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A C*-algebraic Approach to Interacting Quantum Field Theories

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Abstract: A novel C*-algebraic framework is presented for relativistic quantum field theories, fixed by a Lagrangean. It combines the postulates of local quantum physics, encoded in the Haag–Kastler axioms, with insights gained in the perturbative approach to quantum field theory. Key ingredients are an appropriate version of Bogolubov's relative S-operators and a reformulation of the Schwinger–Dyson equations. These are used to define for any classical relativistic Lagrangean of a scalar field a non-trivial local net of C*-algebras, encoding the resulting interactions at the quantum level. The construction works in any number of space-time dimensions. It reduces the longstanding existence problem of interacting quantum field theories in physical spacetime to the question of whether the C*-algebras so constructed admit suitable states, such as stable ground and equilibrium states. The method is illustrated on the example of a non-interacting field and it is shown how to pass from it within the algebra to interacting theories by relying on a rigorous local version of the interaction picture.

1. Introduction

Quantum field theory aims to reconcile the principles of quantum physics, governing the microcosmos, with those of relativistic causality, regulating all physical processes. It was conceived immediately after the advent of quantum mechanics as a framework for the quantization of the electromagnetic field. Yet, whereas quantum mechanics quickly matured into a meaningful theory with solid mathematical foundations, the consolidation of quantum field theory took several decades and, as a matter of fact, has not yet come to a fully satisfactory end. In the course of these endeavors it became clear that the framework of quantum field theory reaches far beyond electromagnetism. In fact, it covers all fundamental forces known to date which, with the exception of gravity, are subsumed in the standard model of particle physics.

On the mathematical side there exist two complementary attempts towards mastering the theory. With regard to its computational aspects, one proceeds usually from a specific

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classical Lagrangean, encoding the postulated field content and its interactions. One then "quantizes" the theory, commonly by writing down path integrals or relying on canonical quantization schemes. This informal starting point acquires some precise meaning in the form of calculational rules, ranging from Feynman graphs in renormalized perturbation theory to lattice approximations. It leads to a multitude of theoretical predictions which are in solid agreement with experimental results. Yet the mathematical status of the starting point, *i.e.* the existence of the conceived quantized theory, is not touched upon by these investigations and, most likely, cannot be clarified in this manner.

It is the latter issue which is in the focus of attempts to put quantum field theory on firm mathematical grounds. There one proceeds from the conceptual foundations of the theory, such as its probabilistic interpretation and its causal structure, and casts them into proper mathematical conditions. In this manner one obtains a general framework for quantum field theory, such as the Wightman axioms and their Euclidean ramifications or the Haag–Kastler postulates of algebraic quantum field theory [23,24]. These settings have been the basis for the explanation of distinctive features of particle physics, such as the possible manifestations of particle statistics, the existence of anti-particles and the appearance of internal symmetry groups. Moreover, they form the arena for the rigorous construction of quantum field theoretic models. Yet, disregarding examples in a low dimensional model world or non-interacting theories, these constructive attempts have not yet succeeded in establishing the existence of quantum field theories in real spacetime, which comply with all basic constraints put forward in the general framework [22,32].

It is the aim of the present article to combine these two attempts. Our construction relies on insights gained in a perturbative approach to quantum field theory, which can be traced back to some seminal work of Bogolubov [5,6]. The essential ingredient in this approach are unitary S-operators, which may be regarded as local versions of a scattering matrix. In order to pass from one theory to another one proceeds from them to relative S-operators, depending on the interaction. The latter unitary operators satisfy causality relations which are model independent. Moreover, for given Lagrangean, they allow to describe the effect of local changes of the underlying field by corresponding variations of the action, fixed by the Lagrangean. This feature is closely related to the Schwinger–Dyson equations, which comprise the equations of motion of the theory. In the simple closed form given here, the presentation of these equations seems to be new.

The relative S-operators are constructed in the perturbative approach as formal power series, leaving aside questions of convergence. In lieu thereof we introduce in the present article a unitary group that is generated by abstract S-operators, encoding the above-mentioned causal and dynamical constraints. These unitaries are labelled by local functionals, mapping classical field configurations into real numbers. In order to simplify the discussion, we restrict our attention to d-dimensional Minkowski space which carries a scalar field, being described by smooth, real-valued functions. The functionals which we consider are determined by polynomials formed out of the field and its derivatives, which are integrated with test functions. Let us emphasize that we are not introducing "quantization rules" for the underlying classical theory. The classical theory primarily serves to describe the localization properties of the S-operators and to indicate which particular observable we have in mind, without trying to specify its concrete quantum realization. Thus, in accord with the doctrine of Niels Bohr, we are using "common language" in order to describe observables and operations relating to the quantum world.

Making use of a standard construction method, we shall extend the unitary group so defined to a C*-algebra. This algebra is shown to be the inductive limit of a local net of algebras on Minkowski space which comply with the condition of locality (Einstein causality). Moreover, the spacetime symmetry group, the Poincaré group, acts by automorphisms on this net, in accordance with the Haag–Kastler postulates. Having established the general framework, we will illustrate its usefulness by considering the algebra determined by the Lagrangean of a non-interacting field. It turns out that it contains the Weyl operators of a free field, satisfying the Klein–Gordon equation and having c-number commutation relations. This proves that the algebra is non-trivial and encodes specific dynamical information. We will therefore refer to it as "dynamical algebra". We then discuss the case of interacting theories and show that the corresponding operators are related to those of the non-interacting theory by the adjoint action of S-operators which involve functionals describing the suitably localized interaction. This result justifies within the present setting the interpretation of the S-operators as localized scattering matrices.

The dynamical algebra has all properties which are needed to identify vacuum states or thermal equilibrium states in its dual space. These are commonly taken as characteristics for the stability of the theory. The question of whether such states exist is expected to depend on the form of the Lagrangean entering in the definition of the underlying group and the dimension d of Minkowski space. As a matter of fact, the existence of such states may not always be expected in theories of physical interest, such as in massless theories in low spacetime dimensions; there one has to rely on milder stability conditions. The existence of vacuum and thermal equilibrium states in physical spacetime has been established in interacting theories in the perturbative approach to the S-operators [15–17]. But these encouraging results do not yet settle the problem for the dynamical algebras in the present C*-algebraic setting. An affirmative solution would be a vital step in the consolidation of the mathematical foundations of quantum field theory.

Our article is organized as follows. In the subsequent section we introduce the concepts used in the framework of classical field theory, which enter in our construction. Section 3 contains the definition of the dynamical algebra and the discussion of its general properties. In Sect. 4 we elaborate on the case of non-interacting theories and in Sect. 5 on theories involving interactions. The article concludes with a summary and outlook on generalizations of the present results. In the "Appendix", the dynamical relations used in our approach are derived from the Schwinger–Dyson equations.

2. Classical Field Theory

In order to simplify the discussion, we restrict our attention to fields on Minkowski space; yet the present framwork can be extended to fields on arbitrary Lorentzian manifolds. So let $\mathscr{M}$ be d-dimensional Minkowski space with its standard metric $g(x,x) \doteq x_0^2 - \boldsymbol{x}^2$, where $x_0, \boldsymbol{x}$ denote the time and space components of $x \in \mathbb{R}^d$. The symmetry group of $\mathscr{M}$ is the Poincaré group $\mathscr{P} = \mathbb{R}^d \rtimes \mathscr{L}$, consisting of the semi-direct product of translations and (proper, orthochronous) Lorentz transformations.

We consider a scalar field on $\mathscr{M}$. Its configuration space $\mathscr{E}$ is the real vector space of smooth functions $\phi : \mathscr{M} \to \mathbb{R}$ on which the Poincaré transformations $P \in \mathscr{P}$ act by automorphisms, $P\phi(\,\cdot\,) \doteq \phi(P\,\cdot\,)$. Note that the field is not assumed to satisfy some field equation, it is "off shell" according to standard terminology.

For the sake of simplicity, we restrict our attention to functionals $F : \mathscr{E} \to \mathbb{R}$ of the specific form

$$F[\phi] = \int dx \sum_{n=0}^{N} g_n(x)\,\phi(x)^n\,,$$

where $g_n \in \mathcal{D}(\mathbb{R}^d)$ are arbitrary test functions. If $N > 2$, the sum contains terms describing some self-interaction of the field. The resulting space $\mathscr{F}$ is sufficiently big in order to deal with the Lagrangeans of interest here, cf. below. Moreover, given any field $\phi_0 \in \mathscr{E}$, $\mathscr{F}$ is stable under the shifts $F \mapsto F^{\phi_0}$, defined by $F^{\phi_0}[\phi] \doteq F[\phi + \phi_0]$, $\phi \in \mathscr{E}$. Wheras the functionals F are in general not linear, one easily checks that they satisfy the additivity relation

$$F[\phi_1 + \phi_2 + \phi_3] = F[\phi_1 + \phi_3] - F[\phi_3] + F[\phi_2 + \phi_3]$$

for arbitrary ϕ_3, provided the supports of the fields ϕ_1 and ϕ_2 are disjoint. This feature is a consequence of the locality properties of the functionals.

The support of functionals on $\mathscr{E}$ can be intrinsically defined [7]. For the present family of functionals $F \in \mathscr{F}$, it can be identified with the union of the supports of the underlying test functions g_n for $n \geq 1$. The action of the Poincaré transformations $P \in \mathscr{P}$ on $\mathscr{E}$ can be transferred to the functionals $\mathscr{F}$ by shifting them to the underlying test functions,

$$F_P[\phi] \doteq F[P\phi] = \int dx \sum_{n=0}^{N} g_n(P^{-1}x)\, \phi(x)^n \, .$$

Thus, if F has support in some region $\mathscr{O} \subset \mathscr{M}$, then F_P has support in $P\,\mathscr{O}$.

The Lagrangean densities of the field which we consider here have the customary form

$$x \mapsto L(x)[\phi] = 1/2\,(\partial_\mu \phi(x)\, \partial^\mu \phi(x) - m^2\, \phi(x)^2) - \sum_{n=0}^{N} g_n(x)\, \phi(x)^n \, , \qquad (2.1)$$

where $m \geq 0$ is the mass and $g_n \in \mathbb{R}$ are fixed coupling constants. If $N > 2$, they describe some self-interaction of the field. Other local interaction potentials can be treated in a similar manner. We regard these densities as distributions L on the space of test functions $\mathcal{D}(\mathscr{M})$ with values in functionals, *viz.*

$$L(f)[\phi] \doteq \int dx\, f(x)\, L(x)[\phi] \in \mathbb{R}\, , \quad f \in \mathcal{D}(\mathscr{M})\, , \; \phi \in \mathscr{E} \, .$$

Given a Lagrangean density L, these integrals define localized versions of a corresponding action, which informally corresponds to the constant function $f = 1$. In spite of the fact that we do not have at our disposal the full action, field equations can be derived in the present setting in the sense of distributions by proceeding to relative actions. Denoting the subspace of compactly supported fields by $\mathscr{E}_0 \subset \mathscr{E}$, the family of relative actions fixed by L consists of the maps $\delta L : \mathscr{E}_0 \times \mathscr{E} \to \mathbb{R}$ given by

$$\delta L(\phi_0)[\phi] \doteq L(f_0)^{\phi_0}[\phi] - L(f_0)[\phi] = L(f_0)[\phi + \phi_0] - L(f_0)[\phi]\, , \quad \phi_0 \in \mathscr{E}_0 \, ;$$

the test function f_0 has to be equal to 1 on the support of ϕ_0. Because of the local structure of the Lagrangean density, the relative actions do not depend on the particular choice of f_0 satisfying this condition. Morover, they belong to the space of functionals $\mathscr{F}$, defined above. This is so since the terms involving the kinetic energy of the field cancel each other and terms which are linear in derivatives of the field can be transformed into terms which are linear in the field by partial integration. So the relative actions depend only on powers of the field ϕ.

The derivative of a relative action with regard to ϕ_0 defines the Euler–Lagrange derivative $\epsilon L : \mathscr{E}_0 \times \mathscr{E} \to \mathbb{R}$,

$$\epsilon L(\phi_0)[\phi] \doteq \frac{d}{du}\,\delta L(u\,\phi_0)[\phi]\Big|_{u=0}.$$

A field ϕ is said to satisfy the Euler–Lagrange equation in the sense of distributions, *i.e.* it is "on shell", if $\epsilon L(\phi_0)[\phi] = 0$ for all $\phi_0 \in \mathscr{E}_0$.

3. The Dynamical C*-algebra

We turn now to the construction of the dynamical C*-algebra and the discussion of its general properties. As already mentioned, this algebra has its conceptual roots in the perturbative approach to quantum field theory. We present here the essential elements of our approach; the underlying arguments, motivating the proposed structures, are explained in the "Appendix".

Given a Lagrangean L, we construct in a first step a corresponding group $\mathcal{G}_L$. Its elements are abstract S-operators $S(F)$, which are labelled by functionals $F \in \mathscr{F}$. As already mentioned, the functionals can be shifted by the fields $\phi_0 \in \mathscr{E}_0$, putting $F^{\phi_0}[\phi] \doteq F[\phi + \phi_0]$, $\phi \in \mathscr{E}$. Utilizing the localization properties of the functionals, we say that the support of a functional F_1 is later than that of F_2 if there exists some Cauchy surface C such that the support of F_1 lies above and that of F_2 below that surface relative to the time orientation of $\mathscr{M}$ (Fig. 1). We also recall that $\delta L(\phi_0)$ denotes the relative action for given field $\phi_0 \in \mathscr{E}_0$. With these ingredients, the group $\mathcal{G}_L$ is defined as follows.

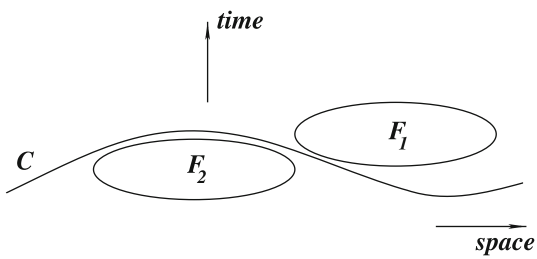

Fig. 1. The support of functional F_1 is later than that of F_2

Definition. Given a Lagrangean L, the corresponding group $\mathcal{G}_L$ is the free group generated by elements $S(F)$, $F \in \mathscr{F}$, modulo the relations

(i) $S(F)\,S(\delta L(\phi_0)) = S(F^{\phi_0} + \delta L(\phi_0)) = S(\delta L(\phi_0))\,S(F)$ for $\phi_0 \in \mathscr{E}_0$, $F \in \mathscr{F}$,

(ii) $S(F_1 + F_2 + F_3) = S(F_1 + F_3)\,S(F_3)^{-1}\,S(F_2 + F_3)$ for any $F_3 \in \mathscr{F}$, provided the support of $F_1 \in \mathscr{F}$ is later than that of $F_2 \in \mathscr{F}$.

Relation (i) describes the dynamics incorporated in $\mathcal{G}_L$. Putting $\phi_0 = 0$, one finds that $S(0) = 1$. The factorization relation (ii) comprises the causal properties of $\mathcal{G}_L$. Putting $S_3 = 0$, one obtains in particular $S(F_1)\,S(F_2) = S(F_1 + F_2)$ if the support of F_1 is later than that of F_2. If the supports of F_1 and F_2 are spacelike separated, this condition implies that the corresponding elements commute since then there exist Cauchy surfaces separating the supports of the functionals in either temporal order, cf. Fig. 2 below.

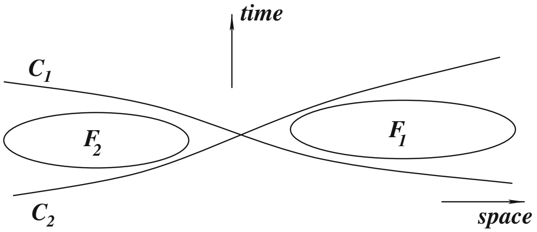

Fig. 2. Functionals F_1 and F_2 with spacelike separated supports

The preceding relations imply that the group $\mathcal{G}_L$ has a center. A basic central subgroup is determined by the constant functionals F_c which, for $c \in \mathbb{R}$, are given by $F_c[\phi] = c$, $\phi \in \mathscr{E}$. Since their support is empty, it follows from the causal factorization property of the S-operators that $S(F + F_c) = S(F)S(F_c) = S(F_c)S(F)$, $F \in \mathscr{F}$. Hence $c \mapsto S(F_c)$ defines a unitary representation of $\mathbb{R}$ in the center of $\mathcal{G}_L$.

Another interesting subgroup in the center of $\mathcal{G}_L$ is related to the dynamics. It is determined by the S-operators fixed by the relative actions, $S(\delta L(\phi_0))$, $\phi_0 \in \mathscr{E}_0$. They lie in the center according to the dynamical equations. To see that they form a group, note that according to these equations one has

$$S(\delta L(\phi_1))\, S(\delta L(\phi_2)) = S(\delta L(\phi_1)^{\phi_2} + \delta L(\phi_2)) = S(\delta L(\phi_1 + \phi_2))\,, \quad \phi_1, \phi_2 \in \mathscr{E}_0\,.$$

These S-operators allow to discriminate off-shell from on-shell fields.

We also note that $\mathcal{G}_L$ is stable under the action of the Poincaré transformations $P \in \mathscr{P}$, inducing the maps $S(F) \mapsto S(F_P)$. In case of relation (ii), this is obvious since the causal order of the supports of functionals remains unaffected by the action of the elements of $\mathscr{P}$, which do not change the time direction. With regard to relation (i), this is a consequence of the equality $F^{\phi_0}{}_P[\phi] = F[P\phi + \phi_0] = F_P^{P^{-1}\phi_0}[\phi]$ and the fact that the Lagrangean density transforms as a scalar under Lorentz transformations. It implies that $\delta L(\phi_0)_P[\phi] = \delta L(\phi_0)[P\phi] = \delta L(P^{-1}\phi_0)[\phi]$, $\phi \in \mathscr{E}$.

As is common practice, we use units, where Planck's constant has the value 1. Since it is of interest to study the effect of hypothetical changes of this fundamental constant, we also consider the groups, where this constant is scaled by some factor $h > 0$. It amounts to proceeding to the scaled Lagrangean $L_h \doteq h^{-1}L$ and scaled S-operators given by $S_h(F) \doteq S(h\,F)$, $F \in \mathscr{F}$. This scaling neither affects the localization properties nor the Poincaré covariance of the functionals.

We proceed now from $\mathcal{G}_L$ to the corresponding group algebra $\mathcal{A}_L$ over $\mathbb{C}$. This is a known procedure, which we briefly recall here. The algebra $\mathcal{A}_L$ is by definition the complex linear span of the elements $S \in \mathcal{G}_L$. We also fix the central group elements $S(F_c)$, corresponding to the constant functionals, putting $S(F_c) = e^{ic}\,1$, $c \in \mathbb{R}$. The adjoint operators are defined by $(\sum c\,S)^* \doteq \sum \overline{c}\,S^{-1}$ and the multiplication in $\mathcal{A}_L$ is inherited from $\mathcal{G}_L$ by the distributive law.

On $\mathcal{A}_L$ there exists a functional ω, which is obtained by linear extension from the defining equalities $\omega(S) = 0$ for $S \in \mathcal{G}_L \backslash \{1\}$ and $\omega(1) = 1$, cf. [11]. Thus, for any choice of a finite number of different elements $S_i \in \mathcal{G}_L$, $i = 1, \ldots n$, one has

$$\omega\Big(\big(\sum_{i=1}^{n} c_i S_i\big)^* \big(\sum_{j=1}^{n} c_j S_j\big)\Big) = \sum_{i,j=1}^{n} \overline{c}_i c_j\, \omega(S_i^{-1} S_j) = \sum_{i=1}^{n} |c_i|^2 \geq 0\,.$$

So, disregarding the zero element, the functional ω has positive values on positive operators in $\mathcal{A}_L$, *i.e.* it is a faithful state. Whence, proceeding to the corresponding GNS-representation, the operator norm of the elements of $\mathcal{A}_L$ in that representation defines a C*-norm on $\mathcal{A}_L$. We denote by $\| \cdot \|$ the supremum of all C*-norms, obtained in this manner by states on $\mathcal{A}_L$. (Note that this supremum exists since each element of $\mathcal{A}_L$ is a finite sum of unitary operators.) Completing $\mathcal{A}_L$ in this norm topology, we obtain a C*-algebra, which we denote by the same symbol.

Definition. Given a Lagrangean L, the dynamical algebra $\mathcal{A}_L$ is the C*-algebra determined by the group $\mathcal{G}_L$, as outlined above.

3.1. Haag–Kastler postulates. The dynamical algebra $\mathcal{A}_L$ complies with the Haag–Kastler postulates of local quantum field theory [24] for any choice of Lagrangean L. In order to verify this assertion, we first need to specify local subalgebras $\mathcal{A}_L(\mathscr{O})$ for each bounded, causally closed spacetime region $\mathscr{O} \subset \mathscr{M}$. This is accomplished by making use of the support properties of the underlying functionals. We denote by $\mathscr{F}(\mathscr{O}) \subset \mathscr{F}$ the subspace of all functionals having support in $\mathscr{O}$. It determines a corresponing subgroup $\mathcal{G}_L(\mathscr{O}) \subset \mathcal{G}_L$, generated by all $S(F)$ with $F \in \mathscr{F}(\mathscr{O})$. From there one proceeds to the norm-closed subalgebra

$$\mathcal{A}_L(\mathscr{O}) \doteq \Big\{ \sum c\, S : c \in \mathbb{C},\ S \in \mathcal{G}_L(\mathscr{O}) \Big\}^{\| \cdot \|} \subset \mathcal{A}_L\,, \quad \mathscr{O} \subset \mathscr{M}\,.$$

By construction, $\mathcal{A}_L(\mathscr{O}_1) \subset \mathcal{A}_L(\mathscr{O}_2)$ if $\mathscr{O}_1 \subset \mathscr{O}_2$, *i.e.* the assignment $\mathscr{O} \mapsto \mathcal{A}_L(\mathscr{O})$ satisfies the condition of isotony and thus defines a net of C*-algebras on $\mathscr{M}$. Since all functionals $F \in \mathscr{F}$ underlying the construction of $\mathcal{A}_L$ have compact supports, the C*-inductive limit of this net coincides with $\mathcal{A}_L$.

Next, if $\mathscr{O}_1$ and $\mathscr{O}_2$ are spacelike separated regions, then the elements of $\mathcal{G}_L(\mathscr{O}_1)$ commute with those of $\mathcal{G}_L(\mathscr{O}_2)$, as was explained above. This feature is passed on to the corresponding algebras $\mathcal{A}_L(\mathscr{O}_1)$ and $\mathcal{A}_L(\mathscr{O}_2)$, whose elements also commute with each other. Thus the net $\mathscr{O} \mapsto \mathcal{A}_L(\mathscr{O})$ satisfies the condition of locality (Einstein causality).

As was also explained, each Poincaré transformation $P \in \mathscr{P}$ determines an automomorphism $\alpha_P : \mathcal{G}_L \to \mathcal{G}_L$, fixed by the maps $S(F) \mapsto S(F_P)$, $F \in \mathscr{F}$. It straightforwardly extends to the linear span of the elements of $\mathcal{G}_L$. Now, $\| \cdot \|_P \doteq \| \alpha_P(\,\cdot\,) \|$ defines a C*-norm on this space. Since $\| \cdot \|$ is, by definition, the (unique) supremum of its C*-norms, it implies that $\| \alpha_P(\,\cdot\,) \| = \| \cdot \|$. Thus α_P extends as an automorphism to the C*-algebra $\mathcal{A}_L$. Moreover, the automorphisms α_P, $P \in \mathscr{P}$, act covariantly on the local algebras. This is a consequence of the fact that if $F \in \mathscr{F}$ has support in $\mathscr{O}$, then F_P has support in $P\mathscr{O}$; it entails

$$\alpha_P\big(\mathcal{A}_L(\mathscr{O})\big) = \mathcal{A}_L(P\mathscr{O})\,, \quad P \in \mathscr{P}\,,\ \mathscr{O} \subset \mathscr{M}\,.$$

Thus the local net $\mathscr{O} \mapsto \mathcal{A}_L(\mathscr{O})$ has the fundamental properties postulated by Haag and Kastler for any physically meaningful quantum field theory on Minkowski space. In addition, these authors require that the global algebra generated by a net should be primitive, *i.e.* have some faithful irreducible representation. This condition is motivated by their principle of physical equivalence according to which the states in any faithful representation should be weakly dense in the state space of any other representation. The algebra $\mathcal{A}_L$, however, does not have this property since it has a non-trivial center (containing for example the operators corresponding to the relative actions). This problem is solved by picking some irreducible representation of $\mathcal{A}_L$ and taking the quotient with regard to its kernel, being a primitive ideal of the algebra. In this way one abtains a primitive algebra, where as to yet unspecified physical data of the underlying quantum theory, such as the field equation, the specific values of coupling constants and the mass are fixed. If the kernel of the representation is stable under the automorphic action of the Poincaré transformations, this quotient still defines a net with the preceding desirable properties. So there arises the question of determining such representations of physical interest. This is discussed in the subsequent subsection.

3.2. States of interest. Given a Lagrangean L, all possible states of $\mathcal{A}_L$ appear as elements of its dual space and determine corresponding representations by the GNS-construction. Pure states give rise to irreducible representations. In view of its manifold applications, the theory ought to describe states with a definite physical interpretation. These are primarily stable elementary systems and their excitations. On the other hand, the theory should reproduce quantitative results, obtained in the perturbative treatment of quantum field theory. As a matter of fact, these two issues are related.

In order to exhibit this relation, let us consider the Epstein–Glaser method of renormalized perturbation theory [16]; it is based on power series expansions of correlation functions in terms of the scaled Planck constant. There one succeeds in constructing formal states on the linear span of operators, generating $\mathcal{A}_L$, *i.e.* linear functionals which take values in the space of formal power series in h,

$$\omega[h] = \sum h^k \omega_k \, .$$

The functionals $\omega[h]$ satisfy in the sense of formal power series the positivity condition

$$\omega[h](A^*A) = |\sum a_n \, h^n|^2 \, ,$$

expressing the fact that it is a series with real coefficients whose lowest non-vanishing term is positive and of even order in h. Their construction relies on local stability properties of states on the sub-algebra of bounded functions generated by smeared non-interacting fields, cf. the subsequent sections.

In general, one may not expect that these series converge. But in view of the empirical success of perturbative quantum field theory, one may hope that there exist states ω on $\mathcal{A}_L$ which determine corresponding formal states. In more detail, let

$$\omega_h(S(F_1)^{\sigma_1} \dots S(F_n)^{\sigma_n}) \doteq \omega(S(hF_1)^{\sigma_1} \dots S(hF_n)^{\sigma_n})$$

with $S(F_1), \dots, S(F_n) \in \mathcal{A}_L$ and $\sigma_i \in \{\pm 1\}$, $i = 1, \dots, n$. Then $\omega[h]$ should ideally describe the Taylor series at $h = 0$ corresponding to the function $h \mapsto \omega_h$. At present it is not known which precise conditions a state ω on $\mathcal{A}_L$ must satisfy in order to combine the desired features. So we have to remain somewhat sketchy at this point. Based on insights gained in perturbation theory and basic properties of the algebras $\mathcal{A}_L$, we will indicate some promising conditions and call pure states satisfying any one of them "principal states".

The most prominent examples of principal states are vacuum states. They can be identified as follows.

Definition. A pure state ω_0 on $\mathcal{A}_L$ is said to be a vacuum state if (i) $\omega_0 \circ \alpha_P = \omega_0$ and $P \mapsto \omega_0(A_1 \alpha_P(A_2))$, $P \in \mathscr{P}$, is continuous for all $A_1, A_2 \in \mathcal{A}_L$; (ii) the Fourier transforms (in the sense of distributions) of $x \mapsto \omega_0(A_1 \alpha_x(A_2))$, $x \in \mathbb{R}^d$, have support in the forward lightcone V_+.

It is a basic result in algebraic quantum field theory [2, Sect. 4.2] that these conditions imply that (i) the Poincaré transformations are unitarily implemented in the GNS representation π_0 induced by ω_0, (ii) the generators of the space-time translations (energy and momentum) have joint spectrum in V_+, and (iii) ω_0 is their ground state. Thus the kernel of π_0 is Poincaré invariant, so the net $\mathscr{O} \mapsto \mathcal{A}_L(\mathscr{O})/\ker \pi_0$ complies with the Haag–Kastler postulates and $\mathcal{A}_L/\ker \pi_0$ is a primitive C*-algebra.

As already mentioned, there exist theories of interest, such as the free massless field in $d = 2$ dimensions, where such vacuum states do not exist. But one can relax the condition of Poincaré invariance of principal states and also drop the assumption that the full Poincaré group is unitarily represented in the corresponding representations. In order to establish the required stability, it would suffice to exhibit principal states, where only the space-time translations are unitarily implemented in the corresponding representations, having generators with spectral properties as stated above. Or one may even be content with principal states satisfying a microlocal version of the spectral condition [9], which does not require the existence of generators.

The condition that the kernels of the resulting irreducible representations π are stable under Poincaré transformations seems, however, to be inevitable on physical grounds. For, otherwise, the Poincaré group would not act on the resulting algebras $\mathcal{A}_L/\ker \pi$; it would be truly (not only spontaneously) broken. The condition is satisfied by a principal state ω if all Poincaré transformed states $\omega \circ \alpha_P$ are locally normal with respect to each other, *i.e.* if the restrictions of the resulting representations π_P to any given local algebra $\mathcal{A}_L(\mathscr{O})$ are quasi-equivalent, $P \in \mathscr{P}$, cf. [23, Def. III.2.2.15]. In the non-interacting case, these conditions are satisfied by so-called infra-vacuum states.

Having chosen a principal state ω, one can determine the equation of motion of the underlying field. It is encoded in the operators $S(\delta L(\phi_0))$, depending on the relative actions, which form a unitary group in the center of $\mathcal{A}_L$. Since the representation π, fixed by ω, is irreducible, they are represented by phases, $\pi(S(\delta L(\phi_0)) \in \mathbb{T}\,1$. If the functions $u \mapsto \pi(S(\delta L(u\phi_0)), u \in \mathbb{R}$, are continuous one can proceed to their derivatives. In the absence of external sources, one then obtains the quantum analogue of the classical Euler–Lagrange equation,

$$\frac{d}{du}\,\pi(S(\delta L(u\phi_0))\Big|_{u=0} = 0\,, \quad \phi_0 \in \mathscr{E}_0\,.$$

It expresses the fact that the underlying quantum field corresponds to a saddle point of the action. In cases, where the field couples to an external (classical) source, this source manifests itself on the right hand side of this equality in the form of non-vanishing c-number contributions.

Given a Lagrangean L, it is, however, not clear whether the corresponding C*-algebra $\mathcal{A}_L$ has any principal state in its dual space. This issue in the representation theory of C*-algebras is, from the present point of view, the remaining fundamental problem of constructive quantum field theory. Thinking for example of Lagrangeans with the common interaction potential ϕ^4, one expects on the basis of previous constructive results that principal states (even vacua) can be found in $d = 2$ and $d = 3$ dimensions [22]. Yet there are also indications that in physical spacetime $d = 4$ and in higher dimensions such states do not exist [1,20]. A proof of the presence or absence of physically acceptable principal states within our algebraic setting would settle this matter, independently of any particular constructive scheme.

4. Non-interacting Theories

As a first application of our approach, we discuss the case of non-interacting scalar quantum fields in d dimensions with masses $m \geq 0$. The corresponding classical Lagrangeans are given by

$$x \mapsto L_0(x)[\phi] \doteq 1/2\,(\partial_\mu \phi(x)\,\partial^\mu \phi(x) - m^2\,\phi(x)^2)\,, \quad \phi \in \mathscr{E}\,.$$

It is our goal to determine the algebraic properties of the quantum fields in the corresponding algebras $\mathcal{A}_{L_0}$.

We consider functionals containing the sum of a linear term involving the underlying field and a constant functional. Let $K \doteq -(\Box + m^2)$ be the Klein–Gordon operator, fixed by the Lagrangean L_0, and let Δ_R, Δ_A be the corresponding retarded and advanced propagators; their mean is the Dirac propagator $\Delta_D \doteq 1/2\,(\Delta_A + \Delta_R)$. These propagators define maps of the test function space $\mathcal{D}(\mathscr{M})$ into its dual space of distributions. Making use of standard notation, the functionals have the form

$$F_f[\phi] \doteq \phi(f) + 1/2\langle f, \Delta_D\, f\rangle\,, \quad f \in \mathcal{D}(\mathscr{M})\,,\ \phi \in \mathscr{E}\,.$$

Picking any field $\phi_0 \in \mathscr{E}_0 \simeq \mathcal{D}(\mathbb{R}^d)$, one obtains for test functions of the special form $f = K\phi_0$

$$F_f[\phi] = \phi(K\phi_0) + 1/2\langle K\phi_0,\ \Delta_D\, K\phi_0\,\rangle = \phi(K\phi_0) + 1/2\langle \phi_0, K\phi_0\,\rangle = \delta L_0(\phi_0)[\phi]\,.$$

Thus for these special test functions the functionals coincide with the relative actions.

We proceed now to the corresponding unitary operators $W(f) \doteq S(F_f)$ for arbitrary test functions $f \in \mathcal{D}(\mathscr{M})$. As we shall see, these operators have the algebraic properties of exponentials of a free field (Weyl operators). Given f, let $f = f_0 + K\phi_0$ be any decomposition with $f_0, \phi_0 \in \mathcal{D}(\mathbb{R}^d)$. Then

$$\begin{aligned} F_f[\phi] &= \phi(f_0 + K\phi_0) + 1/2\langle (f_0 + K\phi_0),\, \Delta_D\,(f_0 + K\phi_0)\rangle \\ &= \big((\phi + \phi_0)(f_0) + 1/2\langle f_0, \Delta_D\, f_0\rangle\big) + \big(\phi(K\phi_0) + 1/2\langle \phi_0, K\phi_0\,\rangle\big) \\ &= F_{f_0}^{\phi_0}[\phi] + \delta L_0(\phi_0)[\phi]\,. \end{aligned}$$

Thus, making use of the dynamical relations between S-operators, we obtain

$$W(f) = S(F_{f_0}^{\phi_0} + \delta L_0(\phi_0)) = S(F_{f_0})\,S(\delta L_0(\phi_0)) = W(f_0)W(K\phi_0)\,.$$

Given a second test function $g \in \mathcal{D}(\mathscr{M})$, we choose a decomposition of f such that the support of f_0 is later than the support of g. That such a decomposition exists is a known fact; we briefly recall the argument [21]. Let C be a Cauchy surface lying in the future of the support of g and let χ be a smooth function which is equal to 1 in the future of C and tends to 0 in its past at sufficiently small distance. Making use of the equality $K\,\Delta_R f = f$, we define $f_0 \doteq K\chi\Delta_R f$ and $\phi_0 \doteq (1-\chi)\Delta_R f$. Due to the support properties of Δ_R, both expressions are test functions, $f = f_0 + K\phi_0$, and f_0 has its support in the future of g, cf. Fig. 3.

We exploit now these support properties of f_0, g. Since $S(F_{K\phi_0}) = S(\delta L_0(\phi_0))$, the defining properties of the S-operators imply

$$\begin{aligned} W(f)W(g) &= W(f_0)W(K\phi_0)W(g) = S(F_{f_0})S(F_{K\phi_0})S(F_g) \\ &= S(F_{f_0} + F_g)S(\delta L_0(\phi_0)) = S(F_{f_0}^{\phi_0} + F_g^{\phi_0} + \delta L_0(\phi_0))\,. \end{aligned}$$

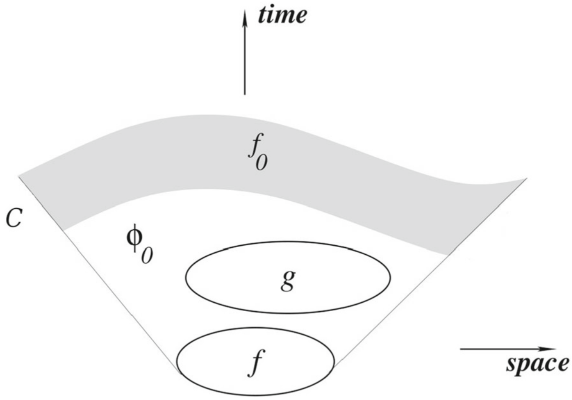

Fig. 3. Split $f = f_0 + K\phi_0$: f_0 has support in C and ϕ_0 in the entire region

Now, for $\phi \in \mathscr{E}$,

$$\begin{aligned} & F^{\phi_0}_{f_0}[\phi] + F^{\phi_0}_{g}[\phi] + \delta L_0(\phi_0)[\phi] \\ & \quad = (\phi{+}\phi_0)(f_0{+}g){+}1/2\langle f_0, \Delta_D f_0\rangle + 1/2\langle g, \Delta_D g\rangle + \phi(K\phi_0) - 1/2\langle K\phi_0, \Delta_D K\phi_0\rangle \\ & \quad = \phi(f+g) + 1/2\langle (f+g), \Delta_D(f+g)\rangle + \langle f_0, \Delta_D g\rangle\,. \end{aligned}$$

In view of the support properties of Δ_A, we have supp $f_0 \bigcap$ supp $\Delta_A g = \emptyset$. Hence $\langle f_0, \Delta_D g\rangle = 1/2\,\langle f_0, \Delta_R g\rangle = 1/2\,\langle f_0, \Delta g\rangle$, where $\Delta = \Delta_R - \Delta_A$ is the commutator function, which is a bi-solution of the Klein–Gordon equation. Thus $\langle K\phi_0, \Delta g\rangle = 0$, which altogether gives $\langle f_0, \Delta_D g\rangle = 1/2\,\langle f, \Delta g\rangle$. Since $1/2\,\langle f, \Delta g\rangle$ is independent of $\phi \in \mathscr{E}$, it defines the constant functional $F_{1/2\,\langle f,\Delta g\rangle}$ on $\mathscr{E}$, hence

$$F^{\phi_0}_{f_0}[\phi] + F^{\phi_0}_{g}[\phi] + \delta L_0(\phi_0)[\phi] = F_{f+g} + F_{1/2\,\langle f,\Delta g\rangle}\,.$$

Plugging this relation into the preceding equality of S-operators, we arrive at

$$W(f)W(g) = W(f+g)\,e^{-i/2\,\langle f,\Delta g\rangle}\,, \quad f, g \in \mathcal{D}(\mathbb{R}^d)\,.$$

These are the Weyl relations of a free field; since $\langle f, \Delta f\rangle = 0$, they imply in particular that $W(f)^{-1} = W(-f)$.

By similar arguments one can also compute the product of Weyl operators with arbitrary S-operators $S(F)$, $F \in \mathscr{F}$, cf. also the discussion in [28, Sect. 4]. There one obtains the equalities

$$W(f)S(F) = S(F_f + F^{\Delta_R f})\,, \quad S(F)W(f) = S(F_f + F^{\Delta_A f})\,.$$

They imply that the Weyl operators induce specific automorphisms of the S-operators,

$$W(f)S(F)W(f)^{-1} = S(F^{\Delta f})\,, \quad f \in \mathscr{D}(\mathbb{R}^d)\,.$$

The Weyl operators form an infinite dimensional non-commutative subalgebra (Weyl algebra) of $\mathcal{A}_{L_0}$. In view of the preceding relations, involving also functionals F depending on higher powers of the field ϕ, it is apparent that the dynamical algebra $\mathcal{A}_{L_0}$ has an even more complex structure. As a matter of fact, one can establish a natural correspondence between local algebras in the interacting theories and subalgebras of $\mathcal{A}_{L_0}$. This fact will be explained in the subsequent section.

On the Weyl algebra there exist pure vacuum states. (In case of the massless free field in $d = 2$ dimensions, there exist other principal states.) Moreover, in the corresponding GNS-representations, a perturbative expansion of all S_h-operators into formal power series in h has been established. It is not known, however, whether the full (unitary) S_h-operators can also be accommodated in these representations. This issue is reminiscent of non-commutative moment problems, where an affirmative solution often requires an extension of the given Hilbert space. Since $\mathcal{A}_{L_0}$ is a C*-algebra, the pure states on its Weyl subalgebra can be extended to pure states on the full algebra by the Hahn-Banach theorem. So there remains the question of whether there exist extensions which are still principal states.

5. Interacting Theories

Given a Lagrangean L_0 (which may differ from the Lagrangean of a free field), we study now the effect of perturbations of the dynamics, obtained by changes of the mass, the coupling constants, and the degree N of the polynomial appearing in the interaction potential; the kinetic energy remains unaffected. The perturbed Lagrangean is denoted by $L_V \doteq L_0 + V$, where the perturbation V has the form

$$x \mapsto V(x)[\phi] \doteq \sum_{n=0}^{N} \Delta g_n \, \phi^n(x) \,, \quad \phi \in \mathscr{E} \,,$$

with fixed variations $\Delta g_n \in \mathbb{R}$ of the coupling constants. Thus, by our constructive scheme, we are dealing now with two algebras, $\mathcal{A}_{L_0}$ and $\mathcal{A}_{L_V}$; the corresponding S-operators will be denoted by S_0 and S_V, respectively.

We want to show that the effect of the perturbations on the local algebraic properties of $\mathcal{A}_{L_V}$ can be fully described within the unperturbed algebra $\mathcal{A}_{L_0}$ [8,29]. More precisely, given any bounded, causally closed region $\mathscr{O} \subset \mathscr{M}$ and any larger region $\widehat{\mathscr{O}}$, containing the closure of $\mathscr{O}$ in its interior, we will exhibit some subalgebra $\mathcal{A}_{L_0+V(\chi)}(\mathscr{O}) \subset \mathcal{A}_{L_0}(\widehat{\mathscr{O}})$ which is isomorphic to $\mathcal{A}_{L_V}(\mathscr{O})$. These subalgebras are not unique, but different choices are related by inner automorphisms of $\mathcal{A}_{L_0}(\widehat{\mathscr{O}})$. Because of the latter fact, there exists a homomorphic picture of the net $\mathscr{O} \mapsto \mathcal{A}_{L_V}(\mathscr{O})$ within the algebra $\mathcal{A}_{L_0}$ for any choice of interaction potential V.

Turning to the proof of these assertions, we choose for given pair $\mathscr{O} \subset \widehat{\mathscr{O}}$ some smooth characteristic function χ of $\mathscr{O}$ which has support in $\widehat{\mathscr{O}}$ and is equal to 1 in an open neighbourhood of the closure of $\mathscr{O}$. Integrating V with this test function, we obtain a functional $V(\chi) \in \mathscr{F}(\widehat{\mathscr{O}})$, describing a perturbation which is localized in $\widehat{\mathscr{O}}$. We also put $\delta L_{V(\chi)}(\phi_0) \doteq \delta L_0(\phi_0) + V(\chi)^{\phi_0} - V(\chi)$; in view of the properties of χ, this functional coincides with $\delta L_V(\phi_0)$ for fields $\phi_0 \in \mathscr{E}_0$ having support in $\mathscr{O}$. Adopting basic ideas of Bogolubov on the incorporation of interaction in quantum field theory, we define operators (corresponding to the relative S-operators in the perturbative setting)

$$\mathcal{B}_{V(\chi)}(F) \doteq S_0(V(\chi))^{-1} S_0(F + V(\chi)) \,, \quad F \in \mathscr{F} \,.$$

In view of the dynamical relations in $\mathcal{A}_{L_0}$, they satisfy the equalities

$$\begin{aligned}
&\mathcal{B}_{V(\chi)}(F)\,\mathcal{B}_{V(\chi)}(\delta L_{V(\chi)}(\phi_0)) \\
&\quad = S_0(V(\chi))^{-1} S_0(F + V(\chi))\; S_0(V(\chi))^{-1} S_0(\delta L_0(\phi_0) + V(\chi)^{\phi_0}) \\
&\quad = S_0(V(\chi))^{-1} S_0(F+V(\chi))\; S_0(\delta L_0(\phi_0)) = S_0(V(\chi))^{-1} S_0(F^{\phi_0} + V(\chi)^{\phi_0} + \delta L_0(\phi_0)) \\
&\quad = \mathcal{B}_{V(\chi)}(F^{\phi_0} + \delta L_{V(\chi)}(\phi_0))\,.
\end{aligned}$$

Moreover, for any functional F_1 having support in the future of F_2 and any $F_3 \in \mathscr{F}$, the causal relations in $\mathcal{A}_{L_0}$ imply

$$\begin{aligned}
&\mathcal{B}_{V(\chi)}(F_1 + F_3)\,\mathcal{B}_{V(\chi)}(F_3)^{-1}\mathcal{B}_{V(\chi)}(F_2 + F_3) \\
&\quad = S_0(V(\chi))^{-1} S_0(F_1 + (F_3 + V(\chi)))\; S_0(F_3 + V(\chi))^{-1}\; S_0(F_2 + (F_3 + V(\chi))) \\
&\quad = S_0(V(\chi))^{-1} S_0(F_1 + F_2 + F_3 + V(\chi)) \;=\; \mathcal{B}_{V(\chi)}(F_1 + F_2 + F_3)\,.
\end{aligned}$$

We restrict now the maps $\mathcal{B}_{V(\chi)} : \mathscr{F} \to \mathcal{A}_{L_0}$ to functionals F having support in the region $\mathscr{O}$. Then $\delta L_{V(\chi)}(\phi_0) = \delta L_V(\phi_0)$ and the preceding two equations for the unitaries $\mathcal{B}_{V(\chi)}(F)$ coincide with the defining relations of the perturbed S-operators $S_V(F)$, $F \in \mathscr{F}(\mathscr{O})$. Let $\mathcal{G}_{L_0,\chi}(\mathscr{O}) \subset \mathcal{G}_{L_0}(\widehat{\mathscr{O}})$ be the group, which is generated by $\mathcal{B}_{V(\chi)}(F)$, $F \in \mathscr{F}(\mathscr{O})$, and let $\mathcal{A}_{L_0+V(\chi)}(\mathscr{O}) \subset \mathcal{A}_{L_0}(\widehat{\mathscr{O}})$ be the corresponding C*-algebra. Identifying the operators $\mathcal{B}_{V(\chi)}(F)$ with $S_V(F)$, $F \in \mathscr{F}(\mathscr{O})$, establishes an isomorphism $\beta_{\mathscr{O},\chi}$ between the groups $\mathcal{G}_{L_0,\chi}(\mathscr{O})$ and $\mathcal{G}_{L_V}(\mathscr{O})$. This isomorphism extends to congruent linear combinations of the group elements, which form norm dense subalgebras of $\mathcal{A}_{L_0+V(\chi)}(\mathscr{O})$ and $\mathcal{A}_{L_V}(\mathscr{O})$, respectively. Denoting by $\|\cdot\|_0$ and $\|\cdot\|_V$ their original C*-norms, one obtains new C*-norms on these subalgebras, defined by $\|\beta_{\mathscr{O},\chi}(\,\cdot\,)\|_V$ and $\|\beta_{\mathscr{O},\chi}^{-1}(\,\cdot\,)\|_0$, respectively. In view of the maximality of the original C*-norms, one has on the respective subalgebras

$$\|\beta_{\mathscr{O},\chi}(\,\cdot\,)\|_V \le \|\cdot\|_0\,, \quad \|\beta_{\mathscr{O},\chi}^{-1}(\,\cdot\,)\|_0 \le \|\cdot\|_V\,.$$

It implies $\|\beta_{\mathscr{O},\chi}(\,\cdot\,)\|_V = \|\cdot\|_0$ and $\|\beta_{\mathscr{O},\chi}^{-1}(\,\cdot\,)\|_0 = \|\cdot\|_V$, so the isomorphism $\beta_{\mathscr{O},\chi}$ extends to the full C*-algebras. Moreover, the interpretation of the operators, based on the underlying classical functionals, does not change under its action. This establishes our first assertion, saying that the perturbed theory can locally be described in terms of the unperturbed one. Interchanging the role of L_0 and L_V, the converse is also true.

Before we enter into the discussion of the dependence of our construction on the choice of χ, let us briefly comment on its physical interpretation. To this end we return to the operators $\mathcal{B}_{V(\chi)}(F)$ for arbitrary $F \in \mathscr{F}$. If F has its support in the past of $V(\chi)$, it follows from the causal relations that

$$\mathcal{B}_{V(\chi)}(F) = S_0(V(\chi))^{-1} S_0(F + V(\chi)) = S_0(F)\,.$$

Similarly, if F has its support in the future of $V(\chi)$, one gets

$$\mathcal{B}_{V(\chi)}(F) = S_0(V(\chi))^{-1} S_0(F + V(\chi)) = S_0(V(\chi))^{-1} S_0(F)\, S_0(V(\chi))\,.$$

Thus the map $\mathcal{B}_{V(\chi)}$ describes a perturbation of the original theory in the region $\widehat{\mathscr{O}}$, leaving it unaffected in the past and spacelike complement of $\widehat{\mathscr{O}}$. Due to the interaction, the unperturbed theory is affected, however, in $\widehat{\mathscr{O}}$ and its causal future. Alluding

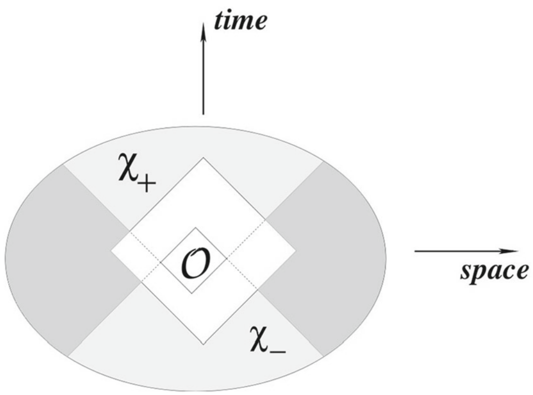

Fig. 4. The supports of $\chi_+,\ \chi_-$ are empty in the white and overlapping in the dark regions

to a dynamical picture, one may think of the unperturbed system as coming in from large negative times and, eventually, reaching the localized potential in $\widehat{\mathscr{O}}$. The overall effects of this perturbation become visible in the future of $\widehat{\mathscr{O}}$ and can be reinterpreted in terms of the original (similarity transformed) theory. Thus the similarity transformation $\mathrm{Ad}\, S(V(\chi))^{-1}$ has the meaning of a scattering automorphism.

Depending on the choice of χ, the inclusion $\mathcal{A}_{L_0+V(\chi)}(\mathscr{O}) \subset \mathcal{A}_{L_0}(\widehat{\mathscr{O}})$ holds for regions $\widehat{\mathscr{O}}$ which are arbitrarily close to the given region $\mathscr{O}$. And for any such choice of χ, the algebra $\mathcal{A}_{L_0+V(\chi)}(\mathscr{O})$ is isomorphic to $\mathcal{A}_{L_V}(\mathscr{O})$. But this does *not* imply, that the Haag–Kastler nets $\mathscr{O} \mapsto \mathcal{A}_{L_0}(\mathscr{O})$ and $\mathscr{O} \mapsto \mathcal{A}_{L_V}(\mathscr{O})$ are isomorphic. That is, in general there does not exist a global isomorphism between $\mathcal{A}_{L_0}$ and $\mathcal{A}_{L_V}$, which also maps the local subalgebras onto each other for all regions $\mathscr{O} \subset \mathscr{M}$. At best, one can hope that algebras corresponding to causally closed regions, having their base in a common Cauchy surface, can be identified in the two theories. This identification of local algebras does not persist, however, at other times due to the differing interactions. Nevertheless, the perturbed net $\mathscr{O} \mapsto \mathcal{A}_{L_V}(\mathscr{O})$ can be accommodated in the unperturbed global algebra $\mathcal{A}_{L_0}$, as we shall show next.

Given a region $\mathscr{O} \subset \mathscr{M}$, we pick any two test functions $\chi_1, \chi_2 \in \mathscr{D}(\mathbb{R}^d)$, which, both, are equal to 1 on $\mathscr{O}$ and have support in a region $\widehat{\mathscr{O}} \supset \mathscr{O}$. In a first step we show that the two subalgebras $\mathcal{A}_{L_0+V(\chi_1)}(\mathscr{O}),\ \mathcal{A}_{L_0+V(\chi_2)}(\mathscr{O}) \subset \mathcal{A}_{L_0}(\widehat{\mathscr{O}})$ are related by an inner automorphism of $\mathcal{A}_{L_0}(\widehat{\mathscr{O}})$. To this end we choose a decomposition of the difference $\chi_2 - \chi_1 = \chi_+ + \chi_-$, such that the support of χ_+ does not intersect the past of $\mathscr{O}$ and that of χ_- its future. (This is possible since the regions considered here are causally closed, cf. Fig. 4).

Picking any $F \in \mathscr{F}$ which has support in $\mathscr{O}$ and making use of the fact that χ_+ has its support in the future of $\mathscr{O}$, we get from the causal relation for S-operators

$$S_0(V(\chi_2) + F) = S_0(V(\chi_+) + V(\chi_1 + \chi_-))\, S_0(V(\chi_1 + \chi_-))^{-1}\, S_0(V(\chi_1 + \chi_-) + F)\,.$$

Similarly, since χ_- has its support in the past of $\mathscr{O}$, we obtain

$$S_0(V(\chi_1 + \chi_-) + F)) = S_0(F + V(\chi_1))\, S_0(V(\chi_1))^{-1}\, S_0(V(\chi_1 + \chi_-))\,.$$

Combining these relations and inserting the result into the corresponding Bogolubov operators gives

$$\mathcal{B}_{V(\chi_2)}(F) = \mathcal{B}_{V(\chi_1)}(V(\chi_-))^{-1} \mathcal{B}_{V(\chi_1)}(F)\, \mathcal{B}_{V(\chi_1)}(V(\chi_-)) \in \mathcal{A}_{L_0}(\widehat{\mathscr{O}})\,.$$

It implies that the two isomorphisms $\beta^{-1}_{\mathscr{O},\chi_1}$, $\beta^{-1}_{\mathscr{O},\chi_2}$, mapping $\mathcal{A}_{L_V}(\mathscr{O})$ into corresponding subalgebras of $\mathcal{A}_{L_0}$, are related by

$$\mathrm{Ad}\left(\mathcal{B}_{V(\chi_1)}(V(\chi_-))^{-1}\right)\circ \beta^{-1}_{\mathscr{O},\chi_1} = \beta^{-1}_{\mathscr{O},\chi_2} .$$

It shows that, disregarding inner automorphisms, the embeddings of the algebra $\mathcal{A}_{L_V}(\mathscr{O})$ into $\mathcal{A}_{L_0}$ do not depend on the choice of test functions χ whose restrictions to $\mathscr{O}$ are equal to 1.

For the proof that the entire C*-algebra $\mathcal{A}_{L_V}$ can consistently be accommodated in $\mathcal{A}_{L_0}$, one has to adjust the embeddings of the local algebras $\mathcal{A}_{L_V}(\mathscr{O})$, $\mathscr{O} \subset \mathscr{M}$, into $\mathcal{A}_{L_0}$; we adopt here arguments given in [27, Prop. 3.1]. Let $\mathscr{O}_n \subset \widehat{\mathscr{O}_n} \subset \mathscr{O}_{n+1}$, $n \in \mathbb{N}$, be a sequence of bounded regions, such that the closure of $\mathscr{O}_n$ is contained in the interior of $\widehat{\mathscr{O}_n}$ and $\lim_n \mathscr{O}_n \nearrow \mathscr{M}$. Furthermore, let $\chi_n \in \mathcal{D}(\mathbb{R}^d)$ be a sequence such that $\chi_n \upharpoonright \mathscr{O}_n = 1$ and $\mathrm{supp}\,\chi_n \subset \widehat{\mathscr{O}_n}$, $n \in \mathbb{N}$. As we shall see, the corresponding isomorphisms $\beta_{\mathscr{O}_n,\chi_n}$, mapping $\mathcal{A}_{L_V}(\mathscr{O}_n)$ into $\mathcal{A}_{L_0}(\widehat{\mathscr{O}_n})$, can be transformed into a coherent sequence of isomorphisms $\gamma_{\mathscr{O}_n}$; their restrictions, respectively ranges, satisfy $\gamma_{\mathscr{O}_{n+1}} \upharpoonright \mathcal{A}_{L_V}(\mathscr{O}_n) = \gamma_{\mathscr{O}_n}$ and $\gamma_{\mathscr{O}_n}(\mathcal{A}_{L_V}(\mathscr{O}_n)) \subset \mathcal{A}_{L_0}(\widehat{\mathscr{O}_{n+1}})$, $n \in \mathbb{N}$. Since $\mathcal{A}_{L_V}$ is the C*-inductive limit of its local subalgebras, this implies that there exist in the sense of pointwise norm convergence the limits

$$\gamma(A_{L_V}) \doteq \lim_n \gamma_n(A_{L_V}) , \quad A_{L_V} \in \mathcal{A}_{L_V} .$$

The limit $\gamma : \mathcal{A}_{L_V} \to \mathcal{A}_{L_0}$ is a monomorphism (injective homomorphism), mapping the local net $\mathscr{O} \mapsto \mathcal{A}_{L_V}(\mathscr{O})$ in the perturbed theory, into a net of subalgebras of $\mathcal{A}_{L_0}$. Moreover, for the given sequence of regions, the algebras $\mathcal{A}_{L_V}(\mathscr{O}_n)$ are mapped into $\mathcal{A}_{L_0}(\widehat{\mathscr{O}_n})$, $n \in \mathbb{N}$. As already mentioned, this does not imply that a similar (fuzzy) identification of local subalgebras holds for all regions $\mathscr{O} \subset \mathscr{M}$.

Turning to the construction of the sequence $\gamma_{\mathscr{O}_n}$, $n \in \mathbb{N}$, we proceed from the isomorphism $\beta_{\mathscr{O}_n,\chi_n}$ for given n. As we have shown, replacing in this isomorphism the underlying test function χ_n by χ_{n+1} amounts to transforming it by some inner automorphism of $\mathcal{A}_{L_0}(\widehat{\mathscr{O}}_{n+1})$. For the sake of brevity, these inner automorphism are denoted by $\mathrm{Ad}\, V_{n+1,n}$, where $V_{n+1,n} \in \mathcal{A}_{L_0}(\widehat{\mathscr{O}}_{n+1})$, *i.e.*

$$\mathrm{Ad}\, V_{n+1,n} \circ \beta_{\mathscr{O}_n,\chi_n} = \beta_{\mathscr{O}_n,\chi_{n+1}} , \quad n \in \mathbb{N} .$$

By construction, we also have $\beta_{\mathscr{O}_{n+1},\chi_{n+1}} \upharpoonright \mathcal{A}_{L_V}(\mathscr{O}_n) = \beta_{\mathscr{O}_n,\chi_{n+1}}$. We define now

$$\gamma_{\mathscr{O}_n} \doteq \mathrm{Ad}\left(V_{n,n-1}\ldots V_{2,1}\right)^{-1} \circ \beta_{\mathscr{O}_n,\chi_n} , \quad n \in \mathbb{N} ,$$

where $V_{1,0} \doteq 1$. The preceding preparations imply that $\gamma_{\mathscr{O}_n}(\mathcal{A}_{L_V}(\mathscr{O}_n) \subset \mathcal{A}_{L_0}(\widehat{\mathscr{O}}_n)$. Moreover, we obtain for $n \in \mathbb{N}$

$$\begin{aligned}
\gamma_{\mathscr{O}_{n+1}} \upharpoonright \mathcal{A}_{L_V}(\mathscr{O}_n) &= \mathrm{Ad}\left(V_{n,n-1}\ldots V_{2,1}\right)^{-1} \circ \mathrm{Ad}\, V^{-1}_{n+1,n} \circ \beta_{\mathscr{O}_{n+1},\chi_{n+1}} \upharpoonright \mathcal{A}_{L_V}(\mathscr{O}_n) \\
&= \mathrm{Ad}\left(V_{n,n-1}\ldots V_{2,1}\right)^{-1} \circ \mathrm{Ad}\, V^{-1}_{n+1,n} \circ \beta_{\mathscr{O}_n,\chi_{n+1}} \upharpoonright \mathcal{A}_{L_V}(\mathscr{O}_n) \\
&= \mathrm{Ad}\left(V_{n,n-1}\ldots V_{2,1}\right)^{-1} \circ \beta_{\mathscr{O}_n,\chi_n} \upharpoonright \mathcal{A}_{L_V}(\mathscr{O}_n) \;=\; \gamma_{\mathscr{O}_n} ,
\end{aligned}$$

establishing the existence of monomorphisms with the desired properties.

This completes our proof that for any choice of Lagrangean L, the corresponding local nets $\mathscr{O} \mapsto \mathcal{A}_L(\mathscr{O})$ can be embedded into a fixed dynamical C*-algebra $\mathcal{A}_{L_0}$. For given $\mathcal{A}_{L_0}$, the nets induced by different theories differ by the assignment of subalgebras of $\mathcal{A}_{L_0}$ to a given spacetime region. The global algebra $\mathcal{A}_{L_0}$ does not contain any specific dynamical information by itself.

The existence of a global algebra containing the local nets of a large family of theories has also been established in the axiomatic framework of algebraic quantum field theory. For mathematical convenience, one works there with local von Neumann algebras. Whenever a theory has the so-called split property [14], which amounts to restrictions on the number of degrees of freedom of the theory in finite volumes of phase space [23], the corresponding nets can isomorphically be embedded into a unique global C*-algebra, the "proper sequential type I_∞ funnel", invented by Takesaki [33]. In the present approach we work in the setting of C*-algebras. The passage to local von Neumann algebras would require to select some states of interest, inducing a weak topology on the algebra. Apart from this technical point, the existence of a universal global algebra does not come as a surprise in the present approach.

It might perhaps be more surprising that, using the interaction picture, free and interacting theories can be placed into one and the same algebra. This seems to contradict Haag's theorem, which says that the combination of free and interacting Hamiltonians into a single interaction operator is impossible in quantum field theory. This obstruction is avoided in the present approach by dealing with a local version of the interaction picture, based on perturbations of the dynamics in finite spacetime regions. In this way large volume as well as ultraviolet singularities are avoided and a consistent theory results.

6. Summary and Outlook

We have constructed in this article for any given Lagrangean of a real scalar field in Minkowski space a corresponding dynamical C*-algebra. In order not to obscure the underlying ideas, we have restricted our attention to interactions given by polynomials of the field; but our construction works for quite arbitrary interaction potentials. The novel input in our construction is an integrated version of the Schwinger–Dyson equation in terms of unitary S-operators, containing information encoded in the field equations. The second ingredient are causal factorization rules for these operators. Both features have been established in the perturbative setting of quantum field theory, based on formal power series in Planck's constant. The resulting equalities, established in the latter approach, were taken as input in our construction of the C*-algebras. Let us emphasize that these algebras exist for any number of spacetime dimensions, any degree of the interaction polynomial and any choice of coupling constants, irrespective of their signs. It is the existence of states of physical interest on these algebras, named principal states in the present article, which is expected to depend on the choice of these parameters. Thus the question of whether there are such states in a particular model has been traced back to a problem in the representation theory of C*-algebras.

The C*-algebras, obtained by our construction, have a quite non-trivial structure. Without having to impose from the outset any quantization rules, they are intrinsically non-commutative. Taking as input a non-interacting Lagrangean, the exponentials of the underlying field were shown to satisfy the Weyl relations, and there exist unitary operators in the algebra which can be interpreted as time ordered exponentials of its normal ordered powers, integrated with test functions. More importantly, we have shown that the resulting C*-algebra is universal in the sense that (up to isomorphisms) it coincides with the C*-algebras obtained from Lagrangeans describing arbitrary local interactions.

The latter result is in accord with the known fact that the physical information encoded in a theory is not contained in the global algebra, generated by the fields, but in the assignment of its subalgebras to spacetime regions, *i.e.* in the corresponding net of local algebras. This basic insight found its expression in the postulates of algebraic quantum field theory, formulated by Haag and Kastler 60 years ago. As a matter of fact, based on findings in his work on collision theory, Haag was convinced that one can recover from a given net the entire physical content of the underlying theory. But his program to specify a specific theory through properties of the corresponding net of local algebras remained unfinished [25].

We have shown that the dynamical C*-algebras, constructed here, comply with all Haag–Kastler postulates. Yet, in contrast to the ideas of Haag, we have used a bottom-up approach, where the operators, generating the algebra, are labelled from the outset by physical quantities, such as "field", "interaction potential", "relative action" *etc*. In the spirit of Bohr, we regard these labels as notions in the framework of classical field theory, which are merely used to describe which kind of objects in the quantum world we have in mind. There is no *a priori* quantization rule for them. The realization of the corresponding operators in the mathematical setting of quantum theory is fixed by the notion of causality, involving time ordering; the actual results depend on the chosen Lagrangean. These observations solve the longstanding problem of incorporating a dynamical principle into the Haag–Kastler framework.

The present results seem to suggest, however, a change of paradigm in the interpretation of the Haag–Kastler framework. Originally, it was proposed to interpret the selfadjoint elements of the local algebras as observables. Yet such an interpretation does not fit well with our construction. To explain this point, consider for example the unitary S-operator, labelled by a localized interaction potential. It should *not* be interpreted as (function of) a quantum observable, describing the potential in the sense of the statistical interpretation of quantum physics. In view of the time ordering involved in the construction of S-operators, such an interpretation would be meaningful if measurements could be performed instantaneously; only then time ordered and unordered operators do coincide. Yet, restricting observables to a Cauchy surface makes them in general ill-defined and requires some smoothing in time, surpressing their infinite fluctuations. In more physical terms, reliable measurements require time, and this fact is taken into account in the time-ordered S-operators. Unfortunately, this step blurs in general the interpretation of the envisaged observables due to perturbations caused by the interaction.

As a way out of this conceptual dilemma, we propose to interpret the unitary operators as operations, describing the impact of measurements of the conceived observables on quantum states in the given spacetime regions. According to this view, a net of local algebras subsumes these operations. It is of interest in this context that in principal states, complying with the split property mentioned above, one can recover from the operations the standard interpretation of states in terms of "primitive observables", which have a consistent statistical interpretation in accordance with basic principles of quantum physics [12].

In the context of the present family of models, there remain some issues of physical interest which were not discussed in this article. First, besides the observables considered here, there exist other prominent examples, such as the kinetic energy, the stress energy tensor, the full Lagrangean *etc*. They have in common that they involve derivatives of the underlying classical field. The resulting functionals therefore require some qualifications in applications of the causal factorization relation, which go beyond formal perturbation theory. One has to ensure that the pertubations admitted in these relations are compatible with the causal structure of Minkowski space. In case of local functions of the field, considered here, this condition is automatically satisfied and was therefore not mentioned. In the general case, the constraints on the functionals can be expressed by conditions on the correspondingly perturbed Euler–Lagrange derivatives. The dynamical C*-algebras then exist for the enlarged set of observables. Second, in this enlarged framework one can study the behaviour of symmetry transformations, such as the space-time translations, under perturbations of the dynamics. They determine cocycles in the C*-algebra. It is an interesting, but more difficult problem, whether there exists also an analogue of Noether's theorem in the present setting, establishing the existence of currents inducing these symmetry transformations. We will return to these topics in a future publication.

The remaining major problem concerns the existence of principal states. At present, there are two strategies visible towards its solution. The first one is based on perturbation theory, where one can try to prove convergence of the formal power series of the S-operators. This works indeed in some exceptional cases, as for example for quadratic functionals of the field [4,31] and in models in two dimensional Minkowski space, such as the ϕ^4 and Sine-Gordon theories, cf. [34] and [3]. The second one is based on the development of criteria in the C*-algebraic setting, implying the existence of principal states. Such a criterion was proposed by Doplicher in [13]. It works in case of C*-dynamical systems; but it is not applicable here, since the present algebras are lacking the continuity properties under the action of symmetry transformations, required in the criterion. So some progress is needed on this algebraic side.

Further steps in this program are the extension of our construction to theories with Lagrangeans, involving several Bose and Fermi fields, vector fields and, ultimately, also local gauge groups. We believe that in spite of the indefinite metric entering in the conventional perturbative treatment of such theories, they can be transferred into a C*-algebraic setting. Limitations on a meaningful physical interpretation of the underlying fields will manifest themselves in the absence of principal states. These will in general only exist on (gauge invariant) subalgebras. In the construction of the corresponding C*-algebras, there appear several new problems, such as the formulation of Ward identities, the characterization of anomalies and the description of BRS-transformations. They all have to be cast into a proper C*-algebraic form, in analogy to the Schwinger–Dyson equations used in the present article. The present C*-algebraic approach can also be extended to interacting quantum field theories on curved spacetimes. There the generally covariant locality principle [10], which is basic in the treatment of these theories, finds its most natural formulation. Even though we are only at the very beginning of these developments, we are confident that the present novel approach will contribute to the long hoped-for mathematical consolidation of quantum field theory.

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Appendix

In this appendix we show that the dynamical relation, taken as an input in the present C*-algebraic setting, are valid in the sense of formal power series in renormalized perturbation theory. We also give a brief account of the mathematical framework underlying our arguments.

There exist several rigorous approaches to the perturbative treatment of quantum field theories. The framework used here, based on classical off-shell fields, is related to the path integral formulation of quantum field theory and to deformation quantization. The former approach is successful for perturbative computations in Euclidean space, based on expansions in terms of Planck's constant; but the interpretation of the results in Minkowski space requires analytic continuations. The latter approach works in physical spacetime and generates power series in Planck's constant of the quantized fields. But it is usually restricted to on-shell fields of the classical theory, which hampers the comparison of different theories.

We therefore work in the framework of algebraic perturbation theory, based on Minkowski space, which is conceptually very close to algebraic quantum field theory, cf. for example [19,30]. The starting point are functionals F on the underlying classical configuration space. In the case of a real scalar field this is the space of smooth real functions $\mathscr{E}$ on Minkowski space $\mathscr{M}$; the functionals map arbitrary field configurations to complex numbers. The vector space of functionals is equipped with three associative products: the pointwise product, $\odot$, the non-commutative product $\circledstar$, which corresponds to the operator product in quantum theory (used in the main text), and the time ordered product $\textcircled{\scriptsize T}$.

For functionals of the form $e^{i\phi(f)}$ where $f \in \mathscr{D}(\mathbb{R}^d)$, $\phi \in \mathscr{E}$, these products are defined by

$$
\begin{aligned}
e^{i\phi(f)} \odot e^{i\phi(g)} &\doteq e^{i\phi(f+g)} \\
e^{i\phi(f)} \circledstar e^{i\phi(g)} &\doteq e^{i\phi(f+g)} e^{-i/2\langle f, \Delta g\rangle} \\
e^{i\phi(f)} \textcircled{\scriptsize T} e^{i\phi(g)} &\doteq e^{i\phi(f+g)} e^{-i\langle f, \Delta_D g\rangle}
\end{aligned}
$$

with the commutator function Δ and the Dirac propagator $\Delta_D = (1/2)(\Delta_R + \Delta_A)$ as in Sect. 4. For later use we also define the time ordered exponential of the field, characterized by the functional equation

$$
e_T^{i\phi(f)} \textcircled{\scriptsize T} e_T^{i\phi(g)} = e_T^{i\phi(f+g)}
$$

and given by

$$
e_T^{i\phi(f)} \doteq e^{i\phi(f)}\, e^{-i/2\langle f, \Delta_D f\rangle} .
$$

In a similar manner one can define $\circledstar$ -exponentials of the field which coincide with the $\odot$ -exponentials because of the antisymmetry of the commutator function Δ.

Due to the singularities of the propagators, the $\circledstar$ -product and the $\textcircled{\scriptsize T}$ -product cannot directly be extended to all functionals of interest here. In particular, they are undefined for non-linear local functionals occuring in typical interaction Lagrangeans, such as $V(f)[\phi] = \int dx\, f(x)\phi^4(x)$. The ill-posed problem of defining the $\circledstar$ -product can be circumvented, however, by *normal ordering*. It is defined as follows.

Let F be functionals of the form

$$
F[\phi] = \sum_{k=0}^{n} \int dx_1 \ldots dx_n\; f_k(x_1, \ldots, x_k)\, \phi(x_1) \ldots \phi(x_k)\,, \quad \phi \in \mathscr{E}\,, \tag{A.1}
$$

where $f_k, k = 0, \dots, n$, are test functions. Their normal ordering is defined by a linear invertible map $F \mapsto :F:$ with generating function

$$:e^{i\phi(f)}: \doteq e^{i\phi(f)} \, e^{(1/2)\,||f||_1^2} .$$

Here $||f||_1$ is the familiar single-particle norm in d spacetime dimensions for particles with mass $m \geq 0$; it is given by

$$||f||_1^2 = (2\pi)^{-(d-1)} \int dp \, \theta(p_0)\delta(p^2 - m^2) \, |\widetilde{f}(p)|^2 ,$$

where $\widetilde{f}$ denotes the Fourier transform of f. (If $m = 0$, this expression is in general undefined in $d = 2$ dimensions; yet there exist other suitable Hilbert norms.) One thus obtains the relation

$$:e^{i\phi(f)}: \circledast :e^{i\phi(g)}: = :e^{i\phi(f+g)}: \, e^{\langle f,g \rangle_1} \doteq : e^{i\phi(f)} \; ⓦ \; e^{i\phi(g)} :$$

with the scalar product $\langle \cdot \, , \, \cdot \rangle_1$ in the 1-particle space. The new product ⓦ is called Wick-star product.

The Wick-star-product can be extended from functionals of the form (A.1) to so-called microcausal functionals [8]. These are functionals as in equation (A.1), where the test functions $f_k, k \in \mathbb{N}$, are replaced by the larger set of distributions, whose wave front sets $WF(f_k)$ (cf. [26]) are restricted by the following condition,

$$WF(f_k) \cap \big((\mathscr{M}^k, V_+^k) \cup (\mathscr{M}^k, V_-^k) \big) = \emptyset , \quad k \in \mathbb{N} ;$$

here $V_\pm$ are the closed forward, respectively backward, lightcones. This condition includes in particular translationally invariant distributions, multiplied with test functions. For the corresponding functionals, the product was shown to be well defined in [16, Thm. 0]. The algebra of regular functionals with respect to the product $\circledast$ can then be extended to normal ordered microcausal functionals, denoted by $:F:$. They satisfy

$$:F: + c \, :G: = :F + c \, G: , \quad c \in \mathbb{C}$$
$$:F: \circledast :G: = :F \; ⓦ \; G: .$$

An analogous extension of the time ordered product would be more complicated since the Dirac propagator is not a solution of the Klein Gordon equation. Using the normal ordering procedure amounts to transforming the time ordered product in a manner such that the Dirac propagator is replaced by the Feynman propagator. But also after normal ordering, the time ordered product is not always well defined on microcausal functionals. In order to cure this defect some further steps are necessary.

One first observes that the n-fold products of local functionals are well defined whenever these functionals have disjoint supports. These products admit extensions to local functionals with arbitrary support in compact regions. But these extensions are not unique; they can be classified and parametrized by renormalization conditions, related to those used in other perturbative schemes [16]. Proceeding to the resulting multilinear products on the space of normal ordered local functionals, one finds [18] that these can be understood as iterated binary products Ⓣ on a subspace of normal ordered microcausal functionals. That space also contains the normal ordered local functionals and their time ordered products. The Ⓣ product is commutative and associative on this space.

One of the renormalization conditions which fix the time ordered products is the Schwinger–Dyson equation. It characterizes the field equation of the interacting quantum fields and has the form, K being the Klein–Gordon operator,

$$:F: \circledT \ \phi(K\phi_0) \ =:F: \circledast \ \phi(K\phi_0) + i \ :\epsilon F(\phi_0): \ . \tag{A.2}$$

Here the functional derivative ϵF of F in the direction of ϕ_0 enters, cf. Sect. 2. For regular functionals F, equation (A.2) is a direct consequence of the definition of the various products considered here. The equation still holds for microcausal functionals after the extension procedure, described above.

The Schwinger–Dyson equation has the form of a differential equation. We can integrate it in the following way. Let

$$:F_\lambda: \doteq :F^{\lambda\phi_0}: \ \circledT \ e_T^{i\phi(K\lambda\phi_0)} \ e^{i(\lambda^2/2)\langle\phi_0, K\phi_0\rangle}$$

where $F^{\lambda\phi_0}[\phi] \doteq F[\phi + \lambda\phi_0]$. Then

$$\begin{aligned}\frac{d}{d\lambda} :F_\lambda: \ &=:\epsilon F^{\lambda\phi_0}(\phi_0): \circledT \ e_T^{i\phi(K\lambda\phi_0)} e^{i(\lambda^2/2)\langle\phi_0, K\phi_0\rangle} + i \ :F_\lambda: \ \circledT \ (\phi(K\phi_0) + \lambda\langle\phi_0, K\phi_0\rangle) \\ &= \epsilon :F_\lambda: (\phi_0) + i \ :F_\lambda: \circledT \ \phi(K\phi_0) \ = \ i \ :F_\lambda: \circledast \ \phi(K\phi_0)\,,\end{aligned}$$

where we inserted the Schwinger–Dyson equation (A.2) in the last step. Integrating this differential equation, we conclude that

$$:F_\lambda: = :F: \circledast \ e^{i\lambda\phi(K\phi_0)}\,, \quad \lambda \in \mathbb{R}\,.$$

This relation is the integrated form of the Schwinger–Dyson equation in the perturbative setting.

We can proceed now to the time ordered exponentials of the functionals $:F:$ in the non-interacting theory with Lagrangean L_o,

$$S_0(F) \ \doteq \ e_T^{i:F:}\,.$$

In case of nonlinear functionals, these time ordered exponentials are defined as formal power series with regard to Planck's constant. They can be interpreted as scattering matrices associated with the localized interaction induced by $:F:$. Noticing that

$$\delta L_0(\phi_0) = (L_0^{\phi_0} - L_0) = \phi(K\phi_0) + 1/2\,\langle\phi_0, K\phi_0\rangle\,,$$

where K is the Klein Gordon operator, the Schwinger Dyson equation implies

$$S_0(F^{\phi_0} + \delta L_0(\phi_0)) = S_0(F) \circledast \ S_0(\delta L_0(\phi_0)) = S_0(\delta L_0(\phi_0)) \circledast \ S_0(F)\,; \tag{A.3}$$

the second equality follows from the fact that the commutator function Δ, appearing in the $\circledast$-product, is a bi-solution of the Klein–Gordon equation.

Next, we turn to the relative S-operators for interaction potentials V of the type considered in this article. For given microcausal functional F and any test function χ which is equal to 1 on supp F, they are defined by

$$S_{V(\chi)}(F) \doteq S_0(V(\chi))^{-1} \circledast \ S_0(V(\chi) + F)\,.$$

Now let ϕ_0 be any test function which also has support in the region where χ equals 1. Denoting by $L = L_0 + V$ the full Lagrangean for the given interaction potential, we have

$$S_{V(\chi)}(F) \circledast S_{V(\chi)}(\delta L(\phi_0))$$
$$= S_0(V(\chi))^{-1} \circledast S_0(V(\chi) + F) \circledast S_0(V(\chi))^{-1} \circledast S_0(V(\chi) + \delta L(\phi_0)) .$$

Since $V(\chi) + \delta L(\phi_0) = V(\chi)^{\phi_0} + \delta L_0(\phi_0)$, relation (A.3) implies

$$S_0(V(\chi) + \delta L(\phi_0)) = S_0(V(\chi)^{\phi_0} + \delta L_0(\phi_0)) = S_0(V(\chi)) \circledast S_0(\delta L_0(\phi_0)) .$$

It follows that

$$S_{V(\chi)}(F) \circledast S_{V(\chi)}(\delta L(\phi_0)) = S_0(V(\chi))^{-1} \circledast S_0(V(\chi) + F) \circledast S(\delta L_0(\phi_0)) .$$

Furthermore, the support properties of ϕ_0 imply $\delta V(\chi)(\phi_0) = \delta V(\phi_0)$, hence

$$S_0(V(\chi) + F) \circledast S_0(\delta L_0(\phi_0))$$
$$= S_0(V(\chi)^{\phi_0} + F^{\phi_0} + \delta L_0(\phi_0)) = S_0(V(\chi) + F^{\phi_0} + \delta L(\phi_0)) .$$

Multiplying this equality from the left by $S_0(V(\chi)^{-1}$ finally gives

$$S_{V(\chi)}(F) \circledast S_{V(\chi)}(\delta L(\phi_0)) = S_{V(\chi)}(F^{\phi_0} + \delta L(\phi_0)) = S_{V(\chi)}(\delta L(\phi_0)) \circledast S_{V(\chi)}(F) , \tag{A.4}$$

where the second equality follows from (A.3) and the preceding relation. Since, the chosen test function χ is equal to 1 on, both, the supports of F and ϕ_0, we can omit this spacetime cutoff of the potential. The equation resulting from (A.4) is the integrated Schwinger–Dyson equation in the presence of interaction.

Recalling that the $\circledast$-product corresponds to the operator product, used in the main text, we have established in the perturbative framework the dynamical relations, which were taken as input in our C*-algebraic approach. In a similar manner one can also justify the causal factorization relations. Since these are widely discussed in the literature [16], we refrain from reconsidering them here.

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