q -Virasoro Modular Triple

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Abstract: Inspired by 5d supersymmetric Yang–Mills theories placed on the compact space S^5 , we propose an intriguing algebraic construction for the q -Virasoro algebra. We show that, when multiple q -Virasoro “chiral” sectors have to be fused together, a natural $SL(3, \mathbb{Z})$ structure arises. This construction, which we call the modular triple, is consistent with the observed triple factorization properties of supersymmetric partition functions derived from localization arguments. We also give a 2d CFT-like construction of the modular triple, and conjecture for the first time a (non-local) Lagrangian formulation for a q -Virasoro model, resembling ordinary Liouville theory.

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1. Introduction

The discovery of algebraic structures in quantum field theories has always led to dramatic improvements of our understanding thereof. The hidden Yangian symmetry of planar 4d

$\mathcal{N} = 4$ Yang–Mills [1] and the Virasoro/W symmetry of 4d $\mathcal{N} = 2$ class $\mathcal{S}$ theories [2], a.k.a. the AGT correspondence [3], are perhaps the most famous examples. In the former case, the infinite dimensional symmetry algebra points towards the complete integrability of the model and has helped in computing the exact S-matrix (see e.g. [4] for a review). In the latter case, the 2d CFT Liouville/Toda description of the gauge theory placed on the compact space $\mathbb{S}^4$ [5] has allowed to establish a powerful dictionary between observables of the two sides. More generally, the circle of ideas around the BPS/CFT correspondence (see e.g. [6] for a recent review) has led to an enormous amount of works relating supersymmetric gauge theories and (quantum) algebras (see e.g. [7, 8]). For the purposes of this paper, the most interesting aspects are the identification of 5d Nekrasov partition functions on $\mathbb{R}_{q,t}^4 \times \mathbb{S}^1$ [9–14] with “chiral” q -Virasoro/W correlators [15–26], as well as the extension of the AGT duality to 5d $\mathcal{N} = 1$ supersymmetric gauge theories on compact spaces and q -deformed CFTs with multiple “chiral” sectors [27–30].

While the original 4d AGT setup is now quite well understood [31] thanks to the 6d (2, 0) SCFT construction of class $\mathcal{S}$ theories and the Lagrangian formulation of Liouville/Toda theory, the 5d picture is less developed. Among the main obstacles, we can recall that the origin of the q -deformation has been proposed to lie in the 6d (2, 0) Little String Theory [32], much less under control than its conformal limit, and the lack of any Lagrangian q -CFT model with underlying q -Virasoro/W symmetry. In this paper, we will focus on the algebraic side of the BPS/CFT correspondence, trying to shed some new light on q -Virasoro systems by taking inspiration from the gauge theory results.

Our strategy is to mimic as much as possible the structure of ordinary 2d CFTs, including the simplest real free boson theory. An important feature of these models is the holomorphic/anti-holomorphic factorization of physical (monodromy invariant) correlators, which is due to the existence of two chiral sectors (left and right moving) each associated to an independent copy of the Virasoro algebra. It is natural to expect a similar structure in the q -deformed case too, in which multiple q -Virasoro “chiral” sectors are suitably glued together. Here it is where the gauge theory inspiration comes into play. Among all the possible pairings, the study of 5d supersymmetric gauge theories on $\mathbb{S}^5$ [28, 33–40] (and/or co-dimension 2 defects on $\mathbb{S}^3$ [41–43]) suggests that one should consider 2 or 3 “chiral” sectors glued together in an $\mathrm{SL}(2, \mathbb{Z})$ or $\mathrm{SL}(3, \mathbb{Z})$ fashion. In this case, it has also been shown that the resulting correlators are consistent with a q -deformed version of the bootstrap equations in Liouville theory [27].

The $\mathrm{SL}(2, \mathbb{Z})$ -pairing of two q -Virasoro sectors has been studied in detail in [44], and the emerging algebraic structure has been given the name *modular double*, in analogy with the $\mathcal{U}_q(\mathfrak{sl}_2)$ modular double introduced in [45–48]. In the case of major interest to us, the $\{q_i, t_i\}_{i=1,2}$ deformation parameters of the two commuting q -Virasoro algebras are related by the $S \in \mathrm{SL}(2, \mathbb{Z})$ element

$$(q_1 = e^{2\pi i \tau}, t_1 = e^{2\pi i \sigma}) \xrightarrow{-S} (q_2 = e^{2\pi i/\tau}, t_2 = e^{2\pi i \sigma/\tau}),$$

and it has been shown that there exist two commuting sets of Ward Identities that correlators of the modular double must satisfy. The whole structure can be compactly encoded into the existence of a *single screening current* $\mathbf{S}(X)$ defined by (roughly)

$$[\mathbf{T}_{m,i}, \mathbf{S}(X)] = \text{total } X\text{-difference,}$$

where $\{\mathbf{T}_{m,i}, m \in \mathbb{Z}\}_{i=1,2}$ are two commuting copies of q -Virasoro generators. This is strongly reminiscent of what happens in ordinary 2d CFT, where a primary operator

(screening current) $\mathbf{s}(z, \bar{z})$ of conformal dimension $(1, 1)$ satisfies

$$[\mathbf{L}_m, \mathbf{s}(z, \bar{z})] = \text{total } z\text{-derivative}, \quad [\bar{\mathbf{L}}_m, \mathbf{s}(z, \bar{z})] = \text{total } \bar{z}\text{-derivative},$$

where $\{\mathbf{L}_m \mid m \in \mathbb{Z}\}$ and $\{\bar{\mathbf{L}}_m \mid m \in \mathbb{Z}\}$ are the Virasoro generators of the two chiral sectors of the conformal algebra.

One of the main results of this paper is that the above $\text{SL}(2, \mathbb{Z})$ -pairing can (in fact, must) be enhanced to an $\text{SL}(3, \mathbb{Z})$ -pairing, in which *three* q -Virasoro “chiral” sectors with deformation parameters $\{q_i, t_i\}_{i=1,2,3}$ are related as follows

$$\begin{array}{ccc} & (q_1 = e^{2\pi i \tau}, t_1 = e^{2\pi i \sigma}) & \\ \swarrow & & \nwarrow \\ (q_2 = e^{-2\pi i \sigma/\tau}, t_2 = e^{-2\pi i/\tau}) & \xrightarrow{\quad\quad\quad} & (q_3 = e^{-2\pi i/\sigma}, t_3 = e^{2\pi i \tau/\sigma}). \end{array}$$

We can call this structure the q -Virasoro *modular triple*, and it can be compactly encoded into the existence of three screening currents $\mathbf{S}_{12}(X)$, $\mathbf{S}_{23}(X)$, $\mathbf{S}_{31}(X)$ satisfying the relations (for all $i = 1, 2, 3$)

$$[\mathbf{T}_{m,i}, \mathbf{S}_{12}(X)] = [\mathbf{T}_{m,i}, \mathbf{S}_{23}(X)] = [\mathbf{T}_{m,i}, \mathbf{S}_{31}(X)] = \text{total } X\text{-difference},$$

eventually leading to the existence of three commuting sets of Ward Identities that correlators of the modular triple must satisfy. It is clear that any two pairs among the above deformation parameters are related by the same $S \in \text{SL}(2, \mathbb{Z})$ element as before, up to $q \leftrightarrow t^{-1}$ exchanges. Therefore, one can extract three different modular doubles out of the triple by simply neglecting a “chiral” sector, in a cyclic order. Vice versa, the triple can be constructed by gluing three doubles upon identification of common “chiral” sectors. This reasoning also reveals the need, origin and uniqueness of the $\text{SL}(3, \mathbb{Z})$ structure: the $q \leftrightarrow t^{-1}$ exchange symmetry of the q -Virasoro algebra combines with the $\text{SL}(2, \mathbb{Z})$ structure of the double and enhances to the $\text{SL}(3, \mathbb{Z})$ structure in the triple, establishing full democracy among the (τ, σ) parameters which otherwise would be on a different footing. Hopefully, our algebraic construction can serve as a basis for a better understanding of the triple factorization structure of the $\mathbb{S}^5$ partition function of supersymmetric Yang–Mills theories derived from localization, as well as for developing further the proposal of [33] (see also [49]) for the non-perturbative completion of refined topological strings [50–52].

The rest of this paper is organized as follows. In Sect. 2, we review the basics of the q -Virasoro algebra to set the notations, as well as the modular double construction. In Sect. 3, we fuse three modular doubles together and introduce the modular triple. In Sect. 4, we give a 2d CFT-like construction of the modular triple, conjecturing for the first time a Lagrangian formulation for a q -Virasoro system, resembling a non-local version of ordinary Liouville theory. In Sect. 5, we further our discussion and outline possible applications and directions for future research. We supplement the paper with two appendices, where we collect the definition of special functions used in the main body and some technical computation.

2. q -Virasoro and Its Modular Double

2.1. q -Virasoro algebra and notations. We start by reviewing basics of the q -Virasoro algebra and its free boson realization through the Heisenberg algebra [53–55]. We will also introduce some notations for later convenience.

The q -Virasoro algebra is the associative algebra generated by $\{\mathbf{T}_m, m \in \mathbb{Z}\}$ ¹ with parameters $q, t \in \mathbb{C}$, $p \equiv qt^{-1}$, satisfying the commutation relations

$$\sum_{k \geq 0} f_k (\mathbf{T}_{m-k} \mathbf{T}_{n+k} - \mathbf{T}_{n-k} \mathbf{T}_{m+k}) = -\frac{(1-q)(1-t^{-1})}{1-p} (p^m - p^{-m}) \delta_{m+n,0}. \quad (2.1)$$

The constants f_k are encoded into the series expansion of the structure function $f(z) \equiv \exp[\sum_{m>0} \frac{(1-q^m)(1-t^{-m})}{m(1+p^m)} z^m]$, namely $f(z) = \sum_{k=0}^{+\infty} f_k z^k$. The above commutation relations can be reorganized as

$$\begin{aligned} & f\left(\frac{w}{z}\right) \mathbf{T}(z) \mathbf{T}(w) - f\left(\frac{z}{w}\right) \mathbf{T}(w) \mathbf{T}(z) \\ &= -\frac{(1-q)(1-t^{-1})}{1-p} \left(\delta\left(p \frac{w}{z}\right) - \delta\left(p^{-1} \frac{z}{w}\right) \right), \end{aligned} \quad (2.2)$$

where we defined the q -Virasoro current $\mathbf{T}(z) \equiv \sum_{m \in \mathbb{Z}} \mathbf{T}_m z^{-m}$ and the multiplicative δ function $\delta(z) \equiv \sum_{m \in \mathbb{Z}} z^m$. We notice that the q -Virasoro algebra is invariant under $q \leftrightarrow t^{-1}$, which is also known as quantum q -geometric Langlands equivalence [23, 56]. Under this swapping p is invariant, namely $p \leftrightarrow p$.

The q -Virasoro algebra admits a representation in terms of the Heisenberg algebra, where the latter is generated by oscillators $\{\mathbf{a}_m, m \in \mathbb{Z}_{\neq 0}\}$ and zero modes $\mathbf{P}, \mathbf{Q}$, with nontrivial commutation relations (here we follow the conventions of [16])

$$[\mathbf{a}_m, \mathbf{a}_n] = -\frac{1}{m} (q^{m/2} - q^{-m/2}) (t^{-m/2} - t^{m/2}) C^{[m]}(p) \delta_{m+n,0}, \quad [\mathbf{P}, \mathbf{Q}] = 2, \quad (2.3)$$

where $C^{[m]}(p) = (p^{m/2} + p^{-m/2})$ is the deformed Cartan matrix of the A_1 algebra. The q -Virasoro current $\mathbf{T}(z)$ can be represented as

$$\mathbf{T}(z) = \mathbf{Y}(p^{-1/2}z) + \mathbf{Y}(p^{1/2}z)^{-1}, \quad \mathbf{Y}(z) = \exp \left[\sum_{m \neq 0} \frac{\mathbf{a}_m z^{-m}}{p^{m/2} + p^{-m/2}} \right] q^{\sqrt{\beta} \mathbf{P}/2} p^{1/2}, \quad (2.4)$$

where $\beta \in \mathbb{C}$ is such that $q^\beta = t$ and normal ordering of the modes is implicitly assumed. Note that the $q \leftrightarrow t^{-1}$ requires $\sqrt{\beta} \leftrightarrow -1/\sqrt{\beta}$ under this swapping.

A central object in q -Virasoro theory is played by the *screening current* $\mathbf{S}_+(x)$. If we introduce the ξ -shift operator $\widehat{T}_\xi$ acting in the multiplicative notation on x as

$$\widehat{T}_\xi f(x) = f(\xi x), \quad (2.5)$$

the defining property of the screening current is

$$[\mathbf{T}_m, \mathbf{S}_+(x)] = \frac{\widehat{T}_q - 1}{x} (\dots) \quad \Rightarrow \quad \left[\mathbf{T}_m, \oint dx \mathbf{S}_+(x) \right] = 0, \quad (2.6)$$

where the dots represent some (uninteresting) operator (we refer to [53] for details).

¹ We will use straight letters in boldface, like $\mathbf{T}, \mathbf{a}, \mathbf{P}, \mathbf{Q}$ and so, to denote operators.

Remark. A suitable specification of integration contour is to be understood. For multiple screening currents, the actual choice depends on the particular correlator one wishes to compute. Here, we can take it to be a small circle around the origin which simply extracts the constant term of $x \mathbf{S}_+(x)$. Generally, one has to consider an invariant contour w.r.t. multiplicative q -shifts avoiding singularities of the integrand.

The contour integral of $\mathbf{S}_+(x)$ is called the *screening charge*, and the q -Virasoro algebra can be indeed defined as the commutant of the screening charge in the Heisenberg algebra. Concretely, the screening current can be realized by the following operator

$$\mathbf{S}_+(x) \equiv \exp \left[- \sum_{m \neq 0} \frac{\mathbf{a}_m x^{-m}}{q^{m/2} - q^{-m/2}} + \sqrt{\beta} \mathbf{Q} \right] \frac{\Theta(x q^{-\sqrt{\beta} \mathbf{P}}; q)}{\Theta(x; q) \Theta(q^{-\sqrt{\beta} \mathbf{P}}; q)}. \quad (2.7)$$

Remark. The dependence on the momentum operator $\mathbf{P}$ adopted here is different from the usual definition in existing literature, which is simply $x^{\sqrt{\beta} \mathbf{P}}$. Our redefinition is possible because $[\mathbf{T}(z), \mathbf{P}] = 0$ in the free boson realization, and the substitution $\mathbf{S}(x) \rightarrow \Omega(x) \mathbf{S}(x)$ does not spoil the property (2.9) provided that $\Omega(x)$ is an operator containing only $\mathbf{P}$ with the property $\Omega(qx) = \Omega(x)$. Even though this replacement is not necessary for our purposes, it will be useful later on for emphasizing the similarities with the undeformed setup.

A crucial observation is that the screening current above is not the only operator that commutes with $\mathbf{T}_m$. In fact, one can define a similar screening current $\mathbf{S}_-(x)$ and charge using the $q \leftrightarrow t^{-1}$ symmetry, namely

$$\mathbf{S}_-(x) \equiv \exp \left[- \sum_{m \neq 0} \frac{\mathbf{a}_m x^{-m}}{t^{-m/2} - t^{m/2}} - \frac{\mathbf{Q}}{\sqrt{\beta}} \right] \frac{\Theta(x t^{-\sqrt{\beta}^{-1} \mathbf{P}}; t^{-1})}{\Theta(x; t^{-1}) \Theta(t^{-\sqrt{\beta}^{-1} \mathbf{P}}; t^{-1})}, \quad (2.8)$$

with the property that

$$[\mathbf{T}_m, \mathbf{S}_-(x)] = \frac{\widehat{T}_{t^{-1}} - 1}{x} (\dots) \Rightarrow \left[\mathbf{T}_m, \oint dx \mathbf{S}_-(x) \right] = 0. \quad (2.9)$$

2.2. q -Virasoro modular double. We recall that in standard 2d free boson theory, or more general physical 2d CFTs, the degrees of freedom factorize into holomorphic and anti-holomorphic sectors, each having an independent copy of Virasoro symmetry algebra generated by $\{\mathbf{L}_m, m \in \mathbb{Z}\}$ and $\{\bar{\mathbf{L}}_m, m \in \mathbb{Z}\}$ respectively. However, the two chiral sectors do not decouple completely as they share the same zero modes. Concretely, the real free boson $\phi(z, \bar{z})$ has on-shell mode expansion

$$\phi(z, \bar{z}) = \sum_{m \neq 0} a_m z^{-m} + \sum_{m \neq 0} \bar{a}_m \bar{z}^{-m} - P \ln z \bar{z} - Q, \quad (2.10)$$

while for most of the purposes one can work with either one of the two chiral fields

$$\begin{aligned} \phi(z) &\equiv \sum_{m \neq 0} a_m z^{-m} - P \ln z - Q, \\ \bar{\phi}(\bar{z}) &\equiv \sum_{m \neq 0} \bar{a}_m \bar{z}^{-m} - P \ln \bar{z} - Q, \end{aligned} \quad (2.11)$$

and restore the physical dependence on z and $\bar{z}$ successively. Upon quantization, a_m , $\bar{a}_m$ and P , Q are promoted to operators. Exponential operators of the fundamental field $\phi(z, \bar{z})$ are primary operators, and amongst them there are two with conformal dimension $(1, 1)$

$$\mathbf{s}_{\pm}(z, \bar{z}) \equiv e^{\mp \sqrt{\beta^{\pm 1}} \phi(z, \bar{z})}. \quad (2.12)$$

These operators satisfy the crucial relations

$$[\mathbf{L}_m, \mathbf{s}_{\pm}(z, \bar{z})] = \partial_z(\cdots), \quad [\bar{\mathbf{L}}_m, \mathbf{s}_{\pm}(z, \bar{z})] = \partial_{\bar{z}}(\cdots), \quad (2.13)$$

and hence their holomorphic and anti-holomorphic components

$$\mathbf{s}_{\pm}(z) \equiv e^{\mp \sqrt{\beta^{\pm 1}} \phi(z)}, \quad \bar{\mathbf{s}}_{\pm}(\bar{z}) \equiv e^{\mp \sqrt{\beta^{\pm 1}} \bar{\phi}(\bar{z})} \quad (2.14)$$

can be used to define the screening currents and charges of each copy of the Virasoro algebra. It is worth noting that the exchange symmetry $\sqrt{\beta} \leftrightarrow -1/\sqrt{\beta}$ is the analogous of the $q \leftrightarrow t^{-1}$ symmetry of the q -Virasoro algebra, while the existence of two (holomorphic and anti-holomorphic) chiral sectors should be accounted by multiple q -Virasoro sectors. How this can consistently be done in a non-trivial way was explored in [44], and here we recall the basic ideas before generalizing that construction.

Since the q -Virasoro algebra is as a deformation of the ordinary Virasoro algebra, it is natural to seek the q -deformed counterpart of (2.13), given two commuting q -Virasoro algebras. In doing so, we can consider (loosely speaking) deformations of the form

$$z \rightarrow x^{(\text{I})} = e^{2\pi i \frac{X}{\omega_1}}, \quad \bar{z} \rightarrow x^{(\text{II})} = e^{2\pi i \frac{X}{\omega_2}}, \quad \mathbf{L}_m \rightarrow \mathbf{T}_m^{(\text{I})}, \quad \bar{\mathbf{L}}_m \rightarrow \mathbf{T}_m^{(\text{II})}, \quad (2.15)$$

where $\omega_{i=1,2}$ are two (possibly complex) generic² *equivariant parameters* and X a (possibly complex) coordinate. We stress that the “chiral” variables $x^{(\text{I,II})}$ are neither independent or complex conjugate to each other. Rather, they introduce two different periodicities in the X -direction. More precisely, let us consider two q -Virasoro algebras labeled by $A = \text{I, II}$ and with deformation parameters

$$q^{(\text{I})} \equiv e^{2\pi i \frac{\omega_{12}}{\omega_1}}, \quad t^{(\text{I})} \equiv e^{-2\pi i \frac{\omega_3}{\omega_1}}, \quad q^{(\text{II})} \equiv e^{2\pi i \frac{\omega_{12}}{\omega_2}}, \quad t^{(\text{II})} \equiv e^{-2\pi i \frac{\omega_3}{\omega_2}}, \quad (2.16)$$

$$\omega_{12} \equiv \omega_1 + \omega_2, \quad \omega_3 \equiv -\beta \omega_{12}, \quad (2.17)$$

which can be realized by two commuting Heisenberg algebras $\{\mathbf{a}_m^{(A)}, m \in \mathbb{Z}/\{0\}\}_{A=\text{I,II}}$ together with the common zero modes $\mathbf{P}, \mathbf{Q}$. Notice that we can trade β for the third equivariant parameter ω_3 , which is however on different footing w.r.t. ω_1 and ω_2 (for the moment). Then one can verify that the *modular double screening current* $\mathbf{S}(X)$ given by³

² If not stated otherwise, throughout this paper we will assume the equivariant parameters to be generic, namely for any $i \neq j$, ω_i/ω_j will be an irrational number which can be given an imaginary part if needed.

³ For later convenience, we have extracted a square root of $e^{2\pi i m}$, resulting in the $(-1)^m$ factor.

$$\begin{aligned} \mathbf{S}(X) \equiv \exp \left[- \sum_{m \neq 0} \frac{(-1)^m \mathbf{a}_m^{(\text{I})} e^{-m \frac{2\pi i}{\omega_1} X}}{e^{i\pi m \omega_2 / \omega_1} - e^{-i\pi m \omega_2 / \omega_1}} \right. \\ \left. - \sum_{m \neq 0} \frac{(-1)^m \mathbf{a}_m^{(\text{II})} e^{-m \frac{2\pi i}{\omega_2} X}}{e^{i\pi m \omega_1 / \omega_2} - e^{-i\pi m \omega_1 / \omega_2}} \right. \\ \left. + \sqrt{\beta} \mathbf{Q} + \frac{2\pi i \omega_{12} \sqrt{\beta} \mathbf{P}}{\omega_1 \omega_2} X \right], \end{aligned} \quad (2.18)$$

satisfies the crucial relations

$$[\mathbf{T}_m^{(\text{I}, \text{II})}, \mathbf{S}(X)] = e^{-\frac{2\pi i \omega_{12} X}{\omega_1 \omega_2}} (T_{\lambda_{\text{I}, \text{II}}} - 1) (\cdots), \quad \lambda_{\text{I}, \text{II}} = \omega_{2, 1}, \quad (2.19)$$

where we have defined the ε -shift operator T_ε acting in the additive notation on X as

$$T_\varepsilon f(X) = f(X + \varepsilon). \quad (2.20)$$

Remark. Notice that the screening current encodes two independent sets of oscillators, but the zero mode operators are shared, as in standard 2d CFT (2.10). This is a more precise construction of the modular double compared to [44].

Remark. The name modular double comes from the fact that the screening current $\mathbf{S}(X)$ shows a manifest $\text{SL}(2, \mathbb{Z})$ structure. In fact, let $S \in \text{SL}(2, \mathbb{Z})$ act on certain modular parameters τ, σ, χ by

$$(\tau, \sigma, \chi) \xrightarrow{S} (-1/\tau, -\sigma/\tau, -\chi/\tau). \quad (2.21)$$

Then we can define $\tau \equiv \omega_2/\omega_1, \sigma \equiv -\omega_3/\omega_1, \chi \equiv X/\omega_1$ and $q^{(\text{I})} = e^{2\pi i \tau}, t^{(\text{I})} = e^{2\pi i \sigma}, x^{(\text{I})} = e^{2\pi i \chi}$, and see that

$$(q^{(\text{I})}, t^{(\text{I})}, x^{(\text{I})}) \xrightarrow{S} (q^{(\text{II})}, t^{(\text{II})}, x^{(\text{II})}). \quad (2.22)$$

The parametrization in terms of the equivariant parameters ω_i is convenient because the $S \in \text{SL}(2, \mathbb{Z})$ symmetry is simply reflected in the permutation symmetry $\omega_1 \leftrightarrow \omega_2$.

The oscillator part of the screening current is clearly factorized into two “chiral” sectors, while it does not seem so for the zero mode part.⁴ However, analogous to the holomorphic/anti-holomorphic splitting $\ln(z\bar{z}) = \ln z + \ln \bar{z}$, using (A.11) the momentum factor can also be factorized as

$$\begin{aligned} \exp \left[\frac{2\pi i \omega_{12} \sqrt{\beta} \mathbf{P}}{\omega_1 \omega_2} X \right] \\ = \prod_{i=1,2} \frac{\Theta(e^{\frac{2\pi i}{\omega_i} X} e^{-2\pi i \frac{\omega_{12}}{\omega_i} \sqrt{\beta} \mathbf{P}}; e^{\frac{2\pi i \omega_{12}}{\omega_i}})}{\Theta(e^{\frac{2\pi i}{\omega_i} X}; e^{\frac{2\pi i \omega_{12}}{\omega_i}}) \Theta(e^{-2\pi i \frac{\omega_{12}}{\omega_i} \sqrt{\beta} \mathbf{P}}; e^{\frac{2\pi i \omega_{12}}{\omega_i}})} e^{-\frac{i\pi}{6} \left(\frac{\omega_{12}^2}{\omega_1 \omega_2} + 1 \right)}. \end{aligned} \quad (2.23)$$

Thanks to this factorization, one can extract the A^{th} “chiral” component $\mathbf{S}^{(\text{A})}(X)$ of the modular double screening current depending only on the “chiral” variable $x^{(\text{A})}$

⁴ The naive splitting $\frac{\omega_{12} \sqrt{\beta} \mathbf{P} X}{\omega_1 \omega_2} = \frac{\sqrt{\beta} \mathbf{P} X}{\omega_1} + \frac{\sqrt{\beta} \mathbf{P} X}{\omega_2}$ does not respect the ω_i -periodicity in the i -th sector.

$$\begin{aligned}
& \mathbf{S}^{(A)}(X) \\
& \equiv \exp \left[- \sum_{m \neq 0} \frac{\mathbf{a}_m^{(A)} (x^{(A)})^{-m}}{(q^{(A)})^{m/2} - (q^{(A)})^{-m/2} + \sqrt{\beta} \mathbf{Q}} \right] \frac{\Theta(x^{(A)}(q^{(A)})^{-\sqrt{\beta} \mathbf{P}}; q^{(A)})}{\Theta(x^{(A)}; q^{(A)}) \Theta((q^{(A)})^{-\sqrt{\beta} \mathbf{P}}; q^{(A)})},
\end{aligned} \tag{2.24}$$

coinciding with (2.7) upon the identifications $q \sim q^{(A)}$, $t \sim t^{(A)}$, $\mathbf{a}_m \sim \mathbf{a}_m^{(A)}$, $x \sim x^{(A)}$ for fixed $A = \text{I, II}$, and to be compared with the holomorphic/anti-holomorphic components in (2.11), (2.14). In fact, the form of the dependence on $\mathbf{P}$ in (2.7) is useful precisely for thinking about $\mathbf{S}_+(x)$ as a “chiral” component of the modular double screening current. The relation (2.19) immediately follows from this interpretation and (2.9). Remarkably, we can now consider not only the integrals of the “chiral” components $\mathbf{S}^{(A)}(X)$ defining the usual screening charges (as in the ordinary undeformed case), but also (and more interestingly) the integral of $\mathbf{S}(X)$ defining the screening charge of the modular double (of course, a similar remark below (2.9) applies here as well).

To summarize, two independent q -Virasoro algebras with deformation parameters coming in the specific form (2.16) related by $\text{SL}(2, \mathbb{Z})$ can be fused into what we called the modular double, which can be denoted by $\mathcal{MD}(\omega_1, \omega_2; \beta)$. In the equivariant parametrization (2.16), the $S \in \text{SL}(2, \mathbb{Z})$ symmetry simply amounts to the permutation symmetry $\omega_1 \leftrightarrow \omega_2$. The whole structure can be encoded into the existence of a single screening current $\mathbf{S}(X)$ (2.18) which simultaneously commutes (up to total differences) with two independent sets of q -Virasoro generators whose deformation parameters are related by $\text{SL}(2, \mathbb{Z})$.

Before ending this section, we observe that, among other things, the modular double has allowed us to consider two “chiral” sectors resembling the holomorphic/anti-holomorphic structure of ordinary 2d CFTs. However, the analogous of the $\sqrt{\beta} \leftrightarrow -1/\sqrt{\beta}$ symmetry (2.12) has now disappeared because the simple $q \leftrightarrow t^{-1}$ exchange of the “chiral” theory is not allowed anymore. The reason is that in deriving the property (2.19), it is crucial to use the $\omega_{1,2}$ -periodicity in the I, II sectors, which is not respected by the equivariant parameter ω_3 parametrizing $t^{(\text{I,II})}$. It is therefore natural to ask whether it is possible to restore this symmetry by considering a third “chiral” sector with ω_3 -periodicity and such that there is complete democracy among the equivariant parameters ω_1, ω_2 and ω_3 . Remarkably, we give a positive answer to this question in the next section, the key being to combine the “old” $q \leftrightarrow t^{-1}$ symmetry of each “chiral” sector and the “new” $\text{SL}(2, \mathbb{Z})$ structure of the modular double into an $\text{SL}(3, \mathbb{Z})$ action on (τ, σ) .

3. q -Virasoro Modular Triple

In this section, we will generalize the above results and “glue” multiple copies of modular doubles. Let us specify our conventions and introduce some useful notations. Since we will be dealing with multiple commuting copies of q -Virasoro algebras with different parameters, we use the lower case letters $i, j = 1, 2, 3$ with cyclic identification $i \sim i+3$ to label them. Generic equivariant parameters $\omega_{i=1,2,3} \in \mathbb{C}$ are used to parametrize the q -Virasoro parameters q, t in different copies, and we find it useful to also define the combinations

$$\omega_{i,i+1} \equiv \omega_i + \omega_{i+1}, \quad \omega \equiv \omega_1 + \omega_2 + \omega_3. \tag{3.1}$$

We start our discussion by “non-trivially” gluing two modular doubles. The basic requirement is that all the q -Virasoro generators have to commute, up to total differences,

Fig. 1. A modular double is represented by an oriented edge, with the end-points representing the I, II q -Virasoro sectors. Successive modular doubles are glued along a common vertex, where the generators are identified. The two screening currents associated to each vertex behave like $\mathbf{S}_{\pm}$ for the q -Virasoro algebra of that vertex

with all the screening currents. As shown in Fig. 1, we can conveniently represent a modular double by an oriented edge $(i, i + 1)$, with the end-points i and $i + 1$ representing the I and II-th q -Virasoro sectors respectively. In this notation, “non-trivially” means that at the common vertex $i + 1$, sector II of the $(i, i + 1)$ double and sector I of the $(i + 1, i + 2)$ double are identified/glued. At the level of the q -Virasoro generators meeting at that vertex, we will impose the natural constraints

$$\mathbf{T}_{i,i+1}^{(\text{II})}(z) = \mathbf{T}_{i+1,i+2}^{(\text{I})}(z). \quad (3.2)$$

We now investigate the constraints on the q -Heisenberg algebra following from (3.2). For concreteness, let us consider two modular doubles $\mathcal{MD}(\omega_1, \omega_2; \beta_{12})$ and $\mathcal{MD}(\omega_2, \omega_3; \beta_{23})$ generated by the currents $\{\mathbf{T}_{i,i+1}^{(\text{A})}(z)\}_{i=1,2}^{\text{A=I,II}}$ respectively, with two special pairs of parameters

$$\left(q_{12}^{(\text{I})}, t_{12}^{(\text{I})}; q_{12}^{(\text{II})}, t_{12}^{(\text{II})}\right) = \left(e^{2\pi i \frac{\omega_{12}}{\omega_1}}, e^{2\pi i \beta_{12} \frac{\omega_{12}}{\omega_1}}; e^{2\pi i \frac{\omega_{12}}{\omega_2}}, e^{2\pi i \beta_{12} \frac{\omega_{12}}{\omega_2}}\right), \quad (3.3)$$

$$\left(q_{23}^{(\text{I})}, t_{23}^{(\text{I})}; q_{23}^{(\text{II})}, t_{23}^{(\text{II})}\right) = \left(e^{2\pi i \frac{\omega_{23}}{\omega_2}}, e^{2\pi i \beta_{23} \frac{\omega_{23}}{\omega_2}}; e^{2\pi i \frac{\omega_{23}}{\omega_3}}, e^{2\pi i \beta_{23} \frac{\omega_{23}}{\omega_3}}\right). \quad (3.4)$$

Note that we have chosen the q -Virasoro sectors labeled by (II)₁₂ and (I)₂₃, associated to vertex 2, to have the same period ω_2 , which we are about to use for the gluing procedure. Concretely, we identify

$$e^{2\pi i \frac{\omega_3}{\omega_2}} = e^{-2\pi i \beta_{12} \frac{\omega_{12}}{\omega_2}}, \quad e^{2\pi i \beta_{23} \frac{\omega_{23}}{\omega_2}} = e^{-2\pi i \frac{\omega_1}{\omega_2}}, \quad (3.5)$$

which, taking into account the ω_2 -periodicity, is also equivalent to

$$q_{23}^{(\text{I})} = 1/t_{12}^{(\text{II})}, \quad t_{23}^{(\text{I})} = 1/q_{12}^{(\text{II})}, \quad p_{23}^{(\text{I})} = p_{12}^{(\text{II})} = e^{2\pi i \frac{\omega}{\omega_2}}. \quad (3.6)$$

Combining the above parameter identification with $\mathbf{T}_{12}^{(\text{II})}(z) = \mathbf{T}_{23}^{(\text{I})}(z)$, we observe that the Heisenberg operators are constrained by

$$\mathbf{a}_{23,m}^{(\text{I})} = \mathbf{a}_{12,m}^{(\text{II})}, \quad \omega_{12}\sqrt{\beta_{12}}\mathbf{P}_{12} = \omega_{23}\sqrt{\beta_{23}}\mathbf{P}_{23}, \quad \frac{\mathbf{Q}_{12}}{\omega_{12}\sqrt{\beta_{12}}} = \frac{\mathbf{Q}_{23}}{\omega_{23}\sqrt{\beta_{23}}}, \quad (3.7)$$

where the last identification ensures that $[\mathbf{P}_{12}, \mathbf{Q}_{12}] = [\mathbf{P}_{23}, \mathbf{Q}_{23}] = 2$.

The above sequence of identification is possible thanks to the $q \leftrightarrow t^{-1}$ invariance of the q -Virasoro algebra. See Fig. 1 for a visual illustration of the gluing. The invariance further implies that, from the viewpoint of $\mathbf{T}_{23}^{(I)}$, $\mathbf{S}_{12}$ behaves like a $(\mathbf{S}_-)^{(I)}_{23}$ screening current and thus without surprise it commutes with the $\mathbf{T}_{23}^{(I)}$ up to a total difference. More explicitly, we have the relations

$$\begin{aligned}
 [\mathbf{T}_{12,m}^{(I)}, \mathbf{S}_{12}(X)] &= e^{-\frac{2\pi i \omega_{12} X}{\omega_1 \omega_2}} (T_{\omega_2} - 1)(\cdots), \\
 [\mathbf{T}_{12,m}^{(II)}, \mathbf{S}_{12}(X)] &= e^{-\frac{2\pi i \omega_{12} X}{\omega_1 \omega_2}} (T_{\omega_1} - 1)(\cdots), \\
 [\mathbf{T}_{23,m}^{(I)}, \mathbf{S}_{23}(X)] &= e^{-\frac{2\pi i \omega_{23} X}{\omega_2 \omega_3}} (T_{\omega_3} - 1)(\cdots), \\
 [\mathbf{T}_{23,m}^{(II)}, \mathbf{S}_{23}(X)] &= e^{-\frac{2\pi i \omega_{23} X}{\omega_2 \omega_3}} (T_{\omega_2} - 1)(\cdots), \\
 [\mathbf{T}_{12,m}^{(II)}, \mathbf{S}_{23}(X)] &= e^{-\frac{2\pi i \omega_{23} X}{\omega_2 \omega_3}} (T_{\omega_3} - 1)(\cdots), \\
 [\mathbf{T}_{23,m}^{(I)}, \mathbf{S}_{12}(X)] &= e^{-\frac{2\pi i \omega_{12} X}{\omega_1 \omega_2}} (T_{\omega_1} - 1)(\cdots).
 \end{aligned} \tag{3.8}$$

What remain to be studied are $[\mathbf{T}_{12,m}^{(I)}, \mathbf{S}_{23}(X)]$ and $[\mathbf{T}_{23,m}^{(II)}, \mathbf{S}_{12}(X)]$, which we postulate to vanish identically. The only non-trivial contributions to the commutators come from the zero modes,

$$\begin{aligned}
 \left[e^{i\pi \frac{\omega_{12} \sqrt{\beta_{12}} \mathbf{P}_{12}}{\omega_1}}, e^{\sqrt{\beta_{23}} \mathbf{Q}_{23}} \right] &= e^{\sqrt{\beta_{23}} \mathbf{Q}_{23}} e^{i\pi \frac{\omega_{12} \sqrt{\beta_{12}} \mathbf{P}_{12}}{\omega_1}} \left(e^{2\pi i \beta_{23} \frac{\omega_{23}}{\omega_1}} - 1 \right), \\
 \left[e^{i\pi \frac{\omega_{23} \sqrt{\beta_{23}} \mathbf{P}_{23}}{\omega_3}}, e^{\sqrt{\beta_{12}} \mathbf{Q}_{12}} \right] &= e^{\sqrt{\beta_{12}} \mathbf{Q}_{12}} e^{i\pi \frac{\omega_{23} \sqrt{\beta_{23}} \mathbf{P}_{23}}{\omega_3}} \left(e^{2\pi i \beta_{12} \frac{\omega_{12}}{\omega_3}} - 1 \right),
 \end{aligned} \tag{3.9}$$

and their vanishing leads to the final constraints

$$\beta_{12} = -\frac{\omega_3}{\omega_{12}}, \quad \beta_{23} = -\frac{\omega_1}{\omega_{23}}, \tag{3.10}$$

consistent with the identifications made in (3.5).

Now it is natural to glue more than two copies of modular doubles one after another, while requiring all generators to commute with all modular double screening currents up to total differences. This turns out to be quite restrictive, and *at most three copies* of q -Virasoro algebras can be glued in the aforementioned way. Moreover, the third copy needs to be glued onto the first. More details about this rigidity can be found in “Appendix C”. Pictorially, the three modular doubles represented by the edges $(i, i + 1) = (12), (23), (31)$ have to form a closed triangle in which each vertex $i = 1, 2, 3$ corresponds to one q -Virasoro algebra with generating current $\mathbf{T}_i(z)$ ⁵, see Fig. 2 for an illustration. More concretely, maximally three modular doubles can be glued together in the following way,

$$\begin{array}{ccc}
 (23) : \mathcal{MD}(\omega_2, \omega_3; \beta_{23} \sim \omega_1) & \longrightarrow & (31) : \mathcal{MD}(\omega_3, \omega_1; \beta_{31} \sim \omega_2) \\
 & \swarrow \quad \searrow & \\
 & (12) : \mathcal{MD}(\omega_1, \omega_2; \beta_{12} \sim \omega_3), &
 \end{array} \tag{3.11}$$

⁵ We will take it to be the I copy in each double, so that we can neglect the index I, II.

Fig. 2. The result of gluing three modular doubles. For it to work, the parameters in the doubles must conspire in the form of (3.12). Each **vertex** represents a q -Virasoro algebra, whose q, t parameters can be chosen to be either of the adjacent blue ones; the two are related by $q \leftrightarrow t^{-1}$ leaving the algebra invariant. Each edge represents a modular double, which accepts those q, t -parameters along the edge, and a unique β value. In the **interior**, the zero modes in green are identified up to numerical constants

Table 1. Relations between ω_i and (q_i, t_i) in various copies of q -Virasoro algebras

Vertex i	1	2	3
q_i	$e^{\frac{2\pi i}{\omega_1} \frac{\omega_1 + \omega_2}{\omega_1}}$	$e^{\frac{2\pi i}{\omega_2} \frac{\omega_2 + \omega_3}{\omega_2}}$	$e^{\frac{2\pi i}{\omega_3} \frac{\omega_1 + \omega_3}{\omega_3}}$
t_i	$e^{-\frac{2\pi i}{\omega_1} \frac{\omega_3}{\omega_1}}$	$e^{-\frac{2\pi i}{\omega_2} \frac{\omega_1}{\omega_2}}$	$e^{-\frac{2\pi i}{\omega_3} \frac{\omega_2}{\omega_3}}$

with the parameters of the q -Virasoro algebras at each vertex given by (see also Table 1)

$$q_i \equiv e^{\frac{2\pi i}{\omega_i} \frac{\omega_{i+1}}{\omega_i}}, \quad t_i \equiv e^{-\frac{2\pi i}{\omega_i} \frac{\omega_{i+2}}{\omega_i}} = q_i^{\beta_{i,i+1}}, \quad \beta_{i,i+1} \equiv -\frac{\omega_{i+2}}{\omega_{i,i+1}}. \quad (3.12)$$

The zero modes are identified up to numerical constants,

$$\omega_{i,i+1} \sqrt{\beta_{i,i+1}} \mathbf{P}_{i,i+1} = \omega_{i+1,i+2} \sqrt{\beta_{i+1,i+2}} \mathbf{P}_{i+1,i+2},$$

$$\frac{\mathbf{Q}_{i,i+1}}{\omega_{i,i+1} \sqrt{\beta_{i,i+1}}} = \frac{\mathbf{Q}_{i+1,i+2}}{\omega_{i+1,i+2} \sqrt{\beta_{i+1,i+2}}}. \quad (3.13)$$

For this maximal case, we say that the three q -Virasoro algebras with generating currents $\mathbf{T}_i(z)$ have fused into the *modular triple*. Therefore, the main result of this section is the following

Proposition. *Using the previous notation, we define*

$$\mathbf{S}_{i,i+1}(X) \equiv \exp \left[- \sum_{m \neq 0} \frac{(-1)^m \mathbf{a}_{i,i+1,m}^{(I)} e^{-m \frac{2\pi i}{\omega_i} X}}{e^{i\pi m \omega_{i+1}/\omega_i} - e^{-i\pi m \omega_{i+1}/\omega_i}} \right]$$

$$\begin{aligned}
& - \sum_{m \neq 0} \frac{(-1)^m \mathbf{a}_{i,i+1,m}^{(II)} e^{-m \frac{2\pi i}{\omega_{i+1}} X}}{e^{i\pi m \omega_i / \omega_{i+1}} - e^{-i\pi m \omega_i / \omega_{i+1}}} + \\
& + \left[\sqrt{\beta_{i,i+1}} \mathbf{Q}_{i,i+1} + \frac{2\pi i \omega_{i,i+1} \sqrt{\beta_{i,i+1}} \mathbf{P}_{i,i+1}}{\omega_i \omega_{i+1}} X \right]. \quad (3.14)
\end{aligned}$$

Then the commutators $[\mathbf{T}_{j,m}, \mathbf{S}_{i,i+1}(X)]$ for any $m \in \mathbb{Z}$ and $i, j = 1, 2, 3$ all vanish up to total differences as given in (3.8). The operators $\mathbf{S}_{i,i+1}(X)$ can thus be called the screening currents of the q -Virasoro modular triple.

Let us summarize what we have proposed in this section. Given three commuting q -Virasoro algebras with deformation parameters as in (3.12), one can construct three modular doubles by fusing any two of them. Finally, with properly chosen β 's and identifications of the zero modes, the three modular doubles fuse in a circular fashion into one modular triple construction, utilizing the $q \leftrightarrow t^{-1}$ invariance of each q -Virasoro algebra. It turns out that no more than three copies can be glued in this manner.

Remark. Each modular double $\mathcal{MD}(\omega_i, \omega_{i+1}; \beta_i \sim \omega_{i+2})$ carries only an $\mathrm{SL}(2, \mathbb{Z})$ structure realized in the $\omega_i \leftrightarrow \omega_{i+1}$ exchange symmetry, while the last equivariant parameter ω_{i+2} is on a different footing. The modular triple, which we can denote by $\mathcal{MT}(\omega_1, \omega_2, \omega_3)$, establishes full democracy among all the equivariant parameters due to the manifest and complete permutation symmetry. In fact, each $\mathrm{SL}(2, \mathbb{Z})$ structure is part of a more fundamental $\mathrm{SL}(3, \mathbb{Z})$ structure. This is more manifest if we recall the parametrization (2.22) for, say, the $i = 1$ vertex of the modular triple

$$q_1 = e^{2\pi i \tau}, \quad t_1 = e^{2\pi i \sigma}. \quad (3.15)$$

Then the other two vertices have deformation parameters (q_2, t_2) and (q_3, t_3) which are simply related to (τ, σ) by $\mathrm{SL}(3, \mathbb{Z})$ transformations

$$q_2 = e^{-2\pi i \sigma / \tau}, \quad t_2 = e^{-2\pi i / \tau}, \quad q_3 = e^{-2\pi i / \sigma}, \quad t_3 = e^{2\pi i \tau / \sigma}. \quad (3.16)$$

In this sense, the $q \leftrightarrow t^{-1}$ symmetry of each individual q -Virasoro algebra, together with the new $\mathrm{SL}(2, \mathbb{Z})$ structure of the modular double, is enhanced to the $\mathrm{SL}(3, \mathbb{Z})$ structure in the modular triple.

4. Formal 2d CFT-Like Construction

In the previous section, we have shown that it is algebraically possible to fuse multiple q -Virasoro algebras when the deformation parameters are related by $\mathrm{SL}(2, \mathbb{Z})$ or $\mathrm{SL}(3, \mathbb{Z})$, and hence we have called the resulting constructions (q -Virasoro) modular double and triple respectively. These resemble ordinary structures of physical 2d CFTs, where the conformal algebra splits into holomorphic and anti-holomorphic parts giving rise to two (essentially independent) chiral sectors. It is therefore natural to ask whether our constructions admit a 2d CFT-like interpretation, possibly deriving some of the results from an action.

For any given $i \in \{1, 2, 3\}$ we will abbreviate the ordered sequence $(i, i+1, i+2)$ as (i, j, k) to help shorten formulas. We also remind the cyclic identification $i+1 \sim i$. Given

three q -Virasoro algebras with parameters (3.12) in terms of the generic equivariant parameters $\omega_i \in \mathbb{C}$, we define the *triple formal boson*

$$\Phi(X) \equiv \sum_{i=1}^3 \sum_{m \neq 0} \frac{\mathbf{h}_{i,m} e^{\pi i m \omega / \omega_i} e^{-m \frac{2\pi i}{\omega_i} X}}{(1 - e^{2\pi i m \omega_j / \omega_i})(1 - e^{2\pi i m \omega_k / \omega_i})} - \frac{i\pi \mathbf{C}_2}{\omega_1 \omega_2 \omega_3} X^2 + \frac{\mathbf{C}_1}{\omega_1 \omega_2 \omega_3} X + \mathbf{C}_0, \quad (4.1)$$

with the operators $\mathbf{h}_{i,m}$, $\mathbf{C}_2$, $\mathbf{C}_1$, $\mathbf{C}_0$ satisfy the commutation relations (we display the non-trivial relations only)

$$[\mathbf{h}_{i,m}, \mathbf{h}_{i,n}] = -\frac{e^{-\pi i m \omega / \omega_i}}{m} (1 - e^{2\pi i m \omega_j / \omega_i})(1 - e^{2\pi i m \omega_k / \omega_i}) \delta_{n+m,0}, \quad [\mathbf{C}_2, \mathbf{C}_1] = \hbar, \quad (4.2)$$

where $\hbar$ is a constant (to be determined later). As we will see momentarily, this object is a useful and elegant device for compactly encoding the structure of the q -Virasoro modular double and triple. In order to show that, let us recall the definition of the shift operator $T_\varepsilon = e^{\varepsilon \partial_X}$ acting as

$$T_\varepsilon f(X) \equiv f(X + \varepsilon), \quad (4.3)$$

and the difference operator d_ε defined as

$$d_\varepsilon f(X) \equiv (T_{\varepsilon/2} - T_{-\varepsilon/2})f(X) = f(X + \varepsilon/2) - f(X - \varepsilon/2). \quad (4.4)$$

Then, let us define the bosons

$$\begin{aligned} \phi_{ij}(X) &\equiv d_{\omega_k} \Phi(X) \\ &= - \sum_{m \neq 0} \left[\frac{(-1)^m \mathbf{h}_{i,m} e^{-m \frac{2\pi i}{\omega_i} X}}{e^{i\pi m \omega_j / \omega_i} - e^{-i\pi m \omega_j / \omega_i}} + \frac{(-1)^m \mathbf{h}_{j,m} e^{-m \frac{2\pi i}{\omega_j} X}}{e^{i\pi m \omega_i / \omega_j} - e^{-i\pi m \omega_i / \omega_j}} \right] \\ &\quad - \frac{2\pi i \mathbf{C}_2}{\omega_i \omega_j} X + \frac{\mathbf{C}_1}{\omega_i \omega_j}. \end{aligned} \quad (4.5)$$

The modes of these bosons resemble the ones appearing in the screening currents of the modular triple (3.14), except that the we cannot readily identify the *fundamental* oscillators $\mathbf{h}_{i,m}$, $\mathbf{h}_{i+1,m}$ here with the *root* oscillators $\mathbf{a}_{i,i+1,m}^{(I)}$, $\mathbf{a}_{i,i+1,m}^{(II)}$ there, the commutation relations differing by the appearance of the deformed Cartan matrix $C^{[m]}(p) = e^{i\pi \omega m / \omega_i} + e^{-i\pi \omega m / \omega_i}$ of the A_1 algebra, which is non-trivial. The precise map is

$$\mathbf{a}_{i-1,i,m}^{(II)} = \mathbf{a}_{i,i+1,m}^{(I)} = \sqrt{-i}(e^{\pi i m \omega / 2\omega_i} + i e^{-\pi i m \omega / 2\omega_i}) \mathbf{h}_{i,m}, \quad (4.6)$$

which can be implemented by acting on $\phi_{ij}(X)$ with the bracket $(\omega/2, \cdot)$ defined by

$$(\varepsilon, f(X)) \equiv \sqrt{-i}(T_{\varepsilon/2} + iT_{-\varepsilon/2})f(X). \quad (4.7)$$

Similarly, we can identify the zero modes $\mathbf{P}_{ij}$, $\mathbf{Q}_{ij}$ of the screening currents with $\mathbf{C}_2$, $\mathbf{C}_1$ as follows

$$\sqrt{\beta_{ij}} \mathbf{P}_{ij} = \frac{\sqrt{2}}{\omega_j} \mathbf{C}_2, \quad \sqrt{\beta_{ij}} \mathbf{Q}_{ij} = -\frac{\sqrt{2}}{\omega_i \omega_j} (\mathbf{C}_1 - \pi \omega \mathbf{C}_2/2), \quad (4.8)$$

implying also $\hbar = \omega_1 \omega_2 \omega_3$ so that $[\mathbf{P}_{ij}, \mathbf{Q}_{ij}] = 2$. Now, we can suggestively rewrite all the screening currents and charges of the modular triple in terms of the formal bosons $\Phi(X)$ or $\phi_{ij}(X)$ as

$$\mathbf{S}_{ij}(X) = e^{(\omega/2, \phi_{ij}(X))}, \quad (4.9)$$

resembling the usual 2d CFT definition (2.12). We have focused on the construction of the screening currents because these are the most important operators for computational purposes. In “Appendix D”, we also discuss the construction of the q -Virasoro currents.

Let us summarize what we have constructed so far: (i) we have defined a formal boson $\Phi(X)$ (4.1) which uniformly encodes three deformed “chiral” bosons related by $\mathrm{SL}(3, \mathbb{Z})$; (ii) we have acted on $\Phi(X)$ with a simple difference operator d_{ω_k} to single out the bosons $\phi_{ij}(X)$ (4.5), each of which is closely related to a copy of the modular double; (iii) we have used these fields to construct all the three screening currents of the modular triple by introducing a sort of *quantization* of the Killing form (4.9). In view of the analogies with standard 2d CFT, in the rest of this section we will try to speculate on possible Lagrangian QFT constructions.

First of all, we would like to try to explain the origin of the three “chiral” sectors encoded into $\Phi(X)$ and its mode expansion. We are thus led to postulate the equation of motion

$$\square_E \Phi(X) = 0, \quad \square_E = \frac{1}{2\pi} \partial_X \prod_{i=1}^3 d_{\omega_i}, \quad (4.10)$$

which can be derived from the (non-local) action

$$S_0[\Phi] = \int dX \partial_X d_{\omega_i} \Phi(X) d_{\omega_j} d_{\omega_k} \Phi(X), \quad (4.11)$$

where symmetrization in the indexes is left implicit. If we restrict to $X \in \mathbb{R}$ and $\omega_i \in \mathbb{R}$ such that the ratios ω_i/ω_{i+1} are irrational $\forall i$, solutions to (4.10) are of the form

$$\Phi(X) = \sum_{i=1,2,3} f_i(X) + p_3(X), \quad (4.12)$$

where $f_i(X)$ is a periodic function with period ω_i and $p_3(X)$ is a cubic polynomial. We can therefore match this on-shell field with the mode expansion of the formal boson in (4.1), provided that we neglect the cubic coefficient of the zero mode polynomial

$$p_3(X) = C_3 X^3 - \frac{i\pi C_2}{\omega_1 \omega_2 \omega_3} X^2 + \frac{C_1}{\omega_1 \omega_2 \omega_3} X + C_0. \quad (4.13)$$

This can be justified, for example, by requiring an asymptotic behavior that forbids cubic terms.

Next, we would like to quantize the theory and compute the 2-point function of the formal boson, using the ansatz (4.1), (4.2). As usual, let us postulate a vacuum state $|0\rangle$ annihilated by $\mathbf{h}_{i,m>0}$, $\mathbf{C}_0$, and a dual vacuum state $\langle 0|$ annihilated by $\mathbf{h}_{i,m<0}$, with pairing $\langle 0|0\rangle = 1$. We can now easily compute the formal series

$$\Delta_+(X) \equiv \langle 0|\Phi(X)\Phi(0)|0\rangle = - \sum_{i=1}^3 \sum_{m>0} \frac{e^{\pi i m \omega/\omega_i} e^{-\frac{2\pi i}{\omega_i} X}}{m(1 - e^{2\pi i m \omega_j/\omega_i})(1 - e^{2\pi i m \omega_k/\omega_i})}, \quad (4.14)$$

$$\Delta_{-}(X) \equiv \langle 0 | \Phi(0) \Phi(X) | 0 \rangle = - \sum_{i=1}^3 \sum_{m>0} \frac{e^{\pi i m \omega / \omega_i} e^{\frac{2\pi i}{\omega_i} X}}{m(1 - e^{2\pi i m \omega_j / \omega_i})(1 - e^{2\pi i m \omega_k / \omega_i})} . \quad (4.15)$$

In order to make more sense of these formal expressions, we have to consider an analytic continuation off the real line for X and ω_i . In this case we recognize that, when $\text{Im}(\omega_i / \omega_{i+1}) \neq 0 \forall i$, $\Delta_{\pm}(X)$ are nothing but combinations of log of triple Sine functions and cubic Bernoulli polynomials (see [57] and “Appendix A”)

$$\Delta_{\pm}(X) = \log S_3(\omega/2 \mp X | \omega) \mp \frac{i\pi}{6} B_{33}(\omega/2 \mp X | \omega), \quad (4.16)$$

which are well-defined on the entire complex plane as long as X does not hit any zero of the triple Sine function. We then define the propagator

$$G_E(X) = \langle 0 | T \Phi(X) \Phi(0) | 0 \rangle = \begin{cases} \Delta_{+}(X) & \text{if } \text{Re}(X) \geq 0 \\ \Delta_{-}(X) & \text{if } \text{Re}(X) < 0 \end{cases}, \quad (4.17)$$

where T denotes X -ordering. For consistency of the proposed action at the quantum level and of our quantization procedure, we would expect this propagator to be a Green’s function of the kinetic operator $\square_E$ arising from the equation of motion. As shown in “Appendix B”, $G(X) \equiv G_E(-iX)$ is in fact a Green’s function for the “Wick” rotated theory, namely

$$\square G(X) = \delta(X), \quad \square = -\frac{i}{2\pi} \partial_X \prod_{i=1}^3 d_{-i\omega_i}. \quad (4.18)$$

We speculate that a proper quantization of the proposed action would lead to the commutation relations (4.2), which have been so far axiomatically given. Also, notice that the non-trivial part of the Green’s function is captured by the log of the triple Sine function, as the kinetic operator annihilates any cubic polynomial. Such ambiguity is probably related to different boundary conditions or regularization prescriptions.

Finally, by analogy with ordinary Liouville theory, we are led to propose the interacting theory

$$S_{\mu}[\Phi] = S_0[\Phi] + \sum_{i=1,2,3} \mu_i \int dX e^{\frac{2\pi i \omega_{i+1} X}{\omega_i \omega_{i+1}}} e^{(\omega/2, d_{\omega_i} \Phi(X))}, \quad (4.19)$$

where $\mu \equiv (\mu_1, \mu_2, \mu_3)$ are coupling constants. At the quantum level, the q -Virasoro modular triple symmetry of the interacting model may be argued by considering the formal perturbative expansion of its correlation functions

$$\langle \cdots \rangle_{\mu} = \sum_{n_1, n_2, n_3 \geq 0} \langle \cdots \prod_{i=1,2,3} \frac{\mu_i^{n_i}}{n_i!} \left(\int dX e^{\frac{2\pi i \omega_{i+1} X}{\omega_i \omega_{i+1}}} e^{(\omega/2, d_{\omega_i} \Phi(X))} \right)^{n_i} \rangle_0, \quad (4.20)$$

where $\langle \cdots \rangle_{\mu}$ denotes the quantum v.e.v. in the theory described by $S_{\mu}[\Phi]$. At any finite order in the coupling constants, the above quantity involve insertions of the modular triple screening charges in the free theory, and hence by construction it should satisfy three commuting copies of q -Virasoro constraints with deformation parameters related by $\text{SL}(3, \mathbb{Z})$, in the spirit of conformal matrix model technology [58].

5. Discussion

In this note, we have reviewed the q -Virasoro modular double originally introduced in [44], and showed how it is possible to consistently fuse three copies of it into what we have called the q -Virasoro modular triple. Essentially, the $q \leftrightarrow t^{-1}$ symmetry of an individual q -Virasoro algebra and the $\mathrm{SL}(2, \mathbb{Z})$ structure of the modular double can be combined into the $\mathrm{SL}(3, \mathbb{Z})$ structure of the modular triple. Under certain natural assumptions, we have also shown that no more than three copies can be glued in this way. Finally, we have given a 2d CFT-like construction of the modular triple and proposed for the first time a (non-local) Lagrangian formulation of a q -Virasoro system.

There are many directions worth pursuing for future research. For instance, it is natural to generalize the modular triple to quiver $W_{q,t}$ algebras of [23], including the elliptic case [59–64]. Also, although the formal 2d CFT-like construction of the modular triple is rather elegant, it is incomplete. A more satisfactory understanding of the proposed (non-local) q -Virasoro model would be highly desirable, including a proper quantization procedure. New methods similar to the Ostrogradsky formalism [65] for higher order Lagrangians may be needed.

5.1. SUSY gauge theories on odd dimensional spheres. This paper is strongly motivated by gauge theoretical considerations. Ultimately, we hope to recover BPS observables, or even better, their duality properties, from the q -Virasoro algebra and related algebraic structures. Concretely, there are a few observations that hint at several possible future directions. It is shown in [23] that the *time-extended* Nekrasov partition function of pure 5d $U(N)$ supersymmetric Yang–Mills theory on $\mathbb{R}_{q,t}^4 \times \mathbb{S}^1$ can be written in terms of *infinitely-many* q -Virasoro screening charges, naturally leading to q -Virasoro Ward identities because of the defining relation (2.9). Moreover, the Coulomb branch expression of the (squashed) $\mathbb{S}^5$ partition function can be constructed by gluing three copies of the $\mathbb{R}_{q,t}^4 \times \mathbb{S}^1$ partition function in the very same $\mathrm{SL}(3, \mathbb{Z})$ fashion [33, 37] considered in this paper. Therefore, the $\mathbb{S}^5$ partition function (or its time-extended version) should satisfy the Ward identities of the modular triple, and we expect our algebraic construction to be able to describe expectation values and properties of BPS observables of gauge theories on $\mathbb{S}^5$. This is in fact part of the motivation behind the proposal (4.19), (4.20) and the whole paper, but we leave this topic for future work [66]. However, we can observe here that if we consider a suitable linear combination of products of an arbitrary number of screening charges of the modular triple, we can formally obtain a solution to those identities. This reasoning supports the results of [67] for the *Higgs branch expression* of the $\mathbb{S}^5$ partition function. In fact, the vigilant readers may have noticed the striking resemblance between Fig. 2 and the toric diagram of $\mathbb{S}^5$, represented as the $\mathbb{T}^3$ fibration over a filled triangle. Actually, we can literally identify the two as shown Fig. 3 and observe some deeper connections. In [44], it was shown that one can construct $\mathbb{S}^3$ partition functions out of the q -Virasoro modular double. It is well-known that an $\mathbb{S}^3$ is represented by an edge in the $\mathbb{S}^5$ toric diagram, just as a modular double is an edge in Fig. 2. This analogy goes beyond just one $\mathbb{S}^3$ as one can consider up to three *intersecting* $\mathbb{S}^3$'s inside $\mathbb{S}^5$, corresponding to the three edges in the toric diagram. On such intersecting space, it is possible to define certain gauge theory glued out of three individual unitary gauge theories on each $\mathbb{S}^3$ and mutually interacting at the $\mathbb{S}^1$ intersections. The partition function (Z) of such theory can be calculated through supersymmetric localization and, quite remarkably, it can be explicitly shown (e.g. for $U(N)$ SQCD) to assume the form of a correlator involving a finite number n_i of screening currents $\mathbf{S}_{i,i+1}$ of the modular

Fig. 3. The identification between the q -Virasoro modular triple and the toric diagram of the (squashed) 5-sphere

triple (fitting with the model (4.19)), whereas the $\mathbb{S}^5$ partition function is recovered by summing over all the insertion numbers n_i , schematically

$$Z[\mathbb{S}^5] = \sum_{\alpha} \sum_{\{n_i \geq 0\}} \langle \xi | (\cdots)_{\alpha} \prod_{i=1}^3 \frac{1}{n_i!} \left(\int dX e^{\frac{2\pi i \omega_{i,i+1} X}{\omega_i \omega_{i+1}}} S_{i,i+1}(X) \right)^{n_i} | \xi_{\{n_i\}} \rangle. \quad (5.1)$$

Here $(\langle \xi |) | \xi_{\{n_i\}} \rangle$ denotes a charged vacuum state of the (dual) Fock module constructed through the (right) left action of the (positive) negative Heisenberg oscillators of the q -Virasoro modular triple, while $(\cdots)_{\alpha}$ represents the insertion of additional vertex operators (whose explicit form is not particularly important for our purposes) generically labeled by a discrete choice (permutation of parameters) denoted by α . This expression neatly agrees with partition function in the so-called *Higgs branch localization scheme*, and on the gauge theory side the difference of the vacuum momenta encodes the coupling constant, the additional vertex operators describe fundamental matter and the finite range discrete index α denotes the selected Higgs vacuum. For details, we refer to the short note [66] (see also [28, 68] for a related discussion in the context of 4d and 5d AGT).

Now that we have uncovered the algebraic structure corresponding to the gauge theories on $\mathbb{S}^5$ and the codimension two defects supported on the intersecting $\mathbb{S}^3$'s, it is natural to wonder if there exist similar structures corresponding to gauge theories on arbitrary toric Sasaki–Einstein spaces [36, 39, 69] and the codimension two defects thereof. Such structures, should they exist, must involve more than three modular doubles, one for every edge of the toric diagram. As we have seen in the main text, at most three modular doubles can be glued together through our criteria based on the $S \in \text{SL}(3, \mathbb{Z})$ element. Therefore, in order to acquire more general structures, those criteria need to be relaxed in a consistent manner, presumably using other $\text{SL}(3, \mathbb{Z})$ elements dictated by the toric geometry. It would be interesting to study the modular properties [70–72] of the partition functions from this perspective.

Finally, it is worth noting that our results may represent a basis for developing further non-perturbative refined topological strings and large N open/closed duality in the spirit of [33], as well as for considering Little String Theories [73–75] at the origin of the q -deformation [32] from a novel perspective.

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A. Special Functions

In this appendix, we recall the definition of several special functions which we use in the main body. Below, r is a positive integer, and $\omega \equiv (\omega_1, \dots, \omega_r)$ is a collection of non-zero complex parameters. We frequently take $r = 2$ or 3 for concreteness. We refer to [57] for further details.

The **multiple Bernoulli polynomials** $B_{rn}(Z|\omega)$ are defined by the generating function

$$\frac{t^r e^{Xt}}{\prod_{i=1}^r e^{\omega_i t} - 1} = \sum_{m \geq \mathbb{N}} B_{rn}(X|\omega) \frac{t^m}{m!}. \quad (\text{A.1})$$

In particular, we use $B_{22}(X|\omega)$ and $B_{33}(X|\omega)$ in this note, and they are given explicitly by

$$B_{22}(X|\omega) = \frac{X^2}{\omega_1 \omega_2} - \frac{\omega_1 + \omega_2}{\omega_1 \omega_2} X + \frac{\omega_1^2 + \omega_2^2 + 3\omega_1 \omega_2}{6\omega_1 \omega_2}, \quad (\text{A.2})$$

$$\begin{aligned} B_{33}(X|\omega) = & \frac{X^3}{\omega_1 \omega_2 \omega_3} - \frac{3(\omega_1 + \omega_2 + \omega_3)}{2\omega_1 \omega_2 \omega_3} X^2 \\ & + \frac{\omega_1^2 + \omega_2^2 + \omega_3^2 + 3\omega_1 \omega_2 + 3\omega_2 \omega_3 + 3\omega_3 \omega_1}{2\omega_1 \omega_2 \omega_3} X \\ & - \frac{(\omega_1 + \omega_2 + \omega_3)(\omega_1 \omega_2 + \omega_2 \omega_3 + \omega_3 \omega_1)}{4\omega_1 \omega_2 \omega_3}. \end{aligned} \quad (\text{A.3})$$

The **q -Pochhammer symbols** are defined as

$$(x; q_1, \dots, q_r)_\infty \equiv \prod_{n_1, \dots, n_r=0}^{\infty} (1 - x q_1^{n_1} \dots q_r^{n_r}) \quad \text{when all } |q_i| < 1. \quad (\text{A.4})$$

Other regions in the q -planes are defined through relations

$$(x; q_1, \dots, q_r)_\infty = \frac{1}{(q_i^{-1} x; q_1, \dots, q_i^{-1}, \dots, q_r)_\infty}. \quad (\text{A.5})$$

They have exponentiated series expansion

$$(x; q_1, \dots, q_r)_\infty = \exp \left[- \sum_{n>0} \frac{1}{n(1-q_1^n) \dots (1-q_r^n)} x^n \right]. \quad (\text{A.6})$$

The **multiple Sine functions** $S_r(X|\omega)$ can be defined by the ζ -regularized product

$$S_r(X|\omega) \sim \prod_{m_1, \dots, m_r \in \mathbb{N}} \left(X + \sum_{i=1}^r m_i \omega_i \right)^{(-1)^{r+1}} \left(-X + \sum_{i=1}^r (m_i + 1) \omega_i \right). \quad (\text{A.7})$$

$S_r(X|\omega)$ is symmetric in all ω_i , has the reflection property $S_r(X|\omega) = S_r(\omega - X|\omega)^{(-1)^{r+1}}$ for $\omega \equiv \omega_1 + \dots + \omega_r$, and the shift property

$$S_r(X + \omega_i|\omega) = \frac{S_r(X|\omega)}{S_{r-1}(X|\widehat{\omega})}, \quad \widehat{\omega} \equiv (\omega_1, \omega_{i-1}, \omega_{i+1}, \dots, \omega_r). \quad (\text{A.8})$$

The **triple Sine function** $S_3(X|\omega)$ has a useful factorization property. When $\text{Im}(\omega_i/\omega_j) \neq 0$ for all $i \neq j$, $S_3(X|\omega)$ factorizes

$$S_3(X|\omega) = e^{-\frac{i\pi}{6} B_{33}(X|\omega)} \prod_{i=1,2,3} \left(e^{\frac{2\pi i}{\omega_i} X}; q_i, t_i^{-1} \right)_\infty, \quad (\text{A.9})$$

where q_i and t_i are given in (3.12) or Table 1 in terms of ω_i .

When $\text{Re } \omega_i > 0$ and $\text{Re } \omega > \text{Re } X > 0$, the $S_3(X|\omega)$ admits the integral representations

$$\log S_3(X|\omega) = \int_{\mathbb{R} \pm i0} \frac{dt}{t} \frac{e^{Xt}}{\prod_{i=1}^3 (e^{\omega_i t} - 1)} \mp \frac{i\pi}{6} B_{33}(X|\omega). \quad (\text{A.10})$$

The **Θ function** is defined as $\Theta(x; q) \equiv (x; q)_\infty (qx^{-1}; q)_\infty$. Its modular properties are closely related to the $B_{22}(X|\omega)$ polynomial

$$\Theta(e^{\frac{2\pi i}{\omega_1} X}; e^{2\pi i \frac{\omega_2}{\omega_1}}) \Theta(e^{\frac{2\pi i}{\omega_2} X}; e^{2\pi i \frac{\omega_1}{\omega_2}}) = \exp \left[-i\pi B_{22}(X|\omega) \right]. \quad (\text{A.11})$$

As one can construct linear functions of X using $B_{22}(X|\omega)$, one can factorize $e^{(\dots)X}$ in terms of products of Θ functions. Also, q -constants can be defined by using (products of) Θ functions, for instance

$$c_\alpha(x; q) \equiv \frac{\Theta(q^\alpha x; q)}{\Theta(x; q)} x^\alpha \quad \Rightarrow \quad c_\alpha(qx; q) = c(x; q). \quad (\text{A.12})$$

B. Green's Functions

In this appendix, we show that the logarithm of the triple Sine function can be viewed as a Green's function of the operator (4.18). Let us consider $G(X) \equiv \ln S_3(\omega/2 + iX|\omega)$. Applying repeatedly (A.8), we have for $0 < \epsilon \ll 1$ ⁶

$$\begin{aligned} & -i\partial_X d_{-i\omega_1 - i\epsilon} d_{-i\omega_2} d_{-i\omega_3} G(X) \\ &= \pi \omega_1^{-1} \left[\frac{\cos \pi \omega_1^{-1} (iX + \epsilon)}{\sin \pi \omega_1^{-1} (iX + \epsilon)} - \frac{\cos \pi \omega_1^{-1} (iX - \epsilon)}{\sin \pi \omega_1^{-1} (iX - \epsilon)} \right], \quad \forall X \in \mathbb{C}. \end{aligned} \quad (\text{B.1})$$

⁶ It is crucial to introduce the regularization by ϵ , as $1/\sin(\pi \omega_1^{-1} iX)$ is singular at $X = 0$, and we need to extract the difference between two singular functions.

Fig. 4. The integration contour when showing $G(X)$ is a Green's function of $\square$

Let us now specialize $X \in \mathbb{R}$, and observe that the two terms on the r.h.s. have a pole at $X = \pm i\epsilon$ respectively. For any test function $f(X)$ that decays fast enough as $|X| \rightarrow +\infty$, we can compute its integral with the kernel (B.1) by residues, and in the $\epsilon \rightarrow 0^+$ limit we get

$$\begin{aligned}
 & \lim_{\epsilon \rightarrow 0^+} -\frac{i}{2\pi} \int_{\mathbb{R}} dX f(X) d_{-i\omega_1 - i\epsilon} d_{-i\omega_2} d_{-i\omega_3} \partial_X G(X) \\
 &= \frac{i\pi}{\omega_1} \lim_{\epsilon \rightarrow 0^+} \left(\frac{1}{2} \operatorname{Res}_{X \rightarrow +i\epsilon} \frac{f(X)}{\sin \pi \omega_1^{-1}(iX + \epsilon)} - \left(-\frac{1}{2} \right) \operatorname{Res}_{X \rightarrow -i\epsilon} \frac{f(X)}{\sin \pi \omega_1^{-1}(iX - \epsilon)} \right) \\
 &= f(0).
 \end{aligned} \tag{B.2}$$

Here, when evaluating the two integrals for $\pm\epsilon$, one can deform respectively the integration contours to $\mathbb{R} \pm i\epsilon$ but going around the pole at $X = \pm i\epsilon$ in a small half circle of radius ϵ from below/above, see Fig. 4. In the second line, the principal values of the two integrals, *i.e.* the integration in the regions $(-\infty \pm i\epsilon, -\epsilon \pm i\epsilon)$ and $(+\epsilon \pm i\epsilon, +\infty \pm i\epsilon)$ cancel, while the two half-residues from the integration over the two half-circles add. Therefore, we conclude that for $X \in \mathbb{R}$

$$\square G(X) = \delta(X). \tag{B.3}$$

In the above computation, we have just used the shift properties of the triple Sine function and a regularization prescription. Alternatively, we can take advantage of the integral representation (A.10), and bringing the kinetic operator $\square$ under the integral we immediately conclude that

$$\square G(X) = \frac{1}{2\pi} \int dt e^{iXt} = \delta(X). \tag{B.4}$$

Notice that, formally, the choice of different integration contours can lead to different Green's functions.

C. Gluing Multiple Modular Doubles

We now show how to glue multiple modular doubles and derive the maximal number of modular doubles that can be glued.

Consider a collection of r independent modular doubles labeled by $(i, i + 1)$ with $i \in \mathcal{I} \equiv \{1, \dots, r\}$, and the corresponding Heisenberg algebras generated by $(\mathbf{a}_{i,i+1,m}^{(\text{I})}, \mathbf{a}_{i,i+1,m}^{(\text{II})}, \mathbf{P}_{i,i+1}, \mathbf{Q}_{i,i+1})$ with the equivariant parameters

$$\begin{aligned} (q_{i,i+1}^{(\text{I})}, t_{i,i+1}^{(\text{I})}) &= (\exp\left(2\pi i \frac{\omega_{i,i+1}}{\omega_i}\right), \exp\left(2\pi i \frac{\beta_{i,i+1}\omega_{i,i+1}}{\omega_i}\right)), \\ (q_{i,i+1}^{(\text{II})}, t_{i,i+1}^{(\text{II})}) &= (\exp\left(2\pi i \frac{\omega_{i,i+1}}{\omega_{i+1}}\right), \exp\left(2\pi i \frac{\beta_{i,i+1}\omega_{i,i+1}}{\omega_{i+1}}\right)). \end{aligned} \quad (\text{C.1})$$

Here $\omega_{i,i+1} \equiv \omega_i + \omega_{i+1}$.

We now glue the modular doubles consecutively. For each pair of neighboring modular doubles labeled by $(i, i + 1)$ and $(i + 1, i + 2)$, we identify the generators $\mathbf{T}_{i,i+1,m}^{(\text{II})} = \mathbf{T}_{i+1,i+2,m}^{(\text{I})}$. At the level of Heisenberg operators, the identification implies

$$\begin{aligned} q_{i+1,i+2}^{(\text{I})} &= 1/t_{i,i+1}^{(\text{II})}, \quad t_{i+1,i+2}^{(\text{I})} = 1/q_{i,i+1}^{(\text{II})}, \\ \mathbf{a}_{i,i+1,m}^{(\text{II})} &= \mathbf{a}_{i+1,i+2,m}^{(\text{I})}, \quad (q_{i,i+1}^{(\text{II})})^{\sqrt{\beta_{i,i+1}}\mathbf{P}_{i,i+1}/2} = (q_{i+1,i+2}^{(\text{I})})^{\sqrt{\beta_{i+1,i+2}}\mathbf{P}_{i+1,i+2}/2}. \end{aligned} \quad (\text{C.2})$$

By construction and the identification $\mathbf{T}_{i,i+1,m}^{(\text{II})} = \mathbf{T}_{i+1,i+2,m}^{(\text{I})}$, we obtain for each $i = 1, \dots, r - 1$ a set of commutativity relations up to ω -shifts, namely

$$[\mathbf{T}_{i,i+1,m}^{(\text{I})}, \mathbf{S}_{i,i+1}(X)] = e^{-\frac{2\pi i \omega_{i,i+1} X}{\omega_i \omega_{i+1}}} (T_{\omega_{i+1}} - 1)(\dots), \quad \text{and same for } i \rightarrow i + 1, \quad (\text{C.3})$$

$$[\mathbf{T}_{i,i+1,m}^{(\text{II})}, \mathbf{S}_{i,i+1}(X)] = e^{-\frac{2\pi i \omega_{i,i+1} X}{\omega_i \omega_{i+1}}} (T_{\omega_i} - 1)(\dots), \quad \text{and same for } i \rightarrow i + 1, \quad (\text{C.4})$$

$$[\mathbf{T}_{i+1,i+2,m}^{(\text{I})}, \mathbf{S}_{i,i+1}(X)] = e^{-\frac{2\pi i \omega_{i,i+1} X}{\omega_i \omega_{i+1}}} (T_{\omega_i} - 1)(\dots),$$

$$[\mathbf{T}_{i,i+1,m}^{(\text{II})}, \mathbf{S}_{i+1,i+2}(X)] = e^{-\frac{2\pi i \omega_{i+1,i+2} X}{\omega_i \omega_{i+1}}} (T_{\omega_{i+1}} - 1)(\dots).$$

This is of course not enough as there are more generators $\mathbf{T}$ further away whose commutation relations with $\mathbf{S}_{i,i+1}$ need to be specified. For those, we further impose full commutativity, i.e., $\forall i \in \mathcal{I}$

$$[\mathbf{T}_{j,j+1,m}^{(\text{I})}, \mathbf{S}_{i,i+1}(X)] = 0, \quad \forall j \in \mathcal{I}, \text{ and } j \neq i, i + 1, \quad (\text{C.5})$$

$$[\mathbf{T}_{j,j+1,m}^{(\text{II})}, \mathbf{S}_{i,i+1}(X)] = 0, \quad \forall j \in \mathcal{I}, \text{ and } j \neq i, i - 1. \quad (\text{C.6})$$

Note that the two sets of the involved oscillatory Heisenberg generators are assumed to be independent and hence commute with each other; the zero modes, however, can be seen to have non-trivial commutation relations due to C.2. Hence the full commutativity requirement leads to additional constraints on $\mathbf{P}_{i,i+1}$ and $\mathbf{Q}_{i,i+1}$,

$$\begin{aligned} [(q_{j,j+1}^{(\text{I})})^{\pm\sqrt{\beta_{j,j+1}}\mathbf{P}_{j,j+1}/2}, e^{\sqrt{\beta_{i,i+1}}\mathbf{Q}_{i,i+1}}] &= 0, \quad \forall j \in \mathcal{I}, j \neq i, i + 1, \\ [(q_{j,j+1}^{(\text{II})})^{\pm\sqrt{\beta_{j,j+1}}\mathbf{P}_{j,j+1}/2}, e^{\sqrt{\beta_{i,i+1}}\mathbf{Q}_{i,i+1}}] &= 0, \quad \forall j \in \mathcal{I}, j \neq i, i - 1. \end{aligned} \quad (\text{C.7})$$

Together, constraints (C.2) and (C.7) lead to a set of equations involving undetermined integers $m^{(i)}, n^{(i)}, k^{(i)}, M^{(j,i)}$ and $N^{(j,i)}$,

$$\omega_{i,i+1} + m^{(i)}\omega_{i+1} = -\beta_{i+1,i+2}\omega_{i+1,i+2}, i = 1, \dots, r - 1, \quad (\text{C.8})$$

Fig. 5. Gluing multiple modular double consecutively, where the red arrows indicate the origin of incompatible constraints

$$\beta_{i,i+1}(\omega_{i,i+1}) + n^{(i)}\omega_{i+1} = -(\omega_{i+1,i+2}), i = 1, \dots, r-1, \quad (C.9)$$

$$\omega_{i,i+1} \frac{1}{2} \sqrt{\beta_{i,i+1}} \mathbf{P}_{i,i+1} + k^{(i)}\omega_{i+1} = \omega_{i+1,i+2} \frac{1}{2} \sqrt{\beta_{i+1,i+2}} \mathbf{P}_{i+1,i+2}, i = 1, \dots, r-1, \quad (C.10)$$

$$\frac{\omega_{j,j+1}}{\omega_j} \frac{1}{2} \sqrt{\beta_{j,j+1} \beta_{i,i+1}} [\mathbf{P}_{j,j+1}, \mathbf{Q}_{i,i+1}] = M^{(j,i)}, \quad \forall j = 1, \dots, r, \text{ and } j \neq i, i+1, \quad (C.11)$$

$$\frac{\omega_{j,j+1}}{\omega_{j+1}} \frac{1}{2} \sqrt{\beta_{j,j+1} \beta_{i,i+1}} [\mathbf{P}_{j,j+1}, \mathbf{Q}_{i,i+1}] = N^{(j,i)}, \quad \forall j = 1, \dots, r, \text{ and } j \neq i, i-1. \quad (C.12)$$

Let us analyze these constraints in detail. The equations in the third line relate the zero modes $\mathbf{P}_{i,i+1}$ in different modular double by nonzero proportionality, and therefore we conclude that $[\mathbf{P}_{i,i+1}, \mathbf{Q}_{j,j+1}] \neq 0$ for all i, j , simply because $[\mathbf{P}_{i,i+1}, \mathbf{Q}_{i,i+1}] = 2 \neq 0$.

The equations in the last two lines come from (C.7) by applying the formula $e^{\mathbf{X}} e^{\mathbf{Y}} = e^{\mathbf{Y}} e^{\mathbf{X}} e^{[\mathbf{X}, \mathbf{Y}]}$, valid when $[\mathbf{X}, \mathbf{Y}]$ commute with both $\mathbf{X}, \mathbf{Y}$, to $[e^{\mathbf{X}}, e^{\mathbf{Y}}] = e^{\mathbf{Y}} e^{\mathbf{X}} (e^{[\mathbf{X}, \mathbf{Y}]} - 1)$. As we just saw, the integers $M^{(j,i)}, N^{(j,i)} \neq 0$. For any fixed i and any fixed $j \neq i-1, i, i+1$, these two set of equations are incompatible for generic ω 's, as they require

$$\omega_{j,j+1} \frac{1}{2} \sqrt{\beta_{j,j+1} \beta_{i,i+1}} [\mathbf{P}_{j,j+1}, \mathbf{Q}_{i,i+1}] = M^{(j,i)} \omega_j = N^{(j,i)} \omega_{j+1}. \quad (C.13)$$

In other words, if $r \geq 4$, one can always choose, for instance, $i = 1$ and $j = 3$, or, $i = 2$, and $j = 4$, and render the set of constraints unsolvable. See Fig. 5. When $r = 3$, the constraints are solvable if and only if the three modular doubles are glued in a cyclic fashion, as in this special case, the constraints in the last two lines are absent.⁷ When $r = 2$, it is easy to find solutions as we presented in the main text. Finally, when $r = 3$ and the three modular doubles are glued cyclically, the constraints are solved by $m^{(i)} = n^{(i)} = -1, k^{(i)} = 0$. Consequently,

$$\beta_{i,i+1} = -\frac{\omega_{i+2}}{\omega_i + \omega_{i+1}}, \quad (C.14)$$

$$\begin{aligned} \omega_{i,i+1} \sqrt{\beta_{i,i+1}} \mathbf{P}_{i,i+1} &= \omega_{i+1,i+2} \sqrt{\beta_{i+1,i+2}} \mathbf{P}_{i+1,i+2}, \\ \frac{\mathbf{Q}_{i,i+1}}{\omega_{i,i+1} \sqrt{\beta_{i,i+1}}} &= \frac{\mathbf{Q}_{i+1,i+2}}{\omega_{i+1,i+2} \sqrt{\beta_{i+1,i+2}}}. \end{aligned} \quad (C.15)$$

⁷ In this cyclic case, we can formally set $r = 4$ and identify $i \sim i + 3$.

D. q -Virasoro Currents

In this appendix, we show how it is possible to write the q -Virasoro current $\mathbf{T}_{i,i+1}^{(1)}(x)$ in terms of the fundamental field $\Phi(X)$ introduced in Sect. 4, where $x = e^{2\pi i X/\omega_i}$. Let us start by rewriting the operator (2.4) as

$$\begin{aligned} \mathbf{Y}_{i,i+1}^{(1)}(x) &= \exp \left[\frac{1}{2} \sum_{\pm} \sum_{m \neq 0} \frac{\mathbf{a}_{i,i+1,m}^{(1)} e^{-m \frac{2\pi i}{\omega_i} (X+\omega/4)}}{e^{\pi i m \omega/2\omega_i} \pm i e^{-\pi i m \omega/2\omega_i}} + \frac{i\pi \omega_{i,i+1} \sqrt{\beta_{i,i+1}} \mathbf{P}_{i,i+1}}{\omega_i} + \frac{i\pi \omega}{\omega_i} \right]. \end{aligned} \quad (\text{D.1})$$

Now we introduce two fundamental fields $\Phi(X)$ and $\Phi(X)^\vee$ such that

$$\mathbf{a}_{i,i+1,m}^{(1)} = \sqrt{-i}(e^{\pi i m \omega/2\omega_i} + i e^{-\pi i m \omega/2\omega_i}) \mathbf{h}_{i,m} = \sqrt{i}(e^{\pi i m \omega/2\omega_i} - i e^{-\pi i m \omega/2\omega_i}) \mathbf{h}_{i,m}^\vee. \quad (\text{D.2})$$

We observe that when all the parameters are real, the $^\vee$ operation can be identified with Hermitian conjugation on $\mathbf{a}_{i,i+1,m}^{(1)}$. This double possibility arises from the splitting $C^{[m]}(p) = \prod_{\pm} (p^{m/4} \pm i p^{-m/4})$. This allows us to rewrite the above operator as

$$\begin{aligned} \mathbf{Y}_{i,i+1}^{(1)}(x) &= \exp \left[\sum_{m \neq 0} \left(\frac{\sqrt{-i}}{2} \mathbf{h}_{i,m} + \frac{\sqrt{i}}{2} \mathbf{h}_{i,m}^\vee \right) e^{-m \frac{2\pi i}{\omega_i} (X+\omega/4)} \right. \\ &\quad \left. + \frac{i\pi \omega_{i,i+1} \sqrt{\beta_{i,i+1}} \mathbf{P}_{i,i+1}}{\omega_i} + \frac{i\pi \omega}{\omega_i} \right], \end{aligned} \quad (\text{D.3})$$

or

$$\begin{aligned} \mathbf{Y}_{i,i+1}^{(1)}(x) &= \exp \left[\frac{\sqrt{-i}}{2} T_{\omega_i/2} d_{\omega_j} d_{\omega_k} \Phi(X + \omega/4) + \frac{\sqrt{i}}{2} T_{\omega_i/2} d_{\omega_j} d_{\omega_k} \Phi(X + \omega/4)^\vee + \frac{i\pi \omega}{\omega_i} \right], \end{aligned} \quad (\text{D.4})$$

where we also assumed $\mathbf{C}_2^\vee = -\mathbf{C}_2$.

References

1. Drummond, J.M., Henn, J.M., Plefka, J.: Yangian symmetry of scattering amplitudes in $N = 4$ super Yang–Mills theory. JHEP **05**, 046 [arXiv:0902.2987](#) [hep-th] (2009)
2. Gaiotto, D.: $N = 2$ dualities. JHEP **08**, 034 [arXiv:0904.2715](#) [hep-th] (2012)
3. Alday, L.F., Gaiotto, D., Tachikawa, Y.: Liouville correlation functions from four-dimensional gauge theories. Lett. Math. Phys. **91**, 167–197 [arXiv:0906.3219](#) [hep-th] (2010)
4. Beisert, N. et al.: Review of AdS/CFT integrability: an overview. Lett. Math. Phys. **99**, 3–32 [arXiv:1012.3982](#) [hep-th] (2012)
5. Pestun, V.: Localization of gauge theory on a four-sphere and supersymmetric Wilson loops. Commun. Math. Phys. **313**, 71–129 [arXiv:0712.2824](#) [hep-th] (2012)
6. Nekrasov, N.: BPS/CFT correspondence: non-perturbative Dyson–Schwinger equations and qq-characters. JHEP **03**, 181 [arXiv:1512.05388](#) [hep-th] (2016)
7. Nekrasov, N., Pestun, V.: Seiberg–Witten geometry of four dimensional $N = 2$ quiver gauge theories [arXiv:1211.2240](#) [hep-th]

8. Nekrasov, N., Pestun, V., Shatashvili, S.: Quantum geometry and quiver gauge theories [arXiv:1312.6689](#) [hep-th]
9. Lossev, A., Nekrasov, N., Shatashvili, S.L.: Testing Seiberg–Witten solution. In: Strings, Branes and Dualities. Proceedings, NATO Advanced Study Institute, Cargese, France, May 26–June 14, 1997, pp. 359–372 [arXiv:hep-th/9801061](#) [hep-th] (1997)
10. Moore, G.W., Nekrasov, N., Shatashvili, S.: Integrating over Higgs branches. Commun. Math. Phys. **209**, 97–121 [arXiv:hep-th/9712241](#) [hep-th] (2000)
11. Losev, A., Nekrasov, N., Shatashvili, S.L.: Issues in topological gauge theory. Nucl. Phys. B **534**, 549–611 [arXiv:hep-th/9711108](#) [hep-th] (1998)
12. Moore, G.W., Nekrasov, N., Shatashvili, S.: D particle bound states and generalized instantons. Commun. Math. Phys. **209**, 77–95 [arXiv:hep-th/9803265](#) [hep-th] (2000)
13. Nekrasov, N.A.: Seiberg–Witten prepotential from instanton counting. Adv. Theor. Math. Phys. **7**(5), 831–864 [arXiv:hep-th/0206161](#) [hep-th] (2003)
14. Nekrasov, N., Okounkov, A.: Seiberg–Witten theory and random partitions. Prog. Math. **244**, 525–596 [arXiv:hep-th/0306238](#) [hep-th] (2006)
15. Awata, H., Yamada, Y.: Five-dimensional AGT conjecture and the deformed Virasoro algebra. JHEP **01**, 125 [arXiv:0910.4431](#) [hep-th] (2010)
16. Awata, H., Yamada, Y.: Five-dimensional AGT relation and the deformed beta-ensemble. Prog. Theor. Phys. **124**, 227–262 [arXiv:1004.5122](#) [hep-th] (2010)
17. Awata, H., Feigin, B., Hoshino, A., Kanai, M., Shiraishi, J., Yanagida, S.: Notes on Ding–Iohara algebra and AGT conjecture [arXiv:1106.4088](#) [math-ph]
18. Mironov, A., Morozov, A., Shakirov, S., Smirnov, A.: Proving AGT conjecture as HS duality: extension to five dimensions. Nucl. Phys. B **855**, 128–151 [arXiv:1105.0948](#) [hep-th] (2012)
19. Carlsson, E., Nekrasov, N., Okounkov, A.: Five dimensional gauge theories and vertex operators [arXiv:1308.2465](#) [math.RT]
20. Aganagic, M., Haouzi, N., Kozcz, C., Shakirov, S.: Gauge/Liouville triality [arXiv:1309.1687](#) [hep-th]
21. Aganagic, M., Haouzi, N., Shakirov, S.: A_n -Triality [arXiv:1403.3657](#) [hep-th]
22. Aganagic, M., Shakirov, S.: Gauge/Vortex duality and AGT. In: Teschner, J. (ed.) New Dualities of Supersymmetric Gauge Theories, pp. 419–448 [arXiv:1412.7132](#) [hep-th] (2016)
23. Kimura, T., Pestun, V.: Quiver W-algebras [arXiv:1512.08533](#) [hep-th]
24. Mironov, A., Morozov, A., Zenkevich, Y.: Ding–Iohara–Miki symmetry of network matrix models [arXiv:1603.05467](#) [hep-th]
25. Bourguine, J.-E., Fukuda, M., Matsuo, Y., Zhang, H., Zhu, R.-D.: Coherent states in quantum $\mathcal{W}_{1+\infty}$ algebra and qq-character for 5d Super Yang–Mills. PTEP **2016**(12), 123B05 [arXiv:1606.08020](#) [hep-th] (2016)
26. Bourguine, J.-E., Fukuda, M., Harada, K., Matsuo, Y., Zhu, R.-D.: (p,q)-Webs of DIM representations, 5d $N = 1$ instanton partition functions and qq-characters [arXiv:1703.10759](#) [hep-th]
27. Nieri, F., Pasquetti, S., Passerini, F.: 3d and 5d gauge theory partition functions as q -deformed CFT correlators. Lett. Math. Phys. **105**(1), 109–148 [arXiv:1303.2626](#) [hep-th] (2015)
28. Nieri, F., Pasquetti, S., Passerini, F., Torrielli, A.: 5D partition functions, q-*Virasoro* systems and integrable spin-chains. JHEP **12**, 040 [arXiv:1312.1294](#) [hep-th] (2014)
29. Bao, L., Mitev, V., Pomoni, E., Taki, M., Yagi, F.: Non-Lagrangian theories from Brane junctions. JHEP **01**, 175 [arXiv:1310.3841](#) [hep-th] (2014)
30. Mitev, V., Pomoni, E.: Toda 3-point functions from topological strings. JHEP **06**, 049 [arXiv:1409.6313](#) [hep-th] (2015)
31. Cordova, C., Jafferis, D.L.: Toda theory from six dimensions [arXiv:1605.03997](#) [hep-th]
32. Aganagic, M., Haouzi, N.: ADE little string theory on a Riemann surface (and triality) [arXiv:1506.04183](#) [hep-th]
33. Lockhart, G., Vafa, C.: Superconformal partition functions and non-perturbative topological strings [arXiv:1210.5909](#) [hep-th]
34. Källén, J., Zabzine, M.: Twisted supersymmetric 5D Yang–Mills theory and contact geometry. JHEP **05**, 125 [arXiv:1202.1956](#) [hep-th] (2012)
35. Källén, J., Qiu, J., Zabzine, M.: The perturbative partition function of supersymmetric 5D Yang–Mills theory with matter on the five-sphere. JHEP **08**, 157 [arXiv:1206.6008](#) [hep-th] (2012)
36. Qiu, J., Zabzine, M.: Factorization of 5D super Yang–Mills theory on $Y^{p,q}$ spaces. Phys. Rev. D **89**(6), 065040 [arXiv:1312.3475](#) [hep-th] (2014)
37. Kim, H.-C., Kim, J., Kim, S.: Instantons on the 5-sphere and M5-branes [arXiv:1211.0144](#) [hep-th]
38. Kim, H.-C., Kim, S.: M5-Branes from gauge theories on the 5-sphere. JHEP **05**, 144 [arXiv:1206.6339](#) [hep-th] (2013)
39. Qiu, J., Tizzano, L., Winding, J., Zabzine, M.: Gluing Nekrasov partition functions. Commun. Math. Phys. **337**(2), 785–816 [arXiv:1403.2945](#) [hep-th] (2015)

40. Pan, Y.: 5d Higgs branch localization, Seiberg–Witten equations and contact geometry. *JHEP* **01**, 145 [arXiv:1406.5236](#) [hep-th] (2015)
41. Pasquetti, S.: Factorisation of $N = 2$ theories on the squashed 3-sphere. *JHEP* **04**, 120 [arXiv:1111.6905](#) [hep-th] (2012)
42. Beem, C., Dimofte, T., Pasquetti, S.: Holomorphic blocks in three dimensions. *JHEP* **12**, 177 [arXiv:1211.1986](#) [hep-th] (2014)
43. Bullimore, M., Kim, H.-C., Koroteev, P.: Defects and quantum Seiberg–Witten geometry. *JHEP* **05**, 095 [arXiv:1412.6081](#) [hep-th] (2015)
44. Nedelin, A., Nieri, F., Zabzine, M.: q -Virasoro modular double and 3d partition functions. *Commun. Math. Phys.* **353**(3), 1059–1102 [arXiv:1605.07029](#) [hep-th] (2017)
45. Faddeev, L.D.: Modular double of quantum group. *Math. Phys. Stud.* **21**, 149–156 [arXiv:math/9912078](#) [math-qa] (2000)
46. Faddeev, L.D.: Modular double of quantum group. *Math. Phys. Stud.* **21**, 149 [arXiv:math/9912078](#) [math-qa] (1999)
47. Ponsot, B., Tschner, J.: Liouville bootstrap via harmonic analysis on a noncompact quantum group [arXiv:hep-th/9911110](#) [hep-th]
48. Ponsot, B., Tschner, J.: Clebsch–Gordan and Racah–Wigner coefficients for a continuous series of representations of $U(q)(sl(2, R))$. *Commun. Math. Phys.* **224**, 613–655 [arXiv:math/0007097](#) [math-qa] (2001)
49. Hatsuda, Y., Marino, M., Moriyama, S., Okuyama, K.: Non-perturbative effects and the refined topological string. *JHEP* **09**, 168 [arXiv:1306.1734](#) [hep-th] (2014)
50. Aganagic, M., Klemm, A., Marino, M., Vafa, C.: The topological vertex. *Commun. Math. Phys.* **254**, 425–478 [arXiv:hep-th/0305132](#) [hep-th] (2005)
51. Iqbal, A., Kozcaz, C., Vafa, C.: The refined topological vertex. *JHEP* **10**, 069 [arXiv:hep-th/0701156](#) [hep-th] (2009)
52. Awata, H., Kanno, H.: Refined BPS state counting from Nekrasov’s formula and Macdonald functions. *Int. J. Mod. Phys. A* **24**, 2253–2306 [arXiv:0805.0191](#) [hep-th] (2009)
53. Shiraishi, J., Kubo, H., Awata, H., Odake, S.: A quantum deformation of the Virasoro algebra and the Macdonald symmetric functions. *Lett. Math. Phys.* **38**, 33–51 [arXiv:q-alg/9507034](#) [q-alg] (1996)
54. Awata, H., Kubo, H., Odake, S., Shiraishi, J.: Quantum $W(N)$ algebras and Macdonald polynomials. *Commun. Math. Phys.* **179**, 401–416 [arXiv:q-alg/9508011](#) [q-alg] (1996)
55. Frenkel, E., Reshetikhin, N.: Deformations of W -algebras associated to simple Lie algebras [arXiv:q-alg/9708006](#) [q-alg]
56. Aganagic, M., Frenkel, E., Okounkov, A.: Quantum q -Langlands correspondence [arXiv:1701.03146](#) [hep-th]
57. Narukawa, A.: The modular properties and the integral representations of the multiple elliptic gamma functions [arXiv:math/0306164](#) [math.QA].
58. Morozov, A.: Integrability and matrix models. *Phys. Usp.* **37**, 1–55 [arXiv:hep-th/9303139](#) [hep-th] (1994)
59. Nieri, F.: An elliptic Virasoro symmetry in 6d. *Lett. Math. Phys.* **107**(11), 2147–2187 [arXiv:1511.00574](#) [hep-th] (2017)
60. Iqbal, A., Kozcaz, C., Yau, S.-T.: Elliptic Virasoro conformal blocks [arXiv:1511.00458](#) [hep-th]
61. Kimura, T., Pestun, V.: Quiver elliptic W -algebras [arXiv:1608.04651](#) [hep-th]
62. Tan, M.-C.: An M-theoretic derivation of a 5d and 6d AGT correspondence, and relativistic and elliptized integrable systems. *JHEP* **12**, 031 [arXiv:1309.4775](#) [hep-th] (2013)
63. Tan, M.-C.: Higher AGT correspondences, W -algebras, and higher quantum geometric langlands duality from M-theory. *Adv. Theor. Math. Phys.* **22**, 429–507 [arXiv:1607.08330](#) [hep-th] (2018)
64. Lodin, R., Nieri, F., Zabzine, M.: Elliptic modular double and 4d partition functions. *J. Phys. A* **51**(4), 045402 [arXiv:1703.04614](#) [hep-th] (2018)
65. Grosse-Knetter, C.: Effective Lagrangians with higher derivatives and equations of motion. *Phys. Rev. D* **49**, 6709–6719 [arXiv:hep-ph/9306321](#) [hep-ph] (1994)
66. Nieri, F., Pan, Y., Zabzine, M.: Bootstrapping the S^5 partition function. In: 20th International Seminar on High Energy Physics (Quarks 2018) Valday (Russia) [arXiv:1807.11900](#) [hep-th] (2018)
67. Pan, Y., Peelaers, W.: Intersecting surface defects and instanton partition functions [arXiv:1612.04839](#) [hep-th]
68. Gomis, J., Le Floch, B., Pan, Y., Peelaers, W.: Intersecting surface defects and two-dimensional CFT [arXiv:1610.03501](#) [hep-th]
69. Schmude, J.: Localisation on Sasaki–Einstein manifolds from holomorphic functions on the cone. *JHEP* **01**, 119 [arXiv:1401.3266](#) [hep-th] (2015)
70. Iqbal, A., Shabbir, K.: Elliptic CY3folds and non-perturbative modular transformation. *Eur. Phys. J. C* **76**(3), 148 [arXiv:1510.03332](#) [hep-th] (2016)
71. Qiu, J., Tizzano, L., Winding, J., Zabzine, M.: Modular properties of full 5D SYM partition function. *JHEP* **03**, 193 [arXiv:1511.06304](#) [hep-th] (2016)

- 72. Benvenuti, S., Bonelli, G., Ronzani, M., Tanzini, A.: Symmetry enhancements via 5d instantons, $q\mathcal{W}$ -algebras and $(1, 0)$ superconformal index. JHEP **09**, 053 [arXiv:1606.03036](#) [hep-th] (2016)
- 73. Witten, E.: Some comments on string dynamics [arXiv:hep-th/9507121](#) [hep-th]
- 74. Seiberg, N.: New theories in six-dimensions and matrix description of M theory on T^5 and $T^5/Z(2)$. Phys. Lett. B **408**, 98–104 [arXiv:hep-th/9705221](#) [hep-th] (1997)
- 75. Losev, A., Moore, G.W., Shatashvili, S.L.: M & m's. Nucl. Phys. B **522**, 105–124 [arXiv:hep-th/9707250](#) [hep-th] (1998)

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