Commun. Math. Phys. 383, 1051–1092 (2021)
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-021-03996-1

Communications in
**Mathematical
Physics**

Complete integrability of the Benjamin–Ono equation on the multi-soliton manifolds

Ruoci Sun

Institute for Analysis, Karlsruhe Institute of Technology (KIT), 76131 Karlsruhe, Germany.
E-mail: ruoci.sun@kit.edu

Received: 20 April 2020 / Accepted: 15 January 2021
Published online: 26 March 2021

Abstract: This paper is dedicated to proving the complete integrability of the Benjamin–Ono (BO) equation on the line when restricted to every N-soliton manifold, denoted by $\mathcal{U}_N$. We construct generalized action–angle coordinates which establish a real analytic symplectomorphism from $\mathcal{U}_N$ onto some open convex subset of $\mathbb{R}^{2N}$ and allow to solve the equation by quadrature for any such initial datum. As a consequence, $\mathcal{U}_N$ is the universal covering of the manifold of N-gap potentials for the BO equation on the torus as described by Gérard–Kappeler (Commun Pure Appl Math, 2020. https://doi.org/10.1002/cpa.21896. arXiv:1905.01849). The global well-posedness of the BO equation on $\mathcal{U}_N$ is given by a polynomial characterization and a spectral characterization of the manifold $\mathcal{U}_N$. Besides the spectral analysis of the Lax operator of the BO equation and the shift semigroup acting on some Hardy spaces, the construction of such coordinates also relies on the use of a generating functional, which encodes the entire BO hierarchy. The inverse spectral formula of an N-soliton provides a spectral connection between the Lax operator and the infinitesimal generator of the very shift semigroup.

Contents

1. Introduction

The Benjamin–Ono (BO) equation on the line reads as

$$\partial_t u = \mathrm{H}\partial_x^2 u - \partial_x(u^2), \qquad (t, x) \in \mathbb{R} \times \mathbb{R}, \tag{1.1}$$

where u is real-valued and $\mathrm{H} = -i\operatorname{sign}(\mathrm{D}) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denotes the Hilbert transform, $\mathrm{D} = -i\partial_x$,

$$\widehat{\mathrm{H}f}(\xi) = -i\operatorname{sign}(\xi)\hat{f}(\xi), \qquad \forall f \in L^2(\mathbb{R}). \tag{1.2}$$

$\operatorname{sign}(\pm\xi) = \pm 1$, for all $\xi > 0$ and $\operatorname{sign}(0) = 0$, $\hat{f} \in L^2(\mathbb{R})$ denotes the Fourier–Plancherel transform of $f \in L^2(\mathbb{R})$. We adopt the convention $L^p(\mathbb{R}) = L^p(\mathbb{R}, \mathbb{C})$. Its $\mathbb{R}$-subspace consisting of all real-valued L^p-functions is specially emphasized as $L^p(\mathbb{R}, \mathbb{R})$ throughout this paper. Equipped with the inner product $(f, g) \in L^2(\mathbb{R}) \times L^2(\mathbb{R}) \mapsto \langle f, g\rangle_{L^2} = \int_{\mathbb{R}} f(x)\overline{g(x)}\mathrm{d}x \in \mathbb{C}$, $L^2(\mathbb{R})$ is a $\mathbb{C}$-Hilbert space. Derived by Benjamin [3] and Ono [17], the BO equation (1.1) describes the evolution of weakly nonlinear internal long waves in a two-layer fluid. Equation (1.1) is globally well-posed in every Sobolev space $H^s(\mathbb{R}, \mathbb{R})$, see Tao [23] for $s \geq 1$, see Ionescu–Kenig [10] for $s \geq 0$, etc. On appropriate Sobolev spaces, equation (1.1) can be written in Hamiltonian form

$$\partial_t u = \partial_x \nabla_u E(u), \qquad E(u) = \frac{1}{2}\langle |\mathrm{D}|u, u\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} - \frac{1}{3}\int_{\mathbb{R}} u^3, \tag{1.3}$$

where $\nabla_u E(u)$ denotes the $L^2(\mathbb{R})$-gradient of E, ∂_x is the Gardner–Faddeev–Zakharov Poisson structure and $X_E(u) = \partial_x \nabla_u E(u)$ is the Hamiltonian vector field of E with respect to the Poisson structure ∂_x. Since $\partial_x = -\partial_x^*$ is an unbounded operator on $L^2(\mathbb{R}, \mathbb{R})$ with domain $H^1(\mathbb{R}, \mathbb{R})$ and range given by

$$\mathcal{W} := \partial_x\Big(H^1(\mathbb{R}, \mathbb{R})\Big) = \{u \in L^2(\mathbb{R}, \mathbb{R}) : \int_{\mathbb{R}} \tfrac{|\hat{u}(\xi)|^2}{|\xi|^2}\mathrm{d}\xi < +\infty\}, \tag{1.4}$$

its inverse $\partial_x^{-1} : \mathcal{W} \to H^1(\mathbb{R}, \mathbb{R})$ is a symplectic structure on $\mathcal{W}$. A 2-covector $\boldsymbol{\omega} \in \boldsymbol{\Lambda}^2(\mathcal{W}^*)$ is defined by $\boldsymbol{\omega}(h_1, h_2) = \langle h_1, \partial_x^{-1}h_2\rangle_{L^2}$, $\forall h_1, h_2 \in \mathcal{W}$. Under appropriate conditions on the functionals F and G, $\boldsymbol{\omega}$ is the symplectic form corresponding to the Gardner bracket, which is defined by

$$\{F, G\}(u) := \langle \partial_x \nabla_u F(u), \nabla_u G(u)\rangle_{L^2}. \tag{1.5}$$

The goal of this paper is to show the complete integrability of equation (1.1) when restricted to every multi-soliton manifold. Recall the scaling and translation invariances of equation (1.1): if $u = u(t, x)$ is a solution, so is the function $u_{c,y} : (t, x) \mapsto cu(c^2 t, c(x - y))$. A smooth solution $u = u(t, x)$ is called a solitary wave of (1.1) if there exists $\mathcal{R} \in C^\infty(\mathbb{R})$ solving the following non local elliptic equation

$$\mathrm{H}\mathcal{R}' + \mathcal{R} - \mathcal{R}^2 = 0, \qquad \mathcal{R}(x) > 0 \tag{1.6}$$

such that $u(t, x) = \mathcal{R}_c(x - y - ct)$, where $\mathcal{R}_c(x) = c\mathcal{R}(cx)$, for some $c > 0$ and $y \in \mathbb{R}$. In [2], Amick and Toland have shown that the unique (up to translation) solution of (1.6) is given by

$$\mathcal{R}(x) = \tfrac{2}{1+x^2}, \qquad \forall x \in \mathbb{R}. \tag{1.7}$$

Definition 1.1. For any positive integer $N \in \mathbb{N}_+ := \mathbb{Z} \bigcap (0, +\infty)$, the set $\mathcal{U}_N$ is defined as follows,

$$\begin{aligned} \mathcal{U}_N := \{u \in L^2(\mathbb{R}, \mathbb{R}) : u(x) = \sum_{j=1}^{N} c_j \mathcal{R}(c_j(x - x_j)), c_j > 0, \quad x_j \in \mathbb{R}, \\ \forall 1 \le j \le N\}. \end{aligned} \tag{1.8}$$

A function $u \in \mathcal{U}_N$ is called an N-soliton of the BO equation (1.1). The set of translation–scaling parameters of u is given by $\mathbf{P}(u) := \{x_1 - c_1^{-1} i, x_2 - c_2^{-1} i, \dots, x_N - c_N^{-1} i\}$ and $\mathbf{m}(z)$ denotes the multiplicity of $z \in \mathbf{P}(u)$ in the expression of u in (1.8). As a consequence, $u(x) = \sum_{z \in \mathbf{P}(u)} \frac{-2\mathbf{m}(z)\mathrm{Im}z}{(x-\mathrm{Re}z)^2 + (\mathrm{Im}z)^2}$ and a polynomial characterization of each N-soliton is given as follows,

$$u(x) = \sum_{z \in \mathbf{P}(u)} \mathrm{Im}\tfrac{2\mathbf{m}(z)}{z - x} = -2\mathrm{Im}\tfrac{Q_u'(x)}{Q_u(x)}, \quad Q_u(X) := \prod_{z \in \mathbf{P}(u)} (X - z)^{\mathbf{m}(z)}, \tag{1.9}$$

where $Q_u \in \mathbb{C}[X]$ is called the characteristic polynomial of u, $\forall u \in \mathcal{U}_N$.

The set $\mathcal{U}_N$ is in one to one correspondance with the set $\mathcal{V}_N$ that consists of all polynomials of degree N with leading coefficient 1, whose roots are contained in the lower half plane $\mathbb{C}_- = \{z \in \mathbb{C} : \mathrm{Im}z < 0\}$. Moreover, the bijection $u \in \mathcal{U}_N \mapsto Q_u \in \mathcal{V}_N$ provides the real analytic structure on $\mathcal{U}_N$.

Proposition 1.2. *Equipped with the subspace topology of the $\mathbb{R}$-Hilbert space $L^2(\mathbb{R}, \mathbb{R})$, the subset $\mathcal{U}_N$ is a simply connected, real analytic, embedded submanifold of $L^2(\mathbb{R}, \mathbb{R})$ and* $\dim_{\mathbb{R}} \mathcal{U}_N = 2N$. *For every $u \in \mathcal{U}_N$, the tangent space to $\mathcal{U}_N$ at u is given by $\mathcal{T}_u(\mathcal{U}_N) = \bigoplus_{z \in \mathbf{P}(u)} (\mathbb{R}^{\mathbf{m}(z)}(\mathrm{Re}\boldsymbol{\phi}_z) \bigoplus \mathbb{R}^{\mathbf{m}(z)}(\mathrm{Im}\boldsymbol{\phi}_z))$, where $\boldsymbol{\phi}_z(x) := (x - z)^{-2}$, $\forall z \in$* $\mathbf{P}(u) \subset \mathbb{C}_-$.

Given $u \in \mathcal{U}_N$, we have $\int_{\mathbb{R}} u = 2\pi N$, so the tangent space $\mathcal{T}_u(\mathcal{U}_N)$ is included in an auxiliary space

$$\mathcal{T} := \{h \in L^2(\mathbb{R}, (1 + x^2)\mathrm{d}x) : h(\mathbb{R}) \subset \mathbb{R}, \quad \hat{h}(0) = 0\}. \tag{1.10}$$

The Hardy's inequality yields that $\mathcal{T}$ is contained in the auxiliary space $\mathcal{W}$ given by (1.4). So $\mathcal{W} \bigcap L^2(x^2\mathrm{d}x) = \mathcal{T}$. We define a real analytic 2-form $\omega : u \in \mathcal{U}_N \mapsto \boldsymbol{\omega} \in \boldsymbol{\Lambda}^2(\mathcal{W}^*)$, i.e.

$$\omega_u(h_1, h_2) = \frac{i}{2\pi} \int_{\mathbb{R}} \frac{\hat{h}_1(\xi)\overline{\hat{h}_2(\xi)}}{\xi} \mathrm{d}\xi = -\mathrm{Im} \int_0^{+\infty} \frac{\hat{h}_1(\xi)\overline{\hat{h}_2(\xi)}}{\pi\xi} \mathrm{d}\xi,$$

$$\forall h_1, h_2 \in \mathcal{T}_u(\mathcal{U}_N). \tag{1.11}$$

Then we show that ω establishes the symplectic structure, which corresponds to the Gardner bracket (1.5), on the N-soliton manifold $\mathcal{U}_N$ defined by (1.8).

Proposition 1.3. *Endowed with ω in (1.11), the real analytic manifold $(\mathcal{U}_N, \omega)$ is a symplectic manifold. For any smooth function $f : \mathcal{U}_N \to \mathbb{R}$, let $X_f \in \mathfrak{X}(\mathcal{U}_N)$ denote its Hamiltonian vector field, then*

$$X_f(u) = \partial_x \nabla_u f(u) \in \mathcal{T}_u(\mathcal{U}_N), \qquad \forall u \in \mathcal{U}_N. \tag{1.12}$$

The Gardner bracket in (1.5) coincides with the Poisson bracket associated to the symplectic form ω, i.e. for another smooth function $g : \mathcal{U}_N \to \mathbb{R}$, we have $\omega_u(X_f(u), X_g(u)) = \{f, g\}(u)$, $\forall u \in \mathcal{U}_N$.

The following result indicates the global well-posedness of the BO equation (1.1) on the manifold $\mathcal{U}_N$.

Proposition 1.4. *For every $N \in \mathbb{N}_+$, the manifold $\mathcal{U}_N$ is invariant under the BO flow.*

Remark 1.5. Since $\mathcal{U}_N \subset H^\infty(\mathbb{R}, \mathbb{R}) \bigcap L^2(\mathbb{R}, x^2 \mathrm{d}x)$ with $H^\infty(\mathbb{R}, \mathbb{R}) := \bigcap_{s \geq 0} H^s(\mathbb{R}, \mathbb{R})$, the energy functional E in (1.3) is well defined on $\mathcal{U}_N$. So equations (1.1) and (1.3) are equivalent on $\mathcal{U}_N$.

Inspired from the construction of Birkhoff coordinates of the space-periodic BO equation in Gérard–Kappeler [8], we want to establish the generalized action–angle coordinates of (1.1) on $\mathcal{U}_N$. Let

$$\Omega_N := \{(r_1, r_2, \ldots, r_N) \in \mathbb{R}^N : r_1 < r_2 < \cdots < r_N < 0\} \tag{1.13}$$

denote the subset of actions. For any $j, k = 1, 2, \ldots, N$, the Kronecker symbol is denoted by δ_{kj}, i.e. $\delta_{kj} = 1$ if $j = k$; $\delta_{kj} = 0$, if $j \neq k$. The main result of this paper is stated as follows.

Theorem 1. *There exists a real analytic diffeomorphism*

$$\Phi_N : u \in \mathcal{U}_N \mapsto (I_1(u), I_2(u), \ldots, I_N(u); \gamma_1(u), \gamma_2(u), \ldots, \gamma_N(u)) \in \Omega_N \times \mathbb{R}^N \tag{1.14}$$

such that the following statements hold:

(i) *The Poisson brackets* (1.5) *between the coordinate functions are well defined and*

$$\{I_j, I_k\} = 0, \quad \{I_j, \gamma_k\} = \delta_{kj}, \quad \{\gamma_j, \gamma_k\} = 0 \quad \text{on} \quad \mathcal{U}_N, \quad \forall j, k = 1, 2, \ldots, N. \tag{1.15}$$

(ii) *The energy functional E defined in* (1.3)*, when expressed in the coordinate functions, is given by*

$$E(u) = -\frac{1}{2\pi} \sum_{j=1}^{N} I_j(u)^2, \qquad \forall u \in \mathcal{U}_N.$$

The coordinates $\{I_j\}_{1 \leq j \leq N}$ are referred to as actions and $\{\gamma_j\}_{1 \leq j \leq N}$ as (generalized) angles.

Corollary 1.6. *When expressed in the generalized action–angle coordinates I_j, γ_j, $1 \leq j \leq N$, the restriction of the BO equation (1.1) to $\mathcal{U}_N$ reads as*

$$\partial_t \left(I_j \circ u\right)(t) = \{E, I_j\}(u(t)) = 0, \quad \partial_t \left(\gamma_j \circ u\right)(t) = \{E, \gamma_j\}(u(t)) = \mathbf{k}_j(u(t)), \quad \forall t \in \mathbb{R}, \tag{1.16}$$

where $\mathbf{k}_j := -\frac{I_j}{\pi}$ is referred to as the j th frequency and $u : t \in \mathbb{R} \mapsto u(t) \in \mathcal{U}_N$ solves equation (1.1). As a consequence, $I_j \circ u(t) = I_j \circ u(0)$ and $\gamma_j \circ u(t) = \gamma_j \circ u(0) + (\mathbf{k}_j \circ u(0))t$. For any $\mathbf{r} \in \Omega_N$, $\Phi_N^{-1}(\{\mathbf{r}\} \times \mathbb{R}^N)$ is a Lagrangian submanifold that is invariant under the flow of (1.1).

Remark 1.7. For any $j = 1, 2, \ldots, N$, the frequency $\mathbf{k}_j : \mathcal{U}_N \to (0, +\infty)$ is a linear function of the action I_j. Hence the motions of the angles are completely decoupled.

Remark 1.8. The image of actions Ω_N is a noncompact convex polytope. As a consequence, the manifold $\mathcal{U}_N$ can be interpreted as the universal covering of the manifold of N-gap potentials $U_N^{\mathbb{T}}$ for the Benjamin–Ono equation on the torus $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, which is introduced in theorem 7.1 of Gérard–Kappeler [8],

$$U_N^{\mathbb{T}} := \{v = 2\mathrm{Re}h \in L^2(\mathbb{T}, \mathbb{R}) : \quad h(y) = -e^{iy} \tfrac{\mathfrak{Q}'(e^{iy})}{\mathfrak{Q}(e^{iy})}, \quad \mathfrak{Q} \in \mathbb{C}_N^+[X]\}, \tag{1.17}$$

where $\mathbb{C}_N^+[X]$ consists of all polynomials $\mathfrak{Q} \in \mathbb{C}[X]$ of degree N with leading coefficient 1, whose roots are contained in the annulus $\mathscr{A} := \{z \in \mathbb{C} : |z| > 1\}$. Since the fundamental group of $U_N^{\mathbb{T}}$ is $(\mathbb{Z}, +)$, the manifold $U_N^{\mathbb{T}}$ is mapped real bi-analytically onto $\mathcal{U}_N/\mathbb{Z}$. We refer to remark 1.13 to see the comparison between the main theorem 1 and theorem 7.1 of [8].

A precise description of Φ_N is given in Definition 5.1 and Theorem 5.2. In order to establish the link between the action–angle coordinates and the translation–scaling parameters of an N-soliton, we introduce the inverse spectral matrix associated to Φ_N, denoted by $M : u \in \mathcal{U}_N \mapsto (M_{kj}(u))_{1 \leq k,j \leq N} \in \mathbb{C}^{N \times N}$, where

$$\begin{aligned} &M_{jj}(u) := \gamma_j(u) + \tfrac{\pi i}{I_j(u)}, \quad \forall 1 \leq j \leq N; \quad M_{kj}(u) := \tfrac{2\pi i}{I_k(u) - I_j(u)} \sqrt{\tfrac{I_k(u)}{I_j(u)}}, \\ &\quad \forall 1 \leq j \neq k \leq N. \end{aligned} \tag{1.18}$$

Proposition 1.9. *Given $u \in \mathcal{U}_N$, the polynomial Q_u in (1.9) is the characteristic polynomial of the inverse spectral matrix $M(u) \in \mathbb{C}^{N \times N}$ defined by (1.18). As a consequence, an N-soliton is expressed by $u(x) = \sum_{j=1}^N c_j \mathcal{R}(c_j(x - x_j))$ if and only if its translation–scaling parameters $\{x_j - c_j^{-1} i\}_{1 \leq j \leq N} \subset \mathbb{C}_-$ are eigenvalues with corresponding multiplicities of the matrix $M(u)$, whose coefficients are expressed in terms of the action–angle coordinates $(I_j(u), \gamma_j(u))_{1 \leq j \leq N} \in \Omega_N \times \mathbb{R}^N$.*

Proposition 1.9 is restated with more details in Theorem 4.8, Proposition 5.4 and Corollary 5.5, which both give a spectral characterization of the N-soliton manifold $\mathcal{U}_N$ and establish a spectral connection between the inverse spectral matrix $M(u) \in \mathbb{C}^{N \times N}$ and the Lax operator L_u, which is given in Definition 2.2, of the BO equation (1.1), for any $u \in \mathcal{U}_N$. Then an explicit expression of solutions of equation (1.1) on the multi-soliton manifolds can be deduced by using Corollary 1.6 and Proposition 1.9.

Corollary 1.10. *If $u : t \in \mathbb{R} \mapsto u(t) \in \mathcal{U}_N$ solves equation (1.1) such that $u(0) = u_0$, then for any $(t, x) \in \mathbb{R} \times \mathbb{R}$, we have*

$$u(t, x) = u(t, x; u_0) = 2\mathrm{Im}\langle\big(M(u_0) - (x + \tfrac{t}{\pi}\mathfrak{V}(u_0))\big)^{-1} X(u_0), Y(u_0)\rangle_{\mathbb{C}^N}, \quad (1.19)$$

where the inner product of $\mathbb{C}^N$ is $\langle X, Y\rangle_{\mathbb{C}^N} = X^T\overline{Y}$; and $\forall u \in \mathcal{U}_N$, the matrix $M(u)$ is given by (1.18), the matrix $\mathfrak{V}(u) \in \mathbb{C}^{N\times N}$ and the vectors $X(u), Y(u) \in \mathbb{C}^N$ are defined by

$$\begin{aligned}\sqrt{2\pi}X(u)^T &= (\sqrt{|I_1(u)|}, \sqrt{|I_2(u)|}, \ldots, \sqrt{|I_N(u)|}),\\ \sqrt{2\pi}^{-1}Y(u)^T &= (\sqrt{|I_1(u)|^{-1}}, \sqrt{|I_2(u)|^{-1}}, \ldots, \sqrt{|I_N(u)|^{-1}}),\end{aligned} \qquad \mathfrak{V}(u) = \begin{pmatrix} I_1(u) & & & \\ & I_2(u) & & \\ & & \ddots & \\ & & & I_N(u)\end{pmatrix}.$$

One application of the explicit formula (1.19) is to describe the asymptotic behavior of the multi-soliton solutions of the BO equation (1.1).

Corollary 1.11. *Given $u_0 \in \mathcal{U}_N$, we set $u_\infty(t, x) = u_\infty(t, x; u_0) := \sum_{j=1}^N \mathcal{R}_{\mathbf{k}_j(u_0)}(x - \gamma_j(u_0) - \mathbf{k}_j(u_0)t)$, where $\mathcal{R}_c(x) = \frac{2c}{1+c^2x^2}$ and $\mathbf{k}_j = -\frac{I_j}{\pi}$. If $u : t \in \mathbb{R} \mapsto u(t) \in \mathcal{U}_N$ solves (1.1) with $u(0) = u_0$, then*

*(**i**) for any $R > 0$, we have $\lim_{t\to\pm\infty} \|u(t) - u_\infty(t)\|_{L^2(-R,R)} = 0$;*

*(**ii**) for any $x \in \mathbb{R}$, we have $\lim_{t\to\pm\infty} \frac{u(t,x)}{u_\infty(t,x)} = 1$.*

When $t \to \pm\infty$, the N-soliton solutions of equation (1.1) can be approximated asymptotically by the superposition of N solitons such that the j th soliton which starts from the point $\gamma_j(u_0)$, moves with constant velocity $\mathbf{k}_j(u_0)$ and constant scaling parameter $\mathbf{k}_j(u_0)$. We refer to Matsuno [14] and the references therein to see another expression of multi-soliton solutions, the soliton interactions, the non linear superposition principle and other asymptotic behaviors of solutions of equation (1.1), which are studied by using Hirota's bilinear transformation, the pole expansion and the Bäcklund transformation. However, it still remains to solve an algebraic equation (see for instance Proposition 1.9 or formula (3.266) in section **3.3** of Matsuno [14]) by radicals in order to express the velocity & scaling parameter $\mathbf{k}_j(u_0)$ and the starting point $\gamma_j(u_0)$ of the asymptotic approximation $u_\infty(u_0)$ in terms of the translation–scaling parameters with corresponding multiplicities of the initial datum $u_0 \in \mathcal{U}_N$. Compared with Matsuno [14], we give a precise and explicit expression of the velocity & scaling parameter $\mathbf{k}_j(u_0) = -\frac{I_j(u_0)}{\pi}$ of $u_\infty(u_0)$, thanks to the min-max formula (4.8) and definition 5.1.

Remark 1.12. When $N = 1$, formula (1.19) has been established in Benjamin [3], Ono [17] and Amick–Toland [2]. Moreover, let $u : t \in \mathbb{R} \mapsto u(t) \in \mathcal{U}_1$ solve the BO equation (1.1), if $u(0, x) = \frac{2c_1}{c_1^2(x-x_1)^2+1}$ for some $x_1 \in \mathbb{R}$ and $c_1 > 0$, then $u_\infty(t, x) = u(t, x) = \frac{2c_1}{c_1^2(x-(x_1+c_1t))^2+1}$, $\forall (t, x) \in \mathbb{R}^2$.

1.1. Notation. Before outlining the construction of action–angle coordinates, we introduce some notations used in this paper. The indicator function of a subset $A \subset X$ is denoted by $\mathbf{1}_A$, i.e. $\mathbf{1}_A(x) = 1$ if $x \in A$ and $\mathbf{1}_A(x) = 0$ if $x \in X\backslash A$. Recall that $\mathrm{H} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denotes the Hilbert transform given by (1.2). Set $\mathrm{Id}_{L^2(\mathbb{R})}(f) = f$, for every $f \in L^2(\mathbb{R})$. Let $\Pi : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the Szegő projector, defined by

$$\Pi := \frac{\mathrm{Id}_{L^2(\mathbb{R})}+i\mathrm{H}}{2} \Leftrightarrow \widehat{\Pi f}(\xi) = \mathbf{1}_{[0,+\infty)}(\xi)\hat{f}(\xi), \quad \forall \xi \in \mathbb{R}, \quad \forall f \in L^2(\mathbb{R}). \quad (1.20)$$

If $\mathfrak{O}$ is an open subset of $\mathbb{C}$, we denote by $\mathrm{Hol}(\mathfrak{O})$ all holomorphic functions on $\mathfrak{O}$. Let the upper half-plane and the lower half-plane be denoted by $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$ and $\mathbb{C}_- = \{z \in \mathbb{C} : \mathrm{Im} z < 0\}$ respectively. For every $p \in (0, +\infty]$, we denote by L^p_+ to be the Hardy space on $\mathbb{C}_+$, which is defined by $L^p_+ = L^p_+(\mathbb{R}) := \{g \in \mathrm{Hol}(\mathbb{C}_+) : \|g\|_{L^p_+} < +\infty\}$, where

$$\|g\|_{L^p_+} = \sup_{y>0} \left(\int_{\mathbb{R}} |g(x+iy)|^p \mathrm{d}x \right)^{\frac{1}{p}}, \qquad \text{if} \quad p \in (0, +\infty), \tag{1.21}$$

and $\|g\|_{L^\infty_+} = \sup_{z \in \mathbb{C}_+} |g(z)|$. A function $g \in L^\infty_+$ is called an inner function if $|g| = 1$ on $\mathbb{R}$. When $p = 2$, the Paley–Wiener theorem yields the identification between L^2_+ and $\Pi[L^2(\mathbb{R})]$:

$$L^2_+ = \mathcal{F}^{-1}[L^2(0, +\infty)] = \{f \in L^2(\mathbb{R}) : \mathrm{supp} \hat{f} \subset [0, +\infty)\} = \Pi(L^2(\mathbb{R})),$$

where $\mathcal{F} : f \in L^2(\mathbb{R}) \mapsto \hat{f} \in L^2(\mathbb{R})$ denotes the Fourier–Plancherel transform. Similarly, we set $L^2_- = (\mathrm{Id}_{L^2(\mathbb{R})} - \Pi)(L^2(\mathbb{R}))$. Let the filtered Sobolev spaces be denoted as $H^s_+ := L^2_+ \bigcap H^s(\mathbb{R})$ and $H^s_- := L^2_- \bigcap H^s(\mathbb{R})$, for every $s \geq 0$. We set $H^\infty(\mathbb{R}, \mathbb{R}) := \bigcap_{s \geq 0} H^s(\mathbb{R}, \mathbb{R})$.

The domain of definition of an unbounded operator $\mathcal{A}$ on some Hilbert space $\mathcal{E}$ is denoted by $\mathbf{D}(\mathcal{A}) \subset \mathcal{E}$. Given another operator $\mathcal{B}$ on $\mathbf{D}(\mathcal{B}) \subset \mathcal{E}$ such that $\mathcal{A}(\mathbf{D}(\mathcal{A})) \subset \mathbf{D}(\mathcal{B})$ and $\mathcal{B}(\mathbf{D}(\mathcal{B})) \subset \mathbf{D}(\mathcal{A})$, their Lie bracket is an operator defined on $\mathbf{D}(\mathcal{A}) \bigcap \mathbf{D}(\mathcal{B}) \subset \mathcal{E}$, which is given by $[\mathcal{A}, \mathcal{B}] := \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}$. If the operator $\mathcal{A}$ is self-adjoint, let $\sigma(\mathcal{A})$ denote its spectrum, $\sigma_{\mathrm{pp}}(\mathcal{A})$ denotes the set of its eigenvalues and $\sigma_{\mathrm{cont}}(\mathcal{A})$ denotes its continuous spectrum. Then $\sigma_{\mathrm{cont}}(\mathcal{A}) \bigcup \overline{\sigma_{\mathrm{pp}}(\mathcal{A})} = \sigma(A) \subset \mathbb{R}$. Given two $\mathbb{C}$-Hilbert spaces $\mathcal{E}_1$ and $\mathcal{E}_2$, let $\mathfrak{B}(\mathcal{E}_1, \mathcal{E}_2)$ denote the $\mathbb{C}$-Banach space that consists of all bounded $\mathbb{C}$-linear transformations $\mathcal{E}_1 \to \mathcal{E}_2$, equipped with the uniform norm. We set $\mathfrak{B}(\mathcal{E}_1) := \mathfrak{B}(\mathcal{E}_1, \mathcal{E}_1)$.

All manifolds introduced in this paper are smooth manifolds without boundary. Given a smooth manifold $\mathbf{M}$ of real dimension N, let $C^\infty(\mathbf{M})$ denote all smooth functions $f : \mathbf{M} \to \mathbb{R}$ and the set of all smooth vector fields is denoted by $\mathfrak{X}(\mathbf{M})$. The tangent (resp. cotangent) space to $\mathbf{M}$ at $p \in \mathbf{M}$ is denoted by $\mathcal{T}_p(\mathbf{M})$ (resp. $\mathcal{T}^*_p(\mathbf{M})$). Given $k \in \mathbb{N}_+$, the $\mathbb{R}$-vector space of smooth k-forms on $\mathbf{M}$ is denoted by $\mathbf{\Omega}^k(\mathbf{M})$. Given a $\mathbb{R}$-vector space $\mathbb{V}$, we denote by $\mathbf{\Lambda}^k(\mathbb{V}^*)$ the vector space of all k-covectors on $\mathbb{V}$. Given a smooth covariant tensor field $\mathbf{A}$ on $\mathbf{M}$ and $X \in \mathfrak{X}(\mathbf{M})$, the Lie derivative of $\mathbf{A}$ with respect to X is denoted by $\mathscr{L}_X(\mathbf{A})$, which is also a smooth tensor field on $\mathbf{M}$. If $\mathbf{N}$ is another smooth manifold, $\mathbf{F} : \mathbf{N} \to \mathbf{M}$ is a smooth map and $\mathbf{A}$ is a smooth covariant k-tensor field on $\mathbf{M}$, the pullback of $\mathbf{A}$ by $\mathbf{F}$, denoted by $\mathbf{F}^*\mathbf{A}$, is a smooth k-tensor field on $\mathbf{N}$ that is defined by $\forall p \in \mathbf{N}$, $\forall j = 1, 2, \ldots, k$,

$$(\mathbf{F}^*\mathbf{A})_p(v_1, v_2, \ldots, v_k) = \mathbf{A}_{\mathbf{F}(p)}\left(\mathrm{d}\mathbf{F}(p)(v_1), \mathrm{d}\mathbf{F}(p)(v_2), \ldots, \mathrm{d}\mathbf{F}(p)(v_k)\right), \quad \forall v_j \in \mathcal{T}_p(\mathbf{N}). \tag{1.22}$$

Given a positive integer N, let $\mathbb{C}_{\leq N-1}[X]$ denote the $\mathbb{C}$-vector space of all polynomials with complex coefficients whose degree is no greater than $N-1$ and $\mathbb{C}_N[X] = \mathbb{C}_{\leq N}[X] \backslash \mathbb{C}_{\leq N-1}[X]$ consists of all polynomials of degree exactly N. Given $Q \in \mathbb{C}_N[X]$, we set $\overline{Q}(X) := \sum_{j=0}^N \overline{a}_j X^j$, if $Q(X) = \sum_{j=0}^N a_j X^j$. We set $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{R}^*_+ = (0, +\infty)$ and $\mathbb{C}^* = \mathbb{C} \backslash \{0\}$. Let $D(z, r) \subset \mathbb{C}$ denote the open disc of radius $r > 0$, whose center is $z \in \mathbb{C}$ and its boundary is denoted by $\mathscr{C}(z, r) = \partial D(z, r) = \{\eta \in \mathbb{C} : |\eta - z| = r\}$.

1.2. Organization of the paper. The construction of action–angle coordinates for the BO equation (1.3) on $\mathcal{U}_N$ mainly relies on the Lax pair formulation $\partial_t L_u = [B_u, L_u]$, discovered by Nakamura [15] and Bock–Kruskal [4]. Section 2 is dedicated to the spectral analysis of the Lax operator $L_u : h \in H^1_+ \mapsto -i\partial_x h - \Pi(uh) \in L^2_+$ given by definition 2.2 for general symbol $u \in L^2(\mathbb{R}, \mathbb{R})$, where Π denotes the Szegő projector given in (1.20) and the Hardy space L^2_+ is given in (1.21). Then L_u is an unbounded self-adjoint operator on L^2_+ that is bounded from below, it has essential spectrum $\sigma_{\mathrm{ess}}(L_u) = [0, +\infty)$. In addition, if $u \in L^2(\mathbb{R}, x^2\mathrm{d}x) \bigcap L^2(\mathbb{R}, \mathbb{R})$, every eigenvalue of L_u is negative and simple, thanks to the following identity,

$$\lambda|\hat{\varphi}(0)|^2 = -2\pi \|\varphi\|^2_{L^2}, \quad \text{if} \quad \lambda \in \mathbb{R} \quad \text{and} \quad \varphi \in \mathrm{Ker}(\lambda - L_u), \tag{1.23}$$

which is firstly found by Wu [24] in the case $\lambda < 0$. Then we introduce a generating functional which encodes the entire BO hierarchy,

$$\mathcal{H}_\lambda(u) = \langle (L_u + \lambda)^{-1}\Pi u, \Pi u\rangle_{L^2}, \qquad \text{if} \quad \lambda \in \mathbb{C}\backslash\sigma(-L_u), \tag{1.24}$$

in Definition 2.14. It provides a sequence of conservation laws controlling every Sobolev norm.

In Sect. 3, we study the shift semigroup $(S(\eta)^*)_{\eta \geq 0}$ acting on the Hardy space L^2_+, where $S(\eta)f = e_\eta f$ and $e_\eta(x) = e^{i\eta x}$. Then a weak version of the Lax Theorem 3.2, which is stated as Lemma 3.3, can be obtained by solving a linear differential equation with constant coefficients. Every N-dimensional subspace of L^2_+ that is invariant under the infinitesimal generator $G = i\frac{\mathrm{d}}{\mathrm{d}\eta}\big|_{\eta=0^+} S(\eta)^*$ is of the form $\frac{\mathbb{C}_{\leq N-1}[X]}{Q}$, for some monic polynomial $Q \in \mathbb{C}_N[X]$ whose roots are contained in the lower half-plane $\mathbb{C}_-$.

In Sect. 4, the real analytic structure and symplectic structure of the N-soliton subset $\mathcal{U}_N$ are established at first. Then we continue the spectral analysis of L_u, $\forall u \in \mathcal{U}_N$. The Lax operator L_u has N simple eigenvalues $\lambda^u_1 < \lambda^u_2 < \cdots < \lambda^u_N < 0$ and the Hardy space L^2_+ splits as

$$\begin{aligned} L^2_+ &= \mathscr{H}_{\mathrm{cont}}(L_u) \bigoplus \mathscr{H}_{\mathrm{pp}}(L_u), \quad \mathscr{H}_{\mathrm{cont}}(L_u) = \mathscr{H}_{\mathrm{ac}}(L_u) = \Theta_u L^2_+, \\ \mathscr{H}_{\mathrm{pp}}(L_u) &= \frac{\mathbb{C}_{\leq N-1}[X]}{Q_u}, \end{aligned} \tag{1.25}$$

where Q_u denotes the characteristic polynomial of u given by (1.9) and $\Theta_u = \frac{\overline{Q_u}}{Q_u}$ is an inner function on the upper half-plane $\mathbb{C}_+$. Proposition 1.9 is proved by identifying $M(u)$ in (1.18) as the matrix of the restriction $G|_{\mathscr{H}_{\mathrm{pp}}(L_u)}$ associated to the spectral basis $\{\varphi^u_1, \varphi^u_2, \ldots, \varphi^u_N\}$, where $\varphi^u_j \in \mathrm{Ker}(\lambda^u_j - L_u)$ such that $\|\varphi^u_j\|_{L^2} = 1$ and $\int_{\mathbb{R}} u\varphi^u_j > 0$. The generating function $\mathcal{H}_\lambda$ in (1.24) can be identified as the Borel–Cauchy transform of the spectral measure of L_u associated to Πu, which yields the invariance of $\mathcal{U}_N$ under the BO flow in $H^\infty(\mathbb{R}, \mathbb{R})$. Hence (1.3) is a globally well-posed Hamiltonian system on $\mathcal{U}_N$.

Section 5 is dedicated to completing the proof of theorem 1. The generalized angle variables are the real parts of the diagonal elements of the matrix $M(u)$, i.e. $\gamma_j : u \in \mathcal{U}_N \mapsto \mathrm{Re}\langle G\varphi^u_j, \varphi^u_j\rangle_{L^2} \in \mathbb{R}$ and the action variables are $I_j : u \in \mathcal{U}_N \mapsto 2\pi\lambda^u_j \in \mathbb{R}$. Thanks to the Lax pair formulation $\mathrm{d}L(u)(X_{\mathcal{H}_\lambda}(u)) = [B^\lambda_u, L_u]$, where $L : u \in \mathcal{U}_N \mapsto L_u \in \mathfrak{B}(H^1_+, L^2_+)$ is $\mathbb{R}$-affine and B^λ_u is some skew-adjoint operator on L^2_+, we have $2\pi\{\lambda_j, \gamma_k\} = \delta_{kj}$ and $\{\lambda_j, \lambda_k\} = 0$ on $\mathcal{U}_N$, $1 \leq j, k \leq N$. Then $\Phi_N : \mathcal{U}_N \to \Omega_N \times \mathbb{R}^N$

is a real analytic immersion. The diffeomorphism property of Φ_N is given by Hadamard's global inverse function theorem. Finally, we show that $\Phi_N : (\mathcal{U}_N, \omega) \to (\Omega_N \times \mathbb{R}^N, \nu)$ is a symplectomorphism by restricting $\omega - \Phi_N^* \nu$ to a special Lagrangian submanifold $\Lambda_N := \bigcap_{j=1}^N \gamma_j^{-1}(0) \subset \mathcal{U}_N$. Corollary 1.11 is proved in Sect. 6.

Remark 1.13. (Comparison with Gérard–Kappeler [8]) The BO equation on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ reads as

$$\partial_t v = \mathrm{H}^{\mathbb{T}} \partial_x^2 v - \partial_x(v^2), \quad (t, x) \in \mathbb{R} \times \mathbb{T}, \tag{1.26}$$

where $\mathrm{H}^{\mathbb{T}}$ denotes the Hilbert transform on $L^2(\mathbb{T}, \mathbb{C})$ that is defined by $\mathrm{H}^{\mathbb{T}} f(x) = -i \sum_{|n|\geq 1} \frac{|n|}{n} \hat{f}(n) e^{inx}$, $\forall f = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{inx} \in L^2(\mathbb{T}, \mathbb{C})$. It can be written in Hamiltonian form on appropriate Sobolev spaces

$$\partial_t v = \partial_x \nabla_v E^{\mathbb{T}}(v), \quad E^{\mathbb{T}}(v) = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1}{2}(|\partial_x|^{\frac{1}{2}} v(x))^2 - \frac{1}{3} v(x)^3 \right) \mathrm{d}x, \tag{1.27}$$

where $X_{E^{\mathbb{T}}}(v) = \partial_x \nabla_v E^{\mathbb{T}}(v)$ is the Hamiltonian vector field of $E^{\mathbb{T}}$ with respect to the symplectic form

$$\omega^{\mathbb{T}}(f_1, f_2) = \sum_{|n|\geq 1} \frac{i}{2\pi n} \hat{f}_1(n) \overline{\hat{f}_2(n)}, \; f_j = \sum_{|n|\geq 1} \hat{f}_j(n) e^{inx} \in L^2_{r,0}(\mathbb{T}) := \{v \in L^2(\mathbb{T}, \mathbb{R}) : \hat{v}(0) = 0\}. \tag{1.28}$$

The global Birkhoff coordinates for (1.26) on $L^2_{r,0}(\mathbb{T})$ described in theorem 1.1 of [8] is denoted by

$$\boldsymbol{\zeta}^{\mathbb{T}} : v \in L^2_{r,0}(\mathbb{T}) \mapsto (\zeta_n(v))_{n\geq 1} \in \mathfrak{h}_+^{\frac{1}{2}}, \tag{1.29}$$

where $\mathfrak{h}_+^{\frac{1}{2}} := \{(z_n)_{n\in\mathbb{N}_+} \subset \mathbb{C} : \|(z_n)_{n\in\mathbb{N}_+}\|^2_{\frac{1}{2}} := \sum_{n\geq 1} |n||z_n|^2 < +\infty\}$ is a weighted ℓ^2-sequence space. Thanks to theorem 7.1 of [8], the N-gap potential manifold $U_N^{\mathbb{T}}$ defined by (1.17) is a connected, real analytic, symplectic submanifold of $(L^2_{r,0}(\mathbb{T}), \omega^{\mathbb{T}})$ given by (1.28) and $U_N^{\mathbb{T}}$ is characterized by

$$U_N^{\mathbb{T}} = \{v \in L^2_{r,0}(\mathbb{T}) \quad : \quad \zeta_N(v) \neq 0, \quad \zeta_j(v) = 0, \quad \forall j > N\}. \tag{1.30}$$

So it is invariant under the flow of equation (1.26) and $\dim_{\mathbb{R}} U_N^{\mathbb{T}} = 2N$. Let $\tilde{\nu} := i \sum_{j=1}^N \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j$ denote the canonical symplectic form on $\mathbb{C}^{N-1} \times \mathbb{C}^*$. The restriction of complex Birkhoff coordinates $\boldsymbol{\zeta}^{\mathbb{T}}$ given by (1.29), to the manifold $U_N^{\mathbb{T}}$ establishes a real analytic diffeomorphism

$$\boldsymbol{\zeta}_N^{\mathbb{T}} := \left(\boldsymbol{\zeta}^{\mathbb{T}}\right)|_{U_N^{\mathbb{T}}} : v \in U_N^{\mathbb{T}} \mapsto (\zeta_1(v), \zeta_2(v), \ldots, \zeta_N(v)) \in \mathbb{C}^{N-1} \times \mathbb{C}^*, \tag{1.31}$$

such that $\boldsymbol{\zeta}_N^{\mathbb{T}}$ preserves the symplectic structure, i.e. $\left(\boldsymbol{\zeta}_N^{\mathbb{T}}\right)^* \tilde{\nu} = \omega^{\mathbb{T}}$, and the energy functional $E^{\mathbb{T}}$ in (1.27), when expressed in the coordinate functions, is given by

$$E^{\mathbb{T}}(v) = \sum_{n=1}^N n^2 |\zeta_n(v)|^2 - \sum_{n=1}^N \left(\sum_{k=n}^N |\zeta_k(v)|^2 \right)^2, \qquad \forall v \in U_N^{\mathbb{T}}.$$

The generating functional defined in (1.24) plays a key role in proving the local diffeomorphism property and the symplectomorphism property of action–angle/Birkhoff map in both theorem 1 of this paper and theorem 7.1 of [8]. The real analytic structure of $\mathcal{U}_N$ (resp. $U_N^{\mathbb{T}}$) is constructed by establishing a real analytic embedding from an open subset of $\mathbb{C}^N$ to $L^2(\mathbb{R}, \mathbb{R})$ (resp. $L^2(\mathbb{T}, \mathbb{R})$) with range given by $\mathcal{U}_N$ (resp. $U_N^{\mathbb{T}}$). A real analytic covering map from the N-soliton manifold $\mathcal{U}_N$ to the N-gap potential manifold $U_N^{\mathbb{T}}$ is established in remark 1.8. However, the construction of the action–angle map Φ_N in (1.14) is quite different from the construction of the Birkhoff map $\boldsymbol{\zeta}^{\mathbb{T}}$ in [8].

1. The symplectic form $\omega^{\mathbb{T}}$ given by (1.28) is well defined on $L^2_{r,0}(\mathbb{T})$, which is a $\mathbb{C}$-Hilbert space that contains every manifold $U_N^{\mathbb{T}}$. So $U_N^{\mathbb{T}}$ is a symplectic submanifold of $(L^2_{r,0}, \omega^{\mathbb{T}})$. The BO equation on the torus (1.26), when restricted to $U_N^{\mathbb{T}}$, is interpreted as an integrable subsystem of equation (1.26) on $(L^2_{r,0}, \omega^{\mathbb{T}})$. On the other hand, in the space non-periodic regime, we do not know whether there exists a large submanifold of $L^2(\mathbb{R}, \mathbb{R})$, denoted by $\mathfrak{L}$, such that $\mathfrak{L}$ contains every multi-soliton manifold $\mathcal{U}_N$, $\mathfrak{L}$ is invariant under the flow of (1.1), and there exist action–angle coordinates for the BO equation (1.1) on $\mathfrak{L}$, whose restriction to $\mathcal{U}_N$ is Φ_N given in (1.14). Evidently, $\mathfrak{L}$ can not be chosen as $\mathcal{W} = \partial_x(H^1(\mathbb{R}, \mathbb{R}))$ given by (1.4), because $\mathcal{U}_N \bigcap \mathcal{W} = \emptyset$. However, the 2-covector $\boldsymbol{\omega} : (h_1, h_2) \in \mathcal{W}^2 \mapsto \langle h_1, \partial_x^{-1} h_2 \rangle_{L^2(\mathbb{R})}$ is defined on $\mathcal{W}$. The extension of the symplectic form $\omega \in \boldsymbol{\Omega}^2(\mathcal{U}_N)$, which is defined by (1.11), to the manifold $\mathfrak{L}$ would be the major difficulty for constructing action–angle coordinates of the BO equation (1.1) on $\mathfrak{L}$. Since $\mathcal{U}_N \bigcap \mathcal{W} = \emptyset$, we have to use Cartan's formula (4.2) in order to prove the closedness of the 2-form $\omega : u \in \mathcal{U}_N \mapsto \omega_u = \boldsymbol{\omega} \in \boldsymbol{\Lambda}^2(\mathcal{W}^*)$, which may not be interpreted as a pullback of $\boldsymbol{\omega}$. Moreover, the simple connectedness of $\mathcal{U}_N$ is established by a special property of the Viète map (4.1).

2. In any case, the Lax operator for the BO equation is self-adjoint and bounded from below. The spectrum of the Lax operator $L_v^{\mathbb{T}}$ in the space-periodic regime consists of a sequence of simple eigenvalues $\sigma(L_v^{\mathbb{T}}) = \{\lambda_0^{\mathbb{T}}(v) < \lambda_1^{\mathbb{T}}(v) < \cdots\} \subset \mathbb{R}$ and the gap between each two of them is at least 1. Then the n th action variable is defined by $|\zeta_n(v)|^2 := \lambda_n^{\mathbb{T}}(v) - \lambda_{n-1}^{\mathbb{T}}(v) - 1$ in [8], $\forall n \geq 1$. However, in order to prove the simplicity and negativeness of eigenvalues of the Lax operator L_u in Definition 2.2 for the BO equation on the line (1.1), we have to introduce the auxiliary identity (1.23). The action variables for equation (1.1) on $\mathcal{U}_N$ are actually the eigenvalues of $2\pi L_u$, $\forall u \in \mathcal{U}_N$.

3. The shift operator $S^{\mathbb{T}} : f \in L^2_+(\mathbb{T}) \mapsto e^{ix} f(x) \in L^2_+(\mathbb{T})$ and its adjoint are bounded operators on the Hardy space $L^2_+(\mathbb{T}) := \Pi^{\mathbb{T}}(L^2(\mathbb{T}, \mathbb{C}))$, where $\Pi^{\mathbb{T}} : \sum_{n \in \mathbb{Z}} g_n e^{inx} \in L^2(\mathbb{T}, \mathbb{C}) \mapsto \sum_{n \geq 0} g_n e^{inx} \in L^2(\mathbb{T}, \mathbb{C})$ denotes the Szegő projector on $L^2(\mathbb{T}, \mathbb{C})$. So both the inverse formula for $v \in L^2(\mathbb{T}, \mathbb{R})$, which is denoted by formula (4.5) in [8], and the spectral characterization of $U_N^{\mathbb{T}}$, which is given by formula (1.30) of this paper and (7.2) in [8], can be directly obtained by computing the 0 th Fourier mode of each eigenfunction of the space-periodic Lax operator $L_v^{\mathbb{T}}$ without using Beurling's theorem that characterizes all the shift-invariant subspaces of $L^2_+(\mathbb{T}) = \mathscr{H}_{\mathrm{pp}}(L_v^{\mathbb{T}})$. On the other hand, in the case of the BO equation on the line (1.1), the Lax operator L_u in definition 2.2 has not only eigenvalues but also continuous spectrum. In order to determine the characteristic polynomial Q_u in (1.9) and prove the spectral characterization Theorem 4.8 for $\mathcal{U}_N$, we have to do the spectral decomposition (1.25) and identify each spectral subspace as the corresponding closed shift-invariant (also called translation-invariant) subspace by both introducing the two shift semigroups $(S(\eta))_{\eta \geq 0}$, $(S(\eta)^*)_{\eta \geq 0}$, and using Lax's scalar representation Theorem 3.2 or its special case stated as Lemma 3.3. In fact, $\forall u \in \mathcal{U}_N$, the spectral subspace $\mathscr{H}_{\mathrm{ac}}(L_u)$ is invariant under $(S(\eta))_{\eta\geq 0}$; the spectral subspace $\mathscr{H}_{\mathrm{pp}}(L_u)$ is invariant under $(S(\eta)^*)_{\eta\geq 0}$ and $\dim_{\mathbb{C}} \mathscr{H}_{\mathrm{pp}}(L_u) = N$. Since the infinitesimal generator $G = i\frac{\mathrm{d}}{\mathrm{d}\eta}\big|_{\eta=0^+} S(\eta)^*$ is an unbounded, densely defined operator on $L^2_+(\mathbb{R})$, given by (3.2), we study its restriction to the N-dimensional spectral subspace $\mathscr{H}_{\mathrm{pp}}(L_u)$. Then Lemma 3.3 yields that $Q_u(X) = \det(X - G|_{\mathscr{H}_{\mathrm{pp}}(L_u)})$.

1.3. Related work. Besides the global well-posedness problem of the BO equation (1.1), various properties of its multi-soliton solutions have been investigated in detail. Both the solitary waves for (1.1) and the internal periodic waves for (1.26) are completely classified in Amick–Toland [2]. The H^1-orbital stability of double solitons of (1.1) is obtained in Neves–Lopes [16]. In Dobrokhotov–Krichever [6], the multi-phase solutions (periodic multi-solitons) for (1.26) are constructed by finite zone integration and they have also established an inversion formula for multi-phase solutions. Compared with their work, we give a geometric description of the inverse spectral transform by proving the real bi-analyticity and the symplectomorphism property of the action–angle map Φ_N given by (1.14). Furthermore, the inverse spectral formula $u = -2\mathrm{Im}\frac{Q_u'}{Q_u}$ with $Q_u(x) = \det(x - G|_{\mathscr{H}_{\mathrm{pp}}(L_u)}) = \det(x - M(u))$ provides a spectral connection between the Lax operator L_u and the operator $G|_{\mathscr{H}_{\mathrm{pp}}(L_u)}$, $\forall u \in \mathcal{U}_N$.

Concerning the investigation of the integrability of the BO equations (1.1) and (1.26), besides the discovery of their Lax pair structures, we mention the pioneering work of Ablowitz–Fokas [1], Coifman–Wickerhauser [5], Kaup–Matsuno [12] and Wu [24,25] about the direct and inverse scattering transform of (1.1). Equations (1.1) and (1.26) both admit an infinite hierarchy of conservation laws that control every H^s-norm of the solutions, see [1] and [5] for the case $2s \in \mathbb{N}$, see Talbut [22] for the case $-\frac{1}{2} < s < 0$ and for conservation laws controlling Besov norms, etc. In the space-periodic regime, Gérard and Kappeler have shown in [8] that (1.26) admits global Birkhoff coordinates on $L^2_{r,0}(\mathbb{T})$, see also remark 1.13 for the comparison between [8] and theorem 1 of this paper. We point out that both Korteweg–de Vries (KdV) equation on $\mathbb{T}$ (see Kappeler–Pöschel [11]) and the cubic defocusing Schrödinger (dNLS) equation on $\mathbb{T}$ (see Grébert–Kappeler [9]) admit global Birkhoff coordinates. The theory of finite-dimensional Hamiltonian system is transferred to BO, KdV and dNLS equations on $\mathbb{T}$ through the submanifolds of finite-gap potentials, which are introduced in order to solve the periodic KdV initial problem. Moreover, the cubic Szegő equations both on $\mathbb{T}$ (see Gérard–Grellier [7]) and on $\mathbb{R}$ (see Pocovnicu [18]) admit global (generalized) action–angle coordinates on all finite-rank generic rational function manifolds, denoted respectively by $\mathcal{M}(N)^{\mathbb{T}}_{\mathrm{gen}}$ and $\mathcal{M}(N)^{\mathbb{R}}_{\mathrm{gen}}$. A real analytic covering map can be established from $\mathcal{M}(N)^{\mathbb{R}}_{\mathrm{gen}}$ to $\mathcal{M}(N)^{\mathbb{T}}_{\mathrm{gen}}$. Moreover, the cubic Szegő equations both on $\mathbb{T}$ and on $\mathbb{R}$ have inverse spectral formulas which permit the Szegő flows to be expressed explicitly in terms of time-variables and initial data without using action–angle coordinates. The shift semigroup $(S(\eta)^*)_{\eta\geq 0}$ and its infinitesimal generator G are also used in [18].

Remark 1.14. The BO equation (1.1) can be interpreted as a Schrödinger-type equation, which is filtered by the Szegő projector $\Pi : L^2(\mathbb{R}) \to L^2_+$. If $u : t \in \mathbb{R} \mapsto u(t) \in H^2(\mathbb{R}, \mathbb{R})$ solves (1.1) and $w : t \in \mathbb{R} \mapsto w(t) := \Pi(u(t)) \in H^2_+$, then equation (1.1) reads as an NLS–Szegő equation

$$i\partial_t w - \partial_x^2 w + i\partial_x(w^2 + 2\Pi(|w|^2)) = 0, \qquad (t, x) \in \mathbb{R} \times \mathbb{R}. \tag{1.32}$$

We refer to Sun [20,21] to see the long time and asymptotic behavior of other NLS–Szegő equations.

2. The Lax Operator

This section is dedicated to studying the Lax operator L_u in the Lax pair formulation of the BO equation (1.1). Then we describe the location of its spectrum and revisit the simplicity of its eigenvalues. At last, we introduce a generating functional $\mathcal{H}_\lambda$ which encodes the entire BO hierarchy. The equation $\partial_t u = \partial_x \nabla_u \mathcal{H}_\lambda(u)$ also enjoys a Lax pair structure. Now, we recall a basic fact concerning unitarily equivalent self-adjoint operators.

Proposition 2.1. *If $\mathcal{E}_1$ and $\mathcal{E}_2$ are two Hilbert spaces, let $\mathcal{A}$ be a self-adjoint operator defined on $\mathbf{D}(\mathcal{A}) \subset \mathcal{E}_1$ and $\mathcal{B}$ be a self-adjoint operator defined on $\mathbf{D}(\mathcal{B}) \subset \mathcal{E}_2$. Both $\mathcal{A}$ and $\mathcal{B}$ have spectral decompositions*

$$\mathcal{E}_1 = \mathscr{H}_{\mathrm{ac}}(\mathcal{A}) \bigoplus \mathscr{H}_{\mathrm{sc}}(\mathcal{A}) \bigoplus \mathscr{H}_{\mathrm{pp}}(\mathcal{A}), \qquad \mathcal{E}_2 = \mathscr{H}_{\mathrm{ac}}(\mathcal{B}) \bigoplus \mathscr{H}_{\mathrm{sc}}(\mathcal{B}) \bigoplus \mathscr{H}_{\mathrm{pp}}(\mathcal{B}). \tag{2.1}$$

If $\mathcal{A}$ and $\mathcal{B}$ are unitarily equivalent i.e. there exists a unitary operator $\mathcal{U} : \mathcal{E}_1 \to \mathcal{E}_2$ such that

$$\mathcal{B}\mathcal{U} = \mathcal{U}\mathcal{A}, \qquad \mathbf{D}(\mathcal{B}) = \mathcal{U}\mathbf{D}(\mathcal{A}), \tag{2.2}$$

then $\sigma_{\mathrm{xx}}(A) = \sigma_{\mathrm{xx}}(B)$ and $\mathcal{U}\mathscr{H}_{\mathrm{xx}}(\mathcal{A}) = \mathscr{H}_{\mathrm{xx}}(\mathcal{B})$, for every $\mathrm{xx} \in \{\mathrm{ac}, \mathrm{sc}, \mathrm{pp}\}$. *Moreover, for every bounded borel function $f : \mathbb{R} \to \mathbb{C}$, $f(\mathcal{A})$ is a bounded operator on $\mathcal{E}_1$, $f(\mathcal{B})$ is a bounded operator on $\mathcal{E}_2$, we have $f(\mathcal{B}) = \mathcal{U} f(\mathcal{A})\mathcal{U}^*$.*

2.1. Spectral analysis I. In this subsection, we study the essential spectrum and discrete spectrum of the Lax operator L_u. The spectral analysis of L_u such that u is a multi-soliton in definition 1.1, will be continued in Sect. 4.2.

Definition 2.2. Given $u \in L^2(\mathbb{R}, \mathbb{R})$, its associated Lax operator L_u is an unbounded operator on L^2_+, given by $L_u := \mathrm{D} - T_u$, where $\mathrm{D} : h \in H^1_+ \mapsto -i\partial_x h \in L^2_+$ and T_u denotes the Toeplitz operator of symbol u, defined by $T_u : h \in H^1_+ \mapsto \Pi(uh) \in L^2_+$, where the Szegő projector $\Pi : L^2(\mathbb{R}) \to L^2_+$ is given by (1.20). If $u \in H^1(\mathbb{R}, \mathbb{R})$ in addition, we define $B_u := i(T_{|\mathrm{D}|u} - T_u^2) \in \mathfrak{B}(H^1_+, L^2_+)$.

Both D and T_u are densely defined symmetric operators on L^2_+ and $\|T_u(h)\|_{L^2} \leq \|u\|_{L^2}\|h\|_{L^\infty}$, for every $h \in H^1_+$ and $u \in L^2(\mathbb{R}, \mathbb{R})$. Moreover, the Fourier–Plancherel transform implies that D is a self-adjoint operator on L^2_+, whose domain of definition is H^1_+.

Proposition 2.3. *If $u \in L^2(\mathbb{R}, \mathbb{R})$, then L_u is an unbounded self-adjoint operator on L^2_+, whose domain of definition is $\mathbf{D}(L_u) = H^1_+$. Moreover, L_u is bounded from below. The essential spectrum of L_u is $\sigma_{\mathrm{ess}}(L_u) = \sigma_{\mathrm{ess}}(\mathrm{D}) = [0, +\infty)$ and its pure point spectrum satisfies $\sigma_{\mathrm{pp}}(L_u) \subset [-\frac{\|u\|^2_{L^2}}{4C^4}, +\infty)$, where $C = \inf_{f \in H^1_+ \setminus \{0\}} \frac{\||\mathrm{D}|^{\frac{1}{4}} f\|_{L^2}}{\|f\|_{L^4}}$ denotes the Sobolev constant.*

Proof. For every $h \in L^2_+$, let μ_h^{D} denote the spectral measure of D associated to h, then we have $\langle f(\mathrm{D})h, h\rangle_{L^2} = \int_0^{+\infty} f(\xi)\frac{|\hat{h}(\xi)|^2}{2\pi}\mathrm{d}\xi$, so $\mathrm{d}\mu_h^{\mathrm{D}}(\xi) = \frac{\mathbf{1}_{[0,+\infty)}(\xi)|\hat{h}(\xi)|^2}{2\pi}\mathrm{d}\xi$. Thus $\sigma(\mathrm{D}) = \sigma_{\mathrm{ess}}(\mathrm{D}) = \sigma_{\mathrm{ac}}(\mathrm{D}) = [0, +\infty)$. If $u \in L^2(\mathbb{R}, \mathbb{R})$, we claim that $\mathcal{P}_u := T_u \circ (\mathrm{D} + i)^{-1}$ is a Hilbert–Schmidt operator on L^2_+.

In fact, let $\mathscr{F} : h \in L^2_+ \mapsto \frac{\hat{h}}{\sqrt{2\pi}} \in L^2(\mathbb{R}_+^*)$ denotes the renormalized Fourier–Plancherel transform, then $\mathcal{A}_u := \mathscr{F} \circ \mathcal{P}_u \circ \mathscr{F}^{-1}$ is an operator on $L^2(\mathbb{R}_+^*)$. Then we have $\mathcal{A}_u g(\xi) = \int_0^{+\infty} K_u(\xi, \eta)g(\eta)\mathrm{d}\eta$, where $K_u(\xi, \eta) := \frac{\hat{u}(\xi-\eta)}{2\pi(\eta+i)}$, $\forall \xi, \eta \in \mathbb{R}_+^*$. So $\|\mathcal{A}_u\|_{\mathcal{HS}(L^2(\mathbb{R}_+^*))} \leq \|K\|_{L^2(\mathbb{R}_+^*\times\mathbb{R}_+^*)} \leq \frac{\|u\|_{L^2}}{2}$. Since $\mathcal{P}_u$ is unitarily equivalent to $\mathcal{A}_u$, we have $\|\mathcal{P}_u\|^2_{\mathcal{HS}(L^2_+)} = \sum_{\lambda\in\sigma(\mathcal{P}_u)} \lambda^2 = \sum_{\lambda\in\sigma(\mathcal{A}_u)} \lambda^2 = \|\mathcal{A}_u\|^2_{\mathcal{HS}(L^2(\mathbb{R}_+^*))} \leq \frac{\|u\|^2_{L^2}}{4}$.

Then the symmetric operator T_u is relatively compact with respect to D and Weyl's essential spectrum theorem (Theorem XIII.14 of Reed–Simon [19]) yields that $\sigma_{\mathrm{ess}}(L_u) = \sigma_{\mathrm{ess}}(\mathrm{D})$ and L_u is self-adjoint with $\mathbf{D}(L_u) = \mathbf{D}(\mathrm{D}) = H^1_+$. Moreover, $|\langle T_u f, f\rangle_{L^2}| = |\int_{\mathbb{R}} u|f|^2| \leq \|u\|_{L^2}\|f\|^2_{L^4} \leq C^{-2}\|u\|_{L^2}\|f\|_{L^2}\||\mathrm{D}|^{\frac{1}{2}} f\|_{L^2}$ holds by Sobolev embedding $\|f\|_{L^4} \leq C^{-1}\||\mathrm{D}|^{\frac{1}{4}} f\|_{L^2}$, for every $f \in H^1_+$. Then L_u is bounded from below, precisely $\langle L_u f, f\rangle_{L^2} = \||\mathrm{D}|^{\frac{1}{2}} f\|^2_{L^2} - \langle T_u f, f\rangle_{L^2} \geq -\frac{\|u\|^2_{L^2}\|f\|^2_{L^2}}{4C^4}$. When $\lambda < -\frac{\|u\|^2_{L^2}}{4C^4}$, the map $L_u - \lambda : H^1_+ \to L^2_+$ is injective. Hence $\sigma_{\mathrm{pp}}(L_u) \subset [-\frac{\|u\|^2_{L^2}}{4C^4}, +\infty)$. □

Proposition 2.4. *Assume that $u \in L^2(\mathbb{R}, (1 + x^2)\mathrm{d}x)$ and u is real-valued. For every $\lambda \in \mathbb{R}$ and $\varphi \in \mathrm{Ker}(\lambda - L_u)$, we have $\widehat{u\varphi} \in C^1(\mathbb{R}) \bigcap H^1(\mathbb{R})$ and the following identity holds,*

$$|\langle u, \varphi\rangle_{L^2}|^2 = -2\pi\lambda\|\varphi\|^2_{L^2}. \tag{2.3}$$

Thus $\sigma_{\mathrm{pp}}(L_u) \subset (-\infty, 0)$ and for every $\lambda \in \sigma_{\mathrm{pp}}(L_u)$, we have

$$\begin{aligned} \mathrm{Ker}(\lambda - L_u) \subset \{&\varphi \in H^1_+ : \hat{\varphi}_{|\mathbb{R}_+} \in C^1(\mathbb{R}_+) \bigcap H^1(\mathbb{R}_+) \quad \text{and} \\ &\xi \mapsto \xi[\hat{\varphi}(\xi) + \partial_\xi\hat{\varphi}(\xi)] \in L^2(\mathbb{R}_+)\}. \end{aligned} \tag{2.4}$$

Before the proof of proposition 2.4, we recall a lemma concerning the regularity of convolutions.

Lemma 2.5. *For any $p \in (1, +\infty)$, we have $W^{m,p}(\mathbb{R}) * W^{n,\frac{p}{p-1}}(\mathbb{R}) \subset C^{m+n}(\mathbb{R}) \bigcap W^{m+n,+\infty}(\mathbb{R})$, $\forall m, n \in \mathbb{N}$. For every $f \in W^{m,p}(\mathbb{R}) * W^{n,\frac{p}{p-1}}(\mathbb{R})$, we have $\lim_{|x|\to+\infty} \partial_x^\alpha f(x) = 0$, $\forall \alpha = 0, 1, \ldots, m + n$.*

Remark 2.6. Identity (2.3) was firstly found by Wu [24] in the case $\lambda < 0$. We show that (2.3) still holds in the case $\lambda \geq 0$. Hence the operator L_u has no eigenvalues in $[0, +\infty)$.

Proof of proposition 2.4. We choose $u \in L^2(\mathbb{R}, (1+x^2)\mathrm{d}x)$ such that $u(\mathbb{R}) \subset \mathbb{R}$, $\lambda \in \mathbb{R}$ and $\varphi \in L^2_+$ such that $L_u(\varphi) = \lambda\varphi$. Applying the Fourier–Plancherel transform, we obtain

$$\widehat{u\varphi}(\xi)\mathbf{1}_{[0,+\infty)}(\xi) = (\xi - \lambda)\hat{\varphi}(\xi) =: g_\lambda(\xi). \tag{2.5}$$

Since $\hat{u} \in H^1(\mathbb{R})$ and $\hat{\varphi} \in L^2(\mathbb{R})$, their convolution $\widehat{u\varphi} = \frac{1}{2\pi}\hat{u} * \hat{\varphi} \in C^1(\mathbb{R}) \bigcap C_0(\mathbb{R})$, where $C_0(\mathbb{R})$ denotes the uniform closure of $C_c(\mathbb{R})$ with respect to the $L^\infty(\mathbb{R})$-norm, by Lemma 2.5. We claim that if $\lambda < 0$, then $\hat{\varphi} \in C^1(\mathbb{R}_+)$; if $\lambda \geq 0$, then $\hat{\varphi} \in C(\mathbb{R}_+) \bigcap C^1(\mathbb{R}_+ \backslash \{\lambda\})$.

In fact, if $\lambda \geq 0$, we have $g_\lambda(\lambda) = 0$. Otherwise, λ would be a singular point of $\hat{\varphi}$ that prevents $\hat{\varphi}$ from being a L^2 function on $\mathbb{R}_+$, because $\xi \to \frac{1}{\xi - \lambda} \notin L^2(\mathbb{R}_+)$. By using the fact $g \in C^1(\mathbb{R}_+)$ (g is right differentiable at $\xi = 0$ and the derivative g' is right continuous at $\xi = 0$), we have

$$\hat{\varphi}(\xi) = \frac{g_\lambda(\xi) - g_\lambda(\lambda)}{\xi - \lambda} \to \begin{cases} g_\lambda'(\lambda), & \text{if} \quad \lambda > 0; \\ g_\lambda'(0^+), & \text{if} \quad \lambda = 0; \end{cases}$$

when $\xi \to \lambda$. So $\hat{\varphi} \in C(\mathbb{R}_+)$ and $\lim_{\xi \to +\infty} \hat{\varphi}(\xi) = 0$. Then we derive (2.5) with respect to ξ to get

$$-i\widehat{xu} * \hat{\varphi}(\xi) = g_\lambda'(\xi) = (\widehat{u\varphi})'(\xi) = \hat{\varphi}(\xi) + (\xi - \lambda)(\hat{\varphi})'(\xi), \qquad \forall \xi \in [0, +\infty) \backslash \{\lambda\}. \tag{2.6}$$

Thus we have

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}\xi}[(\xi - \lambda)|\hat{\varphi}(\xi)|^2] &= |\hat{\varphi}(\xi)|^2 + 2\mathrm{Re}[((\xi - \lambda)(\hat{\varphi})'(\xi))\overline{\hat{\varphi}}(\xi)] \\ &= 2\mathrm{Re}[(\widehat{u\varphi})'(\xi)\overline{\hat{\varphi}}(\xi)] - |\hat{\varphi}(\xi)|^2. \end{aligned} \tag{2.7}$$

• When $\lambda < 0$, it suffices to use the Plancherel formula $\int_0^{+\infty} (\widehat{u\varphi})'(\xi)\overline{\hat{\varphi}}(\xi)\mathrm{d}\xi = -2\pi i \int_{\mathbb{R}} xu(x)|\varphi(x)|^2 \mathrm{d}x$ and to integrate equation (2.7) on $[0, +\infty)$. Since $(\xi - \lambda)|\hat{\varphi}(\xi)|^2 = \widehat{u\varphi}(\xi)\overline{\hat{\varphi}}(\xi) \to 0$, as $\xi \to +\infty$, we have $\lambda|\hat{\varphi}(0)|^2 = \int_0^{+\infty} \frac{\mathrm{d}}{\mathrm{d}\xi}[(\xi - \lambda)|\hat{\varphi}(\xi)|^2]\mathrm{d}\xi = 4\pi \mathrm{Im} \int_{\mathbb{R}} xu(x)|\varphi(x)|^2\mathrm{d}x - \int_0^{+\infty} |\hat{\varphi}(\xi)|^2\mathrm{d}\xi = -2\pi \|\varphi\|_{L^2(\mathbb{R})}^2$.

• When $\lambda > 0$, there may be some problem of derivability of $\hat{\varphi}$ at $\xi = \lambda$. We replace the integral $\int_0^{+\infty}$ by two integrals $\int_0^{\lambda - \epsilon}$ and $\int_{\lambda+\epsilon}^{+\infty}$, for some $\epsilon \in (0, \lambda)$. We set $\mathcal{I}(\epsilon) := \lambda|\hat{\varphi}(0)|^2 - \epsilon|\hat{\varphi}(\lambda - \epsilon)|^2 - \epsilon|\hat{\varphi}(\lambda + \epsilon)|^2$, then $\mathcal{I}(\epsilon) = 2\mathrm{Re}\left(\int_0^{+\infty} (\widehat{u\varphi})'(\xi)\overline{\hat{\varphi}}(\xi)\mathrm{d}\xi - \int_{\lambda-\epsilon}^{\lambda+\epsilon} (\widehat{u\varphi})'(\xi)\overline{\hat{\varphi}}(\xi)\mathrm{d}\xi\right) - \int_0^{+\infty} |\hat{\varphi}(\xi)|^2\mathrm{d}\xi + \int_{\lambda-\epsilon}^{\lambda+\epsilon} |\hat{\varphi}(\xi)|^2\mathrm{d}\xi$. Thanks to the continuity of $\hat{\varphi}$ on $\mathbb{R}_+$, we have $\lambda|\hat{\varphi}(0)|^2 = \lim_{\epsilon \to 0^+} \mathcal{I}(\epsilon) = -2\pi \|\varphi\|_{L^2(\mathbb{R})}^2$.

• When $\lambda = 0$, we use the same idea and integrate (2.7) over interval $[\epsilon, +\infty)$, for some $\epsilon > 0$. Then $\mathcal{J}(\epsilon) := -\epsilon|\hat{\varphi}(\epsilon)|^2 = 2\mathrm{Re} \int_\epsilon^{+\infty} (\widehat{u\varphi})'(\xi)\overline{\hat{\varphi}}(\xi)\mathrm{d}\xi - \int_\epsilon^{+\infty} |\hat{\varphi}(\xi)|^2\mathrm{d}\xi \to 0$, as $\epsilon \to 0$.

So we always have $-2\pi \|\varphi\|_{L^2(\mathbb{R})}^2 = \lambda|\hat{\varphi}(0)|^2$, if $\varphi \in \mathrm{Ker}(\lambda - L_u)$. As a consequence L_u has only negative eigenvalues, if the real-valued function $u \in L^2(\mathbb{R}, (1+x^2)\mathrm{d}x)$. Finally we use $\widehat{u\varphi}(0) = -\lambda\hat{\varphi}(0)$ to get identity (2.3). If $\lambda \in \sigma_{\mathrm{pp}}(L_u)$ and $\varphi \in \mathrm{Ker}(\lambda - L_u) \backslash \{0\}$, we want to prove that

$$\xi \mapsto (1 + |\xi|)\partial_\xi \hat{\varphi}(\xi) \in L^2(0, +\infty). \tag{2.8}$$

In fact, since $\varphi \in H_+^1 \hookrightarrow L^\infty(\mathbb{R})$ and $u \in L^2(\mathbb{R}, (1 + x^2)\mathrm{d}x)$, we have $\widehat{u\varphi} = \frac{\hat{u} * \hat{\varphi}}{2\pi} \in H^1(\mathbb{R})$. Formula (2.5) yields that $\xi \mapsto (|\lambda| + \xi)\hat{\varphi}(\xi) \in L^2(\mathbb{R})$ and we have $\hat{\varphi} \in L^1(\mathbb{R})$. The hypothesis $u \in L^2(\mathbb{R}, x^2\mathrm{d}x)$ implies that the convolution term $\widehat{xu} * \hat{\varphi} \in L^2(\mathbb{R})$. Since $\lambda < 0$, we obtain (2.8) by using formula (2.6). □

Corollary 2.7. *Assume that $u \in L^2(\mathbb{R}, (1+x^2)\mathrm{d}x)$ and u is real-valued. Then every eigenvalue of L_u is simple. If $u \in L^\infty(\mathbb{R})$ in addition, then $\sigma_{\mathrm{pp}}(L_u)$ is a finite subset of $[-\frac{\|u\|_{L^2}^2}{4C^4}, 0)$.*

Proof. Fix $\lambda \in \sigma_{\mathrm{pp}}(L_u)$ and set $V_\lambda = \mathrm{Ker}(\lambda - L_u)$, then $\dim_{\mathbb{C}}(V_\lambda) \geq 1$. We define a linear form $A : V_\lambda \to \mathbb{C}$ such that $A(\varphi) := \int_{\mathbb{R}} u\varphi$. Then identity (2.3) yields that $\mathrm{Ker}(A) = \{0\}$. Thus we have $V_\lambda \cong V_\lambda/\mathrm{Ker}(A) \cong \mathrm{Im}(A) \hookrightarrow \mathbb{C}$. So $\dim_{\mathbb{C}}(V_\lambda) = 1$. When $u \in L^\infty(\mathbb{R})$ in addition, the finiteness of $\sigma_{\mathrm{pp}}(L_u)\bigcap(-\infty, 0)$ is given by Theorem 1.2 of Wu [24]. □

2.2. Lax pair formulation. We recall some known results of global well-posedness of the BO equation on the line.

Proposition 2.8 (Tao [23], Ionescu–Kenig [10], etc.). *Given $s \geq 0$, the Fréchet space $C(\mathbb{R}, H^s(\mathbb{R}))$ is endowed with the topology of uniform convergence on every compact subset of $\mathbb{R}$. There exists a unique continuous mapping $u_0 \in H^s(\mathbb{R}) \mapsto u \in C(\mathbb{R}, H^s(\mathbb{R}))$ such that u solves the BO equation (1.1) with initial datum $u(0) = u_0$.*

Proposition 2.9 (Ablowitz–Fokas [1], Coifman–Wickerhauser [5], etc.). *For every $n \in \mathbb{N} := \mathbb{Z}\bigcap[0, +\infty)$, if $u_0 \in H^{\frac{n}{2}}(\mathbb{R}, \mathbb{R})$, let $u : t \in \mathbb{R} \mapsto u(t) \in H^{\frac{n}{2}}(\mathbb{R}, \mathbb{R})$ solves equation (1.1) with initial datum $u(0) = u_0$, then we have $\mathcal{C}(\|u_0\|_{H^{\frac{n}{2}}}) := \sup_{t\in\mathbb{R}} \|u(t)\|_{H^{\frac{n}{2}}} < +\infty$.*

When $u \in H^2(\mathbb{R}, \mathbb{R})$, the Toeplitz operators $T_{|\mathrm{D}|u}$ and T_u are bounded both on L^2_+ and on H^1_+. So B_u is a bounded skew-adjoint operator both on L^2_+ and on H^1_+.

Proposition 2.10. *Let $u : t \in \mathbb{R} \mapsto u(t) \in H^2(\mathbb{R}, \mathbb{R})$ denote the unique solution of equation (1.1), then*

$$\partial_t L_{u(t)} = [B_{u(t)}, L_{u(t)}] \in \mathfrak{B}(H^1_+, L^2_+), \qquad \forall t \in \mathbb{R}. \tag{2.9}$$

The proof of proposition 2.10 can be found in Gérard–Kappeler [8], Wu [24] etc. In order to make this paper self contained, we recall it here.

Proof. Since $\frac{\mathrm{d}}{\mathrm{d}t}(L \circ u)(t) = -T_{\partial_t u(t)} = -T_{\mathrm{H}\partial_x^2 u(t) - \partial_x(u(t)^2)}$, it suffices to prove $[B_u, L_u] + T_{\mathrm{H}\partial_x^2 u - \partial_x(u^2)} = 0$ for every $u \in H^2(\mathbb{R}, \mathbb{R})$. In fact, we have $\hat{u}(-\xi) = \overline{\hat{u}(\xi)}$, $u = \Pi u + \overline{\Pi u}$ and $|\mathrm{D}|u = \mathrm{D}\Pi u - \mathrm{D}\overline{\Pi u}$. Since both T_u and B_u are bounded both $L^2_+ \to L^2_+$ and $H^1_+ \to H^1_+$, we have

$$\begin{aligned} [B_u, L_u]f &= -\Pi(f\partial_x|\mathrm{D}|u) + i\Pi[u\Pi(f|\mathrm{D}|u) - |\mathrm{D}|u\Pi(uf)] + \Pi[\partial_x u\Pi(uf) + u\Pi(f\partial_x u)] \\ &= -\Pi(f\mathrm{H}\partial_x^2 u) + \mathcal{I}_1 + \mathcal{I}_2 \in L^2_+, \end{aligned} \tag{2.10}$$

for every $f \in H^1_+$, where the terms $\mathcal{I}_1$ and $\mathcal{I}_2$ are given by

$$\begin{aligned} \mathcal{I}_1 &:= i\Pi[u\Pi(f|\mathrm{D}|u) - |\mathrm{D}|u\Pi(uf)] \\ &= \Pi[f\overline{\Pi u}\partial_x\Pi u + f\Pi u\partial_x\overline{\Pi u}] - \Pi u\Pi(f\partial_x\overline{\Pi u}) - \Pi(f\overline{\Pi u})\partial_x\Pi u + \Pi[\Pi(f\overline{\Pi u})\partial_x\overline{\Pi u} - \overline{\Pi u}\Pi(f\partial_x\overline{\Pi u})], \\ \mathcal{I}_2 &:= \Pi[\partial_x u\Pi(uf) + u\Pi(f\partial_x u)] = \Pi(f\overline{\Pi u})\partial_x\Pi u + \Pi u\Pi(f\partial_x\overline{\Pi u}) + \Pi(\overline{\Pi u}\Pi(f\partial_x\overline{\Pi u})) \\ &\quad + 2f\Pi u\partial_x\Pi u + \Pi[f\Pi u\partial_x\overline{\Pi u} + f\overline{\Pi u}\partial_x\Pi u + \Pi(f\overline{\Pi u})\partial_x\overline{\Pi u}]. \end{aligned}$$

Since $\partial_x \overline{\Pi u} \in L^2_-$, we have $\Pi[\Pi(f\overline{\Pi u})\partial_x\overline{\Pi u}] = \Pi[f\overline{\Pi u}\partial_x\overline{\Pi u}]$. Thus,

$$\mathcal{I}_1 + \mathcal{I}_2 = 2f\Pi u\partial_x\Pi u + 2\Pi[f\Pi u\partial_x\overline{\Pi u} + f\overline{\Pi u}\partial_x\Pi u + \Pi(f\overline{\Pi u})\partial_x\overline{\Pi u}] = \Pi[f\partial_x(u^2)] \in H^1_+. \tag{2.11}$$

Formulas (2.10) and (2.11) yield that $[B_u, L_u]f = \Pi[f(\partial_x(u^2) - \mathrm{H}\partial_x^2 u)]$. Thus equation (2.9) holds along the evolution of equation (1.1). $\square$

Remark 2.11. As indicated in Gérard–Kappeler [8], there are many choices of the operator B_u. We can replace B_u by any operator of the form $B_u + P_u$ such that P_u is a skew-adjoint operator commuting with L_u. For instance, we set $C_u := B_u + iL_u^2$ and we obtain $C_u = i\mathrm{D}^2 - 2i\mathrm{D}T_u + 2iT_{\mathrm{D}\Pi u}$. So (L_u, C_u) is also a Lax pair of the BO equation (1.1). The advantage of the operator $B_u = i(T_{|\mathrm{D}|u} - T_u^2)$ is that $B_u : L^2_+ \to L^2_+$ is bounded if u is sufficiently regular. For instance, $u \in H^2(\mathbb{R}, \mathbb{R})$.

Let $U : t \mapsto U(t) \in \mathfrak{B}(L^2_+) := \mathfrak{B}(L^2_+, L^2_+)$ denote the unique solution of the following equation

$$U'(t) = B_{u(t)}U(t), \qquad U(0) = \mathrm{Id}_{L^2_+}, \tag{2.12}$$

if $u : t \in \mathbb{R} \mapsto u(t) \in H^2(\mathbb{R}, \mathbb{R})$ denote the unique solution of equation (1.1). The system (2.12) is globally well-posed in $\mathfrak{B}(L^2_+)$, thanks to Proposition 2.9 and the following estimate

$$\|B_u(h)\|_{L^2} \lesssim (\|u\|_{H^2} + \|u\|^2_{H^1})\|h\|_{L^2}, \qquad \forall h \in L^2_+, \quad \forall u \in H^2(\mathbb{R}, \mathbb{R}).$$

Since $B_u^* = -B_u$, the operator $U(t)$ is unitary for every $t \in \mathbb{R}$. Thus, the Lax pair formulation (2.9) of the BO equation (1.1) is equivalent to $L_{u(t)} = U(t)L_{u(0)}U(t)^* \in \mathfrak{B}(H^1_+, L^2_+)$. On the one hand, the spectrum of L_u is invariant under the BO flow. On the other hand, there exists a sequence of conservation laws controlling every Sobolev norms $H^{\frac{n}{2}}(\mathbb{R})$, $n \geq 0$. Furthermore, the Lax operator in the Lax pair formulation is not unique. If $f \in L^\infty(\mathbb{R})$ and p is a polynomial with complex coefficients, then we have $f(L_{u(t)}) = U(t)f(L_{u(0)})U(t)^* \in \mathfrak{B}(L^2_+)$ and $p(L_{u(t)}) = U(t)p(L_{u(0)})U(t)^* \in \mathfrak{B}(H^N_+, L^2_+)$, where N is the degree of the polynomial p.

Proposition 2.12. *Given $n \in \mathbb{N}$, let $u : t \in \mathbb{R} \mapsto u(t) \in H^{\frac{n}{2}}(\mathbb{R}, \mathbb{R})$ solve equation (1.1), we set*

$$E_n(u) := \langle L_u^n \Pi u, \Pi u\rangle_{H^{-\frac{n}{2}}, H^{\frac{n}{2}}}. \tag{2.13}$$

Then $E_n(u(t)) = E_n(u(0))$, for every $t \in \mathbb{R}$. In particular, $E_1 = E$ on $H^{\frac{1}{2}}(\mathbb{R}, \mathbb{R})$, where the energy functional E is given by (1.3).

In order to prove Proposition 2.12, we need the following result.

Proposition 2.13. *If $u : t \in \mathbb{R} \mapsto u(t) \in H^2(\mathbb{R}, \mathbb{R})$ solve the BO equation (1.1), then we have*

$$\partial_t \Pi u(t) = B_{u(t)}(\Pi u(t)) + iL^2_{u(t)}(\Pi u(t)) \in L^2_+. \tag{2.14}$$

Proof. For every $u \in H^2(\mathbb{R},\mathbb{R})$, B_u is a bounded operator on both L^2_+ and H^1_+, $\Pi u \in \mathbf{D}(L_u) = H^1_+$. We have $\hat{u}(-\xi) = \overline{\hat{u}(\xi)}$, $u = \Pi u + \overline{\Pi u}$ and $|\mathrm{D}|u = \mathrm{D}\Pi u - \mathrm{D}\overline{\Pi u}$. Since $\mathrm{D}\overline{\Pi u} \in L^2_-$, we have $\Pi(\Pi u \mathrm{D}\overline{\Pi u}) = \Pi(u\mathrm{D}\overline{\Pi u})$. Thus the following two formulas hold,

$$\begin{aligned} B_u(\Pi u) &= i(T_{|\mathrm{D}|u} - T_u^2)(\Pi u) = i(\Pi u)(\mathrm{D}\Pi u) - i\Pi(u\mathrm{D}\overline{\Pi u}) - iT_u^2(\Pi u)\\ &= \Pi u\partial_x\Pi u - \Pi(u\partial_x\overline{\Pi u}) - iT_u^2(\Pi u),\\ iL_u^2(\Pi u) &= i\mathrm{D}^2\Pi u - iT_u(\mathrm{D}\Pi u) - i\mathrm{D}\circ T_u(\Pi u) + iT_u^2(\Pi u)\\ &= -i\partial_x^2\Pi u - T_u(\partial_x\Pi u) - \partial_x[T_u(\Pi u)] + iT_u^2(\Pi u). \end{aligned}$$

Then $B_u(\Pi u) + iL_u^2(\Pi u) = -i\partial_x^2\Pi u - 2\Pi[\Pi u\partial_x\Pi u + \Pi u\partial_x\overline{\Pi u} + \overline{\Pi u}\partial_x\Pi u]$. Finally we replace u by $u(t)$, where $u : t \in \mathbb{R} \mapsto u(t) \in H^2(\mathbb{R},\mathbb{R})$ solves equation (1.1) to obtain (2.14). □

Proof of proposition 2.12. It suffices to prove it in the case $u_0 \in H^\infty(\mathbb{R},\mathbb{R})$. Then we use the density argument and the continuity of the flow map $u_0 \in H^s(\mathbb{R}) \mapsto u \in C([-T,T]; H^s(\mathbb{R}))$ in proposition 2.8, where $\forall T > 0$, $s \geq 0$. We choose $u = u(t) \in H^\infty(\mathbb{R},\mathbb{R}) = \bigcap_{s\geq 0} H^s(\mathbb{R},\mathbb{R})$, so $L_u^n\Pi u$, $\partial_t\Pi u$ and $\partial_t(L_u^n)\Pi u = [B_u, L_u^n]\Pi u$ are in $H^\infty(\mathbb{R},\mathbb{C})$. Thus $\partial_t E_n(u) = 2\mathrm{Re}\langle L_u^n\Pi u, \partial_t\Pi u\rangle_{L^2} + \langle\partial_t(L_u^n)\Pi u, \Pi u\rangle_{L^2}$. Since $B_u + iL_u^2$ is skew-adjoint, we have $2\mathrm{Re}\langle L_u^n\Pi u, \partial_t\Pi u\rangle_{L^2} = \langle[L_u^n, B_u + iL_u^2]\Pi u, \Pi u\rangle_{L^2} = \langle[L_u^n, B_u]\Pi u, \Pi u\rangle_{L^2}$ by (2.14). Since (L_u^n, B_u) is also a Lax pair of (1.1), we have $\partial_t E_n(u) = \langle([L_u^n, B_u] + \partial_t(L_u^n))\Pi u, \Pi u\rangle_{L^2} = 0$. In the case $n = 1$, we assume that $u \in H^1(\mathbb{R},\mathbb{R})$. Since $u = \Pi u + \overline{\Pi u}$, $|\mathrm{D}|u = \mathrm{D}\Pi u - \mathrm{D}\overline{\Pi u}$ and $\int_{\mathbb{R}}(\Pi u)^3 = 0$, we have $\langle|\mathrm{D}|u, u\rangle_{L^2} = 2\langle\mathrm{D}\Pi u, \Pi u\rangle_{L^2}$ and $\int_{\mathbb{R}} u^3 = 3\int_{\mathbb{R}}(\Pi u + \overline{\Pi u})|\Pi u|^2 = 3\int_{\mathbb{R}} u|\Pi u|^2$. □

2.3. The generating functional. We introduce a new conservation law of the BO equation (1.1) that encodes the entire BO hierarchy.

Definition 2.14. Given $u \in L^2(\mathbb{R},\mathbb{R})$, $\lambda \in \mathbb{C}\backslash\sigma(-L_u)$, the generating functional of equation (1.1) is defined by $\mathcal{H}_\lambda(u) = \langle(L_u+\lambda)^{-1}\Pi u, \Pi u\rangle_{L^2}$. The subset $\mathcal{X} := \{(\lambda, u) \in \mathbb{R}\times L^2(\mathbb{R},\mathbb{R}) : 4C^4\lambda > \|u\|_{L^2}^2\}$ is open in $\mathbb{R}\times L^2(\mathbb{R},\mathbb{R})$, where the Sobolev constant is given by $C = \inf_{f\in H^1_+\backslash\{0\}} \frac{\||\mathrm{D}|^{\frac{1}{4}}f\|_{L^2}}{\|f\|_{L^4}}$.

Since $\sigma(L_u) \subset [-\frac{\|u\|_{L^2}^2}{4C^4}, +\infty)$, the map $(\lambda, u) \in \mathcal{X} \mapsto \mathcal{H}_\lambda(u) = \langle(L_u + \lambda)^{-1}\Pi u, \Pi u\rangle_{L^2} \in \mathbb{R}$ is real analytic.

Proposition 2.15. *Let $u : t \in \mathbb{R} \mapsto u(t) \in H^\infty(\mathbb{R},\mathbb{R})$ denote the solution of the BO equation (1.1) and we choose $\lambda \in \mathbb{C}\backslash\sigma(-L_{u(0)})$, then $\mathcal{H}_\lambda(u(t)) = \mathcal{H}_\lambda(u(0))$, for every $t \in \mathbb{R}$.*

Proof. Let $u : t \in \mathbb{R} \mapsto u(t) \in H^\infty(\mathbb{R},\mathbb{R})$ solve equation (1.1). Since $\sigma(-L_{u(t)}) = \sigma(-L_{u(0)})$ by Proposition 2.1, the operator $\lambda + L_{u(t)} \in \mathfrak{B}(H^1_+, L^2_+)$ is invertible and we have

$$\partial_t\mathcal{H}_\lambda(u) = 2\mathrm{Re}\langle(L_u + \lambda)^{-1}\Pi u, \partial_t\Pi u\rangle_{L^2} - \langle(L_u + \lambda)^{-1}\partial_t L_u(L_u + \lambda)^{-1}\Pi u, \Pi u\rangle_{L^2}. \tag{2.15}$$

Formula (2.14) yields that

$$
\begin{aligned}
&2\mathrm{Re}\langle (L_u+\lambda)^{-1}\Pi u, \partial_t \Pi u\rangle_{L^2} \\
&\quad = \langle [(L_u+\lambda)^{-1}, B_u + iL_u^2]\Pi u, \Pi u\rangle_{L^2} = \langle [(L_u+\lambda)^{-1}, B_u]\Pi u, \Pi u\rangle_{L^2}, \\
&\langle [(L_u+\lambda)^{-1}, B_u]\Pi u, \Pi u\rangle_{L^2} \\
&\quad = \langle (L_u+\lambda)^{-1}[B_u, L_u+\lambda](L_u+\lambda)^{-1}\Pi u, \Pi u\rangle_{L^2}.
\end{aligned}
$$

Then $\partial_t L_u = [B_u, L_u]$ yields that $\partial_t H_\lambda(u(t)) = 0$. $\square$

Given $(\lambda, u) \in \mathcal{X}$, there exists a neighbourhood of u in $L^2(\mathbb{R}, \mathbb{R})$, denoted by $\mathcal{V}_u$ such that the restriction $\mathcal{H}_\lambda : v \in \mathcal{V}_u \mapsto \mathcal{H}_\lambda(v) \in \mathbb{R}$ can be expressed by power series. Then the Fréchet derivative of $\mathcal{H}_\lambda$ at u is given by $\mathrm{d}\mathcal{H}_\lambda(u)(h) = \langle w_\lambda, \Pi h\rangle_{L^2} + \overline{\langle w_\lambda, \Pi h\rangle_{L^2}} + \langle T_h w_\lambda, w_\lambda\rangle_{L^2} = \langle h, w_\lambda + \overline{w}_\lambda + |w_\lambda|^2\rangle_{L^2}$, $\forall h \in L^2(\mathbb{R}, \mathbb{R})$, where $w_\lambda \in H^1_+$ is defined by $w_\lambda \equiv w_\lambda(u) \equiv w_\lambda(x, u) = [(L_u+\lambda)^{-1} \circ \Pi]u(x)$, $\forall x \in \mathbb{R}$. So

$$
\nabla_u \mathcal{H}_\lambda(u) = |w_\lambda(u)|^2 + w_\lambda(u) + \overline{w}_\lambda(u). \tag{2.16}
$$

Given $(\lambda, u_0) \in \mathcal{X}$ fixed, we consider the following equation

$$
\partial_t u = \partial_x \nabla_u \mathcal{H}_\lambda(u) = \partial_x \left(|w_\lambda(u)|^2 + w_\lambda(u) + \overline{w}_\lambda(u)\right), \qquad u(0) = u_0. \tag{2.17}
$$

There exists an open subset $\mathcal{V}_{u_0}$ of $L^2(\mathbb{R}, \mathbb{R})$ such that $v \in \mathcal{V}_{u_0} \mapsto \partial_x \left(|w_\lambda(v)|^2 + w_\lambda(v) + \overline{w}_\lambda(v)\right) \in L^2(\mathbb{R}, \mathbb{R})$ is real analytic and $u_0 \in \mathcal{V}_{u_0}$. Hence equation (2.17) admits a unique local $L^2(\mathbb{R}, \mathbb{R})$-solution by Cauchy–Lipschitz theorem.

Remark 2.16. In Sect. 4, we show that $u \in \mathcal{U}_N \mapsto \partial_x \nabla_u f(u) \in \mathcal{T}_u(\mathcal{U}_N)$ is exactly the Hamiltonian vector field of the smooth function $f : \mathcal{U}_N \to \mathbb{R}$ with respect to the symplectic form ω on the N-soliton manifold $\mathcal{U}_N$ defined in (1.11).

Proposition 2.17. *Given $(\lambda, u_0) \in \mathcal{X}$ fixed, there exists $\varepsilon > 0$ such that $(\lambda, u(t)) \in \mathcal{X}$, for every $t \in (-\varepsilon, \varepsilon)$, where $u : t \in (-\varepsilon, +\varepsilon) \mapsto u(t) \in L^2(\mathbb{R}, \mathbb{R})$ solves (2.17) with initial datum $u(0) = u_0$. Then*

$$
\partial_t L_{u(t)} = [B^\lambda_{u(t)}, L_{u(t)}], \quad \text{where} \quad B^\lambda_v := i(T_{w_\lambda(v)}T_{\overline{w}_\lambda(v)} + T_{w_\lambda(v)} + T_{\overline{w}_\lambda(v)}), \quad \text{if} \quad (\lambda, v) \in \mathcal{X}. \tag{2.18}
$$

Remark 2.18. For every $u \in H^\infty(\mathbb{R}, \mathbb{R})$ and $\epsilon \in (0, \frac{4C^4}{\|u\|^2_{L^2}})$, we set $\tilde{\mathcal{H}}_\epsilon(u) := \frac{1}{\epsilon}\mathcal{H}_{\frac{1}{\epsilon}}(u)$ and $\tilde{B}_{\epsilon,u} := \frac{1}{\epsilon}B_u^{\frac{1}{\epsilon}}$. Recall that $E_n(u) = \langle L_u^n \Pi u, \Pi u\rangle_{L^2}$, we have the following Taylor expansion

$$
\tilde{\mathcal{H}}_\epsilon(u) = \sum_{k=0}^{K} (-\epsilon)^n E_n(u) - (-\epsilon)^K \langle (L_u + \tfrac{1}{\epsilon})^{-1}\Pi u, L_u^K \Pi u\rangle_{L^2}, \quad \forall K \in \mathbb{N}. \tag{2.19}
$$

Then Proposition 2.17 leads to a Lax pair formulation for the equations corresponding to the conservation laws in the BO hierarchy, $\partial_t L_u = [\frac{\mathrm{d}^n}{\mathrm{d}\epsilon^n}\Big|_{\epsilon=0} \tilde{B}_{\epsilon,u}, L_u]$, where now u evolves according to the Hamiltonian flow of $E_n = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}\epsilon^n}\Big|_{\epsilon=0} \tilde{\mathcal{H}}_\epsilon$ with respect to the Gardner–Faddeev–Zakharov Poisson structure. In the case $n = 1$, we have $E_1 = E$ and $B_u = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \tilde{B}_{\epsilon,u}$.

Before proving Proposition 2.17, we introduce the Hankel operators of symbols in $L^2(\mathbb{R})\bigcup L^\infty(\mathbb{R})$. They are used to calculate the commutators of Toeplitz operators. We notice that the Hankel operators are $\mathbb{C}$-anti-linear and the Toeplitz operators are $\mathbb{C}$-linear. For every symbol $v \in L^2(\mathbb{R})\bigcup L^\infty(\mathbb{R})$, its associated Hankel operator is defined by $H_v(h) = T_{\overline{h}}v = \Pi(v\overline{h})$, $\forall h \in H^1_+$. If $v \in L^\infty(\mathbb{R})$, then $H_v : L^2_+ \to L^2_+$ is a bounded operator. If $v \in L^2(\mathbb{R})$, then H_v may be an unbounded operator on L^2_+ whose domain of definition contains H^1_+. For any $b \in H^1(\mathbb{R})$, $h \in H^1_+$, we have $\|T_b(h)\|_{H^1} + \|H_b(h)\|_{H^1} \lesssim \|b\|_{H^1}\|h\|_{H^1}$, so both T_b and H_b are bounded on L^2_+ and on H^1_+.

Lemma 2.19. *For every $v, w \in L^2_+\bigcap L^\infty(\mathbb{R})$ and $u \in L^2(\mathbb{R})$, we have*

$$[T_v, T_{\overline{w}}] = -H_v \circ H_w \in \mathfrak{B}(L^2_+). \tag{2.20}$$

If $w \in H^1_+$ in addition, then we have $T_u(w) \in L^2_+$ and

$$H_{T_u w} = T_w \circ H_{\Pi u} + H_w \circ T_{\overline{u}} = T_u \circ H_w + H_{\Pi u} \circ T_{\overline{w}} \in \mathfrak{B}(H^1_+, L^2_+). \tag{2.21}$$

Proof. For every $v, w \in L^2_+\bigcap L^\infty(\mathbb{R})$ and $h \in L^2_+$, we have $\overline{w}h = \Pi(\overline{w}h) + \overline{\Pi(w\overline{h})} \in L^2_+$. Thus, we have $[T_v, T_{\overline{w}}]h = \Pi(v\Pi(\overline{w}h) - \overline{w}\Pi(vh)) = \Pi(v\overline{w}h - v\overline{\Pi(w\overline{h})} - v\overline{w}h) = -\Pi(v\overline{\Pi(w\overline{h})}) = -H_v \circ H_w(h) \in L^2_+$. Given $u \in L^2(\mathbb{R})$ and $w \in H^1_+$, for every $h \in H^1_+$, we have $w\overline{h} = \Pi(w\overline{h}) + \overline{\Pi(\overline{w}h)} \in H^1(\mathbb{R})$ and $H_w(h), T_{\overline{w}}(h) \in H^1_+$. So $\Pi(u\overline{\Pi(\overline{w}h)}) = \Pi(\overline{\Pi(\overline{w}h)}\Pi u) = H_{\Pi u} \circ T_{\overline{w}}(h) \in L^2_+$ and we have

$$\begin{aligned} H_{T_u w}(h) &= \Pi(\Pi(uw)\overline{h}) = \Pi(uw\overline{h}) = \Pi(u\Pi(w\overline{h}) + u\overline{\Pi(\overline{w}h)}) \\ &= (T_u \circ H_w + H_{\Pi u} \circ T_{\overline{w}})(h) \in L^2_+. \end{aligned}$$

Similarly, we have $u\overline{h} = \Pi(u\overline{h}) + \overline{\Pi(\overline{u}h)} \in L^2(\mathbb{R})$ and $\Pi(u\overline{h}) = \Pi(\overline{h}\Pi u) = H_{\Pi u}(h) \in L^2_+$. Thus, we have $H_{T_u w}(h) = \Pi(wu\overline{h}) = \Pi(w\Pi(u\overline{h}) + w\overline{\Pi(\overline{u}h)}) = (T_w \circ H_{\Pi u} + H_w \circ T_{\overline{u}})(h) \in L^2_+$. $\square$

Lemma 2.20. *Given $(\lambda, u) \in \mathcal{X}$ given in Definition 2.14, set $w_\lambda(u) = (L_u + \lambda)^{-1} \circ \Pi(u) \in H^1_+$, then*

$$[\mathrm{D} - T_u, T_{w_\lambda(u)}T_{\overline{w}_\lambda(u)} + T_{w_\lambda(u)} + T_{\overline{w}_\lambda(u)}] = T_{\mathrm{D}[|w_\lambda(u)|^2 + w_\lambda(u) + \overline{w}_\lambda(u)]} \in \mathfrak{B}(H^1_+, L^2_+). \tag{2.22}$$

Proof. We use abbreviation $w_\lambda := w_\lambda(u) \in H^1_+$, then $\overline{w}_\lambda \in H^1_-$. If $f^+, g^+ \in H^1_+$ and $f^-, g^- \in H^1_-$, then $[T_{f^+}, T_{g^+}] = [T_{f^-}, T_{g^-}] = 0$, because for every $h \in L^2_+$, we have $T_{f^+}[T_{g^+}(h)] = f^+g^+h = T_{g^+}[T_{f^+}(h)]$ and $T_{f^-}[T_{g^-}(h)] = \Pi(f^-\Pi(g^-h)) = \Pi(f^-g^-h) = \Pi(g^-\Pi(f^-h)) = T_{g^-}[T_{f^-}(h)]$. Since $\Pi u \in L^2_+$ and $\overline{\Pi u} \in L^2_-$, we use Leibnitz's rule and formula (2.20) to obtain that

$$\begin{aligned} [\mathrm{D} - T_u, T_{w_\lambda} + T_{\overline{w}_\lambda}] &= T_{\mathrm{D}w_\lambda} + T_{\mathrm{D}\overline{w}_\lambda} - [T_u, T_{w_\lambda}] - [T_u, T_{\overline{w}_\lambda}] \\ &= T_{\mathrm{D}w_\lambda} + T_{\mathrm{D}\overline{w}_\lambda} - [T_{\overline{\Pi u}}, T_{w_\lambda}] - [T_{\Pi u}, T_{\overline{w}_\lambda}] \\ &= T_{\mathrm{D}w_\lambda} + T_{\mathrm{D}\overline{w}_\lambda} - H_{w_\lambda}H_{\Pi u} + H_{\Pi u}H_{w_\lambda}. \end{aligned} \tag{2.23}$$

Similarly, formula (2.20) implies that

$$\begin{aligned}[T_u, T_{w_\lambda} T_{\overline{w}_\lambda}] &= [T_u, T_{w_\lambda}]T_{\overline{w}_\lambda} + T_{w_\lambda}[T_u, T_{\overline{w}_\lambda}] \\ &= [T_{\overline{\Pi u}}, T_{w_\lambda}]T_{\overline{w}_\lambda} + T_{w_\lambda}[T_{\Pi u}, T_{\overline{w}_\lambda}] = H_{w_\lambda} H_{\Pi u} T_{\overline{w}_\lambda} - T_{w_\lambda} H_{\Pi u} H_{w_\lambda}.\end{aligned} \tag{2.24}$$

For every $h \in H^1_+$, since $\overline{w}_\lambda, \mathrm{D}\overline{w}_\lambda \in L^2_-$, we have

$$\begin{aligned}[\mathrm{D}, T_{\overline{w}_\lambda} T_{w_\lambda}]h &= [\mathrm{D}, T_{\overline{w}_\lambda}]T_{w_\lambda} h + T_{\overline{w}_\lambda}[\mathrm{D}, T_{w_\lambda}]h \\ &= T_{\mathrm{D}\overline{w}_\lambda}(T_{w_\lambda} h) + T_{\overline{w}_\lambda}(T_{\mathrm{D}w_\lambda} h) \\ &= \Pi[\mathrm{D}\overline{w}_\lambda \Pi(w_\lambda h) + \overline{w}_\lambda \Pi(\mathrm{D}w_\lambda h)] = \Pi[(w_\lambda \mathrm{D}\overline{w}_\lambda + \overline{w}_\lambda \mathrm{D}w_\lambda)h] \in L^2_+.\end{aligned}$$

So $[\mathrm{D}, T_{\overline{w}_\lambda} T_{w_\lambda}] = T_{\mathrm{D}|w_\lambda|^2} \in \mathfrak{B}(H^1_+, L^2_+)$. We use formula (2.20) and Leibnitz's Rule to obtain that

$$[\mathrm{D}, T_{w_\lambda} T_{\overline{w}_\lambda}] = [\mathrm{D}, T_{\overline{w}_\lambda} T_{w_\lambda}] - [\mathrm{D}, H^2_{w_\lambda}] = T_{\mathrm{D}|w_\lambda|^2} - H_{\mathrm{D}w_\lambda} H_{w_\lambda} + H_{w_\lambda} H_{\mathrm{D}w_\lambda} \tag{2.25}$$

Recall that $w_\lambda = (\lambda + L_u)^{-1}\Pi u$, then we have

$$\mathrm{D}w_\lambda = T_u(w_\lambda) - \lambda w_\lambda + \Pi u. \tag{2.26}$$

The formulas (2.21) and (2.26) imply the following two identities,

$$\begin{aligned}H_{\mathrm{D}w_\lambda} - T_{w_\lambda} H_{\Pi u} &= H_{T_u w_\lambda} - \lambda H_{w_\lambda} + H_{\Pi u} - T_{w_\lambda} H_{\Pi u} = H_{w_\lambda} T_u - \lambda H_{w_\lambda} + H_{\Pi u}, \\ H_{\mathrm{D}w_\lambda} - H_{\Pi u} T_{\overline{w}_\lambda} &= H_{T_u w_\lambda} - \lambda H_{w_\lambda} + H_{\Pi u} - H_{\Pi u} T_{\overline{w}_\lambda} = T_u H_{w_\lambda} - \lambda H_{w_\lambda} + H_{\Pi u}.\end{aligned} \tag{2.27}$$

We use formulas (2.24), (2.25) and (2.27) to get the following formula

$$\begin{aligned}&[\mathrm{D} - T_u, T_{w_\lambda} T_{\overline{w}_\lambda}] = T_{\mathrm{D}|w_\lambda|^2} - (H_{\mathrm{D}w_\lambda} - T_{w_\lambda} H_{\Pi u}) H_{w_\lambda} + H_{w_\lambda}(H_{\mathrm{D}w_\lambda} - H_{\Pi u} T_{\overline{w}_\lambda}) \\ =&T_{\mathrm{D}|w_\lambda|^2} - (H_{w_\lambda} T_u H_{w_\lambda} - \lambda H^2_{w_\lambda} + H_{\Pi u} H_{w_\lambda}) + (H_{w_\lambda} T_u H_{w_\lambda} - \lambda H^2_{w_\lambda} + H_{w_\lambda} H_{\Pi u}) \\ =&T_{\mathrm{D}|w_\lambda|^2} - H_{\Pi u} H_{w_\lambda} + H_{w_\lambda} H_{\Pi u}.\end{aligned} \tag{2.28}$$

At last, we combine formulas (2.23) and (2.28) to obtain formula (2.22). □

End of the proof of proposition 2.17. For every $u : t \mapsto u(t) \in L^2(\mathbb{R}, \mathbb{R})$ solving equation (2.17), we have $\frac{\mathrm{d}}{\mathrm{d}t}(L \circ u)(t) = -T_{\partial_t u(t)} = -i T_{\mathrm{D}(w_\lambda(u(t))\overline{w}_\lambda(u(t)) + w_\lambda(u(t)) + \overline{w}_\lambda(u(t)))}$. Consequently, the Lax equation (2.18) is obtained by identity (2.22) in Lemma 2.20. □

3. The Action of the Shift Semigroup

In this section, we introduce the semigroup of shift operators $(S(\eta)^*)_{\eta \geq 0}$ acting on the Hardy space L^2_+ and classify all finite-dimensional translation-invariant subspaces of L^2_+. For every $\eta \geq 0$, we define the operator $S(\eta) : L^2_+ \to L^2_+$ such that $S(\eta)f = e_\eta f$, where $e_\eta(x) = e^{i\eta x}$. Then, its adjoint is given by $S(\eta)^* = T_{e_{-\eta}}$. We have $S(\eta)^* \circ L_u \circ S(\eta) = L_u + \eta \mathrm{Id}_{L^2_+}$, $\forall \eta \geq 0$. Since $\|S(\eta)^*\|_{\mathfrak{B}(L^2_+)} = \|S(\eta)\|_{\mathfrak{B}(L^2_+)} = 1$, $(S(\eta)^*)_{\eta \geq 0}$

is a contraction semi-group. Let $-iG$ denote its infinitesimal generator, i.e. $Gf = i\frac{\mathrm{d}}{\mathrm{d}\eta}\big|_{\eta=0^+} S(\eta)^* f \in L^2_+, \forall f \in \mathbf{D}(G)$, where

$$\mathbf{D}(G) := \{f \in L^2_+ : \hat{f}_{|\mathbb{R}_+} \in H^1(0, +\infty)\}, \tag{3.1}$$

because $\lim_{\epsilon\to 0} \|\frac{\psi - \tau_\epsilon \psi}{\epsilon} - \partial_x \psi\|_{L^2(0,+\infty)} = 0$, where $\tau_\epsilon \psi(x) = \psi(x - \epsilon)$ and $\psi \in H^1(0, +\infty)$. Every function $f \in \mathbf{D}(G)$ has bounded Hölder continuous Fourier transform by Morrey's inequality and Sobolev extension operator yields the existence of $\hat{f}(0^+) := \lim_{\xi\to 0^+} \hat{f}(\xi)$. The operator G is densely defined and closed. The Fourier transform of Gf is given by

$$\widehat{Gf}(\xi) = i\partial_\xi \hat{f}(\xi), \qquad \forall f \in \mathbf{D}(G), \qquad \forall \xi > 0. \tag{3.2}$$

The Hille–Yosida theorem implies that $(-\infty, 0) \subset \rho(iG)$ and $\|(G - \lambda i)^{-1}\|_{\mathfrak{B}(L^2_+)} \leq \lambda^{-1}, \forall \lambda > 0$.

Lemma 3.1. *For every $b \in L^2(\mathbb{R}) \bigcap L^\infty(\mathbb{R})$, we have $T_b(\mathbf{D}(G)) \subset \mathbf{D}(G)$ and the following identity*

$$[G, T_b]\varphi = \tfrac{i\hat{\varphi}(0^+)}{2\pi} \Pi b \tag{3.3}$$

holds for every $\varphi \in \mathbf{D}(G)$.

Proof. For every $\eta > 0$ and $\varphi \in \mathbf{D}(G)$, both $S(\eta)^*$ and T_b are bounded operators on L^2_+, so we have $\frac{1}{\eta}\left([S(\eta)^*, T_b]\varphi\right)^\wedge(\xi) = \frac{1}{2\pi\eta}\Big(\hat{b} * \hat{\varphi}(\xi + \eta) - \hat{b} * (\mathbf{1}_{\mathbb{R}_+}(\tau_{-\eta}\hat{\varphi}))(\xi)\Big) = \frac{1}{2\pi\eta}\int_\xi^{\xi+\eta} \hat{b}(\zeta)\hat{\varphi}(\xi + \eta - \zeta)\mathrm{d}\zeta, \forall \xi > 0$, where $\tau_{-\eta}\hat{\varphi}(x) = \hat{\varphi}(x + \eta), \forall x \in \mathbb{R}$. Then we change the variable $\zeta = \xi + t\eta$, for $0 \leq t \leq 1$,

$$\tfrac{1}{\eta}\Big([S(\eta)^* - \mathrm{Id}_{L^2_+}, T_b]\varphi\Big)^\wedge(\xi) = \frac{1}{2\pi}\int_0^1 \hat{b}(\xi + t\eta)\hat{\varphi}((1-t)\eta)\mathrm{d}t = a_\eta \widehat{b}(\xi) + \widehat{\phi_\eta}(\xi), \quad \forall \xi > 0, \tag{3.4}$$

where $a_\eta := \frac{1}{2\pi}\int_0^1 \hat{\varphi}((1-t)\eta)\mathrm{d}t \in \mathbb{C}$ and $\phi_\eta \in L^2_+$ such that $\widehat{\phi_\eta}(\xi) := \frac{1}{2\pi}\int_0^1 [\hat{b}(\xi + t\eta) - \hat{b}(\xi)]\hat{\varphi}((1-t)\eta)\mathrm{d}t, \forall \xi > 0$. Since $\hat{\varphi}|_{\mathbb{R}_+} \in H^1(0, +\infty)$, $\hat{\varphi}$ is bounded and $\lim_{\eta\to 0^+} \hat{\varphi}(\eta) = \hat{\varphi}(0^+)$, Lebesgue's dominated convergence theorem yields that $\lim_{\eta\to 0^+} a_\eta = \frac{\hat{\varphi}(0^+)}{2\pi}$. Since $b \in L^2(\mathbb{R})$, we have $\lim_{\epsilon\to 0}\|\tau_\epsilon \hat{b} - \hat{b}\|_{L^2} = 0$. So $\|\phi_\eta\|^2_{L^2} \lesssim \|\hat{\varphi}\|^2_{L^\infty}\int_0^1\int_0^{+\infty} |\hat{b}(\xi + t\eta) - \hat{b}(\xi)|^2 \mathrm{d}\xi\mathrm{d}t = \|\hat{\varphi}\|^2_{L^\infty}\int_0^1 \|\tau_{-t\eta}\hat{b} - \hat{b}\|^2_{L^2}\mathrm{d}t \to 0$, if $\eta \to 0^+$. Thus (3.4) implies that $\frac{1}{\eta}[S(\eta)^* - \mathrm{Id}_{L^2_+}, T_b]\varphi = a_\eta \Pi b + \phi_\eta \to \frac{\hat{\varphi}(0^+)}{2\pi}\Pi b$ in L^2_+, when $\eta \to 0^+$. Since $\varphi \in \mathbf{D}(G)$ and T_b is bounded, we have $\frac{i}{\eta}T_b[(S(\eta)^* - \mathrm{Id}_{L^2_+})\varphi] \to (T_b G)\varphi$ in L^2_+, when $\eta \to 0^+$. Consequently, $\frac{i}{\eta}(S(\eta)^* - \mathrm{Id}_{L^2_+})(T_b\varphi) \to (T_b G)\varphi + \frac{i\hat{\varphi}(0^+)}{2\pi}\Pi b$ in L^2_+ when $\eta \to 0^+$. So $T_b\varphi \in \mathbf{D}(G)$ and (3.3) holds. □

The following scalar representation theorem discovered by Lax in [13] allows to classify all translation-invariant subspaces of the Hardy space L^2_+, which plays the same role as the Beurling's theorem in the case of Hardy space on the circle.

Theorem 3.2. *(Lax) Every nonempty closed subspace of L^2_+ that is invariant under the semigroup of shift operators $(S(\eta))_{\eta\geq 0}$ is of the form ΘL^2_+, where Θ is a holomorphic function on the upper-half plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$. We have $|\Theta(z)| \leq 1$, for all $z \in \mathbb{C}_+$ and $|\Theta(x)| = 1, \forall x \in \mathbb{R}$. Moreover, Θ is uniquely determined up to multiplication by a complex constant of absolute value* 1.

The following lemma classifies all finite-dimensional subspaces that are invariant under the semi-group $(S(\eta)^*)_{\eta\geq 0}$, which is a weak version of Theorem 3.2.

Lemma 3.3. *Let M be a subspace of $D(G) \subset L^2_+$ of finite dimension $N = \dim_{\mathbb{C}} M \geq 1$ and $G(M) \subset M$. Then there exists a unique monic polynomial $Q \in \mathbb{C}_N[X]$ such that $Q^{-1}(0) \subset \mathbb{C}_-$ and $M = \frac{\mathbb{C}_{\leq N-1}[X]}{Q}$. Moreover, Q is the characteristic polynomial of the operator $G|_M$.*

Proof. We set $\hat{M} = \{\hat{f} \in L^2(0, +\infty) : f \in M\}$, then $\dim_{\mathbb{C}} \hat{M} = N$. Since $\widehat{Gf} = i\partial_\xi \hat{f}$ on $\mathbb{R}\backslash\{0\}$, the restriction $G|_M$ is unitarily equivalent to $i\partial_\xi|_{\hat{M}}$ by the renormalized Fourier–Plancherel transformation. So the characteristic polynomial $Q \in \mathbb{C}_N[X]$ of $i\partial_\xi|_{\hat{M}}$ is well defined, let $\{\beta_1, \beta_2, \ldots, \beta_n\} \subset \mathbb{C}$ denote the distinct roots of Q and m_j denotes the multiplicity of β_j, we have $\sum_{j=1}^n m_j = N$ and there exist $c_0, c_1, \ldots, c_{N-1} \in \mathbb{C}$ such that $Q(z) = \det(z - i\partial_\xi|_{\hat{M}}) = \prod_{j=1}^n (z-\beta_j)^{m_j} = z^N + \sum_{k=0}^{N-1} c_k z^k$. The Cayley–Hamilton theorem implies that $Q(i\partial_\xi) = 0$ on the subspace $\hat{M}$. If $\psi \in \hat{M} \subset L^2(0, +\infty)$, then ψ is a weak-solution of the following differential equation

$$i^{-N} Q(-\mathrm{D})\psi = \partial_\xi^N \psi + \sum_{k=0}^{N-1} i^{k-N} c_k \partial_\xi^k \psi = 0 \quad \text{on} \quad (0, +\infty), \qquad \psi \equiv 0 \quad \text{on} \quad (-\infty, 0), \tag{3.5}$$

where $\mathrm{D} = -i\partial_\xi$. The differential operator $Q(-\mathrm{D})$ is elliptic on the open interval $(0, +\infty)$ i.e. the symbol of the principal part of $Q(-\mathrm{D})$, denoted by $a_Q : (x, \xi) \in (0, +\infty) \times \mathbb{R} \mapsto (-\xi)^N$, does not vanish except for $\xi = 0$. So ψ is a smooth function on $(0, +\infty)$. The solution space

$$\mathrm{Sol}(3.5) = \mathrm{Span}_{\mathbb{C}}\{\hat{f}_{j,l}\}_{0\leq l\leq m_j-1, 1\leq j\leq n}, \qquad \hat{f}_{j,l}(\xi) = \xi^l e^{-i\beta_j\xi} \mathbf{1}_{\mathbb{R}_+}, \tag{3.6}$$

has complex dimension $\sum_{j=1}^n m_j = N$, so we have $\mathrm{Sol}(3.5) = \hat{M} \subset L^2_+$ and $\mathrm{Im}\beta_j < 0$, $\forall j = 1, 2, \ldots, N$. At last, we have $M = \mathrm{Span}_{\mathbb{C}}\{f_{j,l}\}_{0\leq l\leq m_j-1, 1\leq j\leq n} = \frac{\mathbb{C}_{\leq N-1}[X]}{Q}$, where $f_{j,l}(x) = \frac{l!}{2\pi[(-i)(x-\beta_j)]^{l+1}}$, $\forall x \in \mathbb{R}$. The uniqueness is obtained by identifying all the roots. $\square$

Lemma 3.4. *For every monic polynomial $Q \in \mathbb{C}_N[X]$ such that $Q^{-1}(0) \subset \mathbb{C}_-$, the associated inner function is defined by $\Theta = \Theta_Q = \frac{\overline{Q}}{Q}$. The following identity holds for every $\varphi \in \frac{\mathbb{C}_{\leq N-1}[X]}{Q}$,*

$$\hat{\varphi}(\xi) = \langle S(\xi)^*\varphi, 1 - \Theta\rangle_{L^2}, \quad \forall \xi > 0. \tag{3.7}$$

In particular, $\hat{\varphi}(0^+) = \langle \varphi, 1 - \Theta\rangle_{L^2}$.

Proof. Formula (3.6) yields that $\frac{\mathbb{C}_{\leq N-1}[X]}{Q} \subset \mathbf{D}(G)$, $G(\frac{\mathbb{C}_{\leq N-1}[X]}{Q}) \subset \frac{\mathbb{C}_{\leq N-1}[X]}{Q}$ and $\hat{\varphi} \in C^1(\mathbb{R}_+^*)$, for any $\varphi \in \frac{\mathbb{C}_{\leq N-1}[X]}{Q}$. Set $\varphi = \frac{P}{Q}$, for some $P \in \mathbb{C}_{\leq N-1}[X]$, then we have $\overline{\Theta}\varphi = \frac{Q}{\overline{Q}}\frac{P}{Q} = \frac{P}{\overline{Q}} \in L^2_-$. Since $Q(X) = \prod_{j=1}^N (X - \beta_j)$, $\mathrm{Im}\beta_j < 0$, we have $\Theta(x) = 1 + 2i\sum_{j=1}^N \frac{\mathrm{Im}\beta_j}{x-\beta_j} + \mathcal{O}(\frac{1}{x^2})$, when $x \to +\infty$, so $1 - \Theta \in L^2_+$. As a consequence, we have $\hat{\varphi}(\xi) = \int_{\mathbb{R}} \varphi(y)(1 - \overline{\Theta(y)})e^{-iy\xi}\mathrm{d}y = \langle S(\xi)^*\varphi, 1 - \Theta\rangle_{L^2}$, $\forall \xi > 0$. $\square$

4. The Manifold of Multi-solitons

This section is dedicated to a geometric description of every multi-soliton subset given in definition 1.1. Then we give a spectral characterization for the real analytic symplectic manifold $\mathcal{U}_N$ in order to prove the global well-posedness of the BO equation (1.1) on $\mathcal{U}_N$.

4.1. Differential structure and symplectic structure. The real analytic structure of $\mathcal{U}_N$ is constructed at first.

Proof of proposition 1.2. The set $\mathcal{V}_N := \{Q \in \mathbb{C}_N[X] : Q^{-1}(0) \subset \mathbb{C}_-, \lim_{x\to+\infty} \frac{Q(x)}{x^N} = 1\}$ is identified as $\mathbf{V}(\mathbb{C}_-^N)$, where $\mathbf{V} : (\beta_1, \beta_2, \ldots, \beta_N) \in \mathbb{C}^N \mapsto (a_0, a_1, \ldots, a_{N-1}) \in \mathbb{C}^N$ denotes the Viète map, defined by

$$\prod_{j=1}^{N}(X - \beta_j) = \sum_{k=0}^{N-1} a_k X^k + X^N. \tag{4.1}$$

Recall that $\mathbf{V} : \mathbb{C}^N \to \mathbb{C}^N$ is a both open and closed quotient map. For any open simply connected subset $A \subset \mathbb{C}^N$, if A is saturated with respect to $\mathbf{V}$ and $A \bigcap \Delta \neq \emptyset$ with $\Delta := \{(\beta, \beta, \ldots, \beta) \in \mathbb{C}^N : \forall \beta \in \mathbb{C}\}$, then $\mathbf{V}(A)$ is an open simply connected subset of $\mathbb{C}^N$. With the subspace topology of $\mathbb{C}^N$ and the Hermitian form $\mathfrak{H}_{\mathbb{C}^N}(X, Y) = X^T\overline{Y}$, the subset $(\mathbf{V}(\mathbb{C}_-^N), \mathfrak{H}_{\mathbb{C}^N})$ is a simply connected Kähler manifold of complex dimension N. The map $\Gamma_N : (a_0, a_1, \ldots, a_{N-1}) \in \mathbf{V}(\mathbb{C}_-^N) \mapsto \Pi u = i\frac{Q'}{Q} \in L^2_+$, where Q is given by $Q(X) = \sum_{k=0}^{N-1} a_k X^k + X^N$, is both a holomorphic immersion and a topological embedding. So $\Pi(\mathcal{U}_N) = \Gamma_N \circ \mathbf{V}(\mathbb{C}_-^N)$ is an embedded complex analytic submanifold of L^2_+ and $\dim_{\mathbb{C}}(\Pi(\mathcal{U}_N)) = N$. The map $\Gamma_N : \mathbf{V}(\mathbb{C}_-^N) \to \Pi(\mathcal{U}_N)$ is a biholomorphism and $\mathcal{T}_{\Pi u}(\Pi(\mathcal{U}_N)) = \bigoplus_{z\in\mathbf{P}(u)} \mathbb{C}^{\mathbf{m}(z)}\boldsymbol{\phi}_z$, where $\boldsymbol{\phi}_z(x) = (x-z)^{-2}$, $\forall z \in \mathbf{P}(u)$, $\forall u \in \mathcal{U}_N$. The proof is completed by using the isometry property of the $\mathbb{R}$-linear isomorphism $\sqrt{2}\Pi : u \in L^2(\mathbb{R}, \mathbb{R}) \mapsto \sqrt{2}\Pi u \in L^2_+$. In fact, we have $2\mathrm{Re}|_{L^2_+} = (\Pi_{|L^2(\mathbb{R},\mathbb{R})})^{-1}$ and $\|u\|_{L^2} = \sqrt{2}\|\Pi u\|_{L^2}$. $\square$

We set $\mathcal{E} := L^2(\mathbb{R}, \mathbb{R}) \bigcap L^2(\mathbb{R}, x^2\mathrm{d}x)$, $\mathcal{E}_c := \{u \in \mathcal{E} : \int_{\mathbb{R}} u = c\}$, $\forall c \in \mathbb{R}$. Then we have $\mathcal{U}_N \subset \mathcal{E}_{2\pi N}$, $\mathcal{T}_u(\mathcal{U}_N) \subset \mathcal{E}_0 = \mathcal{T}$ defined in (1.10), $\forall u \in \mathcal{U}_N$. Moreover, $\mathcal{T}$ is included in $\mathcal{W} = \partial_x(H^1(\mathbb{R}, \mathbb{R}))$, which is defined in (1.4), thanks to the following lemma.

Lemma 4.1 (Hardy). *For every $f \in H^1(\mathbb{R})$ such that $f(0) = 0$, we have $\int_{\mathbb{R}} \frac{|f(x)|^2}{|x|^2}\mathrm{d}x \leq 4\|\partial_x f\|^2_{L^2}$.*

So the 2-form ω in (1.11) is well defined. Then we show that ω is a real analytic symplectic form on $\mathcal{U}_N$.

Proof of proposition 1.3. Given any smooth vector field $X \in \mathfrak{X}(\mathcal{U}_N)$, let $X \lrcorner \omega \in \mathbf{\Omega}^1(\mathcal{U}_N)$ denote the interior multiplication by X, i.e. $(X \lrcorner \omega)(Y) = \omega(X, Y)$, for every $Y \in \mathfrak{X}(\mathcal{U}_N)$. The first step is prove that $\mathrm{d}\omega = 0$ on $\mathcal{U}_N$ by using the following Cartan's formula:

$$\mathscr{L}_X \omega = X \lrcorner(\mathrm{d}\omega) + \mathrm{d}(X \lrcorner \omega). \tag{4.2}$$

Let ϕ denote the smooth maximal flow of X. If t is sufficiently close to 0, then $\phi_t : u \in \mathcal{U}_N \mapsto \phi(t,u) \in \mathcal{U}_N$ is a local diffeomorphism by the fundamental theorem on flows. For any $u \in \mathcal{U}_N$, $h_1, h_2 \in \mathcal{T}_u(\mathcal{U}_N)$, we compute the Lie derivative of ω with respect to X,

$$(\mathscr{L}_X\omega)_u(h_1,h_2) = \lim_{t\to 0} \frac{\omega_{\phi_t(u)}(\mathrm{d}\phi_t(u)h_1, \mathrm{d}\phi_t(u)h_2) - \omega_u(h_1,h_2)}{t}$$
$$= \lim_{t\to 0} \boldsymbol{\omega}\left(\frac{\mathrm{d}\phi_t(u)h_1 - h_1}{t}, \mathrm{d}\phi_t(u)h_2\right) + \lim_{t\to 0} \boldsymbol{\omega}\left(h_1, \frac{\mathrm{d}\phi_t(u)h_2 - h_2}{t}\right).$$

So $(\mathscr{L}_X\omega)_u(h_1,h_2) = (h_1\omega(X,h_2))(u) - (h_2\omega(X,h_1))(u)$. We choose (V, x^i) a smooth local chart for $\mathcal{U}_N$ such that $u \in V$ and the tangent vector h_k has the coordinate expression $h_k = \sum_{j=1}^{2N} h_k^{(j)} \frac{\partial}{\partial x^j}\big|_u$, for some $h_k^{(j)} \in \mathbb{R}$, $j = 1, 2\ldots, 2N$ and $k = 1, 2$. The tangent vector h_k can be identified as some locally constant vector field $Y_k \in \mathfrak{X}(\mathcal{U}_N)$, which is defined by $Y_k : v \in V \mapsto \sum_{j=1}^{2N} h_k^{(j)} \frac{\partial}{\partial x^j}\big|_v \in \mathcal{T}_v(\mathcal{U}_N)$, $Y_k : u \mapsto (Y_k)_u = h_k$, $\forall k = 1, 2$. Then the vector field $[Y_1, Y_2]$ vanishes in the open subset V. The exterior derivative of the 1-form $\beta = X \lrcorner \omega$ is computed as $\mathrm{d}\beta(Y_1, Y_2) = Y_1(\beta(Y_2)) - Y_2(\beta(Y_1)) + \beta([Y_1, Y_2])$. Thus $(\mathrm{d}(X\lrcorner\omega))_u(h_1,h_2) = (\mathscr{L}_X\omega)_u(h_1,h_2)$. Then Cartan's formula (4.2) yields that $X\lrcorner(\mathrm{d}\omega) = 0$. Since $X \in \mathfrak{X}(\mathcal{U}_N)$ is arbitrary, we have $\mathrm{d}\omega = 0$.

Given $u \in \mathcal{U}_N$, we claim that the linear map $\Upsilon_u^\omega : h \in \mathcal{T}_u(\mathcal{U}_N) \mapsto h\lrcorner\omega_u \in \mathcal{T}_u^*(\mathcal{U}_N)$ is injective.

In fact, for any $h \in \mathrm{Ker}\Upsilon_u^\omega$, we define $h^\sharp := 2\mathrm{Re}(i\Pi h) \in \mathcal{T}_u(\mathcal{U}_N)$. Then the second expression of (1.11) yields that $0 = (h\lrcorner\omega_u)(h^\sharp) = \int_0^{+\infty} \frac{|\hat{h}(\xi)|^2}{\pi\xi}\mathrm{d}\xi$ and hence $h = 2\mathrm{Re}\circ\Pi(h) = 0$. So ω is nondegenerate and it is a real analytic symplectic form on $\mathcal{U}_N$. For any smooth function $f : \mathcal{U}_N \to \mathbb{R}$, its Hamiltonian vector field $X_f \in \mathfrak{X}(\mathcal{U}_N)$ is given by $X_f(u) := -(\Upsilon_u^\omega)^{-1}(\mathrm{d}f(u))$. Since $\mathrm{d}f(u)(h) = \langle h, \nabla_u f(u)\rangle_{L^2} = \frac{i}{2\pi}\int_{\mathbb{R}} \frac{\hat{h}(\xi)}{\xi}\overline{i\xi(\nabla_u f(u))^\wedge(\xi)}\mathrm{d}\xi$, $\forall h \in \mathcal{T}_u(\mathcal{U}_N)$, formula (1.12) is obtained. □

Corollary 4.2. *Endowed with Hermitian form $\mathfrak{H}$, which is defined by* $\mathfrak{H}_{\Pi u}(h_1,h_2) := \int_0^{+\infty} \frac{\hat{h}_1(\xi)\overline{\hat{h}_2(\xi)}}{\pi\xi}\mathrm{d}\xi$, $\forall h_1, h_2 \in \mathcal{T}_{\Pi u}(\Pi(\mathcal{U}_N))$, $\forall u \in \mathcal{U}_N$, $(\Pi(\mathcal{U}_N), \mathfrak{H})$ *is a Kähler manifold and* $\omega = -\Pi^*(\mathrm{Im}\mathfrak{H})$.

4.2. Spectral analysis II. We continue to study the spectrum of the Lax operator L_u introduced in Definition 2.2. The general case $u \in \mathcal{E} = L^2(\mathbb{R},\mathbb{R}) \bigcap L^2(\mathbb{R}, x^2\mathrm{d}x)$ has been studied in Sect. 2.1. We restrict our study to the case $u \in \mathcal{U}_N$ in this subsection. The operator L_u has the following spectral decomposition

$$L_+^2 = \mathscr{H}_{\mathrm{ac}}(L_u) \bigoplus \mathscr{H}_{\mathrm{sc}}(L_u) \bigoplus \mathscr{H}_{\mathrm{pp}}(L_u). \tag{4.3}$$

Let Q_u denote the characteristic polynomial of u given by (1.9) and $\Theta_u := \Theta_{Q_u} = \frac{\overline{Q_u}}{Q_u}$ denotes the inner function on $\mathbb{C}_+$ associated to Q_u. We have $S(\eta)[\Theta_u h] = \Theta_u[S(\eta)h]$, $\forall h \in L_+^2$, so $\Theta_u L_+^2$ is a closed subspace of L_+^2 that is invariant under the semigroup $(S(\eta))_{\eta\geq 0}$ in section 3. Set $K_{\Theta_u} := (\Theta_u L_+^2)^\perp$. Thus,

$$L_+^2 = \Theta L_+^2 \bigoplus K_{\Theta_u}, \quad S(\eta)^*(K_{\Theta_u}) \subset K_{\Theta_u} \quad \text{and} \quad G(\mathbf{D}(G) \bigcap K_{\Theta_u}) \subset K_{\Theta_u}. \tag{4.4}$$

where G is defined in (3.2). The following proposition identifies the subspaces in (4.3) and (4.4).

Proposition 4.3. *If $u \in \mathcal{U}_N$, then L_u has exactly N simple negative eigenvalues and we have*

$$\mathscr{H}_{\mathrm{ac}}(L_u) = \Theta_u L^2_+, \qquad \mathscr{H}_{\mathrm{sc}}(L_u) = \{0\}, \qquad \mathscr{H}_{\mathrm{pp}}(L_u) = K_{\Theta_u} = \frac{\mathbb{C}_{\leq N-1}[X]}{Q_u}. \quad (4.5)$$

Proof. Fix $u \in \mathcal{U}_N$, we use abbreviation $Q := Q_u$ and $\Theta := \Theta_u$. The first step is to prove $K_\Theta = \frac{\mathbb{C}_{\leq N-1}[X]}{Q}$. In fact, $\forall h \in L^2_+$ and $f = \frac{P}{Q} \in \frac{\mathbb{C}_{\leq N-1}[X]}{Q}$, for some $P \in \mathbb{C}_{\leq N-1}[X]$, we have $\langle f, \Theta h\rangle_{L^2} = \langle \frac{P}{\overline{Q}}, h\rangle_{L^2}$. Since $\overline{Q}(x) = \prod_{j=1}^N (x - \overline{\beta}_j)$ with $\mathrm{Im}(\beta_j) < 0$, the meromorphic function $\frac{P}{\overline{Q}}$ has poles in $\mathbb{C}_+$, so $\frac{P}{\overline{Q}} \in L^2_-$. Thus $\langle f, \Theta h\rangle_{L^2} = \langle \frac{P}{\overline{Q}}, h\rangle_{L^2} = 0$. Thus $\frac{\mathbb{C}_{\leq N-1}[X]}{Q} \subset (\Theta L^2_+)^\perp = K_\Theta$. Conversely, if $f \in K_\Theta$, then $\langle \Theta^{-1} f, h\rangle_{L^2} = \langle f, \Theta h\rangle_{L^2} = 0$, for every $h \in L^2_+$. Thus $g := \frac{Q}{\overline{Q}} f \in L^2_-$. It suffices to prove that $P := Qf = \overline{Q}g \in \mathbb{C}[X]$. In fact, $\widehat{Qf} = Q(i\partial_\xi)\hat{f}$ and $\mathrm{supp}(\hat{f}) \subset [0, +\infty) \Rightarrow \mathrm{supp}(\widehat{Qf}) \subset [0, +\infty)$. Similarly, we have $\mathrm{supp}((\overline{Q}g)^\wedge) \subset (-\infty, 0]$. So $\mathrm{supp}(\hat{P}) \subset \{0\}$ and P is a polynomial. Since $f = \frac{P}{Q} \in L^2(\mathbb{R})$, we have $\deg P \leq N-1$. So $K_\Theta \subset \frac{\mathbb{C}_{\leq N-1}[X]}{Q} \subset K_\Theta$. The second step is to show that

$$L_u(\Theta h) = \Theta \mathrm{D} h, \qquad \forall h \in L^2_+. \quad (4.6)$$

In fact, we have $\frac{\mathbb{C}_{\leq N-1}[X]}{Q} \subset L^2_+$, $\Theta = \frac{\overline{Q}}{Q}$ and $\frac{\mathrm{D}\Theta}{\Theta} = \frac{\mathrm{D}\overline{Q}}{\overline{Q}} - \frac{\mathrm{D}Q}{Q} = i\frac{Q'}{Q} - i\frac{\overline{Q}'}{\overline{Q}} = \Pi u + \overline{\Pi u} = u$ on $\mathbb{R}$, then $L_u(\Theta h) = (\mathrm{D} - T_u)(\Theta h) = \Theta \mathrm{D} h + h\left(\mathrm{D}\Theta - i\frac{Q'}{Q}\Theta + i\frac{\overline{Q}'}{\overline{Q}}\right) = \Theta \mathrm{D} h + h\Theta\left(\frac{\mathrm{D}\Theta}{\Theta} - i\frac{Q'}{Q} + i\frac{\overline{Q}'}{\overline{Q}}\right) = \Theta \mathrm{D} h$. Recall that $L_u = L_u^*$, so we have $L_u(K_\Theta) \subset K_\Theta$. Since $\dim_{\mathbb{C}} K_\Theta = N$, Corollary 2.7 yields that the Hermitian matrix $L_{u|K_\Theta}$ has exactly N distinct eigenvalues. Hence $K_\Theta \subset \mathscr{H}_{\mathrm{pp}}(L_u)$.

We set $U_\Theta : L^2_+ \to \Theta L^2_+$ such that $U_\Theta h = \Theta h$. Then $U_\Theta^{-1} = U_\Theta^* : g \in \Theta L^2_+ \mapsto \Theta^{-1} g \in L^2_+$, i.e $U_\Theta : L^2_+ \to \Theta L^2_+$ is unitary. Moreover, we have $U_\Theta(H^1_+) = \Theta H^1_+ = H^1_+ \bigcap \Theta L^2_+$. Formula (4.6) yields that $U_\Theta[\mathbf{D}(\mathrm{D})] = \Theta H^1_+ = H^1_+ \bigcap \Theta L^2_+ = \mathbf{D}(L_{u|\Theta L^2_+})$ and $U_\Theta^* L_{u|\Theta L^2_+} U_\Theta = \mathrm{D}$. For every bounded Borel function $f : \mathbb{R} \to \mathbb{C}$, we have $f(L_{u|\Theta L^2_+})U_\Theta = U_\Theta f(\mathrm{D})$ by proposition 2.1. Let $\mu_\psi = \mu_\psi^{L_u}$ denote the spectral measure of L_u associated to $\psi \in L^2_+$. Then $\int_{\mathbb{R}} f(\xi)\mathrm{d}\mu_{\Theta h}(\xi) = \langle f(L_u)U_\Theta h, U_\Theta h\rangle_{L^2} = \langle f(\mathrm{D})h, h\rangle_{L^2} = \frac{1}{2\pi}\int_0^{+\infty} f(\xi)|\hat{h}(\xi)|^2 \mathrm{d}\xi$, $\forall h \in L^2_+$. So $2\pi \mathrm{d}\mu_{\Theta h}(\xi) = \mathbf{1}_{\mathbb{R}_+}|\hat{h}(\xi)|^2\mathrm{d}\xi$. The measure $\mu_{\Theta h}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. Thus $\Theta L^2_+ \subset \mathscr{H}_{\mathrm{ac}}(L_u) \subset \mathscr{H}_{\mathrm{cont}}(L_u) = (\mathscr{H}_{\mathrm{pp}}(L_u))^\perp \subset \Theta L^2_+$ and (4.5) is obtained. We have $\mathrm{supp}(\mu_{\Theta h}) \subset [0, +\infty)$. For any $\xi > 0$, there exists $h \in L^2_+ \bigcap L^1(\mathbb{R})$ such that $\hat{h}(\xi) \neq 0$. So we have $\sigma_{\mathrm{ess}}(L_u) = \sigma_{\mathrm{cont}}(L_u) = \sigma_{\mathrm{ac}}(L_u) = [0, +\infty)$. $\square$

Definition 4.4. For every $u \in \mathcal{U}_N$, we have the following spectral decomposition of L_u:

$$\sigma(L_u) = \sigma_{\mathrm{ac}}(L_u) \bigcup \sigma_{\mathrm{sc}}(L_u) \bigcup \sigma_{\mathrm{pp}}(L_u), \quad \text{where } \sigma_{\mathrm{ac}}(L_u) = [0, +\infty), \quad \sigma_{\mathrm{sc}}(L_u) = \emptyset \quad (4.7)$$

and $\sigma_{\mathrm{pp}}(L_u) = \{\lambda^u_1, \lambda^u_2, \ldots, \lambda^u_N\}$ consists of all eigenvalues of L_u. Proposition 2.3 yields that L_u is bounded from below and $-\frac{\|u\|^2_{L^2}}{4C^4} \leq \lambda^u_1 < \cdots < \lambda^u_N < 0$ with $C = \inf_{f \in H^1_+ \setminus \{0\}} \frac{\||\mathrm{D}|^{\frac{1}{4}} f\|_{L^2}}{\|f\|_{L^4}}$.

Hence the min-max principle (Theorem XIII.1 of Reed–Simon [19]) yields that

$$\lambda_n^u = \sup_{\dim_{\mathbb{C}} F = n-1} \mathfrak{I}(F, L_u), \qquad \mathfrak{I}(F, L_u) = \inf\{\langle L_u h, h\rangle_{L^2} : h \in H_+^1 \bigcap F^\perp, \|h\|_{L^2} = 1\} \quad (4.8)$$

where, the above supremum, F describes all subspaces of L_+^2 of complex dimension $n-1$, $1 \leq n \leq N$. When $n \geq N+1$, $\sup_{\dim_{\mathbb{C}} F=n-1} \mathfrak{I}(F, L_u) = \inf \sigma_{\text{ess}}(L_u) = 0$. Given $j = 1, 2, \ldots, N$, Proposition 2.4 and Corollary 2.7 yield that there exists a unique function $\varphi_j : u \in \mathcal{U}_N \mapsto \varphi_j^u \in \mathscr{H}_{\text{pp}}(L_u)$ such that

$$\text{Ker}(\lambda_j^u - L_u) = \mathbb{C}\varphi_j^u, \qquad \|\varphi_j^u\|_{L^2} = 1, \qquad \langle \varphi_j^u, u\rangle_{L^2} = \sqrt{2\pi|\lambda_j^u|}, \quad (4.9)$$

for every $j = 1, 2, \ldots, N$. Then $\{\varphi_1^u, \varphi_2^u, \ldots, \varphi_N^u\}$ is an orthonormal basis of the subspace $\mathscr{H}_{\text{pp}}(L_u)$. Before proving the real analyticity of each eigenvalue, we show its continuity at first.

Lemma 4.5. *For every $j = 1, 2, \ldots, N$, the jth eigenvalue $\lambda_j : u \in \mathcal{U}_N \mapsto \lambda_j^u \in \mathbb{R}$ is Lipschitz continuous on every compact subset of $\mathcal{U}_N$.*

Proof. For every $f \in H^1(\mathbb{R})$, the Sobolev embedding $\|f\|_{L^4} \leq C^{-1}\||\text{D}|^{\frac{1}{4}} f\|_{L^2}$ yields that $\forall u, v \in \mathcal{U}_N$,

$$\left|\langle L_u h, h\rangle_{L^2} - \langle L_v h, h\rangle_{L^2}\right| \leq \|u - v\|_{L^2}\|h\|_{L^4}^2 \leq C^{-2}\|u - v\|_{L^2}\||\text{D}|^{\frac{1}{2}} h\|_{L^2}\|h\|_{L^2},$$
$$\forall h \in H_+^1. \quad (4.10)$$

Given $j = 1, 2, \ldots, N$ and a subspace $F \subset L_+^2$ whose complex dimension is $j-1$, we choose a function $h \in F^\perp \bigcap \bigoplus_{k=1}^j \text{Ker}(\lambda_k^u - L_u) \subset H_+^1$ such that $\|h\|_{L^2} = 1$. We have $h = \sum_{k=1}^j h_k \varphi_k^u$ for some $h_k \in \mathbb{C}$. Then $\langle L_u h, h\rangle_{L^2} = \sum_{k=1}^j |h_k|^2 \lambda_k^u \leq \lambda_j^u < 0$, because $\lambda_k^u < \lambda_{k+1}^u$. We have the following estimate

$$\begin{aligned}\||\text{D}|^{\frac{1}{2}} h\|_{L^2}^2 &= \langle \text{D}h, h\rangle_{L^2} = \langle L_u h, h\rangle_{L^2} + \langle uh, h\rangle_{L^2} \leq \lambda_j^u + \|u\|_{L^2}\|h\|_{L^4}^2 \\ &\leq C^{-2}\|u\|_{L^2}\||\text{D}|^{\frac{1}{2}} h\|_{L^2}\|h\|_{L^2}. \quad (4.11)\end{aligned}$$

So estimates (4.10) and (4.11) yield that $\langle L_v h, h\rangle_{L^2} \leq \lambda_j^u + C^{-4}\|u\|_{L^2}\|u - v\|_{L^2}$. Since F is arbitrary, the max–min formula (4.8) implies that $|\lambda_j^u - \lambda_j^v| \leq C^{-4}(\|u\|_{L^2} + \|v\|_{L^2})\|u - v\|_{L^2}$. Every compact subset $K \subset \mathcal{U}_N$ is bounded in $L^2(\mathbb{R}, \mathbb{R})$. Hence $u \in K \mapsto \lambda_j^u \in \mathbb{R}$ is Lipschitz continuous. □

Proposition 4.6. *For every $j = 1, 2, \ldots, N$, the jth eigenvalue $\lambda_j : u \in \mathcal{U}_N \mapsto \lambda_j^u \in \mathbb{R}$ is real analytic.*

Its proof is based on Kato's perturbation theory for linear operators.

Proof. For every $u \in \mathcal{U}_N$, let $\mathbb{P}_u^j$ denotes the Riesz projector of the eigenvalue λ_j^u. Then there exists $\epsilon_0 > 0$ such that the family of closed discs $\{\overline{D}(\lambda_j^u, \epsilon_0)\}_{1 \leq j \leq N} \bigcup \{\overline{D}(0, \epsilon_0)\}$ is mutually disjoint and for every $j, k = 1, 2 \ldots, N$ and any closed path $\mathbf{\Gamma}_j^u$ (piecewise

C^1 closed curve) in $D(\lambda_j^u, \epsilon_0)$ with respect to which the eigenvalue λ_j^u has winding number 1, we have

$$\mathbb{P}_u^j = \frac{1}{2\pi i}\oint_{\boldsymbol{\Gamma}_j^u}(\zeta - L_u)^{-1}\mathrm{d}\zeta, \qquad \mathbb{P}_u^j \circ \mathbb{P}_u^j = \mathbb{P}_u^j, \qquad \mathbb{P}_u^j\varphi_k^u = \delta_{kj}\varphi_k^u. \tag{4.12}$$

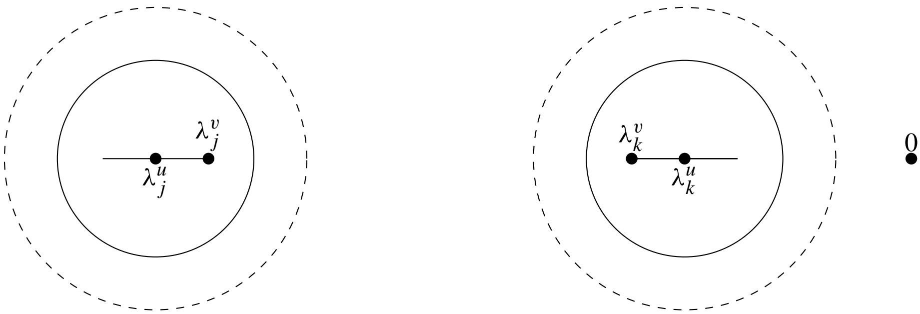
by Theorem XII.5 of Reed–Simon [19]. We choose $\boldsymbol{\Gamma}_j^u$ to be the counterclockwise-oriented circle $\mathscr{C}(\lambda_j^u, \epsilon)$ in (4.12) for some $\epsilon \in (0, \epsilon_0)$. We claim that $\mathrm{Im}\mathbb{P}_u^j = \mathrm{Ker}(\lambda_j^u - L_u) = \mathbb{C}\varphi_j^u$.

It suffices to show that $\mathbb{P}_u^j|_{\mathscr{H}_{\mathrm{ac}}(L_u)} = 0$. In fact the operator $\mathbb{P}_u^j = g_{\lambda_j^u}(L_u)$ is self-adjoint and bounded, where the bounded Borel function $g_\lambda : \mathbb{R} \to \mathbb{R}$ is given by

$$g_\lambda(x) := \frac{1}{2\pi i}\oint_{\mathscr{C}(\lambda,\epsilon)}(\zeta - x)^{-1}\mathrm{d}\zeta = \mathbf{1}_{(\lambda-\epsilon,\lambda+\epsilon)}(x), \qquad \text{a.e. on } \mathbb{R},$$

for every $\lambda \in \mathbb{R}$. Since $\mathbb{P}_u^j(\mathscr{H}_{\mathrm{pp}}(L_u)) \subset \mathbb{C}\varphi_j^u \subset \mathscr{H}_{\mathrm{pp}}(L_u)$, we have $\mathbb{P}_u^j(\mathscr{H}_{\mathrm{ac}}(L_u)) \subset \mathscr{H}_{\mathrm{ac}}(L_u)$. Let $\mu_\psi = \mu_\psi^{L_u}$ denote the spectral measure of L_u associated to $\psi \in \mathscr{H}_{\mathrm{ac}}(L_u)$, whose support is included in $[0, +\infty)$ by (4.7), so $\langle \mathbb{P}_u^j\psi, \psi\rangle_{L^2} = \frac{1}{2\pi i}\oint_{\mathscr{C}(\lambda_j^u,\epsilon)}\langle(\zeta - L_u)^{-1}\psi, \psi\rangle_{L^2}\mathrm{d}\zeta = \frac{1}{2\pi i}\int_0^{+\infty}\left(\oint_{\mathscr{C}(\lambda_j^u,\epsilon)}(\zeta - \xi)^{-1}\mathrm{d}\zeta\right)\mathrm{d}\mu_\psi(\xi) = 0$. Set $\tilde{\psi} = \mathbb{P}_u^j\psi \in \mathscr{H}_{\mathrm{ac}}(L_u)$, then $\|\tilde{\psi}\|_{L^2}^2 = \langle \mathbb{P}_u^j\tilde{\psi}, \tilde{\psi}\rangle_{L^2} = 0$. So the claim is obtained.

For every fixed $j = 1, 2, \ldots N$, we have $\lambda_j^u = \mathrm{Tr}(L_u \circ \mathbb{P}_u^j)$. Since every eigenvalue $\lambda_k : v \in \mathcal{U}_N \mapsto \lambda_k^v \in \mathbb{R}$ is continuous, there exists an open subset $\mathcal{V} \subset \mathcal{U}_N$ containing u such that $\sup_{v\in\mathcal{V}}\sup_{1\le k\le N}|\lambda_k^v - \lambda_k^u| < \frac{\epsilon_0}{3}$. We set $\epsilon = \frac{2\epsilon_0}{3}$, then $\lambda_j^v \in D(\lambda_j^u, \epsilon)\backslash\overline{D}(\lambda_k^u, \epsilon_0)$, for every $v \in \mathcal{V}$ and $k \ne j$. For example, in the next picture, the dashed circles denote respectively $\mathscr{C}(\lambda_j^u, \epsilon_0)$ and $\mathscr{C}(\lambda_k^u, \epsilon_0)$; the smaller circles denote respectively $\mathscr{C}(\lambda_j^u, \epsilon)$ and $\mathscr{C}(\lambda_k^u, \epsilon)$ with $j < k$. The segments inside small circles denote the possible positions of λ_j^v and λ_k^v.



Then $\sigma(L_v)\bigcap D(\lambda_j^u, \epsilon_0) = \{\lambda_j^v\}$ and $\mathscr{C}(\lambda_j^u, \epsilon)$ is a closed path in $D(\lambda_j^u, \epsilon_0)$ with respect to which λ_j^v has winding number 1. Thus,

$$\mathbb{P}_v^j = \frac{1}{2\pi i}\oint_{\mathscr{C}(\lambda_j^u,\epsilon)}(\zeta - L_v)^{-1}\mathrm{d}\zeta, \qquad \lambda_j^v = \mathrm{Tr}(L_v \circ \mathbb{P}_v^j), \qquad \forall v \in \mathcal{V}. \tag{4.13}$$

Since $v \in \mathcal{V} \mapsto L_v \in \mathfrak{B}(H^1_+, L^2_+)$ is $\mathbb{R}$-affine and $\mathbf{i} : \mathcal{A} \in \mathfrak{B}_{\mathfrak{I}}(H^1_+, L^2_+) \mapsto \mathcal{A}^{-1} \in \mathfrak{B}(L^2_+, H^1_+)$ is complex analytic, where $\mathfrak{B}_{\mathfrak{I}}(H^1_+, L^2_+) \subset \mathfrak{B}(H^1_+, L^2_+)$ denotes the open subset of all bijective bounded $\mathbb{C}$-linear transformations $H^1_+ \to L^2_+$, we have the real analyticity of the following map

$$(\zeta, v) \in \left(D(\lambda^u_j, \frac{3}{4}\epsilon_0)\backslash\overline{D}(\lambda^u_j, \frac{1}{2}\epsilon_0)\right) \times \mathcal{V} \mapsto (\zeta - L_v)^{-1} \in \mathfrak{B}(L^2_+, H^1_+). \quad (4.14)$$

Hence the maps $\mathbb{P}^j : v \in \mathcal{V} \mapsto \mathbb{P}^j_v \in \mathfrak{B}(L^2_+, H^1_+)$ and $\lambda_j : v \in \mathcal{V} \mapsto \mathrm{Tr}(L_v \circ \mathbb{P}^j_v) \in \mathbb{R}$ are both real analytic by composing (4.13) and (4.14). $\square$

Recall that $\mathscr{H}_{\mathrm{pp}}(L_u) = \frac{\mathbb{C}_{\leq N-1}[X]}{Q_u} \subset \mathbf{D}(G)$ is given by (3.6), $\forall u \in \mathcal{U}_N$. We have the following consequence.

Corollary 4.7. *For every $j = 1, 2, \ldots, N$, both the map $\varphi_j : u \in \mathcal{U}_N \mapsto \varphi^u_j \in H^1_+$ and the map $\mho_j : u \in \mathcal{U}_N \mapsto \langle G\varphi^u_j, \varphi^u_j\rangle_{L^2} \in \mathbb{C}$ are real analytic.*

Proof. Given $u, v \in \mathcal{U}_N$, we have $\mathbb{P}^j_v\varphi^u_j = \langle\varphi^u_j, \varphi^v_j\rangle_{L^2}\varphi^v_j$. Since the Riesz projector $\mathbb{P}^j : v \in \mathcal{U}_N \mapsto \mathbb{P}^j_v \in \mathfrak{B}(L^2_+, H^1_+)$ is real analytic in the proof of proposition 4.6 and $\|\mathbb{P}^j_u\varphi^u_j\|_{L^2} = 1$, there exists a neighbourhood of u, denoted by $\mathcal{V}$, such that $\|\mathbb{P}^j_v\varphi^u_j\|_{L^2} > \frac{1}{2}$ for every $v \in \mathcal{V}$ and $\mathbb{P}^j : v \in \mathcal{V} \mapsto \mathbb{P}^j_v \in \mathfrak{B}(L^2_+, H^1_+)$ can be expressed by power series. Then we have $\varphi^v_j = \frac{\mathbb{P}^j_v\varphi^u_j}{\langle\varphi^u_j, \varphi^v_j\rangle_{L^2}}$ and $\mho_j(v) = \frac{\langle G\circ\mathbb{P}^j_v(\varphi^u_j), \mathbb{P}^j_v(\varphi^u_j)\rangle_{L^2}}{\|\mathbb{P}^j_v(\varphi^u_j)\|^2_{L^2}}$. Hence the restriction $\mho_j : v \in \mathcal{V} \mapsto \|\mathbb{P}^j_v(\varphi^u_j)\|^{-2}_{L^2}\langle G \circ \mathbb{P}^j_v(\varphi^u_j), \mathbb{P}^j_v(\varphi^u_j)\rangle_{L^2} \in \mathbb{C}$ is real analytic. Since (4.9) yields that $\langle\mathbb{P}^j_v\varphi^u_j, v\rangle_{L^2} = \sqrt{-2\pi\lambda^v_j}\langle\varphi^u_j, \varphi^v_j\rangle_{L^2}$, the restriction $\varphi_j : v \in \mathcal{V} \mapsto \frac{\sqrt{-2\pi\lambda^v_j}}{\langle\mathbb{P}^j_v\varphi^u_j, v\rangle_{L^2}}\mathbb{P}^j_v\varphi^u_j \in H^1_+$ is real analytic. $\square$

4.3. Characterization theorem. This subsection is dedicated to proving the following spectral characterization theorem for multi-solitons.

Theorem 4.8. *Given $N \in \mathbb{N}_+$, a function $u \in \mathcal{U}_N$ if and only if $u \in L^2(\mathbb{R}, (1+x^2)\mathrm{d}x)$ is real-valued, $\dim_{\mathbb{C}} \mathscr{H}_{\mathrm{pp}}(L_u) = N$ and $\Pi u \in \mathscr{H}_{\mathrm{pp}}(L_u)$. Moreover, we have the following inverse formula*

$$\Pi u(x) = i \det(x - G|_{\mathscr{H}_{\mathrm{pp}}(L_u)})^{-1} \frac{\mathrm{d}}{\mathrm{d}x}\left(\det(x - G|_{\mathscr{H}_{\mathrm{pp}}(L_u)})\right), \qquad \forall x \in \mathbb{R}. \quad (4.15)$$

The direct sense is given by Proposition 4.3. Before proving the converse sense of Theorem 4.8, we need to prove the invariance of $\mathscr{H}_{\mathrm{pp}}(L_u)$ under G, if $u \in L^2(\mathbb{R}, (1+x^2)\mathrm{d}x)$ is real-valued, $\Pi u \in \mathscr{H}_{\mathrm{pp}}(L_u)$ and $\dim_{\mathbb{C}} \mathscr{H}_{\mathrm{pp}}(L_u) = N \geq 1$. We give another version of formula of commutators (see also Lemma 3.1).

Lemma 4.9. *For $u \in L^2(\mathbb{R}, (1 + x^2)\mathrm{d}x)$, u is real-valued, $\forall\varphi \in \mathrm{Ker}(\lambda - L_u)$ for some $\lambda \in \sigma_{\mathrm{pp}}(L_u)$, then we have $\varphi, T_u\varphi, L_u\varphi \in \mathbf{D}(G)$ and*

$$[G, T_u]\varphi = \frac{i\hat{\varphi}(0^+)}{2\pi}\Pi u, \qquad [G, L_u]\varphi = i\varphi - \frac{i\hat{\varphi}(0^+)}{2\pi}\Pi u. \quad (4.16)$$

Proof. In Proposition 2.4, we have shown that $\widehat{u\varphi} \in H^1(\mathbb{R})$, so $(T_u\varphi)^\wedge = \widehat{u\varphi}\mathbf{1}_{\mathbb{R}_+} \in H^1(0,+\infty)$ and $T_u\varphi \in \mathbf{D}(G)$. So $G\varphi \in H^1_+ = \mathbf{D}(L_u) = \mathbf{D}(T_u)$. Moreover, we have $\hat{\varphi}$ is right-continuous at $\xi = 0^+$ and $\hat{\varphi} \in C^1(0,+\infty)$. The weak-derivative of $\hat{\varphi}$ is denoted by $\partial_\xi^w\hat{\varphi}$, δ_0 denotes the Dirac measure with support $\{0\}$, then $\partial_\xi^w\hat{\varphi} = \mathbf{1}_{\mathbb{R}_+^*}\frac{\mathrm{d}}{\mathrm{d}\xi}\hat{\varphi} + \hat{\varphi}(0^+)\delta_0$ and $\partial_\xi(\hat{u} * \hat{\varphi}) = \partial_\xi^w(\hat{u} * \hat{\varphi}) = \hat{u} * \partial_\xi^w\hat{\varphi}$ by Lemma 2.5. Since $\hat{\varphi} = \mathbf{1}_{\mathbb{R}_+^*}\hat{\varphi}$ a.e. in $\mathbb{R}$ and $\hat{u} \in H^1(\mathbb{R})$, we have $\hat{u} * \widehat{G\varphi}(\xi) = \hat{u} * [\mathbf{1}_{\mathbb{R}_+^*}\widehat{G\varphi}](\xi)$, for every $\xi > 0$ and $([G,T_u]\varphi)^\wedge(\xi) = \frac{i}{2\pi}\partial_\xi(\hat{u} * \hat{\varphi})(\xi) - \frac{i}{2\pi}\hat{u} * [\mathbf{1}_{\mathbb{R}_+^*}\frac{\mathrm{d}}{\mathrm{d}\xi}\hat{\varphi}](\xi) = \frac{i}{2\pi}\hat{\varphi}(0^+)\widehat{u}(\xi)$. The first formula of (4.16) is obtained. Since $L_u = \mathrm{D} - T_u$, we claim that $\mathrm{D}\varphi \in \mathbf{D}(G)$. In fact, $\partial_\xi(\mathrm{D}\varphi)^\wedge(\xi) = \hat{\varphi}(\xi) + \xi\partial_\xi\hat{\varphi}(\xi)$, $\forall \xi > 0$. Thus (2.4) implies that $\widehat{\mathrm{D}\varphi} \in H^1(0,+\infty)$. Then $([G,\mathrm{D}]\varphi)^\wedge(\xi) = i\partial_\xi(\xi\hat{\varphi})(\xi) - \xi \cdot i\partial_\xi\hat{\varphi}(\xi) = i\hat{\varphi}(\xi)$, $\forall \xi > 0$. So we have $[\partial_x, G] = \mathrm{Id}_{L^2_+}$. The second formula of (4.16) holds. □

Proposition 4.10. *If $u \in L^2(\mathbb{R},(1+x^2)\mathrm{d}x)$ is real-valued,* $\dim_{\mathbb{C}}\mathscr{H}_{\mathrm{pp}}(L_u) = N \geq 1$ *and $\Pi u \in \mathscr{H}_{\mathrm{pp}}(L_u)$, then we have $\mathscr{H}_{\mathrm{pp}}(L_u) \subset \mathbf{D}(G)$ and $G(\mathscr{H}_{\mathrm{pp}}(L_u)) \subset \mathscr{H}_{\mathrm{pp}}(L_u)$.*

Proof. There exists an orthonormal basis of the vector space $\mathscr{H}_{\mathrm{pp}}(L_u)$, denoted by $\{\psi_1, \psi_2, \ldots, \psi_N\}$, such that $L_u\psi_j = \lambda_j\psi_j$, where $\sigma_{\mathrm{pp}}(L_u) = \{\lambda_1, \lambda_2, \ldots, \lambda_N\} \subset (-\infty, 0)$ and $\lambda_j < \lambda_{j+1}$. Since (2.4) implies that $\mathscr{H}_{\mathrm{pp}}(L_u) \subset G^{-1}(H^1_+)\bigcap\mathbf{D}(G)$, formula (4.16) gives that $f_j := [L_u, G]\psi_j = -i\psi_j + \frac{i\hat{\psi}_j(0^+)}{2\pi}\Pi u \in \mathscr{H}_{\mathrm{pp}}(L_u)$. So we have $\langle f_j, \psi_j\rangle_{L^2} = \langle G\psi_j, L_u\psi_j\rangle_{L^2} - \langle GL_u\psi_j, \psi_j\rangle_{L^2} = \lambda(\langle G\psi_j, \psi_j\rangle_{L^2} - \langle G\psi_j, \psi_j\rangle_{L^2}) = 0$. For every $j = 1, 2, \ldots, N$, we set $g_j := \sum_{1\leq k\leq N, k\neq j}\frac{\langle f_j, \psi_k\rangle_{L^2}}{\lambda_k - \lambda_j}\psi_k$. Since $f_j = \sum_{1\leq k\leq N, k\neq j}\langle f_j, \psi_k\rangle_{L^2}\psi_k$, we have $(L_u - \lambda_j)g_j = f_j = (L_u - \lambda_j)G\psi_j$. Then $G\psi_j - g_j \in \mathrm{Ker}(L_u - \lambda_j) = \mathbb{C}\psi_j$ and $G\psi_j \in g_j + \mathbb{C}\psi_j \subset \mathscr{H}_{\mathrm{pp}}(L_u)$. We conclude by $\mathscr{H}_{\mathrm{pp}}(L_u) = \mathrm{Span}_{\mathbb{C}}\{\psi_1, \psi_2, \ldots, \psi_N\}$. □

Now, we perform the proof of converse sense of Theorem 4.8 and give the explicit formula of Q_u.

End of the proof of theorem 4.8. $\Leftarrow$: Proposition 4.10 yields that $G(\mathscr{H}_{\mathrm{pp}}(L_u)) \subset \mathscr{H}_{\mathrm{pp}}(L_u)$. Let Q denote the characteristic polynomial of the operator $G|_{\mathscr{H}_{\mathrm{pp}}(L_u)}$, then we have $\mathscr{H}_{\mathrm{pp}}(L_u) = \frac{\mathbb{C}_{\leq N-1}[X]}{Q}$ by Lemma 3.3. So $\Pi u = \frac{\mathrm{P}_0}{Q}$, for some $\mathrm{P}_0 \in \mathbb{C}[X]$ such that $\deg \mathrm{P}_0 \leq N-1$. It remains to show that $\mathrm{P}_0 = iQ'$.
In fact, we have $L_u(\frac{P}{Q}) = (\mathrm{D} - T_{\frac{\mathrm{P}_0}{Q}} - T_{\frac{\overline{\mathrm{P}}_0}{\overline{Q}}})(\frac{P}{Q}) = \frac{\mathrm{D}P}{Q} - \Pi(\frac{\overline{\mathrm{P}}_0 P}{\overline{Q}Q}) + \frac{(iQ' - \mathrm{P}_0)P}{Q^2} \in \frac{\mathbb{C}_{\leq N-1}[X]}{Q}$, for every $P \in \mathbb{C}_{\leq N-1}[X]$, thanks to the invariance of $\mathscr{H}_{\mathrm{pp}}(L_u)$ under L_u. Partial-fraction decomposition implies that $\Pi(\frac{\overline{\mathrm{P}}_0 P}{\overline{Q}Q}) \in \frac{\mathbb{C}_{\leq N-1}[X]}{Q}$. So $\frac{(iQ' - \mathrm{P}_0)P}{Q} \in \mathbb{C}_{\leq N-1}[X]$ for every $P \in \mathbb{C}_{\leq N-1}[X]$. Choose $P = \mathbf{1}$, since $\deg(iQ' - \mathrm{P}_0) \leq N-1$, we have $\mathrm{P}_0 = iQ'$, so $u \in \mathcal{U}_N$. Since $Q \in \mathbb{C}_N[X]$ is monic and $Q^{-1}(0) \subset \mathbb{C}_-$, we have $Q_u(x) = Q(x) = \det(x - G|_{\mathscr{H}_{\mathrm{pp}}(L_u)})$. □

We refer to Proposition 4.10 and formula (4.4) to see the invariance of $\mathscr{H}_{\mathrm{pp}}(L_u) \subset \mathbf{D}(G)$ under G, $\forall u \in \mathcal{U}_N$. The translation–scaling parameters of u can be identified as the spectrum of $G|_{\mathscr{H}_{\mathrm{pp}}(L_u)}$. The matrix representation of $G|_{\mathscr{H}_{\mathrm{pp}}(L_u)}$ with respect to the orthonormal basis $\{\varphi_1^u, \varphi_2^u, \ldots, \varphi_N^u\}$ is given in Proposition 5.4.

4.4. The invariance under the Benjamin–Ono flow. Proposition 1.4 is proved in this subsection. At first, we show the invariance of the property $x \mapsto xu(x) \in L^2(\mathbb{R})$ under the BO flow. Then the spectral characterization Theorem 4.8 is used to establish the global well-posedness of the Hamiltonian system (1.3) on $\mathcal{U}_N$.

Lemma 4.11. *If $u_0 \in H^2(\mathbb{R}, \mathbb{R}) \bigcap L^2(\mathbb{R}, x^2\mathrm{d}x)$, let $u = u(t,x)$ solves the BO equation (1.1) with initial datum $u(0) = u_0$, then $u(t) \in L^2(\mathbb{R}, x^2\mathrm{d}x)$, for every $t \in \mathbb{R}$.*

Remark 4.12. This result can be strengthened by replacing the assumption $u_0 \in H^2(\mathbb{R}, \mathbb{R})$ by a weaker assumption $u_0 \in H^{\frac{3}{2}+}(\mathbb{R}, \mathbb{R}) = \bigcup_{s>\frac{3}{2}} H^s(\mathbb{R}, \mathbb{R})$, because one can construct a conservation law of (1.1), which controls the H^s-norm of solution, $\forall s > -\frac{1}{2}$, by using the method of perturbation of determinants. We refer to Talbut [22] to see details. It suffices to use Lemma 4.11 to prove Proposition 1.4.

Before proving Lemma 4.11, we need some commutator estimates.

Lemma 4.13. *For a general locally Lipschitz function $\chi : \mathbb{R} \to \mathbb{R}$ such that $\partial_x\chi, \partial_x^3\chi, \partial_x^5\chi \in L^1(\mathbb{R})$, we have the following commutator estimates*

$$\begin{aligned} &\|[|\mathrm{D}|, \chi]g\|_{L^2} + \|[\partial_x, \chi]g\|_{L^2} \lesssim (\|\partial_x\chi\|_{L^1}\|\partial_x^3\chi\|_{L^1})^{\frac{1}{2}}\|g\|_{L^2}, \qquad \forall g \in L^2(\mathbb{R}), \\ &\||\mathrm{D}|[\partial_x, \chi]g\|_{L^2} \lesssim (\|\partial_x\chi\|_{L^1}\|\partial_x^3\chi\|_{L^1})^{\frac{1}{2}}\|\partial_x g\|_{L^2} + (\|\partial_x\chi\|_{L^1}\|\partial_x^5\chi\|_{L^1})^{\frac{1}{2}}\|g\|_{L^2}, \qquad \forall g \in H^1(\mathbb{R}). \end{aligned} \tag{4.17}$$

Proof. Since $2\pi\left|([|\mathrm{D}|, \chi]g)^\wedge(\xi)\right| \leq \int_{\eta\in\mathbb{R}} \left||\xi| - |\eta|\right||\hat{\chi}(\xi - \eta)||\hat{g}(\eta)|\mathrm{d}\eta \leq |\widehat{\partial_x\chi}| * |\hat{g}|(\xi)$, Young's convolution inequality yields that $\|[|\mathrm{D}|, \chi]g\|_{L^2} \lesssim \|\widehat{\partial_x\chi}\|_{L^1}\|g\|_{L^2}$. We set $\mathcal{R}_1 = \|\partial_x\chi\|_{L^1}^{-\frac{1}{2}}\|\partial_x^3\chi\|_{L^1}^{\frac{1}{2}}$, then $\|\widehat{\partial_x\chi}\|_{L^1} \leq \|\widehat{\partial_x\chi}\|_{L^\infty}\int_{|\xi|\leq\mathcal{R}_1}\mathrm{d}\xi + \int_{|\xi|>\mathcal{R}_1}\frac{\|\widehat{\partial_x^3\chi}\|_{L^\infty}}{|\xi|^2}\mathrm{d}\xi \lesssim \|\partial_x\chi\|_{L^1}\mathcal{R}_1 + \frac{\|\partial_x^3\chi\|_{L^1}}{\mathcal{R}_1} = 2(\|\partial_x\chi\|_{L^1}\|\partial_x^3\chi\|_{L^1})^{\frac{1}{2}}$. Similarly, we have $\|[\partial_x, \chi]g\|_{L^2} \lesssim \|\widehat{\partial_x\chi}\|_{L^1}\|g\|_{L^2} \lesssim (\|\partial_x\chi\|_{L^1}\|\partial_x^3\chi\|_{L^1})^{\frac{1}{2}}\|g\|_{L^2}$, so the first inequality of (4.17) is obtained. Since $2\pi\left|(|\mathrm{D}|[\partial_x, \chi]g)^\wedge(\xi)\right| \leq |\xi|\int_{\eta\in\mathbb{R}}|\xi - \eta||\hat{\chi}(\xi - \eta)||\hat{g}(\eta)|\mathrm{d}\eta \leq |\widehat{\partial_x^2\chi}| * |\hat{g}|(\xi) + |\widehat{\partial_x\chi}| * |\widehat{\partial_x g}|(\xi)$, then $\||\mathrm{D}|[\partial_x, \chi]g\|_{L^2} \lesssim \|\widehat{\partial_x^2\chi}\|_{L^1}\|g\|_{L^2} + \|\widehat{\partial_x\chi}\|_{L^1}\|\partial_x g\|_{L^2}$. We set $\mathcal{R}_2 := \|\partial_x\chi\|_{L^1}^{-\frac{1}{4}}\|\partial_x^5\chi\|_{L^1}^{\frac{1}{4}}$, then $\|\widehat{\partial_x^2\chi}\|_{L^1} \leq \|\widehat{\partial_x\chi}\|_{L^\infty}\int_{|\xi|\leq\mathcal{R}_1}|\xi|\mathrm{d}\xi + \int_{|\xi|>\mathcal{R}_1}\frac{\|\widehat{\partial_x^5\chi}\|_{L^\infty}}{|\xi|^3}\mathrm{d}\xi \lesssim \|\partial_x\chi\|_{L^1}\mathcal{R}_2^2 + \frac{\|\partial_x^5\chi\|_{L^1}}{\mathcal{R}_2^2} = 2(\|\partial_x\chi\|_{L^1}\|\partial_x^5\chi\|_{L^1})^{\frac{1}{2}}$. Finally, we add them together to get the second estimate of (4.17). $\square$

Now we prove the invariance of the property $x \mapsto xu(x) \in L^2(\mathbb{R})$ is invariant under the BO flow.

Proof of lemma 4.11. We choose a cut-off function $\chi \in C_c^\infty(\mathbb{R})$ such that χ decreases in $[0, +\infty)$, χ is even, $0 \leq \chi \leq 1$, $\chi \equiv 1$ on $[-1, 1]$ and $\mathrm{supp}(\chi) \subset [-2, 2]$. If $u_0 \in H^2(\mathbb{R}, \mathbb{R}) \bigcap L^2(\mathbb{R}, x^2\mathrm{d}x)$, let $u : t \in \mathbb{R} \mapsto u(t) \in H^2(\mathbb{R}, \mathbb{R})$ solves the BO equation (1.1) with initial datum $u(0) = u_0$, we claim that there exists a constant $\mathcal{C} = \mathcal{C}(\|u(0)\|_{H^1})$ such that

$$\begin{aligned} I(R, t) &:= \int_{\mathbb{R}} \chi^2(\tfrac{x}{R})|x|^2|u(t, x)|^2\mathrm{d}x \\ &\leq \mathcal{C}e^{|t|}(\int_{\mathbb{R}} |x|^2|u(0, x)|^2\mathrm{d}x + 1), \qquad \forall t \in \mathbb{R}, \quad \forall R > 1, \end{aligned} \tag{4.18}$$

In fact, we define $\rho(x) := x\chi(x)$. For every $R > 0$, we set $\rho_R(x) := R\rho(\frac{x}{R}) = x\chi(\frac{x}{R})$. Thus

$$\partial_t I(R,t) = 2\mathrm{Re}\langle \rho_R^2 |\mathrm{D}|\partial_x u(t) - 2\rho_R^2 u(t)\partial_x u(t),\\ u(t)\rangle_{L^2} = \mathcal{J}_1(u(t)) + \mathcal{J}_2(u(t)),$$

where for every $u \in H^2(\mathbb{R})$, we define

$$\mathcal{J}_1(u) := -4\mathrm{Re}\langle \rho_R^2 u\partial_x u, u\rangle_{L^2} \Longrightarrow\\ |\mathcal{J}_1(u)| \leq 4\|\partial_x u\|_{L^\infty}\|\rho_R u\|_{L^2}^2 \lesssim \|u\|_{H^2}\|\rho_R u\|_{L^2}^2 \tag{4.19}$$

and $\mathcal{J}_2(u) := 2\mathrm{Re}\langle \rho_R^2|\mathrm{D}|\partial_x u, u\rangle_{L^2} = \langle [\rho_R^2, |\mathrm{D}|\partial_x]u, u\rangle_{L^2}$. Since $[\rho_R^2, |\mathrm{D}|\partial_x] = \rho_R[\rho_R, |\mathrm{D}|\partial_x] + [\rho_R, |\mathrm{D}|\partial_x]\rho_R$ and $[\rho_R, |\mathrm{D}|\partial_x] = [\rho_R, |\mathrm{D}|\partial_x]^* = [\rho_R, |\mathrm{D}|]\partial_x + |\mathrm{D}|[\rho_R, \partial_x]$, we have

$$\mathcal{J}_2(u) = 2\mathrm{Re}\langle [\rho_R, |\mathrm{D}|]\partial_x u, \rho_R u\rangle_{L^2} + 2\mathrm{Re}\langle |\mathrm{D}|[\rho_R, \partial_x]u, \rho_R u\rangle_{L^2}.$$

Since $\|\partial_x\rho_R\|_{L^1} = R\|\partial_x\rho\|_{L^1}$, $\|\partial_x^3\rho_R\|_{L^1} = R^{-1}\|\partial_x\rho\|_{L^1}$ and $\|\partial_x^5\rho_R\|_{L^1} = R^{-3}\|\partial_x\rho\|_{L^1}$, the commutator estimates (4.17) yield that if $u \in H^2(\mathbb{R})$, then

$$\begin{aligned}|\mathcal{J}_2(u)| &\leq 2\|\rho_R u\|_{L^2}^2 + \|[\rho_R, |\mathrm{D}|]\partial_x u\|_{L^2}^2 + \||\mathrm{D}|[\rho_R, \partial_x]u\|_{L^2}^2\\ &\lesssim \|\rho_R u\|_{L^2}^2 + \|\partial_x\rho_R\|_{L^1}\|\partial_x^3\rho_R\|_{L^1}\|\partial_x u\|_{L^2}^2 + \|\partial_x\rho_R\|_{L^1}\|\partial_x^5\rho_R\|_{L^1}\|u\|_{L^2}^2\\ &\lesssim \|\rho_R u\|_{L^2}^2 + \|\partial_x\rho\|_{L^1}\|\partial_x^3\rho\|_{L^1}\|\partial_x u\|_{L^2}^2 + R^{-2}\|\partial_x\rho\|_{L^1}\|\partial_x^5\rho\|_{L^1}\|u\|_{L^2}^2\\ &\lesssim \|\rho_R u\|_{L^2}^2 + \|u\|_{H^1}^2\end{aligned} \tag{4.20}$$

for every $R \geq 1$. Proposition 2.9 and 2.12 yield that there exists a conservation law of (1.1) controlling H^2-norm of the solution. Let $u : t \in \mathbb{R} \mapsto u(t) \in H^2(\mathbb{R})$ denote the solution of the BO equation (1.1). Then $\sup_{t\in\mathbb{R}}\|u(t)\|_{H^2} \lesssim_{\|u_0\|_{H^2}} 1$. Since $I(R,t) = \|\rho_R u(t)\|_{L^2}^2$, estimates (4.19) and (4.20) imply that $|\partial_t I(R,t)| \leq \mathcal{C}(I(R,t)+1)$, $\forall t \in \mathbb{R}$, for some constant $\mathcal{C} = \mathcal{C}(\|u_0\|_{H^2})$. Thus (4.18) is obtained by Gronwall's inequality. Let $R \to +\infty$, we conclude by Lebesgue's monotone convergence theorem. □

Since the generating function $\lambda \in \mathbb{C}\backslash\sigma(-L_u) \mapsto \mathcal{H}_\lambda(u) \in \mathbb{C}$ is the Borel–Cauchy transform of the spectral measure of L_u, the invariance of $\mathcal{U}_N$ under the BO flow is obtained by the inverse spectral transform.

End of the proof of proposition 1.4. If $u_0 \in \mathcal{U}_N \subset H^\infty(\mathbb{R},\mathbb{R}) \bigcap L^2(\mathbb{R}, x^2\mathrm{d}x)$, let $u = u(t,x)$ denote the solution of the BO equation (1.1) with initial datum $u(0) = u_0$, then $u(t) \in H^\infty(\mathbb{R},\mathbb{R}) \bigcap L^2(\mathbb{R}, x^2\mathrm{d}x)$ by Proposition 2.8 and Lemma 4.11. Given $\lambda \in \mathbb{C}\backslash\mathbb{R}$, the generating function $\mathcal{H}_\lambda : u \in L^2(\mathbb{R},\mathbb{R}) \to \mathbb{R}$ reads as $\mathcal{H}_\lambda(u) = \langle (\lambda + L_u)^{-1}\Pi u, \Pi u\rangle_{L^2} = \int_{\mathbb{R}} \frac{\mathrm{d}\mathbf{m}_u(\xi)}{\xi+\lambda}$ with $\mathbf{m}_u := \mu_{\Pi u}^{L_u}$, where $\mu_\psi^{L_u}$ denotes the spectral measure of L_u associated to the function $\psi \in L_+^2$. So the holomorphic function $\lambda \in \mathbb{C}\backslash\mathbb{R} \mapsto \mathcal{H}_\lambda u$ is the Borel–Cauchy transform of the positive Borel measure $\mathbf{m}_u$. The total variation $\mathbf{m}_u(\mathbb{R}) = \|\Pi u\|_{L^2}^2$ is a conservation law of the BO equation (1.1) by Proposition 2.12 and formula (2.14). Thanks to the Stieltjes inversion formula, every finite Borel measure is uniquely determined by its Borel–Cauchy transform. For every $t \in \mathbb{R}$, we have $\mathcal{H}_\lambda[u(t)] = \mathcal{H}_\lambda[u(0)]$ by proposition 2.15. Since $u(0) \in \mathcal{U}_N$, we have $\Pi[u(0)] \in \mathscr{H}_{\mathrm{pp}}(L_{u(0)})$ by Proposition 4.3. Consequently, there exist $c_1, c_2, \ldots, c_N \in$

$\mathbb{R}\backslash\{0\}$ such that $\mu^{L_{u(t)}}_{\Pi[u(t)]} = \mathbf{m}_{u(t)} = \mathbf{m}_{u(0)} = \mu^{L_{u(0)}}_{\Pi[u(0)]} = \sum_{j=1}^{N} c_j \delta_{\lambda_j^{u(0)}}$. Then $\Pi[u(t)] \in \mathscr{H}_{\mathrm{pp}}(L_{u(t)})$, $\forall t \in \mathbb{R}$. The Lax pair structure yields the unitary equivalence between $L_{u(t)}$ and $L_{u(0)}$. So $\dim_{\mathbb{C}} \mathscr{H}_{\mathrm{pp}}(L_{u(t)}) = \dim_{\mathbb{C}} \mathscr{H}_{\mathrm{pp}}(L_{u(0)}) = N$ by Proposition 2.1. We conclude by Theorem 4.8. $\square$

5. The Generalized Action–Angle Coordinates

In this section, we construct the global (generalized) action–angle coordinates Φ_N in Theorem 1 of the Hamiltonian system (1.3) with solutions in the real analytic symplectic manifold $(\mathcal{U}_N, \omega)$ of real dimension $2N$ given in Proposition 1.2. The goal of this section is to establish the diffeomorphism property and the symplectomorphism property of Φ_N. Proposition 1.3 yields that the Poisson bracket of two smooth functions $f, g : \mathcal{U}_N \to \mathbb{R}$ is given by

$$\{f, g\} : u \in \mathcal{U}_N \mapsto \omega_u(X_f(u), X_g(u)) = \langle \partial_x \nabla_u f(u), \nabla_u g(u)\rangle_{L^2} \in \mathbb{R}. \tag{5.1}$$

Given $u \in \mathcal{U}_N$, Proposition 4.3 yields that there exist $\lambda_1^u < \lambda_2^u < \cdots < \lambda_N^u < 0$ and $\varphi_j^u \in \mathrm{Ker}(\lambda_j^u - L_u) \subset \mathbf{D}(G)$ such that $\|\varphi_j^u\|_{L^2} = 1$ and $\langle u, \varphi_j^u\rangle_{L^2} = \sqrt{2\pi|\lambda_j^u|}$, thanks to the spectral analysis in Sect. 4.2.

Definition 5.1. For every $j = 1, 2, \ldots, N$, the map $I_j : u \in \mathcal{U}_N \mapsto 2\pi\lambda_j^u \in \mathbb{R}$ is called the j th action. The map $\gamma_j : u \in \mathcal{U}_N \mapsto \mathrm{Re}\langle G\varphi_j^u, \varphi_j^u\rangle_{L^2} \in \mathbb{R}$ is called the j th (generalized) angle.

The set Ω_N is defined by (1.13) and we adopt the superscript instead of the subscript in this section: $\Omega_N = \{(r^1, r^2, \ldots, r^N) \in \mathbb{R}^N : r^1 < r^2 < \cdots < r^N < 0\}$. Then the real analytic manifold $(\Omega_N \times \mathbb{R}^N, \nu)$ is a symplectic manifold of real dimension $2N$, where $\nu = \sum_{j=1}^{N} \mathrm{d}r^j \wedge \mathrm{d}\alpha^j$. The action–angle map is given by $\Phi_N : u \in \mathcal{U}_N \mapsto (I_1(u), I_2(u), \ldots, I_N(u); \gamma_1(u), \gamma_2(u), \ldots, \gamma_N(u)) \in \Omega_N \times \mathbb{R}^N$. Theorem 1 is restated here.

Theorem 5.2. *The map Φ_N has following properties:*
(a). *The map $\Phi_N : \mathcal{U}_N \to \Omega_N \times \mathbb{R}^N$ is a real analytic diffeomorphism.*
(b). *The pullback of ν by Φ_N is ω, i.e. $\Phi_N^* \nu = \omega$.*
(c). *We have $E \circ \Phi_N^{-1} : (r^1, r^2, \ldots, r^N; \alpha^1, \alpha^2, \ldots, \alpha^N) \in \Omega_N \times \mathbb{R}^N \mapsto -\frac{1}{2\pi}\sum_{j=1}^{N} |r^j|^2 \in (-\infty, 0)$.*

Remark 5.3. The real analyticity of $\Phi_N : \mathcal{U}_N \to \Omega_N \times \mathbb{R}^N$ is given by Proposition 4.6 and Corollary 4.7. The symplectomorphism property (b) is equivalent to the Poisson bracket characterization (1.15). The family $(X_{I_1}, X_{I_2}, \ldots, X_{I_N}; X_{\gamma_1}, X_{\gamma_2}, \ldots, X_{\gamma_N})$ is linearly independent in $\mathfrak{X}(\mathcal{U}_N)$ and we have

$$\mathrm{d}\Phi_N(u) : X_{I_k}(u) \mapsto \tfrac{\partial}{\partial \alpha^k}\big|_{\Phi_N(u)}, \qquad \mathrm{d}\Phi_N(u) : X_{\gamma_k}(u) \mapsto -\tfrac{\partial}{\partial r^k}\big|_{\Phi_N(u)}.$$

The assertion (c) is obtained by a direct calculus: $\Pi u = \sum_{j=1}^{N} \langle \Pi u, \varphi_j^u\rangle_{L^2}\varphi_j^u$, formula (4.9) yields that $E(u) = \langle L_u(\Pi u), \Pi u\rangle_{L^2} = \sum_{j=1}^{N} |\langle \Pi u, \varphi_j^u\rangle_{L^2}|^2 \lambda_j^u = -\sum_{j=1}^{N} \frac{I_j(u)^2}{2\pi}$.

This section is organized as follows. The matrix associated to $G|_{\mathscr{H}_{\mathrm{pp}}(L_u)}$ is expressed in terms of actions and angles in Sect. 5.1. In Sect. 5.2, the Poisson brackets of actions and angles are used to prove the local diffeomorphism property of Φ_N. The bijectivity of Φ_N is obtained by Hadamard's global inverse function theorem in Sect. 5.3. Finally, we use Sects. 5.4 and 5.5 to prove that $\Phi_N : (\mathcal{U}_N, \omega) \to (\Omega_N \times \mathbb{R}^N, \nu)$ preserves the symplectic structure.

5.1. The inverse spectral matrix. We continue to study the infinitesimal generator G defined in (3.2) when restricted to the invariant subspace $\mathscr{H}_{\mathrm{pp}}(L_u)$ with complex dimension N. Then we state a general linear algebra lemma that describes the location of eigenvalues of the operator $G|_{\mathscr{H}_{\mathrm{pp}}(L_u)}$.

Proposition 5.4. *For every $u \in \mathcal{U}_N$, let $M(u) = (M_{kj}(u))_{1\leq k,j\leq N} \in \mathbb{C}^{N\times N}$ denote the inverse spectral matrix defined by (1.18) and Definition 5.1. Then $M(u)$ is the matrix associated to the operator $G|_{\mathscr{H}_{\mathrm{pp}}(L_u)}$ with respect to the basis $\{\varphi_1^u, \varphi_2^u, \dots, \varphi_N^u\}$, i.e. $M_{kj}(u) = \langle G\varphi_j^u, \varphi_k^u\rangle_{L^2}$, $1 \leq k, j \leq N$.*

Proof. Since $L_u = L_u^*$ and $\mathscr{H}_{\mathrm{pp}}(L_u) \subset \mathbf{D}(G)$, we have $(\lambda_j^u - \lambda_k^u)\langle G\varphi_j^u, \varphi_k^u\rangle_{L^2} = \langle [G, L_u]\varphi_j^u, \varphi_k^u\rangle_{L^2}$. Since formulas (2.5) and (4.9) imply that $-\lambda_j^u\widehat{\varphi_j^u}(0) = \widehat{u\varphi_j^u}(0) = \sqrt{2\pi|\lambda_j^u|}$, we use formula (4.16) to obtain that if k and j are different, then $(\lambda_j^u - \lambda_k^u)\langle G\varphi_j^u, \varphi_k^u\rangle_{L^2} = \langle i\varphi_j^u - \frac{i}{2\pi}\widehat{\varphi_j^u}(0^+)\Pi u, \varphi_k^u\rangle_{L^2} = -\frac{i}{2\pi}\widehat{\varphi_j^u}(0^+)\overline{\widehat{u\varphi_k^u}(0)} = -i\sqrt{\frac{\lambda_k^u}{\lambda_j^u}}$. In the case $k = j$, we have $\langle G^* f, g\rangle_{L^2} = -\frac{i}{2\pi}\int_0^{+\infty} \hat{f}(\xi)\partial_\xi \overline{\hat{g}}(\xi)\mathrm{d}\xi = \frac{i}{2\pi}\left[\hat{f}(0^+)\overline{\hat{g}}(0^+) + \int_0^{+\infty} \partial_\xi \hat{f}(\xi)\overline{\hat{g}}(\xi)\mathrm{d}\xi\right]$ and $\langle G^* f, g\rangle_{L^2} = \langle Gf, g\rangle_{L^2} + \frac{i}{2\pi}\hat{f}(0^+)\overline{\hat{g}}(0^+)$, for any $f, g \in \mathscr{H}_{\mathrm{pp}}(L_u)$ by using formula (3.2). Consequently, we have $\mathrm{Im}\langle G\varphi_j^u, \varphi_j^u\rangle_{L^2} = -\frac{1}{4\pi}|\widehat{\varphi_j^u}(0)|^2 = -\frac{1}{2|\lambda_j^u|} = \mathrm{Im} M_{jj}(u)$. $\square$

Corollary 5.5. *For every $u \in \mathcal{U}_N$, we define two vectors $X(u), Y(u) \in \mathbb{R}^N$ as*

$$X(u)^T = (\sqrt{|\lambda_1^u|}, \sqrt{|\lambda_2^u|}, \dots, \sqrt{|\lambda_N^u|}), \quad Y(u)^T = (\sqrt{|\lambda_1^u|^{-1}}, \sqrt{|\lambda_2^u|^{-1}}, \dots, \sqrt{|\lambda_N^u|^{-1}}), \tag{5.2}$$

Then we have the following inverse spectral formula

$$\Pi u(x) = -i\langle (M(u) - x)^{-1}X(u), Y(u)\rangle_{\mathbb{C}^N}, \qquad \forall x \in \mathbb{R}. \tag{5.3}$$

Hence, the map $\Phi_N : \mathcal{U}_N \to \Omega_N \times \mathbb{R}^N$ is injective.

Proof. For any $k, j = 1, 2, \dots, N$, let $K_{kj}^u(x)$ denote the $(N-1)\times(N-1)$ submatrix obtained by deleting the k th row and j th column of the matrix $M(u) - x$, for all $x \in \mathbb{R}$. The Cramer's rule yields that $\langle (M(u) - x)^{-1}X(u), Y(u)\rangle_{\mathbb{C}^N} = \sum_{1\leq k,j\leq N} \frac{(-1)^{k+j}\det(K_{kj}^u(x))}{\det(M(u)-x)}\sqrt{\frac{\lambda_k^u}{\lambda_j^u}} = \frac{\sum_{j=1}^N \det(K_{jj}^u(x)) + R}{\det(M(u)-x)}$, where $R := \sum_{1\leq k\neq j\leq N}(-1)^{k+j}\det(K_{kj}^u(x))\sqrt{\frac{\lambda_k^u}{\lambda_j^u}} = i(\sum_{j=1}^N \lambda_j^u - \sum_{k=1}^N \lambda_k^u)\det(M(u) - x) = 0$ by (1.18) and Definition 5.1. If $Q_u(x) = \det(x - M(u))$, then $Q_u'(x) = (-1)^{N-1}\sum_{j=1}^N \det(K_{jj}^u(x))$. Then formula (5.3) is obtained by formula (4.15). $\square$

The next lemma describes the location of spectrum of all matrices of the form defined as (1.18).

Lemma 5.6. *For every $N \in \mathbb{N}_+$, we choose N negative numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_N < 0$ and $\gamma_1, \gamma_2, \ldots, \gamma_N \in \mathbb{R}$. The matrix $\mathcal{M} = (\mathcal{M}_{kj})_{1\le k,j\le N} \in \mathbb{C}^{N\times N}$ is defined as $\mathcal{M}_{jj} = \gamma_j + \frac{i}{2\lambda_j}$ and $\mathcal{M}_{kj} = \frac{i}{\lambda_k-\lambda_j}\sqrt{\frac{\lambda_k}{\lambda_j}}$, if $k \neq j$. Then $\Im(\mathcal{M}) := \frac{\mathcal{M}-\mathcal{M}^*}{2i}$ is negative semi-definite and $\sigma_{\mathrm{pp}}(\mathcal{M}) \subset \mathbb{C}_-$.*

Proof. The vector $V_\lambda \in \mathbb{R}^N$ is defined as $V_\lambda^T := ((2|\lambda_1|)^{-\frac{1}{2}}, (2|\lambda_2|)^{-\frac{1}{2}}, \ldots, (2|\lambda_N|)^{-\frac{1}{2}})$. So we have $\Im(\mathcal{M}) = \left(-\frac{1}{2\sqrt{|\lambda_j||\lambda_k|}}\right)_{1\le k,j\le N} = -V_\lambda \cdot V_\lambda^T$. Thus $\langle(\Im(\mathcal{M}))X, X\rangle_{\mathbb{C}^N} = -|\langle X, V_\lambda\rangle_{\mathbb{C}^N}|^2 \le 0$. So $\Im(\mathcal{M})$ is negative semi-definite. If $\mu \in \sigma_{\mathrm{pp}}(\mathcal{M})$ and $V \in \operatorname{Ker}(\mu - \mathcal{M})\backslash\{0\}$, it suffices to show $\operatorname{Im}\mu < 0$. Since

$$\begin{aligned} -|\langle V, V_\lambda\rangle_{\mathbb{C}^N}|^2 &= \langle \Im(\mathcal{M})V, V\rangle_{\mathbb{C}^N} = \operatorname{Im}\mu\|V\|^2_{\mathbb{C}^N}, \\ \text{where}\quad \|V\|^2_{\mathbb{C}^N} &= \langle V, V\rangle_{\mathbb{C}^N} > 0, \end{aligned} \tag{5.4}$$

we have $\operatorname{Im}\mu \le 0$. Assume that $\mu \in \mathbb{R}$, then formula (5.4) yields that $V \perp V_\lambda$. Moreover, we have $(\mathcal{M}-\mathcal{M}^*)V = -2i\langle V, V_\lambda\rangle_{\mathbb{C}^N} V_\lambda = 0$. We set $D^\lambda \in \mathbb{C}^{N\times N}$ to be the diagonal matrix whose diagonal elements are $\lambda_1, \lambda_2, \ldots, \lambda_N$, i.e. $D^\lambda = (\lambda_k\delta_{jk})_{1\le k,j\le N}$. Then we have the following formula

$$[\mathcal{M}, D^\lambda] = i(\mathrm{I}_N + 2D^\lambda V_\lambda V_\lambda^T). \tag{5.5}$$

So $[\mathcal{M}, D^\lambda]V = iV$ by (5.5). Finally, $i\|V\|^2_{\mathbb{C}^N} = \langle(\mathcal{M}-\mu)D^\lambda V, V\rangle_{\mathbb{C}^N} = \langle D^\lambda V, (\mathcal{M}^*-\mu)V\rangle_{\mathbb{C}^N} = 0$ contradicts the fact that $V \neq 0$. Consequently, we have $\mu \in \mathbb{C}_-$. □

5.2. Poisson brackets. In this subsection, the Poisson bracket defined in (5.1) is generalized in order to obtain the first two formulas of (1.15). It can be defined between a smooth function from $\mathcal{U}_N$ to an arbitrary Banach space and another smooth function from $\mathcal{U}_N$ to $\mathbb{R}$. For every smooth function $f : \mathcal{U}_N \to \mathbb{R}$, its Hamiltonian vector field $X_f \in \mathfrak{X}(\mathcal{U}_N)$ is given by (1.12). For any Banach space $\mathcal{E}$ and any smooth map $F : u \in \mathcal{U}_N \mapsto F(u) \in \mathcal{E}$, we define the Poisson bracket of f and F as follows

$$\{f, F\} : u \in \mathcal{U}_N \mapsto \{f, F\}(u) := \mathrm{d}F(u)(X_f(u)) \in \mathcal{T}_{F(u)}(\mathcal{E}) = \mathcal{E}. \tag{5.6}$$

If $\mathcal{E} = \mathbb{R}$, then the definition in formula (5.6) coincides with (5.1). For every $u \in \mathcal{U}_N$ and $\lambda \in \mathbb{C}\backslash\sigma(-L_u)$, since $\Pi u = \sum_{j=1}^N \langle \Pi u, \varphi_j^u\rangle_{L^2}\varphi_j^u$, the generating functional

$$\mathcal{H}_\lambda(u) = \langle (L_u+\lambda)^{-1}\Pi u, \Pi u\rangle_{L^2} = -\sum_{j=1}^N \frac{2\pi\lambda_j^u}{\lambda+\lambda_j^u} \tag{5.7}$$

is well defined. The analytical continuation allows to extend the map $\lambda \mapsto \mathcal{H}_\lambda(u)$ to the domain $\mathbb{C}\backslash\sigma_{\mathrm{pp}}(-L_u)$, and it has simple poles at every $\lambda = -\lambda_j^u$. Proposition 2.3 yields that $-\frac{\|u\|_{L^2}^2}{4C^4} \le \lambda_1^u < \cdots < \lambda_N^u < 0$, where $C = \inf_{f \in H_+^1 \setminus \{0\}} \frac{\||\mathrm{D}|^{\frac{1}{4}} f\|_{L^2}}{\|f\|_{L^4}}$ denotes the Sobolev constant. So we introduce

$$\mathcal{Y} = \{(\lambda, u) \in \mathbb{R} \times \mathcal{U}_N : 4C^4\lambda > \|u\|_{L^2}^2\} = \mathcal{X} \bigcap (\mathbb{R} \times \mathcal{U}_N), \tag{5.8}$$

where $\mathcal{X}$ is given by Definition 2.14. Then $\mathcal{Y}$ is open in $\mathbb{R} \times \mathcal{U}_N$ and $\mathcal{H} : (\lambda, u) \in \mathcal{Y} \mapsto -\sum_{j=1}^N \frac{2\pi\lambda_j^u}{\lambda+\lambda_j^u} \in \mathbb{R}$ is real analytic by Proposition 4.6. The Fréchet derivative of $\mathcal{H}_\lambda$ is given by (2.16), so

$$X_{\mathcal{H}_\lambda}(u) = \partial_x \nabla_u \mathcal{H}_\lambda(u) = \partial_x(|w_\lambda(u)|^2 + w_\lambda(u) + \overline{w}_\lambda(u)), \qquad \forall (\lambda, u) \in \mathcal{Y}, \tag{5.9}$$

by formula (1.12), where $w_\lambda(u) = (L_u + \lambda)^{-1}(\Pi u)$. The following proposition restates the Lax pair structure of the Hamiltonian equation associated to $\mathcal{H}_\lambda$. Even though the stability of $\mathcal{U}_N$ under the Hamiltonian flow of $\mathcal{H}_\lambda$ remains as an open problem, the Poisson bracket defined in (5.6) provides an algebraic method to obtain the first two formulas of (1.15).

Proposition 5.7. *Given $(\lambda, u) \in \mathcal{Y}$ defined by (5.8), we have $\{\mathcal{H}_\lambda, L\}(u) = [B_u^\lambda, L_u]$ and*

$$\{\mathcal{H}_\lambda, \lambda_j\}(u) = 0, \qquad \{\mathcal{H}_\lambda, \gamma_j\}(u) = \mathrm{Re}\langle [G, B_u^\lambda]\varphi_j^u, \varphi_j^u\rangle_{L^2} = -\tfrac{\lambda}{(\lambda+\lambda_j^u)^2}, \tag{5.10}$$

for every $j = 1, 2, \ldots, N$, where $B_u^\lambda = i(T_{w_\lambda(u)} T_{\overline{w}_\lambda(u)} + T_{w_\lambda(u)} + T_{\overline{w}_\lambda(u)})$.

Proof. Since $L : u \in L^2(\mathbb{R}, \mathbb{R}) \mapsto L_u = \mathrm{D} - T_u \in \mathfrak{B}(H_+^1, L_+^2)$, $\forall u \in L_+^2$, we have $\mathrm{d}L(u)(h) = -T_h$, $\forall h \in L_+^2$. If $(\lambda, u) \in \mathcal{Y}$, then the $\mathbb{C}$-linear transformation $L_u + \lambda \in \mathfrak{B}(H_+^1, L_+^2)$ is bijective. So formula (5.9) yields that $\{\mathcal{H}_\lambda, L\}(u) = \mathrm{d}L(u)(X_{\mathcal{H}_\lambda}(u)) = -T_{\partial_x(|w_\lambda(u)|^2 + w_\lambda(u) + \overline{w}_\lambda(u))}$. Then identity (2.22) yields the Lax equation for the Hamiltonian flow of the generating function $\mathcal{H}_\lambda$, i.e.

$$\{\mathcal{H}_\lambda, L\}(u) = [B_u^\lambda, L_u] \in \mathfrak{B}(H_+^1, L_+^2). \tag{5.11}$$

Consider the map $L\varphi_j : u \in \mathcal{U}_N \mapsto L_u\varphi_j^u = \lambda_j^u \varphi_j^u \in H_+^1$, for every $(\lambda, u) \in \mathcal{Y}$, we have

$$\{\mathcal{H}_\lambda, L\}(u)\varphi_j^u + L_u\left(\{\mathcal{H}_\lambda, \varphi_j\}(u)\right) = \lambda_j^u \{\mathcal{H}_\lambda, \varphi_j\}(u) + \{\mathcal{H}_\lambda, \lambda_j\}(u)\varphi_j^u \in H_+^1.$$

Then (5.11) yields $(\lambda_j^u - L_u)\left(B_u^\lambda \varphi_j^u - \{\mathcal{H}_\lambda, \varphi_j\}(u)\right) = \{\mathcal{H}_\lambda, \lambda_j\}(u)\varphi_j^u$. Since $\varphi_j^u \in \mathrm{Ker}(\lambda_j^u - L_u)$ and $\|\varphi_j^u\|_{L^2} = 1$ by (4.9), we have $\{\mathcal{H}_\lambda, \lambda_j\}(u) = \langle (\lambda_j^u - L_u)\left(B_u^\lambda \varphi_j^u - \{\mathcal{H}_\lambda, \varphi_j\}(u)\right), \varphi_j^u\rangle_{L^2} = 0$. We define $\mathcal{N}_2 : \varphi \in L^2 \mapsto \|\varphi\|_{L^2}^2$, then $\mathcal{N}_2 \circ \varphi_j \equiv 1$ on $\mathcal{U}_N$. Then we have

$$0 = \mathrm{d}(\mathcal{N}_2 \circ \varphi_j)(u)\left(X_{\mathcal{H}_\lambda}(u)\right) = 2\mathrm{Re}\langle \varphi_j^u, \{\mathcal{H}_\lambda, \varphi_j\}(u)\rangle_{L^2}. \tag{5.12}$$

So there exists $r \in \mathbb{R}$ such that $B_u^\lambda \varphi_j^u - \{\mathcal{H}_\lambda, \varphi_j\}(u) = ir\varphi_j^u$ because $\mathrm{Ker}(\lambda_j^u - L_u) = \mathbb{C}\varphi_j^u$ by corollary 2.7 and formula (5.12). Since B_u^λ is a skew-adjoint operator on L^2_+ and $\gamma_j = \mathrm{Re}\langle G\varphi_j^u, \varphi_j^u\rangle_{L^2}$, we have $\{\mathcal{H}_\lambda, \gamma_j\}(u) = \mathrm{Re}\Big(\langle G\{\mathcal{H}_\lambda, \varphi_j\}(u), \varphi_j^u\rangle_{L^2} + \langle G\varphi_j^u, \{\mathcal{H}_\lambda, \varphi_j\}(u)\rangle_{L^2}\Big) = \mathrm{Re}\langle [G, B_u^\lambda]\varphi_j^u, \varphi_j^u\rangle_{L^2}$. Furthermore, for every $(\lambda, u) \in \mathcal{Y}$, formula (3.3) implies that $[G, T_{\overline{w}_\lambda(u)}] = 0$ and

$$[G, B_u^\lambda]f = i[G, T_{w_\lambda(u)}](T_{\overline{w}_\lambda(u)}(f) + f) = -\tfrac{1}{2\pi}[(\overline{w}_\lambda(u)f)^\wedge(0^+) + \hat{f}(0^+)]w_\lambda(u), \quad \forall f \in \mathbf{D}(G). \tag{5.13}$$

Since $(\overline{w}_\lambda(u)\varphi_j^u)^\wedge(0^+) = \langle \varphi_j^u, w_\lambda(u)\rangle_{L^2} = (\lambda + \lambda_j^u)^{-1}\overline{\langle u, \varphi_j^u\rangle}_{L^2}$ and $\overline{\langle u, \varphi_j^u\rangle}_{L^2} = -\lambda_j^u\widehat{\varphi_j^u}(0^+)$, we replace f by φ_j^u in formula (5.13) to obtain $\langle [G, B_u^\lambda]\varphi_j^u, \varphi_j^u\rangle_{L^2} = -\frac{\lambda}{(\lambda+\lambda_j^u)^2}$, $\forall (\lambda, u) \in \mathcal{Y}$. $\square$

Remark 5.8. Recall that $\tilde{\mathcal{H}}_\epsilon = \frac{1}{\epsilon}\mathcal{H}_{\frac{1}{\epsilon}}$ and $\tilde{B}_{\epsilon,u} := \frac{1}{\epsilon}B_u^{\frac{1}{\epsilon}}$ in Remark 2.18, $\forall(\epsilon^{-1}, u) \in \mathcal{Y}$. In general, the identity $(-1)^n\ \{E_n, \gamma_j\}(u) = \mathrm{Re}\langle [G, \frac{\mathrm{d}^n}{\mathrm{d}\epsilon^n}\big|_{\epsilon=0}\tilde{B}_{\epsilon,u}]\varphi_j^u, \varphi_j^u\rangle_{L^2}$ holds for every conservation law $E_n = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}\epsilon^n}\big|_{\epsilon=0}\tilde{\mathcal{H}}_\epsilon$ in the BO hierarchy, $\forall 1 \le j \le N$.

Corollary 5.9. *For any $j, k = 1, 2, \ldots, N$, we have $2\pi\{\lambda_j, \gamma_k\}(u) = \delta_{kj}$, $\{\lambda_k, \lambda_j\}(u) = 0$, $\forall u \in \mathcal{U}_N$.*

Proof. Given $u \in \mathcal{U}_N$, $\forall \lambda > \frac{\|u\|_{L^2}^2}{4C^4}$, we have $(\lambda, u) \in \mathcal{Y}$, then formula (5.7) and formula (5.10) imply that $-\frac{\lambda}{(\lambda+\lambda_j^u)^2} = \{\mathcal{H}_\lambda, \gamma_j\}(u) = 2\pi\sum_{k=1}^N\{\frac{\lambda}{\lambda+\lambda_k}, \gamma_j\}(u) = -2\pi\lambda\sum_{k=1}^N \frac{\{\lambda_k, \gamma_j\}(u)}{(\lambda+\lambda_k^u)^2}$ and $0 = \{\mathcal{H}_\lambda, \lambda_j\}(u) = 2\pi\lambda\sum_{k=1}^N \frac{\{\lambda_k, \lambda_j\}(u)}{(\lambda+\lambda_k^u)^2}$, $\forall j = 1, 2, \ldots, N$. The uniqueness of analytic continuation yields that $-\frac{z}{(z+\lambda_j^u)^2} = -2\pi z\sum_{k=1}^N \frac{\{\lambda_k, \gamma_j\}(u)}{(z+\lambda_k^u)^2}$ and $\sum_{k=1}^N \frac{\{\lambda_k, \lambda_j\}(u)}{(z+\lambda_k^u)^2} = 0$, $\forall z \in \mathbb{C}\backslash\mathbb{R}$. $\square$

Recall that the actions $I_j : u \in \mathcal{U}_N \mapsto 2\pi\lambda_j^u$ and the generalized angles $\gamma_j : u \in \mathcal{U}_N \mapsto \mathrm{Re}\langle G\varphi_j^u, \varphi_j^u\rangle_{L^2}$ are both real analytic functions by Proposition 4.6 and Corollary 4.7.

Proposition 5.10. *Given $u \in \mathcal{U}_N$, the family $\{\mathrm{d}I_1(u), \mathrm{d}I_2(u), \ldots \mathrm{d}I_N(u); \mathrm{d}\gamma_1(u), \mathrm{d}\gamma_2(u), \ldots \mathrm{d}\gamma_N(u)\}$ is linearly independent in the cotangent space $\mathcal{T}_u^*(\mathcal{U}_N)$. As a consequence, $\Phi_N : \mathcal{U}_N \to \Omega_N \times \mathbb{R}^N$ is a local diffeomorphism.*

Proof. Given $a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N \in \mathbb{R}$ such that $(\sum_{j=1}^N a_j\mathrm{d}I_j(u) + b_j\mathrm{d}\gamma_j(u))(h) = 0$, $\forall h \in \mathcal{T}_u(\mathcal{U}_N)$. Corollary 5.9 yields that $\forall j, k = 1, 2, \ldots, N$, we have $\mathrm{d}I_j(u)(X_{I_k}(u)) = \{I_k, I_j\}(u) = 0$ and $\mathrm{d}\gamma_j(u)(X_{I_k}(u)) = \{I_k, \gamma_j\}(u) = \delta_{jk}$. We replace h by $X_{I_k}(u)$ to obtain that $b_k = 0$. Then set $h = X_{\gamma_k}(u)$, we have $a_k = 0$. $\square$

Since all the actions $(I_j)_{1\le j\le N}$ are in evolution by Corollary 5.9 and the differentials $(\mathrm{d}I_j(u))_{1\le j\le N}$ are linearly independent for any $u \in \mathcal{U}_N$, the level set $\mathcal{L}_{\mathbf{r}} := \bigcap_{j=1}^N I_j^{-1}(r^j)$ is a real analytic Lagrangian submanifold of $\mathcal{U}_N$, $\forall \mathbf{r} = (r^1, r^2, \ldots, r^N) \in \Omega_N$. Moreover, $\mathcal{L}_{\mathbf{r}}$ is invariant under the Hamiltonian flow of I_j, $\forall j = 1, 2, \ldots, N$, by the Arnold–Liouville theorem.

5.3. The diffeomorphism property. This subsection is dedicated to proving the real bi-analyticity of $\Phi_N : \mathcal{U}_N \to \Omega_N \times \mathbb{R}^N$. It remains to show its surjectivity. The proof is based on Hadamard's global inverse function theorem.

Theorem 5.11. (Hadamard) *Suppose X and Y are connected smooth manifolds, then every proper local diffeomorphism $F : X \to Y$ is surjective. If Y is simply connected in addition, then every proper local diffeomorphism $F : X \to Y$ is a diffeomorphism.*

Lemma 5.12. *The map $\Phi_N : \mathcal{U}_N \to \Omega_N \times \mathbb{R}^N$ is proper.*

Proof. If K is compact in $\Omega_N \times \mathbb{R}^N$, we choose $u_n \in \Phi_N^{-1}(K)$, so

$$\Phi_N(u_n) = (2\pi\lambda_1^{u_n}, 2\pi\lambda_2^{u_n}, \ldots, 2\pi\lambda_N^{u_n}; \gamma_1(u_n), \gamma_2(u_n), \ldots, \gamma_N(u_n)) \in K, \qquad \forall n \in \mathbb{N}.$$

We assume that there exists $(2\pi\lambda_1, 2\pi\lambda_2, \ldots, 2\pi\lambda_N; \gamma_1, \gamma_2, \ldots, \gamma_N) \in K$ such that $\lambda_j^{u_n} \to \lambda_j$ and $\gamma_j(u_n) \to \gamma_j$ up to a subsequence. So $(M(u_n))_{n\in\mathbb{N}}$ converges to some matrix $M = (M_{kj})_{1\le k,j\le N} \in \mathbb{C}^{N\times N}$ whose coefficients are defined by $M_{kj} = \frac{i}{\lambda_k-\lambda_j}\sqrt{\frac{|\lambda_k|}{|\lambda_j|}}$, if $k \neq j$; $M_{jj} = \gamma_j - \frac{i}{2|\lambda_j|}$, $\forall 1 \le j,k \le N$. Lemma 5.6 yields that $\sigma_{\mathrm{pp}}(M) \subset \mathbb{C}_-$. We set $Q(x) := \det(x - M)$ and $u = i\frac{Q'}{Q} - i\frac{\overline{Q}'}{\overline{Q}} \in \mathcal{U}_N$. The Viète map $\mathbf{V}$ is defined in (4.1) and $\mathbf{V}(\mathbb{C}_-^N)$ is open in $\mathbb{C}^N$. Then there exists $\mathbf{a}^{(n)} = (a_0^{(n)}, a_1^{(n)}, \ldots, a_{N-1}^{(n)})$, $\mathbf{a} = (a_0, a_1, \ldots, a_{N-1}) \in \mathbf{V}(\mathbb{C}_-^N)$ such that $Q_n(x) = \det(x - M(u_n)) = \sum_{j=0}^{N-1} a_j^{(n)} x^j + x^N$ and $Q(x) = \sum_{j=0}^{N-1} a_j x^j + x^N$. We have $\lim_{n\to+\infty} Q_n(x) = Q(x)$, $\forall x \in \mathbb{R}$. So $\lim_{n\to+\infty} \mathbf{a}^{(n)} = \mathbf{a}$. The continuity of $\Gamma_N : \mathbf{a} = (a_0, a_1, \ldots, a_{N-1}) \in \mathbf{V}(\mathbb{C}_-^N) \mapsto \Pi u = i\frac{Q'}{Q} \in L_+^2$ yields that $\Pi u_n \to \Pi u$ in L_+^2, as $n \to +\infty$. Since $\mathcal{U}_N$ inherits the subspace topology of $L^2(\mathbb{R}, \mathbb{R})$, we have $(u_n)_{n\in\mathbb{N}}$ converges to u in $\mathcal{U}_N$. The continuity of the map Φ_N shows that $\Phi_N(u) = (2\pi\lambda_1, 2\pi\lambda_2, \ldots, 2\pi\lambda_N; \gamma_1, \gamma_2, \ldots, \gamma_N) \in K$. □

Proposition 5.13. *The map $\Phi_N : \mathcal{U}_N \to \Omega_N \times \mathbb{R}^N$ is a real analytic diffeomorphism.*

5.4. A Lagrangian submanifold. In general, the symplectomorphism property of Φ_N is equivalent to its Poisson bracket characterization (1.15). The first two formulas of (1.15), which are given in Corollary 5.9, lead us to focusing on the study of a special Lagrangian submanifold of $\mathcal{U}_N$, denoted by

$$\Lambda_N := \{u \in \mathcal{U}_N : \gamma_j(u) = 0, \quad \forall j = 1, 2, \ldots, N\}. \tag{5.14}$$

Lemma 5.14. *For every $u \in \mathcal{U}_N$, then each of the following four properties implies the others:*

(a) The N-soliton $u \in \Lambda_N$.
(b) For every $x \in \mathbb{R}$, we have $\overline{\Pi u}(x) = \Pi u(-x)$.
(c) The N-soliton u is an even function $\mathbb{R} \to \mathbb{R}$.
(d) The Fourier transform $\hat{u}$ is real-valued.

Proof. (a) $\Rightarrow$ (b) is obtained by (5.3) and (5.2). (b) $\Rightarrow$ (c) is given by the formula $u = \Pi u + \overline{\Pi u}$. (c) $\Rightarrow$ (d) is given by $\overline{u}(x) = u(x) = u(-x)$. Finally, (d) $\Rightarrow$ (a): fix $\lambda \in \sigma_{\mathrm{pp}}(L_u) = \{\lambda_1^u, \lambda_2^u, \ldots, \lambda_N^u\}$ and $\varphi \in \mathrm{Ker}(\lambda - L_u)$. Since both u and its Fourier transform $\hat{u}$ are real-valued, we have $[(\overline{\varphi})^\vee]^\wedge(\xi) = \overline{\hat{\varphi}(\xi)}$, where

$(\overline{\varphi})^{\vee}(x) := \overline{\varphi(-x)}, \forall x, \xi \in \mathbb{R}$. Since $T_u((\overline{\varphi})^{\vee}) = (\overline{T_u\varphi})^{\vee}$, we have $(\overline{\varphi})^{\vee} \in \mathrm{Ker}(\lambda - L_u)$. We choose the orthonormal basis $\{\varphi_1^u, \varphi_2^u, \ldots, \varphi_N^u\}$ in $\mathscr{H}_{\mathrm{pp}}(L_u)$ as in formula (4.9). Corollary 2.7 yields that $\dim_{\mathbb{C}} \mathrm{Ker}(\lambda - L_u) = 1$. There exists $\tilde{\theta}_j \in \mathbb{R}$ such that $(\overline{\varphi_j^u})^{\vee} = e^{i\tilde{\theta}_j}\varphi_j^u \Leftrightarrow \overline{(\varphi_j^u)^{\wedge}}(\xi) = e^{i\tilde{\theta}_j}(\varphi_j^u)^{\wedge}(\xi)$, $\forall \xi \in \mathbb{R}$, $\forall j = 1, 2, \ldots, N$. So we set $\phi_j^u := \exp(\frac{i\tilde{\theta}_j}{2})\varphi_j^u$, then its Fourier transform $(\phi_j^u)^{\wedge}$ is a real-valued function. Hence $\gamma_j(u) = \mathrm{Re}\langle G\phi_j^u, \phi_j^u\rangle_{L^2(\mathbb{R})} = -\frac{1}{2\pi}\mathrm{Im}\langle \partial_\xi[(\phi_j^u)^{\wedge}], (\phi_j^u)^{\wedge}\rangle_{L^2(0,+\infty)} = 0$. □

Lemma 5.15. *The level set Λ_N is a real analytic Lagrangian submanifold of $(\mathcal{U}_N, \omega)$.*

Proof. The map $\gamma : u \in \mathcal{U}_N \mapsto (\gamma_1(u), \gamma_2(u), \ldots, \gamma_N(u)) \in \mathbb{R}^N$ is a real analytic submersion by Proposition 5.10. So the level set Λ_N is a properly embedded real analytic submanifold of $\mathcal{U}_N$ and $\dim_{\mathbb{R}} \Lambda_N = N$. The classification of the tangent space $\mathcal{T}_u(\mathcal{U}_N)$ is given by Proposition 1.2. If $u(x) = \sum_{j=1}^N \frac{2\eta_j}{x^2+\eta_j^2}$, for some $\eta_j > 0$, then we have $\mathcal{T}_u(\Lambda_N) = \bigoplus_{j=1}^N \mathbb{R} f_j^u$, where $f_j^u(x) = \frac{2[x^2-\eta_j^2]}{[x^2+\eta_j^2]^2}$. We have $(f_j^u)^{\wedge}(\xi) = -2\pi|\xi|e^{-\eta_j|\xi|}$. Then by definition of ω, we have $\omega_u(h_1, h_2) = \frac{i}{2\pi}\int_{\mathbb{R}} \frac{\hat{h}_1(\xi)\overline{\hat{h}_2(\xi)}}{\xi}\mathrm{d}\xi = \frac{i}{2\pi}\int_{\mathbb{R}} \frac{\hat{h}_1(\xi)\hat{h}_2(\xi)}{\xi}\mathrm{d}\xi \in i\mathbb{R}$, $\forall h_1, h_2 \in \mathcal{T}_u(\Lambda_N)$. Since the symplectic form ω is real-valued, we have $\omega_u(h_1, h_2) = 0$, for every $h_1, h_2 \in \mathcal{T}_u(\Lambda_N)$. Since $\dim_{\mathbb{R}}(\Lambda_N) = N = \frac{1}{2}\dim_{\mathbb{R}} \mathcal{U}_N$, Λ_N is a Lagrangian submanifold of $\mathcal{U}_N$. □

5.5. The symplectomorphism property. Finally, we prove the assertion (b) in Theorem 5.2, i.e. the map $\Phi_N : (\mathcal{U}_N, \omega) \to (\Omega_N \times \mathbb{R}^N, \nu)$ is symplectic. We set $\Psi_N = \Phi_N^{-1} : \Omega_N \times \mathbb{R}^N \to \mathcal{U}_N$, let $\Psi_N^*\omega$ denote the pullback of the symplectic form ω by Ψ_N which is defined by (1.22). The goal of this subsection is to prove that

$$\tilde{\nu} := \Psi_N^*\omega - \nu = 0. \tag{5.15}$$

Lemma 5.16. *For every $u \in \mathcal{U}_N$, set $p = \Phi_N(u) \in \Omega_N \times \mathbb{R}^N$. Then we have*

$$\mathrm{d}\Phi_N(u)(X_{I_k}(u)) = \frac{\partial}{\partial\alpha^k}\Big|_p, \qquad \forall k = 1, 2, \ldots, N. \tag{5.16}$$

Proof. Fix $u \in \mathcal{U}_N$ and $p = \Phi_N(u)$, $\forall h \in \mathcal{T}_u(\mathcal{U}_N)$, we have $\mathrm{d}\Phi_N(u)(h) \in \mathcal{T}_p(\Omega_N \times \mathbb{R}^N)$. For every smooth function $f : \mathbf{p} = (r^1, r^2, \ldots, r^N; \alpha^1, \alpha^2, \ldots, \alpha^N) \in \Omega_N \times \mathbb{R}^N \mapsto f(\mathbf{p}) \in \mathbb{R}$, we have $(\mathrm{d}\Phi_N(u)(h))f = \mathrm{d}(f \circ \Phi_N)(u)(h) = \sum_{j=1}^N(\mathrm{d}I_j(u)(h)\frac{\partial f}{\partial r^j}\big|_p + \mathrm{d}\gamma_j(u)(h)\frac{\partial f}{\partial\alpha^j}\big|_p)$. For every $k = 1, 2, \ldots, N$, we replace h by $X_{I_k}(u) \in \mathcal{T}_u(\mathcal{U}_N)$, thus Corollary 5.9 yields that $\frac{\partial f}{\partial\alpha^k}\big|_p = \big(\mathrm{d}\Phi_N(u)(X_{I_k}(u))\big)f$. □

Lemma 5.17. *For every $1 \le j < k \le N$, there exists a smooth function $c_{jk} \in C^\infty(\Omega_N \times \mathbb{R}^N)$ such that*

$$\tilde{\nu} = \sum_{1\le j<k\le N} c_{jk}\mathrm{d}r^j \wedge \mathrm{d}r^k, \qquad \frac{\partial c_{jk}}{\partial\alpha^l}\Big|_p = 0, \quad \forall j, k, l = 1, 2, \ldots, N, \tag{5.17}$$

for every $p = (r^1, r^2, \ldots, r^N; \alpha^1, \alpha^2, \ldots, \alpha^N) \in \Omega_N \times \mathbb{R}^N$.

Proof. The proof is divided into three steps. The first step is to prove that for every $p \in \Omega_N \times \mathbb{R}^N$ and every $V \in \mathcal{T}_p(\Omega_N \times \mathbb{R}^N)$,

$$\tilde{\nu}_p(\tfrac{\partial}{\partial \alpha^l}\big|_p, V) = 0, \qquad \forall l = 1, 2, \ldots, N. \tag{5.18}$$

In fact, let $u = \Psi_N(p) \in \mathcal{U}_N$ and $p = (r^1, r^2, \ldots, r^N; \alpha^1, \alpha^2, \ldots, \alpha^N)$, so $r^l = r^l(p) = I_l \circ \Psi_N(p)$. Then we have $(\Psi_N^*\omega)_p(\frac{\partial}{\partial \alpha^l}\big|_p, V) = \omega_u(\mathrm{d}\Psi_N(p)\left(\frac{\partial}{\partial \alpha^l}\big|_p\right), \mathrm{d}\Psi_N(p)(V)) = \omega_u(X_{I_l}(u), \mathrm{d}\Psi_N(p)(V))$ by (5.16). Thus $(\Psi_N^*\omega)_p(\frac{\partial}{\partial \alpha^l}\big|_p, V) = -\mathrm{d}I_l(u)(\mathrm{d}\Psi_N(p)(V)) = -\mathrm{d}(I_l \circ \Psi_N)(p)(V)$. On the other hand, $\nu_p(\frac{\partial}{\partial \alpha^l}\big|_p, V) = \sum_{j=1}^N (\mathrm{d}r^j \wedge \mathrm{d}\alpha^j)(\frac{\partial}{\partial \alpha^l}\big|_p, V) = -\mathrm{d}r^l(p)(V)$. Thus (5.18) is obtained by $\tilde{\nu} = \Psi_N^*\omega - \nu$.

Since we have $\tilde{\nu} = \sum_{1 \le j < k \le N}(a_{jk}\mathrm{d}\alpha^j \wedge \mathrm{d}\alpha^k + b_{jk}\mathrm{d}r^j \wedge \mathrm{d}\alpha^k + c_{jk}\mathrm{d}r^j \wedge \mathrm{d}r^k)$, for some smooth functions $a_{jk}, b_{jk}, c_{jk} \in C^\infty(\Omega_N \times \mathbb{R}^N)$, the second step is to prove that $a_{jk} = b_{jk} = 0$ on $\Omega_N \times \mathbb{R}^N$, for every $1 \le j < k \le N$. In fact, we have $\mathrm{d}r^j \wedge \mathrm{d}r^k(\frac{\partial}{\partial \alpha^l}\big|_p, V) = 0$, $\mathrm{d}r^j \wedge \mathrm{d}\alpha^k(\frac{\partial}{\partial \alpha^l}\big|_p, V) = -\delta_{kl}\mathrm{d}r^j(p)(V)$ and $\mathrm{d}\alpha^j \wedge \mathrm{d}\alpha^k(\frac{\partial}{\partial \alpha^l}\big|_p, V) = \delta_{jl}\mathrm{d}\alpha^k(p)(V) - \delta_{kl}\mathrm{d}\alpha^j(p)(V)$. Let $l \in \{2, \ldots, N\}$ be fixed, $\forall 1 \le j < k \le N$,

$$\begin{aligned}&\sum_{1 \le l < k \le N} a_{lk}\mathrm{d}\alpha^k(p)(V) - \sum_{1 \le j < l \le N}(a_{jl}\mathrm{d}\alpha^j(p)(V) + b_{jl}\mathrm{d}r^j(p)(V))\\ &= \tilde{\nu}_p(\tfrac{\partial}{\partial \alpha^l}\big|_p, V) = 0.\end{aligned} \tag{5.19}$$

Then we replace V by $\frac{\partial}{\partial r^j}\big|_p$ and $\frac{\partial}{\partial \alpha^j}\big|_p$ respectively in (5.19), then $a_{jl} = b_{jl} = 0$, $\forall 1 \le j \le l - 1$.

It remains to show that c_{jk} depends only on $r^1, r^2, \ldots, r^N$, for every $1 \le j < k \le N$. The symplectic form ω is closed by Proposition 1.3 and $\nu = \mathrm{d}\kappa$ is exact, where $\kappa = \sum_{j=1}^N r^j \mathrm{d}\alpha^j$. So $\mathrm{d}\tilde{\nu} = \Psi_N^*(\mathrm{d}\omega) = 0$. Precisely, we have $\sum_{1 \le j < k \le N}\sum_{l=1}^N \left(\frac{\partial c_{jk}}{\partial \alpha^l}\mathrm{d}\alpha^l \wedge \mathrm{d}r^j \wedge \mathrm{d}r^k + \frac{\partial c_{jk}}{\partial r^l}\mathrm{d}r^l \wedge \mathrm{d}r^j \wedge \mathrm{d}r^k\right) = 0$. Since the family $\{\mathrm{d}r^j \wedge \mathrm{d}r^k \wedge \mathrm{d}\alpha^l\}_{1 \le j < k \le N, 1 \le l \le N} \bigcup \{\mathrm{d}r^j \wedge \mathrm{d}r^k \wedge \mathrm{d}r^l\}_{1 \le j < k < l \le N}$ is linearly independent in $\mathbf{\Omega}^3(\mathcal{U}_N)$, we have $\frac{\partial c_{jk}}{\partial \alpha^l} = 0$, for any $1 \le j < k \le N$ and $l = 1, 2, \ldots, N$. $\square$

Since the 2-form $\tilde{\nu}$ is independent of $\alpha^1, \alpha^2, \ldots, \alpha^N$, it suffices to consider points $p = (\mathbf{r}, \boldsymbol{\alpha}) \in \Omega_N \times \mathbb{R}^N$ with $\boldsymbol{\alpha} = 0$. We shall prove $\tilde{\nu} = 0$ by introducing the Lagrangian submanifold $\Omega_N \times \{0_{\mathbb{R}^N}\}$.

End of the proof of formula (5.15). We have $\Omega_N \times \{0_{\mathbb{R}^N}\} = \Phi_N(\Lambda_N)$, where Λ_N is the Lagrangian submanifold of $(\mathcal{U}_N, \omega)$ defined by (5.14). If $q \in \Omega_N \times \{0_{\mathbb{R}^N}\}$, set $v = \Psi_N(q) \in \Lambda_N$, we have

$$\mathcal{T}_q(\Omega_N \times \{0_{\mathbb{R}^N}\}) = \bigoplus_{j=1}^N \mathbb{R}\frac{\partial}{\partial r^j}\Big|_q = \mathrm{d}\Phi_N(v)(\mathcal{T}_v(\Lambda_N)). \tag{5.20}$$

For any point $p = (r^1, r^2, \dots, r^N; \alpha^1, \alpha^2, \dots, \alpha^N) \in \Omega_N \times \mathbb{R}^N$ and $\forall V_1, V_2 \in \mathcal{T}_p(\Omega_N \times \mathbb{R}^N)$, where $V_m = \sum_{j=1}^N \left(a_j^{(m)} \frac{\partial}{\partial r^j}\big|_p + b_j^{(m)} \frac{\partial}{\partial \alpha^j}\big|_p\right)$, $a_j^{(m)}, b_j^{(m)} \in \mathbb{R}$, $m = 1, 2$, we choose $q = (r^1, r^2, \dots, r^N; 0, 0, \dots, 0) \in \Omega_N \times \{0_{\mathbb{R}^N}\}$ and $W_1, W_2 \in \mathcal{T}_q(\Omega_N \times \{0_{\mathbb{R}^N}\})$, where $W_m = \sum_{j=1}^N a_j^{(m)} \frac{\partial}{\partial r^j}\big|_q$, $m = 1, 2$. We set $v = \Psi_N(q) \in \Lambda_N$. Since $c_{jk}(p) = c_{jk}(q)$, then (5.17) yields that $\tilde{\nu}_p(V_1, V_2) = \sum_{1 \le j < k \le N} (a_j^{(1)} a_k^{(2)} - a_k^{(1)} a_j^{(2)}) c_{jk}(p) = \tilde{\nu}_q(W_1, W_2) = \omega_v(\mathrm{d}\Psi_N(v)(W_1), \mathrm{d}\Psi_N(v)(W_2))$, because we have $\nu_q(W_1, W_2) = 0$. The identification (5.20) yields that $h_m := \mathrm{d}\Psi_N(v)(W_m) \in \mathcal{T}_v(\Lambda_N)$, for $m = 1, 2$. Consequently, we have $\tilde{\nu}_p(V_1, V_2) = \omega_v(h_1, h_2) = 0$. $\square$

6. Asymptotic Approximation

This section is dedicated to describing the asymptotic behavior of the multi-soliton solutions of (1.1).

Proof of corollary 1.11. Given $u \in \mathcal{U}_N$, we define $\mathfrak{M}(u) = (M_{jj}(u)\delta_{kj})_{1 \le k, j \le N}$, where M_{jj} is given in (1.18). Given $(t, x) \in \mathbb{R}^2$, we set $\mathfrak{A} = \mathfrak{A}(u, t, x) := \mathfrak{M}(u) - x - \frac{t}{\pi}\mathfrak{V}(u)$, where $\mathfrak{V}$ is given in Corollary 1.10. Then $\mathfrak{A}(u, t, x)^{-1} = (a_j(u, t, x)\delta_{kj})_{1 \le k, j \le N}$, where $a_j(x, t, u)^{-1} := \gamma_j(u) - x - \frac{t}{\pi} I_j(u) + \frac{\pi i}{I_j(u)}$. We set $\mathfrak{K}(u) := M(u) - \mathfrak{M}(u)$, then $\forall u_0 \in \mathcal{U}_N$, we have $u_\infty(t, x, u_0) = 2\mathrm{Im}\langle \mathfrak{A}(u_0, t, x)^{-1} X(u_0), Y(u_0)\rangle_{\mathbb{C}^N}$. If $u : t \in \mathbb{R} \mapsto u(t) \in \mathcal{U}_N$ solves the BO equation (1.1) such that $u(0) = u_0$ and $|t|$ is large, then

$$\begin{aligned} u(t, x) =& u(t, x; u_0) = 2\mathrm{Im}\langle (\mathfrak{A}(u_0, t, x) + \mathfrak{K}(u_0))^{-1} X(u_0), Y(u_0)\rangle_{\mathbb{C}^N} \\ =& u_\infty(t, x; u_0) + 2\mathrm{Im} \sum_{n \ge 2} \langle \left(-\mathfrak{A}(u_0, t, x)^{-1}\mathfrak{K}(u_0)\right)^n \mathfrak{A}(u_0, t, x)^{-1} X(u_0), Y(u_0)\rangle_{\mathbb{C}^N}. \end{aligned} \tag{6.1}$$

Given $R > 0$, we have $\|\mathfrak{A}(u_0, t)^{-1}\|_{L_x^\infty(-R,R)} \le \sum_{j=1}^N \left((\frac{|I_j(u_0)|}{\pi}|t| - R - |\gamma_j(u_0)|)^2 + \frac{\pi^2}{I_j(u_0)^2} \right)^{-\frac{1}{2}} \to 0$, when $|t| \to +\infty$. So there exists $\mathfrak{T}(u_0, R, N) > 0$ such that $2N^2 \|\mathfrak{A}(u_0, t)^{-1}\|_{L_x^\infty(-R,R)} \|\mathfrak{K}(u_0)\|_{\mathbb{C}^{N\times N}} < 1$, if $|t| \ge \mathfrak{T}(u_0, R, N)$. Moreover, $\|\mathfrak{A}(u_0, t)^{-1}\|_{L_x^2(\mathbb{R})}^2 \le \pi \sum_{j=1}^N \mathbf{k}_j(u_0)$. Then (6.1) yields that

$$\begin{aligned} \|u(t) - u_\infty(t)\|_{L_x^2(-R,R)} &\lesssim_{u_0,N} \sum_{n\ge 2} \| \left(-\mathfrak{A}(u_0, t)^{-1}\mathfrak{K}(u_0)\right)^n \mathfrak{A}(u_0, t)^{-1} \|_{L_x^2(-R,R)} \\ &\lesssim_{u_0,N} \|\mathfrak{A}(u_0, t)^{-1}\|_{L_x^\infty(-R,R)}^2 \|\mathfrak{K}(u_0)\|_{\mathbb{C}^{N\times N}}^2 \|\mathfrak{A}(u_0, t)^{-1}\|_{L_x^2(\mathbb{R})} \to 0 \end{aligned}$$

as $|t| \to +\infty$. Given $x \in \mathbb{R}$, similarly, there exists $\mathfrak{T}'(u_0, x, N) > 0$ such that the series of functions $t \in [\mathfrak{T}'(u_0, x, N), +\infty) \mapsto 2t^2\mathrm{Im} \sum_{n\ge 2} \langle \left(-\mathfrak{A}(u_0, t, x)^{-1}\mathfrak{K}(u_0)\right)^n \mathfrak{A}(u_0, t, x)^{-1} X(u_0), Y(u_0)\rangle_{\mathbb{C}^N} \in \mathbb{C}$ converges uniformly. Since $\lim_{t\to\pm\infty} t^2 u_\infty(t, x) = \sum_{j=1}^N \frac{2}{\mathbf{k}_j(u_0)^3}$ and $\lim_{t\to\pm\infty} t\mathfrak{A}(u_0, t, x)^{-1} = -\pi\mathfrak{V}(u_0)^{-1}$, we have $\frac{u(t,x)}{u_\infty(t,x)} = 1 + 2t^2\mathrm{Im} \sum_{n\ge 2} \langle \left(-\mathfrak{A}(u_0, t, x)^{-1}\mathfrak{K}(u_0)\right)^n \mathfrak{A}(u_0, t, x)^{-1} X(u_0), Y(u_0)\rangle_{\mathbb{C}^N} (t^2 u_\infty(t, x))^{-1} \to 1$, as $|t| \to +\infty$ by formula (6.1). $\square$

Acknowledgments. The author would like to express his sincere gratitude towards his PhD advisor Prof. Patrick Gérard for his deep insight, generous advice and continuous encouragement. He is grateful to the editor and the anonymous referee for their careful reading of this manuscript and for their very useful and valuable suggestions. He also would like to thank warmly Prof. Yang Cao and Prof. Dr. Thomas Kappeler for useful thematic and extended discussions. The author thanks the hospitality of Université Paris-Saclay, where much of this research work was carried out during the last year of his PhD studies. He acknowledges the financial support by the grant "ANAÉ" ANR-13-BS01-0010-03 of the 'Agence Nationale de la Recherche' and by the PhD fellowship of École Doctorale de Mathématique Hadamard. The final version of this manuscript was completed while the author was visiting Institute for Analysis and Collaborative Research Center (CRC) 1173 of Karlsruhe Institute of Technology in November 2020. The author is grateful to Prof. Dr. Xian Liao for her welcoming invitation, warm hospitality and valuable comments which make the exposition of this manuscript gain considerably in clarity. It is also a pleasure to acknowledge the financial support by the Deutsche Forschungsgemeinschaft (DFG) through CRC 1173.

Funding Open Access funding enabled and organized by Projekt DEAL.

References

1. Ablowitz, M., Fokas, A.: The inverse scattering transform for the Benjamin–Ono equation, a pivot for multidimensional problems. Stud. Appl. Math. **68**, 1–10 (1983)
2. Amick, C., Toland, J.: Uniqueness and related analytic properties for the Benjamin–Ono equation a nonlinear Neumann problem in the plane. Acta Math. **167**, 107–126 (1991)
3. Benjamin, T.B.: Internal waves of permanent form in fluids of great depth. J. Fluid Mech. **29**, 559–592 (1967)
4. Bock, T., Kruskal, M.: A two-parameter Miura transformation of the Benjamin–Ono equation. Phys. Lett. A **74**, 173–176 (1979)
5. Coifman, R., Wickerhauser, M.: The scattering transform for the Benjamin–Ono equation. Inverse Probl. **6**, 825–861 (1990)
6. Dobrokhotov, S.Y., Krichever, I.M.: Multiphase solutions of the Benjamin–Ono Equation and their averaging. Matematicheskie Zametki **49**(6), 42–50 (1991)
7. Gérard, P., Grellier, S.: Invariant tori for the cubic Szegő equation. Invent. Math. **187**(3), 707–754 (2012)
8. Gérard, P., Kappeler, T.: On the integrability of the Benjamin–Ono equation on the torus, arXiv:1905.01849, to appear in Commun. Pure Appl. Math.https://doi.org/10.1002/cpa.21896 (2020)
9. Grébert, B., Kappeler, T.: The Defocusing NLS Equation and Its Normal Form, Series of Lectures in Mathematics. European Mathematical Society, Zurich (2014)
10. Ionescu, A.D., Kenig, C.E.: Global well-posedness of the Benjamin–Ono equation in low regularity spaces. J. Am. Math. Soc. **20**(3), 753–798 (2007)
11. Kappeler, T., Pöschel, J.: KdV & KAM. Ergeb. der Math. und ihrer Grenzgeb, vol. 45. Springer, Berlin (2003)
12. Kaup, D., Matsuno, Y.: The inverse scattering for the Benjamin–Ono equation. Stud. Appl. Math. **101**, 73–98 (1998)
13. Lax, P.: Translation invariant spaces. Acta Math. **101**, 163–178 (1959)
14. Matsuno, Y.: Bilinear Transformation Method. Academic Press, New York (1984)
15. Nakamura, A.: Backlund transform and conservation laws of the Benjamin–Ono equation. J. Phys. Soc. Jpn. **47**, 1335–1340 (1979)
16. Neves, A., Lopes, O.: Orbital stability of double solitons for the Benjamin–Ono equation. Commun. Math. Phys. **262**, 757–791 (2006)
17. Ono, H.: Algebraic solitary waves in stratified fluids. J. Phys. Soc. Jpn. **39**, 1082–1091 (1975)

18. Pocovnicu, O.: Explicit formula for the solutions of the the cubic Szegő equation on the real line and applications. Discrete Contin. Dyn. Syst. A **31**(3), 607–649 (2011)
19. Reed, M., Simon, B.: Methods of Modern Mathematical Physics. Analysis of Operators, vol. 4. Academic Press, New York (1978)
20. Sun, R.: Long time behavior of the NLS–Szegő equation. Dyn. Partial Differ. Equ. **16**(4), 325–357 (2019)
21. Sun, R.: Filtering the L^2-critical focusing Schrödinger equation. Discrete Contin. Dyn. Syst. A **40**(10), 5973–5990 (2020)
22. Talbut, B.: Low regularity conservation laws for the Benjamin–Ono equation, arXiv:1812.00505 (to appear in Math. Res. Lett)
23. Tao, T.: Global well-posedness of the Benjamin–Ono equation in $H^1(\mathbb{R})$. J. Hyperbol. Differ. Equ. **1**, 27–49 (2004)
24. Wu, Y.: Simplicity and finiteness of discrete spectrum of the Benjamin–Ono scattering operator. SIAM J. Math. Anal. **48**(2), 1348–1367 (2016)
25. Wu, Y.: Jost solutions and the direct scattering problem of the Benjamin–Ono equation. SIAM J. Math. Anal. **49**(6), 5158–5206 (2017)

Communicated by H.-T. Yau