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Correction

Correction to: Thermalization and Canonical Typicality in Translation-Invariant Quantum Lattice Systems

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In our result about dynamical thermalization, the proof of the upper bound on the time average of the distance between the local evolved state $\rho^{(n)}(t)$ and the time-averaged state $\rho^{(n)}_{\rm avg}$ is wrong. While it is correct that this distance tends to zero for block size $|\Lambda_n| \to \infty$ (see corrected proof below), it is unclear whether it can be shown that this happens *exponentially fast* in $|\Lambda_n|$. This affects Theorem 31, and hence also Theorem 3 (the summary of Theorem 31) and Theorem 33 (a small modification of Theorem 31).

This mistake is due to an error in Ref. [3] which we have used in our proof of Lemma 30. Ref. [3] claims that the Rényi entropy H_q is convex in its parameter q, which is incorrect. This claim has been corrected in an erratum published on the author's homepage [4], but we became aware of this only recently.

We give a corrected version of Theorem 31 of our paper [1] in Theorem 4 below. Its summary (and hence the correction of Theorem 3 of our paper) reads as follows.

Theorem 1 (Correction of [1, Theorem 3]). *If there is a unique equilibrium state around inverse temperature $\beta := \lim_{n\to\infty} \beta_n$, if the (possibly pure) initial state has close to maximal population entropy, in the sense that*

$$\bar{S}(\rho_0^{(n)}) \geq S(\gamma^p_{\Lambda_n}(\beta_n)) - o(|\Lambda_n|),$$

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and if each $H^p_{\Lambda_n}$ *is non-degenerate with uniformly bounded gap degeneracy* $\sup_n D_G(H^p_{\Lambda_n}) < \infty$*, then unitary time evolution thermalizes the subsystem* Λ *for most times* t*:*

$$\left\langle \left\| \mathrm{Tr}_{\Lambda_n\setminus\Lambda}\rho^{(n)}(t) - \mathrm{Tr}_{\Lambda_n\setminus\Lambda}\frac{\exp(-\beta_n H^p_{\Lambda_n})}{Z_n} \right\|_1 \right\rangle \stackrel{n\to\infty}{\longrightarrow} 0.$$

The gap degeneracy [5] is defined as $D_G(H^p_{\Lambda_n}) := \max_E |\{(i,j) \mid i \neq j, E_i - E_j = E\}|$*, with* E_i *the eigenvalues of* $H^p_{\Lambda_n}$.

This formulation differs from the old one in the following two ways. First, it does not give concrete bounds on the time-averaged distance between $\rho^{(n)}(t)$ and its time average (it only says that this distance tends to zero for $n \to \infty$); second, it presumes that the gap degeneracy is uniformly bounded.

To prove its formal version (Theorem 4 below), we need two elementary lemmas.

Lemma 2. *Let* Φ *be a translation-invariant finite-range interaction which is not physically equivalent to zero, and let* $\bar{u}$ *be some energy density for which there is a unique Gibbs state at inverse temperature* $\beta(\bar{u})$*. Then the real function* $u \mapsto s(u)$ *defined in [1, Lemma 9] is strictly concave at* $\bar{u}$ *in the following sense: If* $\bar{u} = \lambda u_0 + (1-\lambda)u_1$ *for some* $u_0 < u_1$ *and* $\lambda \in (0,1)$ *then* $s(\bar{u}) > \lambda s(u_0) + (1-\lambda)s(u_1)$.

Proof. Let $u_0 < u_1$ and $u = \lambda u_0 + (1-\lambda)u_1$ for some $\lambda \in (0,1)$. Let $\omega_{\beta(u_0)}$ be an arbitrary Gibbs state with energy density u_0 at inverse temperature $\beta(u_0)$, and similarly $\omega_{\beta(u_1)}$. Set $\omega := \lambda\omega_{\beta(u_0)} + (1-\lambda)\omega_{\beta(u_1)}$, a translation-invariant state. Since the entropy density is affine on the translation-invariant states ([2, Thm. IV.2.4]), we have

$$s(\omega) = \lambda\, s(\omega_{\beta(u_0)}) + (1-\lambda)s(\omega_{\beta(u_1)}) = \lambda s(u_0) + (1-\lambda)s(u_1).$$

By construction, $u(\omega) = u$. Thus, due to [1, Lemma 9], we have $s(\omega) \leq s(u)$, hence $u \mapsto s(u)$ is concave.

Let us now apply the previous argumentation to the special case $u := \bar{u}$, an energy density with a unique Gibbs state. Suppose that $s(\bar{u}) = s(\omega)$. Then the variational principle ([1, Definition 6]) implies that ω is a Gibbs state at inverse temperature $\beta(\bar{u})$. But the set of Gibbs states at inverse temperature $\beta(\bar{u})$ is a face of the set of all translation-invariant states [2, p. 348], hence $\omega_{\beta(u_0)}$ and $\omega_{\beta(u_1)}$ must both be Gibbs states at inverse temperature $\beta(\bar{u})$, too. But these are distinct states, since they have different energy densities, contradicting the uniqueness of the Gibbs state at $\beta(\bar{u})$. Therefore $s(\bar{u}) > s(\omega)$, and we get the statement of strict concavity as claimed. □

Lemma 3. *Let* Φ *be a translation-invariant finite-range interaction which is not physically equivalent to zero. Suppose that the maximal energy degeneracy of* $H^p_{\Lambda_n}$ *grows at most subexponentially in* $|\Lambda_n|$*, i.e.* $\log\max\{\mathrm{tr}(\pi_i^{(n)})\} = o(|\Lambda_n|)$*, where* $(\pi_i^{(n)})_i$ *denotes the eigenprojectors of* $H^p_{\Lambda_n}$*. Let* $(\rho^{(n)})_{n\in\mathbb{N}}$ *be any sequence of* Λ_n*-translation-invariant states with*

$$[\rho^{(n)}, H^p_{\Lambda_n}] = 0, \quad S(\rho^{(n)}) \geq s\cdot|\Lambda_n| + o(|\Lambda_n|), \quad \mathrm{tr}(\rho^{(n)} H^p_{\Lambda_n}) = u\cdot|\Lambda_n| + o(|\Lambda_n|),$$

where $u \in (u_{\min}(\Phi), u_{\max}(\Phi))$ *is an energy density such that there is a unique Gibbs state at inverse temperature* $\beta(u)$*, and* $s = s(u)$*. Then* $\max_i \mathrm{tr}(\rho^{(n)}\pi_i^{(n)}) \stackrel{n\to\infty}{\longrightarrow} 0$.

Proof. We can write u as some convex combination of two distinct energy densities in a small neighborhood of u, and then Lemma 2 implies that $s = s(u) > 0$. Let us now argue by contradiction. Suppose that $\lambda^{(n)} := \max_i \operatorname{tr}(\rho^{(n)}\pi_i^{(n)})$ does not converge to zero. Decompose the state $\rho^{(n)}$ as follows:

$$\rho^{(n)} = \lambda^{(n)}\tau^{(n)} + (1-\lambda^{(n)})\sigma^{(n)}, \tag{1}$$

where $\tau^{(n)} = \pi_i^{(n)}\rho^{(n)}\pi_i^{(n)}/\lambda^{(n)}$ (note that $\lambda^{(n)} > 0$), with $\pi_i^{(n)}$ the maximizing projector. If $\lambda^{(n)} \neq 1$, define $\sigma^{(n)} := \bar{\pi}_i^{(n)}\rho^{(n)}\bar{\pi}_i^{(n)}/(1-\lambda^{(n)})$, where $\bar{\pi}_i^{(n)} := \mathbf{1}-\pi_i^{(n)}$; if $\lambda^{(n)} = 1$, set $\sigma^{(n)} = \bar{\pi}_i^{(n)}/\operatorname{tr}(\bar{\pi}_i^{(n)})$ (if n is large enough, then $\pi_i^{(n)} \neq \mathbf{1}$, hence this is well-defined). It follows that $\tau^{(n)}$ and $\sigma^{(n)}$ are mutually orthogonal Λ_n-translation-invariant states that commute with $H^p_{\Lambda_n}$.

The sequences of real numbers $S(\sigma^{(n)})/|\Lambda_n|$, $\operatorname{tr}(\sigma^{(n)}H^p_{\Lambda_n})/|\Lambda_n|$, $\operatorname{tr}(\tau^{(n)}H^p_{\Lambda_n})/|\Lambda_n|$ and $\lambda^{(n)}$ are all bounded (the latter sequence bounded away from zero by assumption). Thus, we can find a subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$\lambda^{(n_k)} \xrightarrow{k\to\infty} \delta > 0, \quad \frac{1}{|\Lambda_{n_k}|}S(\sigma^{(n_k)}) \xrightarrow{k\to\infty} s_1, \quad \frac{1}{|\Lambda_{n_k}|}\operatorname{tr}(\tau^{(n_k)}H^p_{\Lambda_{n_k}}) \xrightarrow{k\to\infty} u_0,$$
$$\frac{1}{|\Lambda_{n_k}|}\operatorname{tr}(\sigma^{(n_k)}H^p_{\Lambda_{n_k}}) \xrightarrow{k\to\infty} u_1,$$

where s_1, u_0, u_1 are real numbers, and $0 < \delta \leq 1$. Due to (1), computing the von Neumann entropy, we have $S(\rho^{(n_k)}) = \lambda^{(n_k)}S(\tau^{(n_k)}) + (1-\lambda^{(n_k)})S(\sigma^{(n_k)}) + \mathcal{O}(1)$. Since $S(\tau^{(n_k)}) \leq \log\operatorname{tr}(\pi_i^{(n_k)}) = o(|\Lambda_{n_k}|)$, this implies $s \leq (1-\delta)s_1$. Thus, $s > 0$ yields $\delta < 1$. Similarly, computing the energy expectation value, we obtain $u = \delta u_0 + (1-\delta)u_1$.

Suppose that $s_1 \geq s(u_1)$, then $s_1 - \beta u_1 \geq p(\beta, \Phi)$ for $\beta := \beta(u_1)$, hence [1, Lemma 8] implies that we must have equality, i.e. $s_1 = s(u_1)$. In summary, we conclude that $s_1 \leq s(u_1)$. Therefore

$$s(u) = s \leq (1-\delta)s_1 \leq \delta\, s(u_0) + (1-\delta)s(u_1).$$

Since s is strictly concave at u due to Lemma 2 above, this is only possible if $u_0 = u_1 = u$. Hence

$$0 < s(u) \leq (1-\delta)s_1 \leq (1-\delta)s(u_1) = (1-\delta)s(u)$$

which is a contradiction. □

This allows us to obtain a corrected version of [1, Theorem 31].

Theorem 4 (Correction of [1, Theorem 31]: Thermalization, periodic boundary conditions). *Let Φ be a translation-invariant finite-range interaction which is not physically equivalent to zero. Suppose that the maximal energy degeneracy of $H^p_{\Lambda_n}$ grows at most subexponentially in $|\Lambda_n|$, i.e. $\log\max\{\operatorname{tr}(\pi_i^{(n)})\} = o(|\Lambda_n|)$, where $(\pi_i^{(n)})_i$ denotes the eigenprojectors of $H^p_{\Lambda_n}$, and $\sup_n D_G(H^p_{\Lambda_n}) < \infty$. Let $(\rho_0^{(n)})_{n\in\mathbb{N}}$ be some sequence of initial states on Λ_n which have energy expectation value $U_n := \operatorname{tr}(\rho_0^{(n)}H^p_{\Lambda_n})$ with density $U_n/|\Lambda_n|$ converging to some value $u \in (u_{\min}(\Phi), u_{\max}(\Phi))$ as $n \to \infty$, such that there is a unique Gibbs state around inverse temperature $\beta(u)$.*

Define the 'population entropy" $\bar{S}(\rho_0^{(n)}) := S(\lambda_1, \ldots, \lambda_N)$, where S is Shannon entropy, and $\lambda_i := \mathrm{tr}(\rho_0^{(n)}\pi_i^{(n)})$ is the probability that the i-th level is populated. Suppose that for every n large enough, either $H^p_{\Lambda_n}$ is non-degenerate or every $\pi_i^{(n)}\rho_0^{(n)}\pi_i^{(n)}$ is Λ_n-translation-invariant. Then, determine the inverse temperature β_n for which

$$\mathrm{tr}(H^p_{\Lambda_n}\gamma^p_{\Lambda_n}(\beta_n)) = U_n, \quad \text{where } \gamma^p_{\Lambda_n}(\beta_n) := \frac{\exp(-\beta_n H^p_{\Lambda_n})}{Z_n}.$$

If the initial states have close to maximal population entropy in the sense that

$$\bar{S}(\rho_0^{(n)}) \geq S(\gamma^p_{\Lambda_n}(\beta_n)) - o(|\Lambda_n|),$$

then unitary time evolution $\rho^{(n)}(t) := \exp(-itH^p_{\Lambda_n})\rho_0^{(n)}\exp(itH^p_{\Lambda_n})$ thermalizes the subsystem Λ_m for most times t:

$$\lim_{n\to\infty}\left\langle \left\| \mathrm{Tr}_{\Lambda_n\setminus\Lambda_m}\rho^{(n)}(t) - \mathrm{Tr}_{\Lambda_n\setminus\Lambda_m}\frac{\exp(-\beta_n H^p_{\Lambda_n})}{Z_n}\right\|_1\right\rangle = 0,$$

where $Z_n = \mathrm{tr}(\exp(-\beta_n H^p_{\Lambda_n}))$, and $\langle\cdot\rangle$ denotes the average over all times $t \geq 0$. Furthermore, in this statement, β_n can be replaced by $\beta := \beta(u)$.

Proof. The only ingredient in the proof of [1, Theorem 31] that has to be corrected is the argument that lower-bounds the "effective dimension" d_{eff}. The old proof erroneously claimed that d_{eff} grows exponentially in $|\Lambda_n|$, but this relied on a wrong claim about the Rényi entropy of Ref. [3]. We now give a simple alternative argument which makes use of the Rényi entropy $S_\infty(\lambda_1, \ldots, \lambda_N) = -\log\max_i \lambda_i$ and the inequality $S_2 \geq S_\infty$ [4]. Namely,

$$d_{\mathrm{eff}} = \exp(S_2(\lambda_1, \ldots, \lambda_N)) \geq \exp(S_\infty(\lambda_1, \ldots, \lambda_N)) = \left(\max_i \lambda_i\right)^{-1} \stackrel{n\to\infty}{\longrightarrow} \infty$$

according to Lemma 3 above, applied to the sequence of states $\bar{\rho}_0^{(n)} = \sum_i \pi_i^{(n)}\rho_0^{(n)}\pi_i^{(n)}$. Since we have assumed that the gap degeneracy is uniformly bounded, this is enough to show that $\rho^{(n)}(t)$ is close to its time average for most times t if n is large. The rest of the proof works without modification (note that $\rho(\beta_n)$ should read $\gamma^p_{\Lambda_n}(\beta_n)$). □

Finally, [1, Theorem 33] has to be corrected analogously. We omit the obvious details.

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