Cyclic-Homology Chern–Weil Theory for Families of Principal Coactions

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Received: 7 August 2019 / Accepted: 2 May 2020

Published online: 24 December 2020 – © The Author(s) 2020

Abstract: Viewing the space of cotraces in the structural coalgebra of a principal coaction as a noncommutative counterpart of the classical Cartan model, we construct the cyclic-homology Chern–Weil homomorphism. To realize the thus constructed Chern–Weil homomorphism as a Cartan model of the homomorphism tautologically induced by the classifying map on cohomology, we replace the unital subalgebra of coaction-invariants by its natural H-unital nilpotent extension (row extension). Although the row-extension algebra provides a drastically different model of the cyclic object, we prove that, for any row extension of any unital algebra over a commutative ring, the row-extension Hochschild complex and the usual Hochschild complex are chain homotopy equivalent. It is the discovery of an explicit homotopy formula that allows us to improve the homological quasi-isomorphism arguments of Loday and Wodzicki. We work with families of principal coactions, and instantiate our noncommutative Chern–Weil theory by computing the cotrace space and analyzing a dimension-drop-like effect in the spirit of Feng and Tsygan for the quantum-deformation family of the standard quantum Hopf fibrations.

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1. Introduction

Gauge theory of principal bundles very successfully serves particle physics through the celebrated Yang–Mills theory, which is the mathematical backbone of the Standard Model. Yang–Mills theory is also a centerpiece of mathematics used by Simon Donaldson in his famous work on invariants of differentiable structures on four-dimensional manifolds [20], and the subject of one of the Millenium Problems concerning the conjecture that the lowest excitations of Yang–Mills theory without matter fields have a finite mass gap.

Another big virtue of Yang–Mills theory is that it is amenable not only to standard tools of physics such as perturbation and renormalization theory, lattice and numerical approaches, but also to generalizations to noncommutative geometry. Alain Connes, together with his numerous collaborators, exerted an enormous effort to provide a viable noncommutative-geometric extension of the Standard Model by following closely physical predictions necessitated by his theory (see [15,50] and references therein).

Herein, rather than directly following hints from the nature, we conceptualize and abstract the very mathematical foundations of Chern–Weil theory [7], whose objective is to provide characteristic classes of principal bundles taking values in homotopy-invariant cohomology theory, to the general setting of principal coactions of coalgebras [10]. Thus, we arrive at a very general, and rich in potential applications, noncommutative-topological framework for gauge theory based on the concepts of cyclic homology and quantum groups.

Let us now explain how the concept of group actions can be abstracted to coactions of coalgebras. Let G be a compact Hausdorff topological group acting by automorphisms on a unital C^* -algebra A . Then, to define the freeness of such an action α , we need to dualize it to a coaction δ :

$$\delta: A \longrightarrow C(G, A) = A \otimes_{\min} C(G), \quad (\delta(a))(g) := \alpha_g a. \quad (1.1)$$

Here $C(G, A)$ is the C^* -algebra of all norm continuous functions from G to A naturally identified with the complete tensor product C^* -algebra (e.g. see [51, Corollary T.6.17]). Now we can replace $C(G)$ by the C^* -algebra of a compact quantum group [55,56], and define freeness as a density condition [21].

Furthermore, we can define the Peter–Weyl functor [4] from the category of unital C^* -algebras equipped with an action of a compact quantum group (i.e. a coaction of its C^* -algebra) to the category of comodule algebras over the Hopf algebra of regular functions on the compact quantum group [56]. The main theorem of [4] states that the aforementioned density condition defining the freeness of an action of a compact quantum group on a unital C^* -algebra is, via the Peter–Weyl functor, equivalent to the principality of the coaction of the Hopf algebra of the compact quantum group. Thus, the algebraic condition of principality of an appropriate comodule algebra encodes the analytical freeness condition of a compact-quantum-group action on a unital C^* -algebra.

In the commutative case, through the Gelfand–Naimark equivalence between the category of commutative unital C^* -algebras and the category of compact Hausdorff spaces [24], the analytical freeness condition is equivalent to the freeness of a continuous compact-Hausdorff-group action on a compact Hausdorff space. All this gives us a C^* -algebraic and Hopf-algebraic formulation of compact Hausdorff principal bundles [5], which is a main concept of gauge theory.

However, the category of principal comodule algebras, despite being very ample and enjoyable, does not encompass all interesting examples coming from quantizations along Poisson structures. A way to obtain a quantum group is by deforming a Poisson–Lie group along its Poisson structure. It is well known that Poisson–Lie groups admit few Poisson–Lie subgroups, so it is important to consider coisotropic subgroups of Poisson–Lie groups. But the deformation quantization along the Poisson structure of the natural action of a coisotropic subgroup on its Poisson–Lie group leads to a coaction of a coalgebra rather than a Hopf algebra [14, 43]. Such examples motivated the development of general theory of principal coactions $\delta: A \rightarrow A \otimes C$ of coalgebras on algebras [10–12]. Furthermore, since any deformation quantization is a parameterized family of algebras, we consider coactions on algebras over the commutative algebra generated by deformation parameters. Hence, the setting of our paper is based on the concept of coactions of coalgebras on algebras over a commutative algebra R over a field.

A formula computing the Chern character of the finitely generated projective module associated with a given finite-dimensional corepresentation uses a strong connection [11, 19, 31] and an iterated comultiplication applied to the character of the corepresentation to produce a cycle in the complex computing the cyclic homology of the algebra of coaction invariants (fixed-point subalgebra)

$$B := \{b \in A \mid \forall a \in A : \delta(ba) = b\delta(a)\}. \quad (1.2)$$

Its homology class is called the Chern–Galois character [11] of the corepresentation. The Chern–Galois character is a fundamental tool in calculating K_0 -invariants of modules associated to principal coactions of coalgebras on algebras, in particular to principal comodule algebras in Hopf–Galois theory [33].

The goal of this paper is to construct a cyclic-homology Chern–Weil homomorphism from the space of cotraces of the structural coalgebra to the even cyclic homology of the fixed-point subalgebra enjoying analogous properties as its classical counterpart. Our construction works at a very high level of generality of families of principal coactions. In particular, our Chern–Weil homomorphism always exists for compact quantum principal bundles (Hopf–Galois extensions with cosemisimple Hopf algebras), and is independent of the choice of a strong connection, which reflects very well the classical situation. The space of cotraces can be understood as a *cyclic-homology Cartan model* of the generalized classifying space of the structural coalgebra. We justify this point of view by the graded-space construction associated with the Ad-invariant $\mathfrak{m}$ -adic filtration on class functions on a compact Lie group, where $\mathfrak{m}$ is the maximal ideal of all functions vanishing at the neutral element of the group, which produces the classical space of Ad-invariant polynomials on the Lie algebra. We show that this can be related also with the Block–Getzler equivariant cyclic homology. Furthermore, the noncommutative counterpart of homotopy invariance of our cyclic-homological extension of the classical Chern–Weil homomorphism is stability under Connes’ periodicity operator.

In [49], the question of a possibility of a universal characterization of the Chern character as a natural transformation from the even K -group to the even cyclic homology is answered in the positive. We extend the classical relation between the Chern

character and Chern–Weil theory to the noncommutative realm by constructing a commutative diagram which can be interpreted as corresponding to the classifying map of the aforementioned natural transformation from K_0 to HC_{even} .

The classifying space in our approach is replaced by the category of corepresentations. In other words, we think of it as a homotopy quotient stack of the one-point space. Here the Grothendieck group of corepresentations should be understood as the K_0 -group of this classifying space, and the space of cotraces as its even cyclic homology. More precisely, the abelian group completion $\text{Corep}(C)$ of the monoid of finite-dimensional C -comodules can be understood as a generalized K_0 -group of the classifying space of the coalgebra C , the two decompositions of the Chern–Galois character can be understood as the naturality of the Chern character ch_n under the generalized classifying map for a noncommutative principal bundle.

All this can be subsumed by the following commutative diagram:

$$\begin{array}{ccc} \text{Corep}(C) & \xrightarrow{[A\Box^C(-)]} & K_0(B) \\ \chi \downarrow & & \downarrow \text{ch}_n \\ C^{\text{tr}} & \xrightarrow{\text{chw}_n} & HC_{2n}(B \mid R). \end{array} \quad (1.3)$$

Here the map $[A\Box^C(-)]$ associating a finitely generated projective module with a given corepresentation should be understood as the map induced by the classifying map on K -theory, the character χ of a corepresentation should be understood as the Chern character for the classifying space, and the cyclic-homology Chern–Weil homomorphism chw_n should be understood as the map induced by the classifying map on cyclic homology. The Chern–Galois character ch_n is the diagonal composite in this diagram.

The above commutative diagram can be interpreted as a noncommutative counterpart of the naturality of the Chern character under the family $\text{cl} : Y \rightarrow BG$ of classifying maps (parameterized by a space S) of a family $X \rightarrow Y \rightarrow S$ of G -principal bundles of spaces over S . These bundles correspond to a principal G -action $X \times G \rightarrow X$ over S with the family of orbit spaces $Y = X/G$ over S . Thus, we obtain the following commutative diagram:

$$\begin{array}{ccc} K^0(BG) & \xrightarrow{K^0(\text{cl})} & K^0(Y) \\ \text{ch}_n(BG) \downarrow & & \downarrow \text{ch}_n(Y \mid S) \\ H_{dR}^{2n}(BG) & \xrightarrow{H^{2n}(\text{cl})} & H_{dR}^{2n}(Y \mid S). \end{array} \quad (1.4)$$

We achieve our goal of constructing a cyclic-homology Chern–Weil homomorphism by applying an abstract construction which we call *abstract cyclic-homology Chern character*. Just as to express K -theory in terms of matrix idempotents one introduces the functor of forming an H -unital algebra of locally finite matrices $M_\infty(-)$, to embrace the symmetry of a principal bundle in terms of representations of the symmetry, we introduce another H -unital algebra M which is the Ehresmann–Schauenburg quantum groupoid with a non-standard multiplication.

The abstract cyclic-homology Chern character prompts a unified approach to both constructions which can be subsumed in the following commutative diagram:

$$\begin{array}{ccccc}
 \text{Corep}(C) & \xrightarrow{[A\Box^C(-)]} & & K_0(B) & \\
 \downarrow \widetilde{\text{chg}}_n & \searrow \text{chg}_n & & \swarrow \text{ch}_n & \downarrow \widetilde{\text{ch}}_n \\
 \text{HC}_{2n}(M \mid R) & \xrightarrow[\cong]{\text{HC}_{2n}(\epsilon_M)} & \text{HC}_{2n}(B \mid R) & \xleftarrow[\cong]{[\text{tr}_{2n}]} & \text{HC}_{2n}(M_\infty(B) \mid R).
 \end{array} \tag{1.5}$$

Here the bottom horizontal arrows are isomorphisms of H-unital models of cyclic homology of B , and the vertical arrows are tautological constructions. Thus the left-hand-side factorization of the Chern–Galois character becomes analogous to the well-known right-hand-side factorization of the Chern character.

The aforementioned H-unital-algebra construction obtained from the Ehresmann–Schauenburg quantum groupoid leads to a general concept of a *row extension*. We prove that the Hochschild homology is invariant under such extensions by providing an explicit chain homotopy equivalence of complexes. Our consistent use of complexes up to chain homotopy equivalence rather than their homology is motivated by the fact that, although the invariance of Hochschild homology under row extensions can be established by the Wodzicki excision argument [53,54], there is a problem (signalled in [9]) of making this argument explicit in the resulting inverse excision isomorphism. We overcome this difficulty by constructing an explicit chain homotopy compatible with an analogue of the filtration from [26].

The fact that all homotopies we use are natural and explicit suggests a higher homotopy landscape behind our construction, according to the ideas surveyed in [36]. Our Theorem 2.1, replacing the Homological Perturbation Theory evoked in [36], could be of independent interest. Much in the same way as Homological Perturbation Theory is used as a tool in computing Hochschild and cyclic homology and the Chern character [3,39,41], we use our Theorem 2.1 in calculations in the homotopy category of chain complexes. An additional substantiation of homotopical approach is the fact that it is a natural environment for classical Chern–Weil theory [23] as theory induced by the homotopy class of the classifying map.

Further motivation for the homotopical reinforcement of cyclic (and Hochschild) homology is the need for their more subtle version valid for families. For instance, the fact that quasi-isomorphic chain complexes parameterized by a commutative base ring different from a field are not necessarily chain homotopy equivalent could be a potential way of more detailed analysis of the dimension-drop phenomena in deformation quantization [22]. Notice that, although in some cases the dimension drop can be prevented by the use of twisted Hochschild (and cyclic) homology [27–30], the Chern character always takes values in the untwisted cyclic homology.

To put our construction in a historical perspective, let us compare it to other approaches to the Chern–Weil homomorphism in noncommutative geometry. Apparently, the first instance of a connection between cotraces and Chern–Weil theory goes back to Quillen’s work [47]. The coalgebra therein is the bar construction of an algebra, and the analogy with the Chern–Weil homomorphism is explicitly stressed.

Next, in [2,45] Alexeev and Meinrenken introduced noncommutative Chern–Weil theory based on a specific noncommutative deformation of the classical Weil model

aiming to extend the Duflo isomorphism for quadratic Lie algebras to the level of equivariant cohomology. However, they work only with the universal enveloping algebra of a Lie algebra instead of arbitrary Hopf algebras, and without referring to cyclic homology.

Finally, in [17], Crainic considers a Weil model in the context of Hopf-cyclic homology of Hopf algebras. However, his characteristic map based on the characteristic map of Connes and Moscovici takes values in the cyclic homology of a Hopf-module algebra instead of the cyclic homology of the algebra of coaction invariants. As such, it cannot be a noncommutative counterpart of the classical Chern–Weil homomorphism.

2. The Homotopy Category of Chain Complexes

In the present section, we allow complexes in any abelian category, though our main concern is the category of modules over a commutative integral domain R , in which we fix a maximal ideal $\mathfrak{m}$ (a closed point of $\mathrm{Spec}(R)$), and the embedding of R into its field of fractions F (the general point of $\mathrm{Spec}(R)$). For us, the most important examples are the algebra of Laurent polynomials $R = k[q, q^{-1}]$ with $\mathfrak{m} = (q - 1)$ and the field of fractions $F = k(q)$ (rational functions), and the algebra of formal power series $R = k[[h]]$ with $\mathfrak{m} = (h)$ and the field of fractions $F = k((h))$ (formal Laurent series).

A complex over such a ring R can be regarded as a family of complexes parameterized by $\mathrm{Spec}(R)$. At a general point, a quasi-isomorphism of complexes has a chain-homotopy inverse defined over the field of fractions F . Regarded as a rational function on $\mathrm{Spec}(R)$, such an inverse can have a pole at the closed point. Therefore, the minimal possible order of such a pole can measure the degeneracy in the family of quasi-isomorphism classes of complexes parameterized by $\mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$ and degenerating at $\mathfrak{m}$. Note that considering only the quasi-isomorphism class of a complex or its homology is not enough for such an analysis.

An example of a degeneracy of this type is provided by the phenomenon called *dimension drop* discovered by Feng and Tsygan in [22]. Therein, under an h -deformation of the embedding of the normalizer of a Cartan subgroup into a connected complex algebraic group, the induced map of cyclic modules provides a homotopy equivalence of Hochschild and cyclic complexes at the general point of the deformation. It is obviously not even a quasi-isomorphism at the closed point ($h = 0$). In this paper, we analyze another degeneration phenomenon: the infinite number of pairwise distinct components of the cyclic-homology Chern–Weil homomorphism in a q -deformation degenerates to the common value 1 at $q = 1$.

2.1. Killing contractible complexes. The following theorem should have been proved sixty years ago. Strangely enough, the first approximation to it can be found in Loday’s book, without any further reference, under the name “Killing contractible complexes” [42]. Regrettably, the claim therein is about a quasi-isomorphism only instead of a homotopy equivalence. Moreover, that quasi-isomorphism does not respect the obvious structure of the short exact sequence of complexes. The homotopy equivalence was achieved by Crainic [18] only in 2004 by constructing the explicit homotopy inverse using the homological perturbation method. Still, his perturbed maps do not respect the obvious structure of the short exact sequence of complexes. In contrast to these results, in our present approach, we perturb neither the differential nor the structure of the short exact sequence. Instead, we perturb a given splitting in the category of graded objects to make it a splitting in the category of complexes by providing an explicit homotopy

inverse. We focus on split short exact sequences of complexes since only such complexes can produce distinguished triangles in the homotopy category of chain complexes.

Theorem 2.1. *Assume that*

$$0 \longrightarrow X \xrightarrow{\iota} Y \xrightarrow{\pi} Z \longrightarrow 0$$

is a short exact sequence of complexes in an abelian category. Assume also that it is split in the category of graded objects. Then, provided X is contractible, π is a homotopy equivalence.

Proof. For the sake of brevity, in this proof, we will denote all compositions of maps by juxtaposition, all differentials by d , and all identity morphisms by 1. Consider a splitting

$$0 \longrightarrow X \xrightarrow[\iota]{\rho} Y \xrightarrow[\pi]{\sigma} Z \longrightarrow 0 \quad (2.1)$$

and a homotopy h contracting X . (The dashed arrows are not necessarily chain maps.) This is tantamount to the following identities.

$$d^2 = 0, \quad (2.2) \quad \pi\sigma = 1, \quad (2.6)$$

$$d\iota = \iota d, \quad (2.3) \quad \rho\iota = 1, \quad (2.7)$$

$$d\pi = \pi d, \quad (2.4) \quad \sigma\pi + \iota\rho = 1, \quad (2.8)$$

$$\pi\iota = 0, \quad (2.5) \quad \rho\sigma = 0, \quad (2.9)$$

$$hd + dh = 1. \quad (2.10)$$

Next, we define the following expressions

$$\alpha := \sigma d\pi, \quad \beta := \iota\rho d\sigma\pi, \quad \gamma := \iota d\rho, \quad \tilde{h} := \iota h\rho, \quad \tilde{\sigma} := (1 - \tilde{h}d)\sigma. \quad (2.11)$$

By (2.6) and (2.5), we have

$$\pi\tilde{\sigma} = 1. \quad (2.12)$$

Together with (2.9), (2.7) and (2.10) implies that

$$(\alpha + \beta + \gamma)\tilde{\sigma} - \tilde{\sigma}d = \tilde{h}(\beta\alpha + \gamma\beta). \quad (2.13)$$

Now, by (2.4), (2.3) and (2.8), we have

$$\alpha + \beta + \gamma = d. \quad (2.14)$$

After squaring both sides of (2.14), we use (2.2) on the right hand side, and on the left hand side we use the following identities:

$$\begin{aligned} \alpha\beta = \alpha\gamma = \beta^2 = \beta\gamma = 0 & \quad (\text{implied by (2.5)}), \\ \gamma\alpha = 0 & \quad (\text{implied by (2.9)}), \\ \alpha^2 = 0 & \quad (\text{implied by (2.6) and (2.2)}), \\ \gamma^2 = 0 & \quad (\text{implied by (2.7) and (2.2)}), \end{aligned} \quad (2.15)$$

on the left-hand side to obtain

$$\beta\alpha + \gamma\beta = 0. \quad (2.16)$$

Therefore, after substituting (2.14) and (2.16) to (2.13), we obtain that

$$d\tilde{\sigma} - \tilde{\sigma}d = 0, \quad (2.17)$$

so $\tilde{\sigma}$ is a chain map.

Furthermore, by (2.7), (2.10) and (2.8),

$$\tilde{\sigma}\pi + \gamma\tilde{h} + \tilde{h}(\beta + \gamma) = 1. \quad (2.18)$$

Next, using the identities:

$$\begin{aligned} \alpha\tilde{h} = 0, \quad \beta\tilde{h} = 0 & \quad (\text{implied by (2.5)}), \\ \tilde{h}\alpha = 0 & \quad (\text{implied by (2.9)}), \end{aligned} \quad (2.19)$$

we can complete (2.18) to

$$\tilde{\sigma}\pi + (\alpha + \beta + \gamma)\tilde{h} + \tilde{h}(\alpha + \beta + \gamma) = 1 \quad (2.20)$$

By (2.14), the formula (2.20) reads as

$$\tilde{\sigma}\pi + d\tilde{h} + \tilde{h}d = 1. \quad (2.21)$$

Together with (2.12), the above means that $\tilde{\sigma}$ is a homotopy inverse to π . $\square$

Since the above lemma holds in any abelian category, an immediate consequence is its dual version.

Corollary 2.2. *Assume that*

$$0 \longrightarrow X \xrightarrow{\iota} Y \xrightarrow{\pi} Z \longrightarrow 0$$

is a short exact sequence of complexes in an abelian category. Assume that it is split in the category of graded objects. Then, provided Z is contractible, ι is a homotopy equivalence.

2.2. Other homotopy lemmas. The next lemmas are R -balanced and homotopy versions of some homological results collected in [42]. For the convenience of the reader, we sketch their proofs by showing the explicit homotopy inverses and homotopies as in [42].

We consider the first quadrant bicomplex $\text{CC}(B \mid R)$ (see (2.22)), whose total complex computes cyclic homology, and the total complex of the sub-bicomplex $\text{CC}^{(2)}(B \mid R)$

(2.22)

$$\begin{array}{ccccccc}
b \downarrow & & -b' \downarrow & & b \downarrow & & -b' \downarrow \\
B^{\otimes R3} & \xleftarrow{1-t} & B^{\otimes R3} & \xleftarrow{N} & B^{\otimes R3} & \xleftarrow{1-t} & B^{\otimes R3} \xleftarrow{N} \\
b \downarrow & & -b' \downarrow & & b \downarrow & & -b' \downarrow \\
B^{\otimes R2} & \xleftarrow{1-t} & B^{\otimes R2} & \xleftarrow{N} & B^{\otimes R2} & \xleftarrow{1-t} & B^{\otimes R2} \xleftarrow{N} \\
b \downarrow & & -b' \downarrow & & b \downarrow & & -b' \downarrow \\
B & \xleftarrow{1-t} & B & \xleftarrow{N} & B & \xleftarrow{1-t} & B \xleftarrow{N}
\end{array}$$

Lemma 2.3. *The short exact sequence of total complexes*

$$0 \longrightarrow \text{Tot CC}^{\{2\}}(B | R) \longrightarrow \text{Tot CC}(B | R) \longrightarrow \text{Tot CC}(B | R)[2] \longrightarrow 0 \quad (2.23)$$

The next lemma enables, in the special case of our interest, a substantial simplification of the complex computing Hochschild homology to a complex $CC^{\{1\}}(B \mid R)$ consisting of the first column of $CC(B \mid R)$.

$$0 \longrightarrow \mathbf{B}(B \mid R) \longrightarrow \mathrm{Tot} \, \mathbf{CC}^{\{2\}}(B \mid R) \longrightarrow \mathbf{CC}^{\{1\}}(B \mid R) \longrightarrow 0$$
$$\text{Tot CC}^{\{2\}}(B \mid R) \longrightarrow \text{CC}^{\{1\}}(B \mid R).$$
$$h(b^0 \otimes_R \cdots \otimes_R b^n) = e \otimes_R b^0 \otimes_R \cdots \otimes_R b^n. \quad (2.24)$$

Lemma 2.5. *If the R -algebra B is unital, the maps*

$$\text{inc}_n : \mathbf{C}_n(B \mid R) \longrightarrow \mathbf{C}_n(\mathbf{M}_\infty(B) \mid R)$$

$$\text{inc} : B \longrightarrow M_\infty(B), \quad b \longmapsto \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix},$$

form a homotopy equivalence of complexes.

Proof. Following [42], we take an obvious left inverse to inc_n :

$$\begin{aligned} \text{tr}_n : C_n(M_\infty(B) \mid R) &\longrightarrow C_n(B \mid R), \\ \text{tr}_n(\beta^0 \otimes_R \beta^1 \otimes_R \cdots \otimes_R \beta^n) &:= \sum_{i_0, \dots, i_n} \beta_{i_0 i_1}^0 \otimes_R \beta_{i_1 i_2}^1 \otimes_R \cdots \otimes_R \beta_{i_n i_0}^n. \end{aligned} \quad (2.25)$$

Up to the explicit homotopy

$$\begin{aligned} h(\beta^0 \otimes_R \beta^1 \otimes_R \cdots \otimes_R \beta^n) &:= \sum_{m, i_0, \dots, i_m} (-1)^m E_{i_0 1}(\beta_{i_0 i_1}^0) \otimes_R E_{11}(\beta_{i_1 i_2}^1) \otimes_R \cdots \\ &\cdots \otimes_R E_{11}(\beta_{i_m i_{m+1}}^r) \otimes_R E_{1 i_{m+1}}(1) \otimes_R \beta^{m+1} \otimes_R \cdots \otimes_R \beta^n, \end{aligned} \quad (2.26)$$

where $E_{ij}(b)$ denotes the elementary matrix with a single possibly non-zero entry $b \in B$ at the (i, j) -place, this is also a right inverse to inc_n . $\square$

Lemma 2.6. *If B is a unital R -algebra, the conjugation action of the group $\text{GL}_\infty(B)$ on the algebra $M_\infty(B)$*

$$\begin{aligned} \text{GL}_\infty(B) \times M_\infty(B) &\longrightarrow M_\infty(B), \\ (\gamma, \beta) &\longmapsto \gamma \beta \gamma^{-1}, \end{aligned}$$

induces the trivial action on the object $C(M_\infty(B) \mid R)$ of the homotopy category of complexes.

Proof. The action of γ on $M_\infty(B)$ by algebra automorphisms is realized as a simultaneous application of the two well-defined actions: $\beta \mapsto \gamma \beta$ and $\beta \mapsto \beta \gamma^{-1}$. Since R is unital and commutative, they induce the action of γ on $C(M_\infty(B) \mid R)$ given by

$$\gamma(\beta^0 \otimes_R \beta^1 \otimes_R \cdots \otimes_R \beta^n) := \gamma \beta^0 \gamma^{-1} \otimes_R \gamma \beta^1 \gamma^{-1} \otimes_R \cdots \otimes_R \gamma \beta^n \gamma^{-1}, \quad (2.27)$$

and the homotopy between the identity and the action (2.27) provided by

$$\begin{aligned} h(\beta^0 \otimes_R \beta^1 \otimes_R \cdots \otimes_R \beta^n) &:= \sum_{m=0}^n (-1)^m \beta^0 \gamma^{-1} \otimes_R \gamma \beta^1 \gamma^{-1} \otimes_R \cdots \\ &\cdots \otimes_R \gamma \beta^m \gamma^{-1} \otimes_R \gamma \beta^{m+1} \otimes_R \beta^{m+2} \otimes_R \cdots \otimes_R \beta^n, \end{aligned} \quad (2.28)$$

which concludes the proof. $\square$

2.3. The abstract cyclic-homology Chern character. Note first that, for any cyclic object $X = (X_m)$ in a category of modules, we can consider sequences $x = (x_m)$ satisfying the following two conditions when acted on by the cyclic operator t and any face operator d_i , $i = 0, \dots, m$,

$$t(x_m) = (-1)^m x_m, \quad (2.29)$$

$$d_i x_m = x_{m-1}. \quad (2.30)$$

Forming a module $\mathcal{K}(X)$ consisting of such sequences is a functor. For any $x \in \mathcal{K}(X)$, we can construct a natural sequence of even chains in $\text{Tot CC}(X)$ of the form

$$\text{ch}_n(x) := \sum_{m=0}^{2n} (-1)^{\lfloor m/2 \rfloor} \frac{m!}{\lfloor m/2 \rfloor!} x_m \quad (2.31)$$

to obtain a sequence of natural transformations.

Proposition 2.7. *For any $x \in \mathcal{K}(X)$, the chains $\text{ch}_n(x)$ are cycles of degree $2n$ in $\text{Tot CC}(X)$ whose cohomology classes form a sequence stable under the Connes periodicity operator.*

Proof. All formal arguments in the proof of [42, Lemma-Notation 8.3.3] can be adapted to our situation. Namely, from (2.30) followed by (2.29), we obtain

$$b(-2x_{2l}) = -2x_{2l-1} = -(1-t)x_{2l-1}, \quad (2.32)$$

$$b'(-lx_{2l-1}) = -lx_{2l-2} = Nx_{2l-2}. \quad (2.33)$$

This means that the chain $\text{ch}_n(x)$ is a cycle in $\text{Tot CC}_{2n}(X)$. Finally, also the formal argument for stability under Connes' periodicity operator from the proof of [42, Lemma-Notation 8.3.3] is still valid in our situation. $\square$

We call the resulting natural transformation

$$\text{ch}_n(X) : \mathcal{K}(X) \longrightarrow \text{HC}_{2n}(X) \quad (2.34)$$

the *abstract cyclic character*. The motivating example comes from the construction of the Chern character from matrix idempotents.

Let us recall the well-known fact that the Chern character taking values in the cyclic homology of an R -algebra B goes, in fact, to the cyclic homology of a nonunital algebra $M_\infty(B)$ of locally finite matrices. This is due to the fundamental equivalence between the iso-classes of finitely generated projective modules over B and the $\text{GL}_\infty(B)$ -conjugacy classes of idempotents in $M_\infty(B)$. The fact that, for a given idempotent $\mathbf{e} := (e_{ij}) \in M_\infty(B)$, the sequence of elements

$$c_m(\mathbf{e}) := \mathbf{e} \otimes_R \cdots \otimes_R \mathbf{e} \in M_\infty(B)^{\otimes_R(m+1)} \quad (2.35)$$

satisfies the conditions (2.29)–(2.30) follows immediately from the form of $c_m(\mathbf{e})$ and the idempotent property $\mathbf{e}^2 = \mathbf{e}$. Hence, by Proposition 2.7, we infer that the chains

$$\tilde{\text{ch}}_n(\mathbf{e}) := \sum_{m=0}^{2n} (-1)^{\lfloor m/2 \rfloor} \frac{m!}{\lfloor m/2 \rfloor!} c_m(\mathbf{e}) \quad (2.36)$$

are cycles in $\text{Tot CC}_{2n}(M_\infty(B) \mid R)$, and the sequence of their homology classes is stable under Connes' periodicity operator.

Note that, up to this point, the construction of $\tilde{\text{ch}}$ is completely tautological, i.e. $\tilde{\text{ch}}(\mathbf{e})$ contains as much information as $\mathbf{e}$ itself. The next argument, identifying the cyclic homology of the H -unital algebra $M_\infty(B)$ with the cyclic homology of the unital algebra B , uses a specific homotopy equivalence of chain complexes as in Lemma 2.5, and is well defined on the level of $K_0(B)$ by virtue of Lemma 2.6. Notably, applying the $\text{GL}_\infty(B)$ -conjugacy-invariant map

$$\text{Tot CC}_\bullet(M_\infty(B) \mid R) \longrightarrow \text{Tot CC}_\bullet(B \mid R), \quad (2.37)$$

induced by the map defined for all elements $\mathbf{b}^k = (b_{ij}^k) \in M_\infty(B)$ by

$$\text{tr}_n : \mathbf{b}^0 \otimes_R \cdots \otimes_R \mathbf{b}^n \longmapsto \sum_{i_0, \dots, i_n} b_{i_0 i_1}^0 \otimes_R \cdots \otimes_R b_{i_{n-1} i_n}^{n-1} \otimes_R b_{i_n i_0}^n, \quad (2.38)$$

to the element $\widetilde{\text{ch}}_n(\mathbf{e})$, one gets the Chern character $\text{ch}_n(\mathbf{e})$ depending only on the class in $K_0(B)$ defined by the idempotent $\mathbf{e} \in M_\infty(B)$.

Another example of an abstract cyclic-homology Chern–Weil homomorphism will come from a tautological construction of a cyclic-homology Chern–Weil homomorphism. The corresponding model of the cyclic homology of a given H-unital algebra will need another class of H-unital algebra extensions, which we will introduce in the next section.

3. Row Extensions of Algebras

All rings in this section are associative and possibly non-unital. Let R be a commutative ring, B an R -algebra, and M be a (B, R) -bimodule that is symmetric as an R -bimodule. Next, let $\varepsilon_M : M \rightarrow B$ be a (B, R) -bimodule map. We call such a structure an *augmented module* over an R -algebra B . We define a non-unital algebra-over- R structure on M depending on this data as follows. As an R -bimodule, it is the underlying R -bimodule of the (B, R) -bimodule M . The multiplication of elements of M is defined as the (B, R) -bimodule map

$$M \otimes_R M \longrightarrow M, \quad m \otimes_R m' \longmapsto \varepsilon_M(m)m'. \quad (3.1)$$

By the left B -linearity of ε_M , we have the identity

$$\varepsilon_M(\varepsilon_M(m)m')m'' = \varepsilon_M(m)(\varepsilon_M(m')m''), \quad (3.2)$$

which amounts to the associativity of the multiplication.

Proposition 3.1. *The map ε_M is an R -linear algebra map onto a left ideal J , which is also a subalgebra of B . Its kernel is an ideal I in M with the zero right multiplication by the elements of M . In particular, M is a Hochschild extension of J by I ,*

$$0 \longrightarrow I \longrightarrow M \longrightarrow J \longrightarrow 0. \quad (3.3)$$

Proof. To prove that ε_M is an R -algebra map, we check that, by the left B -linearity of ε_M , we have

$$\varepsilon_M(\varepsilon_M(m)m') = \varepsilon_M(m)\varepsilon_M(m'). \quad (3.4)$$

This implies that $I := \ker(\varepsilon_M)$ is an ideal in M . Since ε_M is (B, R) -linear its image $J := \varepsilon_M(M) \subseteq B$ is a (B, R) -sub-bimodule isomorphic to M/I via ε_M . By (3.1), $IM = 0$, whence $I^2 = 0$ and I becomes a J -bimodule such that $IJ = 0$. Therefore, M is a Hochschild extension [34,35] of J by I . $\square$

Proposition 3.2. *Provided the homomorphism $M \rightarrow J$ induced by ε_M admits an R -bimodule splitting, the R -algebra M is isomorphic to the symmetric R -bimodule $I \oplus J$ with the multiplication given by*

$$(i, j)(i', j') := (ji' + \omega(j, j'), jj'). \quad (3.5)$$

Here $\omega : J \otimes_R J \rightarrow I$ is an R -bimodule map satisfying

$$j\omega(j', j'') - \omega(jj', j'') + \omega(j, j'j'') = 0. \quad (3.6)$$

Furthermore, if J has right unit, one can assume that $\omega = 0$, i.e.

$$M \cong \begin{pmatrix} J & I \\ 0 & 0 \end{pmatrix}. \quad (3.7)$$

Proof. We will prove the proposition combining the fact $IM = 0$ with a non-unital version of the relative Hochschild theory [34, 35] of R -bimodule split extensions M of J by ideals I satisfying $I^2 = 0$. An R -bimodule splitting of (3.3) gives an isomorphism of R -bimodules $I \oplus J \cong M$, and amounts to an R -bimodule map $\sigma : J \rightarrow M$ such that $\varepsilon_M \circ \sigma = \text{id}_J$. This allows us to define ω via the formula $\omega(j, j') := \sigma(j)\sigma(j') - \sigma(jj')$. The thus defined ω satisfies (3.6) by the associativity of multiplication in M and the property $IM = 0$. It is, in fact, the Hochschild 2-cocycle condition missing one summand that vanishes by $IM = 0$.

If J has a right unit $e \in J$ ($je = j$ for all $j \in J$), we can define an R -bimodule map $\lambda : J \rightarrow I$ by

$$\lambda(j) := -\omega(j, e). \quad (3.8)$$

Now, putting $j'' := e$ in (3.6), rewriting the result in terms of (3.8) and adding the last summand being zero by $IJ = 0$, we obtain

$$\begin{aligned} \omega(j, j') &= \omega(j, j'e) = -j\omega(j', e) + \omega(jj', e) \\ &= j\lambda(j') - \lambda(jj') + \lambda(j)j'. \end{aligned} \quad (3.9)$$

This means that ω is a Hochschild coboundary of λ . By the theory of Hochschild extensions, this is tantamount to the fact that the automorphism

$$(i, j) \mapsto (i + \lambda(j), j) \quad (3.10)$$

of the R -bimodule $I \oplus J$ transforms the multiplication (3.5) to the multiplication with $\omega = 0$. $\square$

If, in addition, B is unital, $\varepsilon_M(M) = B$, and the left B -module I is free of rank n , then M is isomorphic to the algebra of $(n+1) \times (n+1)$ matrices over B with non-zero entries occurring only in the first row, with ε_M mapping M onto the first entry of the first row. Motivated by this simple case when B is unital and I is a free left B -module of finite rank, we call all split R -algebra extensions of the form

$$M \cong \begin{pmatrix} B & I \\ 0 & 0 \end{pmatrix} \quad (3.11)$$

row extensions. Note that we can think of a row extension as a special, always *non-unital* trivial Hochschild extension [52, p. 312], where I is a *non-unitary* B -bimodule even when B is unital.

For further considerations, it is crucial that a row extension M is an algebra-over- R split extension, so it contains a copy of an R -algebra B as an R -subalgebra with the ideal I as its complement. It is also important that I is not only a square zero ideal, but also $IM = 0$. If e is a left unit of B acting on M on the left as identity, we say that M is a unitary left B -module. Then M becomes a left-unital R -algebra extension with a left unit corresponding under the isomorphism (3.11) to

$$\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}. \quad (3.12)$$

3.1. The periodic cyclic homology of row extensions.

Proposition 3.3. *The R -algebra map ε_M induces an isomorphism of relative periodic cyclic homology of R -algebras*

$$HP_*(M | R) \longrightarrow HP_*(J | R). \quad (3.13)$$

Proof. The claim is a consequence of the Goodwillie theorem [25] (see Theorem 7.3 of [16] for the non-unital case) applied to the Kadison R -algebra periodic cyclic homology [38] of the R -algebra extension (3.3) by the nilpotent ideal I . $\square$

3.2. The Hochschild complex of row extensions. Triangular R -algebras over a unital commutative ring R are of the form

$$T = \begin{pmatrix} B & I \\ 0 & B' \end{pmatrix}, \quad (3.14)$$

where B, B' are unital R -algebras and I is a two-sided unitary (B, B') -bimodule that is symmetric as an underlying R -bimodule. The computation of the Hochschild homology of triangular R -algebras is subsumed by [42, Theorem 1.2.15], which says that the canonical R -algebra map $T \rightarrow B \times B'$ annihilating I induces an isomorphism

$$HH_*(T | R) \cong HH_*(B \times B' | R). \quad (3.15)$$

Note that the pair of two projections onto the factors of the product $B \times B'$ induce another isomorphism:

$$HH_*(B \times B' | R) \cong HH_*(B | R) \oplus HH_*(B' | R). \quad (3.16)$$

Let us observe now that a row extension fits into a ring extension of the form

$$0 \longrightarrow \begin{pmatrix} B & I \\ 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} B & I \\ 0 & R \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 \\ 0 & R \end{pmatrix} \longrightarrow 0. \quad (3.17)$$

Shortening the middle term to T , we can rewrite it as

$$0 \longrightarrow M \longrightarrow T \longrightarrow R \longrightarrow 0. \quad (3.18)$$

Since our extension satisfies the assumption of purity, one can use Wodzicki's excision theorem [53, 54] for a left unital (hence H -unital) R -algebra M to obtain a long exact sequence of Hochschild homology groups which, by (3.15) and (3.16), reads as

$$\begin{array}{ccccccc} HH_{2n+1}(M | R) & \longrightarrow & HH_{2n+1}(B | R) \oplus HH_{2n}(R | R) & \longrightarrow & HH_{2n+1}(R | R) & \longrightarrow & \\ \curvearrowright & & & & & & \\ \longrightarrow & HH_{2n}(M | R) & \longrightarrow & HH_{2n}(B | R) \oplus HH_{2n}(R | R) & \longrightarrow & HH_{2n}(R | R) & \longrightarrow \\ \curvearrowright & & & & & & \\ \longrightarrow & HH_{2n-1}(M | R) & \longrightarrow & HH_{2n-1}(B | R) \oplus HH_{2n-1}(R | R) & \longrightarrow & HH_{2n-1}(R | R). & \end{array}$$

Furthermore, as in every row the first arrow goes into the first direct summand via the map induced by ε_M , and every second arrow is a projection onto a second direct summand, we conclude that ε_M induces a quasi-isomorphism of Hochschild chain complexes.

Our next step is to promote the quasi-isomorphism to a chain homotopy equivalence. Using an R -module splitting $M = I \oplus B$ coming from an R -algebra splitting (3.11), we can form split short exact sequences of symmetric R -bimodules ($\otimes_R$ stands for the R -balanced tensor product of symmetric R -bimodules)

$$0 \rightarrow \bigoplus_{p+q=n} M^{\otimes_R p} \otimes_R I \otimes_R B^{\otimes_R q} \rightarrow M^{\otimes_R n+1} \xrightarrow{\varepsilon_M^{\otimes_R n+1}} B^{\otimes_R n+1} \rightarrow 0. \quad (3.19)$$

Since ε_M is an R -algebra map, and the multiplication of M restricted to the image of B under the R -algebra splitting of M coincides with the original multiplication of B , the collection of the induced maps $\varepsilon_M^{\otimes_R n+1}$ is a morphism of cyclic objects computing the Hochschild, cyclic, periodic and negative cyclic homology of R -algebras. Now we show that, for row extensions of left unital R -algebras, the induced map on the Hochschild chain complexes is a homotopy equivalence. This is an excision-free way to see that the map induces an isomorphism on the Hochschild homology. Hence, by virtue of [42, Proposition 5.1.6], we conclude that the induced maps on cyclic, periodic and negative cyclic homology are also isomorphisms.

We are now ready for a main technical result of this article:

Theorem 3.4. *Let M be a left unitary row extension of a left unital R -algebra B . Then the ε_M -induced map of the Hochschild chain complexes*

$$\mathrm{Tot} \, \mathrm{CC}^{[1]}(M \mid R) \longrightarrow \mathrm{Tot} \, \mathrm{CC}^{[1]}(B \mid R) \quad (3.20)$$

is graded-split surjective and has a contractible kernel. In particular, it is a chain homotopy equivalence.

Proof. We will show that the collection of maps

$$\begin{aligned} h : M^{\otimes_R p} \otimes_R I \otimes_R B^{\otimes_R q} &\longrightarrow M^{\otimes_R p} \otimes_R I \otimes_R B^{\otimes_R q+1} \\ h(m_1, \dots, m_p, i, b_1, \dots, b_q) &:= (-1)^{p+1}(m_1, \dots, m_p, i, e, b_1, \dots, b_q), \end{aligned} \quad (3.21)$$

forms a homotopy contracting the kernel of the map $\varepsilon_M^{\otimes_R n+1}$ of Hochschild complexes. Here, on the right-hand side, e denotes a left unit of B . For the sake of brevity, we replace the tensor sign $\otimes_R$ by a comma.

The kernel of h inherits a boundary from the Hochschild boundary. For $p+q=n$, the boundary on the kernel reads as

$$\begin{aligned} b(m_1, \dots, m_p, i, b_1, \dots, b_q) &= (b'(m_1, \dots, m_p), i, b_1, \dots, b_q) \\ &\quad - (-1)^p(m_1, \dots, m_{p-1}, m_p i, b_1, \dots, b_q) \\ &\quad - (-1)^p(m_1, \dots, m_p, i, b'(b_1, \dots, b_q)) \\ &\quad + (-1)^n(b_q m_1, \dots, m_p, i, b_1, \dots, b_{q-1}). \end{aligned} \quad (3.22)$$

Here, for any (not necessarily unital) associative algebra over R ,

$$b'(a_1, \dots, a_r) := (a_1 a_2, a_3, \dots, a_r) - (a_1, a_2 a_3, \dots, a_r) + \dots + (-1)^r(a_1, \dots, a_{r-1} a_r).$$

Note that here the condition $IM = 0$ shortens the Hochschild boundary by one vanishing summand as it did for the Hochschild cocycle encoding the structure of the extension in the proof of Proposition 3.1.

Now, computing the two compositions $b \circ h$ and $h \circ b$, we get, for $p+q = n$, $p, q > 0$,

$$\begin{aligned} (b \circ h)(m_1, \dots, m_p, i, b_1, \dots, b_q) = & -(-1)^p (b'(m_1, \dots, m_p), i, e, b_1, \dots, b_q) \\ & + (m_1, \dots, m_{p-1}, m_p i, e, b_1, \dots, b_q) \\ & + (m_1, \dots, m_p, i, b_1, \dots, b_q) \\ & - (m_1, \dots, m_p, i, e, b'(b_1, \dots, b_q)) \\ & + (-1)^q (b_q m_1, m_2, \dots, m_p, i, e, b_1, \dots, b_{q-1}), \end{aligned} \quad (3.23)$$

$$\begin{aligned} (h \circ b)(m_1, \dots, m_p, i, b_1, \dots, b_q) = & (-1)^p (b'(m_1, \dots, m_p), i, e, b_1, \dots, b_q) \\ & - (m_1, \dots, m_{p-1}, m_p i, e, b_1, \dots, b_q) \\ & + (m_1, \dots, m_p, i, e, b'(b_1, \dots, b_q)) \\ & - (-1)^q (b_q m_1, m_2, \dots, m_p, i, e, b_1, \dots, b_{q-1}). \end{aligned} \quad (3.24)$$

For $p+q = n$, $q = 0$, we obtain

$$\begin{aligned} (b \circ h)(m_1, \dots, m_n, i) = & -(-1)^n (b'(m_1, \dots, m_n), i, e) \\ & + (m_1, \dots, m_{n-1}, m_n i, e) \\ & + (m_1, \dots, m_n, i), \end{aligned} \quad (3.25)$$

$$\begin{aligned} (h \circ b)(m_1, \dots, m_n, i) = & (-1)^n (b'(m_1, \dots, m_n), i, e) \\ & - (m_1, \dots, m_{n-1}, m_n i, e). \end{aligned} \quad (3.26)$$

Finally, much in the same way, one computes the remaining case $p = 0$. Summarizing, one sees that they add up to give $b \circ h + h \circ b = \text{id}$. $\square$

4. The cyclic-homology Chern–Weil Homomorphism

To help the reader understand what follows, we refer to [10, 11] for basic facts and definitions about coalgebra-Galois extensions. The only modification we need is allowing, instead of a base field k , an integral domain R over k , as the base of unital algebras under consideration, and assuming that the coaction is R -linear. This means that we consider a *family* of coalgebra-Galois extensions parameterized by $\text{Spec}(R)$. The unadorned tensor product symbol denotes the tensor product over k .

Let a unital R -algebra A be a right comodule for a coalgebra C with a group-like element $e \in C$. We will denote the R -linear C -coaction on A using the Heyneman–Sweedler notation

$$A \longrightarrow A \otimes C, \quad a \longmapsto a_{(0)} \otimes a_{(1)}. \quad (4.1)$$

The R -subalgebra of invariants of this coaction we denote by B , i.e.

$$B = A^{\text{co}C} := \{b \in A \mid b_{(0)} \otimes b_{(1)} = b \otimes e\}. \quad (4.2)$$

We assume that $B \subseteq A$ is a C -Galois extension, which means that both the canonical map

$$\text{can} : A \otimes_B A \longrightarrow A \otimes C, \quad a \otimes_B a' \longmapsto aa'_{(0)} \otimes a'_{(1)}, \quad (4.3)$$

and the canonical entwining

$$\psi : C \otimes A \longrightarrow A \otimes C, \quad c \otimes a \longmapsto \text{can}(\text{can}^{-1}(1 \otimes c)a), \quad (4.4)$$

are invertible. Now one can define an R -linear left C -coaction on A as follows:

$$A \longrightarrow C \otimes A, \quad a \longmapsto a_{(-1)} \otimes a_{(0)} := \psi^{-1}(a_{(0)} \otimes a_{(1)}). \quad (4.5)$$

If $C = H$ is a Hopf algebra with invertible antipode, the group-like element e is the unit of H , and A is a right H -comodule algebra, then the coalgebra-Galois extension is called Hopf-Galois (with the Galois Hopf algebra H), and the left H -coaction makes the opposite algebra A^{op} a left comodule algebra over H .

For a C -Galois extension as above, one can define the *translation map*

$$\tau : C \longrightarrow A \otimes_B A, \quad \tau(c) := \text{can}^{-1}(1 \otimes c) \quad (4.6)$$

and prove that it is a C -bilinear map with respect to the left and right coactions of C on the left and right tensorand of $A \otimes_B A$, respectively.

A *strong connection* ℓ [11] is a unital ($\ell(e) = 1 \otimes 1$) C -bilinear lifting of the translation map τ ,

$$\begin{array}{ccc} & & A \otimes_R A \\ & \nearrow \ell & \downarrow \\ C & \xrightarrow{\tau} & A \otimes_B A. \end{array} \quad (4.7)$$

The same arguments as for algebras over a field prove that for a coalgebra-Galois extension, the existence of a non-unital strong connection ℓ is equivalent to the R -relative equivariant projectivity of A viewed as a left B -module right C -comodule [11]. Such coalgebra-Galois extensions are called *principal*. When R is more general than a field, we call them *families of principal coalgebra-Galois extensions*. Finally, note that we will use non-unital strong connections just as in [8] because the unitality will not play a role in the sequel.

4.1. The standard quantum Hopf fibration. Let $R = k[q, q^{-1}]$ and

$$\begin{pmatrix} x & -q^{-1}v^* \\ v & x^* \end{pmatrix} \quad (4.8)$$

be the fundamental matrix of the $*$ -Hopf algebra $\mathcal{O}(SU_q(2))$. The $*$ -algebra $\mathcal{O}(SU_q(2))$ is generated by x and v subject to the relations

$$\begin{aligned} vx &= q xv, & v^*x &= q x v^*, & x^*v &= q v x^*, & x^*v^* &= q v^*v, \\ v v^* &= v^*v, & x^*x + v^*v &= 1, & x x^* + q^{-2} v^*v &= 1. \end{aligned} \quad (4.9)$$

The comultiplication is provided by

$$\Delta \begin{pmatrix} x & -q^{-1}v^* \\ v & x^* \end{pmatrix} := \begin{pmatrix} x \otimes 1 & -q^{-1}v^* \otimes 1 \\ v \otimes 1 & x^* \otimes 1 \end{pmatrix} \begin{pmatrix} 1 \otimes x & -1 \otimes q^{-1}v^* \\ 1 \otimes v & 1 \otimes x^* \end{pmatrix} \quad (4.10)$$

and the right coaction of $\mathcal{O}(U(1)) = k[t, t^{-1}]$ is given by

$$\delta \begin{pmatrix} x & -q^{-1}v^* \\ v & x^* \end{pmatrix} := \begin{pmatrix} x \otimes 1 & -q^{-1}v^* \otimes 1 \\ v \otimes 1 & x^* \otimes 1 \end{pmatrix} \begin{pmatrix} 1 \otimes t & 0 \\ 0 & 1 \otimes t^{-1} \end{pmatrix}. \quad (4.11)$$

The subalgebra $\mathcal{O}(S_q^2)$ of coaction invariants is known as the algebra of the *standard Podleś quantum sphere* [46], and $\mathcal{O}(S_q^2) \subset \mathcal{O}(SU_q(2))$ is an example of a family of $\mathcal{O}(U(1))$ -Hopf–Galois extensions parameterized by $\text{Spec}(R)$.

4.2. The Ehresmann–Schauenburg quantum groupoid. Provided A is a faithfully flat (as a left B -module) C -Galois extension of B , one introduces the Ehresmann–Schauenburg B -coring in terms of the *cotensor product* of two (one right and one left) C -comodules [13, 48]. Since we are interested in families parameterized by $\text{Spec}(R)$, we have to modify the above notion of the cotensor product to allow both C -comodule structures to be R -linear.

Definition 4.1. Let X and Y be, respectively, a right and a left R -module equipped, respectively, with a right and a left C -coaction that is R -linear in the following sense: for any $r \in R$, $x \in X$ and $y \in Y$,

$$(xr)_{(0)} \otimes (xr)_{(1)} = x_{(0)}r \otimes x_{(1)}, \quad (ry)_{(-1)} \otimes (ry)_{(0)} = y_{(-1)} \otimes ry_{(0)}. \quad (4.12)$$

Then we define their *R -balanced cotensor product* as the equalizer (the difference kernel)

$$X \square_R^C Y := \text{Eq}(X \otimes_R Y \rightrightarrows X \otimes_R (R \otimes C) \otimes_R Y),$$

where the pair of parallel maps is given by

$$\begin{aligned} x \otimes_R y &\longmapsto x_{(0)} \otimes_R (1 \otimes x_{(1)}) \otimes_R y, \\ x \otimes_R y &\longmapsto x \otimes_R (1 \otimes y_{(-1)}) \otimes_R y_{(0)}. \end{aligned}$$

Here $R \otimes C$ is equipped with the structure of a symmetric R -bimodule induced from R .

The correctness of the above definition can be seen from the fact that the conditions (4.12) equip respectively X and Y with a right and a left comodule structures over an R -coalgebra $R \otimes C$, and thus make $X \square_R^C Y$ a cotensor product of comodules over this R -coalgebra.

In terms of the R -balanced cotensor product $\square_R^C$, given a C -Galois extension $B \subseteq A$ of R -algebras, one can construct an R -module $M := A \square_R^C A$. It is canonically an R -symmetric B -bimodule with the canonical B -coring structure

$$\begin{aligned} \Delta_M : M &\longrightarrow M \otimes_B M, & \sum_i a_i \otimes_R a'_i &\longmapsto \sum_i a_{i(0)} \otimes_R \tau(a_{i(1)}) \otimes_R a'_i, \\ \varepsilon_M : M &\longrightarrow B, & \sum_i a_i \otimes_R a'_i &\longmapsto \sum_i a_i a'_i. \end{aligned} \quad (4.13)$$

If $C = H$ is a Hopf algebra, and A is a faithfully flat (as a left B -module) H -Galois extension of an R -algebra B , the Ehresmann–Schauenburg coring M is a unital $B \otimes_R B^{\text{op}}$ -subring of $A \otimes_R A^{\text{op}}$ defined as $M := A \square_R^H A^{\text{op}} \subseteq A \otimes_R A^{\text{op}}$. The compatibility of the

B -coring and subring structures makes it a family of *quantum groupoids* parameterized by $\text{Spec}(R)$.

However, in what follows, we will need only the fact that (M, ε_M) is a left B -module with the augmentation equal to the counit of the coring. According to Sect. 3, it leads to a *row extension* of B . Note that the Ehresmann–Schauenburg groupoid equipped with the new multiplication defined by ε_M becomes a row extension, and is not a subring of $A \otimes_R A^{\text{op}}$ anymore.

For Hopf–Galois extensions, the canonical row extension corresponding to the counit of the Ehresmann–Schauenburg quantum groupoid $\varepsilon_M : M \rightarrow B$ can be described as the canonically split extension of left B -modules

$$0 \longrightarrow \Omega^1(A \mid R)^{\text{co}H} \longrightarrow (A \otimes_R A)^{\text{co}H} \xrightarrow[\varepsilon_M]{\sigma} A^{\text{co}H} \longrightarrow 0. \quad (4.14)$$

Here by $\Omega^1(A \mid R)$ we denote the A -bimodule of universal noncommutative R -linear differentials of an R -algebra A , and by σ we denote the canonical left B -linear splitting defined by $\sigma(b) := b \otimes 1$. It is easy to see that σ is an R -algebra map from $B = A^{\text{co}H}$ to the row extension $M = (A \otimes_R A)^{\text{co}H}$. Therefore, we obtain the following complete description of the structure of our row extension

$$M \cong \begin{pmatrix} B & \Omega^1(A \mid R)^{\text{co}H} \\ 0 & 0 \end{pmatrix}, \quad \sum_i a_i \otimes_R a'_i \mapsto \begin{pmatrix} \sum_i a_i a'_i & \sum_i a_i d a'_i \\ 0 & 0 \end{pmatrix}, \quad (4.15)$$

where, on the right-hand side, we use the universal R -linear derivation

$$d : A \longmapsto \Omega^1(A \mid R), \quad da := 1 \otimes_R a - a \otimes_R 1, \quad (4.16)$$

and the A -bimodule structure of $\Omega^1(A \mid R)$.

4.3. The cyclic-homology Chern–Weil homomorphism from a strong connection. For any coalgebra C , we define the subspace

$$C^{\text{tr}} := \text{Eq}(C \rightrightarrows C \otimes C) \quad (4.17)$$

equalizing the comultiplication Δ and the comultiplication Δ composed with the flip. Note that the canonical map $C^{\text{tr}} \rightarrow C$ is the dual counterpart of the universal trace map $A \rightarrow A/[A, A]$ for an algebra A , and an element of C^{tr} defines canonically a trace on the dual convolution algebra C^* . Therefore, we call the elements of C^{tr} *cotraces*.

The classical case, when $C = \mathcal{O}(G)$ is the coordinate algebra of a linear algebraic group G with the comultiplication equivalent to the algebraic group law, sheds some light on the relation between C^{tr} and the Chern–Weil map as follows. First of all, it is easy to see that

$$C^{\text{tr}} = \mathcal{O}(\text{Ad}(G))^G, \quad (4.18)$$

where the right-hand side is the algebra of Ad -invariants (invariants with respect to the action of G on itself by conjugations). In other words, this is the algebra of class functions.

Furthermore, the kernel of the counit ε of C is the maximal ideal $\mathfrak{m}$ corresponding to the neutral element of G , and the $\mathfrak{m}$ -adic filtration of $\mathcal{O}(\mathrm{Ad}(G))$ is G -invariant. Hence, the filtration passes to the Ad -invariants of the associated graded algebra, and one gets the algebra

$$\mathrm{gr}_{\mathfrak{m}} \mathcal{O}(\mathrm{Ad}(G))^G = \left(\bigoplus_{n \geq 0} \mathfrak{m}^n / \mathfrak{m}^{n+1} \right)^G \cong \mathrm{Sym}(\mathfrak{m}/\mathfrak{m}^2)^G = \bigoplus_{n \geq 0} (\mathrm{Sym}^n \mathfrak{g}^*)^G \quad (4.19)$$

of Ad -invariant polynomials on the Lie algebra $\mathfrak{g}$. The latter is the domain of the classical Chern–Weil map and the infinitesimal counterpart of the right-hand side of (4.18). Observe also that replacing the Lie algebra $\mathfrak{g}$ by the G -space $\mathrm{Ad}(G)$ plays a fundamental role in the construction of G -equivariant cyclic homology (see the paper [6] by Block and Getzler).

Below, we will use the associativity of the cotensor product of bicomodules over a coalgebra [13, 11.6] and the tensor-cotensor associativity [13, 10.6]. They both hold because our coalgebra C is defined over a field, so $R \otimes C$ is a flat R -coalgebra.

Lemma 4.2. *For any $m \in \mathbb{N}$, the m -fold comultiplication map of the coalgebra C defines a linear isomorphism*

$$C^{\mathrm{tr}} \xrightarrow{\cong} C \square^{C \otimes C^{\mathrm{op}}} \left(\underbrace{C \square^C \dots \square^C C}_{m+1} \right). \quad (4.20)$$

Here the right coaction of $C \otimes C^{\mathrm{op}}$ on C is given by $c \mapsto c_{(2)} \otimes (c_{(3)} \otimes c_{(1)})$, and the left coaction of $C \otimes C^{\mathrm{op}}$ on the iterated cotensor product is given by $x \mapsto (x_{(-1)} \otimes x_{(1)}) \otimes x_{(0)}$, where $x \mapsto x_{(-1)} \otimes x_{(0)} \otimes x_{(1)}$ denotes the C -bicomodule structure of the iterated cotensor product coming from the standard left and the standard right C -coaction on, respectively, the left-most and the right-most copy of C .

Proof. Since C is the unit object of the monoidal category of bicomodules with respect to the cotensor product, it is enough to prove (4.20) for $m = 0$. Then it is easy to check that the comultiplication restricted to the elements from $C^{\mathrm{tr}} \subseteq C$ lands in $C \square^{C \otimes C^{\mathrm{op}}} C$, and the application of the counit to the first cotensor factor provides the inverse. $\square$

Note that, by applying the counit to the left-most factor C in the cotensor product

$$C \square^{C \otimes C^{\mathrm{op}}} (V_0 \square^C \dots \square^C V_m), \quad (4.21)$$

we can identify it with the circular cotensor product of C -bicomodules $V_0, \dots, V_m$. For example, for $m = 5$ it looks as follows:

$$\quad (4.22)$$

In the following two lemmas, we think of C^{tr} as a circular cotensor product as in (4.20). (A close relation of these lemmas to results in [11] will be explained in Sect. 4.4.)

Lemma 4.3. *A strong connection $\ell : C \longrightarrow A \otimes_R A$ induces the linear map*

$$C^{\text{tr}} \longrightarrow M^{\otimes_R(m+1)}, \quad c \longmapsto c_m(\ell)(c) := \ell(c_{(1)}) \otimes_R \cdots \otimes_R \ell(c_{(m+1)}). \quad (4.23)$$

(Here we think of $M^{\otimes_R(m+1)}$ as a circular tensor power.)

Proof. Since ℓ is a morphism of C -bicomodules, it can be applied to an element of the circular cotensor power of C to get an element of the circular cotensor power of $A \otimes_R A$. By the tensor-cotensor associativity, it can be written as an element of the circular tensor power of $M = A \square_R^C A$. For example, $c_5(\ell)$ reads as the dashed arrow in the following commutative diagram:

$$(4.24) \quad \square$$

Lemma 4.4. *For any $c \in C^{\text{tr}}$ and any strong connection ℓ , the element*

$$c_m(\ell)(c) := \left(c_{(m+1)}^{(2)} \otimes_R c_{(1)}^{(1)} \right) \otimes_R \left(c_{(1)}^{(2)} \otimes_R c_{(2)}^{(1)} \right) \otimes_R \cdots \otimes_R \left(c_{(m)}^{(2)} \otimes_R c_{(m+1)}^{(1)} \right) \quad (4.25)$$

belongs to the cyclic-symmetric part of $M^{\otimes_R m+1}$. Here $c^{(1)} \otimes_R c^{(2)} := \ell(c)$ is the Heyneman–Sweedler-type notation for ℓ .

Proof. Note first that, by the very definition of C^{tr} , applying the comultiplication Δ to any $c \in C^{\text{tr}}$, we obtain a symmetric tensor

$$c_{(1)} \otimes c_{(2)} = c_{(2)} \otimes c_{(1)}. \quad (4.26)$$

Applying the iterated comultiplication Δ^m to c yields the same result as applying $\Delta^{m-1} \otimes \text{id}_C$ to both sides of (4.26). Hence, we obtain

$$\begin{aligned}
 c_{(1)} \otimes c_{(2)} \otimes \cdots \otimes c_{(m)} \otimes c_{(m+1)} \\
 &= c_{(1)(1)} \otimes c_{(1)(2)} \otimes \cdots \otimes c_{(1)(m)} \otimes c_{(2)} \\
 &= c_{(2)(1)} \otimes c_{(2)(2)} \otimes \cdots \otimes c_{(2)(m)} \otimes c_{(1)} \\
 &= c_{(2)} \otimes c_{(3)} \otimes \cdots \otimes c_{(m+1)} \otimes c_{(1)},
 \end{aligned} \tag{4.27}$$

which proves that the right-hand side of (4.25) is a cyclic-symmetric tensor as well. $\square$

Lemma 4.5. *For any face operator d_i coming from the multiplication in M , the elements $c_m := c_m(\ell)(c)$, $m \geq 1$, satisfy the identities*

$$d_i c_m = c_{m-1}. \tag{4.28}$$

Proof. By the cyclic symmetry established in Lemma 4.4, it is enough to check the desired identity only for the 0-th face operator d_0 . This goes as follows:

$$\begin{aligned}
 d_0 c_m &= \left(c_{(m+1)}^{(2)} c_{(1)}^{(1)} c_{(1)}^{(2)} \otimes c_{(2)}^{(1)} \right) \otimes \cdots \otimes \left(c_{(m)}^{(2)} \otimes c_{(m+1)}^{(1)} \right) \\
 &= \left(c_{(m+1)}^{(2)} \varepsilon(c_{(1)}) \otimes c_{(2)}^{(1)} \right) \otimes \cdots \otimes \left(c_{(m)}^{(2)} \otimes c_{(m+1)}^{(1)} \right) \\
 &= \left(c_{(m+1)}^{(2)} \otimes \varepsilon(c_{(1)}) c_{(2)}^{(1)} \right) \otimes \cdots \otimes \left(c_{(m)}^{(2)} \otimes c_{(m+1)}^{(1)} \right) \\
 &= \left(c_{(m)}^{(2)} \otimes c_{(1)}^{(1)} \right) \otimes \cdots \otimes \left(c_{(m-1)}^{(2)} \otimes c_{(m)}^{(1)} \right) \\
 &= c_{m-1}.
 \end{aligned} \tag{4.29}$$

$\square$

Combining above lemmas, we arrive at the centerpiece result of the article:

Theorem 4.6. (Cyclic-homology Chern–Weil homomorphism) *For any $c \in C^{\text{tr}}$ and any strong connection ℓ ,*

$$\widetilde{\text{chw}}_n(\ell, c) := \sum_{m=0}^{2n} (-1)^{\lfloor m/2 \rfloor} \frac{m!}{\lfloor m/2 \rfloor!} c_m(\ell)(c)$$

is a $2n$ -cycle in the total complex $\text{Tot CC}_\bullet(M | R) = \bigoplus_{m=0}^\bullet M^{\otimes m+1}$ computing the cyclic homology $\text{HC}_\bullet(M | R)$. Its homology class is stable under Connes' periodicity operator.

Proof. By Lemma 4.4 and Lemma 4.5, the chains $c_m(\ell)(c)$ satisfy assumptions of Proposition 2.7, which proves the claim. $\square$

Composing $\widetilde{\text{chw}}_n(\ell, -)$ with the map induced by the algebra map $\varepsilon_M : M \rightarrow B$, we obtain the *noncommutative homotopy Chern–Weil homomorphism*

$$\text{chw}_n(\ell, -) : C^{\text{tr}} \longrightarrow (\text{Tot CC})_{2n}(B | R) \tag{4.30}$$

in the homotopy category of complexes. The composition of $\text{chw}_n(\ell, -)$ with the operation of taking the cyclic-homology class we denote by $\text{chw}_n(\ell)$ and call the *cyclic-homology Chern–Weil homomorphism*.

4.4. A factorization of the Chern–Galois character. For any coalgebra C , we consider the group completion $\text{Corep}(C)$ of the monoid of the isomorphism classes of finite-dimensional left C -comodules, which we call *corepresentations*. If V is a corepresentation, then, given a basis $(v_i)_{i \in I}$ of V , the left C -comodule structure $V \rightarrow C \otimes V$ is equivalent to a finite matrix $(c_{ij})_{i,j \in I}$ with entries in C . It is defined by $v_i \mapsto \sum_j c_{ij} \otimes v_j$ and satisfies

$$\Delta(c_{ik}) = \sum_j c_{ij} \otimes c_{jk}, \quad \varepsilon(c_{ij}) = \delta_{ij}. \quad (4.31)$$

Clearly, the element

$$\chi(V) := \sum_i c_{ii} \quad (4.32)$$

is independent of the choice of basis, and hence depends only on the isomorphism class $[V]$ of V . We will call it the *character* of the corepresentation V . Since

$$\chi(V) = \chi(V') + \chi(V'') \quad (4.33)$$

for any short exact sequence of corepresentations

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0, \quad (4.34)$$

we conclude that χ factorizes through $\text{Corep}(C)$. By the obvious symmetry property

$$\chi(V)_{(1)} \otimes \chi(V)_{(2)} = \sum_{i,j} c_{ij} \otimes c_{ji} = \sum_{i,j} c_{ji} \otimes c_{ij} = \chi(V)_{(2)} \otimes \chi(V)_{(1)}, \quad (4.35)$$

the character of a corepresentation defines a map

$$\chi : \text{Corep}(C) \longrightarrow C^{\text{tr}}, \quad [V] \longmapsto \chi(V). \quad (4.36)$$

Furthermore, using (4.25) and (4.32), we compute the composition

$$\begin{aligned} c_m(\ell)(\chi(V)) \\ = \sum_{i_1, \dots, i_{m+1}} \left(c_{i_1 i_2}^{(2)} \otimes_R c_{i_2 i_3}^{(1)} \right) \otimes_R \left(c_{i_2 i_3}^{(2)} \otimes_R c_{i_3 i_4}^{(1)} \right) \otimes_R \cdots \otimes_R \left(c_{i_{m+1} i_1}^{(2)} \otimes_R c_{i_1 i_2}^{(1)} \right). \end{aligned}$$

Now, after applying the map induced by the algebra map $\varepsilon_M : M \rightarrow B$, we obtain

$$\sum_{i_1, \dots, i_{m+1}} c_{i_1 i_2}^{(2)} c_{i_2 i_3}^{(1)} \otimes_R c_{i_2 i_3}^{(2)} c_{i_3 i_4}^{(1)} \otimes_R \cdots \otimes_R c_{i_{m+1} i_1}^{(2)} c_{i_1 i_2}^{(1)}. \quad (4.37)$$

(For $R = k$, this coincides with the expression appearing in the definition of the Chern–Galois character in [11].) The linear combination, as in Theorem 4.6, of elements in the formula (4.37) defines a map $\text{chg}_n(\ell) : \text{Corep}(C) \rightarrow (\text{Tot CC})_{2n}(B | R)$ whose composition with the operation of taking the cyclic-homology class defines the *Chern–Galois character* chg_n (no longer dependent on ℓ). From Theorem 4.6, we now infer:

Corollary 4.7. *The Chern–Galois character chg_n decomposes as the diagonal composition in the following commutative diagram*

$$\begin{array}{ccc} \text{Corep}(C) & \xrightarrow{[A \square^C -]} & K_0(B) \\ \chi \downarrow & & \downarrow \text{ch}_n \\ C^{\text{tr}} & \xrightarrow{\text{chw}_n(\ell)} & \text{HC}_{2n}(B | R). \end{array}$$

4.5. The independence of the choice of a strong connection. The fundamental property of the classical Chern–Weil homomorphism is its independence of the choice of a connection. As we do not know how to reproduce the classical argument in the noncommutative context, herein we use the independence of the Chern–Galois character of the choice of a strong connection to argue such an independence for the cyclic-homology Chern–Weil homomorphism.

We will say that C has *enough characters* iff C^{tr} is linearly spanned by the characters of its corepresentations. Note that the algebra of class functions on a semi-simple connected algebraic group has a linear basis consisting of characters of irreducible rational representations [37, 3.2]. It is well known that the same is true for finite groups.

In fact, both cases are particular cases of cosemisimple coalgebras by Woronowicz–Peter–Weyl theory rewritten in purely coalgebraic language as in [1]. (The theory covers the Hopf algebras corresponding to compact quantum groups in the sense of Woronowicz.) Cosemisimplicity of C means that it is a direct sum of finite-dimensional cosimple sub-coalgebras C_V

$$C = \bigoplus_V C_V, \quad (4.38)$$

where $C_V = V^* \otimes V$ is a co-matrix coalgebra corresponding to an irreducible left C -comodule V . Then

$$C^{\text{tr}} = \bigoplus_V C_V^{\text{tr}}, \quad (4.39)$$

where

$$C_V^{\text{tr}} = \text{span}(\chi(V)). \quad (4.40)$$

Indeed, given a dual basis $(v_i, v_i^*)_{i \in I}$ of any finite-dimensional vector space V , the co-matrix coalgebra $C_V = V^* \otimes V$ satisfies

$$C_V^{\text{tr}} = (V^* \otimes V)^{\text{tr}} = \text{span} \left(\sum_i v_i^* \otimes v_i \right), \quad (4.41)$$

and any left C -comodule structure on V is equivalent to the coalgebra map

$$V^* \otimes V \longrightarrow C, \quad v_j^* \otimes v_i \longmapsto c_{ij}, \quad v_{i(-1)} \otimes v_{i(0)} =: \sum_j c_{ij} \otimes v_j. \quad (4.42)$$

For instance, the characters $\chi^{(n)}$ and $\chi^{(n+\frac{1}{2})}$ of integer and half-integer spin corepresentations in the direct-sum decomposition of the Hopf algebra of $SU_q(2)$ into cosimple coalgebras, using the notation as in Section 4.1, can be expressed in terms of the little q -Jacobi polynomials [44, 2.2] as follows:

$$\chi^{(n)} = P_n^{(0,0)}(v^*v; q^2) + \sum_{i=1}^n x^{2i} P_{n-i}^{(0,2i)}(v^*v; q^2) + P_{n-i}^{(0,2i)}(v^*v; q^2) x^{*2i}, \quad (4.43)$$

$$\chi^{(n+\frac{1}{2})} = \sum_{i=0}^n x^{2i+1} P_{n-i}^{(0,2i+1)}(v^*v; q^2) + P_{n-i}^{(0,2i+1)}(v^*v; q^2) x^{*2i+1}. \quad (4.44)$$

Here

$$P_n^{(0,k)}(v^*v; q) := 1 + \sum_{m=1}^n q^m \prod_{j=1}^m \frac{(1 - q^{j-n-1})(1 - q^{j-n-k})}{(1 - q^j)^2} (v^*v)^m \quad (4.45)$$

are the little q -Jacobi polynomials $P_n^{(l,k)}$ in the variable v^*v at the special value $l = 0$.

Hence, the domain of our cyclic-homology Chern–Weil homomorphism for a quantum principal $SU_q(2)$ -bundle ($\mathcal{O}(SU_q(2))$ -Galois extension) reads as

$$\mathcal{O}(SU_q(2))^{\text{tr}} = \text{span} \left\{ \chi^{(n)}, \chi^{(n+\frac{1}{2})} \mid n \in \mathbb{N} \right\}. \quad (4.46)$$

Note that all the coefficients are defined at $q = 1$, so we have

$$P_n^{(0,k)}(v^*v; 1) = \sum_{m=0}^n \binom{n}{m} \binom{n+k-1}{m} (v^*v)^m. \quad (4.47)$$

Now, by restricting to the maximal torus of $SU(2)$, one can easily see that for $q = 1$ the above characters coincide with the classical ones.

The following proposition provides a noncommutative analog of the independence of the classical Chern–Weil homomorphism of the choice of connection.

Proposition 4.8. *If C has enough characters, the cyclic-homology Chern–Weil homomorphism $\text{chw}_n(\ell)$ is independent of the choice of a strong connection ℓ .*

Proof. By the results of [11], the Chern–Galois character of a corepresentation V computes the Chern character of a finitely generated projective B -module $A \square^C V$ associated through a given corepresentation V with a principal C -Galois extension $B \subseteq A$. Hence the Chern–Galois character is independent of the choice of the strong connection ℓ . Combining Corollary 4.7 with the assumption that C^{tr} is linearly spanned by the characters of corepresentations, we infer that $\text{chw}_n(\ell)$ is independent of the choice of the strong connection ℓ as claimed. $\square$

The above proposition allows us to skip ℓ in the notation for our cyclic-homology Chern–Weil homomorphism: $\text{chw}_n := \text{chw}_n(\ell)$.

4.6. The case of the family of standard Podleś quantum spheres. Recall first that if $ba = qab$, then

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k}_q a^k b^{n-k}, \quad (4.48)$$

where

$$\binom{n}{k}_q := \frac{(q-1)\dots(q^n-1)}{(q-1)\dots(q^k-1)(q-1)\dots(q^{n-k}-1)} \quad (4.49)$$

are the q -binomial coefficients (e.g., see Section IV.2 in [40]). Let also define q -shifted binomial coefficients by the formula

$$\sum_{i=0}^j \left[\begin{matrix} j \\ i \end{matrix} \right]_q t^i := \prod_{i=0}^{j-1} (1 + q^i t), \quad (4.50)$$

where t is a formal variable.

For C being the cocommutative coalgebra underlying the Hopf algebra $\mathbb{C}[t, t^{-1}]$ corresponding to $U(1)$, we have (see [32])

$$\text{chw}_0(t^n) = \sum_{m=0}^n (vv^*)^m \sum_{k=0}^m \binom{n}{k}_{q^2} q^{-2k(n-k-1)-2m} (-1)^k \begin{bmatrix} n-k \\ m-k \end{bmatrix}_{q^{-2}}, \quad (4.51)$$

$$\text{chw}_0(t^{-n}) = \sum_{m=0}^n (vv^*)^m \sum_{k=0}^m \binom{n}{k}_{q^2} q^{-2k} (-1)^{m-k} \begin{bmatrix} n-k \\ m-k \end{bmatrix}_{q^{-2}}. \quad (4.52)$$

Note that, for $q = 1$ and $m > 0$, the coefficients at $(vv^*)^m$ on the right-hand side of both equations (4.52) and (4.52) are, up to a sign, equal to

$$\begin{aligned} \sum_{k=0}^m \binom{n}{k} (-1)^k \binom{n-k}{m-k} &= \sum_{k=0}^m (-1)^k \frac{n!}{k!(n-k)!} \frac{(n-k)!}{(m-k)!(n-m)!} \\ &= \binom{n}{m} \sum_{k=0}^m (-1)^k \binom{m}{k} \\ &= \binom{n}{m} (1-1)^m \\ &= 0. \end{aligned} \quad (4.53)$$

Hence, for $q = 1$ and all $v \in \mathbb{Z}$, we have

$$\text{chw}_0(t^v) = 1. \quad (4.54)$$

On the other hand, by the index pairing argument [32, Theorem 2.1], all $\text{chw}_0(t^v)$ are pairwise distinct for $q \neq 1$. This means that the family of cyclic-homology Chern–Weil homomorphisms parameterized by q degenerates at a special point $q = 1$. Although it is not so obvious from the algebraic form of coefficients of the right-hand side in the formulas (4.51)–(4.52), the same index pairing argument implies that, in (4.54), the order of the common value at $q = 1$ is equal to one.

Interestingly enough, contrary to the Feng–Tsygan dimension-drop phenomenon, passing from $q = 1$ to the general point of the deformation family of the standard quantum Hopf fibrations, blows up the single value of chw_0 on all t^v to the infinite set of pairwise distinct values.

Acknowledgements The authors are very grateful to Gabriella Böhm for her initial work on this paper, to Kaveh Mousavand for his help with calculational experiments, to Christian Voigt for his help in proving the invariance under row extensions, and to Atabey Kaygun for sharing his insight on Wodzicki’s excision. It is also a pleasure to thank Alexander Gorokhovskiy, Masoud Khalkhali, Ryszard Nest and Hugh Thomas for discussions. This work was partially supported by NCN Grant UMO-2015/19/B/ST1/03098.

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Communicated by Y. Kawahigashi