algebras.

Proof. (i) Since, $[\cdot, \Theta]_{\mathfrak{B}}$ is a locally nilpotent derivation of $(\mathbf{C}^\bullet(\mathfrak{g}^\bullet \oplus V^\bullet, \mathfrak{g}^\bullet \oplus V^\bullet), [\cdot, \cdot]_{\mathfrak{B}})$, we deduce that $e^{[\cdot, \Theta]_{\mathfrak{B}}}$ is an automorphism of $(\mathbf{C}^\bullet(\mathfrak{g}^\bullet \oplus V^\bullet, \mathfrak{g}^\bullet \oplus V^\bullet), [\cdot, \cdot]_{\mathfrak{B}})$. Moreover, we have

$$\begin{aligned} & [e^{[\cdot, \Theta]_{\mathfrak{B}}} \left(\sum_{k=1}^{+\infty} (l_k \boxplus \rho_k)\right), e^{[\cdot, \Theta]_{\mathfrak{B}}} \left(\sum_{k=1}^{+\infty} (l_k \boxplus \rho_k)\right)]_{\mathfrak{B}} \\ &= e^{[\cdot, \Theta]_{\mathfrak{B}}} \left[\sum_{k=1}^{+\infty} (l_k \boxplus \rho_k), \sum_{k=1}^{+\infty} (l_k \boxplus \rho_k)\right]_{\mathfrak{B}} = 0, \end{aligned}$$

which implies that $e^{[\cdot, \Theta]_{\mathfrak{B}}} \left(\sum_{k=1}^{+\infty} (l_k \boxplus \rho_k)\right)$ is a Maurer–Cartan element of the graded Lie algebra $(\mathbf{C}^\bullet(\mathfrak{g}^\bullet \oplus V^\bullet, \mathfrak{g}^\bullet \oplus V^\bullet), [\cdot, \cdot]_{\mathfrak{B}})$.

¹ It is a filtered L_∞ -algebra [11]. The condition of being filtered ensures convergence of the series figuring in the definition of Maurer–Cartan elements.

- (ii) By (38), $e^{[\cdot, \Theta]_{\mathbb{B}}} \sum_{k=1}^{+\infty} (l_k \boxplus \rho_k)|_{V^\bullet}$ is a Leibniz_∞ -algebra structure on $V^\bullet$.
- (iii) Let $\Theta \in \text{Hom}(\bar{\mathbb{T}}(V^\bullet), \mathfrak{g}^\bullet)$ and $\Theta' \in \text{Hom}(\bar{\mathbb{T}}(V^\bullet), \mathfrak{g}^\bullet)$ be homotopy embedding tensors and $(\phi_{\mathfrak{g}^\bullet}, \phi_{V^\bullet})$ a strict homomorphism from Θ' to Θ . For any $n \geq 1$ and homogeneous elements $v_1, \dots, v_n \in V^\bullet$, we have

$$\begin{aligned} \phi_{V^\bullet}(\theta'_n(v_1, \dots, v_n)) &= \phi_{V^\bullet}\left(e^{[\cdot, \Theta']_{\mathbb{B}}} \sum_{k=1}^{+\infty} (l_k \boxplus \rho_k)\right)(v_1, \dots, v_n) \\ &\stackrel{(39), (40)}{=} \left(e^{[\cdot, \Theta]_{\mathbb{B}}} \sum_{k=1}^{+\infty} (l_k \boxplus \rho_k)\right)(\phi_{V^\bullet}(v_1), \dots, \phi_{V^\bullet}(v_n)) \\ &= \theta_n(\phi_{V^\bullet}(v_1), \dots, \phi_{V^\bullet}(v_n)). \end{aligned}$$

Therefore, $\phi_{V^\bullet}$ is a strict homomorphism from the Leibniz_∞ -algebra $(V^\bullet, \{\theta'_k\}_{k=1}^{+\infty})$ to $(V^\bullet, \{\theta_k\}_{k=1}^{+\infty})$. Then it is straightforward to see that it is actually a functor. $\square$

6.2. Leibniz $_\infty$ -algebras and A_∞ -algebras. In this subsection, first we recall the Brjeson products on graded associative algebras which is a useful tool to construct A_∞ -algebras. Then by the bar construction, we show that a Leibniz_∞ -algebra $\mathfrak{G}^\bullet$ gives rise to an A_∞ -algebra $\bar{\mathbb{T}}(\mathfrak{G}^\bullet)$.

Definition 6.8 ([47]). An A_∞ -algebra is a $\mathbb{Z}$ -graded vector space $A^\bullet = \bigoplus_{k \in \mathbb{Z}} A^k$ endowed with a family of graded maps $m_i : \bigotimes^i A^\bullet \rightarrow A^\bullet$, $\deg(m_i) = 1$, $i \geq 1$ satisfying the **Stasheff identities**

$$\sum_{i=1}^n \sum_{k=1}^{n-i+1} (-1)^{a_1 + \dots + a_{k-1}} m_{n-i+1}(a_1, \dots, a_{k-1}, m_i(a_k, \dots, a_{k+i-1}), a_{k+i}, \dots, a_n) = 0,$$

for $n \geq 1$ and any homogeneous elements $a_1, \dots, a_n \in A^\bullet$.

We recall the definition of Brjeson products on graded associative algebras, which give rise to A_∞ -algebras.

Definition 6.9 ([8, 12, 36]). Let $A^\bullet$ be a graded associative algebra, and let $\nabla : A^\bullet \rightarrow A^\bullet$ be a degree 1 linear map such that $\nabla \circ \nabla = 0$. The sequence of **Brjeson products** $b_k^\nabla : \bigotimes^k A^\bullet \rightarrow A^\bullet$ are defined as follows:

$$\begin{aligned} b_1^\nabla(a_1) &= \nabla(a_1), \\ b_2^\nabla(a_1, a_2) &= \nabla(a_1 a_2) - \nabla(a_1) a_2 - (-1)^{a_1} a_1 \nabla(a_2), \\ b_3^\nabla(a_1, a_2, a_3) &= \nabla(a_1 a_2 a_3) - \nabla(a_1 a_2) a_3 - (-1)^{a_1} a_1 \nabla(a_2 a_3) + (-1)^{a_1} a_1 \nabla(a_2) a_3, \\ &\vdots \\ b_k^\nabla(a_1, \dots, a_k) &= \nabla(a_1 \cdots a_k) - \nabla(a_1 \cdots a_{k-1}) a_k \\ &\quad - (-1)^{a_1} a_1 \nabla(a_2 \cdots a_k) + (-1)^{a_1} a_1 \nabla(a_2 \cdots a_{k-1}) a_k, \end{aligned}$$

for any homogeneous elements $a_1, \dots, a_k \in A^\bullet$.

Theorem 6.10. ([8, 12, 36]) *With the above notations, $(A^\bullet, \{b_k^\nabla\}_{k=1}^{+\infty})$ is an A_∞ -algebra.*

Let $(\mathfrak{G}^\bullet, \{\theta_k\}_{k=1}^{+\infty})$ be a Leibniz $_\infty$ -algebra. By the bar construction [2] of a Leibniz $_\infty$ -algebra (see also [35, Sect.11.4.3, Sect.13.5]), we have a codifferential cofree conilpotent coZinbiel coalgebra $(\bar{\mathfrak{T}}(\mathfrak{G}^\bullet), \Delta, d = \sum_{k=1}^{+\infty} d_k)$ as following:

$$\Delta(x_1 \otimes \cdots \otimes x_n) = \begin{cases} 0, & n = 1, \\ \sum_{i=1}^{n-1} \sum_{\sigma \in \mathbb{S}_{(i, n-i-1)}} \varepsilon(\sigma)(x_{\sigma(1)} \\ \otimes \cdots \otimes x_{\sigma(i)}) \otimes (x_{\sigma(i+1)} \otimes \cdots \otimes x_{\sigma(n-1)} \otimes x_n), & n \geq 2, \end{cases}$$

for $n < k$, $d_k(x_1 \otimes \cdots \otimes x_n) = 0$ and for $n \geq k$

$$d_k(x_1 \otimes \cdots \otimes x_n) = \sum_{j=1}^{n+1-k} \sum_{\sigma \in \mathbb{S}_{(j-1, k-1)}} (-1)^{x_{\sigma(1)} + \cdots + x_{\sigma(j-1)}} \varepsilon(\sigma; x_1, \dots, x_{j+k-2}) \\ x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(j-1)} \otimes \theta_k(x_{\sigma(j)}, \dots, x_{\sigma(j+k-2)}, x_{j+k-1}) \otimes x_{j+k} \otimes \cdots \otimes x_n.$$

We recall that the graded vector space $\bar{\mathfrak{T}}(\mathfrak{G}^\bullet)$ was equipped with the tensor product $\otimes : \bar{\mathfrak{T}}(\mathfrak{G}^\bullet) \otimes \bar{\mathfrak{T}}(\mathfrak{G}^\bullet) \rightarrow \bar{\mathfrak{T}}(\mathfrak{G}^\bullet)$ given, for $x_1, \dots, x_{m+n} \in \mathfrak{g}$, by

$$(x_1 \otimes \cdots \otimes x_n) \otimes (x_{n+1} \otimes \cdots \otimes x_{m+n}) = x_1 \otimes \cdots \otimes x_{m+n}.$$

Moreover, $(\bar{\mathfrak{T}}(\mathfrak{G}^\bullet), \otimes)$ is a free graded nonunital associative algebra. Thus, $(\bar{\mathfrak{T}}(\mathfrak{G}^\bullet), \otimes, d)$ is a graded associative algebra with a linear map d such that $d \circ d = 0$. By Theorem 6.10, we have

Theorem 6.11. *Let $(\mathfrak{G}^\bullet, \{\theta_k\}_{k=1}^{+\infty})$ be a Leibniz $_\infty$ -algebra. Then $(\bar{\mathfrak{T}}(\mathfrak{G}^\bullet), \{b_k^d\}_{k=1}^{+\infty})$ is an A_∞ -algebra.*

Remark 6.12. If the Leibniz $_\infty$ -algebra $(\mathfrak{G}^\bullet, \{\theta_k\}_{k=1}^{+\infty})$ reduces to a Leibniz algebra $(\mathfrak{G}, [\cdot, \cdot]_\mathfrak{G})$, there is an A_∞ -algebra structure on $\bar{\mathfrak{T}}(\mathfrak{G})$. More precisely, the linear maps m_k are given by

$$m_1(x_1 \otimes \cdots \otimes x_n) = \begin{cases} 0, & n = 1, \\ \sum_{1 \leq i < j \leq n} (-1)^i x_1 \otimes \cdots \otimes x_{i-1} \otimes x_{i+1} \\ \otimes \cdots \otimes x_{j-1} \otimes [x_i, x_j]_\mathfrak{g} \otimes \cdots \otimes x_n, & n \geq 2, \end{cases}$$

and for $k \geq 2$

$$m_k(x_1 \otimes \cdots \otimes x_{n_1}, x_{n_1+1} \otimes \cdots \otimes x_{n_1+n_2}, \dots, x_{n_1+\cdots+n_{k-1}+1} \otimes \cdots \otimes x_{n_1+\cdots+n_{k-1}+n_k}) \\ = \sum_{\substack{1 \leq i \leq n_1 \\ 1 \leq j \leq n_k}} (-1)^i x_1 \otimes \cdots \otimes x_{i-1} \otimes x_{i+1} \otimes \cdots \otimes x_{n_1} \otimes x_{n_1+1} \otimes \cdots \otimes x_{n_1+\cdots+n_{k-1}+j-1} \otimes \\ [x_i, x_{n_1+\cdots+n_{k-1}+j}]_\mathfrak{g} \otimes x_{n_1+\cdots+n_{k-1}+j+1} \otimes \cdots \otimes x_{n_1+\cdots+n_{k-1}+n_k}.$$

6.3. Leibniz $_{\infty}$ -algebras and L_{∞} -algebras. There is a procedure to associate an L_{∞} -algebra to a Leibniz algebra [24]. In this section, we extend this construction to a functor from the category of Leibniz $_{\infty}$ -algebras to that of L_{∞} -algebras. Thus we arrive at a functor from the category of homotopy embedding tensors to that of L_{∞} -algebras.

Let $V^{\bullet}$ be a $\mathbb{Z}$ -graded vector space. Then the tensor algebra $(T(V^{\bullet}), \otimes)$ is a free graded unital associative algebra. The freeness implies the uniqueness of the graded unital algebra morphism $\Delta^{\cosh} : T(V^{\bullet}) \rightarrow T(V^{\bullet}) \otimes T(V^{\bullet})$ such that

$$\Delta^{\cosh}(x) = 1 \otimes x + x \otimes 1, \quad \forall x \in V^{\bullet}.$$

More precisely, $\Delta^{\cosh}$ is given by

$$\begin{aligned} \Delta^{\cosh}(x_1 \otimes \cdots \otimes x_n) &= \sum_{i=0}^n \sum_{\sigma \in \mathbb{S}_{(i, n-i)}} \varepsilon(\sigma; x_1, \dots, x_n) \\ &\quad (x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(i)}) \otimes (x_{\sigma(i+1)} \otimes \cdots \otimes x_{\sigma(n)}). \end{aligned}$$

It is immediate to check that it is coassociative and counital. Hence $(T(V^{\bullet}), \otimes, \Delta^{\cosh})$ is a graded bialgebra. We call $(T(V^{\bullet}), \otimes, \Delta^{\cosh})$ the (graded) coshuffle bialgebra.

Let $\text{Lie}(V^{\bullet})$ be the free graded Lie algebra generated by the graded vector space $V^{\bullet}$. In fact, $\text{Lie}(V^{\bullet})$ is the intersection of all the Lie subalgebras of the (graded) commutator Lie algebra $T(V^{\bullet})_{\text{Lie}} = (T(V^{\bullet}), [\cdot, \cdot]_C)$ containing $V^{\bullet}$. Note that the space of primitive elements of $(T(V^{\bullet}), \otimes, \Delta^{\cosh})$ is $\text{Lie}(V^{\bullet})$. Thus the grading on $T(V^{\bullet})$ induces a natural grading on $\text{Lie}(V^{\bullet})$ making it a graded Lie algebra. We have an embedding $V^{\bullet} \subset \text{Lie}(V^{\bullet})$ of graded vector spaces. This induces $i : T(V^{\bullet}) \rightarrow T(\text{Lie}(V^{\bullet}))$ a grading preserving inclusion of graded coshuffle bialgebras. Recall that the universal enveloping algebra $U(\text{Lie}(V^{\bullet})) = T(\text{Lie}(V^{\bullet}))/I_U$, where the two sided ideal I_U is generated by $a \otimes b - b \otimes a - [a, b]$. Moreover, we have

$$\Delta^{\cosh}(I_U) \subset I_U \otimes T(\text{Lie}(V^{\bullet})) + T(\text{Lie}(V^{\bullet})) \otimes I_U.$$

Thus I_U is a homogeneous ideal of the bialgebra $(T(V^{\bullet}), \otimes, \Delta^{\cosh})$. Then the natural projection $p : T(\text{Lie}(V^{\bullet})) \rightarrow U(\text{Lie}(V^{\bullet}))$ is a surjective homomorphism of graded bialgebras.

Lemma 6.13. *With the above notations, $\Phi = p \circ i : T(V^{\bullet}) \rightarrow U(\text{Lie}(V^{\bullet}))$ is an isomorphism of graded bialgebras. More precisely, the isomorphism Φ is given by*

$$\Phi(1) = 1, \quad \Phi(v_1 \otimes \cdots \otimes v_n) = v_1 * \cdots * v_n, \quad \forall v_1, \dots, v_n \in V^{\bullet},$$

where $*$ denotes the multiplication in $U(\text{Lie}(V^{\bullet}))$.

Proof. Since Φ is a homomorphism of graded bialgebras, we only need to prove that Φ is an isomorphism. Let A be a unital associative algebra, A_{Lie} the commutator Lie algebra of A and $f : \text{Lie}(V^{\bullet}) \rightarrow A_{\text{Lie}}$ a Lie algebra homomorphism. Since $T(V^{\bullet})$ is a free graded unital associative algebra, we have a unique associative algebra homomorphism $\bar{f} : T(V^{\bullet}) \rightarrow A$ such that $f = \bar{f} \circ i$, where $i : \text{Lie}(V^{\bullet}) \rightarrow T(V^{\bullet})_{\text{Lie}}$ is the inclusion of graded Lie algebras. Thus $T(V^{\bullet})$ satisfies the universal property of the universal enveloping algebra of the free graded Lie algebra $\text{Lie}(V^{\bullet})$. Set $A = U(\text{Lie}(V^{\bullet}))$, by the universal property, we deduce that $p \circ i : T(V^{\bullet}) \rightarrow U(\text{Lie}(V^{\bullet}))$ is an isomorphism of graded associative algebras. Thus, Φ is an isomorphism of bialgebras. $\square$

Remark 6.14. Since $(T(V^\bullet), \otimes, \Delta^{\cosh})$ is a conilpotent cocommutative bialgebra, Lemma 6.13 is a special case of the Cartier-Milnor-Moore theorem [37] (see also [35, Sect.1.3.2]). For generalized bialgebras and an operadical version of Cartier-Milnor-Moore theorem, please see the monograph [32].

By Lemma 6.13, Φ is an isomorphism from the graded coaugmented coalgebra $(T^c(V^\bullet), \Delta^{\cosh})$ to the graded coaugmented coalgebra $U^c(\text{Lie}(V^\bullet))$. Thus Φ is also an isomorphism from graded noncounital coalgebra $(\bar{T}^c(V^\bullet), \Delta^{\cosh})$ to the graded noncounital coalgebra $U^c(\text{Lie}(V^\bullet))$.

Theorem 6.15 (Poincaré–Birkhoff–Witt). *Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ be a graded Lie algebra. Then the symmetrization map $\Psi : S^c(\mathfrak{g}) \rightarrow U^c(\mathfrak{g})$*

$$\Psi(x_1 \odot \cdots \odot x_m) = \frac{1}{m!} \sum_{\sigma \in \bar{S}_m} \varepsilon(\sigma; x_1, \dots, x_m) x_{\sigma(1)} * \cdots * x_{\sigma(m)},$$

$$\Psi(1) = 1, \quad \forall x_1, \dots, x_m \in \mathfrak{g}$$

is an isomorphism of graded coaugmented coalgebras.

By Theorem 6.15, Ψ is an isomorphism from the graded coaugmented coalgebra $S^c(\text{Lie}(V^\bullet))$ to the graded coaugmented coalgebra $U^c(\text{Lie}(V^\bullet))$. Thus Ψ is an isomorphism from the graded noncounital coalgebra $\bar{S}^c(\text{Lie}(V^\bullet))$ to the graded noncounital coalgebra $\bar{U}^c(\text{Lie}(V^\bullet))$. Let $(\mathfrak{G}^\bullet, \{\theta_k\}_{k=1}^{+\infty})$ be a Leibniz $_\infty$ -algebra. The bar construction of a Leibniz $_\infty$ -algebra gives us a codifferential cofree conilpotent coZinbiel coalgebra $(\bar{T}(\mathfrak{G}^\bullet), \Delta, d = \sum_{k=1}^{+\infty} d_k)$. Moreover, we have

Lemma 6.16. *Let $(\mathfrak{G}^\bullet, \{\theta_k\}_{k=1}^{+\infty})$ be a Leibniz $_\infty$ -algebra. Then $(\bar{T}(\mathfrak{G}^\bullet), \Delta + \tau_{12} \circ \Delta, d = \sum_{k=1}^{+\infty} d_k)$ is a codifferential coshuffle coalgebra.*

Proof. By the definition of Δ , we deduce that $\Delta^{\cosh} = \Delta + \tau_{12} \circ \Delta$. Since $d = \sum_{k=1}^{+\infty} d_k$ is a codifferential of the cofree conilpotent coZinbiel coalgebra $(\bar{T}(\mathfrak{G}^\bullet), \Delta)$, we have

$$\begin{aligned} \Delta^{\cosh} \circ d &= (\Delta + \tau_{12} \circ \Delta) \circ d \\ &= (d \otimes \text{Id} + \text{Id} \otimes d) \circ \Delta + \tau_{12} \circ (d \otimes \text{Id} + \text{Id} \otimes d) \circ \Delta \\ &= (d \otimes \text{Id} + \text{Id} \otimes d) \circ \Delta + (d \otimes \text{Id} + \text{Id} \otimes d) \circ (\tau_{12} \circ \Delta) \\ &= (d \otimes \text{Id} + \text{Id} \otimes d) \circ \Delta^{\cosh}. \end{aligned}$$

Therefore, $(\bar{T}(\mathfrak{G}^\bullet), \Delta + \tau_{12} \circ \Delta, d = \sum_{k=1}^{+\infty} d_k)$ is a codifferential coshuffle coalgebra. $\square$

Theorem 6.17. *With the above notations, $\Psi^{-1} \circ \Phi \circ d \circ \Phi^{-1} \circ \Psi$ is a codifferential of $S^c(\text{Lie}(\mathfrak{G}^\bullet))$. It gives an L_∞ -algebra structure on the graded vector space $\text{Lie}(\mathfrak{G}^\bullet)$.*

Proof. Since Φ and Ψ are coalgebra isomorphisms, we transfer the codifferential d on $\bar{T}^c(\mathfrak{G}^\bullet)$ to $S^c(\text{Lie}(\mathfrak{G}^\bullet))$. Thus, $\Psi^{-1} \circ \Phi \circ d \circ \Phi^{-1} \circ \Psi$ is a codifferential of $\bar{S}^c(\text{Lie}(\mathfrak{G}^\bullet))$, which gives an L_∞ -algebra structure on the graded vector space $\text{Lie}(\mathfrak{G}^\bullet)$. $\square$

Remark 6.18. This generalizes Kotov and Strobl's construction of an L_∞ -algebra from a Leibniz algebra [24]. In particular, by truncation, one can obtain a Lie 2-algebra, which was applied to the supergravity theory. See also [44] for a direct construction of a Lie 2-algebra from a Leibniz algebra.

We denote the category of Leibniz_∞ -algebras and the category of L_∞ -algebras by $\text{Leibniz}_\infty\text{-Alg}$ and $\text{Lie}_\infty\text{-Alg}$ respectively. We show that the above construction is actually a functor.

Let $f = \{f_k\}_{k=1}^{+\infty}$ be a Leibniz_∞ -algebra homomorphism from $(\mathfrak{G}^\bullet, \{\theta_k\}_{k=1}^{+\infty})$ to $(\mathfrak{G}'^\bullet, \{\theta'_k\}_{k=1}^{+\infty})$. By the bar construction of a Leibniz_∞ -algebra, we have a homomorphism of the codifferential cofree conilpotent coZinbiel coalgebras

$$F : (\bar{\mathbf{T}}(\mathfrak{G}^\bullet), \Delta, d = \sum_{k=1}^{+\infty} d_k) \rightarrow (\bar{\mathbf{T}}(\mathfrak{G}'^\bullet), \Delta, d' = \sum_{k=1}^{+\infty} d'_k),$$

which is defined by

$$\begin{aligned} F(x_1 \otimes \cdots \otimes x_n) &= \sum_{p=1}^n \sum_{\substack{\sigma \in \mathbb{E}_{(k_1, \dots, k_p)} \\ k_1 + \cdots + k_p = n}} \varepsilon(\sigma) \\ &\quad f_{k_1}(x_{\sigma(1)}, \dots, x_{\sigma(k_1)}) \otimes \cdots \otimes f_{k_p}(x_{\sigma(k_1 + \cdots + k_{p-1} + 1)}, \dots, x_{\sigma(n)}). \end{aligned}$$

Lemma 6.19. *With the above notation, then F is a homomorphism from the codifferential coshuffle coalgebra $(\bar{\mathbf{T}}(\mathfrak{G}^\bullet), \Delta + \tau_{12} \circ \Delta, d = \sum_{k=1}^{+\infty} d_k)$ to $(\bar{\mathbf{T}}(\mathfrak{G}'^\bullet), \Delta + \tau_{12} \circ \Delta, d' = \sum_{k=1}^{+\infty} d'_k)$.*

Proof. Since F is a homomorphism of codifferential cofree conilpotent coZinbiel coalgebras, we have

$$\begin{aligned} \Delta^{\text{cosh}} \circ F &= (\Delta + \tau_{12} \circ \Delta) \circ F \\ &= (F \otimes F) \circ \Delta + \tau_{12} \circ (F \otimes F) \circ \Delta \\ &= (F \otimes F) \circ \Delta + (F \otimes F) \circ (\tau_{12} \circ \Delta) \\ &= (F \otimes F) \circ \Delta^{\text{cosh}} \end{aligned}$$

and $d' \circ F = F \circ d$. Thus, F is a homomorphism of codifferential coshuffle coalgebras. $\square$

Theorem 6.20. *With the above notations, $\Psi^{-1} \circ \Phi \circ F \circ \Phi^{-1} \circ \Psi$ is a homomorphism from the codifferential cocommutative coalgebra $\bar{\mathbf{S}}^c(\text{Lie}(\mathfrak{G}^\bullet))$ to $\bar{\mathbf{S}}^c(\text{Lie}(\mathfrak{G}'^\bullet))$. It gives an L_∞ -algebra homomorphism from $\text{Lie}(\mathfrak{G}^\bullet)$ to $\text{Lie}(\mathfrak{G}'^\bullet)$.*

Proof. Since Φ and Ψ are coalgebra isomorphisms, we transfer the homomorphism $F : \bar{\mathbf{T}}^c(\mathfrak{G}^\bullet) \rightarrow \bar{\mathbf{T}}^c(\mathfrak{G}'^\bullet)$ to

$$\Psi^{-1} \circ \Phi \circ F \circ \Phi^{-1} \circ \Psi : \bar{\mathbf{S}}^c(\text{Lie}(\mathfrak{G}^\bullet)) \rightarrow \bar{\mathbf{S}}^c(\text{Lie}(\mathfrak{G}'^\bullet)).$$

Thus, $\Psi^{-1} \circ \Phi \circ F \circ \Phi^{-1} \circ \Psi$ gives an L_∞ -algebra homomorphism from $\text{Lie}(\mathfrak{G}^\bullet)$ to $\text{Lie}(\mathfrak{G}'^\bullet)$. $\square$

Now summarise the results, we generalize Kotov–Strobl’s construction to a functor from the category of homotopy embedding tensors to that of L_∞ -algebras.

Theorem 6.21. *Theorems 6.17 and 6.20 give us a functor $K : \text{Leibniz}_\infty\text{-Alg} \rightarrow \text{Lie}_\infty\text{-Alg}$. Then together with Theorem 6.7, $\text{KS} := K \circ S$ is a functor from the category of homotopy embedding tensors to that of L_∞ -algebras.*

Acknowledgements. Y. Sheng is supported by National Science Foundation of China (11922110). R. Tang is supported by National Science Foundation of China (12001228) and China Postdoctoral Science Foundation (2020M670833). C. Zhu is funded by Deutsche Forschungsgemeinschaft (ZH 274/1-1, ZH 243-3-1, RTG 2491). We thank warmly Florian Naef, Dmitry Roytenberg, Jim Stasheff, and Bruno Vallette for very helpful discussions and suggestions. We also thank ESI, Vienna, for their invitation to present a preliminary version of this work during the Programme on Higher Structures and Field Theory.

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