Correction

Correction to: The Full Ward-Takahashi Identity for Colored Tensor Models

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The author wishes to rectify some oversights and, mainly, to amend an erroneous term of the analytic 2-point Schwinger–Dyson Equation (SDE) for the rank-3, complex tensor field ϕ_3^4 -theory, derived in [Pér18a]. We tag the corrections using primed numeration.

Lemma 4 is stronger than it was stated, and this stronger version is needed. The correct (and rephrased) version is:

Lemma 4'. *Let $\mathcal{R}, \mathcal{P} \in \text{Grph}_{c,3}$ be connected graphs with $2p = |V(\mathcal{P})|$ and $2r = |V(\mathcal{R})|$ vertices, respectively. Let $\mathbf{z}^1, \dots, \mathbf{z}^r \in \mathbb{Z}^3$ be arbitrary momenta, and let $\mathbf{x}^1, \dots, \mathbf{x}^p \in \mathbb{Z}^3$ be colorwise, pairwise different momenta. That is, for each $\alpha, v = 1, \dots, p$, and for each color $c = 1, 2, 3$, one imposes $x_c^\alpha \neq x_c^v$ if $\alpha \neq v$. Then, the derivative of $\mathcal{R}$ with respect to $\mathcal{P}$ satisfies*

$$\frac{\partial \mathcal{R}(\mathbf{z}^1, \dots, \mathbf{z}^r)}{\partial \mathcal{P}(\mathbf{x}^1, \dots, \mathbf{x}^p)} = \begin{cases} \delta_p^r \times \left(\sum_{\sigma \in \text{Aut}_c(\mathcal{R})} \delta_{\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^r}^{\mathbf{z}^{\sigma(1)}, \dots, \mathbf{z}^{\sigma(r)}} \right) & \text{if } \mathcal{R} \cong \mathcal{P}, \\ 0 & \text{if } \mathcal{R} \not\cong \mathcal{P}. \end{cases} \quad (27')$$

That is, the imposed restriction¹ $z_c^\alpha \neq z_c^v$, for $\alpha \neq v$ and $c = 1, 2, 3$, was superfluous (and is not used in the proof). Lemma 4' ensures that one can derive the free energy $\log Z$ with respect to graphs. The reason is that in the boundary-graph expansion [Eq. (35)] appear terms $\mathbb{J}(\mathcal{R})(\mathbf{z}^1, \dots, \mathbf{z}^r)$ of unrestricted momenta, which correspond to $\mathbf{z}^1, \dots, \mathbf{z}^r \in \mathbb{Z}^3$ in Eq. (27') here.

For example, recalling that momenta label white vertices, the equation [Pér18a, before Eq. (33)]

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¹ Appearing in [Pér18a, Lemma 4] as “We impose analogous conditions on $\mathbf{c}^1, \dots, \mathbf{c}^r \in \mathbb{Z}^3$ ”.

$$\frac{\partial}{\partial \begin{array}{c} \text{a} \\ \text{b} \end{array} \begin{array}{c} \text{c} \end{array}} \left(\begin{array}{c} \text{e} \\ \text{f} \end{array} \begin{array}{c} \text{g} \end{array} \right) = \delta_a^e \delta_b^f \delta_c^g + \delta_a^g \delta_b^e \delta_c^f + \delta_a^f \delta_b^g \delta_c^e$$

holds everywhere where $\{a_d \neq b_d \neq c_d \neq a_d\}_{d=1,2,3}$, that is, with no restriction on $\mathbf{e}, \mathbf{f}, \mathbf{g} \in \mathbb{Z}^3$.

The main correction concerns the 2-point SDE [Pér18a, Eq. (51)], which should be instead:

Claim 1. *The exact, analytic Schwinger–Dyson equation for the 2-point function $G^{(2)}$ of the φ_3^4 -theory, which is defined by the action*

$$S[\varphi, \bar{\varphi}] = \sum_{\mathbf{x}} \bar{\varphi}_{\mathbf{x}} \cdot (x_1^2 + x_2^2 + x_3^2 + m^2) \cdot \varphi_{\mathbf{x}} + \lambda (1 \square_1 + 2 \square_2 + 3 \square_3),$$

is given by

$$\begin{aligned} G^{(2)}(\mathbf{x}) = & \frac{1}{m^2 + |\mathbf{x}|^2} + \frac{(-2\lambda)}{m^2 + |\mathbf{x}|^2} \sum_{c=1}^3 \left\{ \sum_{q_a, q_b} G^{(2)}(q_a, q_b, x_c) \times G^{(2)}(\mathbf{x}) \right. \\ & + \sum_{d=a,b} \sum_{q_c} G_{V_d}^{(4)}(x_a, x_b, q_c, \mathbf{x}) + \sum_{q_a, q_b} G_{m|m}^{(4)}(q_a, q_b, x_c, \mathbf{x}) \\ & \left. - \sum_{q_c} \frac{1}{q_c^2 - x_c^2} \left[G^{(2)}(x_a, x_b, q_c) - G^{(2)}(\mathbf{x}) \right] + G_{V_c}^{(4)}(\mathbf{x}, \mathbf{x}) \right\}, \end{aligned} \quad (51')$$

for each $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3$. In (51'), the set equality $\{a, b, c\} = \{1, 2, 3\}$ is assumed.

Here, we conveniently abbreviated:

$$\begin{aligned} G^{(2)} &= G_{\ominus}^{(2)} = \frac{\partial \log Z}{\partial \ominus}, & G_{V_c}^{(4)} &= G_{c|\square c}^{(4)} = \frac{\partial \log Z}{\partial c|\square c}, \\ G_{m|m}^{(4)} &= G_{\ominus|\ominus}^{(4)} = \frac{\partial \log Z}{\partial (\ominus|\ominus)}. \end{aligned} \quad (a)$$

Unfolding the colors, the correct 2-point SDE (51') reads²:

$$\begin{aligned} G_{\ominus}^{(2)}(\mathbf{x}) = & \frac{1}{m^2 + |\mathbf{x}|^2} \\ & + \frac{(-2\lambda)}{m^2 + |\mathbf{x}|^2} \left\{ G_{\ominus}^{(2)}(\mathbf{x}) \cdot \left[\sum_{q,r \in \mathbb{Z}} G_{\ominus}^{(2)}(x_1, q, r) + G_{\ominus}^{(2)}(q, x_2, r) + G_{\ominus}^{(2)}(q, r, x_3) \right] \right. \\ & + \sum_{q \in \mathbb{Z}} \left(G_{1\square_1}^{(4)}(x_1, q, x_3; \mathbf{x}) + G_{1\square_1}^{(4)}(x_1, x_2, q; \mathbf{x}) + G_{2\square_2}^{(4)}(q, x_2, x_3; \mathbf{x}) \right. \\ & \left. \left. + G_{2\square_2}^{(4)}(x_1, x_2, q; \mathbf{x}) + G_{3\square_3}^{(4)}(q, x_2, x_3; \mathbf{x}) + G_{3\square_3}^{(4)}(x_1, q, x_3; \mathbf{x}) \right) \right\} \end{aligned} \quad (51'')$$

² Here, given in the same notation of [Pér18a] to ease comparison.

$$\begin{aligned}
 & + \sum_{q,r \in \mathbb{Z}} \left[G_{\ominus|\ominus}^{(4)}(x_1, q, r; \mathbf{x}) + G_{\ominus|\ominus}^{(4)}(q, x_2, r; \mathbf{x}) + G_{\ominus|\ominus}^{(4)}(q, r, x_3; \mathbf{x}) \right] \\
 & - \sum_{q_1 \in \mathbb{Z}} \left[\frac{1}{q_1^2 - x_1^2} \cdot (G_{\ominus}^{(2)}(q_1, x_2, x_3) - G_{\ominus}^{(2)}(\mathbf{x})) \right] \\
 & - \sum_{q_2 \in \mathbb{Z}} \left[\frac{1}{q_2^2 - x_2^2} \cdot (G_{\ominus}^{(2)}(x_1, q_2, x_3) - G_{\ominus}^{(2)}(\mathbf{x})) \right] \\
 & - \sum_{q_3 \in \mathbb{Z}} \left[\frac{1}{q_3^2 - x_3^2} \cdot (G_{\ominus}^{(2)}(x_1, x_2, q_3) - G_{\ominus}^{(2)}(\mathbf{x})) \right] + \sum_{c=1,2,3} G_{c\Box c}^{(4)}(\mathbf{x}, \mathbf{x}) \Big\}.
 \end{aligned}$$

for all $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3$.

The mistake, which is fixed by replacing the second row of Eq. (51) by the second row of Eq. (51') (or by the third and fourth lines of Eq. (51'')), is a consequence of a computational oversight in page 626 while directly calculating $\delta_{a_1 b_1} \times (\delta^2 Y^{(1)}[J, \bar{J}] / \delta J_a \delta \bar{J}_{b_1 a_2 a_3})$. In fact, if one uses Eq. (48)|_{a=1} for the Y -term instead, one gets the correct result for this double derivative, after using the symmetry of each 4-point function:

$$G_B^{(4)}(\mathbf{x}; \mathbf{y}) = G_B^{(4)}(\mathbf{y}; \mathbf{x}), B \in \{\ominus \sqcup \ominus, 1\Box_1, 2\Box_2, 3\Box_3\}, \text{ with } \mathbf{x}, \mathbf{y} \in \mathbb{Z}^3.$$

To wit:

$$\begin{aligned}
 \frac{\delta^2 Y_{x_1}^{(1)}[J, \bar{J}]}{\delta J_{x_1 x_2 x_3} \delta \bar{J}_{x_1 x_2 x_3}} \Big|_{J=\bar{J}=0} &= \frac{\partial Y_{x_1}^{(1)}[J, \bar{J}]}{\partial \ominus(\mathbf{x})} \\
 &= (\Delta_{x_1,1} G_{\ominus|\ominus}^{(4)} + \Delta_{x_1,1} G_{1\Box_1}^{(4)} + \Delta_{x_1,1} G_{2\Box_2}^{(4)} + \Delta_{x_1,1} G_{3\Box_3}^{(4)})(\mathbf{x}) \\
 &= \sum_{q_2, q_3} G_{\ominus|\ominus}^{(4)}(x_1, q_2, q_3; \mathbf{x}) + G_{1\Box_1}^{(4)}(\mathbf{x}, \mathbf{x}) \\
 &\quad + \sum_{q_3} G_{2\Box_2}^{(4)}(x_1, x_2, q_3; \mathbf{x}) + \sum_{q_2} G_{3\Box_3}^{(4)}(x_1, q_2, x_3; \mathbf{x}),
 \end{aligned} \tag{b}$$

with analogous results for $\partial Y_{x_2}^{(2)} / \partial \ominus$ and $\partial Y_{x_3}^{(3)} / \partial \ominus$. The second row of Eq. (51) derives from terms in the sum $\sum_{c=1,2,3} \partial Y_{x_c}^{(c)} / \partial \ominus(\mathbf{x})$, and the correct result is concretely obtained by the last line of Eq. (b) plus cyclic permutations, namely $\sum_{c=1,2,3} \sum_{d \neq c} \sum_{q_c} G_{V_d}^{(4)}$

$(x_a, x_b, q_c, \mathbf{x})$, after some rearrange. Claim 1 follows.

The extraction of the coefficients of the Y -term via graph derivatives is explored in depth in [PPW17], whose advantage over direct computations, we hope, is now clear.

To conclude this corrigendum, in page 625 I stated that if for certain color $a = 1, \dots, D$ the subgraph

is contained in *each* one of the interaction vertices of a complex tensor model, then one could obtain the Schwinger–Dyson equations departing from the present Ward–Takahashi identity. As it reads, this implies that our techniques work for the interaction vertex

Nevertheless, as it was pointed out to me (see acknowledgements), the previous condition turns out to be necessary but insufficient. The interaction vertices should be furthermore restricted to graphs in which *each* (graph-)vertex sits in such a subgraph. For instance, on top of the ‘pillow-like’ (quartic) vertices treated here, one can derive the tower of analytic Schwinger–Dyson equations for interactions spanned by

and higher-rank analogues. In fact, the complete tower of SDE for connected correlation functions with connected boundary graphs has been obtained for the arbitrary-rank ‘pillow-like’ vertices in [PPW17] (for disconnected boundary graphs cf. [Pér18b]). Their large- N limit was later reported in [PPTW18] for rank-3 theories. Moreover, using the Ward–Takahashi identity and extending the approach [PW19] to solve a quartic model in noncommutative field theory in terms of the Lambert W function, the solution of a single pillow-vertex model [Pas19] was found.

On that basis, the SDE for tensor models with the φ^n -interaction vertices ($n \geq 6$) listed in (e) could be also derived. How to get the analytic SDE-tower for interactions like (d) is still unknown.

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