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The Generalised Complex Geometry of (p, q) Hermitian Geometries

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Abstract: We define (p, q) Hermitian geometry as the target space geometry of the two dimensional (p, q) supersymmetric sigma model. This includes generalised Kähler geometry for (2, 2), generalised hyperkähler geometry for (4, 2), strong Kähler with torsion geometry for (2, 1) and strong hyperkähler with torsion geometry for (4, 1). We provide a generalised complex geometry formulation of hermitian geometry, generalising Gualtieri's formulation of the (2, 2) case. Our formulation involves a chiral version of generalised complex structure and we provide explicit formulae for the map to generalised geometry.

Contents

1. Introduction

Complex geometries with torsion arise in the study of supersymmetric sigma models and in generalised complex geometry. The bihermitean geometry that arises in two-dimensional supersymmetric sigma models [1] was formulated in the framework of generalised complex geometry [2] as generalised Kähler geometry by Gualtieri [3], with the relation between the sigma model geometry and generalised complex geometry given by the Gualtieri map.

The supersymmetry algebras in two dimensions are labelled by two integers p, q. Requiring a non-linear sigma model to be invariant under (p, q) supersymmetry places strong restrictions on the target space geometry, with various geometries arising for different values of p, q [8]. We will refer to the target space geometry of the (p, q) supersymmetric sigma model as (p, q) hermitian geometry; this will be defined in Sect. 2. The models of [1] giving rise to generalised Kähler geometry have (2, 2) supersymmetry while (2, 1) supersymmetry gives an interesting geometry [4,5] that is sometimes referred to as Strong Kähler with Torsion (SKT). SKT geometry was discussed within the formalism of generalised complex geometry in [6,7]. Our purpose here is to find the generalised complex geometry formulation of the all (p, q) hermitian geometries. This will require the definition of a chiral form of generalised complex structure that we will refer to as a half generalised complex structure. We also aim to give a presentation that is closely related to the sigma model geometry, making the map to generalised complex geometry manifest.

2. (p, q) Hermitian Geometry

The (p, q) supersymmetry algebra in two dimensions has p right-handed supercharges and q left-handed ones [4]. The general supersymmetric sigma models with (1, 1), (2, 2) and (4, 4) supersymmetry were constructed in [1], the ones with (1, 0) and (2, 0) supersymmetry were constructed in [4], the one with (2,1) supersymmetry was constructed in [5], while the remaining cases were given in [8]. The (1, 1) supersymmetric sigma model has a target space $(\mathcal{M}, g, H)$ which is a manifold $\mathcal{M}$ with a metric g and a closed 3-form H. This can be given locally in terms of a 2-form potential b, $H = db$. Conversely, given such a geometry one may construct a (1, 1) supersymmetric sigma model with that target space. The (1, 1) model will in fact have (p, q) supersymmetry with $p, q = 1, 2$ or 4 if it has a special geometry that we will call a (p, q) *hermitian geometry*, which has $p-1$ complex structures J_+^a $(a = 1, \dots p-1)$ and $q-1$ complex structures $J_-^{a'}$ $(a' = 1, \dots q-1)$. The space $(\mathcal{M}, g, H, J_+^a, J_-^{a'})$ is a (p, q) hermitian geometry if

1. J_+^a $(a = 1, \dots p-1)$ and $J_-^{a'}$ $(a' = 1, \dots q-1)$ are complex structures on $\mathcal{M}$.
2. The metric g is hermitian with respect to all complex structures
$$g(JX, JY) = g(X, Y), \quad \forall J \in \{J_+^a, J_-^{a'}\}.$$
3. The $J_\pm$ are covariantly constant
$$\nabla^{(\pm)} J_{(\pm)} = 0, \quad \forall J_+ \in \{J_+^a\} \; and \; \forall J_- \in \{J_-^{a'}\}$$
with respect to the connections
$$\nabla^{(\pm)} := (\nabla^{(0)} \pm \tfrac{1}{2} g^{-1} H) \tag{2.1}$$
with torsion $\pm\frac{1}{2} g^{il} H_{ljk}$. Here $\nabla^{(0)}$ is the Levi-Civita connection and the (3, 0) part of H vanishes with respect to each complex structure $J \in \{J_+^a, J_-^{a'}\}$, so that
$$H = H^{(2,1)} + H^{(1,2)} \tag{2.2}$$
with respect to each J.

4. If $p = 4$ then J^1_+, J^2_+, J^3_+ satisfy a quaternion algebra and if $q = 4$ then J^1_-, J^2_-, J^3_- satisfy a quaternion algebra.

For each complex structure $J \in (J^a_+, J^{a'}_-)$, there is a differential operator $d^c = i(\partial - \bar\partial)$ and the corresponding 2-form $\omega = gJ$ which satisfies $dd^c\omega = 0$. The condition (3) can be replaced with the following condition:

- For each $J_+ \in (J^a_+)$, $d^c\omega = H$ while for each $J_- \in (J^{a'}_-)$, $d^c\omega = -H$.

The (2, 2) geometries were called generalised Kähler geometries in [3] while the (4, 2) gives geometries that were called generalised hyperkähler geometries in [9]. The name strong Kähler with torsion (SKT) was proposed for (2, 1) (or (1, 2)) geometry and strong hyperkähler with torsion was proposed for (4, 1) (or (1, 4)) geometry in [10]. If $H = 0$, then $p = q$ and the (2, 2) case gives Kähler geometry in which case ω is the Kähler form, while the (4, 4) case gives hyperkähler geometries. If $p > 4$ then the holonomy of $\nabla^{(+)}$ is trivial while if $q > 4$ then the holonomy of $\nabla^{(-)}$ is trivial; we will restrict ourselves to the cases $p \leq 4$, $q \leq 4$ here.

There is a rich interplay between (p, q) hermitian geometry and the superspace formulation of the (p, q) supersymmetric sigma model. For the (2, 2) case, if the two complex structures J_+, J_- commute, then the corresponding sigma model is formulated in terms of chiral and twisted chiral superfields [1]. On the other hand, if the commutator $[J_+, J_-]$ has trivial kernel, then the sigma model is formulated in terms of semichiral superfields [11]. It was shown in [12] that the general (2, 2) sigma model can be formulated in terms of chiral, twisted chiral and semichiral superfields, giving a complete characterisation of generalised Kähler geometry (away from irregular points) with the geometry given locally by a scalar potential. This will be discussed further in Sect. 10.

3. Generalised Geometry

For a d-dimensional manifold $\mathcal{M}$, the generalised tangent space is the sum of the tangent bundle T and the cotangent bundle T^*

$$\mathbb{T} := T \oplus T^*. \tag{3.1}$$

This has a natural $O(d, d)$ invariant metric η defined by

$$\eta(X + \xi, Y + \sigma) = \xi(Y) + \sigma(X) \tag{3.2}$$

where X, Y are vector fields and and ξ, σ are one-form fields. There is a natural projection

$$\rho : \mathbb{T} \to T, \tag{3.3}$$

with $\rho(X + \xi) = X$.

If g is a metric on $\mathcal{M}$, then this gives a metric g^{-1} on T^* and a metric $\mathcal{H}$ on $\mathbb{T}$ with

$$\mathcal{H}(X + \xi, Y + \sigma) = g(X, Y) + g^{-1}(\xi, \sigma). \tag{3.4}$$

There is also a natural map

$$\mathcal{G} : \mathbb{T} \to \mathbb{T}, \tag{3.5}$$

with

$$\mathcal{G}^2 = 1, \tag{3.6}$$

defined in terms of the lowering and raising maps $g : T^* \to T$ and $g^{-1} : T \to T^*$ by

$$\mathcal{G}(X + \xi) = g\xi + g^{-1}X. \tag{3.7}$$

A 2-form B on $\mathcal{M}$ defines the B-*transformation* which is a map

$$e^B : X + \xi \mapsto X + \xi + i_X B \tag{3.8}$$

that preserves the metric η.

The Courant bracket is defined on smooth sections of $T \oplus T^*$, and is given by

$$[\![X + \xi, Y + \sigma]\!]_0 = [X, Y] + \mathcal{L}_X\sigma - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\sigma - i_Y\xi), \tag{3.9}$$

where $X + \xi, Y + \sigma \in C^\infty(T \oplus T^*)$ and $\mathcal{L}_X$ is the Lie derivative with respect to X. Given a 3-form H, the H-twisted Courant bracket $[\![,]\!]_H$ is

$$[\![X + \xi, Y + \sigma]\!]_H = [\![X + \xi, Y + \sigma]\!]_0 + i_Y i_X H. \tag{3.10}$$

If b is a 2-form then

$$[\![e^b(W), e^b(Z)]\!]_H = e^b[\![W, Z]\!]_{H+db} \quad \forall\, W, Z \in C^\infty(T \oplus T^*), \tag{3.11}$$

so that e^b is a symmetry of $[\![,]\!]_H$ if and only if $db = 0$.

The map $\mathcal{G}$ has eigenvalues ± 1 and the +1 and -1 eigenspaces both have dimension d. It then defines a splitting

$$\mathbb{T} = \mathbb{T}_+ \oplus \mathbb{T}_-, \tag{3.12}$$

of $\mathbb{T}$ into the ± 1 eigenspaces

$$\mathbb{T}_\pm := P_\pm \mathbb{T} \tag{3.13}$$

defined by the projection operators

$$P_\pm := \tfrac{1}{2}\left(1 \pm \mathcal{G}\right). \tag{3.14}$$

The spaces

$$\mathbb{T}_\pm = \{X + \xi \in C^\infty(T \oplus T^*) : \xi = \pm g X\} \tag{3.15}$$

have natural identifications with the tangent bundle

$$\rho_\pm : \mathbb{T}_\pm \to T \tag{3.16}$$

given by

$$\rho_\pm(X \pm gX) = X\,. \tag{3.17}$$

Using the natural matrix notation in which $X + \xi$ is written

$$\mathbb{X} = \begin{pmatrix} X \\ \xi \end{pmatrix} \tag{3.18}$$

the metrics are represented by the matrices

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathcal{H} = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}. \tag{3.19}$$

The map $\mathcal{G}$ acts through the matrix

$$\mathcal{G} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \tag{3.20}$$

satisfying

$$\mathcal{G} = \eta^{-1}\mathcal{H}; \qquad \mathcal{G}^2 = 1 \tag{3.21}$$

while the B-map is represented by

$$\exp(B) = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}. \tag{3.22}$$

The projection ρ is

$$\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.23}$$

Given a closed 3-form H, in each patch there is a 2-form b such that $H = db$. If H represents an integral cohomology class, b is a gerbe connection. Then a B-map using b takes a section W of $T \oplus T^*$ to a local section

$$\tilde{\mathbb{X}} = e^b \mathbb{X} = \begin{pmatrix} X \\ \xi + bX \end{pmatrix} \tag{3.24}$$

of a Courant algebroid E with the short exact sequence

$$0 \to T^* \to E \to T \to 0 \tag{3.25}$$

and anchor map $\tilde{\rho} : E \to T$. See [2,3,6,7,9] for further discussion. The H-twisted Courant bracket on $T \oplus T^*$ is mapped to the untwisted Courant bracket on E:

$$[\![\tilde{\mathbb{X}}, \tilde{\mathbb{Y}}]\!] = [\![e^b(\mathbb{X}), e^b(\mathbb{Y})]\!] = e^b [\![\mathbb{X}, \mathbb{Y}]\!]_H. \tag{3.26}$$

The map then gives $\tilde{\eta}, \tilde{\mathcal{H}}, \tilde{\mathcal{G}}$ on E given by $\tilde{\eta} = \eta$ and

$$\tilde{\mathcal{G}} = e^b \mathcal{G} e^{-b}, \qquad \tilde{\mathcal{H}} = e^b \mathcal{H} e^{-b} \tag{3.27}$$

so that

$$\tilde{\mathcal{G}} = \begin{pmatrix} -g^{-1}b & g^{-1} \\ g - bg^{-1}b & bg^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \\ b & 1 \end{pmatrix} \begin{pmatrix} & g^{-1} \\ g & \end{pmatrix} \begin{pmatrix} 1 & \\ -b & 1 \end{pmatrix} \tag{3.28}$$

and

$$\tilde{\mathcal{H}} = \begin{pmatrix} g - bg^{-1}b & bg^{-1} \\ -g^{-1}b & g^{-1} \end{pmatrix} = \eta \tilde{\mathcal{G}}. \tag{3.29}$$

$\tilde{\mathcal{H}}$ is often referred to as the generalised metric.

The map $\tilde{\mathcal{G}}$ has eigenvalues ± 1 and defines a splitting

$$E = E_+ \oplus E_- \tag{3.30}$$

of E into the ± 1 eigenspaces defined by the projection operators

$$\tilde{P}_\pm := \tfrac{1}{2}\left(1 \pm \tilde{\mathcal{G}}\right). \tag{3.31}$$

In discussing generalised complex geometry, we can either work on E with integrability defined with respect to the untwisted Courant bracket (as in [2,3]), or equivalently on $T \oplus T^*$ with integrability defined with respect to the H-twisted Courant bracket. We will adopt the latter strategy. Our results can be transferred to E using the B-map.

4. Generalised Complex Geometry

A generalised almost complex structure $\mathcal{J}$ is a bundle endomorphism of $\mathbb{T}$ which squares to minus the identity and preserves the metric η,

$$\mathcal{J}^2 = -1$$
$$\eta(\mathcal{J}\mathbb{X}, \mathcal{J}\mathbb{Y}) = \eta(\mathbb{X}, \mathbb{Y}).$$

The generalised almost complex structure $\mathcal{J}$ splits the complexified generalised tangent bundle into the $+i$ eigenspace $\mathbb{L}$ and the $-i$ eigenspace $\overline{\mathbb{L}}$

$$\mathbb{T} \otimes \mathbb{C} = \mathbb{L} \oplus \overline{\mathbb{L}}. \tag{4.1}$$

A *generalised complex structure* [2] is a generalised almost complex structure for which the subspace $\mathbb{L}$ is involutive under the H-twisted Courant bracket, i.e. one for which

$$[\![\mathbb{X}, \mathbb{Y}]\!]_H \in \mathbb{L} \text{ if } \mathbb{X}, \mathbb{Y} \in \mathbb{L}. \tag{4.2}$$

A *generalised Kähler structure* [3] on a manifold $\mathcal{M}$ with metric g can be defined as a generalised complex structure $\mathcal{J}_1$ that commutes with the map $\mathcal{G}$ defined in (3.5):

$$\mathcal{J}_1\mathcal{G} = \mathcal{G}\mathcal{J}_1. \tag{4.3}$$

Then

$$\mathcal{J}_2 = \mathcal{G}\mathcal{J}_1 \tag{4.4}$$

defines a second generalised complex structure that commutes with $\mathcal{J}_1$.

Gualtieri [3] showed that a generalised Kähler structure on $\mathcal{M}$ is equivalent to a (2, 2) or bihermitian geometry $(\mathcal{M}, g, H, J_\pm)$ with complex structures $J_\pm$ on $\mathcal{M}$. The Gualtieri map gives the generalised complex structures in terms of the complex structures $J_\pm$

$$\mathcal{J}_{1/2} = \frac{1}{2}\begin{pmatrix} J_+ \pm J_- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ \omega_+ \mp \omega_- & -(J_+^t \pm J_-^t) \end{pmatrix}. \tag{4.5}$$

Here $\omega_\pm$ are the Kähler forms $\omega_\pm = gJ_\pm$.

A generalised complex structure $\mathcal{J}$ on $\mathbb{T}$ gives a generalised complex structure $\tilde{\mathcal{J}}$ on E

$$\tilde{\mathcal{J}} = e^b \mathcal{J} e^{-b} \tag{4.6}$$

such that the $+i$ eigenspace L of $\tilde{\mathcal{J}}$ is involutive with respect to the untwisted Courant bracket, i.e. for which

$$[\![\tilde{\mathbb{X}}, \tilde{\mathbb{Y}}]\!]_0 \in L \text{ if } \tilde{\mathbb{X}}, \tilde{\mathbb{Y}} \in L. \tag{4.7}$$

A generalised Kähler structure on E consists of a $\tilde{\mathcal{J}}_1$ commuting with $\tilde{\mathcal{G}}$, and the Gualtieri map is now

$$\tilde{\mathcal{J}}_{1/2} = \frac{1}{2}\begin{pmatrix} 1 & \\ b & 1 \end{pmatrix}\begin{pmatrix} J_+ \pm J_- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ \omega_+ \mp \omega_- & -(J_+^t \pm J_-^t) \end{pmatrix}\begin{pmatrix} 1 & \\ -b & 1 \end{pmatrix}. \tag{4.8}$$

5. Half Generalised Complex Structures

The Gualtieri map can be written suggestively as

$$\mathcal{J}_1 = P_+ \begin{pmatrix} J_+ & 0 \\ 0 & -J_+^t \end{pmatrix} P_+ + P_- \begin{pmatrix} J_- & 0 \\ 0 & -J_-^t \end{pmatrix} P_- \tag{5.1}$$

so that

$$\mathcal{J}_1 = \mathcal{J}_+ + \mathcal{J}_- \tag{5.2}$$

where $\mathcal{J}_\pm$ are endomorphisms of $\mathbb{T}_\pm$:

$$\mathcal{J}_\pm : \mathbb{T}_\pm \to \mathbb{T}_\pm \tag{5.3}$$

with

$$\mathcal{J}_+ = P_+ \begin{pmatrix} J_+ & 0 \\ 0 & -J_+^t \end{pmatrix} P_+, \qquad \mathcal{J}_- = P_- \begin{pmatrix} J_- & 0 \\ 0 & -J_-^t \end{pmatrix} P_-. \tag{5.4}$$

This motivates defining structures on $\mathbb{T}_\pm$ instead of on $\mathbb{T}$.

Definition 1. A *positive chirality half generalised almost complex structure* $\mathcal{J}_+$ is a bundle endomorphism of $\mathbb{T}_+$

$$\mathcal{J}_+ : \mathbb{T}_+ \to \mathbb{T}_+ \tag{5.5}$$

which vanishes on $\mathbb{T}_-$, squares to minus the identity on $\mathbb{T}_+$ and preserves the metric η,

$$\mathcal{J}_+^2 = -1_{\mathbb{T}_+}$$
$$\eta(\mathcal{J}_+\mathbb{X}, \mathcal{J}_+\mathbb{Y}) = \eta(\mathbb{X}, \mathbb{Y}) \text{ for } \mathbb{X}, \mathbb{Y} \in \mathbb{T}_+.$$

Definition 2. A *negative chirality half generalised almost complex structure* $\mathcal{J}_-$ is a bundle endomorphism of $\mathbb{T}_-$ which vanishes on $\mathbb{T}_+$, squares to minus the identity on $\mathbb{T}_-$ and preserves the metric η.

Using $\mathcal{J}_+$, we define another two projection operators

$$\Pi_\pm := \tfrac{1}{2}(1 \mp i\mathcal{J}_+). \tag{5.6}$$

This allows a further split

$$\mathbb{T}_+ \otimes \mathbb{C} = \mathbb{L}_+ \oplus \overline{\mathbb{L}}_+ \tag{5.7}$$

into the $+i$ eigenspace $\mathbb{L}_+$ and the $-i$ eigenspace $\overline{\mathbb{L}}_+$. We define a positive chirality half generalised complex structure $\mathcal{J}_+$ as a positive chirality half generalised almost complex structure for which $\mathbb{L}_+$ is involutive with respect to the H-twisted Courant bracket, i.e. one for which

$$[\![\mathbb{X}, \mathbb{Y}]\!]_H \in \mathbb{L}_+ \text{ if } \mathbb{X}, \mathbb{Y} \in C^\infty(\mathbb{L}_+). \tag{5.8}$$

We will refer to this as the condition that $\mathcal{J}_+$ is integrable.

There is a similar construction for negative chirality. A negative chirality half generalised almost complex structure $\mathcal{J}_-$ gives a split

$$\mathbb{T}_- \otimes \mathbb{C} = \mathbb{L}_- \oplus \overline{\mathbb{L}}_- \tag{5.9}$$

into the $+i$ eigenspace $\mathbb{L}_-$ and the $-i$ eigenspace $\overline{\mathbb{L}}_-$ and it will be a negative chirality half generalised complex structure if $\mathbb{L}_-$ is involutive with respect to the H-twisted Courant bracket.

Note that half generalised complex structures automatically commute with $\mathcal{G}$. A generalised Kähler structure corresponds to two half generalised almost complex structure $\mathcal{J}_\pm$. The maps $\rho_\pm : \mathbb{T}_\pm \to T$ take the half generalised complex structures $\mathcal{J}_\pm$ on $\mathbb{T}_\pm$ to the complex structures $J_\pm$ on T.

There is a similar construction on the Courant algebroid E. A *positive chirality half generalised almost complex structure* $\tilde{\mathcal{J}}_+$ is a bundle endomorphism of E_+

$$\tilde{\mathcal{J}}_+ : E_+ \to E_+ \tag{5.10}$$

which squares to minus the identity on E_+ and preserves the metric η,

$$\begin{aligned}&\tilde{\mathcal{J}}_+^2 = -1_{E_+}\\ &\eta(\tilde{\mathcal{J}}_+ W, \tilde{\mathcal{J}}_+ Z) = \eta(W, Z) \text{ for } W, Z \in C^\infty(E_+).\end{aligned}$$

With the decomposition

$$E_+ = L_+ \oplus \bar{L}_+ \tag{5.11}$$

where L is the $+i$ eigenspace of $\tilde{\mathcal{J}}_+$, we define a positive chirality half generalised complex structure $\tilde{\mathcal{J}}_+$ on E as a positive chirality half generalised almost complex structure which is integrable, i.e. for which L is involutive with respect to the Courant bracket

$$[\![\tilde{\mathbb{X}}, \tilde{\mathbb{Y}}]\!] \in L_+ \text{ if } \tilde{\mathbb{X}}, \tilde{\mathbb{Y}} \in C^\infty(L_+). \tag{5.12}$$

There is a similar construction for negative chirality.

The structures on $E_\pm$ are the B-transforms of the structures on $\mathbb{T}_\pm$:

$$\tilde{\mathcal{J}}_\pm = e^b \mathcal{J}_\pm e^{-b} \tag{5.13}$$

so that explicitly

$$\tilde{\mathcal{J}}_+ = e^b P_+ \begin{pmatrix} J_+ & 0 \\ 0 & -J_+^t \end{pmatrix} P_+ e^{-b}, \qquad \tilde{\mathcal{J}}_- = e^b P_- \begin{pmatrix} J_- & 0 \\ 0 & -J_-^t \end{pmatrix} P_- e^{-b}. \tag{5.14}$$

6. Algebraic Structure

If $\mathbb{X} \in C^\infty(\mathbb{T}_+)$, then it takes the form

$$\mathbb{X} = \begin{pmatrix} X \\ gX \end{pmatrix} \tag{6.1}$$

for some vector field $X \in C^\infty(T)$, and the isomorphism $\rho_+ \mathbb{T}_+ \to T$ takes $\rho_+ : \mathbb{X} \to X$,

$$\rho_+ : \mathbb{X} = \begin{pmatrix} X \\ gX \end{pmatrix} \to X. \tag{6.2}$$

The identification of $\mathbb{T}_+$ with T leads to the identification of an automorphism $\mathcal{A}$ of $\mathbb{T}_+$ with an automorphism A of T. In this section, we will develop some useful formulas making this identification explicit.

An automorphism of $\mathbb{T}$ has the general form

$$\mathcal{A} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \tag{6.3}$$

for some automorphisms

$$\begin{aligned} a &: T \to T, \quad & c &: T \to T^*, \\ b &: T^* \to T, \quad & d &: T^* \to T^*. \end{aligned} \tag{6.4}$$

This will be an automorphism of $\mathbb{T}_+$ that leaves $\mathbb{T}_-$ invariant if

$$P_- \mathcal{A} = 0, \qquad \mathcal{A} P_- = 0 \tag{6.5}$$

which implies $\mathcal{A}$ must take the form

$$\mathcal{A} = \frac{1}{2} \begin{pmatrix} A & Ag^{-1} \\ gA & gAg^{-1} \end{pmatrix} \tag{6.6}$$

and A is an automorphism of T:

$$A : T \to T. \tag{6.7}$$

The automorphism $\mathcal{A}$ can be rewritten as

$$\mathcal{A} = P_+ \begin{pmatrix} A & 0 \\ 0 & gAg^{-1} \end{pmatrix} P_+. \tag{6.8}$$

Then

$$\mathcal{A} \begin{pmatrix} X \\ gX \end{pmatrix} = \begin{pmatrix} AX \\ gAX \end{pmatrix} \cdot \tag{6.9}$$

For $\mathbb{X}, \mathbb{Y} \in C^\infty(\mathbb{T}_+)$ with

$$\mathbb{X} = \begin{pmatrix} X \\ gX \end{pmatrix}, \qquad \mathbb{Y} = \begin{pmatrix} Y \\ gY \end{pmatrix}, \tag{6.10}$$

we have

$$\eta(\mathbb{X}, \mathbb{Y}) = \mathcal{H}(\mathbb{X}, \mathbb{Y}) = 2g(X, Y). \tag{6.11}$$

Then $\mathcal{A}$ is orthogonal

$$\eta(\mathcal{A}\mathbb{X}, \mathcal{A}\mathbb{Y}) = \eta(\mathbb{X}, \mathbb{Y}) \tag{6.12}$$

if and only if A satisfies the orthogonality condition

$$g(AX, AY) = g(X, Y). \tag{6.13}$$

This is equivalent to

$$gAg^{-1} = (A^t)^{-1} \tag{6.14}$$

so that an orthogonal transformation $\mathcal{A}$ takes the form

$$\mathcal{A} = P_+ \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} P_+. \tag{6.15}$$

A positive chirality half generalised almost complex structure $\mathcal{J}_+$ is then an orthogonal automomorhism of $\mathbb{T}_+$ satisfying $\mathcal{J}_+^2 = -1_{\mathbb{T}_+}$. It then takes the form

$$\mathcal{J}_+ = P_+ \begin{pmatrix} J_+ & 0 \\ 0 & (J_+^t)^{-1} \end{pmatrix} P_+ \tag{6.16}$$

for an automorphism $J_+ : T \to T$ satisfying

$$(J_+)^2 = -1 \tag{6.17}$$

and

$$g(J_+X, J_+Y) = g(X, Y) \tag{6.18}$$

so that J_+ is an hermitian almost complex structure. As $(J_+)^{-1} = -J_+$, (6.16) can be rewritten as

$$\mathcal{J}_+ = P_+ \begin{pmatrix} J_+ & 0 \\ 0 & -J_+^t \end{pmatrix} P_+. \tag{6.19}$$

Similar formulae apply for $\mathbb{T}_-$. If $\mathbb{X} \in C^\infty(\mathbb{T}_-)$ then

$$\mathbb{X} = \begin{pmatrix} X \\ -gX \end{pmatrix} \tag{6.20}$$

for some $X \in C^\infty(T)$ and an automorphism of $\mathbb{T}_-$ takes the form

$$\mathcal{A} = \frac{1}{2} \begin{pmatrix} A & -Ag^{-1} \\ -gA & gAg^{-1} \end{pmatrix} = P_- \begin{pmatrix} A & 0 \\ 0 & gAg^{-1} \end{pmatrix} P_-. \tag{6.21}$$

For $\mathbb{X}, \mathbb{Y} \in C^\infty(\mathbb{T}_-)$, η is negative definite

$$-\eta(\mathbb{X}, \mathbb{Y}) = \mathcal{H}(\mathbb{X}, \mathbb{Y}) = 2g(X, Y) \tag{6.22}$$

and $\mathcal{A}$ is orthogonal if and only A is, as before. A negative chirality half generalised almost complex structure $\mathcal{J}_-$ is then

$$\mathcal{J}_- = P_- \begin{pmatrix} J_- & 0 \\ 0 & -J_-^t \end{pmatrix} P_- \tag{6.23}$$

for an hermitian almost complex structure $J_- : T \to T$.

If $\mathbb{T}$ has a positive chirality half generalised almost complex structure $\mathcal{J}_+$ and a negative chirality half generalised almost complex structure $\mathcal{J}_-$, then T has hermitian almost complex structures J_+, J_- and there are two commuting generalised almost complex structures

$$\mathcal{J}_1 = P_+ \begin{pmatrix} J_+ & 0 \\ 0 & -J_+^t \end{pmatrix} P_+ + P_- \begin{pmatrix} J_- & 0 \\ 0 & -J_-^t \end{pmatrix} P_- \tag{6.24}$$

and

$$\mathcal{J}_2 = P_+ \begin{pmatrix} J_+ & 0 \\ 0 & -J_+^t \end{pmatrix} P_+ - P_- \begin{pmatrix} J_- & 0 \\ 0 & -J_-^t \end{pmatrix} P_- \tag{6.25}$$

which can be rewritten as (4.5).

7. Integrability

An almost complex structure J on T splits the tangent bundle into $+i$ eigenspace ℓ and the $-i$ eigenspace $\bar{\ell}$

$$T \otimes \mathbb{C} = \ell \oplus \bar{\ell} \tag{7.1}$$

and can be used to define the projectors

$$\pi_{\pm} := \tfrac{1}{2}(1 \mp iJ). \tag{7.2}$$

The almost complex structure J is integrable if ℓ is involutive with respect to the Lie bracket, i.e. if

$$[X, Y] \in \ell \text{ if } X, Y \in C^{\infty}(\ell). \tag{7.3}$$

If it is integrable, then it is a complex structure on T. J is integrable if and only if the Nijenhuis tensor defined by

$$N(X, Y) = \pi_{-}([\pi_{+}X, \pi_{+}Y]) \tag{7.4}$$

vanishes for all vector fields X, Y.

From Sect. 5, a positive chirality half generalised almost complex structure $\mathcal{J}_{+}$ splits the generalised tangent space

$$\mathbb{T}_{+} \otimes \mathbb{C} = \mathbb{L}_{+} \oplus \overline{\mathbb{L}}_{+} \tag{7.5}$$

into the $+i$ eigenspace $\mathbb{L}_{+}$ and the $-i$ eigenspace $\overline{\mathbb{L}}_{+}$ and defines the projection operators $\Pi_{\mp} := \frac{1}{2}(1 \pm i\mathcal{J}_{+})$. It is a positive chirality half generalised complex structure if the subspace $\mathbb{L}_{+}$ is involutive with respect to the H-twisted Courant bracket, i.e. if

$$[\![\mathbb{X}, \mathbb{Y}]\!]_{H} \in \mathbb{L}_{+} \text{ if } \mathbb{X}, \mathbb{Y} \in C^{\infty}(\mathbb{L}_{+}). \tag{7.6}$$

This requires the vanishing of the *generalised Nijenhuis tensor*

$$\mathcal{N}(\mathbb{X}, \mathbb{Y}) = \Pi_{-}[\![\Pi_{+}\mathbb{X}, \Pi_{+}\mathbb{Y}]\!]_{H} \tag{7.7}$$

and the vanishing of

$$\mathcal{M}(\mathbb{X}, \mathbb{Y}) = P_{-}[\![\Pi_{+}\mathbb{X}, \Pi_{+}\mathbb{Y}]\!]_{H} \tag{7.8}$$

which is required for $[\![\Pi_{+}\mathbb{X}, \Pi_{+}\mathbb{Y}]\!]_{H}$ to be a section of $\mathbb{T}_{+}$.

For $\mathbb{X} \in C^{\infty}(\mathbb{T}_{+})$ with

$$\mathbb{X} = \begin{pmatrix} X \\ gX \end{pmatrix} \tag{7.9}$$

it follows from (6.19) that

$$\mathcal{J}_{+}\mathbb{X} = \begin{pmatrix} J_{+}X \\ gJ_{+}X \end{pmatrix} \tag{7.10}$$

so that the automorphism $\mathbb{X} \to \mathcal{J}_{+}\mathbb{X}$ of $\mathbb{T}_{+}$ maps to the automorphism $X \to J_{+}X$ of T. Then

$$\Pi_{\pm}\mathbb{X} = \begin{pmatrix} \pi_{\pm}X \\ g\pi_{\pm}X \end{pmatrix}. \tag{7.11}$$

We have that

$$\mathbb{X} = \begin{pmatrix} X \\ gX \end{pmatrix} \in C^\infty(\mathbb{L}_+) \quad \textit{if and only if} \quad X \in C^\infty(\ell). \tag{7.12}$$

We now turn to the form of the brackets on $\mathbb{T}_+$. For $\mathbb{X}, \mathbb{Y} \in C^\infty(\mathbb{T}_+)$ with

$$\mathbb{X} = \begin{pmatrix} X \\ gX \end{pmatrix}, \qquad \mathbb{Y} = \begin{pmatrix} Y \\ gY \end{pmatrix}, \tag{7.13}$$

the Courant bracket takes the simple form

$$[\![\mathbb{X}, \mathbb{Y}]\!]_C = \begin{pmatrix} [X, Y] \\ g[X, Y] \end{pmatrix} + \begin{pmatrix} 0 \\ S(X, Y) \end{pmatrix} \tag{7.14}$$

where $[X, Y]$ is the Lie bracket. Here the map $S : C^\infty(T \otimes T) \to C^\infty(T^*)$ is defined by

$$i_Z S(X, Y) = g(X, \nabla^{(0)}_Z Y) - g(Y, \nabla^{(0)}_Z X) \tag{7.15}$$

where $\nabla^{(0)}$ is the Levi-Civita connection. In index notation,

$$S_\mu(X, Y) = X_\nu \nabla^{(0)}_\mu Y^\nu - Y_\nu \nabla^{(0)}_\mu X^\nu. \tag{7.16}$$

For the H-twisted Courant bracket, S is replaced in these formulae by

$$S^{(+)}(X, Y) = S(X, Y) + i_X i_Y H \tag{7.17}$$

which has the effect of replacing the connection $\nabla^{(0)}$ with the connection with torsion $\nabla^{(+)}$ given in (2.1). Then

$$i_Z S^{(+)}(X, Y) = g(X, \nabla^{(+)}_Z Y) - g(Y, \nabla^{(+)}_Z X) \tag{7.18}$$

or

$$S^{(+)}_\mu(X, Y) = X_\nu \nabla^{(+)}_\mu Y^\nu - Y_\nu \nabla^{(+)}_\mu X^\nu. \tag{7.19}$$

Then the H-twisted Courant bracket takes the form

$$[\![\mathbb{X}, \mathbb{Y}]\!]_H = \begin{pmatrix} [X, Y] \\ g[X, Y] \end{pmatrix} + \begin{pmatrix} 0 \\ S^{(+)}(X, Y) \end{pmatrix}. \tag{7.20}$$

We now consider the conditions $[\![\mathbb{X}, \mathbb{Y}]\!]_H \in C^\infty(\mathbb{L}_+)$ if $\mathbb{X}, \mathbb{Y} \in C^\infty(\mathbb{L}_+)$ for $\mathcal{J}_+$ to be a positive chirality half generalised complex structure. Now from (7.11), (7.14) and (7.20);

$$[\![\Pi_+\mathbb{X}, \Pi_+\mathbb{Y}]\!]_H = \begin{pmatrix} [\pi_+X, \pi_+Y] \\ g[\pi_+X, \pi_+Y] \end{pmatrix} + \begin{pmatrix} 0 \\ S^{(+)}(\pi_+X, \pi_+Y) \end{pmatrix}. \tag{7.21}$$

This will be in $\mathbb{L}_+$ if and only if the following two conditions hold

1. $[\pi_+X, \pi_+Y] \in C^\infty(\ell)$, i.e. J_+ is a complex structure
2. $S^{(+)}(\pi_+X, \pi_+Y) = 0$ for all vector fields X, Y.

The second condition is the condition that $[\![\Pi_+\mathbb{X}, \Pi_+\mathbb{Y}]\!]_H \in C^\infty(\mathbb{T}_+)$ and is equivalent to the vanishing of $\mathcal{M}(\mathbb{X}, \mathbb{Y})$ in (7.8), as

$$\mathcal{M}(\mathbb{X}, \mathbb{Y}) = P_- \begin{pmatrix} 0 \\ S^{(+)}(X, Y) \end{pmatrix}. \tag{7.22}$$

From (7.11) and (7.19)

$$S^{(+)}_\mu(\pi_+X, \pi_+Y) = -i(\pi_+X)_\nu(\nabla^{(+)}_\mu J_+{}^\nu{}_\rho)Y^\rho + i(\pi_+Y)_\nu(\nabla^{(+)}_\mu J_+{}^\nu{}_\rho)X^\rho. \tag{7.23}$$

Then the imaginary part of

$$S^{(+)}_\mu(\pi_+X, \pi_+Y) = 0 \tag{7.24}$$

gives

$$X^\nu Y^\rho \nabla^{(+)}_\mu J_{+\,\nu\rho} = 0. \tag{7.25}$$

For this to hold for all X, Y requires that J_+ is covariantly constant with respect to $\nabla^{(+)}$

$$\nabla^{(+)}_\mu J_{+\,\nu\rho} = 0. \tag{7.26}$$

If this holds, then it follows from (7.7) and (7.21) that the generalised Nijenhuis tensor is given in terms of the Nijenhuis tensor

$$\mathcal{N}(\mathbb{X}, \mathbb{Y}) = \begin{pmatrix} N(X, Y) \\ gN(X, Y) \end{pmatrix} \tag{7.27}$$

and this will vanish if and only if $N(X, Y) = 0$.

We then have the result that

Proposition. *A positive chirality half generalised complex structure $\mathcal{J}_+$ is equivalent to a hermitian complex structure J_+ that is covariantly constant with respect to $\nabla^{(+)}$, $\nabla^{(+)}J_+ = 0$.*

Similar arguments lead to the result that a negative chirality half generalised complex structure $\mathcal{J}_-$ is equivalent to a hermitian complex structure J_- that is covariantly constant with respect to $\nabla^{(-)}$, $\nabla^{(-)}J_- = 0$.

To see this, note that for $\mathbb{X}, \mathbb{Y} \in C^\infty(\mathbb{T}_-)$ with

$$\mathbb{X} = \begin{pmatrix} X \\ -gX \end{pmatrix}, \qquad \mathbb{Y} = \begin{pmatrix} Y \\ -gY \end{pmatrix} \tag{7.28}$$

the Courant bracket takes the simple form

$$[\![\mathbb{X}, \mathbb{Y}]\!]_C = \begin{pmatrix} [X, Y] \\ -g[X, Y] \end{pmatrix} - \begin{pmatrix} 0 \\ S(X, Y) \end{pmatrix} \tag{7.29}$$

so that

$$[\![\mathbb{X}, \mathbb{Y}]\!]_H = \begin{pmatrix} [X, Y] \\ -g[X, Y] \end{pmatrix} - \begin{pmatrix} 0 \\ S^{(-)}(X, Y) \end{pmatrix} \tag{7.30}$$

where

$$S^{(-)}(X, Y) = S(X, Y) - i_X i_Y H \tag{7.31}$$

so that

$$S^{(-)}_\mu(X, Y) = X_\nu \nabla^{(-)}_\mu Y^\nu - Y_\nu \nabla^{(-)}_\mu X^\nu. \tag{7.32}$$

Then this leads to the condition $\nabla^{(-)}J_- = 0$.

8. Strong Kähler with Torsion and Half Generalised Complex Structures

We can now give the formulation of Strong Kähler with Torsion geometry in terms of generalised complex geometry. Using the results of the previous sections, we have that a (2, 1) SKT geometry $(\mathcal{M}, g, H, J_+)$ is equivalent to a positive chirality half generalised complex structure $\mathcal{J}_+$ and metric structure $\mathcal{G}$ on $\mathbb{T}$. The analogue of the Gualtieri map is

$$\mathcal{J}_+ = P_+ \begin{pmatrix} J_+ & 0 \\ 0 & -J_+^t \end{pmatrix} P_+ \tag{8.1}$$

which can be written as

$$\mathcal{J}_+ = \tfrac{1}{2} \begin{pmatrix} J_+ & -(\omega_+)^{-1} \\ \omega_+ & -J_+^t \end{pmatrix} \tag{8.2}$$

where $\omega_+ = gJ_+$. Similarly, a (1, 2) SKT geometry $(\mathcal{M}, g, H, J_-)$ is precisely a negative chirality half generalised complex structure $\mathcal{J}_-$ and metric structure $\mathcal{G}$ on $\mathbb{T}$, with

$$\mathcal{J}_- = \tfrac{1}{2} \begin{pmatrix} J_- & -(\omega_-)^{-1} \\ \omega_- & -J_-^t \end{pmatrix} = P_- \begin{pmatrix} J_- & 0 \\ 0 & -J_-^t \end{pmatrix} P_- . \tag{8.3}$$

9. (p, q) Generalised Complex Geometry

From Sect. 2, a (p, q) hermitian geometry $(\mathcal{M}, g, H, J_+^a, J_-^{a'})$ has $p-1$ complex structures J_+^a $(a = 1, \ldots p-1)$ and $q-1$ complex structures $J_-^{a'}$ $(a' = 1, \ldots q-1)$ and for each complex structure J, $(\mathcal{M}, g, H, J)$ is an SKT space. Then each complex structure corresponds to a half generalised complex structure, with $p-1$ half positive chirality generalised complex structures $\mathcal{J}_+^a$ on $\mathbb{T}_+$, and $q-1$ negative chirality half generalised complex structures $\mathcal{J}_-^{a'}$ on $\mathbb{T}_-$. This motivates the definition of a (p, q) generalised complex geometry $(\mathbb{T}, \mathcal{G}, H, \mathcal{J}_+^a, \mathcal{J}_-^{a'})$ as

1. The generalised tangent bundle $\mathbb{T}$ with metric $\mathcal{G}$.
2. $\mathbb{T}$ has $p-1$ positive chirality half generalised complex structures $\mathcal{J}_+^a$ and $q-1$ negative chirality half generalised complex structures $\mathcal{J}_-^{a'}$ that are each integrable with respect to the H-twisted Courant bracket.
3. If $p = 4$, the $\mathcal{J}_+^a$ satisfy the quaternion algebra, and if $q = 4$ the $\mathcal{J}_-^{a'}$ satisfy the quaternion algebra.

The maps $\rho_\pm : \mathbb{T}_\pm \to T$ take each half generalised complex structure $\mathcal{J}_\pm$ on $\mathbb{T}_\pm$ to a complex structure $J_\pm$ on T. This can be made explicit as:

$$\begin{aligned} \mathcal{J}_+^a &= \tfrac{1}{2} \begin{pmatrix} J & -(\omega)^{-1} \\ \omega & -J^t \end{pmatrix}_+^a \\ \mathcal{J}_-^{a'} &= \tfrac{1}{2} \begin{pmatrix} J & -(\omega)^{-1} \\ \omega & -J^t \end{pmatrix}_-^{a'} . \end{aligned} \tag{9.1}$$

This then gives us a precise correspondence between a (p, q) generalised complex geometry $(\mathbb{T}, \mathcal{G}, H, \mathcal{J}_+^a, \mathcal{J}_-^{a'})$ and a (p, q) hermitian geometry $(\mathcal{M}, g, H, J_+^a, J_-^{a'})$.

If $p \geq 2$ and $q \geq 2$, any positive chirality half generalised complex structure can be combined with any negative chirality one to form a generalised complex structure, so that we have a basis of $(p-1)(q-1)$ generalised complex structures

$$\mathcal{J}^{ab'} = (\mathcal{J}^a_+, \mathcal{J}^{b'}_-) \tag{9.2}$$

together with a further set of generalised complex structures $\mathcal{G}\mathcal{J}^{ab'}$. This gives a generalised Kähler structure for $p = q = 2$, 3 generalised complex structures $\mathcal{J}^{a1'}$ for $(p, q) = (4, 2)$ and 9 generalised complex structures $\mathcal{J}^{ab'}$ for $(p, q) = (4, 4)$. For $(p, q) = (4, 2)$, the space of generalised complex structures is a 2-sphere while for $(p, q) = (4, 4)$ it is $S^2 \times S^2$. The case of 3 generalised complex structures satisfying a quaternion algebra is a generalised hyperkähler structure [9].

10. Discussion

The (p, q) hermitian geometries are characterised by the holonomy groups $Hol(\nabla^{(\pm)})$ of the connections with torsion $\nabla^{(\pm)}$. If the dimension d of the manifold is even, then $p \geq 2$ if $Hol(\nabla^{(+)}) \subseteq U(d/2)$ and $q \geq 2$ if $Hol(\nabla^{(-)}) \subseteq U(d/2)$, while if d is a multiple of 4, then $p \geq 4$ if $Hol(\nabla^{(+)}) \subseteq Sp(d/4)$ and $q \geq 4$ if $Hol(\nabla^{(-)}) \subseteq Sp(d/4)$. The cases $p > 4$ or $q > 4$ are only possible with trivial holonomy.

There are some features of (p, q) hermitian geometries that are particular to certain values of p and q. One interesting aspect of $(2, 2)$ hermitian geometry, i.e., of generalised Kähler geometry, is that there are three Poisson structures on T,

$$\sigma_\pm := (J_+ \pm J_-)g^{-1}, \quad \sigma := [J_+, J_-]g^{-1}, \tag{10.1}$$

which can have irregular points that form loci in $\mathcal{M}$ where the Poisson structures change rank, giving what has been called type change in [3]. This implies that for $(4, 2)$ hermitian geometry there are then three sets of three Poisson structures

$$\sigma^a_\pm := (J^a_+ \pm J_-)g^{-1}, \quad \sigma^a := [J^a_+, J_-]g^{-1}, \tag{10.2}$$

while for $(4, 4)$ hermitian geometry there are then nine sets of three Poisson structures

$$\sigma^{ab'}_\pm := (J^a_+ \pm J^{b'}_-)g^{-1}, \quad \sigma^{ab'} := [J^a_+, J^{b'}_-]g^{-1}, \tag{10.3}$$

Another interesting feature of generalised Kähler geometry is that it has a generalised potential K that determines all geometric quantities. For commuting complex structures,

$$[J_+, J_-] = 0, \tag{10.4}$$

the expressions for the metric and b-field are linear in the 2nd derivatives of the scalar function K [1], while they are non linear when the complex structures do not commute [13–15]. A proof of this relies on the superspace formulation of the sigma model away from irregular points. An alternative proof of the existence of K for a class of GKGs that *does* include irregular points was recently given in [16]. For (p, q) models with $p, q \geq 2$, the scalar potential governing the geometry is further constrained to allow for enhanced supersymmetry. In certain classes of models for which the supersymmetry algebra closes without use of equations of motion (i.e. off-shell), the general model can be found explicitly, giving a general construction of the corresponding potential and hence the local geometry [1,17,18].

For SKT geometry there is only one complex structure and so no Poisson structures or irregular points. The potential that determines the geometry again follows from the sigma model and is a complex vector potential $(k_\alpha, k_{\bar{\alpha}})$ [4,5]. For recent discussions of such (2, 0) and (2, 1) models, see [17,18]. For (4, 1) hermitian geometry (hyperkähler with torsion) the vector potential is further constrained, and has been studied in [17,18].

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References

1. Gates Jr., S.J., Hull, C.M., Roček, M.: Twisted multiplets and new supersymmetric nonlinear sigma models. Nucl. Phys. B **248**, 157 (1984)
2. Hitchin, N.: Generalized Calabi–Yau manifolds. Q. J. Math. Oxf. Ser. **54**, 271 (2003). arXiv:math/0209099 [math-dg]
3. Gualtieri, M.: Generalized complex geometry. arXiv:math/0401221 [math-dg]
4. Hull, C.M., Witten, E.: Supersymmetric sigma models and the heterotic string. Phys. Lett. B **160**, 398 (1985)
5. Hull, C.M.: σ model beta functions and string compactifications. Nucl. Phys. B **267**, 266 (1986)
6. Cavalcanti, G.R.: Reduction of metric structures on Courant algebroids. arXiv:1203.0497v1 [math.DG] (2012)
7. Cavalcanti, G.R.: Hodge theory and deformations of SKT manifolds. arXiv:1203.0493 [math.DG]
8. Hull, C.M.: Lectures on Nonlinear Sigma Models and Strings. Lectures Given at the Vancouver Advanced Research Workshop, published in *Super Field Theories* (Plenum, New York), edited by H. Lee and G. Kunstatter (1988)
9. Bursztyn, H., Cavalcanti, G.R., Gualtieri, M.: "Generalized Kähler and hyper-Kähler quotients" Poisson geometry in mathematics and physics. Contemp. Math. **450**, 61–77, American Mathematical Society, Providence, RI (2008) arXiv:math/0702104
10. Howe, P.S., Papadopoulos, G.: Twistor spaces for HKT manifolds. Phys. Lett. B **379**, 80 (1996). arXiv:hep-th/9602108
11. Buscher, T., Lindström, U., Roček, M.: New supersymmetric σ models with Wess–Zumino terms. Phys. Lett. B **202**, 94 (1988)
12. Lindström, U., Roček, M., von Unge, R., Zabzine, M.: Generalized Kähler manifolds and off-shell supersymmetry. Commun. Math. Phys. **269**, 833 (2007). arXiv:hep-th/0512164
13. Lindström, U., Roček, M., von Unge, R., Zabzine, M.: Linearizing generalized Kähler geometry. JHEP **0704**, 061 (2007). arXiv:hep-th/0702126
14. Lindström, U., Roček, M., von Unge, R., Zabzine, M.: A potential for generalized Kähler geometry. IRMA Lect. Math. Theor. Phys. **16**, 263 (2010). arXiv:hep-th/0703111
15. Bredthauer, A., Lindström, U., Persson, J., Zabzine, M.: Generalized Kähler geometry from supersymmetric sigma models. Lett. Math. Phys. **77**, 291 (2006). arXiv:hep-th/0603130
16. Bischoff, F., Gualtieri, M., Zabzine, M.: Morita equivalence and the generalized Kähler potential, arXiv:1804.05412 [math.DG]
17. Hull, C., Lindström, U.: All (4, 1): sigma models with (4, q) off-shell supersymmetry. JHEP **1703**, 042 (2017). arXiv:1611.09884 [hep-th]
18. Hull, C., Lindström, U.: All (4, 0): sigma models with (4, 0) off-shell supersymmetry. JHEP **1708**, 129 (2017). arXiv:1707.01918 [hep-th]

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