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Correction to: The Dependence on the Monodromy Data of the Isomonodromic Tau Function

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The note corrects the aforementioned paper (Bertola in Commun Math Phys 294(2):539–579, 2010). The consequences of the correction are traced and the examples updated.

1. Introduction

With this note I wish to correct two mistakes of loc. cit.; one of them (and its consequences) is of trivial nature and it is quickly disposed of. The second is significant, but not fatal, and requires rather extensive additional material, which is the reason of the relatively large size of this erratum.

I also wish to thank Oleg Lisovyy for pointing out certain inconsistencies that convinced me that indeed there was a mistake in need of fixing. The mistake was observed during the preparation of [8] and [9] because it was not consistent with their explicit computations: this prompted one of the authors to contact me.

The correction only affects certain types of setup; if the contours supporting the jumps of the Riemann–Hilbert Problem do not intersect, then the result is correct as it stands. In the subsequent works of my collaborators and myself only this type of problems were actually considered and therefore the following papers are essentially unaffected (although any reporting of the original formula is not correct in the stated generality): [2–6].

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2. Minimal Setup

The original setup requires to consider a Riemann–Hilbert Problem (RHP) with the following data (here reformulated with greater detail than in the original paper)
The Riemann–Hilbert data

1. a finite collection of smooth oriented arcs $\gamma_\nu,\ \nu = 1\dots K$, possibly meeting at a finite number of points but always in non-tangential way. We denote collectively these arcs by the symbol $\Sigma\gamma = \bigcup \gamma_\nu$.
2. a collection of $r \times r$ matrices $M_\nu(z)$, each of which analytic at each interior point of its corresponding arc γ_ν. We will denote collectively by $M(z)$ the matrix defined on $\Sigma\gamma$ that coincides with $M_\nu(z)$ on γ_ν,
$$\begin{aligned} M : \Sigma\gamma &\to SL_r(\mathbb{C}) \\ z &\mapsto \sum_\nu M_\nu(z)\chi_{\gamma_\nu}(z) \end{aligned} \tag{2.1}$$
where, for a set S, χ_S denotes its indicator function.
3. At each point c where several arcs meet, denoting by $\gamma_1, \dots, \gamma_\ell$ the arcs entering a suitably small disk at c, we require that the arcs approach c along distinct, well-defined directions and we impose that the jump matrices along its corresponding arc either
   - admit local analytic extension within said disk. In this case, if we denote by $\gamma_1, \dots, \gamma_n$ the contours incident at c, oriented outwards, and labelled counter-clockwise, and $M_\ell(z) = M(z)\Big|_{z\in\gamma_\ell}$, we require that the aforementioned analytic extensions satisfy
$$M_1(z) \cdot M_2(z) \cdots M_n(z) \equiv \mathbf{1}, \tag{2.2}$$
and this equality holds (locally) identically also with respect to the deformation parameters. Such an intersection point will be referred to as “essential" later on, for lack of better word.
   - tend to the identity matrix as $\mathcal{O}((z-c)^\infty)$ (faster than any power) in an open sector containing the direction of approach (this applies also to any jump matrix on contours extending to infinity, where $(z-c)$ is replaced by $1/z$) and admit analytic continuation on the universal cover of the punctured disk around c. Such an intersection point will be referred to as “inessential".

Problem 2.1 *(RHP). Find a holomorphic matrix* $\Gamma : \mathbb{C}\setminus\Sigma\gamma \to GL_n(\mathbb{C})$ *such that*

- $\Gamma_+(z) = \Gamma_-(z)M(z)\ z \in \Sigma\gamma$;
- $\Gamma(z),\ \Gamma^{-1}(z)$ *are uniformly bounded in* $\mathbb{C}$;
- $\Gamma(z_0) = \mathbf{1}$

Assuming that the solution exists for given initial data, [1] considered the *deformations* of the jump matrices (respecting the conditions listed above).

Remark 2.1. The conditions on the jump matrices laid out above ensure that the solution $\Gamma(z)$ admits analytic continuation in a neighbourhood of the intersection point c, or at least in the open sector around the direction of approach mentioned above. In this latter case the decay condition guarantees that the solution admits an asymptotic expansion near c in the same sector, and that the expansion coefficients do not depend on the sector. The conditions are modelled upon the case of RHPs associated to rational ODEs in the complex plane.

2.1. Corrections. The overall minus sign in (2.7) of Def. 2.1 [1] should be removed. While this is a definition, the purpose was to extend the Jimbo-Miwa-Ueno definition, and the correct sign should have been the opposite one. For convenience, here is the corrected definition. Of course the sign should be changed also in the subsequent formulæ.

Definition 2.1. (Def 2.1 in [1]) Let ∂ denote the derivative w.r.t. one of the parameters s and assume that the Riemann–Hilbert Problem 2.1 admits a solution in an open subset of the deformation-parameter space.[1] Then we define Malgrange's form ω_M

$$\omega_M(\partial) = \omega_M(\partial; [\Gamma]) := \int_{\Sigma\gamma} \mathrm{Tr}\left(\Gamma_-^{-1}(z)\Gamma_-'(z)\Xi_\partial(z)\right)\frac{\mathrm{d}z}{2i\pi}$$
$$\Xi_\partial(z) := \partial M(z)M^{-1}(z)\,. \tag{2.3}$$

New notation In order to deal more expeditiously with the correction we shall also use the matrix-valued forms (Maurer-Cartan like) $\Xi(z) := \delta M(z)M^{-1}(z)$, where δ shall denote henceforth the exterior derivative with respect to the deformation parameters $\boldsymbol{t}$ (not to confuse it with $\mathrm{d}z$ of the spectral variable). We shall also retain the notation Ξ_∂ for the contraction of said form with a vector field ∂.

Proposition 2.1 in [1] offers an incomplete formula for the exterior derivative of ω_M and we correct it now. The additional term in the following Theorem is present only when there are points of $\Sigma\gamma$ with several incident arcs; we call this the "set of vertices" of $\Sigma\gamma$ and denote it by $\mathfrak{V}$. If $\Sigma\gamma$ consists in the union of smooth disjoint arcs, or all the jump matrices tend to the identity at all the vertices, then the original statement stands correct.

Theorem 2.1 (Replaces Prop. 2.1 of [1]). *Denote by $\mathfrak{V} \ni v$ the vertices of the graph $\Sigma\gamma$; let $\mathcal{E}_v = \bigcup_{j=1}^{n_v} \gamma_j^{(v)}$ be the set of arcs incident to v, oriented outwards and enumerated counterclockwise. Then exterior derivative of ω_M is*

$$\delta\omega_M = -\frac{1}{2}\int_{\Sigma\gamma}\frac{\mathrm{d}z}{2i\pi}\mathrm{Tr}\left(\frac{\mathrm{d}}{\mathrm{d}z}\Xi(z)\wedge\Xi(w)\right)\bigg|_{w=z} + \eta_{\mathfrak{V}} \tag{2.4}$$

with

$$\eta_{\mathfrak{V}} := \frac{-1}{4i\pi}\sum_{v\in\mathfrak{V}}\sum_{\ell=2}^{n_v}\sum_{m=1}^{\ell-1}\mathrm{Tr}\left(M^{(v)}_{[1:m-1]}\Xi^{(v)}_m M^{(v)}_{[m:\ell-1]}\wedge\Xi^{(v)}_\ell M^{(v)}_{[\ell:n_v]}\right) \tag{2.5}$$

$$\Xi_\ell^{(v)} = \lim_{z\to v}\delta M_\ell^{(v)}(M_\ell^{(v)})^{-1}\Big|_{z\in\gamma_\ell^{(v)}} \qquad M_\ell^{(v)} = \lim_{\substack{z\to v\\ z\in\gamma_\ell^{(v)}}} M^{\epsilon_\ell}(z). \tag{2.6}$$

where the power $\epsilon_\ell = 1$ if the contour $\gamma_\ell^{(v)}$ is oriented away from v and $\epsilon_\ell = -1$ if oriented towards. Here the subscript $_{[m:\ell-1]}$ is a shorthand to signify the product of the corresponding matrices over the range of indices $m, m+1, \ldots, \ell-1$.

The complete proof is reported in Sect. 3. The "modified Malgrange form" Ω (Def. 2.2 in [1]) is (with the corrected sign)

[1] The small-norm theorem for Riemann–Hilbert problems implies that if a RHP is solvable, then any sufficiently small deformation (in L^2 and L^∞ norms) of the jump matrices leads to a solvable RHP. With our assumptions on the s-dependence of the jump matrices this implies that the subset of solvable RHP is an open set (if non-empty).

Definition 2.2 The modified Malgrange differential is defined as $\Omega := \omega_M + \vartheta$ with

$$\vartheta(\partial) := \frac{1}{2}\int_{\Sigma\gamma} \mathrm{Tr}\left(M'(z)M^{-1}(z)\partial M(z)M^{-1}(z)\right)\frac{\mathrm{d}z}{2i\pi} \tag{2.7}$$

Equivalently (see (2.32) in [1])

$$\Omega(\partial;[\Gamma]) = \frac{1}{2}\int_{\Sigma\gamma} \mathrm{Tr}\left(\Gamma_-^{-1}(z)\Gamma_-'(z)\partial M(z)M^{-1}(z) + \Gamma_+^{-1}(z)\Gamma_+'(z)M^{-1}(z)\partial M(z)\right)\frac{\mathrm{d}z}{2i\pi} \tag{2.8}$$

Consequent to the correction of Prop. 2.1, the ancillary result below (Prop. 2.2 in [1]) is also similarly modified

Proposition 2.1 *The curvature of the modified Malgrange form is*

$$\delta\Omega = -\frac{1}{2}\int_{\Sigma\gamma} \mathrm{Tr}\left(M'(z)M^{-1}(z)\Xi(z)\wedge\Xi(z)\right)\frac{\mathrm{d}z}{2i\pi} + \eta_{\mathfrak{V}} \tag{2.9}$$

2.2. Rational differential equations (amended). In the setup of Sec. 2.2, the term $\eta_{\mathfrak{V}}$ is closed and admits a potential $\theta_{\mathfrak{V}}$; consequently in Sec. 2.2 of [1] the title should read: "Submanifolds of $\mathcal{G}$ where $\delta\Omega - \eta_{\mathfrak{V}} = 0$". Sections 2.3, 2.4 are unaffected.

The main application of the original paper was to Riemann–Hilbert problems related to the setting of [10], i.e. the generalized monodromy data associated to a (generic) rational connection on $\mathbb{CP}^1$.

The statement that Ω is a closed one-form is incorrect in the stated generality and needs to be corrected.

To explain the necessary modifications we keep the same setup of Sections 3,4,5 (and Fig. 5, 6 of [1]).

Theorem 2.2 (Replaces Thm. 5.1 in [1]) *There exists a locally defined one form $\theta_{\mathfrak{V}} = \theta_{\mathfrak{V}}(\boldsymbol{L}, \boldsymbol{S}, \boldsymbol{C})$ on the manifold of generalized monodromy data (independent of the Birkhoff invariants and the position of the poles) such that the Jimbo–Miwa–Ueno tau function satisfies*

$$\delta \ln \tau(\boldsymbol{T}, \boldsymbol{a}, \boldsymbol{L}, \boldsymbol{S}, \boldsymbol{C}) = \omega_M - \theta_{\mathfrak{V}}, \tag{2.10}$$

where $\delta\theta_{\mathfrak{V}} = \eta_{\mathfrak{V}}$ in (2.5). This function is defined up to nonzero multiplicative constant and it vanishes precisely when the Riemann–Hilbert problem is not solvable, namely, on the Malgrange Theta-divisor.

Remark 2.2 Note that the theorem is now stated directly in terms of ω_M rather than the "modified" form Ω used in the original paper. The two forms differ by an explicit one form so there is little simplification in choosing one over the other, since neither is closed by itself in the relevant case.

Remark 2.3 Since $\theta_{\mathfrak{V}}$ is only locally defined on the monodromy manifold, the formula (2.10) allows to identify τ as a section of a line bundle on said manifold. The transition functions are given by $\delta \ln g = \widetilde{\theta_{\mathfrak{V}}} - \theta_{\mathfrak{V}}$ on the overlap of two open charts. This observation, which stems from the correction term in Thm. 2.1 seems to be of interest in applications that are arising from recent works [8] and deserves further study.

Rather than chasing a complete generality we illustrate the statement in several significant cases.

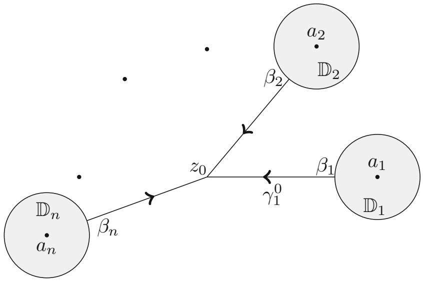

Fig. 1. The arrangement of disks for the scalar case

2.3. Example 0: Scalar Fuchsian case. Consider the scalar RHP (see Fig. 1)

$$\Gamma(z) = \begin{cases} \prod_{j=1}^{n} \left(z - a_j\right)^{\theta_j} & \mathbb{C}\setminus \bigcup \mathbb{D}_j \\ \prod_{j=1, j\neq k}^{n} \left(z - a_j\right)^{\theta_j} & z \in \mathbb{D}_k \\ & \sum \theta_j = 0. \end{cases} \tag{2.11}$$

$$\Gamma_-^{-1}\Gamma_+ = \begin{cases} (z - a_k)^{-\theta_k} & z \in \gamma_k = \partial\mathbb{D}_k \\ \mathrm{e}^{-2i\pi\theta_k} & z \in \gamma_k^0 \end{cases} \tag{2.12}$$

where γ_k^0 is a contour $[\beta_k, z_0]$, with β_k chosen and fixed on γ_k and z_0 a fixed basepoint outside, chosen in such a way that no two points a_j are not on the same ray from z_0. The Malgrange one-form is

$$\begin{aligned} \omega_M &= \int \Gamma_-^{-1}\Gamma_-' \delta M M^{-1} \frac{\mathrm{d}z}{2i\pi} \\ &= \sum_{j=1}^{n} \int \frac{\theta_j}{z - a_j} \sum_k \left(\left(\frac{\theta_k \,\mathrm{d}a_k}{z - a_k} - \ln(z - a_k)\,\mathrm{d}\theta_k \right) \chi_{\partial\mathbb{D}_k} - 2i\pi \,\mathrm{d}\theta_k \chi_{\gamma_k^0} \right) \\ \frac{\mathrm{d}z}{2i\pi} &= \text{(contour deformation)} \\ &= \sum_{j=1}^{n} \left[\sum_{k\neq j} \left(\frac{\theta_j\theta_k \,\mathrm{d}a_k}{a_k - a_j} - \theta_j \,\mathrm{d}\theta_k \int_{a_k}^{z_0} \frac{\mathrm{d}z}{z - a_j} \right) - \frac{1}{4i\pi}\theta_j \,\mathrm{d}\theta_j \ln^2(z - a_j) \Big|_{\beta_j}^{\beta_j+\gamma_j} \right. \\ &\quad \left. -\theta_j \,\mathrm{d}\theta_j \int_{\beta_j}^{z_0} \frac{\mathrm{d}z}{(z - a_j)} \right] \\ &= \sum_{j=1}^{n} \left[\sum_{k\neq j} \left(\frac{\theta_j\theta_k \,\mathrm{d}a_k}{a_k - a_j} - \theta_j \,\mathrm{d}\theta_k \big(\ln(z_0 - a_j) - \ln(a_k - a_j) \big) \right) \right. \\ &\quad \left. + - \frac{1}{4i\pi}\theta_j \,\mathrm{d}\theta_j (4i\pi \ln(\beta_j - a_j) + (2i\pi)^2) - \theta_j \,\mathrm{d}\theta_j (\ln(z_0 - a_j) - \ln(\beta_j - a_j)) \right] \end{aligned} \tag{2.13}$$

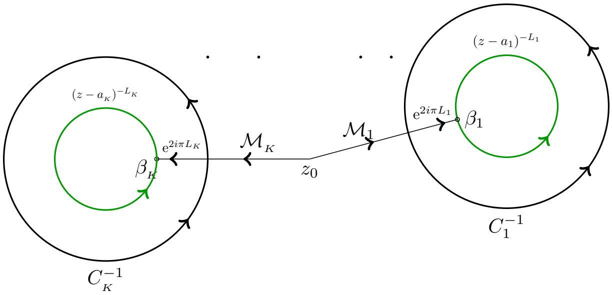

Fig. 2. The arrangement of jumps for a generic Fuchsian system

Here all logarithms are principal; the term involving z_0 drop out because $\sum \mathrm{d}\theta_k = 0$, as well as the dependence on β_j. We are left with

$$\omega_M = \sum_{j=1}^{n} \left[\sum_{k\neq j} \left(\frac{\theta_j \theta_k \, \mathrm{d}a_k}{a_k - a_j} + \theta_j \, \mathrm{d}\theta_k \ln(a_k - a_j) \right) - i\pi \theta_j \, \mathrm{d}\theta_j \right] \tag{2.14}$$

Here the logarithms are all principal. The exterior derivative of the above expression is

$$\begin{aligned} \delta\omega_M = \eta_{\mathfrak{V}} = \sum_j \sum_{k\neq j} \mathrm{d}\theta_j \wedge \mathrm{d}\theta_k (\ln(a_k - a_j) \\ - \ln(a_j - a_k)) = i\pi \sum_j \sum_{k<j} \mathrm{d}\theta_j \wedge \mathrm{d}\theta_k = i\pi \, \mathrm{d} \left(\sum_j \sum_{k<j} \theta_j \, \mathrm{d}\theta_k \right) \end{aligned} \tag{2.15}$$

In this case the Tau function is explicit

$$\delta \ln \tau = \omega_M + i\pi \sum_{k<\ell} \theta_k \, \mathrm{d}\theta_\ell \ , \quad \tau(\boldsymbol{a}, \boldsymbol{\theta}) = \prod_{\ell=1}^{n} \prod_{k<\ell} (a_k - a_\ell)^{\theta_k \theta_\ell} \prod_{k=1}^{n} \mathrm{e}^{-\frac{i\pi}{2}\theta_k^2} \tag{2.16}$$

To be noted, there is an ambiguity in the above writing because of the determinations of the logarithm; the ambiguity is what defines the line bundle of which τ is a section.

Suppose that the RHP corresponds to the solution of a generic Fuchsian ODE with simple poles at $a_1, \ldots, a_K$ of the form

$$\Psi'(z) = \sum_{j=1}^{K} \frac{A_j}{z - a_j} \Psi(z) \ , \quad A_j = O_j L_j O_j^{-1}, \quad L_j = \text{ diagonal} \, . \tag{2.17}$$

We set $\Lambda_j = \mathrm{e}^{2i\pi L_j}$ (diagonal) and the monodromy matrices are $\mathcal{M}_j := C_j^{-1} \Lambda_j C_j$. The enumeration is counterclockwise from the basepoint z_0 as indicated in Fig. 2. We have the condition

$$\mathcal{M}_1 \cdots \mathcal{M}_K = \mathbf{1} \tag{2.18}$$

Then a direct computation using Thm. 2.1 yields

$$\delta\omega_M = \frac{-1}{4i\pi}\left(\sum_{\ell=2}^{K}\sum_{1\le\ell<k}\mathrm{Tr}\Big(\mathcal{M}_{[1:\ell-1]}\delta\mathcal{M}_\ell\mathcal{M}_{[\ell+1:k-1]}\wedge\delta\mathcal{M}_k\mathcal{M}_{[k+1:K]}\Big)\right.$$
$$\left.+\sum_{\ell=1}^{K}\mathrm{Tr}\Big(\Lambda_\ell\delta C_\ell C_\ell^{-1}\wedge\Lambda_\ell^{-1}\delta C_\ell C_\ell^{-1}+2\Lambda_\ell^{-1}\delta\Lambda_\ell\wedge\delta C_\ell C_\ell^{-1}\Big)\right) \tag{2.19}$$

As announced in Thm. 2.2, $\delta\omega_M$ is *independent of the poles' positions*. It is a closed two-form on the monodromy variety (2.18) itself. It is not immediate to verify directly from the formula that the two-form in (2.19) is actually closed, but it is a consequence of the Theorem 2.1.[2]

On the other hand, since it is a closed two-form on the monodromy variety (2.18), it follows that there is a locally defined one-form on the monodromy variety, which we shall denote $\theta_{\mathfrak{V}}$, such that $\delta\theta_{\mathfrak{V}} = \delta\omega_M$.

We remark for the reader that the left hand side of (2.19) is the result of a partial cancellation of terms between the two terms in (2.4).

Example 2.1 The simplest example of Painlevé VI requires to describe explicitly the one-form $\theta_{\mathfrak{V}}$.

We shall assume that the monodromies $\mathcal{M}_{1,..4}$ are non-resonant (i.e. the eigenvalues of L_j do not differ by integers). This, however, proves to be too complicated to handle explicitly in complete generality, so we consider, by the way of example, the following particular submanifold of (2.18);

$$C_1 = \begin{bmatrix}1 & s_1\\ 0 & 1\end{bmatrix},\quad C_2 = \begin{bmatrix}1 & 0\\ s_2 & 1\end{bmatrix},\quad C_3 = \begin{bmatrix}1 & s_3\\ 0 & 1\end{bmatrix},\quad C_4 = \begin{bmatrix}1 & 0\\ s_4 & 1\end{bmatrix} \tag{2.20}$$

We set $\Lambda_j = \mathrm{diag}(\lambda_j, \lambda_j^{-1})$; then we can solve the condition (2.18) for λ_4, s_2, s_4 on a suitable open subset of the above submanifold of the monodromy variety

$$\left\{\lambda_4 = -\frac{s_3\lambda_1\lambda_2\left(\lambda_3^2-1\right)}{\lambda_3 s_1\left(\lambda_1^2-1\right)},\quad s_2 = \frac{s_3\lambda_1{}^2\lambda_2{}^2\lambda_3{}^2 - s_3\lambda_1{}^2\lambda_2{}^2 + \lambda_1{}^2 s_1 - s_1}{s_3 s_1\left(\lambda_3^2-1\right)\left(\lambda_2^2-1\right)\left(\lambda_1^2-1\right)},\right.$$
$$\left. s_4 = -\frac{\left(s_3\lambda_1{}^2\lambda_2{}^2\lambda_3{}^2 - s_3\lambda_1{}^2\lambda_2{}^2 + \lambda_1{}^2 s_1 - s_1\right)\lambda_3{}^2}{\left(\lambda_1{}^2\lambda_3 s_1 - \lambda_3 s_1 + s_3\lambda_1\lambda_2 - s_3\lambda_1\lambda_2\lambda_3{}^2\right)\left(\lambda_1{}^2\lambda_3 s_1 - \lambda_3 s_1 + s_3\lambda_1\lambda_2\lambda_3{}^2 - s_3\lambda_1\lambda_2\right)}\right\} \tag{2.21}$$

so that the monodromy variety is now locally coordinatized $\lambda_1, \lambda_2, \lambda_3, s_1, s_3$. Then the two form $\delta\omega_M$ is

$$\delta\omega_M = \frac{(\lambda_1{}^2+1)d\lambda_1\wedge d\lambda_2}{2i\pi\lambda_2\left(\lambda_1{}^2-1\right)\lambda_1}$$
$$+\frac{(\lambda_1{}^2\lambda_3{}^2+\lambda_3{}^2+\lambda_1{}^2+1)d\lambda_1\wedge d\lambda_3}{2i\pi\left(\lambda_3{}^2-1\right)\lambda_3\lambda_1\left(\lambda_1{}^2-1\right)} - \frac{d\lambda_1\wedge ds_1}{2i\pi\lambda_1 s_1} + \frac{(\lambda_1{}^2+1)d\lambda_1\wedge ds_3}{2i\pi\lambda_1 s_3\left(\lambda_1{}^2-1\right)} +$$
$$-\frac{(\lambda_3{}^2+1)d\lambda_2\wedge d\lambda_3}{2i\pi\lambda_2\lambda_3\left(\lambda_3{}^2-1\right)} - \frac{d\lambda_2\wedge ds_1}{2i\pi s_1\lambda_2}$$

[2] The verification that the expression (2.19) is indeed closed on the manifold (2.18) was performed also directly with the aid of a computer in small cases; a direct proof would be desirable.

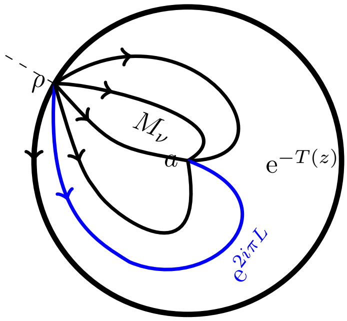

Fig. 3. The contours of the RHP within a "toral circle" (in the terminology of [1]). In the figure, $M_\nu = (z-a)^L \mathrm{e}^{T(z)} S_\nu \mathrm{e}^{-T(z)} (z-a)^{-L}$, the cut of the function $(z-a)^L$ is along the blue contour, $T(z)$ is of the form $T(z) = \sum_{j=1}^{r+1} T_j (z-a)^{-j} + T_0$ and T_0 is a constant diagonal matrix chosen so that $(\rho - a)^L \mathrm{e}^{T(\rho)} = \mathbf{1}$, where ρ is the point on the boundary of the toral circle where the various Stokes' contours meet

$$-\frac{d\lambda_2 \wedge ds_3}{2i\pi s_3 \lambda_2} - \frac{({\lambda_3}^2+1) d\lambda_3 \wedge ds_1}{2i\pi s_1 \left({\lambda_3}^2 - 1\right)\lambda_3} - \frac{d\lambda_3 \wedge ds_3}{2i\pi s_3 \lambda_3} + \frac{ds_1 \wedge ds_3}{2i\pi s_1 s_3} \tag{2.22}$$

and then a direct computation using the DeRham homotopy operator (after checking that the form above is indeed closed), we obtain

$$\begin{aligned}\theta_{\mathfrak{V}} = &\frac{\left({\lambda_3}^2 \ln(s_1 s_3) + \ln\left(\frac{s_1}{s_3}\right)\right) \mathrm{d}\lambda_3}{2i\pi \lambda_3 \left({\lambda_3}^2 - 1\right)}\\ &- \frac{\ln(s_3)\, \mathrm{d}s_1}{2i\pi s_1}\\ &+ \frac{\left(\ln\left(\frac{s_3 s_1 (\lambda_3^2 - 1)}{\lambda_3}\right)\right)\mathrm{d}\lambda_2}{2i\pi\lambda_2} - \frac{\left({\lambda_1}^2 \ln\left(\frac{s_3(\lambda_3^2-1)\lambda_2}{s_1\lambda_3}\right) + \ln\left(\frac{s_3 s_1 \lambda_2 (\lambda_3^2-1)}{\lambda_3}\right)\right)\mathrm{d}\lambda_1}{2i\pi\left({\lambda_1}^2 - 1\right)\lambda_1}\end{aligned}$$

2.5. Higher Poincaré rank singularities: the case of Painlevé II. If the system has also poles of higher order (under the same original genericity assumption that the leading coefficient matrix of the singular part of the connection is semi-simple), then the corresponding RHP has additional contours of jumps to account for the Stokes' phenomenon. In view of the correction in Thm. 2.1 we slightly modify their definition within the "toral circle" (Fig. 5 in [1]) as in Fig. 3. We consider the case of only one singularity of higher Poincaré rank for clarity. Supposing that $S_1, \ldots S_{2r}$ are the Stokes' matrices and L the diagonal matrix of the exponents of formal monodromy at a the singularity, they must satisfy

$$S_1 \cdots S_{2r} \mathrm{e}^{2i\pi L} = \mathbf{1} \tag{2.23}$$

and there is a contribution to $\eta_{\mathfrak{V}}$ in the form (we denote $S_{2r+1} = \mathrm{e}^{2i\pi L}$)

$$\eta_{\mathfrak{V}} = \frac{-1}{4i\pi} \sum_{\ell=1}^{2r+1} \sum_{1\le k<\ell} \mathrm{Tr}\left(S_{[1:k-1]} \delta S_k S_{[k+1;\ell-1]} \wedge \delta S_\ell S_{[\ell+1:2r+1]}\right) \tag{2.24}$$

which is a closed two-form on the manifold (2.23). This type of contributions comes one for each higher Poincaré rank singularity. As an illustration, the example of Painlevé II is instructive. We follow the general formulation of ([11], App.C); in this case we have six rays along $\varpi_\ell = \mathrm{e}^{(\ell-1)i\pi/3}\mathbb{R}_+$ and $\varpi_7 = \mathrm{e}^{-i\pi/6}\mathbb{R}_+$ with jumps

$$M_{1,3,5} = \begin{bmatrix} 1 & s_{1,3,5} \\ 0 & 1 \end{bmatrix}, \quad M_{2,4,6} = \begin{bmatrix} 1 & 0 \\ s_{2,4,6} & 1 \end{bmatrix}, \quad M_7 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \tag{2.25}$$

subject to the condition (2.2). We can solve for s_2, s_4, s_6 in in the subset $\{s_1 s_3 s_5 \lambda \neq 0\}$ to give

$$\left\{ s_2 = -\frac{s_5 + s_3\lambda + \lambda\, s_1}{s_3 \lambda\, s_1}, s_4 = -\frac{\lambda\, s_1 + s_5 + s_3}{s_3 s_5}, s_6 = -\frac{s_5 + s_3\lambda + s_1\lambda^2}{s_1 \lambda^2 s_5} \right\} \tag{2.26}$$

Then $\eta_{\mathfrak{V}}$ becomes

$$\begin{aligned}\eta_{\mathfrak{V}} = &\frac{\mathrm{d}s_1 \wedge \mathrm{d}\lambda}{2i\pi s_1 \lambda} - \frac{\mathrm{d}s_3 \wedge \mathrm{d}\lambda}{2i\pi\lambda\, s_3} \\ &+ \frac{\mathrm{d}s_5 \wedge \mathrm{d}\lambda}{2i\pi\lambda\, s_5} - \frac{\mathrm{d}s_3 \wedge \mathrm{d}s_1}{2i\pi s_3 s_1} + \frac{\mathrm{d}s_5 \wedge \mathrm{d}s_1}{2i\pi s_1 s_5} - \frac{\mathrm{d}s_5 \wedge \mathrm{d}s_3}{2i\pi s_3 s_5}\end{aligned} \tag{2.27}$$

Note that $\eta_{\mathfrak{V}}$ defines a *symplectic form* on the Stokes' manifold, whereby all coordinates λ, s_1, s_3, s_5 are log-canonical and the form $\theta_{\mathfrak{V}}$ is

$$\begin{aligned}\theta_{\mathfrak{V}} = &\ln\left(\frac{s_5 s_1}{s_3}\right)\frac{\mathrm{d}\lambda}{4i\pi\lambda} - \ln\left(\frac{s_3\lambda}{s_5}\right) \\ &\frac{\mathrm{d}s_1}{4i\pi s_1} + \ln\left(\frac{s_1\lambda}{s_5}\right)\frac{\mathrm{d}s_3}{4i\pi s_3} - \ln\left(\frac{s_1\lambda}{s_3}\right)\frac{\mathrm{d}s_5}{4i\pi s_5}\end{aligned} \tag{2.28}$$

The special case of PII as reported in [1] consists in

$$\begin{aligned}&\left\{\lambda = 1, s_1 = s_1, s_2 = -\frac{s_3 + s_1}{s_1 s_3 + 1}, s_3\right. \\ &\left.= s_3, s_4 = s_1, s_5 = -\frac{s_3 + s_1}{s_1 s_3 + 1}, s_6 = s_3\right\}\end{aligned} \tag{2.29}$$

$$\eta_{\mathfrak{V}} = \frac{\mathrm{d}s_1 \wedge \mathrm{d}s_3}{i\pi(s_1 s_3 + 1)}, \qquad \theta_{\mathfrak{V}} = \frac{\ln(s_1 s_3 + 1)}{2i\pi}\left(\frac{\mathrm{d}s_3}{s_3} - \frac{\mathrm{d}s_1}{s_1}\right) \tag{2.30}$$

3. Proof of Thm. 2.1

Lemma 3.1 *Let γ be an oriented smooth arc without self-intersection and let $\varphi : \gamma \times \gamma \to \mathbb{C}$ be a function which is locally analytic in each variable and such that $\varphi(z,w) = -\varphi(w,z)$. Then*

$$\int_\gamma \frac{\mathrm{d}z}{2i\pi}\int_\gamma \frac{\mathrm{d}w}{2i\pi}\frac{\varphi(w,z)}{(w-z_-)^2} = -\frac{1}{2}\int_\gamma \frac{\mathrm{d}z}{2i\pi}\partial_w \varphi(w,z)\Big|_{w=z}. \tag{3.1}$$

Proof It is shown in Section 7 of [7] that if $A(w,z)$ is Hölder (jointly) for $z, w \in \gamma$, then

$$\int_\gamma \frac{\mathrm{d}z}{2i\pi} \fint_\gamma \frac{\mathrm{d}w}{2i\pi} \frac{A(w,z)}{(w-z)} = \int_\gamma \frac{\mathrm{d}w}{2i\pi} \fint_\gamma \frac{\mathrm{d}z}{2i\pi} \frac{A(w,z)}{(w-z)}. \tag{3.2}$$

where $\fint$ denotes the Cauchy's principal value integral. We will use Sokhotski-Plemelj's formula

$$\int_\gamma \frac{\mathrm{d}w}{2i\pi} \frac{\Phi(w)}{w-z_\pm} = \fint_\gamma \frac{\mathrm{d}w}{2i\pi} \frac{\Phi(w)}{w-z} \pm \frac{1}{2}\Phi(z)\,, \quad z \in \gamma. \tag{3.3}$$

Since $\varphi(w,z)$ is (locally) jointly analytic in z, w, we can use Gakhov's result (3.2) with $A(w,z) := \frac{\varphi(w,z)}{(w-z)}$, which is now also a jointly analytic function of its variables in a neighbourhood of γ. Note that $A(z,z) = \partial_w \varphi(w,z)|_{w=z}$ is well defined and $A(z,w) = A(w,z)$. Now

$$\begin{aligned} &\int_\gamma \frac{\mathrm{d}z}{2i\pi} \int_\gamma \frac{\mathrm{d}w}{2i\pi} \left(\frac{\varphi(w,z)}{(w-z_-)^2} \right) \overset{(3.3)}{=} \\ &\quad \int_\gamma \frac{\mathrm{d}z}{2i\pi} \fint_\gamma \frac{\mathrm{d}w}{2i\pi} \frac{A(w,z)}{(w-z)} - \frac{1}{2} \int_\gamma \frac{\mathrm{d}z}{2i\pi} \partial_w \varphi(w,z) \bigg|_{w=z}. \end{aligned} \tag{3.4}$$

We now show that the principal value integral is zero; to this end we use $A(z,w) = A(w,z)$, which holds for our case. Then

$$\begin{aligned} &\int_\gamma \frac{\mathrm{d}z}{2i\pi} \fint_\gamma \frac{\mathrm{d}w}{2i\pi} \frac{A(w,z)}{(w-z)} = \int_\gamma \frac{\mathrm{d}z}{2i\pi} \fint_\gamma \frac{\mathrm{d}w}{2i\pi} \frac{A(z,w)}{(w-z)} \overset{(3.2)}{=} \\ &\quad \int_\gamma \frac{\mathrm{d}w}{2i\pi} \fint_\gamma \frac{\mathrm{d}z}{2i\pi} \frac{A(z,w)}{(w-z)} \overset{z\leftrightarrow w}{=} - \int_\gamma \frac{\mathrm{d}z}{2i\pi} \fint_\gamma \frac{\mathrm{d}w}{2i\pi} \frac{A(w,z)}{(w-z)} \end{aligned} \tag{3.5}$$

Thus the contribution of the principal value integral in (3.4) is zero. □

Proof of Thm. 2.1 We shall denote by $\Sigma\gamma_\epsilon = \Sigma\gamma \setminus \bigcup_{v\in\mathfrak{V}} \mathbb{D}_\epsilon^{(v)} =: \Sigma\gamma\setminus\mathbb{D}_\epsilon$ the support of the jumps minus small ϵ-disks around the point of self-intersection of $\Sigma\gamma$; we use the notation

$$\begin{aligned} \varphi(w,z) :&= \mathrm{Tr}\Big(\Gamma_-(w)\Xi(w)\Gamma_-^{-1}(w) \wedge \Gamma_-(z)\Xi(z)\Gamma_-^{-1}(z) \Big) \\ &\Rightarrow \quad \varphi(z,w) = -\varphi(w,z). \end{aligned} \tag{3.6}$$

Lemma 2.1 in [1] yields the formula (in the new notation)

$$\delta\left(\Gamma_-^{-1}(z)\Gamma_-'(z)\right) = \int_{\Sigma\gamma} \frac{\mathrm{d}w}{2i\pi} \frac{\Gamma_-^{-1}(z)\Gamma_-(w)\Xi(w)\Gamma_-^{-1}(w)\Gamma_-(z)}{(z_- - w)^2} \tag{3.7}$$

whence we compute the exterior derivative as follows

$$\begin{aligned} \delta\omega_M &= \delta \int_{\Sigma\gamma} \frac{\mathrm{d}z}{2i\pi} \mathrm{Tr}\left(\Gamma_-^{-1}(z)\Gamma_-'(z)\Xi(z) \right) = \\ &= \int_{\Sigma\gamma} \frac{\mathrm{d}z}{2i\pi} \int_{\Sigma\gamma} \frac{\mathrm{d}w}{2i\pi} \mathrm{Tr}\left(\frac{\Gamma_-^{-1}(z)\Gamma_-(w)\Xi(w)\Gamma_-^{-1}(w) \wedge \Gamma_-(z)\Xi(z)}{(z_- - w)^2} \right) \end{aligned} \tag{3.8}$$

$$+\int_{\Sigma\gamma}\frac{\mathrm{d}z}{2i\pi}\mathrm{Tr}\left(\Gamma_-^{-1}(z)\Gamma_-'(z)\delta\Xi(z)\right) \tag{3.9}$$

Note that $\Xi=\delta MM^{-1}$ satisfies $\delta\Xi=\Xi\wedge\Xi$ (i.e. $\delta\Xi\lfloor_{\partial,\widetilde{\partial}}=\partial\Xi_{\widetilde{\partial}}-\widetilde{\partial}\Xi_\partial=[\Xi_\partial,\Xi_{\widetilde{\partial}}]$) and hence (using the cyclicity of the trace)

$$\begin{aligned}\delta\omega_M=&\int_{\Sigma\gamma}\frac{\mathrm{d}z}{2i\pi}\int_{\Sigma\gamma}\frac{\mathrm{d}w}{2i\pi}\frac{\varphi(w,z)}{(z_--w)^2}\\&+\int_{\Sigma\gamma}\frac{\mathrm{d}z}{2i\pi}\mathrm{Tr}\left(\Gamma_-^{-1}(z)\Gamma_-'(z)\Xi(z)\wedge\Xi(z)\right)\end{aligned} \tag{3.10}$$

The issue is the computation of the iterated integral; as an iterated integral it is convergent but its value depends on the order of integration. To see why it is convergent, we need to make sure that the inner integral does not have too severe singularities; these may occur at the intersection points of the arcs because the derivative of the Cauchy transform may have poles. For an “inessential” intersection point (where the jump matrices tend exponentially to the identity) we can excise a small disk and easily evaluate the contribution to be infinitesimal as the radius tends to zero (this procedure will be considered in more detail later on). So let us consider the issue of local convergence near an “essential" intersection point $v\in\mathfrak{V}$. On the incident arcs $\gamma_1,\ldots,\gamma_n$ (in counterclockwise order) define

$$\Gamma_j(z):=\Gamma_-(z)\Big|_{z\in\gamma_j},\quad \varphi_{kj}(w,z)=\varphi(w,z)\Big|_{\substack{z\in\gamma_j\\ w\in\gamma_k}}\quad\text{etc.} \tag{3.11}$$

Without loss of generality we can assume that all arcs γ_j are oriented away from v. If a ray is incident, then this requires us to locally re-define $\Gamma_j\mapsto\Gamma_jM_j$, $M_j\mapsto M_j^{-1}$ and reverse the orientation in the integral, thanks to the obvious formula

$$\Gamma_k\partial M_kM_k^{-1}\Gamma_k^{-1}=-(\Gamma_kM_k)\partial(M_k^{-1})M_k(\Gamma_kM_k)^{-1} \tag{3.12}$$

(note that $\Gamma_-M=\Gamma_+$ becomes the $-$ boundary value in the reversed orientation).

Under Assumption (2.2), we can locally express the analytic extensions of each $\Gamma_j(z)$ in terms of the analytic extension of $\Gamma_1(z)$ to a full neighbourhood of $v\in\mathfrak{V}$;

$$\Gamma_k(z)=\Gamma_\ell(z)M_{[\ell:k-1]}(z)=\Gamma_1(z)M_{[1:k-1]}(z)\ ,\quad \ell<k. \tag{3.13}$$

where $M_{[a:b]}(z)=M_a(z)M_{a+1}(z)\cdots M_b(z)$. Taking the differential δ of the local condition (2.2) we find (evaluation being understood at $z=v$)

$$\sum_{j=1}^nM_{[1:j-1]}\Xi_jM_{[j:n]}=\sum_{j=1}^n\Gamma_1^{-1}\Gamma_j\Xi_j\Gamma_j^{-1}\Gamma_1\equiv0\ \Rightarrow\ \sum_{j=1}^n\Gamma_j\Xi_j\Gamma_j^{-1}\equiv0 \tag{3.14}$$

$$\Rightarrow\ \forall\ell=1,\ldots,n\quad\sum_{k=1}^n\varphi_{k\ell}(v,w)=\sum_{k=1}^n\varphi_{\ell k}(z,v)\equiv0 \tag{3.15}$$

Now, consider the part of the inner integral along γ_ℓ for $z\notin\Sigma\gamma$ near v; since $\varphi_{\ell,m}(z,w)$ extends to a locally analytic function in the neighborhood of v, the properties of the Cauchy transform immediately imply the following local identity of analytic functions

$$J_{\ell m}(z)=\int_{\gamma_\ell}\frac{\varphi_{\ell m}(w,z)}{(w-z)^2}\frac{\mathrm{d}w}{2i\pi}=\frac{\varphi_{\ell m}(v,v)}{z-v}+\ln_\ell(z-v)F_{\ell m}(z)+G_{\ell m}(z). \tag{3.16}$$

Here $F_{\ell m}, G_{\ell m}$ are locally analytic functions and $\ln_\ell(z-v)$ stands for the logarithm with the branch-cut extending along γ_ℓ. Now we see that the total integral over $\Sigma\gamma$ involves summing over the incident arcs at v and then (3.15) implies that the pole in (3.16) cancels out in the summation so that the integral is locally convergent in the ordinary sense (irrespectively of the boundary values of the logarithms). Having established the convergence of the integral, we now choose ϵ sufficiently small so that the various disks $\mathbb{D}_\epsilon^{(v)}$ are disjoint. We then have

$$\int_{\Sigma\gamma}\frac{\mathrm{d}z}{2i\pi}\int_{\Sigma\gamma}\frac{\mathrm{d}w}{2i\pi}\frac{\varphi(w,z)}{(z_- - w)^2} = \underbrace{\int_{\Sigma\gamma_\epsilon}\frac{\mathrm{d}z}{2i\pi}\int_{\Sigma\gamma_\epsilon}\frac{\mathrm{d}w}{2i\pi}\frac{\varphi(w,z)}{(z_- - w)^2}}_{A_\epsilon}$$
$$+\underbrace{\left(\int_{\Sigma\gamma_\epsilon}\frac{\mathrm{d}z}{2i\pi}\int_{\Sigma\gamma\cap\mathbb{D}_\epsilon}\frac{\mathrm{d}w}{2i\pi}+\int_{\Sigma\gamma\cap\mathbb{D}_\epsilon}\frac{\mathrm{d}z}{2i\pi}\int_{\Sigma\gamma_\epsilon}\frac{\mathrm{d}w}{2i\pi}\right)\frac{\varphi(w,z)}{(z-w)^2}}_{B_\epsilon}$$
$$+\underbrace{\int_{\Sigma\gamma\cap\mathbb{D}_\epsilon}\frac{\mathrm{d}z}{2i\pi}\int_{\Sigma\gamma\cap\mathbb{D}_\epsilon}\frac{\mathrm{d}w}{2i\pi}\frac{\varphi(w,z)}{(z_- - w)^2}}_{C_\epsilon} \tag{3.17}$$

The expression A_ϵ consists only of integrations over non-intersecting arcs and thus we can apply Lemma 3.1

$$A_\epsilon = -\frac{1}{2}\int_{\Sigma\gamma_\epsilon}\frac{\mathrm{d}z}{2i\pi}\partial_w\varphi(w,z)\bigg|_{w=z} \tag{3.18}$$

This term clearly admits a limit as $\epsilon = 0$ equal to A_0; a short computation gives

$$\partial_w\varphi(w,z)\bigg|_{w=z} \overset{(3.6)}{=} 2\mathrm{Tr}\left(\Gamma_-^{-1}(z)\Gamma_-'(z)\Xi(z)\wedge\Xi(z)\right)$$
$$+\mathrm{Tr}\left(\Xi'(z)\wedge\Xi(z)\right) \tag{3.19}$$

and hence

$$A_0 = -\int_{\Sigma\gamma}\frac{\mathrm{d}z}{2i\pi}$$
$$\left\{\mathrm{Tr}\left(\Gamma_-^{-1}(z)\Gamma_-'(z)\Xi(z)\wedge\Xi(z)\right)+\frac{1}{2}\mathrm{Tr}\left(\Xi'(z)\wedge\Xi(z)\right)\right\} \tag{3.20}$$

The remaining issue is the evaluation of B_ϵ, C_ϵ: as for B_ϵ we now show that it is identically zero. The inner integration in w and the outer integration in z have common points at $\Sigma\gamma\cap\partial\mathbb{D}_\epsilon$; let c be one of these points.
The integrand in B_ϵ is actually L^1 integrable over $\Sigma\gamma_\epsilon\times(\Sigma\gamma\cap\mathbb{D}_\epsilon)$ (and the reversed) because near the points common to those two sets (on the boundary of $\mathbb{D}_\epsilon$) the behaviour of the integrand is

$$\frac{\varphi(w,z)}{(z-w)^2} = \frac{C}{(z-w)} + \mathcal{O}(1) \tag{3.21}$$

and hence the local nature of the integral is the same as the convergent integral $\int_0^\epsilon \int_{-\epsilon}^0 \frac{\mathrm{d}x\,\mathrm{d}y}{x+y}$. Thus the interchange of order of integral is allowed by Fubini's theorem and we conclude that $B_\epsilon \equiv 0$ (using $\varphi(z,w) = -\varphi(w,z)$) for all ϵ (sufficiently small).

It remains to analyze the term C_ϵ; it is clear (due to the skew-symmetry of φ) that the only contributions to the double integral may come from (z, w) in a neighborhood of the *same* vertex $v \in \mathfrak{V}$. Moreover, a simple estimate shows that if v is an "inessential" vertex, then the contribution tends to zero as $\epsilon \to 0$. For this reason we focus below only on the "essential" vertices.

Consider one of them and denote the incident arcs in $\Sigma\gamma \cap \{|z-v| < \epsilon\} = \bigcup_{\ell=1}^{n_v} \gamma_\ell^{(v)}$ by $\gamma_\ell,\ \ell = 1,\ldots,n$; denote by σ_ℓ the distal endpoints of the arcs (at distance ϵ from v). We denote with $\varphi_{\ell m}^{(v)} = \lim_{\substack{z,w\to v\\ z\in\gamma_\ell, w\in\gamma_m}} \varphi_{\ell m}(z,w)$ for brevity (we also omit the superscript $^{(v)}$ since we consider one vertex at a time). Then

$$
\begin{aligned}
&\int_{\Sigma\gamma\cap\mathbb{D}_\epsilon^{(v)}} \frac{\mathrm{d}z}{2i\pi} \int_{\Sigma\gamma\cap\mathbb{D}_\epsilon^{(v)}} \frac{\mathrm{d}w}{2i\pi} \frac{\varphi(w,z)}{(z_- - w)^2} \\
&= \sum_{\ell=1}^{n}\sum_{m=1}^{n} \int_{\gamma_\ell} \frac{\mathrm{d}z}{2i\pi} \int_{\gamma_m} \frac{\mathrm{d}w}{2i\pi} \frac{\varphi_{m\ell}(w,z)}{(z_- - w)^2}
\end{aligned} \tag{3.22}
$$

The term with $\ell = m$ yields a convergent integral that is handled by Lemma 3.1 and tends to zero as $\epsilon \searrow 0$ since the length of γ_ℓ is $\mathcal{O}(\epsilon)$ (and the integrand is bounded). Let us now consider the remaining terms;

$$
\begin{aligned}
\mathbf{J} :=& \sum_\ell \int_{\gamma_\ell} \frac{\mathrm{d}z}{2i\pi} \sum_{m\neq\ell} \int_{\gamma_m} \frac{\mathrm{d}w}{2i\pi} \frac{\varphi_{m\ell}(w,z)}{(w-z)^2} = \\
=& \underbrace{\sum_\ell \int_{\gamma_\ell} \frac{\mathrm{d}z}{2i\pi} \sum_{m\neq\ell} \int_{\gamma_m} \frac{\mathrm{d}w}{2i\pi} \frac{\varphi_{m\ell}(w,z) - \varphi_{m\ell}}{(w-z)^2}}_{(\star)} \\
&+ \sum_\ell \int_{\gamma_\ell} \frac{\mathrm{d}z}{2i\pi} \sum_{m\neq\ell} \int_{\gamma_m} \frac{\mathrm{d}w}{2i\pi} \frac{\varphi_{m\ell}}{(w-z)^2}
\end{aligned} \tag{3.23}
$$

Each double integral in the sum marked $(\star)$ is a regularly convergent integral because of the regularization constant that we have added and subtracted; the other integral, on the contrary, is a singular integral and it depends on the order of integration. On the integral $(\star)$ we can swap order of integration and relabel $z \leftrightarrow w$, $\ell \leftrightarrow m$, and then use the skew symmetry $\varphi_{\ell m}(z,w) = -\varphi_{m\ell}(w,z)$, like so

$$
\begin{aligned}
\mathbf{J} =& \sum_m \int_{\gamma_m} \frac{\mathrm{d}w}{2i\pi} \sum_{\ell\neq m} \int_{\gamma_\ell} \frac{\mathrm{d}z}{2i\pi} \frac{\varphi_{m\ell}(w,z) - \varphi_{m\ell}}{(w-z)^2} \\
&+ \sum_\ell \int_{\gamma_\ell} \frac{\mathrm{d}z}{2i\pi} \sum_{m\neq\ell} \int_{\gamma_m} \frac{\mathrm{d}w}{2i\pi} \frac{\varphi_{m\ell}}{(w-z)^2} \overset{\substack{z\leftrightarrow w\\ \ell\leftrightarrow m}}{=} \\
=& \sum_\ell \int_{\gamma_\ell} \frac{\mathrm{d}z}{2i\pi} \sum_{m\neq\ell} \int_{\gamma_m} \frac{\mathrm{d}w}{2i\pi} \frac{\varphi_{\ell m}(z,w) - \varphi_{\ell m}}{(z-w)^2}
\end{aligned} \tag{3.24}
$$

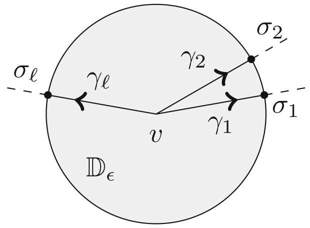

Fig. 4. Arcs near a vertex

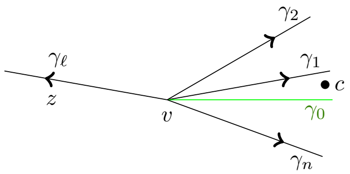

Fig. 5. Vertex contribution to the exterior derivative

$$+\sum_\ell \int_{\gamma_\ell} \frac{\mathrm{d}z}{2i\pi} \sum_{m\neq\ell} \int_{\gamma_m} \frac{\mathrm{d}w}{2i\pi} \frac{\varphi_{m\ell}}{(w-z)^2}. \tag{3.25}$$

In the step above we only have re-labeled the dummy variables of integration. Now we can use the skew-symmetry $\varphi_{\ell m}(z,w) = -\varphi_{m\ell}(w,z)$ and $\varphi_{\ell m} = -\varphi_{m\ell}$ in the first integral. The term containing $\varphi_{\ell m}(z,w)$ yields back $-\mathbf{J}$ while the term containing $\varphi_{m\ell}$ adds to the last integral in (3.25). Thus we continue the chain of equalities:

$$\begin{aligned}&= -\mathbf{J} + 2\sum_\ell \int_{\gamma_\ell} \frac{\mathrm{d}z}{2i\pi} \sum_{m\neq\ell} \int_{\gamma_m} \frac{\mathrm{d}w}{2i\pi} \frac{\varphi_{m\ell}}{(w-z)^2} = -\mathbf{J}\\ &\quad + 2\sum_\ell \int_{\gamma_\ell} \frac{\mathrm{d}z}{2i\pi} \sum_{m\neq\ell} \frac{\varphi_{m\ell}}{2i\pi(z-\sigma_m)}\end{aligned} \tag{3.26}$$

Here $\sigma_m = \gamma_m \cap \mathbb{D}_\epsilon$ (Fig. 4). Solving for $\mathbf{J}$ we obtain finally:

$$\mathbf{J} = \sum_\ell \int_{\gamma_\ell} \frac{\mathrm{d}z}{2i\pi} \sum_{m\neq\ell} \frac{\varphi_{m\ell}}{2i\pi(z-\sigma_m)} = \sum_\ell \sum_{m\neq\ell} \frac{\varphi_{m\ell}}{(2i\pi)^2} \ln_\ell\left(\frac{\sigma_\ell - \sigma_m}{v-\sigma_m}\right) \tag{3.27}$$

To compute the last expression we proceed as follows; first we observe that it is independent of v and σ_ℓ's. Indeed, differentiating we get

$$\begin{aligned}\partial_v \mathbf{J} &= \sum_\ell \sum_{m\neq\ell} \frac{\varphi_{m\ell}}{(2i\pi)^2} \frac{1}{v-\sigma_m} \overset{(3.15)+(\varphi_{\ell\ell}=0)}{=} 0\\ \partial_{\sigma_j} \mathbf{J} &= \sum_{m\neq j} \frac{\varphi_{mj}}{(2i\pi)^2} \frac{1}{\sigma_j - \sigma_m} + \sum_{\ell\neq j} \frac{\varphi_{j\ell}}{(2i\pi)^2} \left(\frac{1}{\sigma_j-\sigma_\ell} - \frac{1}{v-\sigma_j}\right) \overset{(3.15)}{=}\\ &= \sum_{m\neq j} \frac{\varphi_{mj}}{(2i\pi)^2} \frac{1}{\sigma_j - \sigma_m}\end{aligned}$$

$$+\sum_{m\neq j}\frac{\varphi_{jm}}{(2i\pi)^2}\left(\frac{1}{\sigma_j-\sigma_m}\right)\overset{\varphi_{m\ell}=-\varphi_{\ell m}}{=}0 \tag{3.28}$$

Thus we can compute $\mathbf{J}$ by arranging σ_ℓ's as we wish; to do so, we re-write it back as a double integral

$$\begin{aligned}\mathbf{J}&=\sum_\ell\int_{\gamma_\ell}\frac{\mathrm{d}z}{2i\pi}\sum_{m\neq\ell}\int_{\gamma_m}\frac{\mathrm{d}w}{2i\pi}\frac{\varphi_{m\ell}}{(z-w)^2}=\\&=\sum_\ell\int_{\gamma_\ell}\frac{\mathrm{d}z}{2i\pi}\sum_{m\neq\ell}\int_{\gamma_m}\frac{\mathrm{d}w}{2i\pi}\frac{\varphi_{m\ell}\left(1-\frac{v-c}{2(w-c)}-\frac{v-c}{2(z-c)}\right)}{(z-w)^2}\\&\quad+\frac{1}{2}\sum_\ell\int_{\gamma_\ell}\frac{\mathrm{d}z}{2i\pi}\sum_{m\neq\ell}\varphi_{m\ell}\int_{\gamma_m}\frac{\mathrm{d}w}{2i\pi}\frac{\frac{v-c}{w-c}+\frac{v-c}{z-c}}{(z-w)^2}\end{aligned} \tag{3.29}$$

The point c is an arbitrary point not on any of the segments. The first integral is a regular convergent integral (the integrand is in L^1 near $z=w=v$) and exchanging the order of integration and then renaming the variables yields the same expression with a minus sign: hence we conclude that it is zero. We are thus left with the second term, which we now know is also independent of σ_ℓ's. To compute it more conveniently, we send all σ_ℓ's to infinity along distinct directions: the value of the expression, as we know, is independent of σ's, so that now the arcs γ_ℓ are simply pairwise distinct rays issuing from v. In the limit we obtain the following

$$\begin{aligned}\mathbf{J}&=\underbrace{\frac{1}{2}\sum_\ell\int_{\gamma_\ell}\frac{\mathrm{d}z}{2i\pi}\sum_{m\neq\ell}\varphi_{m\ell}\int_{\gamma_m}\frac{\mathrm{d}w}{2i\pi}\frac{\frac{v-c}{z-c}}{(z-w)^2}}_{(\dagger)}\\&\quad+\underbrace{\frac{1}{2}\sum_\ell\int_{\gamma_\ell}\frac{\mathrm{d}z}{2i\pi}\sum_{m\neq\ell}\varphi_{m\ell}\int_{\gamma_m}\frac{\mathrm{d}w}{2i\pi}\frac{\frac{v-c}{w-c}}{(z-w)^2}}_{(\heartsuit)}\end{aligned} \tag{3.30}$$

The term $(\dagger)$ is zero: indeed it is

$$(\dagger)=\frac{1}{2}\sum_\ell\int_{\gamma_\ell}\frac{\mathrm{d}z}{2i\pi}\sum_{m\neq\ell}\varphi_{m\ell}\frac{\frac{v-c}{z-c}}{2i\pi(z-v)}\overset{(3.15)}{=}0. \tag{3.31}$$

In the remaining term $(\heartsuit)$, we write the inner integral in partial fractions

$$\begin{aligned}\int_{\gamma_m}\frac{\mathrm{d}w}{2i\pi}\frac{\frac{v-c}{w-c}}{(w-z)^2}&=\frac{v-c}{(z-c)^2}\int_{\gamma_m}\frac{\mathrm{d}w}{2i\pi}\left(\frac{1}{w-c}-\frac{1}{w-z}\right)\\&\quad+\frac{c-v}{c-z}\underbrace{\int_{\gamma_m}\frac{\mathrm{d}w}{2i\pi}\frac{1}{(w-z)^2}}_{=-\frac{1}{z-v}}\quad z\in\gamma_\ell\end{aligned} \tag{3.32}$$

The first integral depends on m; to make the dependence manifest, we choose c on the right of γ_1 but to the left of γ_n (the final result is independent of this choice) and rotate the contour of integration in (3.32) counterclockwise to a direction between c and γ_n (denoted γ_0, see Fig. 5):

$$\frac{v-c}{(z-c)^2}\int_{\gamma_m}\frac{\mathrm{d}w}{2i\pi}\left(\frac{1}{w-c}-\frac{1}{w-z}\right)$$
$$=\frac{v-c}{(z-c)^2}\int_{\gamma_0}\frac{\mathrm{d}w}{2i\pi}\left(\frac{1}{w-c}-\frac{1}{w-z}\right)-\frac{v-c}{(z-c)^2}\begin{cases}1 & m<\ell\\ 0 & m>\ell\end{cases} \tag{3.33}$$

The last term is due to the residue that we picked up while rotating γ_m counterclockwise, and the fact that $z\in\gamma_\ell$. Thus, summarizing

$$\int_{\gamma_m}\frac{\mathrm{d}w}{2i\pi}\frac{\frac{v-c}{w-c}}{(w-z)^2}=\frac{v-c}{(z-c)^2}\ln_0\left(\frac{v-z}{v-c}\right)+\frac{c-v}{(z-c)(z-v)}-\frac{v-c}{(z-c)^2}\begin{cases}1 & m<\ell\\ 0 & m>\ell\end{cases} \tag{3.34}$$

The first two terms are independent of m and hence they give a zero contribution to the term ($\heartsuit$) in (3.30) because of the condition (3.15). We are thus left with

$$\mathbf{J}=-\frac{1}{2}\sum_\ell\sum_{m<\ell}\varphi_{m\ell}\int_{\gamma_\ell}\frac{\mathrm{d}z}{2i\pi}\frac{v-c}{(z-c)^2}=-\frac{1}{4i\pi}\sum_\ell\sum_{m<\ell}\varphi_{m\ell} \tag{3.35}$$

Finally, one reads off the definition of $\varphi_{m\ell}$,

$$\varphi_{m\ell}=\lim_{\substack{z,w\to v\\ z\in\gamma_m,w\in\gamma_\ell}}\mathrm{Tr}\left(\Gamma_m\Xi_m\Gamma_m^{-1}\wedge\Gamma_\ell\Xi_\ell\Gamma_\ell^{-1}\right) \tag{3.36}$$

and using (3.13) we obtain the terms in the sum appearing in (2.5). $\square$

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