



# Local Nahm Transform and Singularity Formation of ASD Connections

Yang Li 

Department of Mathematics, Imperial College London, 180 Queen's Gate, Kensington, London, UK.  
E-mail: [y121715@ic.ac.uk](mailto:y121715@ic.ac.uk)

Received: 21 November 2018 / Accepted: 6 April 2019  
Published online: 8 June 2019 – © The Author(s) 2019

**Abstract:** This paper develops a local analogue of the ADHM construction, which characterises ASD instantons defined over smooth bounded domains inside Euclidean  $\mathbb{R}^4$  diffeomorphic to the 4-ball, in terms of infinite dimensional Hilbert spaces and bounded Hermitian linear operators satisfying an analogue of the ADHM equation. Moreover, we describe the degeneration of this construction when a family of instantons develops a curvature singularity at the origin.

## 1. Introduction

This paper studies two related problems for ASD connections  $A$  defined on a bounded region  $B \subset \mathbb{R}^4$  with the standard Euclidean metric: **local Nahm transform**, and **singularity formation**. The unifying theme is to understand ASD connections by studying solutions to the coupled Dirac equation (*i.e.* ‘Dirac fields’).

The **Nahm transform** in various contexts is extensively studied [2, 4, 5, 10]. It resembles the Fourier transform in many ways, and is also closely related to the celebrated ADHM construction [7, 10]. The prototype Nahm transform [4] constructs ASD connections over a given 4-torus, starting from coupled Dirac fields attached to ASD connections over the dual torus. It enjoys remarkable properties, such as

- ASD connections without flat factors are preserved under the transform.
- The inverse Nahm transform reproduces the original ASD connection.

It is desirable to establish a local version of Nahm transform; the main motivations are:

---

Y.L. is supported by the Engineering and Physical Sciences Research Council [EP/L015234/1], the EPSRC Centre for Doctoral Training in Geometry and Number Theory (The London School of Geometry and Number Theory), University College London. The author is also funded by Imperial College London for his Ph.D. studies.

- From the viewpoint of *quantum mechanics*, it is natural to expect that gauge fields (*i.e.* connections) can be reconstructed up to gauge equivalence from the information about its interaction with fermions (*i.e.* Dirac fields). Such a description will be *gauge invariant*.
- The local Nahm transform can be recast as an ADHM type construction, which relates ASD instantons on domains to *operator algebras*.
- In algebraic geometry, gauge theory on holomorphic vector bundles is usefully studied via the *sheaf of holomorphic sections*. In the absence of a (preferred) complex structure, we advocate Dirac fields as a substitute of holomorphic sections, and the inverse Nahm transform below presents the vector bundle with ASD connection as a kernel bundle, in a spirit analogous to presentations of coherent sheaves. This idea may be of relevance in higher dimensional gauge theory where there is no complex structure in sight.
- Some motivations from the *twistor space* perspective are discussed in [15].

Our main result in this direction, which borrows techniques from [4], is

**Theorem 1.1** (Local Nahm transform, *cf.* Chapter 2 and 3). *Let  $B$  be a bounded region inside the Euclidean  $\mathbb{R}^4$ , such that  $\bar{B}$  is diffeomorphic to the unit 4-ball with boundary. For a smooth ASD connection on a Hermitian vector bundle  $(E, A)$  over  $\bar{B}$ , there is a bundle of Hilbert spaces with an ASD connection  $(\hat{E}, \hat{A})$  over the dual vector space  $\hat{\mathbb{R}}^4$ , called the local Nahm transform. Moreover, the inverse Nahm transform is canonically isomorphic to  $(E, A)$ .*

The local Nahm transform admits an alternative description analogous to the [10]. The information of the bundle  $\hat{E}$  is equivalent to the Hilbert space of solutions to the coupled Dirac equations

$$H_D^2 = \{s \in L^2(B, E \otimes S_-) : D_A^- s = 0\},$$

which we call the ‘**Bergmann space**’. The information of the connection  $\hat{A}$  is equivalent to 4 bounded Hermitian operators on  $H_D^2$ , defined by

$$\hat{x}_\mu = P_0 \circ x_\mu, \quad \mu = 1, 2, 3, 4,$$

where  $x_\mu$  means multiplication by the coordinate function, and  $P_0$  is the orthogonal projection from  $L^2(B, E \otimes S_-)$  to  $H_D^2$ . One can alternatively think of  $\hat{x}_\mu$  as Hermitian bilinear forms  $\langle s | x_\mu | s' \rangle$  on  $H_D^2$ , in analogy with observables in quantum mechanics. The ASD condition on  $\hat{A}$  is equivalent to

$$\begin{cases} [\hat{x}_1, \hat{x}_2] + [\hat{x}_3, \hat{x}_4] = 0, \\ [\hat{x}_1, \hat{x}_3] + [\hat{x}_4, \hat{x}_2] = 0, \\ [\hat{x}_1, \hat{x}_4] + [\hat{x}_2, \hat{x}_3] = 0, \end{cases}$$

analogous to the **ADHM equation**. Thus  $(\hat{E}, \hat{A})$  is analogous to the ADHM data, albeit  $H_D^2$  is infinite dimensional, which brings forth some nuanced functional analytic features.

The inverse Nahm transform can also be equivalently described in this ADHM language. Given the ADHM data as above, the inverse Nahm transform bundle  $\hat{E}$  is the subbundle of the trivial bundle  $H_D^2 \otimes S_-$ , defined as

$$\hat{E}_y = \ker[H_D^2 \otimes S_- \xrightarrow{-2\pi i \sum (\hat{x}_\mu - y_\mu) \hat{c}_\mu} H_D^2 \otimes S_+], \quad \forall y \in B,$$

and we equip  $\hat{E} \subset H_D^2 \otimes S_-$  with the subbundle connection  $\hat{A}$ , which turns out to be again ASD. The main content of Theorem 1.1 is that  $(\hat{E}, \hat{A})$  reconstructs  $(E, A)$ . Thus we have related ASD connections on domains to operator theory. M. Atiyah suggests to the author that this picture may be related to Alain Connes’s noncommutative geometry [8].

The second problem studied in this paper is **singularity formation**.

**Problem 1.2.** Given a 1-parameter family  $\{A_t\}_{t>0}$  of smooth ASD connections  $A_t$  with uniformly bounded  $L^2$  curvature, on the Hermitian bundle  $E$  over the closed ball  $\bar{B} = \overline{B(R)}$ , such that as  $t \rightarrow \infty$ , the connections converge smoothly away from the origin to  $A_\infty$ , and near the origin they are allowed to develop a curvature singularity. Describe the limiting behaviour of the Bergmann spaces  $H_{D_{A_t}}^2$  of Dirac fields for  $A_t$  as  $t \rightarrow \infty$ .

In the quantum mechanics analogy, this means studying how fermions interact with gauge fields with concentrated curvature. In the algebraic geometry analogy, this situation is akin to a flat family of holomorphic vector bundles degenerating into a coherent sheaf. We wish to capture curvature singularity by the Dirac fields, in the same spirit that singularities of Hermitian Yang-Mills connections are captured by the coherent sheaf of holomorphic sections.

We develop a convergence theory in Chapters 4 and 5 (which can be read independent of Chapters 2 and 3). The Hamiltonian of the harmonic oscillator

$$H(s) = \frac{1}{2R^2} \int_{B(R)} |x|^2 |s|^2$$

on  $H_{D_{A_t}}^2$  induces a Hermitian operator with discrete and non-negative spectrum. The **spectral theory** is our key tool to understand the spaces  $H_{D_{A_t}}^2$ . There is a positive number called the spectral gap, such that for sufficiently concentrated ASD connections, the eigenvalues are either greater than the gap, which form the so called large spectrum, or very close to zero, which form the so called small spectrum, for which the eigenstates are localised near the origin, much like a potential well in quantum mechanics.

The dimension of the small spectrum is constant for large  $t$ . To describe the limiting behaviour of the large spectrum, we recall that Uhlenbeck’s removable singularity theorem allows us to smoothly extend the ASD connection  $A_\infty$  across the origin to a smooth connection on a different bundle  $\tilde{E}$  over  $B$ . Then the large spectrum for  $A_t$  converges in a natural sense to the spectrum of  $A_\infty$ . Moreover,

**Theorem 1.3** (Convergence theory). *There is a natural topological bundle over  $0 < t \leq \infty$ , whose fibres over  $0 < t < \infty$  are  $H_{D_{A_t}}^2$  and the fibre over  $\infty$  is the orthogonal direct sum  $H_{D_{A_\infty}}^2 \oplus V$ , where  $V$  has the dimension of the small spectrum. Moreover, certain natural operators on the spaces  $H_{D_{A_t}}^2$  can be extended continuously to the limit  $H_{D_{A_\infty}}^2 \oplus V$ .*

An important quantity describing the curvature singularity formation is the **instanton number**, i.e. the amount of energy loss

$$k = \lim_{t \rightarrow \infty} \frac{1}{8\pi^2} \int_B |F_{A_t}|^2 - \frac{1}{8\pi^2} \int_B |F_{A_\infty}|^2. \tag{1}$$

This is well known to have a more topological interpretation: for any  $0 < r \leq R$ ,

$$k = \frac{1}{8\pi^2} \int_{B(r)} \text{Tr}(F_{A_t} \wedge F_{A_t}) - \int_{\partial B(r)} CS(A_t) \tag{2}$$

where  $CS(A) = \frac{1}{8\pi^2} \text{Tr}(dA \wedge A + \frac{2}{3}A \wedge A \wedge A)$  is the Chern Simons form, defined up to an integer, and becomes uniquely defined by demanding continuity at  $t = \infty$ :

$$\lim_{t \rightarrow \infty} \int_{\partial B(r)} CS(A_t) = \int_{\partial B(r)} CS(A_\infty) = \frac{1}{8\pi^2} \int_{B(r)} \text{Tr}(F_{A_\infty} \wedge F_{A_\infty}).$$

We combine our theory of the local Nahm transform and our analytic theory of singularity formation, together with some rudiments of K-theory, to show

**Theorem 1.4** (cf. Section 6.2). *The instanton number is equal to the dimension of the small spectrum  $\dim V$ .*

In the standard Uhlenbeck compactification of ASD instanton moduli space [10], one formally puts in ideal instantons, encoding the amount of energy concentrating to points, and the smooth limiting connection after removal of singularity. Our theorems say that both information is captured by the ADHM type data.

*Remark.* In the Nahm transform on  $T^4$  (cf. [4]), if one starts from an ASD connection with flat factors on  $E$ , then the transformed bundle  $\hat{E}$  on  $\hat{T}^4$  will have a local curvature singularity. By the family Atiyah-Singer formula,  $\text{rank}(E)$  (resp.  $c_2(E)$ ) contributes to  $c_2(\hat{E})$  (resp.  $\text{rank}(\hat{E})$ ) of the transformed bundle. In our case we start from an ASD connection with local curvature singularity on a domain, and the local Nahm transform acquires a flat factor corresponding to  $V$ . The instanton number is an analogue of the second Chern class, and  $\dim(V)$  is an analogue of rank.

## 2. The Local Nahm Transform

We define the local Nahm transform, and explain two different perspectives, which exhibit its analogy with the Nahm transform on the 4-torus and the ADHM construction of instantons on  $\mathbb{R}^4$ .

**2.1. Nahm transform perspective.** Suppose we are given a Hermitian bundle  $E$  with a smooth ASD connection  $A$  over the closure of a bounded connected domain  $B \subset \mathbb{R}^4$  with smooth boundary, then we can produce an ASD connection on some (infinite rank) Hilbert bundle  $\hat{E} \rightarrow \hat{\mathbb{R}}^4$  over the dual vector space  $\hat{\mathbb{R}}^4$  as follows. First, we recall the **Poincaré bundle**  $\mathcal{P}$  over  $\mathbb{R}^4 \times \hat{\mathbb{R}}^4$ , given by equipping the trivial line bundle with the connection

$$\omega = 2\pi i \sum_{\mu=1}^4 z_\mu dx_\mu, \tag{3}$$

where  $x_\mu$  are the standard coordinates on  $\mathbb{R}^4$  and  $z_\mu$  are the dual coordinates. For each fixed  $z \in \hat{\mathbb{R}}^4$ , the bundle  $E$  couples to the flat line bundle  $\mathcal{P}|_z$ . Equivalently, one thinks of the coupled connection as a modified connection on  $E$

$$A_z = A + 2\pi i \sum_{\mu=1}^4 z_\mu dx_\mu. \tag{4}$$

These  $A_z$  are all gauge equivalent, but we keep them separate to make apparent the analogy with the Nahm transform on the 4-torus [4]. We introduce the **coupled Dirac operators**:

$$\begin{cases} D_{A_z}^+ : W_0^{1,2}(B, E \otimes S_+) \rightarrow L^2(B, E \otimes S_-), \\ D_{A_z}^- : L^2(B, E \otimes S_-) \rightarrow W^{-1,2}(B, E \otimes S_+). \end{cases} \tag{5}$$

The Sobolev space  $W_0^{1,2}$  imposes a zero Dirichlet boundary condition. The convention in this paper is that the Clifford operators are anti-self-adjoint, with Clifford relations

$$c_\mu c_\nu + c_\nu c_\mu = -2\delta_{\mu\nu},$$

so the Dirac operator is formally self-adjoint.

The Weitzenböck formula implies that  $\ker D_{A_z}^+$  is zero, since parallel coupled spinors with zero boundary condition must vanish. We define a bundle  $\hat{E}$  over  $\mathbb{R}^4$  with fibres given by all the  $L^2$  solutions to the Dirac equation

$$\hat{E}_z = \ker D_{A_z}^-. \tag{6}$$

This sits inside the trivial Hilbert bundle over  $\hat{\mathbb{R}}^4$  with fibre  $L^2(B, E \otimes S_-)$ . Let  $P_z$  be the projections to the kernels according to the orthogonal splitting

$$L^2(B, E \otimes S_-) = \hat{E}_z \oplus D_{A_z}^+ W_0^{1,2}. \tag{7}$$

The projection operator  $P_z$  admits the formula

$$P_z = 1 - D_{A_z}^+ G_z D_{A_z}^-, \tag{8}$$

where  $G_z$  is the **Green’s operator** solving the Dirichlet problem for the coupled Laplacian

$$G_z = (\nabla_{A_z}^* \nabla_{A_z})^{-1} : W^{-1,2}(B, E) \rightarrow W_0^{1,2}(B, E).$$

Then  $\hat{E}$  inherits the subbundle connection  $\hat{A}$ , with covariant derivative given by  $\hat{\nabla} = P_z \hat{d}$ , where  $\hat{d}$  is the trivial connection.

**Definition 2.1.** The **local Nahm transform** of the pair  $(E, A)$  is the Hilbert bundle with a connection  $(\hat{E}, \hat{A})$  over  $\hat{\mathbb{R}}^4$ .

**Proposition 2.2.** *The local Nahm transform remains ASD.*

*Proof.* (Sketch) The same proof in [4], Theorem 1.5 works (but their curvature formula is off by a sign). This goes by taking a local orthonormal framing of  $\hat{E}$ , denoted  $\hat{f}^j(z) = \psi_z^j(x)$ . Then one computes the connection coefficients

$$\hat{A}_{jk} = \langle \hat{f}^j, \hat{d}\hat{f}^k \rangle, \tag{9}$$

where the inner product comes from  $L^2$  integration. The curvature components are

$$\hat{F}_{ij} = \hat{d}\hat{A}_{ij} + \sum_k \hat{A}_{ik} \wedge \hat{A}_{kj} = \langle \Omega \wedge \Omega \cdot \psi_z^i, G_z \psi_z^j \rangle, \tag{10}$$

where

$$\Omega \wedge \Omega \cdot \psi_z^i = (2\pi)^2 \sum_{\mu, \nu} dz_\mu \wedge dz_\nu c_\mu c_\nu \psi_z^i.$$

The curvature formula shows the connection  $\hat{A}$  to be ASD.  $\square$

*2.2. ADHM perspective.* The local Nahm transform can also be cast in a form which makes apparent the analogy with the ADHM data [10].

By gauge equivalence, it is easy to see  $\hat{E}_z = \exp(-2\pi iz)\hat{E}_0$ . This means we can alternatively think about the underlying bundle of  $\hat{E}$  as the trivial bundle with fibre isomorphic to  $H_D^2 = \hat{E}_0 = \ker D_A^-$ ; this assigns a **canonical trivialisation** to  $\hat{E}$ . Notice that for  $s \in H_D^2$ ,

$$\begin{aligned} P_z \hat{d}(\exp(-2\pi iz)s) &= -2\pi i P_z(x_\mu dz_\mu \exp(-2\pi iz)s) \\ &= -2\pi i \exp(-2\pi iz) P_0(x_\mu s) dz_\mu, \end{aligned}$$

so after identifying all fibres with  $H_D^2$ , the information of the connection  $\hat{A}$  is up to some constant given by the operators  $\hat{x}_\mu = P_0 \circ x_\mu$ , defined in essentially the same way as the matrices appearing in the **ADHM data**, except that these operators act on infinite dimensional spaces.

**Lemma 2.3** (Analogue of the ADHM equation). *The ASD condition on  $\hat{A}$  is equivalent to*

$$\begin{cases} [\hat{x}_1, \hat{x}_2] + [\hat{x}_3, \hat{x}_4] = 0 \\ [\hat{x}_1, \hat{x}_3] + [\hat{x}_4, \hat{x}_2] = 0 \\ [\hat{x}_1, \hat{x}_4] + [\hat{x}_2, \hat{x}_3] = 0. \end{cases} \tag{11}$$

*Proof.* By the above discussions, the connection matrix of  $\hat{A}$  in the canonical trivialisation is the constant matrix valued 1-form  $\hat{A} = -2\pi i \sum_\mu \hat{x}_\mu dz_\mu$  on  $\hat{\mathbb{R}}^4$ . Thus the curvature  $\hat{F} = \hat{A} \wedge \hat{A}$ , and the result is clear after expanding this.  $\square$

*Remark.* A useful intuition when one tackles analytical questions concerning the space  $H_D^2$  of solutions to the coupled Dirac equation, is the analogy between the Dirac equation and the Cauchy-Riemann equation. The space of  $L^2$  solutions to the Cauchy-Riemann equation is sometimes called the **Bergmann space**, which will be our terminology for  $H_D^2$ .

*Remark.* In the next Chapter we will be concerned with reconstructing the original ASD connection from these ADHM data. We give some physical intuitions here about why such reconstructions are possible. The connection  $A$  can be thought of as some gauge field, and physically this is detected by observing the motion of fermions inside the gauge field. The space  $H_D^2$ , i.e. solutions to the coupled Dirac equation, can be viewed as the Hilbert space for the fermion, and operators such as  $P_0 \circ x_\mu$  are the natural ingredients to specify the quantum mechanics for the fermion.

### 3. The Inverse Construction

We show how to reconstruct the original ASD connection from its local Nahm transform. We proceed by defining the inverse Nahm transform, interpret it in the frameworks of both the Nahm transform and the ADHM construction, and then define a canonical comparison map between the original bundle and the inverse Nahm transform bundle, which turns out to preserve both the Hermitian structure and the connection matrix.

*3.1. Inverse Nahm transform.* We wish to reconstruct the original bundle by an analogue of the inverse Nahm transform in [4]. This suggests us to look at solutions to the Dirac equation coupled to the infinite rank bundle  $\hat{E}$ . To set up, we take the dual Poincaré bundle  $\hat{\mathcal{P}}$  over  $\mathbb{R}^4 \times \mathbb{R}^4$ , with connection

$$\hat{\omega} = 2\pi i \sum_{\mu=1}^4 x_\mu dz_\mu,$$

and for every  $x \in \mathbb{R}^4$ , we couple  $\hat{E}$  to  $\hat{\mathcal{P}}|_x$ . Alternatively, one thinks of the coupled connection as  $\hat{A}_x = \hat{A} + 2\pi i \sum_{\mu=1}^4 x_\mu dz_\mu$ . Here the connection  $\hat{A}$  on  $\hat{E} \rightarrow \mathbb{R}^4$  is the subbundle connection by embedding  $\hat{E}$  into the trivial bundle with fibre  $L^2(B, E \otimes S_-)$ . We identify the spinors on  $\mathbb{R}^4$  with those on  $\mathbb{R}^4$ .

The sections of  $\hat{E}$  which are constant under the canonical trivialisation  $\hat{E} \simeq H_D^2$  are called **invariant**. One can think of this condition as the analogue of the periodicity condition over the dual 4-torus, when the size of the dual torus has shrunk to zero. It is clear that the covariant derivative on  $\hat{E}$  preserves the invariance condition.

Then we define the fibres of the **inverse Nahm transform bundle**  $\hat{\hat{E}}$  over  $x \in \mathbb{R}^4$  to be the space of invariant solutions to the coupled Dirac operators  $\hat{D}_{\hat{A}_x}^-$ . A priori the inverse Nahm transform bundle may have singularities. An invariant section  $s$  in  $\Gamma(\mathbb{R}^4, \hat{\hat{E}} \otimes S_-)$  has constant modulus for all fibres  $z \in \mathbb{R}^4$ , so the Hermitian metric on any fibre can be taken to give a norm on  $s$ . This equips  $\hat{\hat{E}}$  with a Hermitian structure. There is also a natural connection on  $\hat{\hat{E}}$ , given by the projection of the natural differentiation in the  $x$  variable, where we view a section of  $\hat{\hat{E}}$  as a function from  $\mathbb{R}^4$  to the invariant part of  $\Gamma(\mathbb{R}^4, \hat{\hat{E}} \otimes S_-)$ .

3.2. *A formal comparison with ADHM construction.* The Nahm transform formulation emphasizes the analogy with the torus case, and retains as much symmetry between the Nahm transform with its inverse as is possible. We can also view the inverse Nahm transform from the perspective of the ADHM construction. The dictionary for this interpretation, is that the invariant quantities on  $\widehat{\mathbb{R}}^4$  are replaced by the information on the zero fibre. The correspondence is:

- The space of invariant sections of  $\widehat{E}$  corresponds to  $H_D^2$ .
- The connection  $\widehat{A}$  correspond to four operators on  $H_D^2$ : the covariant derivatives  $\widehat{\nabla}^\mu$  become  $-2\pi i \widehat{x}_\mu = -2\pi i P_0 \circ x_\mu$ , the projection of the multiplication by  $x_\mu$ .
- The fibre of the inverse Nahm transform  $\widehat{E}_y$  becomes

$$\ker[H_D^2 \otimes S_- \xrightarrow{-2\pi i \sum (\widehat{x}_\mu - y_\mu) \widehat{c}_\mu} H_D^2 \otimes S_+].$$

- The Hermitian metric on the inverse Nahm transform bundle is always inherited from the metric on  $H_D^2$ .
- The connection on the inverse Nahm transform is the subbundle connection for  $\widehat{E} \subset H_D^2 \otimes S_-$ , where  $H_D^2 \otimes S_-$  is thought of as the trivial bundle over  $B$ .

**Proposition 3.1** (Analogue of ADHM construction). *Given the Hilbert space  $H_D^2$  and four bounded Hermitian operators  $\widehat{x}_\mu$  on  $H_D^2$  satisfying the ADHM type equation (11). Assume*

$$\widehat{\Delta}_{\widehat{A}_y} = \widehat{\nabla}_{\widehat{A}_y}^* \widehat{\nabla}_{\widehat{A}_y} = 4\pi^2 \sum_{\mu} (\widehat{x}_\mu - y_\mu)^2, \quad \forall y \in B$$

is invertible, then  $\widehat{E}$  is a nonsingular Hermitian bundle over  $B$ , and the subbundle connection  $\widehat{A}$  on  $\widehat{E} \subset H_D^2 \otimes S_-$  is ASD.

For a sketch proof of this fact in the Nahm transform language, see Corollary 3.9 below. The Proposition does not depend on knowing that the ADHM data arise from  $(E, A)$ , although the invertibility of  $\widehat{\Delta}_{\widehat{A}_y}$  turns out to be automatic in that situation.

3.3. *Construction of Dirac fields.* We proceed as in [4]. We aim to produce coupled Dirac fields for  $\widehat{A}$  using the Green’s operator  $G_z$  in order to define a canonical comparison map  $E \rightarrow \widehat{E}$ .

Notice there is a tautological element  $\Psi \in \Gamma(B \times \widehat{\mathbb{R}}^4, \widehat{E}^* \otimes E \otimes S_-)$ , given by evaluating any element of  $\widehat{E}$  on  $E \otimes S_-$ . If we further compose with the Green’s operator  $G_z$ , we obtain a section

$$G\Psi = \sum_{j=1}^{\infty} (G_z \psi_z^j) \widehat{f}^{j*} \in \Gamma(B \times \widehat{\mathbb{R}}^4, \widehat{E}^* \otimes E \otimes S_-),$$

where  $\widehat{f}^{j*}$  stands for the dual basis of  $\widehat{f}^j = \psi_z^j$ . Now suppose we have  $f \in E_x^*$ , then evaluating against  $f$  produces an element in  $\Gamma(\{x\} \times \widehat{\mathbb{R}}^4, \widehat{E}^* \otimes S_-)$ ; denote it as  $G\Psi(f)$ .



We notice that for each  $z \in \mathbb{R}^4$ , the value of  $G\Psi(f)$  at the point  $z$  is finite. This is because if we contract by a unit spinor  $\eta$ ,

$$|\langle G\Psi(f)(z), \eta \rangle|^2 = \sum_j |\langle (G_z \psi_z^j)(f), \eta \rangle|^2$$

is in fact the norm square of the linear functional  $\langle f \circ G_z, \eta \rangle$  on the Hilbert space  $\hat{E}_z$ . Since the connection  $A$  is smooth, this linear functional is bounded by basic elliptic analysis.

**Proposition 3.2.** *The element  $G\Psi(f)$  lies in  $\ker \hat{D}_{\hat{A}^*}^-$ , the kernel of the Dirac operator on the dual bundle of  $\hat{E}$ .*

*Proof.* (Sketch) The formal calculation for the Dirac equation is the same as in [4], Proposition 2.1, which uses explicit expressions for the connection coefficient of  $\hat{A}$ , and reduces the proof to properties of the Green’s operator.  $\square$

**Corollary 3.3.** *The element  $4\pi \exp(2\pi iz(x))G\Psi(f)(z)$  lies in  $\ker \hat{D}_{\hat{A}_x^*}^-$ . Moreover, under the canonical trivialisation  $\hat{E}_z^* = \exp(2\pi iz)(H_D^2)^*$ , this is invariant.*

*Proof.* Using  $\hat{D}_{\hat{A}_x^*}^- = \hat{D}_{\hat{A}^*}^- - 2\pi i \sum_{\mu=1}^4 x_\mu \hat{c}_\mu$ , the first claim is clear. We consider the  $z$  dependence of the construction.

$$\begin{aligned} \exp(2\pi iz(x))G\Psi(f)(z) &= \sum f \circ (G_z \psi_z^j) \hat{f}^{j^*}(z) \exp(2\pi iz(x)) \\ &= \sum f \circ (\exp(-2\pi iz)G_0 \psi_0^j) \hat{f}^{j^*}(z) \exp(2\pi iz(x)) \\ &= \exp(2\pi iz) \sum f \circ (G_0 \psi_0^j) \hat{f}^{j^*}(0), \end{aligned}$$

where we used the fact the  $f \in E_x^*$  is located above the point  $x \in B$ . This means the element is invariant.  $\square$

Recall there is a complex antilinear automorphism of the spin bundle, denoted  $\epsilon : S \rightarrow S$ , which preserves the chiral splitting  $S = S_+ \oplus S_-$  and the Clifford multiplication. Tensoring this with the canonical isomorphism  $E \simeq E^*$ , one obtains an antilinear isomorphism

$$\hat{\epsilon} : E \otimes S \simeq E^* \otimes S, \tag{12}$$

which is checked to preserve the Dirac equation.

This allows us to define a **canonical comparison map**  $\alpha : E \rightarrow \hat{E}$  by

$$\xi \mapsto 4\pi \exp(-2\pi i \langle \cdot, x \rangle) \langle \hat{\epsilon}(G\Psi), \xi \rangle = 4\pi \exp(-2\pi i \langle z, x \rangle) \sum_j \langle \hat{\epsilon}(G_z \psi_z^j), \xi \rangle f^j(z), \tag{13}$$

*i.e.* we follow the composition  $E_x \simeq E_x^* \rightarrow \ker \hat{D}_{\hat{A}_x^*}^- \simeq \ker \hat{D}_{\hat{A}_x}^-$ .

The adjoint operator of this canonical map  $E_y \rightarrow \hat{E}_y \subset H_D^2 \otimes S_-$  can be described as follows. Identify the space of invariant sections of  $\hat{E}$  with  $H_D^2$ , then for  $\zeta \in H_D^2 \otimes S_-$ , the adjoint operator maps it into an element of  $E_y$ , given by

$$\zeta \mapsto 4\pi(\text{Tr}_{S_-}(\epsilon \circ G_0)\zeta)(y), \quad H_D^2 \otimes S_- \rightarrow E_y,$$

where the  $\epsilon \in G_0$  sends the  $H_D^2$  factor to  $\Gamma(E) \otimes S_-$ , so we can then contract the two copies of  $S_-$  to get some element in  $\Gamma(E)$ , and finally evaluate at the point  $y \in B$ .

*3.4. An example: the trivial line bundle.* Let us work out explicitly what this construction means for  $(E, A)$  being the trivial flat line bundle, and  $B = B(R) \subset \mathbb{R}^4$  is a ball. Then we can pick  $\xi$  to be a unit length basis vector, and henceforth suppress it. Modulo the twisting  $\epsilon$ , the canonical map  $E \rightarrow H_D^2 \otimes S_-$  essentially comes down to the following description: calculate the composition

$$H_D^2 \xrightarrow{G_0} W_0^{1,2} \cap C^\infty(B, E \otimes S_-) \xrightarrow{\text{evaluate}} E_y \otimes S_- \simeq S_-$$

and then write this (spinor-valued) functional on  $H_D^2$  using Riesz representation.

The Green’s function for the Euclidean Laplacian on the ball is well known. We recall the formula

$$\begin{cases} G(p, 0) = \frac{1}{4\pi^2} \{|p|^{-2} - R^{-2}\} \\ G(p, q) = \frac{1}{4\pi^2} \{|p - q|^{-2} - |p \frac{|q|}{R} - q \frac{R}{|q|}|^{-2}\} = G(q, p). \end{cases}$$

We treat  $p$  as the independent variable, and  $q$  is a parameter. In the spinorial situation, the Green’s function is merely the above tensored by a parallel spinor field  $\eta$ . We calculate the projection of the Green’s function to the space  $H_D^2$ ,

$$\begin{cases} G(p, 0) \otimes \eta = D\{-\frac{p \cdot \eta}{8\pi^2}(|p|^{-2} - R^{-2})\} + \frac{\eta}{4\pi^2 R^2} \\ G(p, q) \otimes \eta = D\{-\frac{1}{2}(p - q) \cdot \eta G(p, q)\} + \frac{1}{4\pi^2} (\frac{R^2}{|q|^2} - 1) \frac{R^2}{|q|^2} \frac{p - q \frac{R^2}{|q|^2}}{|p - q \frac{R^2}{|q|^2}|^4} \cdot q \cdot \eta. \end{cases}$$

This means for  $s \in H_D^2$ ,

$$\begin{aligned} \hat{\epsilon} Gs(q) &= \hat{\epsilon} \int_{B(R)} G(p, q)s(p)d\text{Vol}_p \\ &= \hat{\epsilon} \sum_{\nu=1}^2 \int_{B(R)} \eta_\nu \otimes \langle G(p, q)\eta_\nu, s(p) \rangle d\text{Vol}_p \\ &= \sum_{\nu=1}^2 \hat{\epsilon} \eta_\nu \otimes \langle \frac{1}{4\pi^2} (\frac{R^2}{|q|^2} - 1) \frac{R^2}{|q|^2} \frac{p - q \frac{R^2}{|q|^2}}{|p - q \frac{R^2}{|q|^2}|^4} \cdot q \cdot \eta_\nu, s(p) \rangle. \end{aligned}$$

Thus the image of the canonical map  $\alpha : E_y \rightarrow \hat{E}_y \subset H_D^2 \otimes S_-$  is

$$\alpha(1) = \begin{cases} \sum_v \epsilon \eta_v \otimes \frac{1}{\pi} \left( \frac{R^2}{|y|^2} - 1 \right) \frac{R^2}{|y|^2} \frac{x-y \frac{R^2}{|y|^2}}{|x-y \frac{R^2}{|y|^2}|^4} \cdot y \cdot \eta_v \in S_- \otimes H_D^2 & 0 < |y| < R, \\ \sum_v \epsilon \eta_v \otimes \frac{\eta_v}{4\pi R^2} & y = 0. \end{cases} \tag{14}$$

Geometrically, the Dirac field  $\frac{x-y \frac{R^2}{|y|^2}}{|x-y \frac{R^2}{|y|^2}|^4} \cdot y \cdot \eta_v$  is proportional to the fundamental solution to the Dirac equation with source at the inversion point of  $y$ ; when  $y \rightarrow 0$  the source moves to infinity and the suitably rescaled solution converges smoothly to the constant Dirac field.

It is interesting to observe the limiting behaviour of this formula near the boundary of  $B(R)$ . For  $|y| \sim R$ , write  $y = Rl \cos \theta$ , where  $\theta \sim 0$ , and  $l$  is a unit vector. Write also  $x = lR + \frac{1}{2}x'R\theta^2$ , then the formula has the asymptote

$$\sum_v \epsilon \eta_v \otimes \frac{1}{\pi} \theta^2 \frac{x - lR - \frac{1}{2}lR\theta^2}{|x - lR - \frac{1}{2}lR\theta^2|^4} \cdot lR \cdot \eta_v \sim \sum_v \epsilon \eta_v \otimes \frac{8}{\pi R^2 \theta^4} \frac{x' - l}{|x' - l|^4} \cdot l \cdot \eta_v. \tag{15}$$

If we view this naively as a function of  $x$ , then the pointwise limit will be zero, because the Dirac field concentrates near  $lR$  with distance scale  $\theta^2$ . If we extract an appropriately rescaled limit, we obtain a Dirac field in some half space, generated by a point charge placed at unit distance outside the boundary plane.

**3.5. Nonsingularity.** This section studies the **Laplacian**  $\hat{\Delta}_{\hat{A}_y} = \hat{\nabla}_{\hat{A}_y}^* \hat{\nabla}_{\hat{A}_y}$  on the local Nahm transform bundle  $\hat{E}$ , where  $y \in \mathbb{R}^4$ . The importance of this operator for our construction is based on the observation that since  $\hat{A}_y$  is ASD, on the positive spin  $\hat{E} \otimes S_+$ , we have

$$\hat{\nabla}_{\hat{A}_y}^* \hat{\nabla}_{\hat{A}_y} = \hat{D}_{\hat{A}_y}^- \hat{D}_{\hat{A}_y}^+. \tag{16}$$

We start with a matrix description of this Laplacian acting on invariant sections of  $\hat{E}$ . Recall that  $s \in H_D^2$  is canonically identified with the invariant section  $se^{-2\pi iz}$ .

**Lemma 3.4.** *Under the identification, the Laplacian can be identified with the operator  $4\pi^2 P_0 \circ \{|x - y|^2 - 4G_0\}$  acting on  $H_D^2$ .*

*Proof.* Let  $s \in H_D^2$  and  $\tau = G_0 s$ . Then by Green’s formula

$$\int_B \langle \tau, s \rangle = \int_B |\nabla \tau|^2.$$

We consider the  $L^2$  decomposition

$$(x_\mu - y_\mu)s = D\tau_\mu + \sigma_\mu, \tag{17}$$

with  $\tau_\mu \in W_0^{1,2}$  and  $D\sigma_\mu = 0$ . Then we observe

$$\nabla^* \nabla \tau_\mu = D^2 \tau_\mu = D((x_\mu - y_\mu)s) = c_\mu s = \nabla^* \nabla c_\mu \tau,$$

so in fact, by the uniqueness of solutions to the Laplace equation,

$$\tau_\mu = c_\mu \tau. \tag{18}$$

Then we compute

$$\|(x_\mu - y_\mu)s\|_{L^2}^2 = \|D\tau_\mu\|_{L^2}^2 + \|\sigma_\mu\|_{L^2}^2 = \|\nabla\tau\|_{L^2}^2 + \|\sigma_\mu\|_{L^2}^2.$$

Summing over  $\mu$ , we get

$$\int_B |x - y|^2 |s|^2 d\text{Vol}_x = 4 \int_B |\nabla\tau|^2 + \sum_\mu \|\sigma_\mu\|_{L^2}^2 = 4\langle\tau, s\rangle + \sum_\mu \|\sigma_\mu\|_{L^2}^2.$$

This equation has the interpretation

$$\langle s, P_0(\{|x - y|^2 - 4G_0\}s) \rangle = \sum_\mu \|\sigma_\mu\|_{L^2}^2 = \sum_\mu \|P_0 \circ (x_\mu - y_\mu)s\|_{L^2}^2. \tag{19}$$

Under the identification, the directional derivatives  $\nabla_{\hat{A}_y}^\mu$  can be identified as  $-2\pi i P_0 \circ (x_\mu - y_\mu)$  (cf. Section 3.2). Thus

$$\langle s, 4\pi^2 P_0(\{|x - y|^2 - 4G_0\}s) \rangle = \sum_\mu \left\| \nabla_{\hat{A}_y}^\mu s \right\|^2 = \langle s, \hat{\Delta}_{\hat{A}_y} s \rangle.$$

Since this holds for all  $s$ , the self adjoint operators  $4\pi^2 P_0 \circ \{|x - y|^2 - 4G_0\}$  and  $\hat{\Delta}_{\hat{A}_y}$  must be the same.  $\square$

*Remark.* One can also derive the Laplacian formula by a calculation similar to Lemma 2.3, Lemma 2.4 in [4].

These preparations lead to

**Proposition 3.5.** *The kernel of  $\hat{\Delta}_{\hat{A}_y}$  on the space of invariant sections of  $\hat{E}$  vanishes for  $y \in \mathbb{R}^4$ .*

*Proof.* An invariant section is the same data as an element  $s \in H_D^2$ . Then the kernel of  $\hat{\Delta}_{\hat{A}_y}$  translates into the condition

$$P_0\left(\sum_\mu (x_\mu - y_\mu)^2 - 4G_0\right)s = 0. \tag{20}$$

Comparing with the proof of Lemma 3.4, this implies (cf. (19))

$$\sum_\mu \|\sigma_\mu\|_{L^2}^2 = 0,$$

hence  $\sigma_\mu = 0$  for all  $\mu$ . Now combining (17), (18),

$$\begin{aligned} D(\{|x - y|^2 - 4G_0\}s) &= 2 \sum_\mu (x_\mu - y_\mu)c_\mu s - 4D\tau = 2 \sum_\mu c_\mu D\tau_\mu - 4D\tau \\ &= 2 \sum_\mu c_\mu D(c_\mu \tau) - 4D\tau = 0, \end{aligned} \tag{21}$$

so  $\{|x - y|^2 - 4G_0\}s$  is both orthogonal to  $H_D^2$  (by equation (20)) and inside  $H_D^2$  (by equation (21)), and therefore

$$|x - y|^2 s = 4\tau.$$

In particular  $s$  has zero boundary data. Substituting this into (17), we get

$$\begin{aligned} (x_\mu - y_\mu)s &= D(c_\mu \frac{1}{4}|x - y|^2 s) = -c_\mu D(\frac{1}{4}|x - y|^2 s) - 2\nabla_\mu(\frac{1}{4}|x - y|^2 s) \\ &= -c_\mu \frac{1}{2}(x - y) \cdot s - (x_\mu - y_\mu)s - \frac{1}{2}|x - y|^2 \nabla_\mu s. \end{aligned}$$

So for any  $\mu$ ,

$$|x - y|^2 \nabla_\mu s = -4(x_\mu - y_\mu)s - \sum_\nu (x_\nu - y_\nu)c_\mu c_\nu s.$$

From this

$$\nabla_A\{|x - y|^2(x - y) \cdot s\} = 0.$$

In other words,

$$s = \frac{(x - y) \cdot \rho}{|x - y|^4} \tag{22}$$

where  $\rho$  is covariant constant in  $E \otimes S^+$ .

This forces  $s = 0$ , by the zero boundary condition.  $\square$

*Remark.* This proof suggests the mechanism of singularity formation: if the discussion is extended to singular connections, then the kernel of  $\hat{\Delta}_{\hat{A}_y}$  can be nonzero, and (22) is expected to describe the asymptotic profile of the kernel elements near the singularity.

Our next aim is to justify the invertibility of  $\hat{\Delta}_{\hat{A}_y}$ , acting on the space of invariant sections of  $\hat{E}$ , identified as  $H_D^2$ . We recall the Laplacian is always self-adjoint and semipositive. In our context it is also bounded by the formula in Lemma 3.4. So it is enough to prove

**Lemma 3.6.** *For  $y \notin \partial B$ , there is a coercive estimate on the space of invariant sections*

$$\left\| \hat{\Delta}_{\hat{A}_y}(s e^{-2\pi i z}) \right\| \geq C \left\| s e^{-2\pi i z} \right\|. \tag{23}$$

*Here the norm of an invariant section is the norm on any fibre  $\hat{E}_z$ . The constant can be taken to be uniform for  $y$  in any compact subset of  $B$ .*

*Proof.* Since we already know the kernel is zero, the strategy for the coercive estimate is a compactness argument, which is delicate due to norm collapsing issues. Suppose we have a sequence  $s_m \in H_D^2$ , with unit norms, satisfying

$$\left\| P_0(\{|x - y|^2 - 4G_0\}s_m) \right\|_{L^2} = \frac{1}{4\pi^2} \left\| \hat{\Delta}_{\hat{A}_y}(s_m e^{-2\pi i z}) \right\| \leq 1/m.$$

By interior regularity, we may assume  $s_m$  converges smoothly to  $s$  inside  $B$ . We associate  $\tau_m = G_0 s_m$  as in the previous proposition. Then  $\tau_m$  satisfies the elliptic equation

$\Delta_A \tau_m = s_m$ , with the zero boundary condition, so for any large Lebesgue exponent  $p$  of our choice,

$$\|\tau_m\|_{L^p} \leq C \|\tau_m\|_{W_0^{2,2}} \leq C \|s_m\|_{L^2} \leq C.$$

Here the first inequality is Sobolev embedding, and the second inequality is elliptic regularity.

For small  $\delta$ , let  $B_\delta = \{x \in B : \text{dist}(x, \partial B) < \delta\}$ . Let  $q$  be the conjugate exponent of  $p$ , then

$$\begin{aligned} & \int_B |x - y|^2 |s_m|^2(x) d\text{Vol}_x \\ &= 4 \int_B \langle \tau_m, s_m \rangle + \langle \{|x - y|^2 - 4G_0\} s_m, s_m \rangle \\ &= 4 \int_B \langle \tau_m, s_m \rangle + \langle P_0(\{|x - y|^2 - 4G_0\} s_m), s_m \rangle \\ &\leq C \|s_m\|_{L^q(B)} + \frac{1}{4\pi^2} \left\| \hat{\Delta}_{\hat{A}_y} (s_m e^{-2\pi iz}) \right\| \\ &\leq C \|s_m\|_{L^q(B \setminus B_\delta)} + C \|s_m\|_{L^q(B_\delta)} + 1/m \\ &\leq C \|s_m\|_{L^q(B \setminus B_\delta)} + C \delta^{-1/2+1/q} \|s_m\|_{L^2(B_\delta)} + 1/m. \end{aligned}$$

On the other hand, interior regularity of Dirac fields implies

$$\text{dist}(y, \partial B)^2 \|s_m\|_{L^2(B)}^2 \leq C \int_B |x - y|^2 |s_m|^2(x) d\text{Vol}_x,$$

so for  $\delta \ll 1$ , we can absorb  $C \delta^{-1/2+1/q} \|s_m\|_{L^2(B_\delta)}$  to deduce

$$\text{dist}(y, \partial B)^2 = \text{dist}(y, \partial B)^2 \|s_m\|_{L^2(B)}^2 \leq C \|s_m\|_{L^q(B \setminus B_\delta)} + 1/m.$$

Hence the norm of  $s_m$  cannot be lost entirely to the boundary:

$$\|s\|_{L^q(B)} \geq \liminf \|s_m\|_{L^q(B \setminus B_\delta)} \geq C \text{dist}(y, \partial B)^2$$

where  $C$  depends only on  $R$  and the connection  $A$ . The moral is that if a higher Lebesgue norm of a function can be controlled by a lower Lebesgue norm, then the function cannot be concentrated in region of small measure.

In particular the limit  $s$  is nontrivial. Now we look at the equation

$$|x - y|^2 s_m - 4\tau_m = P_0(|x - y|^2 s_m - 4\tau_m) + Dt_m, \quad \tau_m = G_0 s_m,$$

where  $t_m$  is just some element in  $W_0^{1,2}$ . When we take the weak limit, we get

$$|x - y|^2 s - 4\tau = Dt, \quad \tau = G_0 s.$$

Thus despite the possibility of partial norm collapsing, the limit  $s$  still satisfies

$$P_0(\{|x - y|^2 - 4G_0\} s) = 0,$$

or in other words,

$$\hat{\Delta}_{\hat{A}_y} (s e^{-2\pi iz}) = 0$$

which implies  $s = 0$ , a contradiction.  $\square$

Hence we finally achieved

**Proposition 3.7.** *For  $y \notin \partial B$ , the Laplacian  $\hat{\Delta}_{\hat{A}_y}$  is invertible on the space of invariant sections of  $\hat{E}$ .*

*Remark.* The subtle argument in the lemma involving norm collapsing reflects the genuine distinction between interior points and the boundary. The coercivity estimate is expected to fail on the boundary, so that  $\hat{E}$  would not extend to a smooth bundle on  $\mathbb{R}^4$ ; otherwise the original ASD connection  $A$  would extend to a global connection on  $\mathbb{R}^4$ , which would be very surprising.

We can therefore define the inverse operator  $\hat{G}_y = (\hat{\Delta}_{\hat{A}_y})^{-1}$ , for  $y \notin \partial B$ , on the space of invariant sections of  $\hat{E}$ . We collect a few formal consequences:

**Corollary 3.8.** *The operator  $\hat{D}_{\hat{A}_y}^- : \Gamma(\mathbb{R}^4, \hat{E} \otimes S_-) \rightarrow \Gamma(\mathbb{R}^4, \hat{E} \otimes S_+)$ , when restricted to the invariant sections, is a surjective map. Moreover, the orthogonal projection from  $H_D^2 \otimes S_-$  to the kernel  $\hat{E}_y$  is given by*

$$\hat{P}_y = 1 - \hat{D}_{\hat{A}_y}^+ \hat{G}_y \hat{D}_{\hat{A}_y}^- . \tag{24}$$

*Proof.* Since  $\hat{A}_y$  is ASD, we have  $\hat{\Delta}_{\hat{A}_y} = \hat{D}_{\hat{A}_y}^- \hat{D}_{\hat{A}_y}^+$ . The surjectivity follows from the explicit formula for a preimage, given by the operator  $\hat{D}_{\hat{A}_y}^+ \hat{G}_y$ . To see the orthogonal projection formula, one also needs to check  $\hat{D}_{\hat{A}_y}^+$  and  $\hat{D}_{\hat{A}_y}^-$  are adjoint on the invariant sections.  $\square$

**Corollary 3.9.** *The bundle  $\hat{E}$  is nonsingular over  $y \notin \partial B$ . Moreover, the natural connection  $\hat{A}$  is ASD.*

*Proof.* The nonsingularity is a formal consequence of a smooth formula for the orthogonal projection  $\hat{P}_y$ ; here the word bundle is interpreted as a Hermitian vector bundle with fibres possibly being Hilbert spaces.

The ASD condition is implied by the following curvature formula for  $\hat{A}$  analogous to (10), which relies on formula (24) for the projection operator. At the point  $y \in B$ , we have  $\hat{F} \in \text{End}(\hat{E}_y) \otimes \Lambda^2 T_y^*(\mathbb{R}^4)$ ,

$$\hat{F} = -(2\pi)^2 \hat{P}_y \hat{G}_y \otimes \sum_{\mu, \nu} \hat{c}_\mu \hat{c}_\nu dx_\mu \wedge dx_\nu, \tag{25}$$

where  $\hat{P}_y \hat{G}_y$  acts on the  $\Gamma(\hat{E})$  factor of  $\hat{E}_y$ , and  $\hat{c}_\mu \hat{c}_\nu$  acts on the spin factor.  $\square$

We next wish to show the inverse Nahm transform vanishes in the exterior of  $B$ .

**Corollary 3.10.** *The inverse Nahm transform  $\hat{E}_y = 0$  for  $y \notin \bar{B}$ .*

*Proof.* Since we know the bundle is nonsingular in the exterior region, the rank is constant, so it is only necessary to show  $\hat{E}_y = 0$  for  $|y| \gg 1$ . But we know

$$\hat{D}_{\hat{A}y}^- = \hat{D}_{\hat{A}}^- + 2\pi i \sum y_\mu \hat{c}_\mu = 2\pi i (1 + \hat{D}_{\hat{A}}^- (2\pi i \sum y_\mu \hat{c}_\mu)^{-1}) \sum y_\mu \hat{c}_\mu,$$

and  $\hat{D}_{\hat{A}}$  acting on invariant sections is a bounded operator, so for large  $|y|$ , the Dirac operator is merely a small perturbation of the invertible operator  $2\pi i \sum y_\mu \hat{c}_\mu$ , hence itself invertible, implying the kernel  $\hat{E}_y = 0$ .  $\square$

**3.6. The canonical map.** To study the canonical map  $\alpha : E \rightarrow \hat{E}$ , we look at

$$v_x(z) = 4\pi \exp(-2\pi i z(x)) \sum_j \hat{e}(G_z \psi_z^j(x)) \otimes \hat{f}^j(z) \in E_x^* \otimes S_- \otimes \hat{E}_z, \tag{26}$$

where  $x \in B$  is a parameter. For  $x_1, x_2 \in B$ , we consider the correlator function  $\langle v_{x_1}(z), v_{x_2}(z) \rangle_{\hat{E}_z \otimes S_-}$ , meaning we contract the  $\hat{E}_z$  and the  $S_-$  factor, to get a matrix in  $\text{Hom}(E_{x_2}, E_{x_1})$ . We aim to derive a formula for the correlators, by adapting arguments in [4].

**Proposition 3.11** (cf. [4], Lemma 2.6). *The correlator is*

$$\langle v_{x_1}(z), v_{x_2}(z) \rangle_{\hat{E} \otimes S_-} = 4\pi^2 G_0(x_1, x_2) |x_1 - x_2|^2 \tag{27}$$

which is independent of  $z$ .

*Proof.* Since  $\langle \hat{e}(v), \hat{e}(w) \rangle = \langle w, v \rangle$ , we calculate

$$\begin{aligned} & \langle v_{x_1}(z), v_{x_2}(z) \rangle_{\hat{E}_z \otimes S_-} \\ &= (4\pi)^2 \exp(2\pi i z(x_1 - x_2)) \langle \sum_j \hat{e} G_z \psi_z^j(x_1) \otimes f^j(z), \\ & \quad \sum_k \hat{e} G_z \psi_z^k(x_2) \otimes f^k(z) \rangle_{\hat{E}_z \otimes S_-} \\ &= (4\pi)^2 \exp(2\pi i z(x_1 - x_2)) \text{Tr}_{S_-} \sum_j |G_z \psi_z^j(x_1)\rangle \langle G_z \psi_z^j(x_2)|. \end{aligned} \tag{28}$$

Here the bra-ket notation indicates  $\langle G_z \psi_z^j(x_2) | \in E_{x_2}^*$  and  $|G_z \psi_z^j(x_1)\rangle \in E_{x_1}$ .

The expression  $\sum_j |G_z \psi_z^j(x_1)\rangle \langle G_z \psi_z^j(x_2)|$  can be viewed as the Schwartz kernel of an operator acting on  $\Gamma(B, E \otimes S_-)$ , which converts a section  $h$  with independent variable  $x_2$  to a section with independent variable  $x_1$ . This can be factorised into several steps. The first step sends  $h$  to

$$\sum_k \hat{f}^k(z) \int_B \langle G_z \psi_z^k(x_2), h(x_2) \rangle d\text{Vol}_{x_2} = P_z G_z(h).$$

Here the equality uses the self-adjointness of  $G_z$ , and the decomposition formula for the projection  $P_z = \sum \hat{f}^k \langle \hat{f}^k, \cdot \rangle$ . The second step contracts the  $\hat{E}_z$  factor with  $\hat{f}^j$  to



get a number, multiplies the result by  $G_z \psi_z^j(x_1)$ , and sums over  $j$ . Using again the decomposition formula for the projection, the result is

$$\int_B \sum_j |G_z \psi_z^j(x_1)\rangle \langle G_z \psi_z^j(x_2)| h(x_2) d\text{Vol}_{x_2} = e v_{x_1} \circ G_z P_z G_z(h). \tag{29}$$

Now we simplify the formula and take the trace over the spinor factor  $S_-$ . The projection can be written as  $P_z = 1 - D_{A_z}^+ G_z D_{A_z}^-$ . Hence using that the Clifford multiplication  $c_\mu c_\nu$  can contribute to the spinor trace if and only if  $\mu = \nu$ , we see that (using summation convention)

$$\text{Tr}_{S_-}(G_z P_z G_z) = \text{Tr}_{S_-}(G_z^2 - G_z c_\mu \nabla_{A_z}^\mu G_z c_\nu \nabla_{A_z}^\nu G_z) = \text{Tr}_{S_-}(G_z^2 + G_z \nabla_{A_z}^\mu G_z \nabla_{A_z}^\mu G_z).$$

Now we use the formula  $G_z \nabla_{A_z}^\mu G_z = \frac{1}{4\pi i} \frac{\partial G_z}{\partial z_\mu}$  (see [4], Lemma 2.2) to simplify:

$$\begin{aligned} G_z \nabla_{A_z}^\mu G_z \nabla_{A_z}^\mu G_z &= \frac{1}{4\pi i} \frac{\partial G_z}{\partial z_\mu} \nabla_{A_z}^\mu G_z \\ &= \frac{1}{4\pi i} \frac{\partial}{\partial z_\mu} (G_z \nabla_{A_z}^\mu G_z) - \frac{1}{4\pi i} G_z \frac{\partial}{\partial z_\mu} (\nabla_{A_z}^\mu G_z) \\ &= \frac{-1}{(4\pi)^2} \frac{\partial^2 G_z}{\partial z_\mu^2} - \frac{1}{4\pi i} G_z \nabla_{A_z}^\mu \frac{\partial G_z}{\partial z_\mu} - \frac{1}{4\pi i} G_z \left[ \frac{\partial}{\partial z_\mu}, \nabla_{A_z}^\mu \right] G_z \\ &= \frac{-1}{(4\pi)^2} \frac{\partial^2 G_z}{\partial z_\mu^2} - G_z \nabla_{A_z}^\mu G_z \nabla_{A_z}^\mu G_z - 2G_z^2. \end{aligned}$$

From this we see

$$\begin{aligned} \text{Tr}_{S_-}(G_z P_z G_z) &= \text{Tr}_{S_-}(G_z^2 + G_z \nabla_{A_z}^\mu G_z \nabla_{A_z}^\mu G_z) \\ &= \frac{-1}{(4\pi)^2} \frac{\partial^2 G_z}{\partial z_\mu^2}. \end{aligned} \tag{30}$$

Here we need to take into account the fact that  $S_-$  is 2-dimensional, so contributes twice to the trace.

Combining the above, we get the formula for the correlator by comparing the Schwartz kernel of operators:

$$\langle v_{x_1}(z), v_{x_2}(z) \rangle_{\hat{E}_z \otimes S_-} = -\exp(2\pi i z(x_1 - x_2)) \sum_\mu \frac{\partial^2 G_z}{\partial z_\mu^2}(x_1, x_2).$$

Furthermore, we have  $\Delta_{A_z} = e^{-2\pi i z} \Delta_A e^{2\pi i z}$ , so the Green’s function is  $G_z(x, y) = \exp(-2\pi i z(x - y)) G_0(x, y)$ . From this we see

$$-\exp(2\pi i z(x - y)) \sum_\mu \frac{\partial^2 G_z}{\partial z_\mu^2}(x, y) = 4\pi^2 G_0(x, y) |x - y|^2.$$

The claim follows.  $\square$

Given  $\xi \in E_x$ , the image under the canonical map  $\alpha : E \rightarrow \hat{E}$  is precisely the invariant section  $\langle v_x, \xi \rangle \in \Gamma(\mathbb{R}^4, \hat{E} \otimes S_-)$ . We show that the canonical map  $\alpha$  preserves the structures (cf. [4], Theorem 2.8 and 2.9).

**Proposition 3.12.** *The canonical map  $\alpha$  preserves the Hermitian metric.*

*Proof.* The Green’s function has the short distance asymptotic expansion

$$G_z(x, y) = \frac{1}{4\pi^2|x - y|^2} \{I - \sum_{\mu} (A_{\mu}(x) + 2\pi i z_{\mu})(x_{\mu} - y_{\mu}) + O(|x - y|^2)\}. \tag{31}$$

From this we see

$$\begin{aligned} \langle v_x(z), v_y(z) \rangle_{\hat{E} \otimes S_-} &= 4\pi^2 G_0(x, y) |x - y|^2 = I \\ &\quad - \sum_{\mu} A_{\mu}(x)(x_{\mu} - y_{\mu}) + O(|x - y|^2). \end{aligned} \tag{32}$$

Thus the norm square of  $\langle v_x, \xi \rangle$  can be calculated by first extending  $\xi$  to be part of a local orthonormal frame, and then taking the limit  $y \rightarrow x$ . This yields  $|\langle v_x, \xi \rangle|^2 = \langle \xi | I | \xi \rangle = |\xi|^2$ , by looking at the lowest order in the expansion.  $\square$

**Proposition 3.13.** *The canonical map  $\alpha$  preserves the connection matrix.*

*Proof.* Let  $\xi$  and  $\xi'$  be two local sections of  $E$ , fitting into a local orthonormal frame of  $E$ . These map to  $v_x^{\xi} = \langle v_x, \xi \rangle$  and  $v_x^{\xi'} = \langle v_x, \xi' \rangle$ . The connection matrix on  $\hat{E}$  at the point  $x \in B$  is specified by knowing  $\langle v_x^{\xi'}, \frac{d}{dy}|_x v_y^{\xi} \rangle(x)$ . But this can be calculated from the expansion of the Green’s function above, which implies the required

$$\langle v_x^{\xi'}, \frac{d}{dy}|_x v_y^{\xi} \rangle(x) = \sum_{\mu} \langle \xi', A_{\mu}(x) \xi \rangle dx_{\mu} = \langle \xi', \nabla_A \xi \rangle.$$

$\square$

**3.7. The reconstruction theorem.** We derive the main reconstruction theorem 1.1 assuming the following Lemma on Fredholm index, which we shall prove later in Section 6.1.

**Lemma 3.14.** *Assume  $\bar{B}$  is diffeomorphic to the unit 4-ball with boundary. The index of the Dirac operator  $\hat{D}_{\bar{A}_y}^-$  acting on the space of invariant sections equals  $\text{rank}(E)$  for  $y \in B$ .*

**Theorem 3.15 (Reconstruction).** *Let  $B$  be a bounded domain in  $\mathbb{R}^4$ , such that  $\bar{B}$  is diffeomorphic to the unit 4-ball with boundary, and  $A$  is a smooth ASD connection on  $\bar{B}$ . The canonical map  $\alpha : E \rightarrow \hat{E}$  is an isomorphism over  $B$ , preserving the Hermitian metric and the connection, and in the exterior of  $B$  the inverse Nahm transform  $\hat{E}$  vanishes.*

*Proof.* By Corollary 3.10, the inverse Nahm transform vanishes in the exterior region. By Corollary 3.8, the operator  $\hat{D}_{\bar{A}_y}^-$  is surjective, so its kernel dimension is equal to the index. For  $y \in B$ , by the Lemma above the index is  $\text{rank}(E)$ , so  $\text{rank}(\hat{E}) =$

rank  $(E)$ . By Proposition 3.12, the canonical map  $\alpha$  is an injective isometry, so must be an isomorphism. The connection  $A$  agrees with  $\hat{A}$  by Proposition 3.13.  $\square$

We now discuss the result from a number of perspectives.

*Remark.* The reconstruction theorem is analogous to the Cauchy integration formula in complex analysis, which says the contour integral

$$\frac{1}{2\pi i} \oint \frac{f(\zeta)}{\zeta - z} d\zeta$$

vanishes for  $z$  in the exterior region of the contour, and reproduces the holomorphic function  $f$  in the interior domain.

*Remark.* Viewed in another way, the reconstruction theorem means there is a canonical embedding of  $E$  inside the trivial Hilbert bundle  $H_D^2 \otimes S_-$ . Then (27) has the interesting interpretation:

**Corollary 3.16.** *The Green’s function  $G_0(x, y)$  on the original bundle is related to the embedding data by*

$$G_0(x, y) = \frac{1}{4\pi^2|x - y|^2} \langle v_x(0), v_y(0) \rangle_{H_D^2 \otimes S_-} \in \text{Hom}(E_y, E_x). \tag{33}$$

Similar results are known in the context of the usual ADHM construction (cf. equation (29) in [6]).

*Remark.* Theorem 1.1 links ASD connections to operator theory via the ADHM interpretation. Donaldson proved in [9] a Runge approximation type theorem, which says that ASD connections on domains can be  $C^\infty$  approximated on compact subsets up to gauge, by restrictions of global ASD connections over  $S^4$  for arbitrarily large second Chern class. This suggests that on the operator theory side, one may approximate  $\hat{x}_\mu$  in some sense by finite rank Hermitian operators satisfying the ADHM equation. It is interesting to ask how this may be proved using purely operator theoretic techniques.

*Remark.* The ADHM data  $(H_D^2, \hat{x}_\mu)$  resembles the key concept of ‘spectral triple’ in noncommutative geometry (NG) [8], and the ADHM type equation (11) fits into the basic philosophy of NG, namely to encode geometry by operator algebras.

### 4. The Spectral Problem

For a smooth ASD connection  $A$  on  $B = B(R) \subset \mathbb{R}^4$ , we wish to understand the Bergmann space  $H_D^2 = \{s \in L^2(B, E \otimes S_-) : D_A^- s = 0\}$ , whose elements are called Dirac fields, by studying the spectrum of a natural operator on  $H_D^2$ , which physically is just the Hamiltonian of a harmonic oscillator (cf. Section 4.1).

We are interested in the behaviour of  $H_{D_A}^2$  as  $A$  develops a curvature singularity at the origin. This means  $A$  is a member of a sequence  $A_i$  (or a 1-parameter family  $A_t$ ), which is uniformly bounded to all orders on the complement of any given neighbourhood of the origin. We derive uniform estimates to control the eigenstates associated to the spectral problem. These are based on the Weitzenböck formula. The picture emerging from the analysis is the following:

The spectrum is divided into 3 characteristic ranges:  $\lambda \ll 1$ , the intermediate range, and  $\lambda \sim 1$ .

- If the eigenvalue is bounded below by a positive constant, then the solution has good interior Morrey type bound. (cf. (44))
- If the eigenvalue is bounded above away from 1, then the solution has good bounds away from the origin (cf. Proposition 4.4 and its ensuing Remark).
- If the eigenvalue is small, then the density of the eigenstate is concentrated near the origin (cf. (36)).
- If the eigenvalue is close to 1, then the density of the eigenstate is concentrated near the boundary (cf. (37)).

4.1. *The spectral problem.* Let  $A$  be an ASD connection on  $B = B(R)$ , which is smooth up to the boundary. We introduce a functional on the Bergmann space  $H_D^2$ ,

$$H(s) = \frac{1}{2R^2} \int_{B(R)} |x|^2 |s|^2 d\text{Vol}. \tag{34}$$

Physically this is the Hamiltonian of a harmonic oscillator.

Now  $H$  defines a Hermitian form, which in the presence of the  $L^2$  inner product defines a bounded self-adjoint operator  $L$  acting on  $H_D^2$ . We observe

**Lemma 4.1.** *The operator  $L = 1/2 + K$  where  $K$  is a compact operator.*

*Proof.* We can write  $L - 1/2$  as the operator norm limit of a sequence of linear operators  $L_i$  corresponding to the Hermitian forms

$$H_i(s) = \frac{1}{2} \int_{B((1-1/i)R)} (|x|^2 R^{-2} - 1) |s|^2 d\text{Vol}.$$

But each  $L_i$  is a compact operator because the  $L^2$  norm of a Dirac field controls all interior higher order derivatives. Compact operators are closed under norm limits, so  $L - 1/2$  must be compact as well.  $\square$

Standard functional analysis implies that  $L$  has **discrete spectrum**; this amounts to the simultaneous diagonalisation of the the Hermitian form  $H$  and the  $L^2$  inner product, and is computable by the Rayleigh-Ritz method.

*Remark.* We have by definition

$$\int_B \frac{|x|^2}{R^2} |s|^2 d\text{Vol} = \lambda^2 \int_B |s|^2 d\text{Vol}, \tag{35}$$

so  $0 < \lambda < 1$ . We make an elementary observation that when  $\lambda \ll 1$ , the eigenstate  $s$  is concentrated near the origin: for any fixed  $0 < r < R$ ,

$$\int_{B \setminus B(r)} |s|^2 \leq \frac{\lambda^2 R^2}{r^2} \int_B |s|^2. \tag{36}$$

Heuristically, the small eigenvalue eigenstates describe particles trapped in the potential well, analogous to bound states in physics. If  $A$  has very concentrated curvature, we expect the characteristic length scale  $\lambda$  of the eigenstates corresponding to the small eigenvalues to be roughly the same as the length scale of the curvature of  $A$ .

On the opposite extreme, if the eigenvalue is close to 1, then the eigenstate  $s$  is concentrated near the boundary: for any fixed  $0 < r < R$ ,

$$\int_{B(r)} |s|^2 \leq \frac{(1 - \lambda^2)R^2}{R^2 - r^2} \int_B |s|^2. \tag{37}$$

We now characterise the eigenstates  $s$  of  $L$ :

**Proposition 4.2** (Characterisation of eigenstates). *There is a unique section  $\zeta$  in  $W_0^{1,2}(B, E \otimes S_+)$ , such that*

$$\begin{cases} D_A^- s = 0 \\ D_A^+ \zeta = (\frac{|x|^2}{2R^2} - \frac{\lambda^2}{2})s. \end{cases} \tag{38}$$

*Proof.* If  $s \in H_D^2$  is an eigenstate of  $L$  with eigenvalue  $\frac{\lambda^2}{2}$ , then  $(\frac{|x|^2}{2R^2} - \frac{\lambda^2}{2})s$  is orthogonal to all elements in the space  $H_D^2$  in the  $L^2$  sense, by linear algebra. This implies the second equation above by using the decomposition

$$L^2(B, E \otimes S_-) = D_A^+ W_0^{1,2}(B, E \otimes S_+) \oplus H_D^2.$$

Here  $\zeta$  is unique because by a standard Weitzenböck formula, there is no coupled Dirac field with positive spin and zero boundary condition.  $\square$

*4.2. The Weitzenböck formula.* The well known Weitzenböck formula says that because  $A$  is ASD,

$$D^2 = D_A^- D_A^+ = \Delta_A$$

acting on the coupled positive spinor  $\zeta$ ; the curvature effect is not directly visible. Thus  $\zeta$  enjoys more favourable analytic properties compared to the negative spinors, so we shall mainly focus on  $\zeta$  when deriving the estimates.

Notice equation (38) implies

$$D^2 \zeta = \sum_i \frac{x_i c_i}{R^2} s = \frac{x \cdot s}{R^2},$$

which, by an application of Weitzenböck formula, gives

$$-\Delta |\zeta|^2 = 2|\nabla \zeta|^2 - 2\text{Re}\langle \zeta, \frac{x \cdot s}{R^2} \rangle, \tag{39}$$

where our convention of the Hodge Laplacian is  $\Delta = -\sum_i \nabla_i \nabla_i$ . Integrating the identity, we obtain:

$$\int_B |\nabla \zeta|^2 = \text{Re} \int_B \langle \zeta, \frac{x \cdot s}{R^2} \rangle d\text{Vol}.$$

Here the boundary term does not appear because  $\zeta$  vanishes on the boundary.

**Lemma 4.3** ( $W^{1,2}$  estimate of  $\zeta$ ).

$$\int_B |\nabla \zeta|^2 \leq C\lambda^2 \int_B |s|^2. \tag{40}$$

As a consequence,

$$\int_B \frac{|\zeta|^2}{|x|^2} d\text{Vol} \leq C\lambda^2 \int_B |s|^2. \tag{41}$$

Here  $C$  is an absolute constant.

*Proof.* By the Poincaré inequality,

$$\begin{aligned} \int_B |\nabla \zeta|^2 &= \operatorname{Re} \int_B \left\langle \zeta, \frac{x \cdot s}{R^2} \right\rangle \leq \frac{1}{R^2} \left( \int_B |\zeta|^2 \right)^{1/2} \left( \int_B |x|^2 |s|^2 \right)^{1/2} \\ &\leq C^{1/2} R^{-1} \left( \int_B |\nabla \zeta|^2 \right)^{1/2} \left( \int_B |x|^2 |s|^2 \right)^{1/2}, \end{aligned}$$

so (40) follows by

$$\int_B |\nabla \zeta|^2 \leq C R^{-2} \int_B |x|^2 |s|^2 = C \lambda^2 \int_B |s|^2.$$

The inequality (41) follows from Hardy’s inequality, by noticing that  $\zeta$  has zero boundary condition.  $\square$

**Proposition 4.4.** *Given constants  $0 < r < R$  and  $0 < C'' < 1$ , in the annulus region  $r \leq |x| \leq R$ , if we normalise  $s$  to have unit  $L^2$  norm, then for eigenstates with eigenvalues satisfying  $\lambda < C''$ , we have smooth estimates on  $s$  and  $\zeta$  to all orders, which are uniform in  $A$  in the setup of this Chapter.*

*Proof.* We can rewrite (38) as

$$D_A \left( \frac{1}{|x|^2 - \lambda^2 R^2} D_A \zeta \right) = 0.$$

This equation is elliptic with uniformly bounded coefficients away from the locus  $\{|x| = \lambda\}$  and the origin, so by the  $L^2$  estimate and the zero boundary condition on  $\zeta$ , we obtain uniform smooth estimates of  $\zeta$  in such region.

Near the locus  $\{|x| = \lambda\}$  we can use the local elliptic regularity of the Dirac equation to estimate  $s$  to all orders, which implies the uniform elliptic estimates on  $\zeta$  to all orders.  $\square$

*Remark.* If we drop the condition  $\lambda < C''$ , the arguments above still imply uniform smooth bounds on  $s$  away from both the origin and  $\partial B$ .

**4.3. Large eigenvalues imply interior control.** We study the situation where the eigenvalue is positively bounded below.

$$\lambda > C' > 0. \tag{42}$$

**Lemma 4.5.** *We have the estimate*

$$\int_{B(R)} \frac{1}{|x|^2} |\nabla \zeta|^2 dVol \leq \frac{C}{R^2} \int_B |s|^2 \tag{43}$$

where  $C$  is an absolute constant.

*Proof.* We interpret (39) as a Poisson equation for  $|\zeta|^2$ , with zero boundary data. The Green representation formula gives

$$|\zeta|^2(0) = \frac{1}{4\pi^2} \int_{B(R)} \left( \frac{1}{R^2} - \frac{1}{|x|^2} \right) \{ 2|\nabla \zeta|^2 - 2\operatorname{Re} \langle \zeta, \frac{x \cdot s}{R^2} \rangle \} dVol.$$

By rearranging terms and applying Cauchy-Schwartz inequality,

$$|\zeta|^2(0) + \frac{1}{4\pi^2} \int_B \left(-\frac{1}{R^2} + \frac{1}{|x|^2}\right) |\nabla\zeta|^2 \leq \frac{C}{R^2} \int_B |x|^{-1} |s| |\zeta| \leq \frac{C}{R^2} \left(\int_B |s|^2\right)^{1/2} \left(\int_B \frac{|\zeta|^2}{|x|^2}\right)^{1/2}.$$

Now we use the consequence of Hardy’s inequality (41) to bound the RHS by an absolute constant. We then use the  $L^2$  gradient estimate on  $\zeta$  (cf. (40)) to drop  $\int_B \frac{1}{R^2} |\nabla\zeta|^2$  from LHS, and drop the term involving  $|\zeta|^2(0)$  to see the claim.  $\square$

**Proposition 4.6** (Interior Morrey estimate for large eigenvalues). *We have*

$$\int_B \frac{1}{|x|^2} |s|^2 dVol \leq \frac{C}{\lambda^4 R^2} \int_B |s|^2 \tag{44}$$

where  $C$  is an absolute constant. In particular, if  $\lambda$  is bounded positively below, then for any  $0 < r < R$ , we have the Morrey decay estimate

$$\int_{B(r)} |s|^2 \leq \frac{Cr^2}{R^2} \int_B |s|^2. \tag{45}$$

*Proof.* We apply (38) and the above Lemma to see

$$\int_B \left(\frac{\lambda^2}{2} - \frac{|x|^2}{R^2}\right)^2 \frac{|s|^2}{|x|^2} dVol \leq \frac{C}{R^2} \int_B |s|^2.$$

The result is then clear.  $\square$

*Remark.* This estimate, ultimately due to the Weitzenböck formula, is highly **non-perturbative** because it holds even when there can be an arbitrarily large amount of curvature concentrating to the origin. The intuition is that if  $\lambda$  is bounded below by some positive constant, then the characteristic length scale of the Dirac field is much larger than the scale of the concentrated curvature, hence the curvature singularity is not very visible to the solution.

### 5. Convergence Theory for Bergmann Spaces

Given a sequence  $A_i$  or a one-paramter family  $\{A_t\}_{t>0}$  of smooth ASD connections on  $\bar{B}$  forming a curvature singularity at the origin (cf. the setup in Chapter 4) with uniformly bounded  $L^2$  curvature. Away from the origin, we assume the connections converge in  $C_{loc}^\infty$  to a connection  $A_\infty$  on  $E|_{B \setminus \{0\}}$ , which is necessarily smooth and ASD with finite  $L^2$  curvature, so by the removable singularity theorem  $(E|_{B \setminus \{0\}}, A_\infty)$  extends to  $(\tilde{E}, A_\infty)$ . Here  $\tilde{E}$  is conceptually a different topological bundle from  $E$ , although their  $L^2$  sections can be identified. We may assume  $A_\infty$  has small  $L^2$  curvature, by possibly shrinking  $B$ . The **convergence problem** asks for a limiting description of the corresponding Bergmann spaces  $H_{D_A}^2$ .

The basic picture is that the part of the spectrum for  $A$  above a threshold value converges to the spectrum for  $A_\infty$  (cf. Theorem 5.2, Proposition 5.3); the 1-parameter family of Bergmann spaces  $H_{D_{A_t}}^2$  converge in a natural way to  $H_{D_{A_\infty}}^2 \oplus V$  where  $V$  is a finite dimensional space (cf. Theorem 5.14); the natural operators on Bergmann spaces extend naturally to the limit space (cf. Proposition 5.15, 5.16). This picture is somewhat analogous to a flat family of holomorphic vector bundles degenerating into a coherent sheaf.

*5.1. Convergence of eigenstates.* We consider a sequence of eigenstates  $s_i$  associated to the connections  $A_i$ , with  $\|s_i\|_{L^2} = 1$ , the eigenvalues  $0 < \lambda_i < 1$  and corresponding coupled positive spinor fields  $\zeta_i$  solving (38), and we ask for a convergence theory of  $s_i$ . Without loss of generality  $\lambda_i \rightarrow \lambda_\infty$  converges, after taking subsequence. The basic picture of Chapter 4 implies:

- If  $\lambda_i \rightarrow 0$ , then the density of  $s_i$  is concentrated to the origin, so  $s_i$  converges to zero weakly.
- If  $\lambda_i \rightarrow 1$ , then the density of  $s_i$  is concentrated to the boundary, so  $s_i$  converges to zero weakly.

Now let us assume the uniform two sided eigenvalue bound  $0 < C' < \lambda_i < C'' < 1$ . By the main results of Chapter 4, in any given annulus region  $0 < r \leq |x| \leq R$ , we have uniform smooth estimates on all the data  $\lambda_i, s_i, \zeta_i$ , so we can extract a subsequence to ensure smooth convergence, by standard compactness arguments. A standard diagonal argument implies we can assume  $C_{loc}^\infty$  **convergence away from the origin**, to the limiting data  $\lambda_\infty, s_\infty$  and  $\zeta_\infty$ , which satisfy the limiting version of (38) on  $B \setminus \{0\}$ .

Since we have uniform estimates  $\|s_i\|_{L^2} = 1$  and  $\|\zeta_i\|_{W^{1,2}} \leq C$  (cf. (40)), and the norms cannot increase in the limit, so the same estimates hold for  $s_\infty$  and  $\zeta_\infty$ . By elliptic regularity of this limiting PDE system, the limiting data extend smoothly across the origin to give a solution of (38).

Crucially, we claim **strong convergence** of  $s_i$  to  $s_\infty$  inside  $L^2$ . The only possible issue is to lose  $L^2$  mass to the origin. But this cannot happen thanks to the uniform interior Morrey estimate (44).

In particular, the norm of  $s_\infty$  does not collapse to zero, so  $s_\infty \in H_{D_{A_\infty}}^2$  is an eigenstate with eigenvalue  $\lambda_\infty$ .

*5.2. The spectral gap.* A rather striking consequence of Section 5.1 is

**Corollary 5.1** (Spectral gap). *Suppose the minimal eigenvalue  $\lambda$  for the system (38) corresponding to the limiting connection  $A_\infty$  is  $\lambda_0 > 0$ . Then either  $\lambda_\infty = 0$  or  $\lambda_\infty \geq \lambda_0$ .*

*Proof.* If the limit of eigenvalues  $\lambda_\infty \neq 0$ , then we can assume a positive lower bound on  $\lambda_i$ . Unless  $\lambda_\infty = 1$ , we can also assume an upper bound smaller than 1. So we are in the situation above and we see the result from the good convergence theory.  $\square$

*Remark.* If  $A_\infty$  is the trivial flat connection, then the associated minimal eigenvalue  $\lambda_0 = \sqrt{2/3}$ . This is because in the flat case any Dirac field satisfies the standard Laplace equation, so  $|s|^2$  is subharmonic, hence the mean value inequality implies a lower bound on the functional (34) on Dirac fields with unit norm, which achieves equality exactly when  $s$  is parallel.

A standard compactness argument shows a more effective version of the spectral gap:

**Theorem 5.2** (Effective spectral gap). *For  $0 < \epsilon \ll 1$ , there is a large  $N$ , such that for  $i \geq N$ , any eigenvalue  $\lambda_i$  in the system (38) corresponding to the connection  $A_i$  satisfies the dichotomy*

$$0 < \lambda_i < \epsilon, \text{ or } \lambda_i > \lambda_0 - \epsilon.$$

*Remark.* It is curious what the physical interpretation of the spectral gap should be.



The spectral gap theorem suggests us to separate the spectrum associated to  $A_i$  into two parts: the **large spectrum** with  $\lambda > \lambda_0 - \epsilon$ , and the **small spectrum** with  $\lambda < \epsilon$ . Correspondingly, the Bergmann spaces decompose as

$$H^2_{D_{A_i}} = (H^2_{D_{A_i}})_{large} \oplus (H^2_{D_{A_i}})_{small}.$$

5.3. *Convergence in the large spectrum.* The convergence theory for eigenstates can be rather formally extended to a convergence theory for the large spectrum. For any given  $A_i$ , let  $s_i^j$  be an orthonormal basis of eigenstates belonging to the large spectrum, where  $j$  is arranged in increasing order of eigenvalues (counting multiplicity). We associate the data  $\zeta_i^j$  and  $\lambda_i^j$  in a self explanatory way. We will show

**Proposition 5.3** (Spectral convergence). *The large spectrum for  $A_i$  converges to the spectrum of the smooth limit  $A_\infty$ , i.e. if we index the eigenvalues associated to  $A_\infty$  in increasing order (counting multiplicity) as  $\lambda_0^k$ , then*

$$\lambda_0^k = \lim_{i \rightarrow \infty} \lambda_i^k. \tag{46}$$

*Remark.* We will henceforth often suppress mentioning taking subsequences, and we shall tacitly use diagonal arguments. A posteriori we shall see that this is not necessary due to the uniqueness of limit.

We first introduce an **algorithm**. By previous work in this Chapter, either  $\lambda_i^1 \rightarrow 1$ , or they satisfy two sided bounds so that  $s_i^1$  converges strongly to  $s_\infty^1$ , which is some eigenstate in  $H^2_{A_\infty}$  with eigenvalue  $\lambda_\infty^1$ . In the first case, we terminate and define  $\lambda_\infty^k = 1$  for all  $k$ . In the second case, we proceed with  $\lambda_i^2$ . We either terminate after a finite stage (which a posteriori does not happen), or continue indefinitely to achieve a sequence of limiting eigenstates  $s_\infty^k$  with eigenvalues  $\lambda_\infty^k$ , which must be orthonormal by strong convergence. The algorithm implies

**Lemma 5.4** (Generalised spectral gap).

$$\lambda_\infty^k \geq \lambda_0^k. \tag{47}$$

*The equality is achieved precisely if every eigenvalue  $\lambda_0^k$  for  $A_\infty$  arises as subsequential limits of eigenvalues, including multiplicity.*

Our next step is to show

**Lemma 5.5.** *If  $s \in H^2_{D_{A_\infty}}$ , then it is a strong  $L^2$  limit of sections of  $H^2_{D_{A_i}}$ .*

*Proof.* As preparation, we study the decomposition of  $L^2(E \otimes S_-) = D^+_{A_i} W_0^{1,2} \oplus H^2_{D_{A_i}}$  for varying connections  $A_i$ . Let  $s \in L^2(E \otimes S_-)$  be a smooth section, with unique decomposition

$$s = D_{A_i} \tau_i + \sigma_i. \tag{48}$$

Notice by orthogonality

$$\|s\|_{L^2}^2 = \|D_{A_i} \tau_i\|_{L^2}^2 + \|\sigma_i\|_{L^2}^2. \tag{49}$$

Now since  $\tau$  has positive spin and zero boundary condition, the Weitzenböck formula implies  $\|D_{A_i} \tau\|_{L^2}^2 = \|\nabla_{A_i} \tau_i\|_{L^2}^2$ . By Poincaré inequality for functions with zero boundary data,

$$\|\tau_i\|_{L^2} \leq CR \|d|\tau_i|\|_{L^2} \leq CR \|\nabla_{A_i} \tau_i\|_{L^2}.$$

Thus  $\|s_i\|_{L^2}$  controls both  $\|\tau_i\|_{W_0^{1,2}}$  and  $\|\sigma_i\|_{L^2}$ .

By the norm control, on every annulus  $0 < r \leq |x| \leq R$  we can extract smoothly convergent subsequences for  $\tau_i$  and  $\sigma_i$ . Hence we have smooth convergence on  $0 < |x| \leq R$  to the limiting data  $(\tau_\infty, \sigma_\infty)$ , for which the limiting version of (48) holds, and moreover  $\|\tau_\infty\|_{W_0^{1,2}} \leq C, \|\sigma_\infty\|_{L^2} \leq C$ . Notice the equation

$$D_{A_\infty}^2 \tau_\infty = D_{A_\infty} s$$

implies that  $\tau_\infty$  extends to a smooth section, so  $\sigma_\infty$  is also smooth. It is clear that the norm identity (49) holds in the limit, namely

$$\|s\|_{L^2}^2 = \|\nabla_{A_\infty} \tau_\infty\|_{L^2}^2 + \|\sigma_\infty\|_{L^2}^2.$$

But Fatou’s lemma implies that

$$\liminf \|\tau_i\|^2 \geq \|\tau_\infty\|^2, \quad \liminf \|\sigma_i\|^2 \geq \|\sigma_\infty\|^2.$$

So the only possibility is for equalities to be achieved everywhere, *i.e.*

$$\lim \|\tau_i\|^2 = \|\tau_\infty\|^2, \quad \lim \|\sigma_i\|^2 = \|\sigma_\infty\|^2.$$

The non-collapsing of norms imply that  $\sigma_i$  converges strongly to  $\sigma_\infty$  in  $L^2$ . Notice also that since the subsequential limit is unique, in fact the whole sequence has to converge.

Now by an approximation argument on  $s$ , it is clear that the smoothness of  $s$  is not essential. We specialise to the case  $s \in H_{D_{A_\infty}}^2$ . Then  $s = \sigma_\infty$  and the claim follows.  $\square$

**Corollary 5.6.** *If moreover  $s$  is an eigenstate for  $A_\infty$ , for which the eigenvalue  $\lambda_0^m$  has multiplicity one, then we may assume the sequence consists of eigenstates as well. If the eigenvalue is degenerate, we need to take linear combinations of eigenstates with approximately the same eigenvalue. In particular  $\lambda_0^m$  arises as a limit of eigenvalues.*

*Proof.* We argue in the nondegenerate case (the degenerate case has only a little more combinatorial complexity). The crucial point is that the spectrum is discrete. There are only finitely many eigenvalues below  $\lambda_0^m$ :

$$\lambda_0^1 \leq \lambda_0^2 \leq \dots \leq \lambda_0^{m-1} < \lambda_0^m < \lambda_0^{m+1} \leq \dots$$

Consider the spectral decomposition

$$\sigma_i = \sum_j a_j s_i^j + (\text{small spectrum contribution}).$$

By the generalised spectral gap lemma, there are essentially at most  $m - 1$  eigenvalues  $\lambda_i^k$  bounded above by  $\lambda_0^{m-1} + \epsilon < \lambda_0^m$ . For these eigenvalues the eigenstates are almost orthogonal to  $s$  when  $i$  is large, because they converge to eigenstates with lower eigenvalues. So their corresponding Fourier coefficients  $a_j$  in the spectral decomposition must go to zero. But once we are not allowed to have contributions from low eigenvalues, then for overall  $L^2$  mass reasons, neither are we allowed to have contributions from eigenvalues larger than  $\lambda_0^{m+1} - \epsilon > \lambda_0^m$ .  $\square$

Once we know  $\lambda_0^k$  arises as limits of eigenvalues, Proposition 5.3 follows from the generalised spectral gap Lemma. We leave the reader to ponder the issue of multiplicity. As remarked earlier, the limit of  $\lambda_i^k$  turns out a posteriori to be independent of the subsequence.

5.4. *The limit of the Bergmann spaces.* From last section, one has the appealing picture that  $H_{D_{A_\infty}}^2$  is the limit of  $(H_{D_{A_i}}^2)_{large}$ , while the small spectrum is lost in the naïve smooth limit. In this Section we give a more operator theoretic perspective; the main result is Proposition 5.14.

The first step is to study a comparison map between  $(H_{D_{A_i}}^2)_{large}$  and  $H_{D_{A_\infty}}^2$ . To save some writing, we will often suppress sequential indices  $i$ . Recall  $D_A$  and  $D_{A_\infty}$  induce two decompositions of  $L^2$ . In particular, the two Bergmann spaces  $H_{D_A}^2$  and  $H_{D_{A_\infty}}^2$  project to each other, via the operator  $\mathcal{P} : H_{D_{A_\infty}}^2 \rightarrow H_{D_A}^2$  and its adjoint  $\mathcal{P}^\dagger$ . Thus there are canonical maps

$$\pi\mathcal{P} : H_{D_{A_\infty}}^2 \rightarrow H_{D_A}^2 \xrightarrow{\text{project}} (H_{D_A}^2)_{large} \tag{50}$$

and its adjoint

$$(H_{D_A}^2)_{large} \rightarrow H_{D_A}^2 \rightarrow H_{D_{A_\infty}}^2. \tag{51}$$

**Lemma 5.7.** *The canonical map  $\pi\mathcal{P} : H_{D_{A_\infty}}^2 \rightarrow (H_{D_{A_i}}^2)_{large} \subset L^2$  converges to the identity operator  $I_{H_{D_{A_\infty}}^2}$  in the operator norm, as  $i \rightarrow \infty$ .*

*Proof.* Pick any  $s \in H_{D_{A_\infty}}^2$ , which is normalised to  $\|s\|_{L^2} = 1$ . We have the decomposition

$$s = D_A\tau + \sigma + \sigma' \tag{52}$$

where  $\tau \in W_0^{1,2}$ ,  $\sigma \in (H_{D_A}^2)_{large}$  and  $\sigma' \in (H_{D_A}^2)_{small}$ . Here  $\sigma$  is the image of the canonical map. Then

$$\|\nabla_A\tau\|_{L^2}^2 + \|\sigma\|_{L^2}^2 + \|\sigma'\|_{L^2}^2 = \|s\|_{L^2}^2. \tag{53}$$

In particular,  $\tau$  is bounded in  $L^2$ . We also have

$$D_A^2\tau = D_A s = (D_A - D_{A_\infty})s,$$

so  $D_A^2\tau$  is  $L^2$  small away from the origin, whereby  $\tau$  is  $W_{loc}^{2,2}(B \setminus \{0\})$  bounded. We can therefore assume  $\tau$  to converge strongly in  $W_{loc}^{1,2}(B \setminus \{0\})$  as  $i \rightarrow \infty$ . The limit  $\tau_\infty$  is globally bounded in  $L^2$ , has zero boundary condition, and satisfies the Laplace equation, so must be zero. Thus in the sequence,  $\tau$  must be  $W^{1,2}$  close to 0 away from the origin, with bounds independent of  $s$ . In particular,  $D_A\tau$  is  $L^2$  small away from the origin.

Notice also

$$\int_B |x|^2 R^{-2} |\sigma'|^2 d\text{Vol} \ll \int_B |\sigma'|^2 \leq 1.$$

Therefore  $s$  is  $L^2$  close to  $\sigma$  away from the origin. In particular, since the Dirac equation implies  $s$  has pointwise bound near the origin, we have for fixed  $\delta \ll 1$ ,

$$\int_{B(R) \setminus B(\delta R)} |s|^2 \geq (1 - C\delta^4) \int_{B(R)} |s|^2,$$

so by the above argument, when  $i$  is sufficiently large depending only on  $\delta$ ,

$$\int_{B(R) \setminus B(\delta R)} |\sigma|^2 \geq (1 - C\delta^4) \int_{B(R)} |s|^2.$$

Comparing with the partition of norm (53),

$$\|D_A \tau\|_{L^2} \leq C\delta^2 \|s\|_{L^2}, \quad \|\sigma'\|_{L^2} \leq C\delta^2 \|s\|_{L^2}$$

so  $\|s - \sigma\|_{L^2} \leq C\delta^2 \|s\|_{L^2}$ . The constants are independent of  $s$ . Since  $\delta$  is arbitrary, this proves the result.  $\square$

**Lemma 5.8.** *The canonical comparison map  $\pi\mathcal{P}$  for  $A_i$  is surjective for  $i \gg 1$ .*

*Proof.* Since Lemma 5.7 implies the coercivity of  $\pi\mathcal{P}$ , the image is closed. The cokernel of  $\pi\mathcal{P}$  is the kernel of (51), which means  $s \in (H^2_{D_A})_{large}$ , and  $s = D_{A_\infty} \tau$ , for  $\tau \in W^{1,2}_0$ . If the cokernel does not vanish for a subsequence of connections  $A_i$ , we normalise  $s$  to  $\|s\|_{L^2} = 1$ , and derive a contradiction by a compactness argument as follows.

First,  $A_\infty$  is ASD, so Weitzenböck formula implies  $\|\nabla_{A_\infty} \tau\|_{L^2} = \|D_{A_\infty} \tau\|_{L^2} = \|s\|_{L^2} = 1$ , hence by Poincaré inequality  $\|\tau\|_{L^2}$  is controlled. Now  $\tau$  satisfies  $D_{A_i} D_{A_\infty} \tau = 0$ , with the zero boundary condition, hence it is controlled to any order away from the origin. Any subsequential limit of  $\tau$  must be zero, by the same argument as in last Lemma. This implies  $\tau$  converges to zero smoothly in the punctured disc; so must  $s$ , which means the  $L^2$  mass of  $s$  is concentrated at the origin. But  $s$  lives in the large spectrum, so  $\int_{B(R)} |x|^2 R^{-2} |s|^2 \geq C \|s\|_{L^2}^2$ , contradiction.  $\square$

**Corollary 5.9.** *For large  $i$ , the canonical map  $\pi\mathcal{P}$  is an isomorphism. Moreover, it is close to being a unitary equivalence:*

$$\left\| (\pi\mathcal{P})^\dagger - (\pi\mathcal{P})^{-1} \right\| \rightarrow 0, \text{ as } i \rightarrow \infty.$$

We next study the small spectrum  $H^2_{D_A}$ . Here it is more convenient to set up the problem in terms of a **one-parameter family**  $(A_t)_{t>0}$  of ASD connections converging smoothly away from the origin. This is more suited to continuity arguments.

**Proposition 5.10.** *For large  $t$  so that the curvature is sufficiently concentrated, the dimension of the small spectrum is constant in the family.*

*Proof.* Imagine  $t$  to flow from a large number to  $\infty$ . By general functional analysis, the eigenvalues flow continuously. But spectral gap (cf. Theorem 5.2) prevents the spectral flow between the large spectrum and the small spectrum, so the dimension of the small spectrum must be a constant finite number.  $\square$

*Remark.* The concept of small spectrum is only meaningful in the large  $t$  limit.

By examining the comparison map  $\pi\mathcal{P}$  in the limit  $t \rightarrow \infty$ , we see

**Lemma 5.11.** *For large  $t$ , the operator  $\mathcal{P} : H^2_{D_{A_\infty}} \rightarrow H^2_{D_{A_t}}$  is Fredholm, with index being the negative of the dimension of the small spectrum.*

**Lemma 5.12.** *The Bergmann spaces  $H^2_{D_{A_t}}$  fit into a Hilbert bundle over the half line  $0 < t < \infty$ .*

*Proof.* Consider  $t$  near  $t_0$ , and the natural projection operator  $H^2_{D_{A_t}} \rightarrow H^2_{D_{A_{t_0}}}$ . This is an isomorphism by the argument proving  $\pi\mathcal{P}$  to be an isomorphism.  $\square$

**Lemma 5.13.** *For large  $t < \infty$ , the subspaces  $(H^2_{D_{A_t}})_{small}$  fit together into a vector bundle.*

*Proof.* As a general fact, consider a family of operators depending smoothly on a parameter  $t$ , with discrete spectrum consisting of eigenvalues. If we fix a spectral domain  $\Omega \subset \mathbb{C}$ , and consider the projection operator to the the span of the eigenspaces for all the eigenvalues inside  $\Omega$ , then as long as no eigenvalue crosses the boundary of  $\Omega$ , the projection operator depends smoothly on  $t$ , and the image has constant dimension. (cf. Appendix of [14]).  $\square$

**Theorem 5.14** (Natural limit of Bergmann spaces). *Let  $V$  be an inner product space with the dimension of the small spectrum. Then there is a natural topological bundle over  $0 < t \leq \infty$ , whose fibres over  $0 < t < \infty$  agree with the Bergmann space  $H^2_{D_{A_t}}$ , and the fibre over  $\infty$  is  $H^2_{D_{A_\infty}} \oplus V$ .*

*Proof.* It suffices to assign a trivialisation near  $\infty$ . One can use the isomorphism provided by the canonical comparison map (50) to deal with the large spectrum part. To trivialise the small spectrum part, one notices there is a canonical connection on the bundle  $\bigsqcup_{\infty > t > t_0 \gg 1} (H^2_{D_{A_t}})_{small}$  coming from the embedding into  $L^2$ . Over the one dimensional base  $(t_0, \infty)$ , this connection gives a parallelisation of this finite rank bundle compatible with the Hermitian structure, which can be extended to the  $\infty$  fibre by formally adding a copy of  $V$ .  $\square$

5.5. *Natural operators on the Bergmann space.* In the setup of Theorem 5.14, we can ask whether natural operators on  $H^2_{D_{A_t}}$  extend continuously to operators on the limit Bergmann space  $H^2_{D_{A_\infty}} \oplus V$ . Here we use the trivialisation near  $\infty$  in Theorem 5.14 to regard  $H^2_{D_{A_t}}$  as a fixed Hilbert space, and the convergence of operators is reduced to the usual definition of norm convergence.

Given a smooth function  $f$  on  $\bar{B}$ , the **Toeplitz operator**  $\hat{f} = P_A \circ f$  on  $H^2_{D_A}$  means multiplying by the function  $f$ , composed with the orthogonal projection of  $L^2$  to  $H^2_{D_A}$ . Physically, the Bergmann space is the state space of a fermion, and then these correspond to quantum observables of the shape  $\langle s|f|s' \rangle$ . The Toeplitz operators are Hermitian.

**Proposition 5.15** (Limit of Toeplitz type operators). *In the setup of Theorem 5.14, if  $f(0) = 0$ , then the Toeplitz operator  $\hat{f}$  on the Bergmann spaces  $H^2_{D_{A_t}}$  converge strongly to the operator*

$$H^2_{D_{A_\infty}} \oplus V \xrightarrow{\hat{f} \oplus 0} H^2_{D_{A_\infty}} \oplus V \tag{54}$$

on the limit Bergmann space, where  $\hat{f}$  also denotes the corresponding Toeplitz operator on  $H^2_{D_{A_\infty}}$ .

*Proof.* We use the trivialisation near  $\infty$  described in the proof of 5.14. The limit operator is zero on the  $V$  factor, because on the small spectrum for  $A_t$ ,

$$\|\widehat{f}s\|_{L^2}^2 \leq \int_{B(R)} f^2 |s|^2 \leq C \int_{B(R)} |x|^2 |s|^2 \leq C\lambda^2 \int_{B(R)} |s|^2 \ll \|s\|_{L^2}^2.$$

Hence we only need to consider the large spectrum for  $A_t$ . Then the result follows from the norm convergence  $\pi\mathcal{P} \rightarrow 1$  as  $t \rightarrow \infty$ .  $\square$

*Remark.* It is unclear to the author how to extend the matrix describing Clifford multiplications  $\langle s|c_j c_k|s' \rangle$  for  $s, s' \in (H^2_{D_{A_t}})_{small}$  to the limit. This question is intimately tied to compactification of ASD moduli spaces. It is interesting to extend the local Nahm transform of Chapters 2, 3 to singularities. On the ASD side, the Uhlenbeck compactification involves ideal instantons. On the operator theory side, the ideal instanton corresponds to  $(H^2_{D_{A_\infty}} \oplus V, \hat{x}_\mu)$ , where  $H^2_{D_{A_\infty}}$  encodes the smooth limit  $A_\infty$  and  $\dim V$  encodes the delta mass of curvature. But if we enlarge the operator algebra to include secondary operators induced by  $\langle s|c_j c_k|s' \rangle$  and ask how they converge, then we may obtain a more refined compactification.

Next we consider the Green operator  $G_A : H^2_{D_A} \subset L^2 \rightarrow W_0^{1,2} \subset L^2$ , defined by solving the  $\Delta_A$ -Poisson equation with zero boundary condition. For a point  $x$  inside the annulus  $0 < r < |x| < r' < R$ , elliptic regularity gives the pointwise estimate

$$|G_A s(x)| \leq C \|G_A s\|_{L^2}.$$

Thus the evaluation map

$$ev_x \circ G_A : H^2_{D_A} \rightarrow E_x \otimes S_-, \quad s \mapsto G_A s(x)$$

is a bounded  $E_x \otimes S_-$  valued linear functional.

Now we vary the connection, so the Green’s operator depends on the parameter  $t$ , and we study its limit.

**Proposition 5.16** (Limit Green operator). *On the annulus  $0 < r < |x| < r' < R$ , as  $t \rightarrow \infty$ , the evaluation maps for the Green operators converge strongly and uniformly in  $x$  to some  $E_x \otimes S_-$  valued bounded linear functional on the limit Bergmann space  $H^2_{D_{A_\infty}} \oplus V$ . The limit functional vanishes on  $V$  and agrees with  $ev_x \circ G_{A_\infty}$  on the  $H^2_{D_{A_\infty}}$  factor.*

*Proof.* We first consider any  $s \in (H^2_{D_{A_t}})_{small}$ . Consider the  $L^2$  decomposition

$$x_\mu s = D_A \tau_\mu + \sigma_\mu,$$

where  $\tau \in W_0^{1,2}$ ,  $\sigma_\mu \in H^2_D$ . Then

$$D_A^2 \tau_\mu = D_A(x_\mu s) = c_\mu s,$$

so  $\tau_\mu = c_\mu(G_A s)$ . Hence

$$C \|G_A s\|_{L^2}^2 \leq \|\nabla(G_A s)\|_{L^2}^2 = \|\nabla \tau_\mu\|_{L^2}^2 = \|D \tau_\mu\|_{L^2}^2 \leq \|x_\mu s\|_{L^2}^2 \ll \|s\|_{L^2}^2,$$

where  $C$  is an absolute constant. Thus the norm of the evaluation functional on the small spectrum converges to zero.

On the large spectrum, we use a compactness argument. Take some  $L^2$  normalised counterexample sequence  $s_i \in H^2_{D_{A_\infty}}$ , and let  $s'_i = \mathcal{P}_i s_i \in H^2_{D_{A_i}}$  be their corresponding elements via the trivialisation. We have points  $x_i$  inside the annulus, such that  $|G_{A_i} s'_i(x_i) - G_{A_\infty} s_i(x_i)| \geq \epsilon$ . We will freely pass to subsequences.

Due to uniform  $W_0^{1,2}$  bounds, we can extract a weak limit for  $G_{A_i} s'_i$ . By elliptic estimates in the annulus region, we can assume the convergence to be uniform in  $0 < r < |x| < r' < R$ , and  $s_i$  converges weakly to  $s_\infty \in H^2_{D_{A_\infty}}$ . Since  $\mathcal{P}_i \rightarrow 1$ , the sequence  $s'_i$  has the same limit  $s_\infty$ . Then using the weak equation for the limit, one sees the weak limit of  $G_{A_i} s'_i$  must be  $G_{A_\infty} s_\infty$ . The same discussions apply to  $G_{A_\infty} s_i$ . Thus the difference  $G_{A_i} s'_i - G_{A_\infty} s_i$  has weak limit zero, the convergence is uniform in the annulus, but  $|G_{A_i} s'_i(x_i) - G_{A_\infty} s_i(x_i)| \geq \epsilon$ , contradiction.  $\square$

*Remark.* These linear functionals can be Riesz represented as elements of  $H^2_{D_A} \otimes (E_x \otimes S_-)$ , which also converge strongly as  $t \rightarrow \infty$ .

### 6. Index Theory and Chern Numbers

*6.1. The rank of  $\hat{E}$  and index computation.* We prove Lemma 3.14, which amounts to computing the index of an operator. It is convenient to work in the ADHM formulation, which links more easily to operator theory. The discussions below are self contained, although the main results are likely to be known in the literature of Toeplitz operators and index theory.

The index problem fits into a more general picture. Consider the operator

$$T = \sum_i \hat{f}_i \hat{c}_i : H^2_D \otimes S_- \rightarrow H^2_D \otimes S_+,$$

where  $\hat{c}_i$  is the Clifford multiplication on the spin factor, and the Toeplitz operator  $\hat{f}_i = P_0 \circ f_i$  is the projection of the multiplication by a smooth real valued function  $f_i$  on  $\bar{B}$ ,  $i = 1, 2, 3, 4$ . We put on the assumption that the quaternion valued function  $f_T = f_1 + if_2 + jf_3 + kf_4$ , which we think of as the **symbol** of the operator, does not vanish anywhere on  $\partial B$ . In the generality of this Section, the connection  $A$  needs not to be ASD.

**Lemma 6.1.** *The operator  $T$  is Fredholm.*

*Proof.* The operator is clearly bounded. We show the finite dimensionality of the kernel. First write down the  $L^2$  decomposition

$$f_i s = D\tau_i + \sigma_i,$$

where  $s \in H^2_D \otimes S_-$ ,  $\tau_i \in W_0^{1,2} \otimes S_-$ , and  $D\sigma_i = 0$ . Thus using  $Ds = 0$ ,

$$D^2\tau_i = \sum_j (c_j \partial_j f_i) s.$$

This means by elliptic regularity,  $\tau_i$  can be chosen to satisfy

$$\|\tau_i\|_{W_0^{2,2}} \leq C \|s\|_{L^2},$$

hence

$$\|D\tau_i\|_{W^{1,2}} \leq C \|s\|_{L^2}.$$

As a remark, for general connections  $A$ , we need to impose  $\tau_i \in (\ker D)^\perp$  to make  $\tau_i$  satisfy the required estimates. Combining the above,

$$\sum f_i \hat{c}_i s = \sum \hat{c}_i D\tau_i + \sum \hat{c}_i \sigma_i = \sum \hat{c}_i D\tau_i + Ts. \tag{55}$$

The matrix  $\sum f_i \hat{c}_i$  can be viewed as a quaternion valued function, so it is invertible near the boundary by the nonvanishing assumption. Thus in a small neighbourhood of the boundary  $B_\delta = \{x \in B : \text{dist}(x, \partial B) < \delta\}$ , if  $Ts = 0$ , then  $\|s\|_{W^{1,2}(B_\delta)} \leq C \|s\|_{L^2}$ . But we also have  $\|s\|_{W^{1,2}(B \setminus B_\delta)} \leq C \|s\|_{L^2}$  by the interior regularity of the Dirac equation, so  $\|s\|_{W^{1,2}(B)} \leq C \|s\|_{L^2}$ , which forces the kernel to be finite dimensional by compactness.

A completely symmetric argument proves the finite dimensionality of the cokernel.

We also need  $T$  to have closed range. It is enough to prove that for  $s \in (\ker D)^\perp$ , we have  $\|s\|_{L^2} \leq C \|Ts\|_{L^2}$ . This follows from a compactness argument similar to the above. The key is to invoke (55), and suppose for contradiction take a sequence of  $s$  to converge smoothly in the interior, and  $\tau_i$  to converge strongly in  $W_0^{1,2}$ , such that  $Ts$  converges strongly to zero.  $\square$

Our next aim is to describe the index of this operator. We proceed by a sequence of observations:

1. The index depends only on the boundary value of  $f_i$ .  
This is because if the boundary value is zero, then  $T$  is a compact operator, by the interior regularity of Dirac fields.
2. The map  $f_T = f_1 + if_2 + jf_3 + kf_4 : \partial B \rightarrow \mathbb{H} \setminus \{0\}$  defines a degree (recall we assume  $B$  is homeomorphic to a ball), which classifies its homotopic type. Homotopic Fredholm operators define the same index. Hence the index depends on  $f$  only through its degree.
3. For the zero degree case, we consider the special operator  $T = \hat{c}_1$ , which clearly gives an isomorphism, so the index is zero.
4. Behaviour under multiplication.

Notice  $\text{index}(T) = \text{index}(\hat{c}_1 T)$ , but  $\hat{c}_1 T$  is an endomorphism of  $H_D^2 \otimes S_-$ , so can be composed. Observe  $1, \hat{c}_1 \hat{c}_2, \hat{c}_1 \hat{c}_3, \hat{c}_1 \hat{c}_4$  is a standard quaternionic basis. Moreover, for the multiplication operators  $\hat{g}$  and  $\hat{g}'$  acting on the  $H_D^2$  factor, where  $g$  and  $g'$  are real valued functions on  $\bar{B}$ , the composite  $\hat{g}\hat{g}'$  agrees with  $\hat{g}\hat{g}'$  up to a compact correction. Combining these, the quaternionic multiplication of the symbol functions and the composition of the operators are related as follows:

$$f_T \cdot f_{T'} = -f_{\hat{c}_1^{-1}(\hat{c}_1 T')(\hat{c}_1 T)}.$$

But the degree is additive with respect to symbol multiplication, and the index is additive with respect to composition, so

**Lemma 6.2.** *For a fixed connection  $A$ , the index is proportional to the degree.*

5. Independence of connection  $A$ .  
We wish to remove the dependence on the background connection  $A$ . For this, introduce the Bergmann space  $H_{\bar{D}}^2$  corresponding to a trivial flat connection  $\bar{A}$  on the original vector bundle  $E$ , which exists because  $B$  is contractible. This induces another



orthogonal decomposition of  $L^2$ . The two Bergmann spaces project onto each other, via operators  $\mathcal{P} : H_D^2 \rightarrow H_D^2$  and its adjoint  $\mathcal{P}^\dagger$ . Another viewpoint is that  $H_D^2$  and  $H_{\bar{D}}^2$  have a natural  $L^2$  pairing, so induce two linear operators  $\mathcal{P}$  and  $\mathcal{P}^\dagger$ . It is clear that  $\mathcal{P}$  and  $\mathcal{P}^\dagger$  are Fredholm; in fact, inside  $L^2$  they are compact perturbations of the identity operator. For example,

$$\mathcal{P} = 1 - D_A^+ G_{\bar{A}} D_A^- = 1 - D_A^+ G_{\bar{A}} (D_A^- - D_{\bar{A}}^-),$$

where  $(D_A^- - D_{\bar{A}}^-)$  is bounded on  $L^2$ , so  $D_A^+ G_{\bar{A}} (D_A^- - D_{\bar{A}}^-) : L^2 \rightarrow L^2$  is compact. We can then consider the composition

$$H_D^2 \otimes S_- \xrightarrow{\mathcal{P}^\dagger} H_D^2 \otimes S_- \xrightarrow{T} H_D^2 \otimes S_+ \xrightarrow{\mathcal{P}} H_D^2 \otimes S_+.$$

This has the same index as  $T$ , because the index of  $\mathcal{P}$  cancels with that of its adjoint. The composite operator is a compact perturbation of

$$\sum \hat{f}_i \hat{c}_i : H_D^2 \otimes S_- \rightarrow H_D^2 \otimes S_+.$$

This shows the index is the same as that of the corresponding problem in the flat case.

6. In the case of the trivial connection, it is clear that  $H_D^2$  decomposes according to the rank of the vector bundle, so the index is proportional to the rank of  $E$ .

Now we treat the standard case of the trivial flat line bundle, with  $T = \sum \hat{x}_\mu \hat{c}_\mu$ . We assume without loss of generality that the origin is an interior point of  $B$ , so the symbol function  $f_T$  is non-vanishing on  $\partial B$ , and moreover  $f_T : \partial B \rightarrow \mathbb{H} \setminus 0$  has degree 1.

The cokernel of  $T$  vanishes, thanks to our non-singularity discussion (cf. Corollary 3.8). The condition for the kernel is

$$\sum x_\mu \hat{c}_\mu s = D\tau, \tag{56}$$

where  $s \in H_D^2 \otimes S_-$  and  $\tau \in W_0^{1,2}(E \otimes S_+) \otimes S_+$ , and  $E$  is the trivial flat line bundle. We observe

$$\Delta \tau = D^2 \tau = \sum c_\mu \hat{c}_\mu s.$$

The significance of this comes from representation theory. Notice

$$S_- \otimes S_- \xrightarrow{\sum c_\mu \hat{c}_\mu} S_+ \otimes S_+$$

is a  $Spin(4)$ -equivariant map. We can decompose the representations

$$S_- \otimes S_- = \Lambda^2 S_- \oplus (\text{3D rep of } su(2)_-), \quad S_+ \otimes S_+ = \Lambda^2 S_+ \oplus (\text{3D rep of } su(2)_+).$$

By Schur’s lemma, the map factors through the one dimensional representation  $\Lambda^2 S_-$ , and the image lands inside the line spanned by  $\eta'_1 \otimes \epsilon(\eta'_1) + \eta'_2 \otimes \epsilon(\eta'_2) \in \Lambda^2 S_+ \subset S_+ \otimes S_+$ , where  $\eta'_1, \eta'_2$  form an orthonormal basis of  $S_+$ . Thus  $\sum c_\mu \hat{c}_\mu s$  is a scalar function times  $\eta'_1 \otimes \epsilon(\eta'_1) + \eta'_2 \otimes \epsilon(\eta'_2) \in S_+ \otimes S_+$ ; therefore there is some scalar function  $\rho$ , such that

$$\tau = \rho(\eta'_1 \otimes \epsilon(\eta'_1) + \eta'_2 \otimes \epsilon(\eta'_2)). \tag{57}$$

This means

$$\sum c_\mu \hat{c}_\mu s = \Delta \rho (\eta'_1 \otimes \epsilon(\eta'_1) + \eta'_2 \otimes \epsilon(\eta'_2)).$$

On the other hand, direct differentiation shows

$$D\tau = \nabla \rho \cdot \eta'_1 \otimes \epsilon(\eta'_1) + \nabla \rho \cdot \eta'_2 \otimes \epsilon(\eta'_2),$$

hence using the defining equation (56),

$$s = -\frac{1}{|x|^2} \{ \nabla \rho \cdot \eta'_1 \otimes x \cdot \epsilon(\eta'_1) + \nabla \rho \cdot \eta'_2 \otimes x \cdot \epsilon(\eta'_2) \},$$

whereby

$$\sum c_\mu \hat{c}_\mu s = -\frac{4}{|x|^2} \langle \nabla \rho, x \rangle (\eta'_1 \otimes \epsilon(\eta'_1) + \eta'_2 \otimes \epsilon(\eta'_2)).$$

Comparing the above,  $\Delta \rho = -\frac{4}{|x|^2} \langle \nabla \rho, x \rangle$ , or

$$\Delta \left( \frac{\rho}{|x|^2} \right) = 0.$$

Recall that  $\rho$  is a smooth function with zero boundary condition, so  $\frac{\rho}{|x|^2}$  has zero boundary condition, and the only possible singularity is a pole of order 2 at the origin, which by our assumption is an interior point of  $B$ . It has to be proportional to the Dirichlet Green's function with delta mass placed at the origin. This proves the kernel dimension is 1, so the index is 1 in the special case.

*Example 6.3.* Consider the special case where  $B = B(R) \subset \mathbb{R}^4$ . Up to a constant,

$$\rho = \frac{1}{2} (|x|^2 - R^2), s = -(\eta_1 \otimes \epsilon(\eta_1) + \eta_2 \otimes \epsilon(\eta_2)),$$

where  $\eta_1, \eta_2$  form an orthonormal basis of  $S_-$ .

To summarise the results of this Section,

**Proposition 6.4.** *The index of  $T$  equals  $\text{rank}(E) \deg(f_T)$ .*

*Remark.* Compare this with the non-singularity discussion (cf. Corollary 3.8). This shows the inverse Nahm transform bundle  $\hat{E}$  has rank equal to  $\text{rank}(E)$  in the interior of  $B$  and vanishes in the exterior, thus proving Lemma 3.14. In particular there is a jump of index when we cross the boundary, so the operator  $T = \sum_\mu (\hat{x}_\mu - y_\mu) \hat{c}_\mu$  fails to be Fredholm when  $y \in \partial B$ .

6.2. *Singularity formation and small spectrum.* The aim of this Section is to show Theorem 1.4. The key input is the following result, which combines our local Nahm transform theory with our analytic convergence theory for singularity formation.

**Lemma 6.5.** *In the setup of a 1-parameter family of ASD connections  $A_t$  developing a curvature singularity at the origin, we have the exact sequence*

$$0 \rightarrow E \xrightarrow{\alpha_t} H^2_{D_{A_t}} \otimes S_- \xrightarrow{\sum_{\mu} (\hat{x}_{\mu} - y_{\mu}) \hat{c}_{\mu}} H^2_{D_{A_t}} \otimes S_+ \rightarrow 0. \tag{58}$$

In the limit  $t \rightarrow \infty$ , the operator  $\sum_{\mu} (\hat{x}_{\mu} - y_{\mu}) \hat{c}_{\mu}$  tends to

$$(H^2_{D_{A_{\infty}}} \oplus V) \otimes S_- \xrightarrow{\sum_{\mu} (\hat{x}_{\mu} \oplus 0 - y_{\mu}) \hat{c}_{\mu}} (H^2_{D_{A_{\infty}}} \oplus V) \otimes S_+,$$

where convergence holds on the whole ball, and  $\alpha_t$  tends to

$$E \xrightarrow{\alpha_{\infty} \oplus 0} (H^2_{D_{A_{\infty}}} \oplus V) \otimes S_-,$$

where convergence holds on  $\{0 < |x| < R\}$ , and is uniform on compact subsets.

*Proof.* The existence of the exact sequence is rephrasing the reconstruction theorem 3.15.

The convergence of  $(\hat{x}_{\mu} - y_{\mu}) \hat{c}_{\mu}$  follows from the convergence theory of Toeplitz operators (cf. Proposition 5.15), applied to  $f = x_{\mu}$ . The convergence of the canonical comparison maps  $\alpha_t$  follows from Proposition 5.16, because  $\alpha_t$  is essentially defined as the evaluation map of the Green operator.  $\square$

*Remark.* The heuristic idea of Theorem 1.4 is that, by (2) we can think of the instanton number as a ‘Chern class’. Since we have the exact sequence (58), we think of the ‘Chern class’ of  $E$  as a ‘difference of the Chern classes for infinite rank bundles’, and compute it by topological manipulations. This idea involves several difficulties:

- This involves infinite rank bundles.
- The domain is not a closed manifold.
- In the limit the exactness fails.

The main idea to remedy these, is to replace vector bundles by relative K-theory classes, and work in the Fredholm setting.

We sketch the following relative version of the index bundle construction, using only first principles in K-theory.

**Lemma 6.6.** *Let  $X$  be a compact connected manifold with boundary  $Y$ . Given the data*

- A family of Fredholm maps between (possibly finite rank) Hilbert bundles, defined over  $X$ :

$$\mathcal{H}_1 \xrightarrow{\mathcal{F}} \mathcal{H}_2.$$

- A trivial finite rank vector bundle  $\tilde{E}|_Y$  over  $Y$ , with a morphism  $\alpha$  into  $\ker \mathcal{F}|_Y$ , such that the complex

$$0 \rightarrow \tilde{E}|_Y \xrightarrow{\alpha} \mathcal{H}_1|_Y \xrightarrow{\mathcal{F}} \mathcal{H}_2|_Y \rightarrow 0$$

is a short exact sequence.

Then we obtain a natural class in the relative K-theory  $K(X, Y)$ . Denote this as  $\text{Ind} [\mathcal{H}_1 \xrightarrow{\mathcal{F}} \mathcal{H}_2, \tilde{E}|_Y, \alpha]$ . Moreover, this class is invariant under small collective deformations of all the defining data, and the construction is naturally additive.

*Proof.* (Sketch) We follow the following steps:

1. Since  $X$  is connected, the index of  $\mathcal{F}$  over each point in  $X$  is constant. So if  $\mathcal{F}$  is fibrewise surjective, then the kernel bundle is a finite rank vector bundle over  $X$  with a trivialisation over  $Y$ , so defines a relative K-theory class. Moreover, the construction is invariant under small deformations of  $\mathcal{F}$  away from  $Y$ .
2. The surjectivity of  $\mathcal{F}$  is satisfied near  $Y$ . If it fails somewhere away from  $Y$ , we can replace  $\mathcal{H}_1$  by  $\mathcal{H}_1 \oplus \mathbb{C}^N$ , where  $\mathbb{C}^N$  is a trivial vector bundle with sufficiently large rank. Now perturb  $\mathcal{F}$  by adding a morphism  $p : \mathbb{C}^N \rightarrow \mathcal{H}_2$ . Generically this will force surjectivity to hold. We require  $p$  to vanish near  $Y$ ; this uses the existence of local cutoff functions. We define the class by

$$\text{Ind} [\mathcal{H}_1 \oplus \mathbb{C}^N \xrightarrow{\mathcal{F} \oplus p} \mathcal{H}_2, \tilde{E}|_Y \oplus \mathbb{C}^N|_Y, \alpha \oplus \text{Id}_{\mathbb{C}^N}].$$

3. We check the well definition of the above construction. Clearly

$$\begin{aligned} &\text{Ind} [\mathcal{H}_1 \oplus \mathbb{C}^N \xrightarrow{\mathcal{F} \oplus p} \mathcal{H}_2, \tilde{E}|_Y \oplus \mathbb{C}^N|_Y, \alpha \oplus \text{Id}_{\mathbb{C}^N}] \\ &= \text{Ind} [\mathcal{H}_1 \oplus \mathbb{C}^{N+N'} \xrightarrow{\mathcal{F} \oplus p \oplus 0} \mathcal{H}_2, \tilde{E}|_Y \oplus \mathbb{C}^{N+N'}|_Y, \alpha \oplus \text{Id}_{\mathbb{C}^{N+N'}}]. \end{aligned}$$

Moreover any two choices of perturbations  $p$  and  $p'$  are connected by some path. This path may pass through some elements  $p''$  which do not make  $\mathcal{F} \oplus p''$  surjective. But there is some  $N'$  such that  $p \oplus 0 : \mathbb{C}^{N+N'} \rightarrow \mathcal{H}_2$  is connected to  $p' \oplus 0$  by some non-singular path, because we can always use the extra degrees of freedom to remove any failure of transversality. The upshot is that by homotopy invariance the class is independent of the perturbation  $p$ , and is clearly independent of  $N$ .

4. Suppose we have a one parameter family of the data  $(\mathcal{H}_1 \xrightarrow{\mathcal{F}} \mathcal{H}_2, \tilde{E}|_Y, \alpha)$ , parametrised by  $t \in [0, 1]$ , with constraints as given in the theorem. We claim they define the same class. This is because without loss of generality  $\mathcal{F}$  is surjective, and then the claim reduces to the usual homotopy invariance property.
5. The additivity is obvious.  $\square$

Back to our context, the characteristic number description (2) gives

**Lemma 6.7.** *The instanton number is equal to the second Chern class of the relative K-theory class defined by the pair  $(E, \tilde{E}|_{\partial B(\mathbb{R}/2)})$ , under the Chern-Weil homomorphism*

$$c_2 : K(B(\frac{\mathbb{R}}{2}), \partial B(\frac{\mathbb{R}}{2})) \rightarrow H^4(B(\frac{\mathbb{R}}{2}), \partial B(\frac{\mathbb{R}}{2})).$$

**Lemma 6.8.** *The class defined by the pair  $(E, \tilde{E}|_{\partial B(\mathbb{R}/2)})$  inside the relative K-theory  $K(B(\frac{\mathbb{R}}{2}), \partial B(\frac{\mathbb{R}}{2}))$  is*

$$(\dim V) \text{Ind} [S_- \xrightarrow{\sum y_\mu \hat{c}_\mu} S_+, 0, 0].$$

*Proof.* Using the exact sequence (58), this class is

$$\text{Ind} [H^2_{D_{A_t}} \otimes S_- \rightarrow H^2_{D_{A_t}} \otimes S_+, \tilde{E}|_{\partial B(R/2)}, \alpha_t],$$

which by continuity is

$$\text{Ind} [(H^2_{D_{A_\infty}} \oplus V) \otimes S_- \rightarrow (H^2_{D_{A_\infty}} \oplus V) \otimes S_+, \tilde{E}|_{\partial B(R/2)}, \alpha_\infty].$$

By the additivity of the index bundle construction, this is

$$\text{Ind} [H^2_{D_{A_\infty}} \otimes S_- \rightarrow H^2_{D_{A_\infty}} \otimes S_+, \tilde{E}|_{\partial B(R/2)}, \alpha_\infty] + \text{Ind} [V \otimes S_- \rightarrow V \otimes S_+, 0, 0].$$

Now by the analogue of the exact sequence (58) applied to  $A_\infty$ , the first summand is the relative K-theory class defined by the pair  $(\tilde{E}, \tilde{E}|_{\partial B(R/2)})$ , which is trivial. So the class simplifies to  $(\dim V)\text{Ind} [S_- \xrightarrow{\sum y_\mu \hat{c}_\mu} S_+, 0, 0]$ .  $\square$

**Lemma 6.9.** *The  $c_2$  of the relative K-theory class  $\text{Ind} [S_- \xrightarrow{\sum y_\mu \hat{c}_\mu} S_+, 0, 0]$  is 1.*

*Proof.* Using the construction of the index bundle,

$$\text{Ind} [S_- \xrightarrow{\sum y_\mu \hat{c}_\mu} S_+, 0, 0] = \text{Ind} [\mathbb{C}^2 \oplus S_- \xrightarrow{0 \oplus \sum y_\mu \hat{c}_\mu} S_+, \mathbb{C}^2|_{\partial B(R/2)}, Id_{\mathbb{C}^2} \oplus 0].$$

To make sense of this, we need to apply a perturbation to the map

$$\mathbb{C}^2 \oplus S_- \xrightarrow{0 \oplus \sum y_\mu \hat{c}_\mu} S_+$$

to make it surjective. A particular choice is given by the map appearing in the ADHM construction of the standard 1-instanton (cf. [10], Section 3.4.1). Thus the relative K-theory class is given by the class of the 1-instanton, together with a trivialisation near infinity. This has charge  $c_2 = 1$ .  $\square$

Combining these Lemmas give Theorem 1.4.

*Acknowledgements.* This work is part of the author’s PhD first year minor projects, supervised by S.K. Donaldson and J.D. Lotay. The author also thanks S. Salamon for discussions, and the referees for useful comments.

**Open Access** This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

## References

1. Atiyah, M.F.: *K-Theory*. W. A. Benjamin, New York (1967)
2. Bartocci, C., Bruzzo, U., Hernández Ruipérez, D.: A hyper-Kähler Fourier transform. *Differ. Geom. Appl.* **8**(3), 239–249 (1998)
3. Böttcher, A., Silbermann, B.: *Analysis of Toeplitz Operators*. Springer Monographs in Mathematics, 2nd edn. Springer, Berlin (2006)
4. Braam, P.J., van Baal, P.: Nahm's transformation for instantons. *Commun. Math. Phys.* **122**(2), 267–280 (1989)
5. Cherkis, S.A., Hurtubise, J.: Monads for instantons and bows. [arXiv:1709.00145](https://arxiv.org/abs/1709.00145)
6. Christ, N.H.: Self-dual Yang–Mills solutions. Complex manifold techniques in theoretical physics. In: *Proceedings of Workshop, Lawrence, Kansas, 1978*, pp. 45–54, *Research Notes in Mathematics*, 32. Pitman, Boston (1979)
7. Corrigan, E., Goddard, P.: Construction of instanton and monopole solutions and reciprocity. *Ann. Phys.* **154**(1), 253–279 (1984)
8. Connes, A.: *Geometry and the Quantum. Foundations of Mathematics and Physics One Century After Hilbert*, pp. 159–196. Springer, Cham (2018)
9. Donaldson, S.K.: The approximation of instantons. *Geom. Funct. Anal.* **3**(2), 179–200 (1993)
10. Donaldson, S.K., Kronheimer, P.B.: *The Geometry of Four-Manifolds*. Oxford Mathematical Monographs. Oxford Science Publications, Oxford (1990)
11. Donaldson, S., Segal, E.: Gauge theory in higher dimensions, II. *Surveys in differential geometry. Volume XVI. Geometry of special holonomy and related topics*, pp. 1–41, *Surveys in Differential Geometry*, 16. Int. Press, Somerville, MA, (2011)
12. Donaldson, S., Sun, S.: Gromov-Hausdorff limits of Kähler manifolds and algebraic geometry. II. *J. Differ. Geom.* **107**(2), 327–371 (2017)
13. Jardim, M.: A survey on Nahm transform. *J. Geom. Phys.* **52**(3), 313–327 (2004)
14. Kodaira, K.: *Complex Manifolds and Deformation of Complex Structures*. Translated from the 1981 Japanese original by Kazuo Akao. Reprint of the 1986 English Edition. *Classics in Mathematics*. Springer, Berlin (2005)
15. Witten, E.: Some comments on the recent twistor space constructions. Complex manifold techniques in theoretical physics. In: *Proceedings of Workshop, Lawrence, Kansas, 1978*, pp. 207–218, *Research Notes in Mathematics*, 32. Pitman, Boston (1979)

Communicated by N. Nekrasov