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Positive Lyapunov Exponent for Some Schrödinger Cocycles Over Strongly Expanding Circle Endomorphisms

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Abstract: We show that for a large class of potential functions and big coupling constant λ the Schrödinger cocycle over the expanding map $x \mapsto bx \pmod 1$ on $\mathbb{T}$ has a Lyapunov exponent $> (\log \lambda)/4$ for all energies, provided that the integer $b \geq \lambda^3$.

1. Introduction

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and let $T : \mathbb{T} \to \mathbb{T}$ be the expanding map $T(x) = bx \pmod 1$, where $b \geq 2$ is an integer. In this note we consider the Schrödinger cocycle on $\mathbb{T} \times \mathbb{R}^2$ defined by

$$F_E : (x, y) \mapsto (T(x), A_E(x)y)$$

where

$$A_E(x) = \begin{pmatrix} 0 & 1 \\ -1 & \lambda v(x) - E \end{pmatrix} \in SL(2, \mathbb{R})$$

and $v : \mathbb{T} \to \mathbb{R}$ is a continuous function, $\lambda \in \mathbb{R}$ is a coupling constant and $E \in \mathbb{R}$ is the energy parameter.

We let[1]

$$A_E^n(x) = A_E(T^{n-1}(x)) \cdots A_E(x), \ n \geq 1,$$

and define the (maximal) Lyapunov exponent by

$$L(E) = \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{T}} \log \|A_E^n(x)\| dx \ (\geq 0).$$

[1] We are interested in the time-evolution of F_E for fixed λ, v and b. Therefore we only indicate the dependence on E and n.

Recall that the Lebesgue measure on $\mathbb{T}$ is an invariant measure for T. Since T is ergodic with respect to this measure we have

$$\lim_{n\to\infty} \frac{1}{n} \log \|A_E^n(x)\| = L(E) \text{ for a.e. } x \in \mathbb{T}. \tag{1.1}$$

For the important connection to the discrete Schrödinger operator we refer to the articles [4–6, 12] and references therein.

A natural question is to ask under which conditions (on λ, v, b and E) we have $L(E) > 0$. Here we are especially interested in conditions on λ, v and b which guarantees $L(E) > \text{const.} > 0$ for all $E \in \mathbb{R}$. Besides the problem in itself, which has a general interest in the theory of non-uniformly hyperbolic dynamical systems, such uniform lower bounds are many times important for deriving finer properties of the associated Schrödinger operator (see, e.g., [3]).

Next follows a brief summary of previous results. It should be stressed that all the results hold for any $b \geq 2$.

In [6] it is shown that if v is measurable, bounded and non-constant, and $\lambda > 0$, then $L(E) > 0$ for a.e. $E \in \mathbb{R}$. Moreover, for small λ and smooth non-constant v one has $L(E) \approx \lambda^2$ for $\sqrt{\lambda} < |E| < 2 - \sqrt{\lambda}$ [4]; and for large λ, and under quite general conditions on v, one has $L(E) \gtrsim \log \lambda$ for all E outside an exponentially small (in λ) set [10, 12].

Furthermore, from Herman's subharmonic argument [7] it follows that if v is a non-constant trigonometric polynomial, then $L(E) \gtrsim \log \lambda$ for all $E \in \mathbb{R}$ and all large λ. However, whether the corresponding result holds for v a non-constant real-analytic function does not seem to be known (if instead $T(x) = x + \omega$, $\omega \in \mathbb{R} \setminus \mathbb{Q}$, this is a well-known result [11]). Steps towards a proof of this result are taken in [9]. Another situation where one has $L(E) \gtrsim \log \lambda$ for all $E \in \mathbb{R}$ and all large λ is when v is C^1 and monotone on (0, 1), with a discontinuity at $x = 0$ [15].

On the other hand, if $\varphi : \mathbb{T} \to \mathbb{R}$ is any bounded and measurable function such that $\int_{\mathbb{T}} \varphi(x)dx = 0$, and we let $v(x) = \exp(\varphi(T(x))) + \exp(-\varphi(x))$ and take $\lambda = 1$, then $L(0) = 0$ (see [2]). Thus there are obstacles for obtaining uniform (in E) lower bounds on $L(E)$.

The aim of this paper is to extend the above results to new situations. We first define the collection of potential functions v with which we shall work.

Definition 1.1. Let $\mathcal{V}^1(\mathbb{T}, \mathbb{R})$ denote the class of C^1-functions $v : \mathbb{T} \to \mathbb{R}$ which satisfy the following condition: there exist $\varepsilon_0 > 0$, $\beta > 0$ and an integer $s \geq 1$ such that for all $0 < \varepsilon \leq \varepsilon_0$ and all $a \in \mathbb{R}$ the set $\{x \in \mathbb{T} : |v(x) - a| < \varepsilon\}$ consists of at most s intervals, each of length at most ε^β.

Remark 1. It is easy to check that all non-constant real-analytic functions $v : \mathbb{T} \to \mathbb{R}$ belongs to $\mathcal{V}^1(\mathbb{T}, \mathbb{R})$. Note also that the assumption on v is very similar to [8, Definition 2.2].

Our main result is the following:

Theorem 1. *Assume that* $v \in \mathcal{V}^1(\mathbb{T}, \mathbb{R})$. *Then there is a* $\lambda_0 = \lambda_0(v) > 0$ *such that for all* $\lambda \geq \lambda_0$ *we have* $L(E) > (\log \lambda)/4$ *for all* $E \in \mathbb{R}$ *provided that* $b \geq \lambda^3$.

Remark 2. (a) We do not aim for optimal conditions on any of the constants or required size of b (as function of λ).

(b) It would be very interesting to know if the statement of Theorem 1 holds true for a fixed (large) b independent of λ. Unfortunately the method we use in the proof requires b to be (much) larger than λ. (Heuristically it should be more likely to have $L(E) > 0$ the bigger is λ.)
(c) We would like to stress that the proof of Theorem 1 does not use the fact that we have a linear system. In fact one can extend it to non-linear systems (as, e.g., we did in [1]) since what we analyze is the dynamics of forced circle diffeomorphisms (see the map (3.1)). However, since there is an interest in the Schrödinger cocycle, and for (hopefully!) transparency, we perform our analysis for this explicit system.
(d) The proof is based on ideas developed in [13]. Related problems (for systems which are not homotopic to the identity) are investigated in [1,14].

2. Preliminaries

We adopt the following convention on the coupling constant λ: In the statements of the lemmas below (where applicable) we always assume (without explicitly stating so) that $\lambda > 0$ is sufficiently large. There are only finitely many largeness conditions on λ, and they only depend on v. This will yield the constant λ_0 in the statement of Theorem 1.

2.1. Assumption on v. We assume from now on that $v \in \mathcal{V}(\mathbb{T}, \mathbb{R})$ is fixed, and that ε_0, β and s are as in Definition 1.1. Without loss of generality we assume, for simplicity, that $|v(x)| \leq 1/3$ for all $x \in \mathbb{T}$ (this only scales λ).

2.2. Projective action. Since

$$A_E(x)\begin{pmatrix}1\\ r\end{pmatrix} = r\begin{pmatrix}1\\ \lambda v(x) - E - 1/r\end{pmatrix}$$

we see that the cocycle F_E induces an action on the projective space $\mathbb{P}^1(\mathbb{R}^2)$ (with coordinates $\binom{1}{r}$) given by

$$G_E(x, r) = (T(x), \lambda v(x) - E - 1/r).$$

If $r_n = \pi_2(G_E^n(x, r))$, $n \geq 0$, it is easy to verify that[2]

$$A_E^{n+1}(x)\begin{pmatrix}1\\ r\end{pmatrix} = \begin{pmatrix}r_n \cdots r\\ r_{n+1} \cdots r\end{pmatrix}. \tag{2.1}$$

Thus, if there for some parameter $E \in \mathbb{R}$ exists a set $X \subset \mathbb{T}$ of positive measure such that for each $x \in X$ there is an $r \in \mathbb{R}$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log \|r_n \cdots r\| \geq C,$$

then it follows from (1.1) and (2.1) that $L(E) \geq C$.

[2] Here π_2 denotes the projection $\pi_2(x, r) = r$.

2.3. Bounds.

Lemma 2.1. *If* $|E| \geq \lambda/3 + 2\sqrt{\lambda}$, *then* $L(E) \geq (\log \lambda)/2$.

Proof. Take any $x \in \mathbb{T}$ and $|r| > \sqrt{\lambda}$, and define $r_n = \pi_2(G_E^n(x, r))$, $n \geq 1$. We note that $|r_1| = |\lambda v(x) - E - 1/r| \geq |E| - |\lambda v(x)| - |1/r| > \sqrt{\lambda}$. By induction we get $|r \cdot r_1 \cdots r_n| > \sqrt{\lambda}^{n+1}$ for all $n \geq 1$. □

Thus, in order to prove Theorem 1, we only need to consider $|E| < \lambda/3+2\sqrt{\lambda}$ (which of course is the cumbersome region).

The next lemma provides lower bounds on products $|r_1 \cdots r_N|$ under the assumption that we know how many of the iterates r_j that are in the interval $(-\sqrt{\lambda}, \sqrt{\lambda})$.

Lemma 2.2. *Assume that* $|E| \leq \lambda/3 + 2\sqrt{\lambda}$. *For any* $N \geq 1$ *the following holds for all* $(x, r) \in \mathbb{T} \times (\mathbb{R} \cup \{\infty\})$*: Let* $r_j = \pi_2(G_E^j(x, r))$ *and let* k *(*$0 \leq k \leq N$*) be the number of indices* j *in* $[1, N]$ *for which* $|r_j| < \sqrt{\lambda}$. *Then*

$$|r_1 \cdots r_N| \geq \sqrt{\lambda}^{N-3k} \text{ if } |r_N| \geq 1/\lambda, \text{ and } |r_1 \cdots r_{N+1}| \geq \sqrt{\lambda}^{N+1-3k} \text{ if } |r_N| < 1/\lambda.$$

Proof. We first note that if $|r_j| < 1/\lambda$, then $|r_j r_{j+1}| = |r_j(\lambda v(x_j) - E) - 1| > 1/4$; thus we also have $|r_{j+1}| > \sqrt{\lambda}$. From these two facts the statement of the lemma follows easily by induction over N. □

2.4. Elementary probability. We begin by defining some natural partitions of $\mathbb{T}$ (relative the transformation T). First, let

$$I_j = \left[\frac{j-1}{b}, \frac{j}{b}\right), \quad j = 1, 2, \ldots, b.$$

Note that $I_1 \cup I_2 \cup \cdots \cup I_b = \mathbb{T}$ and $T(I_j) = \mathbb{T}$ for all j.

Next we define the intervals[3]

$$I_{j_1 j_2 \cdots j_n} = I_{j_1} \cap T^{-1}(I_{j_2}) \cap \ldots \cap T^{-n+1}(I_{j_n}), \quad n \geq 2.$$

Thus, by definition, if $x \in I_{j_1 j_2 \cdots j_n}$, then $x \in I_{j_1}, T(x) \in I_{j_2}, \ldots, T^{n-1}(x) \in I_{j_n}$; and clearly $I_{j_1 \ldots j_n} \subset I_{j_1 \ldots j_{n-1}} \subset \ldots \subset I_{j_1}$. Note that the length of each interval $I_{j_1 j_2 \cdots j_n}$ is b^{-n}. Note also that for any fixed interval $I_{j_1 j_2 \cdots j_n}$ the intervals $I_{j_1 j_2 \cdots j_n j}$, $1 \leq j \leq b$, form a partition of $I_{j_1 j_2 \cdots j_n}$ into b pieces of equal length $b^{-(n+1)}$.

In the following lemma[4] we use the word "bad" to indicate that an interval does not have a certain property:

[3] Since the map T is conjugated to the one-sided shift map σ on $\Omega_b = \{\omega = (\omega_0, \omega_1, \ldots) : \omega_i \in \{1, 2, \ldots, b\}, i \geq 0\}$, defined by $\sigma(\omega_0, \omega_1, \ldots) = (\omega_1, \omega_2, \ldots)$, the interval I_j corresponds to the cylinder $C_j = \{\omega \in \Omega_b : \omega_0 = j\}$, and the interval $I_{j_1 j_2 \ldots j_n}$ corresponds to the cylinder $C_{j_1 j_2 \ldots j_n} = \{\omega \in \Omega_b : \omega_0 = j_1, \omega_1 = j_2, \ldots, \omega_{n-1} = j_n\}$.

[4] It would be more natural to formulate the statement of the lemma, as well as its proof, in the space Ω_b, with the Bernoulli measure which gives each cylinder $C_{j_1 \ldots j_n}$ the measure b^{-n}. However, since we shall work on $\mathbb{T}$ we formulate the lemma as above.

Lemma 2.3. *Let $1 \le q \le b$ be an integer. Assume that q of the intervals I_j ($j = 1, \ldots, b$) are bad. Furthermore, for each interval $I_{j_1 \ldots j_n}$ assume that q of the intervals $I_{j_1 \ldots j_n j}$ ($j = 1, 2, \ldots, b$) are bad. Then for each $n \ge 1$ and $0 \le m \le n$ the set*

$$\{x \in \mathbb{T} : x \in I_{j_1 \ldots j_n} \text{ and exactly } m \text{ of the intervals } I_{j_1}, I_{j_1 j_2}, \ldots, I_{j_1 \ldots j_n} \text{ are bad}\}$$

has measure $\binom{n}{m} \frac{q^m (b-q)^{n-m}}{b^n}$.

Proof. Easy combinatorics. From the assumption it follows that there are $\binom{n}{m} q^m (b - q)^{n-m}$ n-tuples $(j_1, \ldots, j_n) \in \{1, \ldots, b\}^n$ such that exactly m of the intervals $I_{j_1}, I_{j_1 j_2}, \ldots, I_{j_1 \ldots j_n}$ are bad. Using the fact that $|I_{j_1 \ldots j_n}| = b^{-n}$ yields the result. □

Moreover,

Lemma 2.4. *With the same assumptions as in the previous lemma, the measure of the set*

$$M_n = \{x \in \mathbb{T} : x \in I_{j_1 \ldots j_n} \text{ and at most } [2(q/b)n] \text{ of the intervals } I_{j_1}, I_{j_1 j_2}, \ldots, I_{j_1 \ldots j_n} \text{ are bad}\}$$

goes to 1 as $n \to \infty$.

Proof. By the previous lemma we have (provided that $[2(q/b)n] \le n$)

$$|M_n| = \sum_{m=0}^{[2(q/b)n]} \binom{n}{m} \frac{q^m (b-q)^{n-m}}{b^n} = \sum_{m=0}^{[2(q/b)n]} \binom{n}{m} (q/b)^m (1 - q/b)^{n-m}.$$

Applying the de Moivre-Laplace theorem yields the result. □

3. Geometry

We write the circle $S^1 = [-\pi/2, \pi/2]/\sim$. A point $\binom{1}{r}$ in projective coordinates corresponds to $\arctan(r) \in S^1$. Thus, in these coordinates the map G_E becomes $H_E : \mathbb{T} \times S^1 \to \mathbb{T} \times S^1$ given by

$$H_E(x, y) = (T(x), \arctan(\lambda v(x) - E - 1/\tan(y))). \tag{3.1}$$

The following lemma is crucial for the proof of Theorem 1 (compare with "admissible curves" in [13]; and also the idea in [13] that the image of an admissible curve "spreads out" in the y-direction). Recall that the constants β and s come from the assumption on v.

Lemma 3.1. *Assume that $|E| < \lambda/2$ and $b \ge \lambda^3$. Assume further that the function $\varphi : [0, 1] \to S^1$ is C^1 and satisfies*

$$|\varphi'(x)| < \frac{\lambda \|v'\|}{b} (1 + (2\lambda^2)/b + \cdots + (2\lambda^2)^m / b^m)$$

for all $x \in [0, 1]$ and some integer $m \ge 0$. For $j = 1, 2, \ldots, b$, let $\varphi^j : [0, 1] \to S^1$ be defined by

$$\varphi^j(x) = \arctan\left(\lambda v\left(\frac{x + j - 1}{b}\right) - E - \frac{1}{\tan\left(\varphi\left(\frac{x+j-1}{b}\right)\right)} \right).$$

Then the following hold:

(1) *There are at most* $(s+1)(2+[2^\beta \lambda^{-\beta/2} b])$ *indices* $j \in \{1, \ldots, b\}$ *for which*

$$\min_{x \in [0,1]} |\tan \varphi^j(x)| < \sqrt{\lambda}.$$

(2) *The estimate* $|(\varphi^j)'(x)| < \frac{\lambda \|v'\|}{b}(1 + (2\lambda^2)/b + \cdots + (2\lambda^2)^{m+1}/b^{m+1})$ *holds for all* $x \in [0,1]$ *and all* j.

Remark 3. That the functions φ^j are defined as above implies that if $\Gamma_j = \{(x, \varphi(x)) : x \in I_j\}$, then $H_E(\Gamma_j) = \{(x, \varphi^j(x)) : x \in [0,1)\}$. Thus, if $\Gamma = \{(x, \varphi(x)) : x \in [0,1)\}$ is the graph of φ, we have $H_E(\Gamma) = \bigcup_{j=1}^b \{(x, \varphi^j(x)) : x \in [0,1)\}$, i.e., the union of the graphs of the φ^j.

The statement thus says that we have a bound of the number of indices j for which the graph $\{(x, \varphi^j(x)) : x \in [0,1)\}$ intersects the "bad region" $[0,1] \times (-\arctan\sqrt{\lambda}, \arctan\sqrt{\lambda})$; and the derivative estimate shows that we can iterate this process (iterate each of the j graphs) and still have a good control on the derivative (provided that b is large enough, independently of m).

Proof. By the assumptions on b and φ' we have $|\varphi'| < (\lambda\|v'\|/b)\sum_{i=0}^\infty (2\lambda^2/b)^i < 2\|v'\|/\lambda^2$. We let $g(x) = -1/\tan(\varphi(x))$.

We first prove that the set $B := \{x \in [0,1] : |\lambda v(x) - E + g(x)| < \sqrt{\lambda}\}$ can intersect at most $(s+1)(2+[2^\beta\lambda^{-\beta/2}b])$ of the intervals I_j ($j = 1, \ldots, b$). This clearly gives the first statement of the lemma.

If $|\varphi(x_0)| < 1/\lambda$ for some x_0, then the estimate on $|\varphi'|$ implies that $|\varphi(x)| < 1/\lambda + 2\|v'\|/\lambda^2$ for all x; thus, since $|\lambda v(x) - E| \leq \lambda/3 + \lambda/2$, we get $|\lambda v(x) - E + g(x)| > \sqrt{\lambda}$ for all $x \in [0,1]$. We conclude that $B = \emptyset$ in this case.

Assume now that $|\varphi(x)| \geq 1/\lambda$ for all x. Then we have $|g'(x)| = |\varphi'(x)|/|\sin^2(\varphi(x))| < (2\|v'\|/\lambda^2)(2\lambda^2) = 4\|v'\|$, and therefore $|g(x_1) - g(x_0)| < 4\|v'\|$ for all $x_0, x_1 \in [0,1]$ Hence

$$B \subset \{x \in [0,1] : |\lambda v(x) - E + g(0)| < 2\sqrt{\lambda}\}.$$

By the assumption on v the latter set consists of at most $s+1$ intervals (s intervals on $\mathbb{T}$ can be at most $s+1$ intervals in $[0,1]$), each of a length $< (2/\sqrt{\lambda})^\beta$. Since each interval I_j ($j = 1, \ldots, b$) has length $1/b$, it follows that the set B can intersect at most $(s+1)(2+[2^\beta\lambda^{-\beta/2}b])$ of them.

We turn to the derivative estimate in (2). From the assumptions on v and E we have $|\lambda v(t) - E| < \lambda$ for all t. An easy computation shows that $|(\varphi^j)'(x)| \leq (\lambda\|v'\| + \|\varphi'\|2\lambda^2)/b$, from which the desired bound follows. To obtain the estimate in the second term we have used the fact that if $|a| \leq \lambda$, and λ is sufficiently large (larger than a numerical constant), then $\sin^2 t + (a \sin t - \cos t)^2 > 1/(2\lambda^2)$ for all t. □

4. Proof of Theorem 1

By Lemma 2.1 we only need to consider $|E| < \lambda/2$. We therefore assume that $|E| < \lambda/2$ is fixed. We also fix $b \geq \lambda^3$ (so that we can apply Lemma 3.1). Given a point (x, r) we denote by r_j the iterate $r_j = \pi_2(G_E^j(x, r))$.

Let the intervals $I_{j_1 \ldots j_n}$ be defined as in Sect. 2.4. We say that the interval $I_{j_1 \ldots j_n}$ ($n \geq 1$) is "good" if for each $x \in I_{j_1 \ldots j_n}$ and $r = \lambda$ we have $|r_n| \geq \sqrt{\lambda}$; otherwise the interval is "bad".

Lemma 4.1. *Assume that at most* $q = [b/12]$ *of the intervals* I_j $(j = 1, \dots, b)$ *are bad, and that for each interval* $I_{j_1 \dots j_n}$ *at most* q *of the intervals* $I_{j_1 \dots j_n j}$ $(j = 1, \dots, b)$ *are bad. Then* $L(E) \geq (\log \lambda)/4$.

Proof. Let the sets M_n be defined as in Lemma 2.4, and let $M = \limsup_{n\to\infty} M_n$. Since $|M_n| \to 1$ as $n \to \infty$ we have that $|M| = 1$, i.e., the set M has full measure.

Take $x \in M_n$ $(n \geq 1)$ and $r = \lambda$. Then $x \in I_{j_1 \dots j_n} \subset I_{j_1 \dots j_{n-1}} \subset \dots \subset I_{j_1}$ and at most $[2(q/b)n]$ of the intervals $I_{j_1}, \dots, I_{j_1 \dots j_n}$ are bad. Thus, by definition we get that $|r_k| < \sqrt{\lambda}$ for at most $[2(q/b)n] \leq n/6$ indices $k \in [1, n]$. It hence follows from Lemma 2.2 that $|r_1 \cdots r_n| \geq \sqrt{\lambda}^{\,n-3n/6}$ or $|r_1 \cdots r_{n+1}| \geq \sqrt{\lambda}^{\,n+1-3n/6}$.

Consequently, if $x \in M$ (and thus $x \in M_n$ for infinitely many n) and $r = \lambda$ we have

$$\limsup_{n\to\infty} \frac{1}{n} \log |r_1 \cdots r_n| \geq (1/4) \log \lambda.$$

Recalling (1.1) and (2.1) finishes the proof. □

Combining the previous lemma with the next one finishes the proof of Theorem 1.

Lemma 4.2. *At most* $q = [b/12]$ *of the intervals* I_j $(j = 1, \dots, b)$ *are bad; and for each interval* $I_{j_1 \dots j_n}$ *at most* q *of the intervals* $I_{j_1 \dots j_n j}$ $(j = 1, \dots, b)$ *are bad.*

Proof. The strategy is to apply Lemma 3.1 inductively, and we shall begin by iterating the constant graph $\{(x, \arctan \lambda) : x \in [0, 1]\}$.

Note that $(s + 1)(2 + [2^{\beta} \lambda^{-\beta/2} b]) < [b/12]$ if λ is sufficiently large (recall that $b \geq \lambda^3$), where the left hand side is the quantity in Lemma 3.1(1). Since E is fixed we write $H = H_E$ and $G = G_E$.

For $j = 1, 2, \dots, b$, let $\varphi_j : [0, 1] \to S^1$ be defined by

$$\varphi_j(x) = \pi_2 \left(H((x + j - 1)/b, \arctan \lambda) \right).$$

Applying Lemma 3.1 with $\varphi(x) = \arctan(\lambda)$ and $m = 0$ shows that we for each j have $|\varphi_j'(x)| < \frac{\lambda \|v'\|}{b}(1 + (2\lambda^2)/b)$ and that there are at most $[b/12]$ indices $j \in \{1, \dots, b\}$ for which $\min_{x\in[0,1]} |\tan \varphi_j(x)| < \sqrt{\lambda}$. We note that $H(I_j \times \{\arctan \lambda\}) = \{(x, \varphi_j(x)) : x \in [0, 1)\}$, and thus

$$G(I_j \times \{\lambda\}) = \{(x, \tan(\varphi_j(x))) : x \in [0, 1)\}.$$

Consequently, at most $[b/12]$ of the intervals I_j are bad. This proves the first statement of the lemma.

Inductively (over $n \geq 1$) we define, for fixed $(j_1, \dots, j_n) \in \{1, \dots, b\}^n$, the functions $\varphi_{j_1 \dots j_n j} : [0, 1] \to S^1$ by

$$\varphi_{j_1 \dots j_n j}(x) = \pi_2 \left(H((x + j - 1)/b, \varphi_{j_1 \dots j_n}((x + j - 1)/b)) \right), \quad j = 1, 2, \dots, b.$$

Lemma 3.1 gives $|\varphi'_{j_1 \dots j_n j}(x)| < \frac{\lambda \|v'\|}{b}(1 + (2\lambda^2)/b + \dots + (2\lambda^2)^{n+1}/b^{n+1})$ and that there are at most $[b/12]$ indices $j \in \{1, \dots, b\}$ for which $\min_{x\in[0,1]} |\tan \varphi_{j_1 \dots j_n j}(x)| < \sqrt{\lambda}$.

From the above definition it is easy to verify that for each $n \geq 1$ and each fixed $(j_1, \dots, j_n) \in \{1, \dots, b\}^n$ we have

$$H^{n+1}(I_{j_1 \dots j_n j} \times \{\arctan \lambda\}) = \{(x, \varphi_{j_1 \dots j_n j}(x)) : x \in [0, 1)\}; \text{ and}$$
$$G^{n+1}(I_{j_1 \dots j_n j} \times \{\lambda\}) = \{(x, \tan \varphi_{j_1 \dots j_n j}(x)) : x \in [0, 1)\}.$$

Thus, at most $[b/12]$ of the intervals $I_{j_1 \dots j_n j}$ $(1 \le j \le b)$ are bad. This finishes the proof of the lemma. □

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References

1. Bjerklöv, K.: A note on circle maps driven by strongly expanding endomorphisms on $\mathbb{T}$. Dyn. Syst. **33**(2), 361–368 (2018)
2. Bjerklöv, K.: Explicit examples of arbitrarily large analytic ergodic potentials with zero Lyapunov exponent. Geom. Funct. Anal. **16**(6), 1183–1200 (2006)
3. Bourgain, J., Schlag, W.: Anderson localization for Schrödinger operators on $\mathbb{Z}$ with strongly mixing potentials. Commun. Math. Phys. **215**(1), 143–175 (2000)
4. Chulaevsky, V., Spencer, T.: Positive Lyapunov exponents for a class of deterministic potentials. Commun. Math. Phys. **168**(3), 455–466 (1995)
5. Damanik, D.: Schrödinger operators with dynamically defined potentials. Ergodic Theory Dyn. Syst. **37**(6), 1681–1764 (2017)
6. Damanik, D., Killip, R.: Almost everywhere positivity of the Lyapunov exponent for the doubling map. Commun. Math. Phys. **257**(2), 287–290 (2005)
7. Herman, M.: Une méthode pour minorer les exposants de Lyapounov et quelques exemples montrant le caractère local d'un théorème d'Arnol'd et de Moser sur le tore de dimension 2. Comment. Math. Helv. **58**(3), 453–502 (1983)
8. Krüger, H.: Multiscale analysis for ergodic Schrödinger operators and positivity of Lyapunov exponents. J. Anal. Math. **115**, 343–387 (2011)
9. Metzger, F.: Lyapunov exponents of ergodic Schrödinger operators. Ph.D. thesis 2017, Paris 6
10. Shamis, M., Spencer, T.: Bounds on the Lyapunov exponent via crude estimates on the density of states. Commun. Math. Phys. **338**(2), 705–720 (2015)
11. Sorets, E., Spencer, T.: Positive Lyapunov exponents for Schrödinger operators with quasi-periodic potentials. Commun. Math. Phys. **142**(3), 543–566 (1991)
12. Spencer, T.: Ergodic Schrödinger Operators. Analysis, et Cetera, pp. 623–637. Academic Press, Boston (1990)
13. Viana, M.: Multidimensional nonhyperbolic attractors. Inst. Hautes Études Sci. Publ. Math. No **85**, 63–96 (1997)
14. Young, L.-S.: Some open sets of nonuniformly hyperbolic cocycles. Ergodic Theory Dyn. Syst. **13**(2), 409–415 (1993)
15. Zhang, Z.: Uniform positivity of the Lyapunov exponent for monotone potentials generated by the doubling map. arXiv:1610.02137

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